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Eidgenössische Technische Hochschule Zürich
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Two-Phase Reservoir Flow in porous Media

Bachelor Thesis

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Abstract

The evolution equations of incompressible, two-phase flow in porous media are considered. An introduction to conservation laws and finite volume methods, in order to solve them, are given. And an IMPES-method, for approximating the equations of two-phase flow is studied, which is demonstrated by a numerical experiment.

Contents

Chapter 1

Introduction

In this thesis, a numerical IMPES-method is implemented. It is used for numerically solving a set of partial differential equation, which occurs in the oil production process. The method is based on the approach introduced in the semester-project [8]. This method is a finite volume method, like many other numerical schemes, that are commonly used for solving partial differential equations.

First of all the model, which is used in the reservoir simulation is introduced by investigating the physical underlying laws in chapter 2 . In the following, the case of injecting water in an oil reservoir, two-phase flow, is considers. Second, mathematical properties which apply to the equations of the oil-recovery process, the properties of conservation laws, are studied in chapter 3. This mathematical approach is also significant in order to understand numerical finite volume methods, that are introduced in chapter 4. And finally, an IMPES-method is demonstrated in chapter 5, at the example of a simple oil-production model. The implementation is made in MATLAB.

Chapter 2

Modelling Flow in Porous Media

This chapter gives an overview about the oil recovery process, physical underlying laws, mass accumulation and Darcy's law. They are applied first to the single-phase flow model in section 2.2 in order do maintain the mathematical equations, which cover the two-phase flow problem in section 2.3. It follows the formulation of implicit pressure and explicit saturation in section 2.3.1 and the mathematical statement of the problem in section 2.3.2.

2.1 Reservoir Simulation

The following paragraph is based on [9]. In mathematics, porous media are considered as the closure of an Euclidean space $\mathbb{R}^d$ for $d = 1, 2$ or 3. A *petroleum reservoir* is a porous medium which contains hydrocarbons, mostly known as oil and gas. The goal of reservoir simulation is predicting future performances and the optimization of the recovery of hydrocarbons. Furthermore, a reservoir is usually considered as heterogeneous i.e. properties depend on the space location. On the one hand the *phase* specifies matter with homogeneous chemical composition and physical state. And on the other hand the *component* is a unique chemical species. The components constitute the phases.

There are three stages of oil recovery. During *primary recovery* the oil produces itself by natural decompression. Because of the high pressure in the oil reservoir, no effort like pumping is needed. 25% to 30% of the hydrocarbons can be produced this way.
The second stage, called *secondary recovery* or *water flooding*, enables to produce oil such that 50% still remains in the reservoir. In this stage water is injected to the *injection wells*, while oil is produced at the *production wells*. The purpose of the water is to increase the reservoir pressure and flooding out some of the oil. If the reservoir pressure is above the *bubble point pressure* of the oil phase, there are only the water and the oil phase. This process can

be described as *two phase flow*. If the pressure decreases below the bubble point pressure the oil is split into two phases in thermodynamic equilibrium, the liquid and the gaseous phase. This stage of oil production is termed by *black oil flow*. In contrast to the two-phase flow, a mass exchange between the gas and the oil has to be taken into account, while mass conservation applies to the two-phase, oil and water phase, model.
In order to produce the remaining hydrocarbon from the reservoir *tertiary* or *enhanced recovery* has to be considered. These methods use complex chemical and thermal effects. For example chemicals are injected or the temperature of the reservoir is increased.
In this thesis I will restrict myself to a mathematical model of the secondary oil production stage.

2.2 Single-Phase Flow

In this section the *equation of mass conservation* will be deduced, using *Darcy's law* an the *accumulation of mass*. It will be proceeded as in [9] in order to obtain those equations.

It is assumed that the mass fluxes, due to dispersion on diffusion, are negligible and furthermore no mass of fluid can cross a fluid-solid interface. The spatial variable is given by $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and the temporal variable is given by $t \in \mathbb{R}_{\geq 0}$. ρ is the *density* of the fluid-phase per unit volume and $\mathbf{v} = (v_1, v_2, v_3)$ is the *Darcy velocity*.
Consider a rectangular cube with length in the x_i-coordinate Δx_i for $i = 1, 2, 3$ and centre (x_1, x_2, x_3) (see figure 2.1).

The *mass flux* in the i-direction, which is the mass flow per unit area per unit time, is given by

$$\rho v_i$$

and the mass inflow across the left-hand side surface at $x_1 - \frac{\Delta x_1}{2}$ writes as

$$(\rho v_1)_{x_1 - \frac{\Delta x_1}{2}, x_2, x_3} \Delta x_2 \Delta x_3.$$

The mass outflow, at the opposite site, in the point $x_1 + \frac{\Delta x_1}{2}$ is

$$(\rho v_1)_{x_1 + \frac{\Delta x_1}{2}, x_2, x_3} \Delta x_2 \Delta x_3.$$

Similarly the mass inflow and outflow across the surfaces in x_2- and x_3-direction are

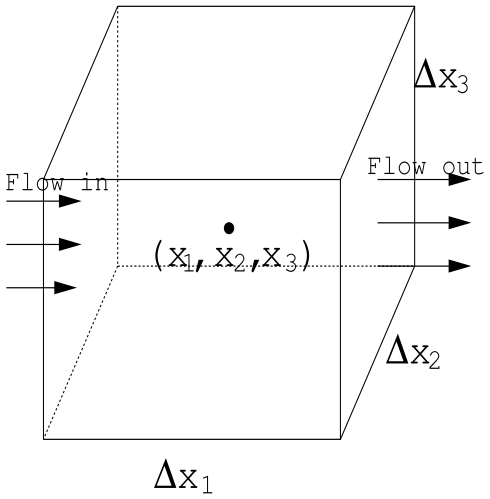


Figure 2.1: A differential Volume. Source : [9]

$$(\rho v_2)_{x_1,x_2-\frac{\Delta x_2}{2},x_3}\Delta x_1\Delta x_3,\ (\rho v_2)_{x_1,x_2+\frac{\Delta x_2}{2},x_3}\Delta x_1\Delta x_3,$$

$$(\rho v_3)_{x_1,x_2,x_3-\frac{\Delta x_3}{2}}\Delta x_1\Delta x_2,\ (\rho v_2)_{x_1,x_2,x_3+\frac{\Delta x_3}{2}}\Delta x_1\Delta x_2.$$

φ is the *porosity* of the medium, which is assumed as constant in this scope. It is the fraction of representative elementary volume available for fluid. The *mass accumulation* due to compressibility writes as

$$\partial_t(\varphi\rho)\Delta x_1\Delta x_2\Delta x_3. \tag{2.1}$$

Mass decay due to a sink of strength q (mass per unit volume per unit time) is

$$-q\Delta x_1\Delta x_2\Delta x_3.$$

Within the cube the difference of the mass inflow and outflow equals the mass accumulation hence the following is obtained:

$$\begin{aligned}\big[(\rho v_1)_{x_1-\frac{\Delta x_1}{2},x_2,x_3} - (\rho v_1)_{x_1+\frac{\Delta x_1}{2},x_2,x_3}\big]\Delta x_2\Delta x_3 \\ + \big[(\rho v_2)_{x_1,x_2-\frac{\Delta x_2}{2},x_3} - (\rho v_2)_{x_1,x_2+\frac{\Delta x_2}{2},x_3}\big]\Delta x_1\Delta x_3 \\ + \big[(\rho v_3)_{x_1,x_2,x_3-\frac{\Delta x_3}{2}} - (\rho v_2)_{x_1,x_2,x_3+\frac{\Delta x_3}{2}}\big]\Delta x_1\Delta x_2 \\ = \big[\partial_t(\varphi\rho) - q\big]\Delta x_1\Delta x_2\Delta x_3.\end{aligned} \tag{2.2}$$

Dividing (2.2) by $\Delta x_1 \Delta x_2 \Delta x_3$ and taking the limit $\Delta x_i \to 0$ for $i = 1, 2$ and 3 yields the *equation of mass conservation*

$$\partial_t(\varphi\rho) = -\mathrm{div}(\rho\mathbf{v}) + q. \tag{2.3}$$

Furthermore, the *momentum conservation* can be stated as Darcy's Law (Darcy, 1856) by the relationship of fluid velocity and the *pressure head gradient*. It states:

$$\mathbf{v} = -\frac{1}{\mu}\mathbf{k}(\nabla p - \rho g_\gamma \nabla z) \tag{2.4}$$

where $\mathbf{k}$ equals the *absolute permeability tensor* of the porous medium, μ is the fluid viscosity, p is the pressure, g_γ is the magnitude of gravitational acceleration and z the depth.

2.3 Two-Phase Flow

Now the equations of two-phase flow in a porous medium should be established. The main goal of this section is to deduce the problem in terms of phase pressure and saturation. It is assumed that the fluids are *immiscible* and that there is no mass transfer between the phases, like in [9]. The first phase, the *wetting phase*, wets the pores more than the other phase ([9]) and is abbreviated by w. Most of the time the wetting phase is the water phase. The other phase, termed *nonwetting phase* ([9]), is indicated by o which is a short-cut for oil.

The saturation of a fluid phase is the fraction of the void volume of a porous medium filled by the phase ([9]). Since the two fluids fill the phase it follows that

$$s_w + s_o = 1, \tag{2.5}$$

where s_w and s_o are the saturations of the wetting and nonwetting phase ([9]).

The *capillary pressure* is given as

$$p_c = p_w - p_o. \tag{2.6}$$

according to ([2]). As a consequence of interfacial tension between the two phases, there exists a discontinuity of pressure across the surface of the

boarders of two immiscible fluids. The capillary pressure refers to this discontinuity ([9]). The capillary pressure is a function of the wetting phase saturation s_w. A typical curve of this function is given in figure 2.2.

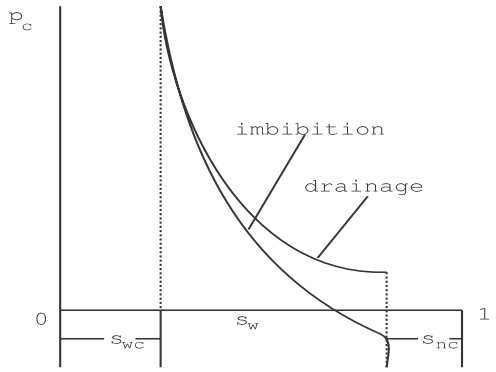


Figure 2.2: Typical capillary pressure curve. Source : [9].

The capillary pressure also depends on the direction of saturation change. This phenomenon is known as *hysteresis*. If the saturation is increasing the procedure is called *imbibition* and in the decreasing case it is called *drainage*. During imbibition the value s_{wc} of s_w, at which water starts to flow, is called *critical saturation*. The value s_{nc} at which oil starts to cease to flow is named *residual saturation*.

Similarly as for the one-phase flow equations (2.1), (2.3) and (2.4) the mass accumulation, the mass conservation equation and Darcy's law can be extended to the two-phase case. The mass accumulation in a differential volume per unit time writes as

$$\partial_t(\varphi\rho_\alpha s_\alpha)\Delta x_1\Delta x_2\Delta x_3 \quad \text{for} \quad \alpha = w, o$$

([9]). And because of the assumption of no mass transfer between the phases, the mass conservation equation is

$$\partial_t(\varphi\rho_\alpha s_\alpha) = -\mathrm{div}(\rho_\alpha \mathbf{v}_\alpha) + q_\alpha \quad \text{for} \quad \alpha = w, o, \tag{2.7}$$

where ρ_α, $\mathbf{v}_\alpha$ and q_α are the density, Darcy velocity and mass flow rate of each phase ([9]). Darcy's law for each phase is (see [2]):

$$\mathbf{v}_\alpha = -\lambda_\alpha(\nabla p_\alpha - \rho_\alpha g_\gamma \nabla z) \quad \text{for} \quad \alpha = w, o, \tag{2.8}$$

where $\lambda_\alpha = \lambda_\alpha(s_\alpha)$ is the *phase mobility*.

2.3.1 Formulation of Pressure and Saturation Equation

In order to apply the IMPES-method in subsection 4.2 the equations (2.5), (2.6), (2.7) and (2.8) have to be split into a saturation equation and a pressure equation.
It is assumed that the densities are constant because incompressible flow is considered.

The *total mobility* is defined as in [2] by

$$\lambda_T = \lambda_w + \lambda_o, \tag{2.9}$$

and the *total velocity* is

$$\mathbf{v} = \mathbf{v}_w + \mathbf{v}_o \tag{2.10}$$

([2]). Inserting (2.8), (2.6) and (2.9) into the above equation gives

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}_w + \mathbf{v}_o \\
&= -\lambda_w \nabla p_w + \lambda_w \rho_w g_\gamma \nabla z \; - \lambda_o \nabla p_o + \lambda_o \rho_o g_\gamma \nabla z \\
&= -\lambda_w \nabla p_w - \lambda_o \nabla p_w + \lambda_o \nabla p_c + (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla z \\
&= -\lambda_T \nabla p_w + \lambda_o \nabla p_c + (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla z.
\end{aligned} \tag{2.11}$$

Substituting (2.8) by (2.11) and inserting (2.9) it is obtained, that

$$\begin{aligned}
\mathbf{v}_w &= -\lambda_w \nabla p_w + \lambda_w \rho_w g_\gamma \nabla z \\
&= \frac{\lambda_w}{\lambda_T}\big(\mathbf{v} - \lambda_o \nabla p_c - (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla z\big) + \lambda_w \rho_w g_\gamma \nabla z \\
&= \frac{\lambda_w}{\lambda_T}\big(\mathbf{v} - \lambda_o \nabla p_c\big) + \frac{\lambda_w}{\lambda_T}\big(- \lambda_w \rho_w - \lambda_o \rho_o + \lambda_T \rho_w\big) g_\gamma \nabla z \\
&= \frac{\lambda_w}{\lambda_T}\big(\mathbf{v} - \lambda_o \nabla p_c\big) + \frac{\lambda_w}{\lambda_T}\big(- \lambda_w \rho_w - \lambda_o \rho_o + (\lambda_w + \lambda_o) \rho_w\big) g_\gamma \nabla z \\
&= \frac{\lambda_w}{\lambda_T}\big(\mathbf{v} - \lambda_o \nabla p_c\big) + \frac{\lambda_w \lambda_o}{\lambda_T}\big(\rho_w - \rho_o\big) g_\gamma \nabla z.
\end{aligned} \tag{2.12}$$

Furthermore, the *fractional flow function* $f(s)$ and the *gravity function* $g(s)$ are introduced as in [2]. They write as

$$f(s) = \frac{\lambda_w(s)}{\lambda_T(s)} = \frac{\lambda_w(s)}{\lambda_w(s) + \lambda_o(s)}, \tag{2.13a}$$

$$g(s) = \frac{\lambda_w(s)\lambda_o(s)}{\lambda_T(s)}(\rho_w - \rho_o) g_\gamma. \tag{2.13b}$$

Inserting the two above equation into (2.12) gives

$$\mathbf{v}_w = f(s)(\mathbf{v} - \lambda_o \nabla p_c) + g(s)\nabla z,$$

like in [2]. Using (2.7) and the assumption that the densities ρ_α for $\alpha = w, o$ are constant it can be written, that

$$\partial_t\big(\varphi(s_w + s_0)\big) + \operatorname{div}(\mathbf{v}_w + \mathbf{v}_o) = \frac{q_w}{\rho_w} + \frac{q_o}{\rho_o}. \tag{2.14}$$

Applying (2.5) and (2.10) forms (2.14) into one of the governing equations for the transport of components in an *incompressible fluid* ([9]). This equation is given by

$$\operatorname{div}_x(\mathbf{v}) = \frac{q_w}{\rho_w} + \frac{q_o}{\rho_o}.$$

Finally, under the assumption $\varphi = 1$, using the mass conservation equation (2.7) and (2.14), the *evolution equations for two-phase flow in a porous medium* similarly as in [2] are

$$\partial_t s + \operatorname{div}\big(f(s)(\mathbf{v} - \lambda_o \nabla p_c) + g(s)\nabla z\big) = \frac{q_w}{\rho_w}, \tag{2.15a}$$

$$\operatorname{div}(\mathbf{v}) = \frac{q_w}{\rho_w} + \frac{q_o}{\rho_o}, \tag{2.15b}$$

$$\mathbf{v} = \lambda_o \nabla p_c - \lambda_T \nabla p + (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla z, \tag{2.15c}$$

for $s = s_w$ and $p = p_w$. (2.15a) is called *saturation equation*. The pressure behaviour of the problem is governed by (2.15b) and (2.15c).

2.3.2 Statement of the Problem

In order to solve the equations of two-phase flow (2.15) some suitable boundary conditions, initial condition and domain of definition have to be chosen.

For the phase mobilities the assumptions are made as in [2]:

$$\lambda_w : [0,1] \mapsto \mathbb{R} \quad \text{is monotone increasing and} \quad \lambda_w(0) = 0, \tag{2.16a}$$

$$\lambda_o : [0,1] \mapsto \mathbb{R} \quad \text{is monotone decresaing and} \quad \lambda_o(1) = 0. \tag{2.16b}$$

And the total mobility is assumed to be strictly positive which writes as

$$\lambda_T \geq \lambda_* > 0 \quad \text{for some} \quad \lambda_*. \tag{2.16c}$$

Together with (2.16), the whole problem can be stated as:

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ or 3. Let ν be the outwards pointing unit normal vector on $\delta\Omega$.

Evolution equations of two-phase flow in a porous medium

$$\partial_t s + \operatorname{div}\big(f(s)(\mathbf{v} - \lambda_o \nabla p_c) + g(s)\nabla z\big) = \frac{q_w}{\rho_w} \quad \text{for} \quad t > 0, \mathbf{x} \in \Omega, \tag{2.17a}$$

$$\operatorname{div}(\mathbf{v}) = \frac{q_w}{\rho_w} + \frac{q_o}{\rho_o} \quad \text{for} \quad t > 0, \mathbf{x} \in \Omega, \tag{2.17b}$$

$$\mathbf{v} = \lambda_o \nabla p_c - \lambda_T \nabla p + (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla z \quad \text{for} \quad t > 0, \mathbf{x} \in \Omega, \tag{2.17c}$$

$$\big(f(s)(\mathbf{v} - \lambda_o \nabla p_c) + g(s)\nabla z\big) \cdot \nu = h(t, \mathbf{x}) \quad \text{for} \quad t > 0, \mathbf{x} \in \delta\Omega, \tag{2.17d}$$

$$\partial_\nu p(t, \mathbf{x}) = \pi(t, \mathbf{x}) \quad \text{for} \quad t > 0, \mathbf{x} \in \delta\Omega, \tag{2.17e}$$

$$\int_\Omega p(t, \mathbf{x}) d\mathbf{x} = 0 \quad \text{for} \quad t > 0, \tag{2.17f}$$

$$s(0, \mathbf{x}) = s_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \tag{2.17g}$$

And due to simplicity (see section 3.5) the capillarity pressure is neglected:

Evolution equations of two-phase flow in a porous medium with vanishing capillary pressure

$$\partial_t s + \operatorname{div}\big(f(s)\mathbf{v} + g(s)\nabla z\big) = \frac{q_w}{\rho_w} \quad \text{for} \quad t > 0, \mathbf{x} \in \Omega, \tag{2.18a}$$

$$\operatorname{div}(\mathbf{v}) = \frac{q_w}{\rho_w} + \frac{q_o}{\rho_o} \quad \text{for} \quad t > 0, \mathbf{x} \in \Omega, \tag{2.18b}$$

$$\mathbf{v} = -\lambda_T \nabla p + (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla z \quad \text{for} \quad t > 0, \mathbf{x} \in \Omega, \tag{2.18c}$$

$$\big(f(s)\mathbf{v} + g(s)\nabla z\big) \cdot \nu = h(t, \mathbf{x}) \quad \text{for} \quad t > 0, \mathbf{x} \in \delta\Omega, \tag{2.18d}$$

$$\partial_\nu p(t, \mathbf{x}) = \pi(t, \mathbf{x}) \quad \text{for} \quad t > 0, \mathbf{x} \in \delta\Omega, \tag{2.18e}$$

$$\int_\Omega p(t, \mathbf{x}) d\mathbf{x} = 0 \quad \text{for} \quad t > 0, \tag{2.18f}$$

$$s(0, \mathbf{x}) = s_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega \tag{2.18g}$$

Chapter 3

Conservation Laws

In this chapter mathematical concepts are given which are needed to understand and treat the two-phase flow problem equations in subsection 2.3.2. General conservation laws are introduced, but it is focused on scalar conservation laws from section 3.1 on. The presentation is guided on the approach in [7, chp. 1, 2 and 3].

As in the previous chapter it is assumed that $\Omega \subset \mathbb{R}^d$ and the quantity of interest $\mathbf{U}(\mathbf{x})$ is defined for all $\mathbf{x} \in \Omega$. The following equation is called *conservation law* ([7]):

$$\mathbf{U}_t + \operatorname{div}(\mathbf{F}) = 0 \quad \forall (\mathbf{x}, t) \in (\Omega, \mathbb{R}_{\geq 0}). \tag{3.1}$$

The underlying, for example physical, problems which can be described by this law can in general be stated as:

> "The time rate of change of $\mathbf{U}$ in any fixed sub-domain $\omega \subset \Omega$ is equal to the total amount of $\mathbf{U}$ produced or destroyed inside ω and the flux of $\mathbf{U}$ across the boundary $\partial\omega$." ([7])

A mathematical formulation of a problem like the above writes as,

$$\frac{d}{dt}\int_{\omega} \mathbf{U} d\mathbf{x} = -\int_{\partial\omega} \mathbf{F} \cdot \nu d\sigma(\mathbf{x}) + \int_{\omega} \mathbf{S} d\mathbf{x}, \tag{3.2}$$

where ν is the outwards pointing normal vector, $d\sigma(\mathbf{x})$ is the surface measure, $\mathbf{F}$ is the flux and $\mathbf{S}$ is the source ([7]). An illustration can be found in figure 3.1.

Using the divergence theorem by Gauss and the fact that above equation holds for every infinitesimal ω contained in Ω, the differential equation, named *balance law*, can be deduced from (3.2). The balance law is

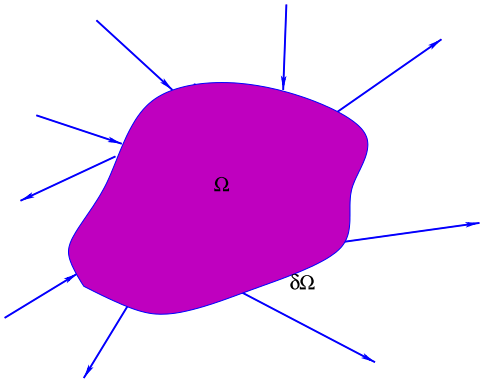


Figure 3.1: An illustration of conservation in a domain with the change being determined by the net flux. Source: [7]

$$\mathbf{U}_t + \text{div}(\mathbf{F}) = \mathbf{S} \quad \forall(\mathbf{x}, t) \in (\Omega, \mathbb{R}_{\geq 0})$$

([7]). If the source term does vanish, which is the case in the conservation law situation, then the conservation law of the form (3.1) is deduced.

3.1 Scalar Conservation Laws

Now the case is studied when $\mathbf{U}$ and $\mathbf{F}$ are scalar functions denoted by U and $f(U)$ respectively, since it is assumed that f is a function of U. Hence (3.1) takes the form of a *scalar conservation law* ([7]):

$$U_t + f(U)_x = 0. \tag{3.3}$$

3.1.1 Burger's Equation

As the "prototype" ([7]) for a scalar conservation law *Burger's equation* is considered. Burger's equation writes as

$$U_t + \left(\frac{U^2}{2}\right)_x = 0. \tag{3.4a}$$

Together with the initial data,

$$U_0(x) = \begin{cases} U_l & \text{if} \quad x < 0 \\ U_r & \text{if} \quad x > 0, \end{cases} \tag{3.4b}$$

(3.4a) is a special initial value problem. Conservation laws (3.3) with initial data of the form (3.4b) are called *Riemann problems* ([7]). A Riemann problem, which can be solved by the *method of characteristics*, has a discontinuous

solution in general. Even in the case of a initial value problem with smooth initial data, the solution $U(x,t)$ of (3.4a) will have a discontinuity at a certain time t.

In the next section discontinuities of solutions to Burger's equation are discuss more precisely.

3.2 Method of Characteristics applied to Burger's Equation

The application of the *method of characteristics* is useful in order to demonstrate not classically expected behaviour like non uniqueness or discontinuities of solutions to scalar conservation laws of the form (3.3).

The linear transport equation is considered. This equation writes as

$$U_t + a(x,t)U_x = 0 \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}. \tag{3.5a}$$

The linear transport equation is a scalar conservation law and can be deduced from the transport of a pollution in a fluid like a river or similar ([7]).

The initial value problem, (3.5a) together with the the initial data

$$U(x,0) = U_0(x) \quad \forall x \in \mathbb{R}, \tag{3.5b}$$

shall be solved. In order to do so, this partial differential equation (PDE) is reduced to an ordinary differential equation (ODE) using the *method of characteristics*. The ansatz is the assumption, that U(x,t) is constant at some curve x(t) in the space domain ([7]). This leads to

$$\begin{aligned} 0 &= \frac{d}{dt} U(x(t),t) \\ &= U_t(x(t),t) + U_x(x(t),t)x'(t) \end{aligned} \tag{3.6}$$

and

$$U(x(t),t) = U_0(x_0).$$

Comparing (3.6) to (3.5a), it is obtained, that the *characteristic curve* $x(t)$ is the solution to the ODE with initial data given by

$$x'(t) = a(x(t),t), \tag{3.7a}$$
$$x(0) = x_0. \tag{3.7b}$$

Through the theorem by Picard-Lindelöf, it is known, that a solution of (3.7) exist, if the right hand side of the ODE (3.7a) is Lipschitz-continuous in both arguments. Furthermore if the characteristic curves go through all (x,t) in $\mathbb{R} \times \mathbb{R}_{\geq 0}$, a solution $U(x,t)$ of (3.5) for all (x,t) in $\mathbb{R} \times \mathbb{R}_{\geq 0}$ might be found ([7]). This case is pictured in figure 3.2.

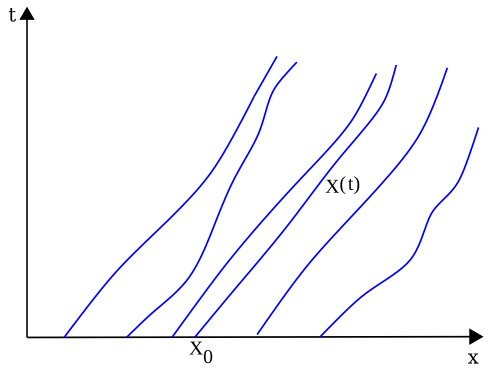


Figure 3.2: Characteristic curves x(t) for (3.5a). Adapted from [7]

The Riemann-problem (3.4) can be rewritten, applying the chain-rule to Burger's-equation, as

$$U_t + UU_x = 0, \tag{3.8a}$$

$$U_0(x) = \begin{cases} U_l & \text{if} \quad x < 0 \\ U_r & \text{if} \quad x > 0. \end{cases} \tag{3.8b}$$

Applying the method of characteristics and taking into account that the solution is constant along the characteristic curve $U(x(t),t) = U_0(x_0)$ the initial value problem

$$x'(t) = U_0(x_0), \tag{3.9a}$$

$$x(0) = x_0 \tag{3.9b}$$

is received.

The solution to (3.9) is

$$x(t) = U_0(x_0)t + x_0.$$

But for example in the case $U_l = 1$ and $U_r = 0$ the characteristics will intersect in the (x,t)-plane (see figure 3.3).

Even in the case of smooth initial data intersecting characteristics can occur after a small time t. An illustration of this case can be found in figure 3.4. Recalling the solution is constant at each characteristic curve, intersecting

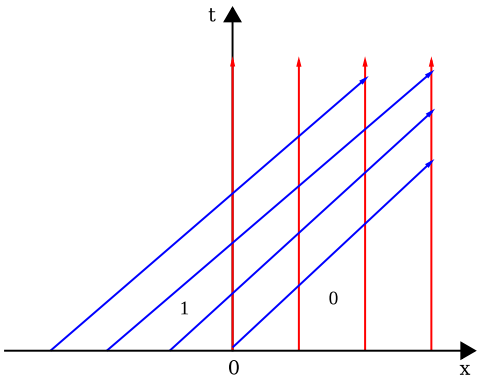


Figure 3.3: Characteristics intersecting for the Riemann problem (3.8) with $(U_l, U_r) = (1, 0)$. Adapted from [7].

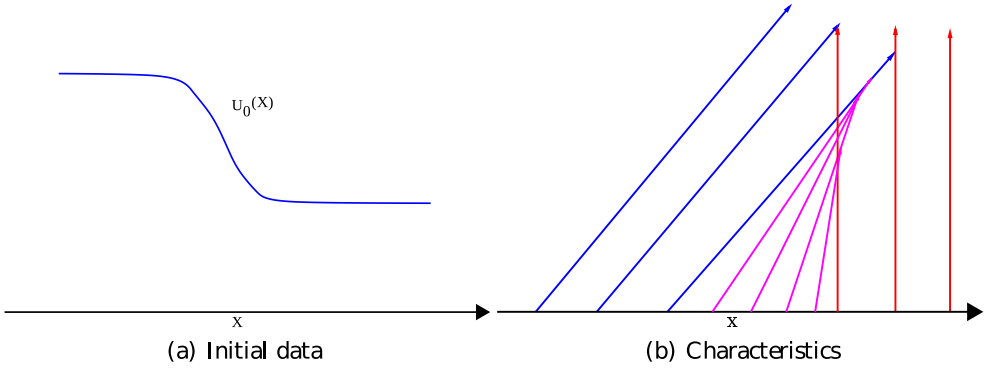


Figure 3.4: Characteristics can even intersect for smooth initial data. Adapted from [7].

characteristics lead to a multivalued solution if the initial data is not constant.

In [7, Exercise 3.1] the following theorem is given:

Theorem 3.1 *Let $U_0(x)$ be differentiable with at least one point x such that $U'(x) < 0$. Then the solution to Burger's equation with initial data U_0 will develop a discontinuity at time*

$$t_{min} = -\frac{1}{\min\limits_{x \in \mathbb{R}} U_0'(x)}. \tag{3.10}$$

So in general the existence of a continuous solution of a scalar conservation law can not be assumed.

3.3 Weak Solutions

As seen in the previous section solving scalar conservation laws demands other types of solutions than a classical, analytical solution. These solutions are *weak solutions*.

Definition 3.2 (Weak solution) *A function $U \in L^1(\mathbb{R} \times \mathbb{R}_{\geq 0})$ is a weak solution of (3.3) with initial data $U(x,0) = U_0(x)$ for all $x \in \mathbb{R}$ if*

$$\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} U\varphi_t + f(U)\varphi_x \quad dxdt + \int_{\mathbb{R}} U_0(x)\varphi(x,0) \quad dx = 0 \tag{3.11}$$

holds for all test functions $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_{\geq 0})$, the space of all continuously differentiable functions on $\mathbb{R} \times \mathbb{R}_{\geq 0}$ with compact support.

Remark 3.3 *If a weak solution U of (3.3) is differentiable, then U satisfies (3.3) pointwise. Hence, the class of weak solutions contains, but is not restricted to classical solutions ([7]).*

It should be noticed, that weak solutions are not necessarily unique as can be shown, in example, for the solution of (3.4a) with initial data (3.15). And neither continuity nor differentiability are required. Discontinuities of weak solutions are called *shock waves*. But shock waves can not be arbitrary curves in the (x,t)-plane, as defined by the Rakine-Hugoniot condition .

Theorem 3.4 (Rakine-Hugoniot condition) *Assume U is a weak solution of (3.3) and it is continuously differentiable on Ω_+ and Ω_-, which are separated by a shock $\sigma(t)$ as in figure 3.5. Let $s(t) = \sigma'(t)$ be the speed of the shock curve. Then the shock speed must satisfy*

$$s(t) = \frac{f(U^+(t)) - f(U^-(t))}{U^+(t) - U^-(t)}, \tag{3.12}$$

where U^+ and U^- are the trace values of U at the right and the left of the discontinuity σ.

Proof This proof is analogously borrowed from [7]. Let $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_{\geq 0})$ be a test function with support contained in Ω. Because $\Omega \cap (\mathbb{R} \times \{0\}) \nsubseteq supp(\varphi)$ it follows from (3.11), that

$$\begin{aligned} 0 &= \int_{\Omega} U\varphi_t + f(U)\varphi_x \quad d\Omega + \int_{\Omega\cap(\mathbb{R}\times\{0\})} U_0(x)\varphi(x,0) \quad dx \\ &= \int_{\Omega} U\varphi_t + f(U)\varphi_x \quad d\Omega. \end{aligned}$$

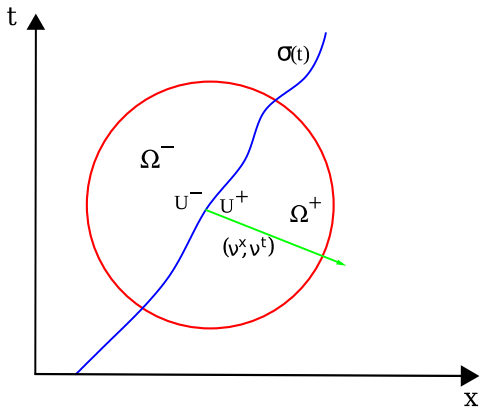


Figure 3.5: What happens across a shock? Adapted from [7]

By assumption $U \in C^1(\Omega_+)$ and $U \in C^1(\Omega_-)$. Making use of integration by parts, it is obtained, that

$$
\begin{aligned}
0 &= \int_\Omega U\varphi_t + f(U)\varphi_x \quad d\Omega \\
&= \int_{\Omega^+} U\varphi_t + f(U)\varphi_x \quad d\Omega + \int_{\Omega_-} U\varphi_t + f(U)\varphi_x \quad d\Omega \\
&= -\int_{\Omega^+} (U_t + f(U)_x)\varphi \quad d\Omega + \int_{\partial\Omega_+} (U^+(t)\nu^t + f(U^+(t))\nu^x)\varphi \quad d\Omega \\
&\quad - \int_{\Omega^-} (U_t + f(U)_x)\varphi \quad d\Omega + \int_{\partial\Omega_-} (U^-(t)\nu^t + f(U^-(t))\nu^x)\varphi \quad d\Omega \\
&= -\int_{\Omega^+\cup\Omega^-} (U_t + f(U)_x)\varphi \quad d\Omega \\
&\quad + \int_{\partial\Omega_+} (U^+(t)\nu^t + f(U^+(t))\nu^x)\varphi \quad d\Omega + \int_{\partial\Omega_-} (U^-(t)\nu^t + f(U^-(t))\nu^x)\varphi \quad d\Omega,
\end{aligned}
$$

where

$$
\begin{aligned}
(\nu^t, \nu^x) &= (s(t), -1) \quad \text{on} \quad \partial\Omega^+, \\
(\nu^t, \nu^x) &= (-s(t), 1) \quad \text{on} \quad \partial\Omega^-,
\end{aligned}
$$

are the outwards pointing normal vectors on σ and $U^+(t)$ and $U^-(t)$ are the right and left trace values of U at each side of the shock σ.

By remark 3.3 U has to satisfy (3.3) on $\Omega^+ \cup \Omega^-$. Hence the first summand in the above integral vanishes:

$$
\int_{\Omega^+\cup\Omega^-} (U_t + f(U)_x)\varphi \quad d\Omega = 0.
$$

And the identity

$$0 = \int_{\partial\Omega^+} (U^+(t)s(t) - f(U^+(t)) - U^-(t)s(t) + f(U^-(t)))\varphi \quad d\Omega$$

can be obtained.

Since φ was an arbitrary test function the integral has to vanish for every Ω, hence the integrand vanishes, which means

$$s(t) = \frac{f(U^+(t)) - f(U^-(t))}{U^+(t) - U^-(t)}. \qquad \square$$

3.3.1 Solutions to Riemann Problems

Although finding a solution to a Riemann problem seems to be not so simple, solutions can be constructed under certain conditions, which are demonstrated in the following.

Definition 3.5 (Oleinik entropy condition) *Let U be a weak solution of (3.3), U^- the left and U^+ be the right trace value at a shock s as stated in the Rakine-Hugioniot condition (3.4).*
The weak solution U satisfies the Oleinik entropy condition if

$$s(t) < \frac{f(k) - f(U^-(t))}{k - U^-(t)} \quad \textit{for all } k \textit{ between} \quad U_l \quad \textit{and} \quad U_r. \tag{3.13a}$$

If the flux function f is strictly convex then the weak solution U satisfies the Oleinik entropy condition, if

$$f'(U^-(t)) > s(t) > f'(U^+(t)). \tag{3.13b}$$

([7])

In the case $U_l > U_r$, a weak solution, which satisfies the Oleinik entropy condition (3.13b), can be constructed for a Riemann Problem (3.3) with initial data (3.8b) and strictly convex flux function f ([7]). (It should be noticed, that the the flux function of Burger's equation (3.4a) is strictly convex.) This solution writes as

$$U(x,t) = \begin{cases} U_l & \text{if} \quad x < st \\ U_r & \text{if} \quad x > st, \end{cases} \tag{3.14a}$$

$$s = \frac{f(U_r) - f(U_l)}{U_r - U_l}. \tag{3.14b}$$

In the case $U_l < U_r$ the above weak solution (3.14) is not satisfying the Oleinik entropy condition (3.13b) ([7]). Because f is strictly convex it can be written

$$f'(U^-(t)) = f'(U_l) < f'(U_r) = f'(U^+(t)) \quad \text{for} \quad U_l < U_r.$$

Let the initial data (3.8b) of the Riemann-problem (3.8) be defined as

$$U_0(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0. \end{cases} \tag{3.15}$$

Then, a weak solution of (3.8a), (3.15) with one shock is

$$U(x,t) = \begin{cases} 0 & \text{if} \quad x < \frac{1}{2}t \\ 1 & \text{if} \quad x > \frac{1}{2}t. \end{cases} \tag{3.16}$$

But on the other hand, a weak solution of (3.8a), (3.15) with two shocks is

$$U(x,t) = \begin{cases} 0 & \text{if} \quad x < \frac{1}{3}t \\ \frac{2}{3} & \text{if} \quad \frac{1}{3}t < x < \frac{5}{6}t, \\ 1 & \text{if} \quad x > \frac{5}{6}t. \end{cases} \tag{3.17}$$

And using the Rakine-Hugoniot condition (3.12) arbitrary many weak solutions can be constructed for $(U_l, U_r) = (0, 1)$ ([7]).
But weak solutions, which satisfy the Oleinik entropy condition, can be constructed by using rarefaction waves.

Definition 3.6 (Rarefaction wave) *Let $U(x,t)$ be a weak solution of a scalar conservation law (3.3) with initial data $U_0(x)$. Then U is called rarefaction wave if*

$$U(x,t) = V(x/t) \quad \textit{for a function} \quad V : \mathbb{R} \to \mathbb{R} \tag{3.18}$$

Theorem 3.7 *If the flux function f in (3.3) is strictly convex and a weak solution U is a rarefaction wave it can be written as*

$$U(x,t) = V(x/t) = \left(f'\right)^{-1}(x/t) \tag{3.19}$$

[7].

Proof (This proof is analogously borrowed from [7].) Let $x/t = \xi$. (3.3) gives

$$\begin{aligned} U(x,t)_t + f(U(x,t))_x &= V(\xi)_t + V(\xi)_x \\ &= V(\xi)_t + f'(V(\xi))V(\xi)_x \\ &= V_\xi \xi_t + f'(V(\xi))V_\xi \xi_x \\ &= -\frac{x}{t^2}V_\xi + f'(V(\xi))\frac{1}{t}V_\xi \\ &= 0. \end{aligned}$$

Hence

$$\left(f'(V(\xi)) - x/t\right) V_\xi = 0.$$

Since $V(\xi)_\xi = 0$ would imply a trivial constant solution $U(x,t)$, it follows

$$f'(V(\xi)) = \frac{x}{t}.$$

The strict convexity of f implies that f' is in invertible. □

In the case $U_l < U_r$ rarefaction waves can be used to construct a Oleinik entropy condition satisfying weak solution of the the Riemann problem (3.3) with initial data (3.8b), if f is strictly convex:

$$U(x,t) = \begin{cases} U_l & \text{if} \quad x < f'(U_l)t \\ (f')^{-1}(x/t) & \text{if} \quad f'(U_l)t < x < f'(U_r)t \\ U_r & \text{if} \quad x > f'(U_r)t, \end{cases}$$

If the flux function f is not strictly convex but has finitely many inflection points f' can be replaced by the derivative of the *lower convex envelope* $f'_\cup$ on the intervals between inflection points where f is not strictly convex in order to construct a weak solution, which satisfies the Oleinik entropy condition (3.13a) ([7]).

Definition 3.8 (Lower convex envelope) *Let $f : [U_l, U_r] \to \mathbb{R}$. The lower convex envelope $f_\cup$ is defined as*

$$f_\cup = \sup\{g(u) | g \leq f \text{ and } g \text{ convex on } [U_l, U_r]\} \tag{3.20}$$

([5]).

An illustration of this solution can be found in figure 3.6. A solution which is made up of a waves followed by shocks is called *compound shock* ([7]).

Furthermore, if $U_l > U_r$ and f is not strictly convex but has finitely many inflection points, the *upper concave envelope* $f_\cap$ can be used in order to find a Oleinik entropy condition satisfying weak solution ([7]). In this case the solution writes as

$$U(x,t) = \begin{cases} U_l & \text{if} \quad x < f'_\cap(U_l)t \\ (f'_\cap)^{-1}(x/t) & \text{if} \quad f'_\cap(U_l)t < x < f'_\cap(U_r)t \\ U_r & \text{if} \quad x > f'_\cap(U_r)t. \end{cases} \tag{3.21}$$

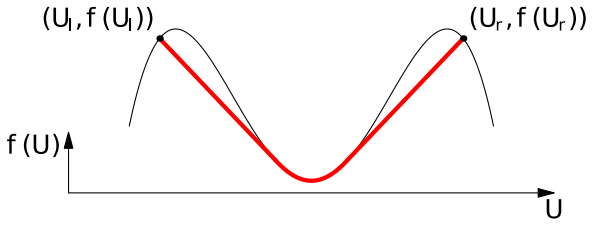
([5]).

The upper concave envelope is analogously defined to the lower convex envelope in definition (3.8).

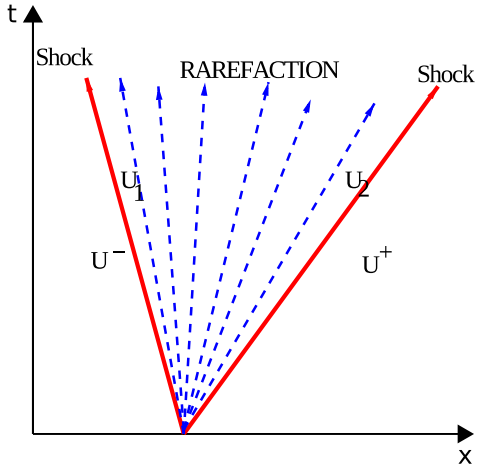
Definition 3.9 (Upper concave envelope) *Let $f : [U_r, U_l] \to \mathbb{R}$. The upper concave envelope $f_\cap$ is defined as*

$$f_\cap = \inf\{g(u) | g \geq f \text{ and } g \text{ concave on } [U_r, U_l]\} \tag{3.22}$$

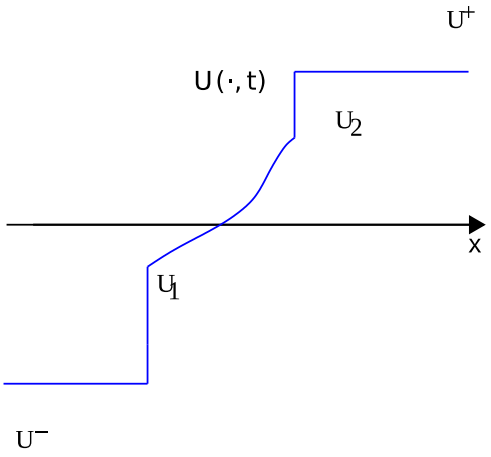
([5]).



(a) The lower convex envelope is the thick red curve.



(b) Solution in the space-time-domain.



(c) A snapshot of the solution illustrating compound shock.

Figure 3.6: The solution of the Riemann problem with a non convex flux. Solutions are constructed as shocks, followed by rarefactions. Adapted from [7].

3.3.2 Entropy Solutions

Because, like mentioned in subsection (3.3.1), weak solutions are not unique, some further regularization approaches, like entropy solutions, have to be introduced in order to turn the solving of a scalar conservation law into a well-posed problem.

Definition 3.10 (Entropy pair) *Let f be a flux function, $\eta = \eta(U)$ be a strictly convex function and let*

$$q(U) = \int_0^U f'(s)\eta'(s)ds, \tag{3.23}$$

then

(i) η is called entropy function and q is called the corresponding entropy flux.

(ii) the pair (η, q) is called entropy pair.

([7]).

Definition 3.11 (Entropy solution) *A function $U(x,t)$ is an entropy solution of (3.3) if it satisfies the following conditions ([7]):*

(i) $U \in L^\infty(\mathbb{R} \times \mathbb{R}_{\geq 0})$.

(ii) U is a weak solution of (3.3), i. e., it satisfies the identity (3.11).

(iii) For all test functions $\varphi \in C_c^1(\mathbb{R} \times (0,\infty))$ with $\varphi \geq 0$, U satisfies

$$\int_{\mathbb{R}\times\mathbb{R}_{\geq 0}} \eta(U)\varphi_t + q(U)\varphi_x dxdt \geq 0, \tag{3.24}$$

for all entropy pairs (η, q)

Entropy solutions are generalizations of the point-wise Oleinik entropy condition (3.5) ([7]). Furthermore entropy solutions satisfy the Oleinik entropy condition, which is proven in example in [3]. Hence satisfying the Oleinik entropy condition is sufficient for a weak solution in order to be an entropy solution.
It is well known that in general the oscillation of a solution is of particular interest. In some cases when numerical schemes produce oscillations, these oscillations do not have a physical origin. In such a case a scheme is called *unstable* (e.g. 'A simple centred finite difference scheme', [7]). In order to measure the oscillation of a solution the *total variation* is introduced.

Definition 3.12 (Total variation) *Let g be a functionn defined on an intervall $[a,b]$. The total variation of g is defined as*

$$\|g\|_{TV} = \sup_{\mathcal{P}} \sum_{j=1}^{N_\mathcal{P}-1} |g(x_{j+1}) - g(x_j)|, \tag{3.25}$$

where the supremum is taken over all partitions $\mathcal{P} = \{a = x_1 < x_2 < \cdots < x_{N_\mathcal{P}} = b\}$ *([7]).*

The total variation defines a semi-norm because the total variation of a constant function vanishes. But by defining the bounded variation an norm can be constructed using the total variation ([7]).

Definition 3.13 (Bounded variation) *Let* g *be a function defined on the interval* $[a, b]$*. The bounded variation of* g *is defined as*

$$\|g\|_{BV} = \|g\|_{TV} + \|g\|_{L^1}. \tag{3.26}$$

The space of functions with bounded variation is

$$BV(\mathbb{R}) = \{g \in L^1(\mathbb{R}) | \|g\|_{BV} < \infty\}. \tag{3.27}$$

([7]).

Existence and uniqueness of an entropy solution is covered by the following theorem, as stated in ([7]) as follows:

Theorem 3.14 *Let* $U_0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$*, then (3.3) has an entropy solution* U *which satisfies the estimates*

$$\|U(\cdot, t)\|_{L^\infty} \leq \|U_0\|_{L^\infty}, \tag{3.28}$$

and

$$\|U(\cdot, t)\|_{TV} \leq \|U_0\|_{TV}. \tag{3.29}$$

Furthermore, if U *and* V *are entropy solutions of (3.3) with initial* U_0 *and* V_0 *respectively, then*

$$\int_{\mathbb{R}} |U(x,t) - V(x,t)| dx \leq \int_{\mathbb{R}} |U_0 - V_0| dx \quad \text{for all } t > 0. \tag{3.30}$$

Hence entropy solutions are unique.

A proof of this theorem can be found in [5, Theorem 2.14] or in [3].

3.4 Hyperbolic and elliptic Systems

In order to investigate solutions of conservation laws more precisely, the concepts of hyperbolicity, parabolicity or elipticity can be introduced.
For scalar conservation laws hyperbolicity is referred to finite speed of propagation ([7]). In [3] hyperbolicity is stated as:

Definition 3.15 (Hyperbolic system) *Let Ω be an open subset of $\mathbb{R}^p$. A system of conservation laws of the form*

$$\mathbf{U}_t + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \mathbf{f}_j(\mathbf{U}) = 0 \quad \textit{for} \quad \mathbf{x} \in \mathbb{R}^d, t \in \mathbb{R}_{>0}, \tag{3.31a}$$

where the quantity of interest $\mathbf{U}$ is a vector-valued function from $\mathbb{R}^d \times \mathbb{R}_{\geq 0}$ into $\Omega \in \mathbb{R}^p$,

$$\mathbf{U} = \begin{pmatrix} U_1 \\ \vdots \\ U_p \end{pmatrix}, \tag{3.31b}$$

and the flux functions,

$$\mathbf{f}_j = \begin{pmatrix} f_{1,j} \\ \vdots \\ f_{p,j} \end{pmatrix} \quad \textit{for} \quad 1 \leq j \leq d, \tag{3.31c}$$

are smooth and vector-valued, is a hyperbolic system if for all $j = 1, \cdots, d$, the Jacobian matrix of $\mathbf{f}_j$ is hyperbolic. The Jacobian matrix of $\mathbf{f}_j$,

$$\mathbf{A}_j(\mathbf{U}) = \left(\frac{\partial f_{ij}}{\partial U_k}(\mathbf{U})\right) \quad \textit{for} \quad 1 \leq i, k \leq p, \tag{3.31d}$$

is called hyperbolic if, for any $\mathbf{U} \in \Omega$ and any $\mathbf{w} \in \mathbb{R}^d$, $\mathbf{w} \neq 0$, the matrix

$$\mathbf{A}(\mathbf{U}, \mathbf{w}) = \sum_{j=1}^{d} w_j \mathbf{A}_j(\mathbf{U}) \tag{3.31e}$$

has p real eigenvalues and p linearly independent corresponding eigenvectors.

As stated in [2] the homogeneous saturation equation,

$$\partial_t s + \operatorname{div}\Big(\frac{\lambda_w(s)}{\lambda_T(s)}\mathbf{v} + \frac{\lambda_w(s)\lambda_o(s)}{\lambda_T(s)}(\rho_w - \rho_o)g_\gamma \nabla z\Big) = 0,$$

assuming the conditions (2.16) for the mobilities, is a hyperbolic scalar conservation law.

Using the velocity $\mathbf{v}$, an elliptic equation for the pressure p can be obtained from (2.18) with vanishing sourc-terms q_w, q_o ([2]). This can be easily verified in the two-dimensional case for $\nabla z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Ellipticity is defined as follows in [4]:

Definition 3.16 (Linear elliptic ordinary differential equation) *The ordinary linear differential equation of second order in two variables*

$$a(x,y)\partial_x^2 u + 2b(x,y)\partial_x\partial_y u + c(x,y)\partial_y^2 u + d(x,y)\partial_x u + e(x,y)\partial_y u + f(x,y)u + g(x,y) = 0 \quad (3.32a)$$

is called elliptic in (x,y), if

$$a(x,y)c(x,y) - b^2(x,y) > 0. \quad (3.32b)$$

(3.32a) is called elliptic on $\Omega \subset \mathbb{R}^2$, if it is elliptic in all $(x,y) \in \Omega$.

Inserting (2.18c) into (2.18b) it is obtained that

$$\begin{aligned}
\operatorname{div}\big(-\lambda_T \nabla p + g_\gamma(\lambda_o\rho_0 + \lambda_w\rho_w)\begin{pmatrix}0\\1\end{pmatrix}\big) &= 0\\
\operatorname{div}\big(-\lambda_T \nabla p\big) + \partial_y\big(g_\gamma(\lambda_o\rho_0 + \lambda_w\rho_w)\big) &= 0\\
-\partial_x(\lambda_T\partial_x p) - \partial_y(\lambda_T\partial_y p) + g_\gamma(\partial_y\lambda_o\rho_0 + \partial_y\lambda_w\rho_w) &= 0\\
-\partial_x\lambda_T\partial_x p - \lambda_T\partial_x^2 p - \partial_y\lambda_T\partial_y p - \lambda_T\partial_x^2 p + g_\gamma(\partial_y\lambda_o\rho_0 + \partial_y\lambda_w\rho_w) &= 0. \quad (3.33)
\end{aligned}$$

So the coefficients from (3.32a) are

$$\begin{aligned}
a &= -\lambda_T, & b &= 0, & c &= -\lambda_T,\\
d &= -\partial_x\lambda_T, & e &= -\partial_y\lambda_T, & f &= 0,\\
& & g &= g_\gamma(\partial_y\lambda_o\rho_o + \partial_y\lambda_w\rho_w). & & \quad (3.34)
\end{aligned}$$

And (3.32b) is satisfied:

$$ac - b^2 = \lambda_T^2 > 0. \quad (3.35)$$

Hence (2.18) with vanishing sourc-terms q_w, q_o is a hyperbolic-elliptic system as written in [2].

3.5 Note on Well-Posedness of Solutions of the Equations of two-phase Flow

In [2] it is stated, that the equations of the two-phase flow problem with a small, non-zero capillary pressure p_c lead to a viscous perturbation of the saturation equation. Revering to [2] this problem has been show to be well-posed in [6]. But there are difficulties in the numerical approximation of the problem with capillary pressure, as written in [2]. Hence due to the simplicity of implementation of a numerical method a problem without capillary pressure is investigated in this thesis.
But proving well-posedness of global weak solutions of the equations of

two-phase flow without capillary pressure (2.18a), (2.18b) and (2.18c) and vanishing source terms q_o, q_w has been open for many decades ([2]). As the state of the art result in 2011 it is stated in [2], that many attempts of showing that the velocity $\mathbf{v}$ in (2.18) is sufficiently regular, for example is BV, had failed. But according to [2] partial results with strong assumptions have been obtained. And evidence for the existence of entropy solutions or weak solutions of modified versions of the problem like in [2] or [1] have been adduced.

Chapter 4

Numerical Methods

In general, no analytic solution to the equations of a model of reservoir simulations can be found, hence numerical methods are needed for their solution ([9]). And as shown in the examples in the previous section 3.3.1, constructing an entropy solution of a scalar conservation law can be a lengthy task. This chapter covers an introduction to the numerical methods of finite volume schemes. And the IMPES-method, which is applied to the two-phase flow equations, is introduced.

4.1 Finite Volume Schemes for scalar Conservation Laws

Assume the scalar conservation law with initial data

$$U_t + f(U)_x = 0 \quad \text{on} \quad \Omega \times \mathbb{R}_{\geq 0}, \tag{4.1a}$$

$$U(x,0) = U_0(x) \quad \text{for} \quad x \in \Omega, \tag{4.1b}$$

should be solved using a numerical method. In this chapter the so called *finite volume methods* for conservation laws are introduced.

4.1.1 Discretisation of the Space-Time Domain and Cell Averages

The domain $(x_L, x_R) \times \mathbb{R}_{\geq 0} = \Omega \times \mathbb{R}_{\geq 0} \subset (\mathbb{R} \times \mathbb{R}_{\geq 0})$ must be discretised in order to do any numerical implementation. For simplicity, the discretization is assumed to be uniform. The space domain (x_L, x_R) is discretised by the grid

$$x_j = x_L + (j + \frac{1}{2})\Delta x \quad \text{for} \quad j = 0, \cdots, N, \tag{4.2a}$$

$$x_{j-\frac{1}{2}} = x_j - \frac{\Delta x}{2} = x_L + j\Delta x \quad \text{for} \quad j = 0, \cdots, N+1, \tag{4.2b}$$

$$\Delta x = \frac{x_R - x_L}{N+1}. \tag{4.2c}$$

The *control volumes* are given by

$$\mathcal{C}_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}). \tag{4.3}$$

An illustration of the spacial descritization can be found in figure 4.1.

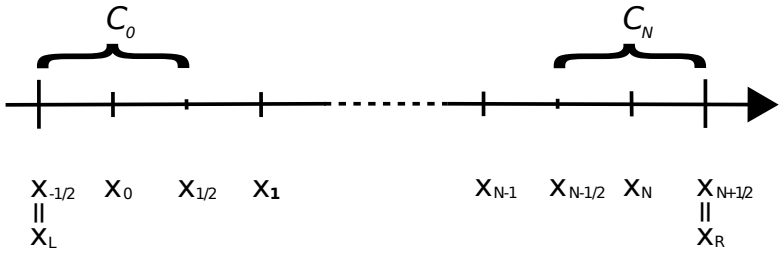


Figure 4.1: Illustration of the spacial discretization.

The time domain $[0, T]$ is discretized by

$$t^n = n\Delta t \quad \text{for} \quad n = 0, \cdots, M, \tag{4.4a}$$

$$\Delta t = \frac{T}{M}. \tag{4.4b}$$

The *cell averages* U_j^n approximate the average of the solution U on the control volume $\mathcal{C}_j$:

$$U_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t^n) dx \tag{4.5}$$

([7]).

The aim is to compute U_j^{n+1}, if U_j^n is known for all $j = 0, \cdots, N$ ([7]). Integrating (4.1a) over a space-time cell gives

$$\int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U_t \quad dxdt + \int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(U)_x \quad dxdt = 0.$$

Applying the fundamental theorem of calculus to the above equation leads to

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t^{n+1}) \quad dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} U(x, t^n) \quad dx \tag{4.6}$$

$$= \int_{t^n}^{t^{n+1}} f(U(x_{j-\frac{1}{2}}, t)) \quad dt - \int_{t^n}^{t^{n+1}} f(U(x_{j+\frac{1}{2}}, t) \quad dt. \tag{4.7}$$

Defining the *numerical flux* like in [7]

$$\bar{F}^n_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(U(x_{j+\frac{1}{2}}, t)) \quad dt \tag{4.8}$$

and comparing (4.7) to (4.5) gives

$$U^{n+1}_j = U^n_j - \frac{\Delta t}{\Delta x}(\bar{F}^n_{j+\frac{1}{2}} - \bar{F}^n_{j-\frac{1}{2}}).$$

Hence a finite volume method consists of finding a solution or approximation of $\bar{F}$, because equation (4.8) requires the exact solution $U(x_{j+\frac{1}{2}}, t)$, which is not given a priori ([7]).

4.1.2 Godunov Method

The idea of the approximation of the numerical fluxes (4.8) via the *Godunov method* is attributable to Gondunov ([7]). The idea is, that the given the cell averages U^n_j in each cell $\mathcal{C}_j$ and the unknown cell averages at the next time step U^{n+1}_j correspond to a Riemann problem with constant initial data on each cell for each time level. The Riemann problem on each cell is formulated by

$$U_t + f(U)_x = 0, \tag{4.9a}$$

$$U(x, t^n) = \begin{cases} U^n_j & \text{if} \quad x < x_{j+\frac{1}{2}} \\ U^n_{j+1} & \text{if} \quad x > x_{j+\frac{1}{2}} \end{cases} \tag{4.9b}$$

([7]).

The problem (4.9) is illustrated in figure 4.2.

All together the cell averages define a superposition of Riemann problems of the form (4.9) at each cell interface $x_{j+\frac{1}{2}}$. In subsection (3.3.1), it was explicated how to solve a Riemann problem.

Let $\bar{U}_j(x, t)$ be the solution of (4.9). It can be shown ([7]) that $\bar{U}$ is self similar, alike a refraction wave (3.18)

$$\bar{U}_j(x, t) = \bar{U}_j\left(\frac{x - x_{j+\frac{1}{2}}}{t}\right) = \bar{U}_j(\xi). \tag{4.10}$$

If the waves from neighbouring Riemann problems do not intersect during the time interval $[t^n, t^{n+1}]$, then $\bar{U}_j$ is constant at the cell interface $\xi = 0$. In order to be confident, that no interference of waves of neighbouring Riemann problems occurs, the *Courant-Friedrichs-Lewy condition (CFL condition)*

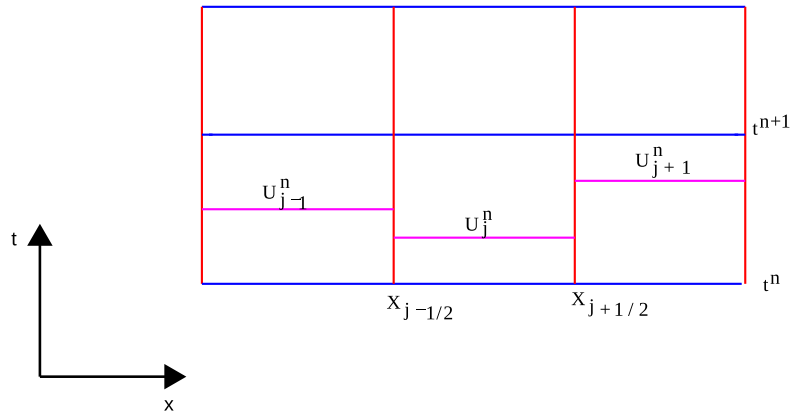


Figure 4.2: Cell averages define Riemann problems at every interface. Adapted from [7].

(4.14) is set like in [5]. It is motivated by the fact, that the maximum wave speed is bounded by $|f'(U_j^n)|$ ([7]).

Under the assumption of the CFL condition it can be shown, that the *edge-centred flux*,

$$F_{j+\frac{1}{2}}^n := f(\bar{U}_j(0+)) = f(\bar{U}_j(0-)), \tag{4.11}$$

is well defined, using either the continuity of $\bar{U}_j$, or else the Rankine-Hugoniot condition like in [7]. And because $\bar{U}_j(0)$ is a constant, it follows that

$$f(U(x_{j+\frac{1}{2}}, t)) = f(\bar{U}_j(0)) \quad \text{for all} \quad t > 0.$$

Inserting the above identity into (4.8) gives

$$\bar{F}_{j+\frac{1}{2}}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(U(x_{j+\frac{1}{2}}, t)) \quad dt = f(\bar{U}_j(0)) = F_{j+\frac{1}{2}}^n.$$

Consequently the following finite volume scheme gives the exact solution to the Riemann problem (4.9), if F is the exact solution of the edge-centred flux

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n). \tag{4.12}$$

Equation (4.12) is the standard form of finite volume schemes for conservation laws ([7]).

The equation (4.11) solving flux is known as the *Godunov flux* and writes as

$$F_{j+\frac{1}{2}}^n = F(U_j^n, U_{j+1}^n) = \begin{cases} \min\limits_{U_j^n \le \theta \le U_{j+1}^n} f(\theta) & \text{if} \quad U_j^n < U_{j+1}^n \\ \min\limits_{U_{j+1}^n \le \theta \le U_j^n} f(\theta) & \text{if} \quad U_{j+1}^n < U_j^n \end{cases}. \tag{4.13}$$

The Gonunov flux can be obtained by a lengthy case by case analysis ([7]). The finite volume scheme (4.12) together with the Godunov flux (4.13) is called *Godunov method* ([7]). It should be mentioned, that computing the Godunov flux requires solving an optimization problem.

Using the form of a finite volume scheme (4.12), the CFL condition can be stated as follows:

Definition 4.1 (CFL condition) *Let U_j^n be the output of a finite volume scheme of the form (4.12) with space step Δx and time step Δt. The scheme satisfies the CFL-conditon if*

$$\max_j |f'(U_j^n)| \frac{\Delta t}{\Delta x} \leq \frac{1}{2} \quad \forall n = 0, \cdots, M, \quad j = 0, \cdots, N. \tag{4.14}$$

4.1.3 Approximate Riemann Solvers

The Godunov flux (4.13) solves the Riemann problem (4.9) exactly but an approximated solution of (4.11) can be more simple to implement or it can be faster in runtime.

The following numerical fluxes are approximated solutions of the Godunov flux and are known under the name *approximate Riemann solvers* ([7]).

Roe Scheme

$$F_{j+\frac{1}{2}}^n = F^{Roe}(U_j^n, U_{j+1}^n) = \begin{cases} f(U_j^n) & \text{if} \quad \hat{A}_{j+\frac{1}{2}} \geq 0 \\ f(U_{j+1}^n) & \text{if} \quad \hat{A}_{j+\frac{1}{2}} < 0, \end{cases} \tag{4.15a}$$

where $\hat{A}$ is called *Roe average* and writes as

$$\hat{A}_{j+\frac{1}{2}} = \begin{cases} \frac{f(U_{j+1}^n) - f(U_j^n)}{U_{j+1}^n - U_j^n} & \text{if} \quad U_{j+1}^n \neq U_j^n \\ f'(U_j^n) & \text{if} \quad U_{j+1}^n = U_j^n \end{cases}. \tag{4.15b}$$

Lax-Friedrichs scheme

$$F_{j+\frac{1}{2}}^n = F^{LxF}(U_j^n, U_{j+1}^n) = \frac{f(U_j^n) + f(U_{j+1}^n)}{2} - \frac{\Delta x}{2\Delta t}(U_{j+1}^n - U_j^n) \tag{4.16}$$

Rusanov scheme

$$F_{j+\frac{1}{2}}^n = F^{Rus}(U_j^n, U_{j+1}^n) \tag{4.17}$$

$$= \frac{f(U_j^n) + f(U_{j+1}^n)}{2} - \frac{\max\{|f'(U_j^n)|, |f'(U_{j+1}^n)|\}}{2}(U_{j+1}^n - U_j^n) \tag{4.18}$$

Enquist-Osher scheme

$$F^n_{j+\frac{1}{2}} = F^{EO}(U^n_j, U^n_{j+1}) \tag{4.19}$$

$$= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{1}{2}\int_{U^n_j}^{U^n_{j+1}} |f'(\theta)| \quad d\theta \tag{4.20}$$

4.1.4 Boundary Conditions

For the implementation of a finite volume scheme (4.12) with a flux function of any of the above mentioned (4.13), (4.15a) – (4.20) the values of U^n_{-1} and U^n_{N+1} need to be specified. This is done via the cell averages on the *ghost cells*

$$\mathcal{C}_{-1} = [x_L - \Delta x, x_l) \tag{4.21}$$

and

$$\mathcal{C}_{N+1} = [x_R, x_R + \Delta x). \tag{4.22}$$

The value of the cell average might be imposed by the boundary condition of the underlying physical problem of (4.1), or, if the spacial domain equals the real line, the domain must be truncated since a computational domain is bounded ([7]).

Dirichlet boundary condition If the conservation law (4.1) is augmented with a Dirichlet boundary condition,

$$U(x_L, t) = g_L(t), \quad U(x_R, t) = g_R(t) \quad \text{for} \quad t \in [0, T], \tag{4.23}$$

then, the numerical boundary condition is

$$U^n_{-1} = g_L(t^n) \quad \text{and} \quad U^n_{N+1} = g_R(t^n) \quad \text{for} \quad n = 0, \cdots, M \tag{4.24}$$

([7]).

Periodic boundary condition For a periodic solution the periodic boundary condition

$$U^n_{-1} = U^n_N \quad \text{and} \quad U^n_{N+1} = U^n_0 \quad \text{for} \quad n = 0, \cdots, M \tag{4.25}$$

can be chosen ([7]).

Artificial boundary condition If non-reflecting Neumann boundary condition is given by the conservation law, the artificial boundary condition is chosen. It writes as

$$U_{-1}^n = U_0^n \quad \text{and} \quad U_{N+1}^n = U_N^n \quad \text{for} \quad n = 0, \cdots, M \tag{4.26}$$

([7]). The non-reflecting Neumann boundary condition on a one-dimensional domain is

$$\partial_x U(x_L, t) = \partial_x U(x_R, t) = 0 \quad \text{for} \quad t \in [0, T]. \tag{4.27}$$

4.2 IMPES-Method

The implicit pressure and explicit saturation (IMPES) method is based on the formulation of the multi phase problem of the form similar to (4.28). It uses an implicit pressure equation, and a explicit saturation equation. The classical IMPES-method is originated in the works of Sheldon et al. (1959) and Stone and Garder (1961) ([9]). The method used in [8], a finite volume scheme, which is based on the approach in [2], will be studied in this work. In subsection 4.2.1 the general method and the algorithm is introduced and in subsection 4.2.2 boundary conditions are examined.

4.2.1 IMPES-Method applied to the Two-Phase Flow Problem

The following problem, the equations of two phase flow with vanishing capillary pressure, as derived in subsection 2.3.2 should be solved on a two dimensional spatial domain:

$$\partial_t s + \operatorname{div}\big(f(s)\mathbf{v} + g(s)\nabla y\big) = q \quad \text{for} \quad t > 0, \quad \mathbf{x} \in (0,a) \times (0,b), \tag{4.28a}$$

$$\operatorname{div}(\mathbf{v}) = q \quad \text{for} \quad t > 0, \quad \mathbf{x} \in (0,a) \times (0,b), \tag{4.28b}$$

$$\mathbf{v} = -\lambda_T \nabla p + (\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla y \quad \text{for} \quad t > 0, \quad \mathbf{x} \in (0,a) \times (0,b), \tag{4.28c}$$

$$\big(f(s)\mathbf{v} + g(s)\nabla z\big) \cdot \nu = 0 \quad \text{for} \quad t > 0, \quad \mathbf{x} \in \partial\big((0,a) \times (0,b)\big), \tag{4.28d}$$

$$\lambda_T(s(t,\mathbf{x}))\partial_\nu p(t,\mathbf{x}) = \pi(t,\mathbf{x}) \quad \text{for} \quad t > 0, \quad \mathbf{x} \in \partial\big((0,a) \times (0,b)\big), \tag{4.28e}$$

$$\int_{(0,a)\times(0,b)} p(t,\mathbf{x}) d\mathbf{x} = 0 \quad \text{for} \quad t > 0, \tag{4.28f}$$

$$s(0,\mathbf{x}) = s_0(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in (0,a) \times (0,b), \tag{4.28g}$$

where $\nabla y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ equals the direction of the action of gravity.

Since finite volume methods are used, the computational domain must be discretized. It is discretized into $N+1 \times M+1$ rectangles and the spatial step sizes are defined as

$$\Delta x = \frac{a}{N+1}, \quad \Delta y = \frac{b}{M+1}. \tag{4.29}$$

The control volumes, cell interfaces $x_{j+\frac{1}{2}}$,$y_{j+\frac{1}{2}}$, grid points x_j and y_j and time steps t^n are defined as in (4.3), (4.2) and (4.4). For reasons of clarity and comprehensibility, the superscrip n, is not written-out, if avoidable.

The forward difference quotient, indicated by $+$, and the backward difference quotient, indicated by $-$, in the x-direction and y-direction respectivly, are defined as in [8] by

$$D_x^{\pm} k_{i,j} = \pm \frac{1}{\Delta x}(k_{i\pm1,j} - k_{i,j}), \tag{4.30a}$$

$$D_y^{\pm} k_{i,j} = \pm \frac{1}{\Delta y}(k_{i,j\pm1} - k_{i,j}). \tag{4.30b}$$

Let φ and ψ be twice continuously differentiable scalar functions on $\mathbb{R}^2$, then

$$\operatorname{div}(\varphi \nabla \psi) = \partial_x(\varphi \partial_x \psi) + \partial_y(\varphi \partial_y \psi). \tag{4.31}$$

The forward and backward difference quotient are approximations of the spatial derivatives:

$$D_x^+ k_{i,j} \approx \partial_x k(x_{i+\frac{1}{2},j}), \quad D_x^- k_{i,j} \approx \partial_x k(x_{i-\frac{1}{2},j}),$$

and analogously for the spacial derivative in the y-direction. (This can be verified by subtracting Taylor expansions of $k(x)$ in the points $x_{i+1,j}$, $x_{i,j}$ or $x_{i,j}$ and $x_{i-1,j}$ respectively under sufficient smoothness assumptions for $k(x)$.) So it can be written, that

$$\operatorname{div}(\varphi \nabla \psi)(x_i, y_j) \approx D_x^-(\varphi_{i+\frac{1}{2},j} D_x^+ \psi_{i,j}) + D_y^-(\varphi_{i,j+\frac{1}{2}} D_y^+ \psi_{i,j}). \tag{4.32}$$

Inserting (4.28c) into (4.28b) gives

$$\begin{aligned} -\operatorname{div}(\lambda_T \nabla p) &= -\operatorname{div}((\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma \nabla y) + q \\ &= -\partial_y((\lambda_w \rho_w + \lambda_o \rho_o) g_\gamma) + q. \end{aligned} \tag{4.33}$$

By using (4.33), the discretized pressure equation can be stated as, alike in [8]:

$$D_x^-(\lambda_{T,i+\frac{1}{2},j}D_x^+ p_{i,j}) + D_y^-(\lambda_{T,i,j+\frac{1}{2}}D_y^+ p_{i,j}) = D_y^-((\rho_o\lambda_{o,i,j+\frac{1}{2}} + \rho_w\lambda_{w,i,j+\frac{1}{2}})g_\gamma) - q_{i,j} \quad \text{for} \quad 0 \le i \le N, \quad 0 \le j \le M. \tag{4.34}$$

The total mobility λ_T and the two phase mobilities λ_α, $\alpha = o, w$ on the cell interfaces are approximated by the harmonic mean like in [8]:

$$\lambda_{\beta,i+\frac{1}{2},j} = \frac{2\lambda_\beta(s_{i,j})\lambda_\beta(s_{i+1,j})}{\lambda_\beta(s_{i,j}) + \lambda_\beta(s_{i+1,j})} \quad \text{for} \quad 0 \le i \le N-1, 0 \le j \le M \tag{4.35a}$$

$$\lambda_{\beta,i,j+\frac{1}{2}} = \frac{2\lambda_\beta(s_{i,j})\lambda_\beta(s_{i,j+1})}{\lambda_\beta(s_{i,j}) + \lambda_\beta(s_{i,j+1})} \quad \text{for} \quad 0 \le j \le M-1, 0 \le i \le N \tag{4.35b}$$

$$\text{for} \quad \beta = T, w, o.$$

And the constraint, as in [1],

$$\Delta x \Delta y \sum_{i=0}^{N}\sum_{j=0}^{M} p_{i,j} = 0, \tag{4.36}$$

is motivated by the condition (4.28f).

Applying the forward and backward difference quotient to the velocity equation (4.28c) results into

$$v_{x,i+\frac{1}{2},j} = -\lambda_{T,i+\frac{1}{2},j}D_x^+ p_{i,j} \tag{4.37a}$$

$$\text{for} \quad 0 \le i \le N-1, \quad 0 \le j \le M,$$

$$v_{y,i,j+\frac{1}{2}} = -\lambda_{T,i,j+\frac{1}{2}}D_y^+ p_{i,j} + (\lambda_{w,i,j+\frac{1}{2}}\rho_w + \lambda_{o,i,j+\frac{1}{2}}\rho_o)g_\gamma \tag{4.37b}$$

$$\text{for} \quad 0 \le i \le N, \quad 0 \le j \le M-1,$$

similarly as in [8].

The saturation equation (4.28a) should be solved using a finite volume scheme of the form (4.12). The flux function in (4.28a) can be split into two parts, the velocity part and the gravitational part:

$$\partial_t s + \operatorname{div}\big(f(s)\mathbf{v}\big) + \partial_y g(s) = q. \tag{4.38}$$

The velocity-numerical flux term is set like in [8] by

$$F^{velo}_{x,i+\frac{1}{2},j} = \begin{cases} v_{x,i+\frac{1}{2},j} f(s_{i,j}) & \text{if} \quad v_{x,i+\frac{1}{2},j} > 0, \\ v_{x,i+\frac{1}{2},j} f(s_{i+1,j}) & \text{if} \quad v_{x,i+\frac{1}{2},j} \leq 0, \end{cases} \tag{4.39a}$$
$$\text{for} \quad 0 \leq i \leq N-1, 0 \leq j \leq M,$$

$$F^{velo}_{y,i,j+\frac{1}{2}} = \begin{cases} v_{y,i,j+\frac{1}{2}} f(s_{i,j}) & \text{if} \quad v_{y,i,j+\frac{1}{2}} > 0, \\ v_{y,i,j+\frac{1}{2}} f(s_{i,j+1}) & \text{if} \quad v_{y,i,j+\frac{1}{2}} \leq 0, \end{cases} \tag{4.39b}$$
$$\text{for} \quad 0 \leq i \leq N, 0 \leq j \leq M-1.$$

The numerical gravitational flux term is given by

$$F^{grav}_{y,i,j+\frac{1}{2}} = g(\min\{s_{i,j+1}, \frac{1}{2}\}) + g(\max\{s_{i,j}, \frac{1}{2}\}) \tag{4.40}$$
$$\text{for} \quad 0 \leq i \leq N, \quad 0 \leq j \leq M-1,$$

like in [2].

The saturation is updated like in [2] by the finite volume scheme

$$s^{n+1}_{i,j} = s^n_{i,j} - h\Big(D^-_x F^{velo,n}_{x,i+\frac{1}{2},j} + D^-_y \left(F^{velo,n}_{y,i,j+\frac{1}{2}} + F^{grav,n}_{y,i,j+\frac{1}{2}}\right)\Big) + hq_{i,j} \tag{4.41}$$
$$\text{for} \quad 0 \leq i \leq N, 0 \leq j \leq M,$$

where h is the current time step (see algorithm 1). Furthermore, a CFL-like condition motivated by (4.14) is chosen alike in [8]. This condition is:

$$\max_{i,j} |v_{x,i+\frac{1}{2},j}| \frac{h}{\Delta x} + \max_{i,j} |v_{y,i,j+\frac{1}{2}}| \frac{h}{\Delta y} \leq 1. \tag{4.42}$$

The aim of this condition is to bound the wave propagation in each direction by the half space-step length.

The mobility functions are defined as in [8] by

$$\lambda_{w,o}(s_{w,o}) = s^2_{w,o}. \tag{4.43a}$$

This choice satisfies the assumptions (2.16) for the mobilities, because using (2.5) results into

$$\lambda_o(s) = (1-s)^2, \quad \lambda_T(s) = (1-s)^2 + s^2. \tag{4.43b}$$

The algorithm, which is used in order to solve the equations of two-phase flow numerically, performs TT time steps for solving until the final time T. It updates the saturation $s_{i,j}$ more often, since is it commonly believed that this is sufficient in reservoir simulations ([2]). This algorithm is given like in [8] by:

Algorithm 1 An IMPES-method

Require: $s_{i,j}$, $q_{i,j}$, T, TT;
  $\Delta t \leftarrow \frac{T}{TT}$;
  for n = 1 to TT **do**
    solve the pressure equation (4.34);
    calculate $v_{x,i+\frac{1}{2},j}$ and $v_{y,i,j+\frac{1}{2}}$ by (4.2.1);
    $t \leftarrow 0$;
    while $t < \Delta t$ **do**
      determine h by the CFL-condition (4.42);
      $t \leftarrow t + h$;
      update $s_{i,j}$ by (4.41);
    end while
  end for

4.2.2 Boundary Conditions for IMPES-Method

Since in [2] and in [8] the boundary conditions for the given IMPES method were not explicitly given for all quantities this section suggests a range of boundary conditions.
First of a all the implicit pressure equation (4.33) must be augmented with a reasonable boundary condition. In [8] and in [2] a zero-boundary condition for the mobilities is suggested:

$$\lambda_{\beta,i+\frac{1}{2},j} = 0 \quad \text{for} \quad i = -1 \text{ or } i = N, \quad 0 \le j \le M, \tag{4.44a}$$

$$\lambda_{\beta,i,j+\frac{1}{2}} = 0 \quad \text{for} \quad j = -1 \text{ or } j = M, \quad 0 \le i \le N, \tag{4.44b}$$

$$\text{for} \quad \beta = T, o, w.$$

And as another possibility it is suggested, that the boundary terms in (4.34) is defined as in [1]:

$$\lambda^n_{T,-\frac{1}{2},j} D_x^+ p^n_{-1,j} = \pi^n_{-\frac{1}{2},j}, \quad \lambda^n_{T,N+\frac{1}{2},j} D_x^+ p^n_{N,j} = \pi^n_{N+\frac{1}{2},j} \tag{4.45a}$$

$$\text{for} \quad i = -1 \text{ or } i = N, 0 \le j \le M,$$

$$\lambda^n_{T,i,-\frac{1}{2}} D_y^+ p^n_{i,-1} = \pi^n_{i,-\frac{1}{2}}, \quad \lambda^n_{T,i,M+\frac{1}{2}} D_y^+ p^n_{i,M} = \pi^n_{i,M+\frac{1}{2}} \tag{4.45b}$$

$$\text{for} \quad j = -1 \text{ or } j = M, 0 \le i \le N,$$

where

$$\pi^n_{-\frac{1}{2},j} = \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \pi(t^n, 0, y)dy,$$

$$\pi^n_{N+\frac{1}{2},j} = \frac{1}{\Delta y}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \pi(t^n, a, y)dy \quad \text{for} \quad 0 \le j \le M,$$

$$\pi^n_{i,-\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \pi(t^n, x, 0)dx,$$

$$\pi^n_{i,M+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \pi(t^n, x, b)dx \quad \text{for} \quad 0 \le i \le N.$$

The discrete total velocity and the numerical flux an the cell-interfaces can be defined by a range of boundary conditions, which all seem to be reasonable. For example: Inserting the zero boundary condition (4.44) for the phase mobilities into (4.2.1) and reusing the boundary values (4.45), it is defined, that

$$v_{x,-\frac{1}{2},j} = -\pi_{-\frac{1}{2},j}, \qquad v_{x,N+\frac{1}{2},j} = -\pi_{N+\frac{1}{2},j} \quad \text{for} \quad 0 \le j \le M, \tag{4.47a}$$

$$v_{y,i,-\frac{1}{2}} = -\pi_{i,-\frac{1}{2}}, \qquad v_{y,i,M+\frac{1}{2}} = -\pi_{i,M+\frac{1}{2}} \quad \text{for} \quad 0 \le i \le N. \tag{4.47b}$$

And of course (4.28d) implies a zero-boundary condition for the velocity flux in the absence of gravity on a subset of the boundary.

In the case of vanishing gravity, all together it is suggested, that the boundary-conditions as listed in table (4.1) should be taken into account, in order to reproduce the scheme, which is introduced in [8].

pressure equation	$\lambda_T = 0$, (4.44)	π, (4.45)	-	mixed
$v_{x,i+\frac{1}{2},j}$	0	$-\pi_{i+\frac{1}{2},j}$, (4.47a)	artificial	mixed
$v_{y,i,j+\frac{1}{2}}$	0	$-\pi_{i,j+\frac{1}{2}}$, (4.47b)	artificial	mixed
$F^v_{x,i+\frac{1}{2},j}$	0	-	artificial	mixed
$F^v_{y,i,j+\frac{1}{2}}$	0	-	artificial	mixed

Table 4.1: Table of suggested boundary conditions for the IMPES-method without garvity.

Chapter 5

Numerical Experiment

The aim of this experiment is reproducing the results in the experiment without capillary pressure and without gravity performed in [8] and the experiment in [2], which was not regularized (regularization parameter $\mu = 0$) and neglects the gravitation.

5.1 Experiment

In this experiment, like it is stated in [2], also known as *quarter-five spot* in the reservoir simulation-community, gravity is neglected ($\rho_w = \rho_o = 0$) while the behaviour of a horizontal reservoir is studied. The quadratic reservoir of the lengths $a = b = 1$ is assumed to be filled with oil at the starting time, $s_{i,j} = 0$ at $t = 0$. The source term, which is given by

$$q_{i,j} = \begin{cases} 1 & \text{for} \quad i = j = 0 \\ -1 & \text{for} \quad i = N \quad \text{and} \quad j = M \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

describes, that water is pumped in the lower left corner and the oil is produced in the upper right corner. The discretization is chosen uniform in both dimensions $N = M = 200$ and 25 time steps, TT, are performed until the final time $T = 0.7$.
$N = 100$ is used in this work, in contrast to the experiment in the literature ([2], [8]). After a rigorous testing of several boundary conditions from table 4.1, the results in figure 5.1 are received. The following boundary conditions were chosen in order to receive this:

$$\pi(t, \mathbf{x}) = -1, \tag{5.2}$$

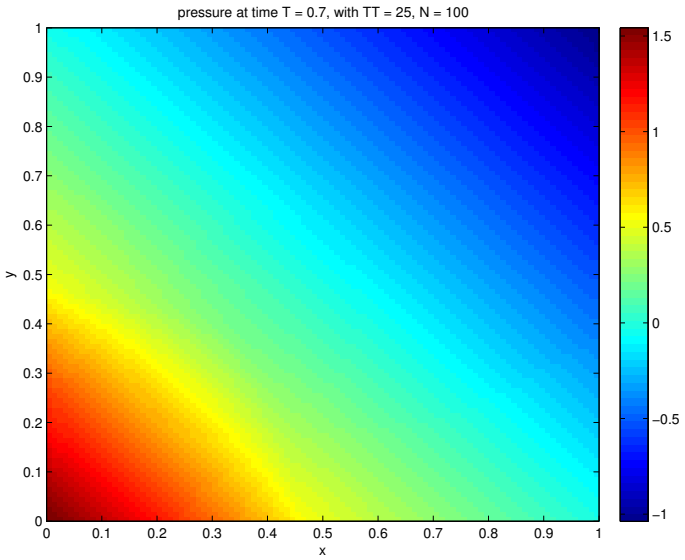
$$v_{x,-\frac{1}{2},j} = v_{x,N+\frac{1}{2},j} = 0 \qquad \text{for} \quad 0 \leq j \leq M, \tag{5.3a}$$

$$v_{y,i,-\frac{1}{2}} = v_{y,i,M+\frac{1}{2}} = 0 \qquad \text{for} \quad 0 \leq i \leq N, \tag{5.3b}$$

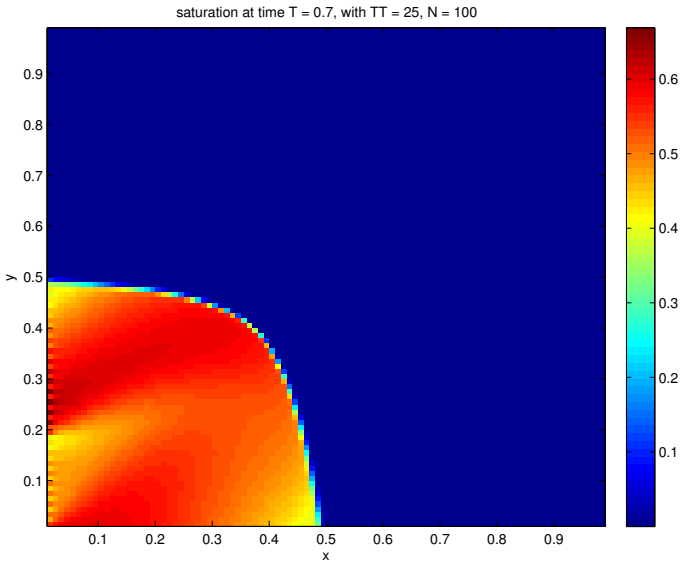
$$F^v_{x,-\frac{1}{2},j} = F^v_{x,\frac{1}{2},j}, \qquad F^v_{x,N+\frac{1}{2},j} = F^v_{x,N-\frac{1}{2},j} \qquad \text{for} \quad 0 \leq j \leq M, \tag{5.3c}$$

$$F^v_{y,i,-\frac{1}{2}} = F^v_{y,i,\frac{1}{2}}, \qquad F^v_{y,i,M+\frac{1}{2}} = F^v_{y,i,M-\frac{1}{2}}, \qquad \text{for} \quad 0 \leq i \leq N. \tag{5.3d}$$

Figure 5.1b shows, that the water is only present in the left lower corner of the reservoir, at the water injection point, and a sharp front is present between the border of water and oil. Looking at figure 5.1a it is obtained, that the pressure is decreasing between the injection and production well and parallel isolines occur at the main diagonal of the reservoir.



(a) Water pressure.



(b) Water saturation.

Figure 5.1: Water pressure and saturation computed with a zero boundary condition for the total mobility, an artificial boundary condition for the fluxes and $\pi(t, \mathbf{x}) = -1$ in the implicit pressure equation.

Chapter 6

Conclusion

In this thesis, the physical underlying model is used in order to obtain the evolution equations of incompressible, two-phase flow in porous media, which are used in modelling oil-recovery. Assuming mass conservation and using Darcy's law the evolution equations are derived. A formulation of implicit pressure and explicit saturation, which is a hyperbolic-elliptic system, is given by substituting the total velocity and the capillary pressure. Conservation laws are examined. By the example of Burger's equation it is investigated, that Riemann problems, do not necessarily have analytical, solutions, but therefore weak solutions. Weak solutions of Riemann problems are constructed by using shock waves and rarefaction waves, taking into account the Rakine-Huginot and the Oleinik-entropy condition. Entropy solutions, which are weak solutions, of conservation laws exist and are unique under the regularity assumptions L^∞ and BV for the initial data.
By the discretization of the space-time domain and the introduction of cell-averages finite volume schemes are deduced and the Godunov method is the solution to the conservation law with Riemann-problem type initial data given by the discretization in each time step.
The IMPES method, which is a finite volume method, for the two-phase flow problem, is investigated for a two-dimensional domain and the quarter-five-spot-experiment, was demonstrated with a small number of grid-points ($N = 100$) compared to the literature ($N = 200$ in [8] and [2]). The typical finger of water pointing at the production well, which can be found in the experminet in [8] and [2] could not be reproduced. However, the shape of the support of the saturation looks reasonable compared to the figure found in [1, figure 1]. The water is present in the lower left corner.

This thesis can be continued at many points. First of all, since the boundary condition for the IMPES-method is not a priori given and does not seem to be clearly reasonable. A more accurate research about the boundary condi-

tion for two-phase problem, explicitly about the behaviour of the pressure-gradient or the total velocity at the boundary, can be done. And the question, of well-posedness of the two-phase flow equations can be investigated. Furthermore, convergence results of numerical methods can be studied. Especially boundary conditions are suggested to be investigated.
On one hand, the numerical experiments can be extended to an experiment which includes gravity or a convergence-analysis experiment can be realised. Further more the development with respect to time of the saturation and pressure can be studied. And on the other, a method with non-vanishing capillary pressure could be performed. Also an extension of the imposed method to three-dimensions or to the black-oil flow can be determined.
The in MATLAB implemented IMPES method can be optimized by researching the field of high-dimensional numerical analysis for a better runtime-performance. Because in the case of a spacial-domain discretization of 200 grid-points ($N = 200$), a 20001×20000 sparse, linear-system has to be solved. A runtime-analysis of the implementation is appropriate.
And finally the method can be compared to other IMPES-like approaches, for example approaches with a regularised two-phase flow problem.

Appendix A

Appendix

A.1 Discrete implicit Pressure Equation as linear System

The discrete implicit pressure equation,

$$D_x^-(\lambda_{T,i+\frac{1}{2},j}D_x^+ p_{i,j}) + D_y^-(\lambda_{T,i,j+\frac{1}{2}}D_y^+ p_{i,j}) = D_y^-((\rho_o\lambda_{o,i,j+\frac{1}{2}} + \rho_w\lambda_{w,i,j+\frac{1}{2}})g_\gamma) - q_{i,j} \quad \text{for} \quad 0 \le i \le N, \quad 0 \le j \le M, \tag{A.1}$$

with the boundary condition

$$\lambda^n_{T,-\frac{1}{2},j}D_x^+ p^n_{-1,j} = \pi^n_{-\frac{1}{2},j}, \quad \lambda^n_{T,N+\frac{1}{2},j}D_x^+ p^n_{N,j} = \pi^n_{N+\frac{1}{2},j} \quad \text{for} \quad 0 \le j \le M, \tag{A.2a}$$

$$\lambda^n_{T,i,-\frac{1}{2}}D_y^+ p^n_{i,-1} = \pi^n_{i,-\frac{1}{2}}, \quad \lambda^n_{T,i,M+\frac{1}{2}}D_y^+ p^n_{i,M} = \pi^n_{i,M+\frac{1}{2}} \quad \text{for} \quad 0 \le i \le N, \tag{A.2b}$$

should be rewritten into a linear system of the form $\mathbf{Ap} = \mathbf{y}$.
For simplification of the implementation in MATLAB it is assumed that $i = 1, \cdots, N$ and $j = 1, \cdots, M$. Then equation (A.1) equals:

$$\frac{\lambda_{T,i+\frac{1}{2},j}\frac{p_{i+1,j}-p_{i,j}}{\Delta x} - \lambda_{T,i-\frac{1}{2},j}\frac{p_{i,j}-p_{i-1,j}}{\Delta x}}{\Delta x} + \frac{\lambda_{T,i,j+\frac{1}{2}}\frac{p_{i,j+1}-p_{i,j}}{\Delta y} - \lambda_{T,i,j-\frac{1}{2}}\frac{p_{i,j}-p_{i,j-1}}{\Delta y}}{\Delta y} = g_\gamma\frac{\rho_o(\lambda_{o,i,j+\frac{1}{2}} - \lambda_{0,i,j-\frac{1}{2}}) + \rho_w(\lambda_{w,i,j+\frac{1}{2}} - \lambda_{w,i,j-\frac{1}{2}})}{\Delta y} - q_{i,j} \quad for \quad 1 \le i \le N, \quad 1 \le j \le M. \tag{A.3}$$

This equation reads as

$$\begin{aligned}\frac{\lambda_{T,i+\frac{1}{2},j}(p_{i+1,j}-p_{i,j})-\lambda_{T,i-\frac{1}{2},j}(p_{i,j}-p_{i-1,j})}{\Delta x^2}\\ +\frac{\lambda_{T,i,j+\frac{1}{2}}(p_{i,j+1}-p_{i,j})-\lambda_{T,i,j-\frac{1}{2}}(p_{i,j}-p_{i,j-1})}{\Delta y^2}\\ =g_\gamma\frac{\rho_o(\lambda_{o,i,j+\frac{1}{2}}-\lambda_{0,i,j-\frac{1}{2}})+\rho_w(\lambda_{w,i,j+\frac{1}{2}}-\lambda_{w,i,j-\frac{1}{2}})}{\Delta y}-q_{i,j}\\ for\quad 1\le i\le N,\quad 1\le j\le M.\end{aligned}\tag{A.4}$$

The left-hand-side is

$$\begin{aligned}\frac{\lambda_{T,i-\frac{1}{2},j}}{\Delta x^2}p_{i-1,j}+\frac{\lambda_{T,i,j-\frac{1}{2}}}{\Delta y^2}p_{i,j-1}-\Big(\frac{\lambda_{T,i+\frac{1}{2},j}+\lambda_{T,i-\frac{1}{2},j}}{\Delta x^2}+\frac{\lambda_{T,i,j+\frac{1}{2}}+\lambda_{T,i,j-\frac{1}{2}}}{\Delta y^2}\Big)p_{i,j}\\ +\frac{\lambda_{T,i,j+\frac{1}{2}}}{\Delta y^2}p_{i,j+1}+\frac{\lambda_{T,i+\frac{1}{2},j}}{\Delta x^2}p_{i+1,j}.\end{aligned}\tag{A.5}$$

In the case $i=1$, replacing $\lambda_{T,\frac{1}{2},j}D_x^+p_{0,j}$ by $\pi_{\frac{1}{2},j}$ in (A.5) gives

$$\begin{aligned}\frac{\lambda_{T,1,j-\frac{1}{2}}}{\Delta y^2}p_{1,j-1}-\Big(\frac{\lambda_{T,\frac{3}{2},j}}{\Delta x^2}+\frac{\lambda_{T,1,j+\frac{1}{2}}+\lambda_{T,1,j-\frac{1}{2}}}{\Delta y^2}\Big)p_{1,j}\\ +\frac{\lambda_{T,1,j+\frac{1}{2}}}{\Delta y^2}p_{1,j+1}+\frac{\lambda_{T,\frac{3}{2},j}}{\Delta x^2}p_{2,j}-\frac{\pi_{\frac{1}{2},j}}{\Delta x}.\end{aligned}\tag{A.6a}$$

And on the other hand the boundary condition in the x-direction gives for $i=N$ the term

$$\begin{aligned}\frac{\lambda_{T,N-\frac{1}{2},j}}{\Delta x^2}p_{N-1,j}+\frac{\lambda_{T,N,j-\frac{1}{2}}}{\Delta y^2}p_{N,j-1}-\Big(\frac{\lambda_{T,N-\frac{1}{2},j}}{\Delta x^2}+\frac{\lambda_{T,N,j+\frac{1}{2}}+\lambda_{T,N,j-\frac{1}{2}}}{\Delta y^2}\Big)p_{N,j}\\ +\frac{\lambda_{T,N,j+\frac{1}{2}}}{\Delta y^2}p_{N,j+1}+\frac{\pi_{N+\frac{1}{2},j}}{\Delta x}.\end{aligned}\tag{A.6b}$$

Similarly as above the implicit pressure equation's left side together with the zero boundary condition in the cases $j=1$ and $j=M$ is

$$\begin{aligned}\frac{\lambda_{T,i-\frac{1}{2},1}}{\Delta x^2}p_{i-1,1}-\Big(\frac{\lambda_{T,i+\frac{1}{2},1}+\lambda_{T,i-\frac{1}{2},1}}{\Delta x^2}+\frac{\lambda_{T,i,\frac{3}{2}}}{\Delta y^2}\Big)p_{i,1}\\ +\frac{\lambda_{T,i,\frac{3}{2}}}{\Delta y^2}p_{i,2}+\frac{\lambda_{T,i+\frac{1}{2},1}}{\Delta x^2}p_{i+1,1}-\frac{\pi_{i,\frac{1}{2}}}{\Delta y},\end{aligned}\tag{A.6c}$$

$$\frac{\lambda_{T,i-\frac{1}{2},M}}{\Delta x^2} p_{i-1,M} + \frac{\lambda_{T,i,M-\frac{1}{2}}}{\Delta y^2} p_{i,M-1} - \Big(\frac{\lambda_{T,i+\frac{1}{2},M} + \lambda_{T,i-\frac{1}{2},M}}{\Delta x^2} + \frac{\lambda_{T,i,M-\frac{1}{2}}}{\Delta y^2} \Big) p_{i,M} + \frac{\lambda_{T,i+\frac{1}{2},M}}{\Delta x^2} p_{i+1,M} + \frac{\pi_{i,M+\frac{1}{2}}}{\Delta y}. \quad \text{(A.6d)}$$

The above constrains, (A.5) and (A.6), can be summarized in a matrix-vector equation of the form

$$\mathbf{A} \begin{pmatrix} p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{i-1,j} \\ \vdots \\ p_{i,j-1} \\ p_{i,j} \\ p_{i,j+1} \\ \vdots \\ p_{i+1,j} \\ \vdots \\ p_{N,M} \end{pmatrix} = \mathbf{y}, \quad \text{(A.7)}$$

where $\mathbf{A}$ is a sparse $NM \times NM$ - dimensional matrix and the non-zero elements are on the diagonal, the first diagonals above and below the main diagonal and the M^{th} diagonals above and below the main diagonal. These elements are

$$A_{k,k-M} = \frac{\lambda_{T,i-\frac{1}{2},j}}{\Delta x^2}, \quad \text{(A.8a)}$$

$$A_{k,k-1} = \frac{\lambda_{T,i,j-\frac{1}{2}}}{\Delta y^2}, \quad \text{(A.8b)}$$

$$A_{k,k} = -\frac{\lambda_{T,i+\frac{1}{2},j} + \lambda_{T,i-\frac{1}{2},j}}{\Delta x^2} - \frac{\lambda_{T,i,j+\frac{1}{2}} + \lambda_{T,i,j-\frac{1}{2}}}{\Delta y^2}, \quad \text{(A.8c)}$$

$$A_{k,k+1} = \frac{\lambda_{T,i,j+\frac{1}{2}}}{\Delta y^2}, \quad \text{(A.8d)}$$

$$A_{k,k+M} = \frac{\lambda_{T,i+\frac{1}{2},j}}{\Delta x^2}, \quad \text{(A.8e)}$$

$$\text{for} \quad k = (i-1)M + j, \quad 1 \le i \le N, \quad 1 \le j \le M, \quad \text{(A.8f)}$$

where

$$\lambda_{T,i+\frac{1}{2},j} = 0 \quad \text{for} \quad i = 0 \text{ or } i = N, 0 \leq j \leq M, \tag{A.9a}$$

$$\lambda_{T,i,j+\frac{1}{2}} = 0 \quad \text{for} \quad j = 0 \text{ or } j = M, 0 \leq i \leq N. \tag{A.9b}$$

The $NM \times 1$-dimensional vector $\mathbf{y}$ is given by

$$\begin{aligned} y_k = g_\gamma \frac{\rho_o(\lambda_{o,i,j+\frac{1}{2}} - \lambda_{o,i,j-\frac{1}{2}}) + \rho_w(\lambda_{w,i,j+\frac{1}{2}} - \lambda_{w,i,j-\frac{1}{2}})}{\Delta y} - q_{i,j} \\ + \frac{\pi_{\frac{1}{2},j}}{\Delta x}\delta_{i,1} - \frac{\pi_{N+\frac{1}{2},j}}{\Delta x}\delta_{i,N} + \frac{\pi_{i,\frac{1}{2}}}{\Delta y}\delta_{j,1} - \frac{\pi_{i,M+\frac{1}{2}}}{\Delta y}\delta_{j,M} \\ \text{for} \quad k = (i-1)M + j, \quad 1 \leq i \leq N, \quad 1 \leq j \leq M, \end{aligned} \tag{A.10a}$$

where $\delta_{i,j}$ denotes the Kronecker delta.

Furthermore, from equations (A.6c) and (A.6d) it follows, that $A_{k-1,k} = 0$ for $j = 1$ and $A_{k,k+1} = 0$ for $j = M$. Hence the matrix $\mathbf{A}$ is a block-tridiagonal matrix, where each block has the size $M \times M$ and is symmetric:

$$\mathbf{A} = \begin{pmatrix} \mathbf{B}^1 & \mathbf{C}^1 & \mathbf{0} & \cdots & & & \cdots & \mathbf{0} \\ \mathbf{D}^1 & \mathbf{B}^2 & \mathbf{C}^2 & \mathbf{0} & \cdots & & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^2 & \mathbf{B}^3 & \mathbf{C}^3 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{D}^{N-3} & \mathbf{B}^{N-2} & \mathbf{C}^{N-2} & \mathbf{0} \\ \mathbf{0} & \cdots & & \cdots & \mathbf{0} & \mathbf{D}^{N-2} & \mathbf{B}^{N-1} & \mathbf{C}^{N-1} \\ \mathbf{0} & \cdots & & & \cdots & \mathbf{0} & \mathbf{D}^{N-1} & \mathbf{B}^N \end{pmatrix}, \tag{A.11}$$

where

$$\begin{aligned} B^i_{j,k} = \delta_{j-1,k}\frac{\lambda_{T,i,j-\frac{1}{2}}}{\Delta y^2} + \delta_{j,k}\Big(- \frac{\lambda_{T,i+\frac{1}{2},j} + \lambda_{T,i-\frac{1}{2},j}}{\Delta x^2} - \frac{\lambda_{T,i,j+\frac{1}{2}} + \lambda_{T,i,j-\frac{1}{2}}}{\Delta y^2}\Big) \\ + \delta_{j+1,k}\frac{\lambda_{T,i,j+\frac{1}{2}}}{\Delta y^2} \\ \text{for} \quad 1 \leq i \leq N, \quad 1 \leq j,k \leq M, \end{aligned}$$

$$C^i_{j,k} = D^i_{j,k} = \delta_{j,k}\frac{\lambda_{T,i+\frac{1}{2},j}}{\Delta x^2} \quad \text{for} \quad 1 \leq i \leq N-1, \quad 1 \leq j,k \leq M.$$

The Matrix A can be initialized in MATLAB using the commands `sparse` and `spdiags` for the speed up of the backslash operator `/`, saving of memory amount and for simplicity of implementation. The backslash operator should be applied to a linear system which takes the constraint (4.36) into account, in order to obtain a matrix of full rank.

Remark A.1 *In the case of the boundary condition (4.44) for the discretized pressure equation (A.4) the left-hand side* **A** *remains the same as in (A.11). The right-hand side then reads as*

$$y_k = g_\gamma \frac{\rho_o(\lambda_{o,i,j+\frac{1}{2}} - \lambda_{o,i,j-\frac{1}{2}}) + \rho_w(\lambda_{w,i,j+\frac{1}{2}} - \lambda_{w,i,j-\frac{1}{2}})}{\Delta y} - q_{i,j} \qquad \text{(A.12)}$$
$$\text{for} \quad k = (i-1)M + j, \quad 1 \le i \le N, \quad 1 \le j \le M.$$

This can be verified by repeating the same lengthy calculation as above.

A.2 The MATLAB-Code

```
function [s,p] = impes(T,TT,a,b,s,q,rho_w,rho_o )
% IMPES-method for two-phase reservoir flow
% IN:  T: final time, double,
%      TT: time steps, integer
%      [0,a]x[0,b]: spacial domain, a, b postive doubles,
%      s: initial data of saturation, NxM matrix of doubles
%      q: source term, NxM matrix of doubles
%      rho_w: desity of water, postive double
%      rho_o: desity of oil, postive double
% OUT: s(i,j)  N x M -dimensional array s(i,j) ~ s(x_i,y_j,T)
%      p(i,j)  N x M -dimensional array p(i,j) ~  p(x_i,y_j,T)

% Author: Tanja Almeroth
% last update 26.04.15


% BC:
%
% lamda_T: 0, lamda_w: 0, lamda_o: 0
% p:    pi = -1
% v_x:  0
% v_y:  0
% F_vx: artificial
% F_vy: artificial
% F_gy: 0

[N,M] = size(q);
q = sparse(q);

% spacial step-size
delta_x = a/N;
delta_y = b/M;
% time-step size
delta_t = T/TT;

% CFL
CFL = 0.95;

```

```
39 % pressure
40 p = zeros(N,M);
41
42 % mobilities
43 l_o = @(x) (1 - x).^2;
44 l_w = @(x) x.^2;
45 l_T = @(x) (1-x).^2 + x.^2;
46
47 % acceleration of gravity
48 g_gam = 1;
49
50 % fractional-flow funktion
51 f = @(x) l_w(x) ./ l_T(x);
52 % gravity function
53 g = @(x) l_w(x) .* l_o(x) * g_gam * (rho_w - rho_o)
       ./ l_T(x);
54
55
56 % allocation of storage location
57 l_o_halfi = zeros(N+1,M);
58 l_w_halfi = zeros(N+1,M);
59 l_T_halfi = zeros(N+1,M);
60 l_o_halfj = zeros(N,M+1);
61 l_w_halfj = zeros(N,M+1);
62 l_T_halfj = zeros(N,M+1);
63
64 v_halfi = zeros(N+1,M);
65 v_halfj = zeros(N,M+1);
66
67 F_v_halfi = zeros(N+1,M);
68 F_v_halfj = zeros(N,M+1);
69
70 F_g_halfj = zeros(N,M+1);
71
72 for nt = 1:TT
73    %% phase-mobilities
74    l_o_halfi(2:end-1,:) = reshape( harmmean( [
         reshape(l_o(s(1:end-1,:))',1,(N-1)*M); reshape(
          l_o(s(2:end,:))',1,(N-1)*M) ] ),M,N-1 )';
75    l_w_halfi(2:end-1,:) = reshape( harmmean( [
         reshape(l_w(s(1:end-1,:))',1,(N-1)*M); reshape(
          l_w(s(2:end,:))',1,(N-1)*M) ] ),M,N-1 )';
76    l_T_halfi(2:end-1,:) = reshape( harmmean( [
         reshape(l_T(s(1:end-1,:))',1,(N-1)*M); reshape(
```

```
        l_T(s(2:end,:))',1,(N-1)*M) ] ),M,N-1 )';
77
78      l_o_halfj(:,2:end-1) = reshape( harmmean( [
           reshape(l_o(s(:,1:end-1))',1,N*(M-1)); reshape(
            l_o(s(:,2:end))',1,N*(M-1)) ] ),M-1,N )';
79      l_w_halfj(:,2:end-1) = reshape( harmmean( [
           reshape(l_w(s(:,1:end-1))',1,N*(M-1)); reshape(
            l_w(s(:,2:end))',1,N*(M-1)) ] ),M-1,N )';
80      l_T_halfj(:,2:end-1) = reshape( harmmean( [
           reshape(l_T(s(:,1:end-1))',1,N*(M-1)); reshape(
            l_T(s(:,2:end))',1,N*(M-1)) ] ),M-1,N )';
81
82      %% implicit pressure equation
83      % diagonal block
84      b_diag = - reshape( l_T_halfi(2:end,:)' +
           l_T_halfi(1:end-1,:)',N*M,1 ) ./ delta_x^2 -
           reshape( l_T_halfj(:,2:end)' + l_T_halfj(:,1:
           end-1)',N*M,1 ) ./ delta_y^2;
85      b_diag_do = reshape(  l_T_halfj(:,2:end)',N*M,1 )
           ./ delta_y^2;
86      b_diag_up = reshape(  l_T_halfj(:,1:end-1)',N*M,1
           ) ./ delta_y^2;
87      % off-diagonal blocks
88      c_diag_do = reshape( l_T_halfi(2:end,:)',N*M,1 )
           ./ delta_x^2;
89      c_diag_up = reshape( l_T_halfi(1:end-1,:)',N*M,1 )
            ./ delta_x^2;
90
91      A = spdiags( [c_diag_do, b_diag_do, b_diag,
           b_diag_up, c_diag_up], [-M,-1,0,1,M], N*M, N*M
           );
92
93    % right-hand side
94      int_pi_i = -1/ delta_x;
95      pi_halfi =  int_pi_i * sparse( [ones(1,M), N*ones
           (1,M)],[1:M, 1:M], [ones(1,M), -ones(1,M)],N,M
           );
96      int_pi_j = -1/ delta_y;
97      pi_halfj =  int_pi_j * sparse( [1:N, 1:N ], repmat
           ([1,M],1,N), repmat([1,-1],1,N),N,M );
98
99      y =  reshape((-q+pi_halfi+pi_halfj)',N*M,1);
100
101     B =[A; ones(1,N*M) * delta_x * delta_y];
```

```
102     x = [y; 0];
103     p = reshape(full(B\x),M,N)';
104
105     %% velocities
106     v_halfi(2:end-1,:) = - l_T_halfi(2:end-1,:) .* ( p
            (2:end,:) - p(1:end-1,:) ) / delta_x;
107     %v_halfi(1,:) = v_halfi(2,:); v_halfi(end,:) =
            v_halfi(end-1,:);
108     v_halfj(:,2:end-1) = - l_T_halfj(:,2:end-1) .* ( p
            (:,2:end) - p(:,1:end-1) ) / delta_y + g_gam *
            ( rho_w * l_w_halfj(:,2:end-1) + rho_o *
            l_o_halfj(:,2:end-1));
109     %v_halfj(:,1) = v_halfj(:,2); v_halfj(:,end) =
            v_halfj(:,end-1);
110
111     %% update of saturation
112     t = 0;
113     while( t < delta_t )
114
115         h = CFL ./ ( max(max(abs(v_halfi))) ./ delta_x
                + max(max(abs(v_halfj))) ./ delta_y );
116         t = t + h;
117
118         F_v_halfi(2:end-1,:) = v_halfi(2:end-1,:) .* (
                (v_halfi(2:end-1,:) > 0) .* f(s(1:end-1,:)
            ) + (v_halfi(2:end-1,:) <= 0) .* f(s(2:end
                ,:)) );
119         F_v_halfi(1,:) = F_v_halfi(2,:); F_v_halfi(end
            ,:) = F_v_halfi(end-1,:);
120         F_v_halfj(:,2:end-1) = v_halfj(:,2:end-1) .* (
                (v_halfj(:,2:end-1) > 0) .* f(s(:,1:end-1)
            ) + (v_halfj(:,2:end-1) <= 0) .* f(s(:,2:
            end)) );
121         F_v_halfj(:,1) = F_v_halfj(:,2); F_v_halfj(:,
            end) = F_v_halfj(:,end-1);
122         max_s = reshape( max([reshape(s(:,1:end-1),1,N
            *(M-1)); ones(1,N*(M-1)) * 0.5 ]),N,M-1 );
123         min_s = reshape( min([reshape(s(:,2:end),1,N*(
            M-1)); ones(1,N*(M-1)) * 0.5 ]),N,M-1 );
124         F_g_halfj(:,2:end-1) = g(max_s) + g(min_s);
125
126         s = s - h * ( ( F_v_halfi(2:end,:) - F_v_halfi
            (1:end-1,:) ) ./ delta_x + ( F_v_halfj(:,2:
            end) -  F_v_halfj(:,1:end-1) ) ./ delta_y )
```

```
            + h * q;
127
128     end
129
130 end
131
132 end
```

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