Shape Recognition and Classification via Anisotropic Polarization Tensors

Master Thesis

Tanja Almeroth

tanja.almeroth@mailbox.org

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Outline

Weakly Electric Fish measure Distance in the Dark

A bio-inspired imaging method: active electro-sensing

behavioral studies:

- **locate** targets
- discriminate between different **shapes**
- discriminate between different **capacitances and conductivies** of objects from surrounding water

application is of interest in:

- neuro-ethology
- signal processing
- applied mathematics

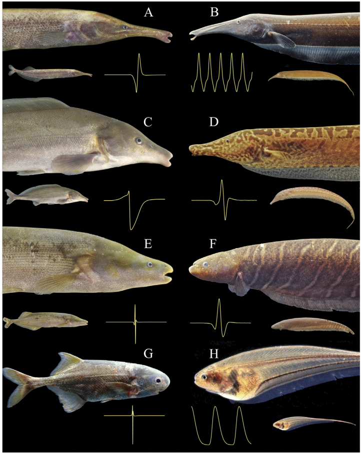

Figure: Mormyroid African electric fishes (left column) and gymnotiform South American electric fishes (right column) [?, fig. 2]

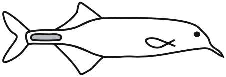

Figure: Elephant Nose Fishes (Mormyriformes). Source: [, fig. 2.3]

electrolocation process including a **self-generated electric field**

- generates a relatively **high frequency** $(0,1 - 10\ kHz)$ **weakly electric** $(\leq 100 \frac{mV}{cm})$ field
- perceives **transdermal potential**
- locates target, identify shape and material parameters by the **electrical distribution on the skin**

mathematical challenges:

- **fundamentally ill-posed** imaging problem ([3],[2]), existence, uniqueness and continuity is not guaranteed ([3])
- **Calderón's problem**, electric field perturbation depend on a complicated highly non-linear function depending on shape, admitivity and distance [36]

Anisotropic and Isotropic Targets

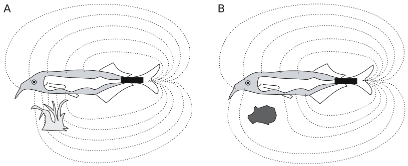

Figure: Distortions of the electric field of Gnathonemus petersii caused by natural objects with impedances differing from that of the surrounding water. A: Anisotropic Target, B: Isotropic target.[**?**, fig. 2.12]

- anisotropic targets
  - capacitive components
  - electromagnetic parameters frequency dependent
  - assume constant conductivity in the cells
- extended targets
- possible application
  - cell culture production

- isotropic targets
  - capacitive components
  - electromagnetic parameters frequency independent
- applications
  - EIT, transdermal scanners of breast tumors, robotics

Anisotropic Transmission Problem

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ bounded Lipschitz domain
- Target: $D \subset \Omega$ a bounded Lipschitz Domain with conductivity $\tilde{A}$
- Background: $\Omega \setminus \overline{D}$ with conductivity A

  $A, \tilde{A}$ constant, positive-definite matrices, $A \neq \tilde{A}$ and $A - \tilde{A}$ positive-definite or negative-definite.
- $g \in L^2_0(\partial\Omega)$ given

u steady-state voltage in the presence of an anisotropic counductifity inclusion D solves

$$\nabla \cdot \left(\chi(\Omega \setminus \overline{D})A + \chi(D)\tilde{A}\right)\nabla u = 0 \quad \text{in} \quad \Omega \tag{1a}$$

$$\nu \cdot A\nabla u|_{\partial\Omega} = g, \tag{1b}$$

$$\int_{\partial\Omega} u(x) \quad d\sigma(x) = 0. \tag{1c}$$

The conductivity profile of Ω is given by

$$\gamma_D = \chi(\Omega \setminus \overline{D})A + \chi(D)\tilde{A}. \tag{2}$$

Definition (The Nemann-Poincaré operator)

[?, p. 56 - p. 57] Let $D \subset \mathbb{R}^d$ be a bounded $\mathcal{C}^2$-domain. The operators $\mathcal{K}_D, \mathcal{K}_D^* : L^2(\partial D) \to L^2(\partial D)$ are given by

$$\mathcal{K}_D[\phi](x) := \frac{1}{\omega_d} \int_{\partial D} \frac{\langle y-x, \nu_y \rangle}{|x-y|^d} \phi(y)\, d\sigma(y) \text{ and} \tag{3a}$$

$$\mathcal{K}_D^*[\phi](x) := \frac{1}{\omega_d} \int_{\partial D} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \phi(y)\, d\sigma(y), \tag{3b}$$

where ω_d denotes the volume of the unit sphere in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ the Euclidean inner product in $\mathbb{R}^d$.

properties:

- $D \subset \mathbb{R}^d$ a **bounded Lipschitz-domain**
  - $\mathcal{K}_D^*$ is the $L^2(\partial D)$-adjoined of $\mathcal{K}_D$, **bounded** and **compact** operators
- For $\lambda \in \mathbb{R}$ and $D \subset \mathbb{R}^d$ a bounded $\mathcal{C}^2$-domain
  - $\lambda I - \mathcal{K}_D^*$ is one to on on $L_0^2(\partial D)$ if $|\lambda| \geq \frac{1}{2}$
  - $\lambda I - \mathcal{K}_D^*$ is one to one on $L^2(\partial D)$ if $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$.

Definition (The Generalized Polarization Tensor)

[?, eq. 4.1] The generalized polarization tensor associated with the bounded $\mathcal{C}^2$-domain $D \subset \mathbb{R}^2$ and the parameter λ, $|\lambda| > \frac{1}{2}$ for the pair of multi-indices $i,j \in \mathbb{N}^2$, $|i|, |j| \geq 1$ is defined by

$$M_{i,j}^G = M_{i,j}^G(\lambda, D) = \int_{\partial D} y^j \phi_i(y)\, d\sigma(y) \tag{4a}$$

$$\text{for} \quad \phi_i(y) = (\lambda I - \mathcal{K}_D^*)^{-1}[\nu_x \cdot \nabla x^i](y), \quad y \in \partial D. \tag{4b}$$

- $\lambda = \frac{k+1}{2(k-1)}$ **contrast**, k conductivity of the inclusion
- The GPT carries **all of the information about the inclusion** D in a **homogeneous background medium** with **unit conductivity** [?, Theorem 4.3].
  - translation, rotation and scaling properties expose invariant descriptors
  - symmetric with respect to harmonic coefficients
  - positivety

Anisotropic Polarization tensor?

- the **single layer** potential associated with A of the density function $\phi \in L^2(\partial D)$

$$\mathcal{S}_D^A[\phi](x) := \int_{\partial D} \Gamma^A(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d \tag{5}$$

- the **double layer** potential

$$\mathcal{D}_D^A[\phi](x) := \int_{\partial D} \nu_y \cdot A\nabla\Gamma^A(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D. \tag{6}$$

- the **fundamental solution** $\Gamma^A(x)$ of the operator $\nabla \cdot A\nabla$

$$\Gamma^A(x) := \begin{cases} \dfrac{1}{2\pi\sqrt{\det(A)}} \log(|A_* x|), & d = 2, \\ -\dfrac{1}{4\pi\sqrt{\det(A)}|A_* x|}, & d = 3, \end{cases} \tag{7}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Anisotropic Polarization Tensors

Definition

Let $i \in \mathbb{N}^d$ be a multi-index with $|i| \geq 1$, assume $(f_i, g_i) \in L^2(\partial D) \times L^2(\partial D)$ is the unique solution to

$$\mathcal{S}_D^{\tilde{A}} f_i - \mathcal{S}_D^{A} g_i = x^i, \tag{8a}$$

$$\nu \cdot \tilde{A} \nabla \mathcal{S}_D^{\tilde{A}} f_i|_- - \nu \cdot A \nabla \mathcal{S}_D^{A} g_i|_+ = \nu \cdot A \nabla x^i \quad \text{on } \partial D, \tag{8b}$$

where $x^i = x_1^{i_1} \cdots x_d^{i_d}$.

The generalized anisotropic polarization tensor associated with the domain D and anisotropic conductivities $\tilde{A}$ and A, respectively the conductivity profile γ_D for the pair of multi-indices $i, j \in \mathbb{N}^d$ is defined by

$$M_{i,j} = M_{i,j}(A, \tilde{A}, D) = \int_{\partial D} x^j g_i(x)\, d\sigma(x). \tag{9}$$

Furthermore, if $i = \mathbf{e}_p$ and $j = \mathbf{e}_q$ for $p, q = 1, ..., d$, where $\{\mathbf{e}_1, ..., \mathbf{e}_d\}$ denotes the standard basis of $\mathbb{R}^d$, we write $M_{i,j} = m := (m_{pq})_{p,q=1}^d$, $g_p = g_i$ and

$$m_{p,q} = \int_{\partial D} x_q g_p(x)\, d\sigma(x). \tag{10}$$

Homogeneous background medium in two-dimensions

$$m(I, \tilde{A}, B) = 2|B|(\tilde{A} + I)^{-1}(\tilde{A} - I) \tag{11}$$

- For $\mathcal{E}$, an ellipse semi-axis aligned with the x_1-x_2-axis of length a and b the GPT is

$$M^G(k,\mathcal{E}) = (k-1)|\mathcal{E}| \begin{pmatrix} \frac{a+b}{a+kb} & 0 \\ 0 & \frac{a+b}{b+ka} \end{pmatrix} \tag{12}$$

- We have $\quad m(I,\tilde{A},B) = M^G(k,\mathcal{E})$

$$\text{for} \quad \tilde{A} = \begin{pmatrix} \frac{(k-1)|\mathcal{E}|(a+b)+2|D|(a+kb)}{(k-1)|\mathcal{E}|(b-a)+2|D|(a+kb)} & 0 \\ 0 & \frac{(k-1)|\mathcal{E}|(a+b)+2|D|(ka+b)}{(k-1)|\mathcal{E}|(a-b)+2|D|(b+ka)} \end{pmatrix}. \tag{13}$$

An **isotropic ellipse** is equivalent to an **anisotropic ball** with **homogeneous background** medium of **unit conductivity**.

Anisotropic Neumann-Poincaré Operator

- Homogenous Background medium with permittivity ϵ_m

$$A := \epsilon_m I \quad \text{and} \quad \tilde{A} := A, \tag{14}$$

- Let $i \in \mathbb{N}^d$ with $|i| > 1$ and $(f_i, g_i) \in L^2(\partial D) \times L^2(\partial D)$ be the solution to (8). Then we have:

Lemma

Let $D \Subset \mathbb{R}^3$, be a bounded simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$, the operator $\mathcal{S}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is invertible. Moreover, we have the jump formula

$$\nu \cdot A \nabla \mathcal{S}_D^A|_{\pm} = \pm \frac{1}{2} I + (\mathcal{K}_D^A)^*, \quad \textit{where} \tag{15}$$

$$(\mathcal{K}_D^A)^*[\phi](x) = \int_{\partial D} \frac{\langle x - y, \nu_x \rangle}{4\pi \sqrt{det(A)} |A_*(x-y)|^3} \phi(y) d\sigma(y). \tag{16}$$

Lemma

For $D \Subset \mathbb{R}^2$ of class $\mathcal{C}^{1+\alpha}$, $0 < \alpha < 1$, the jump relation (15) is satisfied for

$$(\mathcal{K}_D^A)^*[\phi](x) = \int_{\partial D} \frac{\langle x - y, \nu_x \rangle}{2\pi \sqrt{\det(A)} |A_*(x-y)|^2} \phi(y) \, d\sigma(y). \tag{17}$$

Lemma

Let $\lambda \in \mathbb{R}$, $D \Subset \mathbb{R}^d$ be a bounded, simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$, then

i) $\lambda I - (\mathcal{K}_D^A)^*$ *is one to one on* $L_0^2(\partial D)$ *if* $|\lambda| \geq \frac{1}{2}$

ii) $\lambda I - (\mathcal{K}_D^A)^*$ *is one to one on* $L^2(\partial D)$ *for* $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$

- Jump relation
  $D \Subset \mathbb{R}^d$ be a bounded, simply connected domain of class $C^{1,\alpha}$ for $0 < \alpha < 1$ and $\phi \in L^2(\partial D)$ then

$$\mathcal{D}_D^A \phi|_{\pm}(x) = \left(\mp \frac{1}{2} + \mathcal{K}_D^A \right) \phi(x) \quad a.e.\ x \in \partial D. \tag{18}$$

- Symmetrization
  The following Calderón identity holds on $H^{-\frac{1}{2}}(\partial D)$:

$$\mathcal{S}_D^A (\mathcal{K}_D^A)^* = \mathcal{K}_D^A \mathcal{S}_D^A \tag{19}$$

- For $D \Subset \mathbb{R}^3$ simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$ we have
  i) $\mathcal{S}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is self-adjoint and
  ii) $(\mathcal{K}_D^A)^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact.

Anisotropic Neumann-Poincaré Operator

- 3D case
  It follows that $f_i = (\mathcal{S}_D^A)^{-1}\big(\mathcal{S}_D[g_i] + x^i\big)$. And by the jump relation (15) we have [**?**, eq. 5.4]

$$\mathcal{Q}_D^A[g_i] = F_D^A, \quad \text{where} \tag{20a}$$

$$\mathcal{Q}_D^A = \frac{1}{2}\Big(\epsilon_m I + \big(\mathcal{S}_D^A\big)^{-1}\mathcal{S}_D\Big) \quad + \quad \Big(\epsilon_m \mathcal{K}_D^* - (\mathcal{K}_D^A)^*(\mathcal{S}_D^A)^{-1}\mathcal{S}_D\Big)$$

$$F_D^A = -\epsilon_m \nu \cdot \nabla x^i + \big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\big)(\mathcal{S}_D^A)^{-1}[x^i]. \tag{20b}$$

- 2D case
  - Invertibility of $\mathcal{S}_D^A$ is open
  - Invertibility of $\mathcal{S}_D$ is shown in using Fredholm theory and Rellich identiities

Invertibility of $\mathcal{Q}_D^A$

$D \Subset \mathbb{R}^3$ bounded, simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$.

Definition (Fredholm Operator)

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. $A \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is called Fredholm if $\mathrm{Im}(A) \subseteq \mathcal{B}_2$ is closed, $\mathrm{Ker}(A) \subseteq \mathcal{B}_1$ is finite-dimensional and $\mathrm{Im}(A)$ is of finite co-dimension. We write $A \in \mathrm{Fred}(\mathcal{B}_1, \mathcal{B}_2)$.

For $A \in \mathrm{Fred}(\mathcal{B}_1, \mathcal{B}_2)$ the **index of A** is given by

$$\begin{aligned} \mathrm{ind}(A) = \mathrm{dim}(Ker(A)) - \mathrm{codim}(Im(A)) \\ = \mathrm{dim}(Ker(A)) - \mathrm{dim}(\mathcal{B}_2 / Im(A)). \end{aligned} \tag{21}$$

Lemma

$\mathcal{Q}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ *is Fredholm and of index zero.*

For the APT it can be shown:

- translation, rotation and scaling properties expose invariant descriptors
- symmetric with respect to harmonic coefficients

- $\mathcal{D}$ dictionary, collection of standart shapes, centerd at the origin and of characteristic size one
- $B \in \mathcal{D}$ unknown shape
- unknown transformation parameters
  α, β and $\gamma \in [0, 2\pi)$ rotation angles, $s \in \mathbb{R}_{>0}$ scaling parameter and $T \in \mathbb{R}^3$ translation vector
- $D = T_T s R_{\alpha\beta\gamma} B$, $B \in \mathcal{D}$, $R_{\alpha\beta\gamma} \in SO(3)$ given transformed target
- A given anisotropic conductivity of the target and ϵ_m of background

Under the assumption on the previous slide the first order APT satisfies

$$m(\epsilon_m I_{,\alpha\beta\gamma} A_{\alpha\beta\gamma}, D) = \frac{1}{s} R_{\alpha\beta\gamma} m(\epsilon_m I, A, B) R_{-\alpha\beta\gamma}. \tag{22}$$

Moreover, let $i, j \in \mathbb{N}^3$ with $|i|, |j| > 0$ for the general APT the translation formula writes as

$$M_{i,j}(\epsilon_m I, A, B) = \sum_{k,l} c_{j,k}^T c_{i,l}^T M_{l,k}(\epsilon_m I, A, T_T B) \tag{23}$$

$$\text{for} \quad (x - T)^i = \sum_j c_{i,j}^T x^j.$$

It follows that the first order Anisotropic Polarization Tensor is **invariant under translation** .

For any transformed domain $D = T_T s R_{\alpha\beta\gamma} B$, $D \in \mathcal{D}$ we have

$$\mathcal{I}(D) = \frac{tr(m(\epsilon_m I,_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))^3}{\det(m(\epsilon_m I,_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))} = \frac{tr(m(\epsilon_m I, A, B))^3}{\det(m(\epsilon_m I, A, B))}.$$

Algorithm 1 Shape Identification in three Dimensions

Require: Shape descriptor $\mathcal{I}(D)$ of an unknown target D

for $B_n \in \mathcal{D}$ **do**

$e_n \leftarrow |\mathcal{I}(D) - \mathcal{I}(\mathcal{B}_n|$;

$n \leftarrow n + 1$;

end for

return true dictionary element $n^* \leftarrow \arg\min_n e_n$.

- scaling parameter $s \in \mathbb{R}_{>0}$ can be uniquely determined by

$$s = \sqrt[3]{\frac{\det(m(\epsilon_m I, A, B))}{\det(m(\epsilon_m I,_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))}} \tag{24}$$

- rotation Parameters $\alpha, \beta, \gamma \in [0, 2\pi)$ can be uniquely determined be the non-linear least squares problem

$$(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \arg\min_{(\alpha,\beta,\gamma)} \left\| m(\epsilon_m I,_{-\alpha,\beta,\gamma} A_{\alpha\beta\gamma}, D) - \frac{1}{s} R_{\alpha\beta\gamma} m(\epsilon_m I, A, D) R_{-\alpha\beta\gamma} \right\|_F^2 \tag{25}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

- translation parameter $T \in \mathbb{R}_{>0}$ can be uniquely determined by

$$s = \sqrt[3]{\frac{\det(m(\epsilon_m I, A, B))}{\det(m(\epsilon_m I,_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))}} \tag{26}$$

- rotation Parameters $\alpha, \beta, \gamma \in [0, 2\pi)$ can be uniquely determined be the non-linear least squares problem

$$(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \underset{(\alpha,\beta,\gamma)}{\arg\min} \left\| m(\epsilon_m I,_{-\alpha,\beta,\gamma} A_{\alpha\beta\gamma}, D) - \frac{1}{s} R_{\alpha\beta\gamma} m(\epsilon_m I, A, D) R_{-\alpha\beta\gamma} \right\|_F^2 \tag{27}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

Outlook

- 2D case
  - invertibility of $\mathcal{S}_D^A$
  - complex conjugated Anisotropic Polarization Tensors
- 3D case
  - implementation
- Establish more properties of the APT in particular positivity