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Shape Recognition and Classification via Anisotropic Polarization Tensors

Master Thesis

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Abstract

Motivated by the weakly electric fish's electro-sensing skill set we examine the anisotropic transmission problem to identify targets with anisotropic conductivity profile based on anisotropic polarization tensors. To that features of anisotropic polarization tensors and anisotropic layer potentials are shown and invariants are derived. Moreover, common properties of the isotropic and anisotropic single and double layer potential are stated among a spectral decomposition of the anisotropic Neumann-Poincaré operator. Finally a target identification and location procedure for three-dimensional extended targets based on first and second order anisotropic polarization tensors is given.

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Chapter 1

Introduction

There is an interesting history of weakly electric fish in various disciples of science due to their usage of electric fields. The modelling of the underlying physical mechanism inspires applications in robotics and biomedical imaging.

1.1 Shape Recognition and Classification

Using their physical mechanism of shape perception, weakly electric fish can orient themselves in complete darkness by their active electrostatic system [2]. In a stable manner, these fishes use a weak electric field with a relatively high frequency and perceive a transdermal potential to identify targets by their electromagnetic properties differing from the surrounding water. They are able to classify biological organisms. It is known that by capacity effects due to the cell membrane structure of living targets, measurements at multiple frequencies can exploit spectral content of the perceived transdermal potential modulations [30]. The shape recognition procedure is based on identifying targets by classification models, assuming that they belong to a learned dictionary.

1.1.1 Weakly Electric Fish

Serveral fish species and various families are able of electro-sensing. Their orders are *Mormyriforms*, which are living in the turbid rivers of South Africa and *Gymnotiforms* in South America [3]. Furthermore, the species have two time representations of their *Electric Organ Discharges* (EODs) [17]. There are wave-type species such as *Apternotus albifrons*, whose emitted fields are sinosodial, and pulse type eg. *Gnathonemus petersii* [3]. Weakly electric fish use stable, relativley high frequency ($0.1 - 10\ kHz$) and weakly electric ($\leq 100\ \frac{mV}{cm}$) fields for electro-location [2]. These fields are not strong enough for the defense against predators [2]. Weakly electric fish together with strong electric fish contribute to a wider set of species, the electric fish [31]. An illustration of some species of electric fish can be found in figure 1.1.

In 1958 Lissmann and Machin investigated the fish, which emit electrical signals for the purpose of *active electro-sensing* [29]. Thousands of receptors [2], *T-type* and *P-type units*, on the fish's skin sense the perturbation of the transdermal potential [31]. The field is produced in a special organ, compromised by *electrocytes* in the tail region [15]. Targets have different admittivity, in particular electric capacity and conductivity, from the sourrounding water. Moreover, permittivity differences appetite phase shifts [6] and behavioral experiments with *Gymnchus niloticus* demonstrated the fish's ability to determine these [33]. Lissmann and Machin governed the physical phenomena by an electric dipole formula, which did not explain phase shifts [15] as claimed in [6]. In 1996 Rossnow [32] introduced a complex conductivity, for the purpose of increasing the strength of the model. Many more researches contributed to the model as delineated in [6].

1.1.2 Active Electro-Sensing

Today, since the work by Lissmann and Machin there is an increasing number of experimental behavioral, biological and computational studies [2]. Since weakly electric fish can locate targets [39], discriminate between targets with different shapes [38], discriminate differences of electric parameters, i.e., conductivity and permittivity [39] and estimate the distances to objects [39], there is a great interest of understanding the underlying mechanism of electro-sensing. Next to the curiosity of discovering a sixth sense electro sensing is applied in bio-inspired applications in underwater robotics in dark turbid environments (eg. [16]). Furthermore, the principles of electro-sensing are of interest in neuro-ethology, signal processing and applied mathematics [3]. In the literature active electro-sensing is called an electrolocation process including a self-genrated electric field [31]. The effective range is a few centimeter around the weakly electric fish [31].

1.1.3 Mathematical Challenges

Locating targets, identifying shapes and material parameters by the distribution on the skin is an *inverse problem* [3]. The forward problem consists in computing the electric field surrounding the fish, knowing everything about the object [3]. In particular this is a *fundamentaly ill-posed* imaging problem [2], which means the existance, uniqueness and continutiy are not garanteed [3]. Moreover, the electric field pertubation due to the target is a complicated and highly nonlinear function depending on the shape, admittivity and the distance to the fish [2]. In mathematics the problem is known as *Calderón's problem* [36]. An analytic understanding intends an insight into

electro-sensing [2]. Locating targets from potential perturbations is understood now ([28], [1]), but classifying shapes is the most challenging problem in electro-sensing [2]. In [1] a rigorous model for electro-location is derived. Therein, as summarized in [2], it is assumed that, the field emitted by the electric organ is time harmonic and has a known fundamental frequency. Therefore, a space-frequency location search algorithm is applied. It is of interest if this procedure is robust with respect to measurement noise and sensitivity with respect to the number of frequencies, sensors and distance of the target. For disk- and ellipse-shaped targets the conductivity, permittivity and their size are reconstructed separately from multi frequency measurements.
The procedure can be applied in electrical impedance tomography (EIT), a non-invasive imaging technique [3], appplied for transdermal scanners of breast tumours ([10], [25], [34], [30], [4]).

1.1.4 Isotropic and Anisotropic Targets

Living objects have considerable capacity components, while inanimate targets have manely resistive properties [31]. Due to the work by Heiligenberg in 1973 [21] and von der Emde (1990) [37], it is known that the fish can distinguish between these material properties an therefore might use this ability to differ from biological and unlively targets.
Anisotropy and the alignment of cells is of interest in cell culture production. For example in steam-cell-therapy Hyaline cartilage structure differs in superimposed layers [8]. In this work, an algorithm is designed to identify and locate anisotropic targets, and to reconstruct its shape, size and orientation. In the governing model as commonly accepted in EIT community the conductivity is assumed to be constant inside the cells due to thin cell membranes [8], although in general the conductivity depends on the frequency of an injected current. It is focused on extended targets and how the fish look at them.
The aim of this thesis is to investigate the perception of anisotropic targets using the electric sense.

This work is organized as follows. In chapter 2 notations and definitions are listed. Chapter 3 introduces the anisotropic transmission problem, isotropic and anisotropic polarization tensors, such as properties of anisotropic polarization tensors. The following chapter 4 treats isotropic and anisotropic layer potential techniques, the Neumann-Poincaré operator and the anisotropic operator $\mathcal{Q}_D^A$. Translation, rotation and scaling properties of anisotropic polarization tensors are derived in chapter 5 and a target identification and location procedure based on invariants is stated in chapter 6.

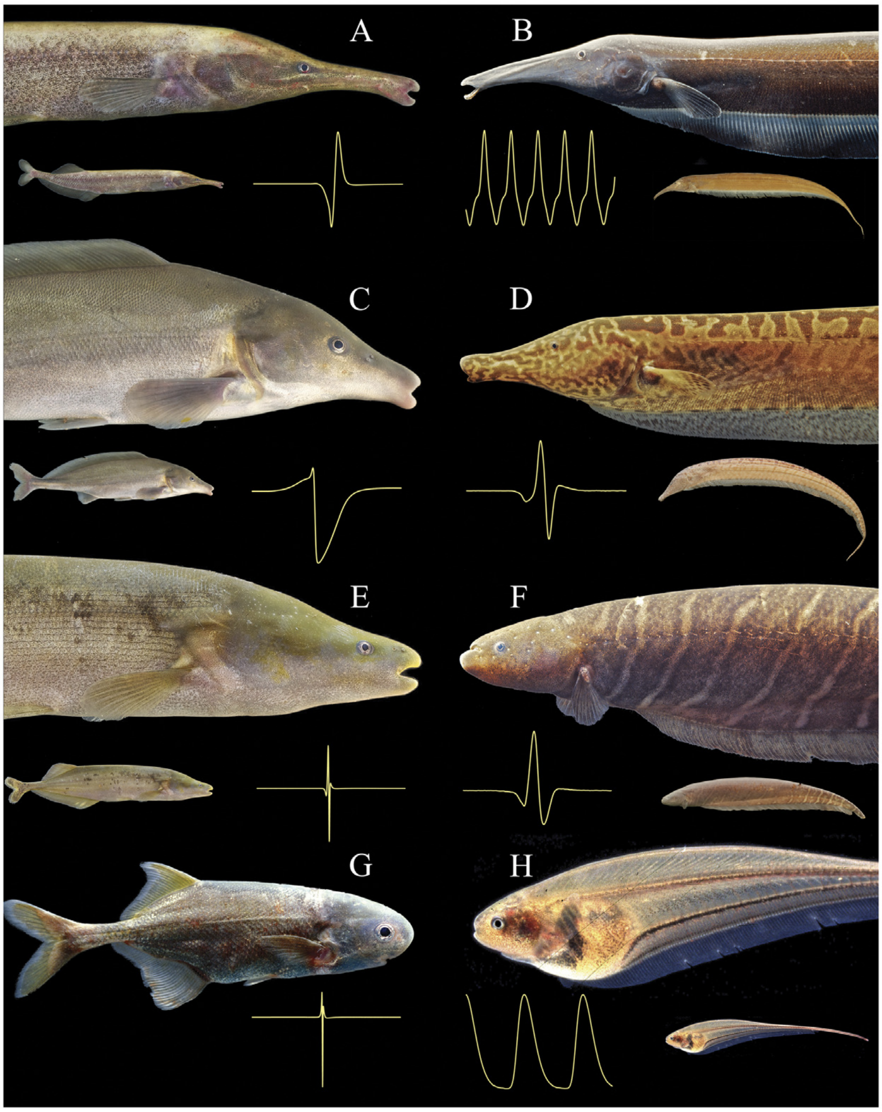

Figure 1.1: Mormyroid African electric fishes (left column) and gymnotiform South American electric fishes (right column); electric organ discharge waveform shown for every species (each trace 5 ms in total duration with head-positivity plotted upwards). (A) Mormyrops zanclirostris, 175 mm standard length (SL), Ivindo River, Gabon, (B) Sternarchorhynchus oxyrhynchus, 220 mm total length (TL), Rio Negro, Brazil; (C) Mormyrus proboscirostris, 232 mm standard length, Ubundu, Congo River, D. R. Congo; (D) Rhamphichthys sp., 305 mm TL, Rio Negro, Brazil; (E) Mormyrops anguilloides, 195 mm SL, Yangambi, Congo River, D. R. Congo; (F) Gymnotus sp., 195 mm TL, Rio Negro, Brazil; and (G) Petrocephalus sullivani, Ogooué River, Gabon; (H) Eigenmannia sp., Apure River, Venezuela. Source: [26, fig. 2]

Chapter 2

Notations and Definitions

The following notations and definitions are used:

$\mathcal{C}^n(D)$, $n \in \mathbb{N}$, D an open subset of $\mathbb{R}^d$, is the space of n times continuously differentiable functions.

For $i = (i_1, \dots, i_d) \in \mathbb{N}^d$ a multi-index, high order derivatives of a $\mathcal{C}^n$ function u are denoted by $\partial^i u = \partial^{i_1} \dots \partial^{i_d} u$.

$\mathcal{C}^{n+\alpha}(D)$, $n \in \mathbb{N}$, $0 < \alpha < 1$, D a bounded Lipschitz domain, denotes the Hölder space of all functions u on D such that for any $i \in \mathbb{N}^d$, $|i| = i_1 + \dots + i_d = n$ we have $|\partial^i u(x) - \partial^i u(y)| \leq C|x-y|^\alpha$ for all $x, y \in D$, where $C > 0$ is a constant depending on u but not x and y.

$L^p(D)$, D an open subset of $\mathbb{R}^d$, is defined by the usual way with the norm $\|u\|_{L^p(D)} := \left(\int_D |u|^p \, dx \right)^{\frac{1}{p}}$.

$L^\infty(D)$, D an open subset of $\mathbb{R}^d$, denotes the space of essentially bounded measurable functions with $\|f\|_\infty := \inf\{C > 0| \; |u(x)| \leq C \text{ a.e. } x \in D\}$.

$L^2_0(\partial D) := \{u \in L^2(\partial D) | \int_{\partial D} \varphi \, d\sigma = 0\}$ denotes the space of square integrable functions with integral mean zero.

$L^2_{loc}(D) := \{u : D \to \mathbb{R} \mid \int_K |u|^2 \, dx < \infty \; \forall \; K \subset D \text{ compact}\}$, D an open subset of $\mathbb{R}^d$, denotes the space of locally square integrable functions on D.

$W^1_p(D) := \{u \in L^p(D) | \int_D |u|^p \, dx + \int_D |\nabla u|^p \, dx < \infty\}$, D an open subset of $\mathbb{R}^d$, $1 < p < \infty$, where ∇u is interpreted as distribution, denotes the Banach space called first order Sobolev space.

$H^1(D) := W^1_2(D)$ is a Hilbert space with the inner product $\langle u, v \rangle = \int_D uv \, dx + \int_D \nabla u \cdot \nabla v \, dx$.

$H^{\frac{1}{2}}(\partial D)$ is the image of the bounded linear trace operator $u \mapsto u|_{\partial D}$ on $H^1(D)$. We have $u \in H^{\frac{1}{2}}(\partial D)$ if and only if $u \in L^2(\partial D)$ and

$$\int_{\partial D} \int_{\partial D} \frac{|u(x) - u(y)|^2}{|x - y|^d} \, d\sigma(x) \, d\sigma(y) < \infty. \tag{2.1}$$

See for example the textbook [20].

$H^{-\frac{1}{2}}(\partial D) := \left(H^{\frac{1}{2}}(\partial D)\right)^*$ is the dual space of the Sobolev space of order $-1/2$ and $\langle u, v \rangle_{\frac{1}{2}, -\frac{1}{2}} := \int_{\partial D} u(x) v(x) \, d\sigma(x)$, $u \in H^{\frac{1}{2}}(\partial D)$, $v \in H^{-\frac{1}{2}}(\partial D)$ the duality pairing.

$H^1_{loc}(D) := \{u \in L^2_{loc}(D) \mid hu \in H^1(D) \ \forall \ h \in C_0^\infty(D)\}$, D an open subset of $\mathbb{R}^d$, is the local space.

For u a function defined on $\mathbb{R}^d \setminus \partial D$ and ν_x the outwards pointing unit normal vector on ∂D at x we denote

$$u|_\pm(x) := \lim_{t \to 0^+} u(x \pm t\nu_x), \quad x \in \partial D \quad \text{and} \tag{2.2}$$

$$\frac{\partial}{\partial \nu_x} u|_\pm(x) := \lim_{t \to 0^+} \nu_x \cdot \nabla u(x \pm t\nu_x), \ x \in \partial D, \tag{2.3}$$

if the limits exist.

Definition 2.1 (Lipschitz Domain) *[9, chp. 2.1.1] Let D be a bounded open connected domain in $\mathbb{R}^d$. D is called Lipschitz domain with Lipschitz character (r, L, N) if for each point x on the boundary ∂D there exists a coordinate system (x', x_d), $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$ such that x is its origin and x_d is the axis of a double truncated cylinder Z (called coordinate cylinder) centred at x and the bottom and top of Z have a positive distance $r < l < 2r$ from ∂D. And there exist a Lipschitz function φ with $\|\nabla \varphi\|_{L^\infty(\mathbb{R}^{d-1})} \leq L$ such that $Z \cap D = Z \cap \{(x', x_d) | \ x_d > \varphi(x')\}$ and $Z \cap \partial D = Z \cap \{(x', x_d) | \ x_d = \varphi(x')\}$.*
The pair (Z, φ) is called coordinate pair.

Definition 2.2 (Variational Solution) *[9, chp. 2.1.6] Let $A \in \mathbb{R}^{d \times d}$, be a symmetric matrix, $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $g \in H^{-\frac{1}{2}}(\partial D)$ with $\langle 1, g \rangle_{\frac{1}{2}, -\frac{1}{2}} = 0$. Then $u \in H^1(D)$ is called the (variational) solution to the Neumann problem*

$$\nabla \cdot (A \nabla u) = 0 \quad \textit{in } D \quad \textit{and} \tag{2.4a}$$

$$\nu \cdot A \nabla u|_{\partial \Omega} = g, \tag{2.4b}$$

where ν denotes the outwards pointing unit normal to ∂D. Furthermore, for $v \in H^1(D)$ we have

$$\langle v, g \rangle_{\frac{1}{2}, -\frac{1}{2}} = \int_D \nabla u \cdot A \nabla v \, dx. \tag{2.5}$$

There exists a unique solution u modulo a constant to the Neumann problem (2.4) due to the Lax-Milgramm Lemma [27].

In this work the following identities are used, which are a consequence of the divergence-theorem:

Lemma 2.3 (Green's first Identity) *[9, chp. 2.1.5] Let D be a bounded Lipschitz domain and ν denote the outwards pointing unit normal vector of ∂D then:*

For $u, v \in H^1(\partial D)$ and $A \in \mathbb{R}^{d\times d}$ symmetric positive-definite we have

$$\int_D \Delta u(x)v(x) + \nabla u(x) \cdot \nabla v(x)\, dx = \int_{\partial D} \frac{\partial u}{\partial \nu} v(x)\, d\sigma \quad \text{and} \tag{2.6}$$

$$\int_D \nabla \cdot (A\nabla u(x))v(x) + \nabla u(x) \cdot A\nabla v(x)\, dx \\ = \int_{\partial D} \nu \cdot A\nabla u(x)\; v(x)\, d\sigma. \tag{2.7}$$

For $u \in H^1_{loc}(D)$, $\Delta u = 0$ on $\mathbb{R}^d \setminus \overline{D}$ and $|u(x)| = O(|x|^{2-d})$ if $d \geq 3$ or $|u(x)| = O(|x|^{-1})$ if $d = 2$ as $|x| \to \infty$ we have

$$\int_{\mathbb{R}^d\setminus\overline{D}} \nabla u(x) \cdot \nabla u(x)\, dx = -\int_{\partial D} \frac{\partial u}{\partial \nu} u(x)\, d\sigma. \tag{2.8}$$

For $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed vector spaces, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from X to Y with the norm $\|A\|_{\mathcal{L}(X,Y)} = \sup\limits_{\|X\|_X \leq 1} \|Ax\|_Y = \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$ for $A \in \mathcal{L}(X,Y)$.

Definition 2.4 *[7, sec. 2.6] Let $\mathcal{B}$ be a Banach space and Y be a normed vector space. A bounded linear operator $A : \mathcal{B} \to Y$ is compact if for every bounded sequence $\{x_j\}_{j\in\mathbb{N}}$ in $\mathcal{B}$ the sequence $\{Ax_j\}_{j\in\mathbb{N}}$ contains a convergent subsequence.*

Definition 2.5 *[20, chp. 7.7] A Banach space $\mathcal{B}_1$ is said to be continuously embedded in a Banach space $\mathcal{B}_2$, if there exists a bounded linear one to one mapping: $\mathcal{B}_1 \to \mathcal{B}_2$.*

Definition 2.6 *[20, chp. 7.10] Let $\mathcal{B}_1$ be a Banach space continuously embeded in the Banach space $\mathcal{B}_2$. $\mathcal{B}_1$ is compactly embeded in $\mathcal{B}_2$ if the embedding operator $I : \mathcal{B}_1 \to \mathcal{B}_2$ is compact. We write $\mathcal{B}_1 \Subset \mathcal{B}_2$.*

For example we have $H^1(\partial D) \Subset L^2(\partial D)$.

Chapter 3

Anisotropic Transmission Problem

The isotropic transmission problem and its relation on the anisotropic polarization tensor is introduced. It is shown that the anisotropic polarization tensor of a disk coincides with the isotropic polarization tensor of an *equivalent* ellipse. Moreover, an asymptotic expansion as the anisotropy ratio goes to one and features of anisotropic polarization tensors such as a rotation formula and symmetry properties are shown.

3.1 Anisotropic Transmission Problem

In this section the anisotropic transmission problem found in [9, chp. 2.7] is stated.
Let Ω, the body, be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, containing an inclusion D, which is a bounded Lipschitz domain. Let A and $\tilde{A}$ be constant, positive-definite matrices and assume that $A \neq \tilde{A}$ and $A - \tilde{A}$ is positive-definite or negative-definite. The conductivity of the background $\Omega \setminus \overline{D}$ is A and the conductivity of D is $\tilde{A}$. The conductivity profile of Ω is given by

$$\gamma_D := \chi(\Omega \setminus \overline{D})A + \chi(D)\tilde{A}. \tag{3.1}$$

Let u denote the steady-stage voltage in the presence of an anisotropic conductivity inclusion D, which is the solution to

$$\nabla \cdot \big(\chi(\Omega \setminus \overline{D})A + \chi(D)\tilde{A}\big)\nabla u = 0 \quad \text{in } \Omega, \tag{3.2a}$$

$$\nu \cdot A\nabla u|_{\delta\Omega} = g, \tag{3.2b}$$

$$\int_{\delta\Omega} u(x)\, d\sigma(x) = 0, \tag{3.2c}$$

for some given $g \in L_0^2(\partial\Omega)$.
Define A_* to be the symmetric positive-definite matrix such that $A^{-1} = A_*^2$.

The fundamental solution $\Gamma^A(x)$ of the operator $\nabla \cdot A\nabla$ is

$$\Gamma^A(x) := \begin{cases} \dfrac{1}{2\pi\sqrt{\det(A)}}\log(|A_* x|), & d = 2, \\ -\dfrac{1}{4\pi\sqrt{\det(A)}|A_* x|}, & d = 3, \end{cases} \tag{3.3}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The single layer potential associated with A of the density function $\varphi \in L^2(\partial D)$ is given by

$$\mathcal{S}_D^A[\varphi](x) := \int_{\partial D} \Gamma^A(x-y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^d. \tag{3.4}$$

Analogously the double layer potential is defined by

$$\mathcal{D}_D^A[\varphi](x) := \int_{\partial D} \nu_y \cdot A\nabla\Gamma^A(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D. \tag{3.5}$$

Layer potentials are well-defined, which is elaborated in section 4.1.1. However, the jump relations

$$\nu_x \cdot A\nabla\mathcal{S}_D^A\varphi(x)|_+ - \nu_x \cdot A\nabla\mathcal{S}_D^A\varphi(x)|_- = \varphi(x) \quad \text{a.e. } x \in \partial D \quad \text{and} \tag{3.6a}$$

$$\mathcal{D}_D^A\varphi(x)|_+ - \mathcal{D}_D^A\varphi(x)|_- = -\varphi(x) \quad \text{a.e. } x \in \partial D \tag{3.6b}$$

are an advantagous tool. Escauriaza and Seo have shown in [19]:

Theorem 3.1 *For each $(F, G) \in H^1(\partial D) \times L^2(\partial D)$, a unique solution $(f, g) \in L^2(\partial D) \times L^2(\partial D)$ of the integral equation*

$$\mathcal{S}_D^{\tilde{A}} f - \mathcal{S}_D^A g = F, \tag{3.7a}$$

$$\nu \cdot \tilde{A}\nabla\mathcal{S}_D^{\tilde{A}} f|_- - \nu \cdot A\nabla\mathcal{S}_D^A g|_+ = G \quad \text{on } \partial D \tag{3.7b}$$

exists. Moreover, there exists a constant C such that

$$\|f\|_{L^2(\partial D)} + \|g\|_{L^2(\partial D)} \leq C(\|F\|_{H^1(\partial D)} + \|G\|_{L^2(\partial D)}), \tag{3.8}$$

where C only depends on the largest and smallest eigenvalues of $\tilde{A}$, A, and $\tilde{A} - A$ and the Lipschitz character of the domain D.

Lemma 3.2 *Let (f, g) be the solution to (3.7). For $G \in L_0^2(\partial\Omega)$ we have $g \in L_0^2(\Omega)$. Moreover, if F is constant and $G = 0$, then $g = 0$.*

As suggested in [9], the first part of the Lemma can be shown by integrating (3.7b). Because for $G \in L_0^2(\partial\Omega)$ the right hand side vanishes, using the jump relation (3.6a), it can be written $g \in L_0^2(\partial\Omega)$. The second part also follows from the jump relation (3.6a) and the equations (3.7).
Furthermore, the solution of (3.2) is given by a representation formula:

Theorem 3.3 *Let $g \in L^2_0(\partial\Omega)$ and $f := u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$. Define*

$$H^A(x) := -\mathcal{S}^A_\Omega g(x) + \mathcal{D}^A_\Omega f(x), \quad x \in \Omega, \tag{3.9a}$$

then the solution u of (3.2) *can be represented as*

$$u(x) = \begin{cases} H^A(x) + \mathcal{S}^A_D \varphi(x), & x \in \Omega \setminus \overline{D}, \\ \mathcal{S}^{\tilde{A}}_D \psi(x), & x \in D, \end{cases} \tag{3.9b}$$

where the pair (φ, ψ) is the unique solution in $L^2_0(\partial D) \times L^2(\partial D)$ to the system of integral equations

$$\mathcal{S}^{\tilde{A}}_D \psi - \mathcal{S}^A_D \varphi = H^A, \tag{3.9c}$$

$$\nu \cdot \tilde{A} \nabla \mathcal{S}^{\tilde{A}}_D \psi|_- - \nu \cdot A \nabla \mathcal{S}^A_D \varphi|_+ = \nu \cdot A \nabla H^A \quad \text{on } \partial D. \tag{3.9d}$$

3.2 Anisotropic Polarization Tensors

As in [9, chp. 4.12] in this section the (generalized) anisotropic polarization tensor (APT) is introduced.

Definition 3.4 *[9, Definition 4.29] Let $i \in \mathbb{N}^d$ be a multi-index with $|i| \geq 1$, assume $(f_i, g_i) \in L^2(\partial D) \times L^2(\partial D)$ is the unique solution to*

$$\mathcal{S}^{\tilde{A}}_D f_i - \mathcal{S}^A_D g_i = x^i, \tag{3.10a}$$

$$\nu \cdot \tilde{A} \nabla \mathcal{S}^{\tilde{A}}_D f_i|_- - \nu \cdot A \nabla \mathcal{S}^A_D g_i|_+ = \nu \cdot A \nabla x^i \quad \text{on } \partial D, \tag{3.10b}$$

where $x^i = x_1^{i_1} \cdots x_d^{i_d}$. The generalized anisotropic polarization tensor associated with the domain D and anisotropic conductivities $\tilde{A}$ and A, respectively the conductivity profile γ_D for the pair of multi-indices $i, j \in \mathbb{N}^d$ is defined by

$$M_{i,j} = M_{i,j}(A, \tilde{A}, D) = \int_{\partial D} x^j g_i(x)\, d\sigma(x). \tag{3.11}$$

Furthermore, if $i = \mathbf{e}_p$ and $j = \mathbf{e}_q$ for $p, q = 1, ..., d$, where $\{\mathbf{e}_1, ..., \mathbf{e}_d\}$ denotes the standard basis of $\mathbb{R}^d$, we write $M_{i,j} = m := (m_{pq})^d_{p,q=1}$, $g_p = g_i$ and

$$m_{p,q} = \int_{\partial D} x_q g_p(x)\, d\sigma(x). \tag{3.12}$$

First order APTs ($|i| = |j| =1$) were first introduced in [23] and it is shown that m is symmetric and positive-definite (negative-definite resp.) if $\tilde{A} - A$ is positive-definite (negative-definite resp.), as mentioned in [9].
APTs are a natural extension of generalized polarization tensors (GPTs), which are the building blocks in representing the perturbation the electric potential in the presence of an inclusion D [6].

Definition 3.5 *[6, p. 56 - p. 57] Let $D \subset \mathbb{R}^d$ be a bounded C^2-domain. The operators $\mathcal{K}_D, \mathcal{K}_D^* : L^2(\partial D) \to L^2(\partial D)$ are given by*

$$\mathcal{K}_D[\varphi](x) := \frac{1}{\omega_d} \int_{\partial D} \frac{\langle y - x, \nu_y \rangle}{|x - y|^d} \varphi(y)\, d\sigma(y) \quad \textit{and} \tag{3.13a}$$

$$\mathcal{K}_D^*[\varphi](x) := \frac{1}{\omega_d} \int_{\partial D} \frac{\langle x - y, \nu_x \rangle}{|x - y|^d} \varphi(y)\, d\sigma(y), \tag{3.13b}$$

where ω_d denotes the volume of the unit sphere in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ the Euclidean inner product in $\mathbb{R}^d$.

$\mathcal{K}_D^*$ is called the *Neumann-Poincaré operator*. Note that it is easy to show, that it is the L^2-adjoint of $\mathcal{K}_D$ [9, p. 56]. Also $\mathcal{K}_D$ and $\mathcal{K}_D^*$ are bounded linear operators. Boundedness is shown in [9]. Furthermore, these operators are compact [6, Lemma 2.4] on $L^2(\partial D)$.

Lemma 3.6 *[6, Lemma 2.9] Let $\lambda \in \mathbb{R}$ and D be a bounded C^2-domain. Then*

i) $\lambda I - \mathcal{K}_D^*$ is one to one on $L_0^2(\partial D)$ if $|\lambda| \geq \frac{1}{2}$ and

ii) $\lambda I - \mathcal{K}_D^*$ is one to one on $L^2(\partial D)$ if $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, +\infty)$.

Definition 3.7 *[6, eq. 4.1] The generalized polarization tensor associated with the bounded C^2-domain $D \subset \mathbb{R}^2$ and the parameter λ, $|\lambda| > \frac{1}{2}$ for the pair of multi-indices i,j $\in \mathbb{N}^2$, $|i|$, $|j| \geq 1$ is defined by*

$$M_{i,j}^G = M_{i,j}^G(\lambda, D) = \int_{\partial D} y^j \varphi_i(y)\, d\sigma(y) \quad \textit{for} \tag{3.14a}$$

$$\varphi_i(y) = (\lambda I - \mathcal{K}_D^*)^{-1}[\nu_x \cdot \nabla x^i](y), \quad y \in \partial D. \tag{3.14b}$$

The parameter λ, referred to as contrast [6, chp. 7], is related to the conductivity k of an inclusion with isotropic conductivity via

$$\lambda = \frac{k+1}{2(k-1)}. \tag{3.15}$$

The GPT carries all of the information about the inclusion D in a homogeneous background medium with conductivity one [6, Theorem 4.3]. On that basis, it is assumed that also the APT carries all information.
The isotropic polarization tensor of an ellipse and first order anisotropic polarization tensor of a disk are the same under certain assumptions on the targets conductivities. For B a disk centred at the origin the first order APT can be computed explicitly by [9, eq. 4.88]

$$m(I, \tilde{A}, B) = 2|B|(\tilde{A} + I)^{-1}(\tilde{A} - I), \tag{3.16}$$

where $|B|$ denotes the volume of $\mathcal{B}$. For $\mathcal{E}$ an ellipse, whose semi-axis are aligned with the x_1- and x_2-axis of length a and b, the GPT can be explicitly calculated to be [9, Proposition 4.6]

$$M^G(k,\mathcal{E}) = (k-1)|\mathcal{E}| \begin{pmatrix} \frac{a+b}{a+kb} & 0 \\ 0 & \frac{a+b}{b+ka} \end{pmatrix}. \tag{3.17}$$

Thus, we have equality if the anisotropic inclusion conductivity satisfies

$$\tilde{A} = \begin{pmatrix} \frac{(k-1)|\mathcal{E}|(a+b)+2|D|(a+kb)}{(k-1)|\mathcal{E}|(b-a)+2|D|(a+kb)} & 0 \\ 0 & \frac{(k-1)|\mathcal{E}|(a+b)+2|D|(ka+b)}{(k-1)|\mathcal{E}|(a-b)+2|D|(b+ka)} \end{pmatrix}. \tag{3.18}$$

3.3 Asymptotic Formula

We state the asymptotic expansion of the potential u as in [9, chp. 5.4] in the following theorem.
The background potential U is defined to be the steady-stage voltage potential in the absence of the conductivity inclusion. Hence U is the solution to the Neumann problem

$$\nabla \cdot AU = 0 \quad \text{in } \Omega, \tag{3.19a}$$
$$\nu \cdot A\nabla U|_{\partial\Omega} = g, \tag{3.19b}$$
$$\int_{\partial\Omega} U(x)\, d\sigma(x) = 0, \tag{3.19c}$$

for $g \in L^2_0(\partial D)$ given. Moreover, we define the Neumann function N^A of the operator $\nabla \cdot A\nabla$ on Ω, i.e.,

$$\nabla \cdot AN^A(x,y) = -\delta_y \quad \text{in } \Omega, \tag{3.20a}$$
$$\nu \cdot A\nabla N^A(x,y)|_{\partial\Omega} = -\frac{1}{|\partial\Omega|}, \tag{3.20b}$$
$$\int_{\partial\Omega} N^A(x,y)\, d\sigma(x) = 0, \tag{3.20c}$$

where $x \in \partial\Omega$ and $y \in \Omega$.

Theorem 3.8 *For $D = \varepsilon B + z$, where B is the unit ball centred at 0 in $\mathbb{R}^d$ and z the location of D, we have for u the solution of (3.2)*

$$u(x) = U(x) - \varepsilon^d \sum_{|i|=1}^{d} \sum_{|j|=1}^{d-|i|+1} \frac{\varepsilon^{|i|+|\beta|-2}}{i!j!} \partial^i U(z) M_{i,j}(A,\tilde{A},B) \partial_z^j N^A(x,z) + O(\varepsilon^{2d}), \tag{3.21}$$

where the remainder $O(\varepsilon^{2d})$ is dominated by $C\varepsilon^{2d}$ for some constant C independent of $x \in \partial\Omega$ and z.

3.4 Properties of Anisotropic Polarization Tensors

This section gives important properties of APTs such as a rotation formula, an explicit formula and symmetry properties.
As proven in [9, Lemma 4.30] for $m = (m_{pq})_{pq=1}^{d}$ the matrix of the first-order APT we have:

Lemma 3.9 *Let $R \in \mathbb{C}^{d\times d}$ be a unitary transformation, then*

$$m(I, \tilde{A}; D) = Rm(R^{\intercal}R, R^{\intercal}\tilde{A}R; R^{-1}D)R^{\intercal}, \tag{3.22}$$

where $^{\intercal}$ denotes the transpose.

Furthermore, there is an explicit formula [9, chp. 4.12], which reads as

$$m(I, \tilde{A}; D) = (\tilde{A} - I)\int_{\partial D} x(-\frac{1}{2}I + \mathcal{K}_D^*)[g](x)\, d\sigma(x) + (\tilde{A} - I)|D|. \tag{3.23}$$

In [9, Therorem 4.32], it is shown that the contracted APT is symmetric for A-harmonic coefficients. We show it is symmetric for harmonic coefficients.

Theorem 3.10 *Let I, J be finite sets of multi-indices and the coefficients $\{a_i | i \in I\}, \{b_j | j \in J\}$ be such that $\sum_{i\in I} a_i x^i$ and $\sum_{j\in J} b_j x^j$ are harmonic, then we have*

$$\sum_{i\in I}\sum_{j\in J} a_i b_j M_{i,j} = \sum_{i\in I}\sum_{j\in J} a_i b_j M_{j,i}. \tag{3.24}$$

Proof Define for f_i, g_i the solution to (3.10)

$$\begin{aligned}
v_1(x) &= \sum_{i\in I} a_i x^i, \quad v_2(x) = \sum_{j\in J} b_j x^j \\
\psi_1(x) &= \sum_{i\in I} a_i f_i(x), \quad \psi_2(x) = \sum_{j\in J} b_j f_j(x) \\
\varphi_1(x) &= \sum_{i\in I} a_i g_i(x) \quad \text{and} \quad \varphi_2(x) = \sum_{j\in J} b_j g_j(x).
\end{aligned}$$

Note that these function are in $L^2(\partial D)$. Since I and J are finite and by the linearity of the integral equations (3.10) we have for $p = 1, 2$

$$\mathcal{S}_D^{\tilde{A}}\psi_p - \mathcal{S}_D^{A}\varphi_p = v_p \tag{3.25}$$

$$\nu \cdot \tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_p|_- - \nu \cdot A\nabla\mathcal{S}_D^{A}\varphi_p|_+ = \nu \cdot A\nabla v_p \quad \text{on } \partial D. \tag{3.26}$$

Furthermore, it can be written

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in J} a_i b_j M_{i,j} &= \int_{\partial D} \sum_{j\in J} b_j x^j \sum_{i\in I} a_i g_i(x)\, d\sigma \\
&= \int_{\partial D} v_2(x)\varphi_1(x)\, d\sigma.
\end{aligned}$$

By the jump relation (3.6a) and equation (3.26) we have

$$\sum_{i\in I}\sum_{j\in J} a_i b_j M_{i,j} = \int_{\partial D} v_2(\nu\cdot\tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_1|_- - \nu\cdot A\nabla v_1) - v_2\nu\cdot A\nabla\mathcal{S}_D^{A}\varphi_1|_-\, d\sigma.$$

By equation (3.25), partial integration and since $\nabla\cdot A\nabla\mathcal{S}_D^{A}\varphi_1 = 0$ on $\mathbb{R}^d\setminus\partial D$, we have the above equals to

$$\begin{aligned}
&= \int_{\partial D}\mathcal{S}_D^{\tilde{A}}\psi_2\,\nu\cdot\tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_1|_- d\sigma - \int_{\partial D}\mathcal{S}_D^{A}\varphi_2\,\nu\cdot\tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_1|_-\, d\sigma\\
&\quad - \int_{\partial D} v_2\,\nu\cdot A\nabla v_1\, d\sigma + \int_{\partial D}\mathcal{S}_D^{A}\varphi_1\,\nu\cdot A\nabla v_2\, d\sigma\\
&= \int_{\partial D}\mathcal{S}_D^{\tilde{A}}\psi_2\,\nu\cdot\tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_1|_-\, d\sigma - \int_{\partial D}\mathcal{S}_D^{A}\varphi_2(\nu\cdot\tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_1|_- + \nu\cdot A\nabla v_2)\, d\sigma\\
&\quad + \int_{\partial D}\mathcal{S}_D^{A}\varphi_1\,\nu\cdot A\nabla v_2\, d\sigma - \int_{\partial D} v_2\,\nu\cdot A\nabla v_1 d\sigma,
\end{aligned}$$

where (3.26) was used in the last line. By Green's first identity we have

$$\begin{aligned}
&= \int_D \tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_1\cdot\nabla\mathcal{S}_D^{\tilde{A}}\psi_2\, dx + \int_{\mathbb{R}^d\setminus\overline{D}} A\nabla\mathcal{S}_D^{A}\varphi_1\cdot\nabla\mathcal{S}_D^{A}\varphi_2 dx - \int_D A\nabla v_1\cdot\nabla v_2\, dx\\
&\quad - \int_{\partial D}\mathcal{S}_D^{A}\varphi_2\,\nu\cdot\tilde{A}\nabla v_1 + \mathcal{S}_D^{A}\varphi_1\,\nu\cdot A\nabla v_2\, d\sigma.
\end{aligned}$$

Moreover, similarly it is obtained that

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in J} a_i b_j M_{j,i} &= \int_D \tilde{A}\nabla\mathcal{S}_D^{\tilde{A}}\psi_2\cdot\nabla\mathcal{S}_D^{\tilde{A}}\psi_1\, dx + \int_{\mathbb{R}^d\setminus\overline{D}} A\nabla\mathcal{S}_D^{A}\varphi_2\cdot\nabla\mathcal{S}_D^{A}\varphi_1\, dx\\
&- \int_D A\nabla v_2\cdot\nabla v_1\, dx - \int_{\partial D}\mathcal{S}_D^{A}\varphi_1\,\nu\cdot A\nabla v_2\, d\sigma + \int_{\partial D}\mathcal{S}_D^{A}\varphi_2\,\nu\cdot A\nabla v_1\, d\sigma.
\end{aligned}$$

Because A and $\tilde{A}$ are symmetric the theorem is shown. □

Chapter 4

Layer Potential Techniques

In this chapter isotropic layer potentials are introduced and properties such as boundedness and invertibility are elaborated.
Anisotropic single layer potentials have common properties. The anisotropic Neumann-Poincaré operator can be symmetrized alike the Neumann-Poincaré operator. And the essential operator in the study of the transmission problem $\mathcal{Q}_D^A$ is shown to be to be invertible in three dimensions.

4.1 Mapping Properties

It turns out that the single layer potential maps $H^{-\frac{1}{2}}(\partial D)$ to $H^{\frac{1}{2}}(\partial D)$ boundedly and has a bouned inverse in three dimensions. In two dimensions, invertibility requires a restriction to a subspace of $H^{-\frac{1}{2}}(\partial D)$.

4.1.1 Isotropic Layer Potentials

The anisotropic single and double layer potentials were introduced in section 3.1. They can be viewed as a generalization of the (isotropic) single and double layer potential, which are define as in [9, eq. (2.12) and (2.13)] for $D \subset \mathbb{R}^d$, $d \geq 2$ a bounded Lipschitz-domain and $\varphi \in L^2(\partial D)$ by

$$\mathcal{S}_D[\varphi](x) = \int_{\partial D} \Gamma(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d \quad \textit{and} \tag{4.1}$$

$$\mathcal{D}_D[\varphi](x) = \int_{\partial D} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D, \tag{4.2}$$

where $\Gamma(x)$ denotes the fundamental solution of the Laplacian, i.e.,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi}\log(|x|), & d = 2, \\ -\dfrac{1}{4\pi|x|}, & d = 3. \end{cases} \tag{4.3}$$

Because $\frac{\partial \Gamma}{\partial \nu_y}(x-y) \in L^\infty(\partial D)$ for $x \in \mathbb{R}^d \setminus \partial D$, the single and double layer potential of square integrable functions are well-defined and harmonic on $\mathbb{R}^d \setminus \partial D$.
In the following we refer to the single and double layer potential as to the isotropic operators.

Lemma 4.1 *[9, p.16] For D a bounded Lipschitz domain and $\varphi \in L^2(\partial D)$, we have*

i) $\mathcal{S}_D\varphi(x) = O(|x|^{2-d})$ *as* $|x| \to \infty$ *for* $d \geq 3$.

ii) If $\int_{\partial D} \varphi(y)\, d\sigma(y) = 0$, *then* $\mathcal{S}_D\varphi(x) = O(|x|^{1-d})$ *as* $|x| \to \infty$ *for* $d \geq 2$.

iii) $\mathcal{D}_D\varphi(x) = O(|x|^{1-d})$ *as* $|x| \to \infty$.

Further properties are not clear among other things, because of the singularity of the kernels [9, chp. 2.3]. Roughly speaking they depend on the domains boundary smoothness.
Because the techniques used in the study of layer potential techniques on Lipschitz domains are above the scope of this work, well-known results are stated in the following. Note that every bounded $\mathcal{C}^{1+\alpha}$-domain is Lipschitz for $0 < \alpha < 1$.

Theorem 4.2 *[9, Theorem 2.17] Let D be a bounded Lipschitz-domain $\varphi \in L^2(\partial D)$, then we have*

$$\mathcal{S}_D\varphi|_+(x) = \mathcal{S}_D\varphi|_-(x) \quad \text{a.e. } x \in \partial D, \tag{4.4a}$$

$$\frac{\partial}{\partial \nu}\mathcal{S}_D\varphi|_\pm(x) = (\pm\frac{1}{2}I + \mathcal{K}_D^*)\varphi(x) \quad \text{a.e. } x \in \partial D \quad \text{and} \tag{4.4b}$$

$$\mathcal{D}_D\varphi|_\pm(x) = (\mp\frac{1}{2}I + \mathcal{K}_D)\varphi(x) \quad \text{a.e. } x \in \partial D, \tag{4.4c}$$

where $\mathcal{K}_D^*$ and $\mathcal{K}_D$ are given by

$$\mathcal{K}_D[\varphi](x) = \frac{1}{\omega_d} p.v. \int_{\partial D} \frac{\langle y-x, \nu_y\rangle}{|x-y|^d}\varphi(y)\, d\sigma(y), \tag{4.5a}$$

$$\mathcal{K}_D^*[\varphi](x) = \frac{1}{\omega_d} p.v. \int_{\partial D} \frac{\langle x-y, \nu_x\rangle}{|x-y|^d}\varphi(y)\, d\sigma(y), \tag{4.5b}$$

where ω_d denotes the volume of the unit sphere in $\mathbb{R}^d$ and p.v. the Chauchy principle value.

Remark 4.3 *[9, p. 25] For D of class $\mathcal{C}^{1+\alpha}$ and $\alpha > 0$, the definition of $\mathcal{K}_D$ respectively $\mathcal{K}_D^*$ by the equations (3.13) and (4.5) are equivalent.*

Due to (4.4b) we have the jump identity $\varphi(x) = \frac{\partial}{\partial_\nu}\mathcal{S}_D\varphi|_+ - \frac{\partial}{\partial\nu}\mathcal{S}_D\varphi|_-$.
Next we ask, if $\mathcal{S}_D$ is a bounded operator on the space $H^{-\frac{1}{2}}(\partial D)$, which is the general Sobolev spaces considered in this work.

Theorem 4.4 *[9, Theorem 2.24] Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz-domain. Then the single layer potential $\mathcal{S}_D : L^2(\partial D) \to H^1(\partial D)$ and $\mathcal{K}_D : H^1(\partial D) \to H^1(\partial D)$ are bounded operators.*

Corollary 4.5 *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, then $\mathcal{S}_D : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ boundedly.*

Proof This proof uses the compact embedding of $H^1(\partial D)$ into $L^2(\partial D)$. By the inclusion $H^{\frac{1}{2}}(\partial D) \subset L^2(\partial D)$, we have $\mathcal{S}_D : H^{\frac{1}{2}}(\partial D) \to H^1(\partial D)$ boundedly due to Theorem 4.4. Let $\varphi \in H^{\frac{1}{2}}(\partial D)$. By [35, Satz 4.2.2] we can write

$$\begin{aligned}
\|\mathcal{S}_D\varphi\|_{H^{-\frac{1}{2}}(\partial D)} &= \sup_{\substack{\psi\in H^{\frac{1}{2}}(\partial D)\\ \|\psi\|_{H^{\frac{1}{2}}}\le 1}} |\langle \mathcal{S}_D\varphi,\psi\rangle_{-\frac{1}{2},\frac{1}{2}}| = \sup_{\substack{\psi\in H^{\frac{1}{2}}(\partial D)\\ \|\psi\|_{H^{\frac{1}{2}}}\le 1}} \left|\int_{\partial D} S_D\varphi\; \psi\; d\sigma\right| \\
&= \sup_{\substack{\psi\in H^{\frac{1}{2}}(\partial D)\\ \|\psi\|_{H^{\frac{1}{2}}}\le 1}} |\langle \mathcal{S}_D\varphi,\psi\rangle_{L^2(\partial D)}| \le \sup_{\substack{\psi\in H^{\frac{1}{2}}(\partial D)\\ \|\psi\|_{H^{\frac{1}{2}}}\le 1}} \|\mathcal{S}_D\varphi\|_{L^2(\partial D)}\|\psi\|_{L^2(\partial D)} \\
&\le \|\mathcal{S}_D\varphi\|_{L^2(\partial D)} \le \|\mathcal{S}_D\varphi\|_{H^1(\partial D)} < \infty,
\end{aligned}$$

where the Cauchy-Schwarz inequality and the compactness of the embedding operator $I : H^{\frac{1}{2}}(\partial D) \to L^2(\partial D)$ was used. □

Further, the question of the invertibility of the single layer potential arises.

Theorem 4.6 *[9, Theorem 2.26] Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz-domain.*

i) For $d \ge 3$, $\mathcal{S}_D : L^2(\partial D) \to H^1(\partial D)$ has a bounded inverse.

ii) For $d = 2$ the operator $A : L^2(\partial D) \times \mathbb{R} \to H^1(\partial D) \times \mathbb{R}$ given by

$$A(\varphi, a) = \big(\mathcal{S}_D\varphi + a, \int_{\partial D} \varphi\; d\sigma\big) \tag{4.6}$$

has a bounded inverse.

iii) For $d = 2$, let $(\varphi_e, a) \in L^2(\partial D) \times \mathbb{R}$ denote the solution of the system

$$\begin{cases} \mathcal{S}_D\varphi_e + a = 0, \\ \int_{\partial D} \varphi_e\; d\sigma = 1, \end{cases} \tag{4.7}$$

then $\mathcal{S}_D : L^2(\partial D) \to H^1(\partial D)$ has a bounded inverse if and only if $a \ne 0$.

Lemma 4.7 *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz-domain, $\varphi \in H^{-\frac{1}{2}}(\partial D)$, such that $\mathcal{S}_D\varphi = 0$ on ∂D.*

i) If $d \geq 3$, then $\varphi = 0$.

ii) If $d = 2$ and $\int_{\partial D} \varphi = 0$, then $\varphi = 0$.

This is [9, Lemma 2.25]. As a consequence the single layer potential is invertible in three dimensions and injective in two on the subspace of integral mean value zero functions.
In fact, in the first instance it is known that $\mathcal{S}_D : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ has a bounded inverse for D simply connected and of class $\mathcal{C}^{1+\alpha}$ for $\alpha > 0$ [5, p. 24]. In two dimensions by Theorem 4.6 and the Range Theorem the following can be shown [11, Lemma A.1]:

Lemma 4.8 *Let D be a bounded Lipschitz domain, then we have for*

$$\mathcal{C} = \{\varphi \in H^{-\frac{1}{2}}(\partial D) | \exists\, \alpha \in \mathbb{C} : \mathcal{S}_D[\varphi] = \alpha\} \quad \textit{that } \dim(\mathcal{C}) = 1. \tag{4.8}$$

Define φ_0 the unique element of $\mathcal{C}$, such that

$$\int_{\partial D} \varphi_0 \, d\sigma = 1. \tag{4.9}$$

For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ it can be written

$$\varphi = \varphi - \varphi_0 \int_{\partial D} \varphi \, d\sigma + \varphi_0 \int_{\partial D} \varphi \, d\sigma := \psi + \varphi_0 \int_{\partial D} \varphi \, d\sigma. \tag{4.10}$$

Note that for all $\varphi \in H^{-\frac{1}{2}}(\partial D)$ the decomposition $\varphi = \psi + \alpha\varphi_0$ is unique. Hence, the Sobolev space of order $-1/2$ decomposes into the direct sum

$$H^{-\frac{1}{2}}(\partial D) = H_0^{-\frac{1}{2}}(\partial D) \oplus \{\mu\varphi_0, \mu \in \mathbb{C}\},$$
$$\text{where } H_0^{-\frac{1}{2}}(\partial D) = H^{\frac{1}{2}}(\partial D) \cap L_0^2(\partial D). \tag{4.11}$$

Theorem 4.9 *$\mathcal{S}_D$ is invertible on $H_0^{-\frac{1}{2}}(\partial D)$.*

This Theorem follows from [11, Theorem A.1].

4.1.2 Anisotropic Layer Potentials

Now consider the anisotropic layer potentials. Recall that

$$\Gamma^A(x) = \begin{cases} \dfrac{1}{2\pi\sqrt{\det(A)}} \log(|A_* x|), & d = 2, \\ -\dfrac{1}{4\pi\sqrt{\det(A)}|A_* x|}, & d = 3, \end{cases} \tag{4.12}$$

where A_* is the symmetric positive-definite matrix, with $A^{-1} = A_*^2$ for $\varphi \in L^2(\partial D)$, D a bounded Lipschitz-domain and

$$\mathcal{S}_D^A[\varphi](x) = \int_{\partial D} \Gamma^A(x-y)\varphi(y)d\sigma(y),\ x \in \mathbb{R}^d, \tag{4.13a}$$

$$\mathcal{D}_D^A[\varphi](x) = \int_{\partial D} \nu_y \cdot A\nabla\Gamma^A(x-y)\varphi(y)\ d\sigma(y), x \in \mathbb{R}^d \setminus \partial D. \tag{4.13b}$$

They are clearly linear and inherit basic properties of the isotropic layer potentials.
Note that $\|\Gamma^A(x)\|_{L^\infty(\partial D)} = 1/\sqrt{det(A)}\|\Gamma(A_* x)\|_{L^\infty(\partial D)} = = 1/\sqrt{det(A)}\|\Gamma(x)\|_{L^\infty(\partial D_*)}$, where $\partial D_* = \{A_* x | x \in \partial D\}$. Hence, $|\mathcal{S}_D^A\varphi(x)|$ can be bounded by $|\mathcal{S}_D\varphi(x)|$, which shows $\mathcal{S}_D^A\varphi$ is well-defined on $\mathbb{R}^d$ for any bounded domain and $\varphi \in L^2(\partial D)$.
For showing the well-defines of $\mathcal{D}_D^A[\varphi](x)$ some more elaborated arguments are needed. Basic properties of the fundamental solution of the operator $\nabla \cdot A\nabla$, which can be found in appendix A are used. Recall that $\partial\Gamma/\partial\nu$ is an essentially bounded function.
In the three-dimensional case for $x \notin \partial D$ the identity (see Lemma A.2, ii))

$$\|\nu_y \cdot A\nabla_y\Gamma^A(x-y)\|_{L^\infty(\partial D)} = \|\frac{\langle y-x, \nu_y\rangle}{4\pi\sqrt{det(A)}|A_*(x-y)|^3}\|_{L^\infty(\partial D)}, \tag{4.14}$$

is satisfied. In two dimensions it can be written (see Lemma A.2, iv))

$$\|\nu_y \cdot A\nabla_y\Gamma^A(x-y)\|_{L^\infty(\partial D)} = \|\frac{\langle y-x, \nu_y\rangle}{2\pi\sqrt{\det(A)}|A_*(x-y)|^2}\|_{L^\infty(\partial D)}. \tag{4.15}$$

Because $A_*^2 = A^{-1}$, we have $|A_*(x-y)| = |x-y|_{A^{-1}}$, the norm induced by the inner product $\langle x, y\rangle_{A^{-1}} = \langle x, A^{-1}y\rangle$ for $x, y \in \mathbb{R}^2$. By the equivalence of norms on finite dimensional vector spaces [35, Satz 2.1.2], there exists a $C > 0$ such that

$$\|\nu \cdot A\nabla\Gamma^A(x-y)\|_{L^\infty(\partial D)} \leq C\|\frac{\partial\Gamma(x-y)}{\partial\nu}\|_{L^\infty(\partial D)}. \tag{4.16}$$

Hence, the double layer potential is well-defined on $\mathbb{R}^d \setminus \partial D$ for $d = 2, 3$.
Moreover, we have $\nabla \cdot A\nabla\Gamma^A(x-y) = \delta(x-y)$ (see Lemma A.3). Therefore, $\nabla \cdot A\nabla\mathcal{S}_D^A\varphi = \nabla \cdot A\nabla\mathcal{D}_D^A\varphi = 0$ on $\mathbb{R}^d \setminus \partial D$.

Lemma 4.10 *For $D \subset \mathbb{R}^d$, a bounded Lipschitz-domain and $\varphi \in L^2(\partial D)$, we have*

i) $\mathcal{S}_D^A\varphi(x) = O(|x|^{-1})$ *as* $|x| \to \infty$ *for* $d = 3$.

ii) If $\int_{\partial D} \varphi(x)\ dx = 0$, *then* $\mathcal{S}_D^A\varphi(x) = O(|x|^{1-d})$ *as* $|x| \to \infty$ *for* $d \geq 2$.

Proof i) follows from the Taylor-expansion of $\Gamma^A(x-y)$ around x. (The interested reader is referred to the proof of Lemma A.4.)
ii) This proof is essentially the proof found in [9, sec. 2.2.1]. It shows the same asymptotic behaviour of the isotropic single layer potential.
Let $\varphi \in L^2(\partial D)$ such that $\int_{\partial D} \varphi(x)\, dx = 0$ and let $y_0 \in D$, then

$$\mathcal{S}_D^A \varphi(x) = \int_{\partial D} \left(\Gamma^A(x-y) - \Gamma^A(x-y_0)\right) \varphi(y)\, d\sigma(y).$$

Because for any $y \in \partial D$, $|\Gamma^A(x-y) - \Gamma^A(x-y_0)| \le C|x|^{1-d}$ as $|x| \to \infty$, for some $C > 0$ (see Lemma A.4), $|\mathcal{S}_D^A \varphi(x)| \le C|x|^{1-d} \|\varphi\|_{L^1(\partial D)}$, which shows the claim.

□

Moreover, the following is well-known [19, p. 408]:

Lemma 4.11 *For $D \subset \mathbb{R}^d$ a Lipschitz-domain, $\varphi \in L^2(\partial D)$ we have that*

$$\mathcal{S}_D^A \varphi|_+(x) = \mathcal{S}_D^A \varphi|_-(x) \quad a.e.\ x \in \partial D. \tag{4.17}$$

Proposition 4.12 *Let $D \Subset \mathbb{R}^3$ be of class $\mathcal{C}^{1,\alpha}$ for $0 < \alpha < 1$. $\mathcal{S}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is a linear bounded operator.*

Proof As used in the proof of [12, Lemma 5.1] it can be written

$$S_D^A = \mathcal{T}_{A_*} \mathcal{S}_{\tilde{D}} \mathcal{T}_{A_*}^{-1} r_\nu^{-1},$$

for the operators $\mathcal{T}_{A_*} \in \mathcal{L}(H^s(\partial\tilde{D}), H^s(\partial D))$ such that $\mathcal{T}_{A_*}[\varphi](x) = \varphi(A_* x)$ for $\varphi \in H^s(\partial\tilde{D})$ and $\tilde{D} = A_* D$, $r_\nu \in \mathcal{L}(H^s(\partial D), H^s(\partial D))$ such that $r_\nu[\varphi](x) = |A_*^{-1}\nu_x|\varphi(x)$, $s \in \mathbb{R}$. Hence by the sub-multiplicativity of operator norms [35, Satz 2.2.3] we have

$$\begin{aligned} &\|\mathcal{S}_D^A\|_{\mathcal{L}(H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D))} \\ &\quad \le \|\mathcal{T}_{A_*}\|_{\mathcal{L}(H^{\frac{1}{2}}, H^{\frac{1}{2}})} \|\mathcal{S}_D\|_{\mathcal{L}(H^{-\frac{1}{2}}, H^{\frac{1}{2}})} \|\mathcal{T}_{A*}^{-1}\|_{\mathcal{L}(H^{-\frac{1}{2}}, H^{-\frac{1}{2}})} \|r_\nu^{-1}\|_{\mathcal{L}(H^{-\frac{1}{2}}, H^{-\frac{1}{2}})}, \end{aligned}$$

which is finite and shows that $\mathcal{S}_D^A$ is a bounded operator. □

4.2 Symmetrization of the Neumann-Poincaré Operator

The mapping properties of the Neumann-Poincaré operator are crucial for deriving the mapping properties of the operators appearing in this work. In the following it is stated how it can be symmetrization by imposing a new

inner product on $H^{-\frac{1}{2}}(\partial D)$. This results into that, the Neumann-Poincaré operator has a discrete spectrum with an accumulation point at zero.
The reader, who might not be familiar to the spectral analysis of self-adjoint operators is refereed to [5, Appendices A].

Lemma 4.13 *[5, Lemma 2.3] Let $d \geq 2$ and $D \subset \mathbb{R}^d$ be a simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $\alpha > 0$. Then $\mathcal{S}_D : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is self-adjoint and $-\mathcal{S}_D \geq 0$ on $L^2(\partial D)$. Furthermore, $\mathcal{K}_D^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact.*

Moreover, it is well-known that the operator $\mathcal{K}_D^*$ can be symmetrized in three dimension using *Calderóns identity*. Assume that $D \subset \mathbb{R}^3$ is given as in Lemma 4.13.

Lemma 4.14 *[5, Lemma 2.4] The following Calderón identity holds on $H^{-\frac{1}{2}}(\partial D)$*

$$\mathcal{S}_D \mathcal{K}_D^* = \mathcal{K}_D \mathcal{S}_D. \tag{4.18}$$

Since $\mathcal{S}_D$ is self-adjoint, $\mathcal{K}_D \mathcal{S}_D$ is self-adjoint on $H^{-\frac{1}{2}}(\partial D)$. Moreover, for $d = 3$ due to the invertibility of the single layer potentail an inner product on $H^{-\frac{1}{2}}(\partial D)$ can be imposed that symmetrizes $\mathcal{K}_D^*$, as stated in [5], refering to [18]. In particular $\mathcal{K}_D^*$ is shown to be self-adjoint.

Theorem 4.15 *[5, Theorem 2.5] Let $\mathcal{H}^*(\partial D)$ be the space $H^{-\frac{1}{2}}(\partial D)$ with the inner product*

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} = -\langle \mathcal{S}_D[\psi], \varphi \rangle_{\frac{1}{2},-\frac{1}{2}} \quad \text{for } \varphi, \psi \in H^{-\frac{1}{2}}(\partial D), \tag{4.19a}$$

which is equivalent to the original one on $H^{-\frac{1}{2}}(\partial D)$.

i) The operator $\mathcal{K}_D^*$ is self-adjoint on the Hilbert space $\mathcal{H}^*(\partial D)$.

ii) For (λ_j, φ_j), $j = 1, 2, \ldots$, the eigenvalues and normalized eigenfunctions of $\mathcal{K}_D^*$ in $\mathcal{H}^*(\partial D)$ with $\lambda_0 = \frac{1}{2}$, we have $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$ for $j \geq 1$, $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq \to 0$ for $j \to \infty$.

iii) For every $\psi \in H^{-\frac{1}{2}}(\partial D)$, we have the spectral representation formula

$$\mathcal{K}_D^*[\psi] = \sum_{j=0}^{\infty} \lambda_j \langle \varphi_j, \psi \rangle_{\mathcal{H}^*} \varphi_j. \tag{4.19b}$$

For a symmetrization in two dimensions, which deals with the technical details of $\mathcal{S}_D$ in two dimensions, it is referred to [11, Appendix A.1].

4.3 Anisotropic Neumann-Poincaré Operator

Now we treat the anisotropic operators.

4.3.1 Homogeneous Background Media

We first start by computing the APT under a simplification for A and $\tilde{A}$ as in [12, sec. 5]. Assume that the background medium Ω is homogeneous and rename $\tilde{A}$, i.e.,

$$A := \varepsilon_m I \quad \text{and} \quad \tilde{A} := A, \tag{4.20}$$

where ε_m denotes the electric permittivity. Let $i \in \mathbb{N}^d$ with $|i| > 1$ and $(f_i, g_i) \in L^2(\partial D) \times L^2(\partial D)$ be the solution to (3.10). Then we have:

Lemma 4.16 *[12, Lemma 5.1] Let $D \Subset \mathbb{R}^3$, be a bounded simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$, the operator $\mathcal{S}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is invertible. Moreover, we have the jump formula*

$$\nu \cdot A\nabla \mathcal{S}_D^A|_\pm = \pm\frac{1}{2}I + \left(\mathcal{K}_D^A\right)^*, \quad \textit{where} \tag{4.21}$$

$$(\mathcal{K}_D^A)^*[\varphi](x) = \int_{\partial D} \frac{\langle x - y, \nu_x \rangle}{4\pi\sqrt{det(A)}|A_*(x-y)|^3}\varphi(y)d\sigma(y). \tag{4.22}$$

It follows that $f_i = (\mathcal{S}_D^A)^{-1}\left(\mathcal{S}_D[g_i] + x^i\right)$. And by the jump relation (4.21) we have [12, eq. 5.4]

$$\mathcal{Q}_D^A[g_i] = F_D^A, \quad \text{where} \tag{4.23a}$$

$$\begin{aligned} \mathcal{Q}_D^A = \frac{1}{2}&\Big(\varepsilon_m I + \left(\mathcal{S}_D^A\right)^{-1}\mathcal{S}_D\Big) \\ &+ \Big(\varepsilon_m \mathcal{K}_D^* - (\mathcal{K}_D^A)^*(\mathcal{S}_D^A)^{-1}\mathcal{S}_D\Big) \end{aligned} \tag{4.23b}$$

$$F_D^A = -\varepsilon_m \nu \cdot \nabla x^i + \left(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\right)(\mathcal{S}_D^A)^{-1}[x^i]. \tag{4.23c}$$

Stating equation (4.23) in two dimensions is costlier.

Remark 4.17 *[19, eq. (1.2)] For $D \Subset \mathbb{R}^2$ of class $\mathcal{C}^{1+\alpha}$, $0 < \alpha < 1$, the jump relation (4.21) is satisfied for*

$$(\mathcal{K}_D^A)^*[\varphi](x) = \int_{\partial D} \frac{\langle x - y, \nu_x \rangle}{2\pi\sqrt{\det(A)}|A_*(x-y)|^2}\varphi(y)\, d\sigma(y). \tag{4.24}$$

Hence it suffices to show the invertibility of $\mathcal{S}_D^A$ in the two-dimensional case. The isotropic single-layer potential can be shown to be invertible by applying *Fredholm theory* (see sec. 4.4) [9, Theorem 2.26] and *Rellich identities* [9, Corollary 2.20]. In this work injectivity of the anisotropic operator is shown by reasons of the limitation of this work.

Lemma 4.18 *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz-domain and $\varphi \in H^{-\frac{1}{2}}(\partial D)$, such that $\mathcal{S}_D^A \varphi = 0$ on ∂D.*

i) If $d \geq 3$, then $\varphi = 0$.

ii) If $d = 2$ and $\int_{\partial D} \varphi = 0$, then $\varphi = 0$.

Proof Assume $\mathcal{S}_D^A \varphi = 0$ on ∂D, then we have for $u = \mathcal{S}_D^A \varphi$, $\nabla \cdot A \nabla u = 0$ on $\mathbb{R}^d \setminus \partial D$ and $u = 0$ on ∂D. Since $u = O(|x|^{2-d})$ for $d \geq 3$ and $u = O(|x|^{1-d})$ for $d = 2$ if $\int_{\partial D} \varphi \, d\sigma = 0$ as $|x| \to \infty$ we have

$$\int_{B_R(0)\setminus \overline{D}} \nabla u \cdot A \nabla u \, dx + \int_{B_R(0)\setminus \overline{D}} \nabla \cdot (A \nabla u) u \, dx = \int_{\partial B_R(0)} \nu \cdot A \nabla u \, u \, d\sigma.$$

Because A is symmetric positive-definite we have for $d \geq 3$ there is a $C > 0$

$$\int_{\partial B_R(0)} |\nu \cdot A \nabla u \, u| \, d\sigma \leq C \int_{\partial B_R(0)} R^{3-2d} \, d\sigma = \int_{S^1} R^{2-d} \, d\Omega = O(R^{2-d}),$$

where S^1 is the unit sphere in $\mathbb{R}^3$ and $d\Omega$ the surface measure. For $d = 2$ it can be written similarly that

$$\int_{\partial B_R(0)} |\nu \cdot A \nabla u \, u| \, d\sigma \leq C \int_{\partial B_R(0)} R^{-2} \, d\sigma = \int_{S^1} R^{-2} \, d\Omega = O(R^{-2}).$$

Taking the limit $R \to 0$ and by the positivity of A we have u is constant on $\mathbb{R}^d \setminus D$. Because it vanishes on ∂D, $u = 0$ on $\mathbb{R}^d \setminus D$. Moreover $\int_D \nabla u \cdot A \nabla u \, dx = 0$, so u is constant on D. But $\mathcal{S}_D^A \varphi|_+ = \mathcal{S}_D^A \varphi|_-$, so $u = 0$ on D. Then $\varphi = \nu \cdot A \nabla \mathcal{S}_D^A \varphi|_+ - \nu \cdot A \nabla \mathcal{S}_D^A \varphi|_- = 0$. □

Theorem 4.19 *Let D be a bounded Lipschitz domain in $\mathbb{R}^2$ and*

i) $A : L^2(\partial D) \times \mathbb{R} \to H^1(\partial D) \times \mathbb{R}$ given by

$$A(\varphi, a) = (\mathcal{S}_D^A \varphi + a, \int_{\partial D} \varphi \, d\sigma) \tag{4.25}$$

is injective.

ii) Let $(\varphi_e, a) \in L^2(\partial D) \times \mathbb{R}$ denote the solution of the system

$$\begin{cases} \mathcal{S}_D^A \varphi_e + a = 0 \\ \int_{\partial D} \varphi_e \, d\sigma = 1 \end{cases} \tag{4.26}$$

then $\mathcal{S}_D^A : L^2(\partial D) \to H^1(\partial D)$ is injective if and only if $a \neq 0$.

Proof i) Assume $\mathcal{S}_D^A \varphi + a = 0$ for $\varphi \in L^2(\partial D)$ with $\int_{\partial D} \varphi \, d\sigma = 0$, then $\int_{\partial D} \mathcal{S}_D^A \varphi \varphi \, d\sigma = 0$. Hence by the jump relation of the normal derivative and because $\nabla \cdot (A\nabla \mathcal{S}_D^A \varphi) = 0$ on $\mathbb{R}^2 \setminus \partial D$, we have

$$\begin{aligned}
0 &= \int_{\partial D} \mathcal{S}_D^A \varphi \varphi \, d\sigma \\
&= \int_{\partial D} \mathcal{S}_D^A \varphi (\nu \cdot A\nabla \mathcal{S}_D^A \varphi|_+ - \nu \cdot A\nabla \mathcal{S}_D^A \varphi|_-) \, d\sigma \\
&= -\int_{\mathbb{R}^2 \setminus \overline{D}} \nabla \cdot (A\nabla \mathcal{S}_D^A \varphi) \mathcal{S}_D^A \varphi \, dx - \int_{\mathbb{R}^2 \setminus \overline{D}} \nabla \mathcal{S}_D^A \varphi \cdot A\nabla \mathcal{S}_D^A \varphi \, dx \\
&\quad - \int_D \nabla \cdot (A\nabla \mathcal{S}_D^A \varphi) \mathcal{S}_D^A \varphi \, dx - \int_D \nabla \mathcal{S}_D^A \varphi \cdot A\nabla \mathcal{S}_D^A \varphi \, dx \\
&= -\int_{\mathbb{R}^2} \nabla \mathcal{S}_D^A \varphi \cdot A\nabla \mathcal{S}_D^A \varphi \, dx.
\end{aligned}$$

Therefore, $\mathcal{S}_D^A \varphi = 0$, because $\mathcal{S}_D^A \varphi \to 0$ as $|x| \to \infty$ and $a = 0$.

ii) On the one hand, if $a = 0$, $\mathcal{S}_D^A$ can not be invertible, because $\mathcal{S}_D^A \varphi_e = 0$ and $\int_{\partial D} \varphi_e \, d\sigma = 1$.
And on the other hand, if $a \neq 0$ and $\varphi \in L^2(\partial D)$ exists such that $\mathcal{S}_D^A \varphi = 0$, define $\varphi_0 = \varphi - \int_{\partial D} \varphi \varphi_e \, d\sigma$. Then $\mathcal{S}_D^A \varphi_0 = -\int_{\partial D} \varphi \, d\sigma \mathcal{S}_D^A \varphi_e = a \int_{\partial D} \varphi \, d\sigma = 0$. Moreover, $\int_{\partial D} \mathcal{S}_D^A \varphi_0 \varphi_0 \, d\sigma = \int_{\partial D} a \int_{\partial D} \varphi \, d\sigma \varphi_0 \, d\sigma = 0$. We have

$$\begin{aligned}
0 &= \int_{\partial D} (\mathcal{S}_D^A \varphi_0) \varphi_0 \, d\sigma \\
&= \int_{\partial D} \mathcal{S}_D^A \varphi_0 (\nu \cdot A\nabla \mathcal{S}_D^A \varphi_0|_+ - \nu \cdot A\nabla \mathcal{S}_D^A \varphi_0|_-) \, d\sigma \\
&= -\int_{\mathbb{R}^2} \nabla \mathcal{S}_D^A \varphi_0 \cdot A\nabla \mathcal{S}_D^A \varphi_0 \, dx.
\end{aligned}$$

Hence, by the asymptotic behaviour of $\mathcal{S}_D^A \varphi_0$, $\mathcal{S}_D^A \varphi_0 = 0$ and $\varphi_0 = 0$. □

4.3.2 Injectivity of the Operator $\lambda I - (\mathcal{K}_D^A)^*$

In general, the set of $\lambda \in \mathbb{C}$ such that the operator $\lambda I - (\mathcal{K}_D^A)^*$ is invertible exposures the eigenvalues of the anisotropic Neumann-Poincaré operator. More particularly, the knowledge of the spectral decomposition of a self-adjoined compact operator is very applicable.
First $\mathcal{K}_D^A[1]$ is figured out building up on the proof of [9, Lemma 2.14], which states that $\mathcal{K}_D[1] = 1/2$.

Lemma 4.20 *Let $D \Subset \mathbb{R}^d$ be a bounded simply connected domain of class $C^{1+\alpha}$, $0 < \alpha < 1$, then $\mathcal{D}_D^A[1](x) = 0$ for $x \in \mathbb{R}^d \setminus \overline{D}$, $\mathcal{D}_D^A[1](x) = 1$ for $x \in D$ and $\mathcal{K}_D^A[1](x) = \frac{1}{2}$ for all $x \in \partial D$.*

Proof By applying Green's first identity on $\mathbb{R}^d \setminus \overline{D}$ and $\nabla \cdot A\nabla \Gamma^A(x-y) = \delta(x-y)$ (see Appendix A, Lemma A.3) it follows $\mathcal{D}_D^A[1](x) = 0$ on $\mathbb{R}^d \setminus \overline{D}$.

Let $x \in D$, $\varepsilon > 0$ such that $\mathcal{E}_\varepsilon^A = A_*^{-1} B_\varepsilon \subset D$, where B_ε denotes the ball of radius ε around x. Note that $\mathcal{E}_\varepsilon^A$ is an ellipsoid around x with semi axes $\lambda_i \varepsilon$, where λ_i, $1 \leq i \leq d$ are the positive eigenvalues of A.
By another application of Green's first identity and the A-harmonicness of $\Gamma^A(x-y)$ on $D \setminus \mathcal{E}_\varepsilon^A$, we have

$$\begin{aligned} 0 &= \int_{\partial D} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y) - \int_{\partial \mathcal{E}_\varepsilon^A} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y) \\ &= \mathcal{D}_D^A[1] - \int_{\mathcal{E}_\varepsilon^A} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y). \end{aligned}$$

Making use of the substitution of variables formula $\tilde{y} = A_* y$ and $\nabla_y = A_* \nabla_{\tilde{y}}$ (see appx. A, eq. (A.1)), $\nu(y) = \frac{A_* \tilde{\nu}(\tilde{y})}{|A_* \tilde{\nu}(\tilde{y})|}$ and $d\sigma(y) = \frac{1}{det(A_*)} |A_* \tilde{\nu}(\tilde{y})|\, d\sigma(\tilde{y})$, it can be written

$$\begin{aligned} \mathcal{D}_D^A[1](x) &= \int_{\partial \mathcal{E}_\varepsilon^A} \nu_y \cdot A \nabla_y \Gamma^A(x-y)\, d\sigma(y) = \int_{\partial \mathcal{E}_\varepsilon^A} \nu_y \cdot A \nabla_y \frac{\Gamma(A_*(x-y))}{\sqrt{\det(A)}}\, d\sigma(y) \\ &= \int_{\partial B_\varepsilon} \tilde{\nu}_{\tilde{y}} \cdot A_* A A_* \nabla_{\tilde{y}} \Gamma(\tilde{x} - \tilde{y})\, d\sigma(\tilde{y}) = \frac{\varepsilon^{1-d}}{\omega_d} \int_{\partial B_\varepsilon} d\sigma(y) = 1, \end{aligned}$$

where the symmetry of A_* and that A_* is the square root of A^{-1} was used.
Next the third identity is shown. Let $x \in \partial D$, $\varepsilon > 0$, $\mathcal{E}_\varepsilon^A := A_*^{-1} B_\varepsilon$, where B_ε denotes the ball of radius ε around x. Furthermore, define $\partial D_\varepsilon^A = \partial D \setminus (\partial D \cap \mathcal{E}_\varepsilon^A)$, $\partial \mathcal{E}_\varepsilon^{A'} = \partial \mathcal{E}_\varepsilon^A \cap D$ and $\partial \mathcal{E}_\varepsilon^{A''} = \{y \in \partial \mathcal{E}_\varepsilon^A |\ \nu_x \cdot y < 0\}$, the hemisphere of $\partial \mathcal{E}_\varepsilon^A$ on the other side of the tangent plane to ∂D at x.
By the Dirac delta property of $\nabla \cdot A \nabla \Gamma^A(x-y)$ and Green's formula we have

$$\begin{aligned} 0 &= \int_{D \setminus \mathcal{E}_\varepsilon^A} \nabla \cdot A \nabla \Gamma^A(x-y)\, dy \\ &= \int_{\partial D_\varepsilon^A} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y) + \int_{\partial \mathcal{E}_\varepsilon^{A'}} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y) \\ &= \int_{\partial D_\varepsilon^A} \frac{\langle y-x, \nu_y \rangle}{\omega_d \sqrt{det(A)} |A_*(x-y)|^d}\, d\sigma(y) + \int_{\partial \mathcal{E}_\varepsilon^{A'}} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y), \end{aligned}$$

where the detailed calculation of the identity used in the last line can be found in appendix A, Lemma A.2, ii) and iv). Clearly, it follows that

$$\int_{\partial D_\varepsilon^A} \frac{\langle y-x, \nu_y \rangle}{\omega_d \sqrt{det(A)} |A_*(x-y)|^d}\, d\sigma(y) = -\int_{\partial \mathcal{E}_\varepsilon^{A'}} \nu \cdot A \nabla \Gamma^A(x-y)\, d\sigma(y)$$

and hence we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon^A} \frac{\langle y-x, \nu_y \rangle}{\omega_d \sqrt{det(A)} |A_*(x-y)|^d}\, d\sigma(y) \\ &\quad = \int_{\partial D} \frac{\langle y-x, \nu_y \rangle}{\omega_d \sqrt{det(A)} |A_*(x-y)|^d}\, d\sigma(y) = \mathcal{K}_D^A[1](x). \end{aligned}$$

Furthermore, applying the substitution of variables formula $\tilde{y} = A_* y$ and $\nabla_y = A_* \nabla_{\tilde{y}}$, $\nu(y) = \frac{A_* \tilde{\nu}(\tilde{y})}{|A_* \tilde{\nu}(\tilde{y})|}$ and $d\sigma(y) = \frac{1}{det(A_*)} |A_* \tilde{\nu}(\tilde{y})| \; d\sigma(\tilde{y})$, it can be written that

$$
\begin{aligned}
-\int_{\partial \mathcal{E}_\varepsilon^{A'}} \nu \cdot A \nabla_y \Gamma^A(x-y) \; d\sigma(y) &= -\int_{\partial \mathcal{E}_\varepsilon^{A'}} \nu \cdot A \nabla_y \frac{\Gamma(A_*(x-y))}{\sqrt{\det(A)}} \; d\sigma(y) \\
&= -\int_{\partial B'_\varepsilon} \frac{\partial \Gamma(\tilde{x}-\tilde{y})}{\partial \tilde{\nu}} \; d\sigma(\tilde{y}) = \frac{\varepsilon^{1-d}}{\omega_d} \int_{\partial B'_\varepsilon} d\sigma(y),
\end{aligned}
$$

where $\partial B'_\varepsilon = \partial B_\varepsilon \cap D$ was used.
Because D is $C^{1+\alpha}$ we have $\int_{\partial B'_\varepsilon} d\sigma \to \frac{1}{2}$ for $\varepsilon \to 0$. In particular, let $\partial B''_\varepsilon = \{y \in \partial B_\varepsilon | \; \nu_x \cdot y < 0\}$, the hemisphere of ∂B_ε on the tangent plant to ∂D. Because the distance between the tangent plane at x and any point on ∂D with distance ε from x is $O(\varepsilon^{1+\alpha})$, we have

$$
\int_{\partial B'_\varepsilon} d\sigma(y) = \int_{\partial B''_\varepsilon} d\sigma(y) + O(\varepsilon^{1+\alpha}) \cdot O(\varepsilon^{d-1}) = \frac{\omega_d \varepsilon^2}{2} + O(\varepsilon^{d+\alpha}). \qquad \square
$$

The following proof of the injectivity of the anisotropic Neumann-Poincaré operator follows the idea of the proof of [9, Lemma 2.18].

Lemma 4.21 *Let D be as in Lemma 4.20 and $\lambda \in \mathbb{R}$. Then we have $\lambda I - (\mathcal{K}_D^A)^*$ is one to one on $L^2_0(\partial D)$ if $|\lambda| \geq \frac{1}{2}$ and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$ the operator $\lambda I - \mathcal{K}_D^A$ is one to one on $L^2(\partial D)$.*

Proof The claim is shown by contradiction. Let $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$, $\varphi \in L^2(\partial D) \setminus \{0\}$ such that $(\lambda I - (\mathcal{K}_D^A)^*)\varphi = 0$.
Due to Green's formula, it can be written that

$$
\begin{aligned}
\int_{\partial D} (\mathcal{K}_D^A)^* \varphi(x) \; d\sigma(x) &= \int_{\partial D} \int_{\partial D} \frac{\langle x-y, \nu_x \rangle}{\omega_d \sqrt{det(A)} |A_*(x-y)|^d} \varphi(y) \; d\sigma(y) \; d\sigma(x) \\
&= \int_{\partial D} \varphi(y) \int_{\partial D} \frac{\langle x-y, \nu_x \rangle}{\omega_d \sqrt{det(A)} |A_*(x-y)|^d} \; d\sigma(x) \; d\sigma(y) \\
&= \int_{\partial D} \varphi(y) \mathcal{K}_D^A[1](y) \; d\sigma(y).
\end{aligned}
$$

Because $\mathcal{K}_D^A[1] = 1/2$, it is concluded that

$$
0 = \int_{\partial D} (\lambda I - (\mathcal{K}_D^A)^*)\varphi \; d\sigma = \int_{\partial D} \lambda \varphi - \varphi \mathcal{K}_D^A[1] \; d\sigma = (\lambda - \frac{1}{2}) \int_{\partial D} \varphi \; d\sigma(y).
$$

Hence $\int_{\partial D} \varphi \; d\sigma(y) = 0$. It follows, $\mathcal{S}_D^A \varphi(x) = O(|x|^{1-d})$ and $\nabla \mathcal{S}_D^A \varphi(x) = O(|x|^{-d})$ for $|x| \to \infty$.
By contradiction using $\varphi \neq 0$ either

$$
c_1 = \int_D \nabla \mathcal{S}_D^A \varphi \cdot A \nabla \mathcal{S}_D^A \varphi \; dx \quad \text{or} \quad c_2 = \int_{\mathbb{R}^d \setminus \overline{D}} \nabla \mathcal{S}_D^A \varphi \cdot A \nabla \mathcal{S}_D^A \varphi \; dx
$$

must not be zero. In particular, assume $c_1 = c_2 = 0$. It follows that $\nabla \mathcal{S}_D^A \varphi = 0$ on D and $\mathbb{R}^d \setminus \overline{D}$, because A is symmetric positive-definite. But by the jump relation (3.6a), $\varphi = \nu \cdot A\nabla \mathcal{S}_D^A \varphi|_+ - \nu \cdot A\nabla \mathcal{S}_D^A \varphi|_- = 0$ on ∂D.
Because $(\lambda I - (\mathcal{K}_D^A)^*)\varphi = 0$, we obtain the identity

$$\frac{1}{2}\frac{c_2 - c_1}{c_2 + c_1} = \lambda.$$

Since c_1 and c_2 are positive, we have $|\lambda| = \frac{1}{2}|\frac{c_2-c_1}{c_2+c_1}| < 1/2$, which is a contradiction.
Assume $\lambda = 1/2$ and $\varphi \in L_0^2(\partial D)$. By the jump relation (4.21) and Green's formula it is obtained that

$$c_1 = \int_{\partial D} \nu \cdot A\nabla \mathcal{S}_D^A \varphi|_- \mathcal{S}_D^A \varphi \; d\sigma = \int_{\partial D} (-\frac{1}{2}I + (\mathcal{K}_D^A)^*)\varphi \mathcal{S}_D^A \varphi \; d\sigma = 0.$$

Hence, $\mathcal{S}_D^A \varphi$ is constant on D. By the jump relation (4.17) this implies $\mathcal{S}_D^A \varphi$ is constant on ∂D. By Lemma 4.10, ii) we have $\mathcal{S}_D^A \varphi(x) = O(|x|^{1-d})$ as $|x| \to \infty$ and

$$c_2 = -\int_{\partial D} (\frac{1}{2}I + (\mathcal{K}_D^A)^*)\varphi \mathcal{S}_D^A \varphi \; d\sigma = -\int_{\partial D} \varphi \mathcal{S}_D^A \varphi \; d\sigma = -\int_{\partial D} C\varphi \; d\sigma = 0,$$

due to the A-harmonicness of $\mathcal{S}_D^A \varphi$ on $\mathbb{R}^d \setminus \overline{D}$. Therefore, $\varphi = 0$ and $\frac{1}{2}I - (\mathcal{K}_D^A)^*$ is one to one on $L_0^2(\partial D)$ for $|\lambda| \geq 1/2$. □

Lemma 4.22 *Let $D \Subset \mathbb{R}^d$ be a bounded simply connected domain of class $\mathcal{C}^{1,\alpha}$ for $0 < \alpha < 1$ and $\varphi \in L^2(\partial D)$, then*

$$\mathcal{D}_D^A \varphi|_\pm(x) = \big(\mp \frac{1}{2}I + \mathcal{K}_D^A\big)\varphi(x) \quad \text{a.e. } x \in \partial D. \tag{4.27}$$

The Lemma can be proven in the same manner as [9, Lemma 2.15].

4.3.3 Symmetrization of the Anisotropic Neumann-Poincaré Operator

The proof of Calderón's identity for the isotropic case given by Khavinson, Putinar and Shapiro in [24, Lemma 2] can be easily modified for the anisotropic case.

Lemma 4.23 *The following Calderón identity holds on $H^{-\frac{1}{2}}(\partial D)$*

$$\mathcal{S}_D^A(\mathcal{K}_D^A)^* = \mathcal{K}_D^A \mathcal{S}_D^A. \tag{4.28}$$

Proof Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$. Due to the jump relations $\nu \cdot A\nabla\mathcal{S}_D^A\varphi|_- = -\frac{1}{2}I + (\mathcal{K}_D^A)^*|_-$, $\mathcal{D}_D^A|_+ = -\frac{1}{2}I + \mathcal{K}_D^A$ and $\mathcal{S}_D^A|_+ = \mathcal{S}_D^A|_-$ Calderón's identity is equivalent to

$$\mathcal{S}_D^A(\nu \cdot A\nabla\mathcal{S}_D^A)\varphi|_+(x) = \mathcal{D}_D^A\mathcal{S}_D^A\varphi|_+(x) \quad \text{on} \quad \partial D.$$

Therefore, it is shown that for $x_0 \in \mathbb{R}^3 \setminus \overline{D}$

$$\mathcal{D}_D^A\mathcal{S}_D^A\varphi(x_0) = \mathcal{S}_D^A(\nu \cdot A\nabla\mathcal{S}_D^A)\varphi(x_0).$$

Due to the well-defines of $\mathcal{D}_D^A\psi(x)$ on $\mathbb{R}^3 \setminus \overline{D}$ Fubini's theorem is applicable, hence we have

$$\begin{aligned}
\mathcal{D}_D^A\mathcal{S}_D^A\varphi(x_0) &= \int_{\partial D} \nu_y \cdot A\nabla_y\Gamma^A(x_0 - y) \int_{\partial D} \Gamma^A(y-z)\varphi(z)\, d\sigma(z)\, d\sigma(y) \\
&= \int_{\partial D} \varphi(z) \int_{\partial D} \nu_y \cdot A\nabla_y\Gamma^A(x_0 - y)\Gamma^A(y-z)\, d\sigma(y)\, d\sigma(z).
\end{aligned}$$

Applying Green's Formula and by $\nabla \cdot A\nabla\Gamma^A(x_0 - y) = \delta(x_0 - y)$ it is obtained that

$$\begin{aligned}
&\int_{\partial D} \nu_y \cdot A\nabla_y\Gamma^A(x_0 - y)\Gamma^A(y-z)\, d\sigma(y) \\
&\quad = \int_D \nabla_y \cdot A\nabla_y\Gamma^A(x_0 - y)\Gamma^A(y-z) + \nabla_y\Gamma^A(x_0 - y) \cdot A\nabla_y\Gamma^A(y-z)\, dy \\
&\quad = \int_D \nabla_y \cdot A\nabla_y\Gamma^A(y-z)\Gamma^A(x_0 - y) + \nabla_y\Gamma^A(x_0 - y) \cdot A\nabla_y\Gamma^A(y-z)\, dy \\
&\quad = \int_{\partial D} \nu_y \cdot A\nabla_y\Gamma^A(y-z)\Gamma^A(x_0 - y)\, d\sigma(y).
\end{aligned}$$

Clearly it follows that

$$\begin{aligned}
\mathcal{D}_D^A\mathcal{S}_D^A\varphi(x_0) &= \int_{\partial D} \Gamma^A(x_0 - y)\nu_y \cdot A\nabla_y \int_{\partial D} \Gamma^A(y-z)\varphi(z)\, d\sigma(z)\, d\sigma(y) \\
&= \mathcal{S}_D^A(\nu_y \cdot A\nabla_y\mathcal{S}_D^A\varphi)(x_0).
\end{aligned}$$

□

Making use of the inner product given in [12, Lemma 5.2] the anisotropic Neumann-Poincaré operator is symmetrized and its spectral representation is stated.

Lemma 4.24 *Let $D \Subset \mathbb{R}^3$ be a simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$. Then $\mathcal{S}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is selfadjoint and $(\mathcal{K}_D^A)^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is compact.*

Proofing the compactness is above the scope of this work. As possible proof the proof of [9, Lemma 2.13] with slight modifictaions is suggested. Moreover, compactness is stated in [12]. Self-adjointness can be easily shown by Fubini's theorem.

Theorem 4.25 *Let $\mathcal{H}_A^*(\partial D)$ be the space $H^{-\frac{1}{2}}(\partial D)$ with the inner product*

$$\langle \varphi, \psi \rangle_{\mathcal{H}_A^*} = -\langle \mathcal{S}_D^A \psi, \varphi \rangle_{\frac{1}{2},-\frac{1}{2}} \quad \text{for } \varphi, \psi \in H^{-\frac{1}{2}}(\partial D), \tag{4.29a}$$

which is equivalent to the original one on $H^{-\frac{1}{2}}(\partial D)$. Moreover, we have

i) The operator $(\mathcal{K}_D^A)^*$ is selfadjoint on the Hilbert space $\mathcal{H}_A^*(\partial D)$.

ii) For $(\lambda_j^A, \varphi_j^A)$, $j = 1, 2, \dots$, the eigenvalues and normalized eigenfunctions of $(\mathcal{K}_D^A)^*$ in $\mathcal{H}_A^*(\partial D)$ with $\lambda_0^A = \frac{1}{2}$, we have $\lambda_j^A \in (-\frac{1}{2}, \frac{1}{2})$ for $j \geq 1$, $|\lambda_1^A| \geq |\lambda_2^A| \geq \dots \geq \to 0$ for $j \to \infty$.

iii) For every $\psi \in H^{-\frac{1}{2}}(\partial D)$ we have the spectral representation formula

$$(\mathcal{K}_D^A)^*[\psi] = \sum_{j=0}^{\infty} \lambda_j^A \langle \varphi_j^A, \psi \rangle_{\mathcal{H}_A^*} \varphi_j^A. \tag{4.29b}$$

Proof In [12] it is proven that $\langle \cdot, \cdot \rangle_{\mathcal{H}_A^*}$ defines an inner product. For the reader's understanding it is rendered here.
Linearity of $\langle \cdot, \cdot \rangle_{\mathcal{H}_A^*}$ can be easily shown and by the self-adjointness of $\mathcal{S}_D^A$ symmetry.
Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$, due to eq. (4.21) and Greens' formula it can be written

$$\begin{aligned}
\langle \varphi, \varphi \rangle_{\mathcal{H}_A^*} &= -\int_{\partial D} \varphi \mathcal{S}_D^A \varphi \, d\sigma \\
&= -\int_{\partial D} \nu \cdot A \nabla \mathcal{S}_D^A \varphi|_+ \mathcal{S}_D^A \varphi \, d\sigma + \int_{\partial D} \nu \cdot A \nabla \mathcal{S}_D^A \varphi|_- \mathcal{S}_D^A \varphi \, d\sigma \\
&= \int_{\mathbb{R}^3 \setminus \overline{D}} \nabla \cdot A \nabla \mathcal{S}_D^A \varphi \; \mathcal{S}_D^A \varphi \, dx + \int_{\mathbb{R}^3 \setminus \overline{D}} \nabla \mathcal{S}_D^A \varphi \cdot A \nabla \mathcal{S}_D^A \varphi \, dx \\
&\quad - \int_{D} \nabla \cdot A \nabla \mathcal{S}_D^A \varphi \; \mathcal{S}_D^A \varphi \, dx + \int_{\partial D} \nabla \mathcal{S}_D^A \varphi \cdot A \nabla \mathcal{S}_D^A \varphi \, dx \\
&= \int_{\mathbb{R}^3} \nabla \mathcal{S}_D^A \varphi \cdot A \nabla \mathcal{S}_D^A \varphi \, dx \geq 0,
\end{aligned}$$

where the A-harmonicness of $\mathcal{S}_D^A \varphi$ and that A is a symmetric positive-definite was used. Note that we have equality if $\mathcal{S}_D^A \varphi$ is constant. Since $\mathcal{S}_D^A \varphi(x) = O(|x|^{-1})$ for $|x| \to \infty$ then it is concluded $\langle \varphi, \varphi \rangle_{\mathcal{H}_A^*} = 0$.
Recall that

$$\|\varphi\|_{\mathcal{H}^*}^2 = -\int_{\partial D} \varphi \mathcal{S}_D \varphi \, d\sigma.$$

Because A is symmetric positive-definite, there exist constants $c, C > 0$ such that

$$c\|\varphi\|_{\mathcal{H}^*}^2 \leq -\int_{\partial D} \varphi \mathcal{S}_D^A \varphi \, d\sigma \leq C\|\varphi\|_{\mathcal{H}^*}^2.$$

Because $\|\cdot\|_{\mathcal{H}^*}$ is equivalent to the norm on $H^{-\frac{1}{2}}(\partial D)$ this shows the equivalence of $\|\cdot\|_{\mathcal{H}_A^*}$ and the original norm on $H^{-\frac{1}{2}}(\partial D)$.

ii) Follows from the self-adjointnes of $\mathcal{S}_D^A$ and Calderón's identity.
iii) From the injectivity on $L^2(\partial D)$ it follows $\lambda I - (\mathcal{K}_D^A)^*$ is one to one on $H^{-\frac{1}{2}}(\partial D)$ for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$. Hence the eigenvalues of $(\mathcal{K}_D^A)^*$ are contained in $(-\frac{1}{2}, \frac{1}{2}]$. Because $\mathcal{K}_D^A[1] = \frac{1}{2}$ it is obtained that the largest eigenvalue is $1/2$. Due to the spectral representation of compact self-adjoint operators, the theorem is shown. □

4.3.4 Asymptotic Expansion

Furthermore, under the assumption of a close to homogeneous conductivity, there is an asymptotic expansion of the anisotropic single layer potential, its inverse and the anisotropic Neumann-Poincaré operators' inverse.

Lemma 4.26 *[12, Lemma 5.3] Assume $A = \varepsilon_c(I + \delta R)$, where $\varepsilon_c \in \mathbb{R}_{>0}$, $R \in \mathbb{R}^{3\times 3}$ is symmetric and $\|R\| = \mathcal{O}(1)$, for $\delta \ll 1$. We have*

$$\mathcal{S}_D^A = \frac{1}{\varepsilon_c}\big(\mathcal{S}_D + \delta \mathcal{S}_{D,1}^R + o(\delta)\big) \qquad \text{as } \delta \to 0, \tag{4.30a}$$

$$(\mathcal{S}_D^A)^{-1} = \varepsilon_c\big(\mathcal{S}_D^{-1} + \delta \mathcal{B}_{D,1}^R + o(\delta)\big) \qquad \text{as } \delta \to 0 \text{ and} \tag{4.30b}$$

$$(\mathcal{K}_D^A)^* = \mathcal{K}_D^* + \delta(\mathcal{K}_{D,1}^R)^* + o(\delta), \qquad \text{as } \delta \to 0, \tag{4.30c}$$

where

$$\begin{aligned} S_{D,1}^R[\varphi](x) &:= -\frac{1}{2}Tr(R)\mathcal{S}_D[\varphi](x) \\ &\quad - \frac{1}{2}\int_{\partial D} \frac{\langle R(x-y), x-y\rangle}{4\pi|x-y|^3}\varphi(y)\, d\sigma(y), \end{aligned} \tag{4.30d}$$

$$\mathcal{B}_{D,1}^R := -\mathcal{S}_D^{-1}\mathcal{S}_{D,1}^R\mathcal{S}_D^{-1}, \tag{4.30e}$$

$$\begin{aligned} (\mathcal{K}_{D,1}^R)^* &:= -\frac{1}{2}Tr(R)\mathcal{K}_D^*[\varphi](x) \\ &\quad - \frac{3}{2}\int_{\partial D} \frac{\langle R(x-y), x-y\rangle\langle x-y, \nu_x\rangle}{4\pi|x-y|^5}\varphi(y)d\sigma(y). \end{aligned} \tag{4.30f}$$

Also, there is an asymptotic expansion of the operator $\mathcal{Q}_D^A$, which writes as ([12, Lemma 5.4])

$$\mathcal{Q}_D^A = \mathcal{Q}_{D,0} + \delta \mathcal{Q}_{D,1}^R + o(\delta) \quad \text{as} \quad \delta \to 0, \quad \text{where} \tag{4.31a}$$

$$\mathcal{Q}_{D,0} = \frac{\varepsilon_m + \varepsilon_c}{2}I + (\varepsilon_m - \varepsilon_c)\mathcal{K}_D^* \quad \text{and} \tag{4.31b}$$

$$\mathcal{Q}_{D,1}^R = \varepsilon_c\big((\frac{1}{2}I - \quad \mathcal{K}_D^*)\mathcal{B}_{D,1}^R\mathcal{S}_D - (\mathcal{K}_{D,1}^R)^*\big). \tag{4.31c}$$

Inserting the above expansions into (4.23a), $\mathcal{Q}_D^A[g_i] = F_D^A$, yields

$$\begin{aligned} &\big(\mathcal{Q}_{D,0} + \delta \mathcal{Q}_{D,1}^R + o(\delta)\big)[g_i] \\ &\quad = -\varepsilon_m \nu \cdot \nabla x^i + \varepsilon_c\big(\big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\big)[(\mathcal{S}_D^{-1} + \delta \mathcal{B}_{D,1}^R + o(\delta))[x^i]. \end{aligned} \tag{4.32}$$

By the linearity of the operators this writes as

$$\begin{aligned}&\big(\mathcal{Q}_{D,0}+\delta\mathcal{Q}^R_{D,1}\big)[g_i]+o(\delta)\\&=-\varepsilon_m\nu\cdot\nabla x^i+\varepsilon_c\big(\big(-\frac{1}{2}I+(\mathcal{K}^A_D)^*\big)\mathcal{S}_D^{-1}+\delta\big(-\frac{1}{2}I+(\mathcal{K}^A_D)^*\big)\mathcal{B}^R_{D,1}\big)[x^i]+o(\delta).\end{aligned} \tag{4.33}$$

Therefore we have the asymptotic expansion formula

$$\begin{aligned}\big(\mathcal{Q}_{D,0}+\delta\mathcal{Q}^R_{D,1}\big)[g_i]&=-\varepsilon_m\nu\cdot\nabla x^i\\&+\varepsilon_c\big(\big(-\frac{1}{2}I+(\mathcal{K}^A_D)^*\big)\mathcal{S}_D^{-1}+\delta\big(-\frac{1}{2}I+(\mathcal{K}^A_D)^*\big)\mathcal{B}^R_{D,1}\big)[x^i]+o(\delta)\end{aligned} \tag{4.34}$$

for g_i. Note that

$$\mathcal{Q}_{D,0}=\frac{\varepsilon_m+\varepsilon_c}{2}I+(\varepsilon_m-\varepsilon_c)\mathcal{K}^*_D=(\varepsilon_m-\varepsilon_c)\big(\frac{\varepsilon_m+\varepsilon_c}{2(\varepsilon_m-\varepsilon_c)}I-\mathcal{K}^*_D\big). \tag{4.35}$$

It follows $\mathcal{Q}_{D,0}:H^{-\frac{1}{2}}(\partial D)\to H^{-\frac{1}{2}}(\partial D)$ is a bounded linear invertible operator for $\frac{\varepsilon_m+\varepsilon_c}{2(\varepsilon_m-\varepsilon_c)}\in(-\infty,-\frac{1}{2}]\cup(\frac{1}{2},\infty)$ due to the inveribility of $\lambda I-\mathcal{K}^*_D$ [9, Theorem 2. 21].

Next, the boundedness of $\mathcal{Q}^R_{D,1}$ is shown using the mapping properties of $\mathcal{S}_D$ and $\mathcal{K}^*_D$.

Proposition 4.27 *Let $D\Subset\mathbb{R}^3$ be a bounded simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0<\alpha<1$. Then $\mathcal{Q}^R_{D,1}:H^{-\frac{1}{2}}(\partial D)\to H^{-\frac{1}{2}}(\partial D)$ is a bounded linear operator.*

Proof By equation (4.31c), the compactness of $\mathcal{K}^*_D$ and the boundedness of $\mathcal{S}_D$ (Lemma 4.13) and $\mathcal{S}_D^{-1}$ it suffices to show that $\mathcal{S}^R_{D,1}$ and $(\mathcal{K}^R_{D,1})^*$ are linear and bounded. By definition they are clearly linear.
On the one hand, due to the definition of $\mathcal{S}^R_{D,1}$ it is bounded if $\tilde{\mathcal{S}}^R_D:H^{-\frac{1}{2}}(\partial D)\to H^{\frac{1}{2}}(\partial D)$ given by

$$\tilde{\mathcal{S}}^R_D[\varphi](x):=\int_{\partial D}\frac{\langle R(x-y),(x-y)\rangle}{4\pi|x-y|^3}\varphi(y)\,d\sigma(y)$$

is bounded. Because R is symmetric by the Rayleigh quotient estimate [13, Theorem 2.17] for all $z\in\mathbb{R}^3\setminus\{0\}$ we have

$$\min\{\lambda_1,\lambda_2,\lambda_3\}\leq\frac{\langle Rz,z\rangle}{|z|^2}\leq\max\{\lambda_1,\lambda_2,\lambda_3\}=C,$$

where $\lambda_1,\lambda_2,\lambda_3$ are the eigenvalues of R. Therefore $\tilde{\mathcal{S}}^R_D\varphi(x)$ is well-defined on $\mathbb{R}^3\setminus\partial D$. Moreover, by the assumption $\max\{|\lambda_1|,|\lambda_2|,|\lambda_3|\}=\|R\|=$

$O(1)$ as $\delta \to 0$ it follows $C < \infty$. Hence, by the estimate $\|\tilde{\mathcal{S}}_D^R\|_{\mathcal{L}(H^{-\frac{1}{2}},H^{\frac{1}{2}})} \leq C\|\mathcal{S}_D\|_{\mathcal{L}(H^{-\frac{1}{2}},H^{\frac{1}{2}})}$, $\tilde{\mathcal{S}}_D^R$ is bounded.

On the other hand, $(\mathcal{K}_{D,1}^R)^*$ is bounded if and only if $\tilde{\mathcal{K}}_D^R : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$,

$$(\tilde{\mathcal{K}}_D^R)^*[\varphi](x) = \int_{\partial D} \frac{\langle R(x-y), x-y\rangle \langle x-y, \nu_x\rangle}{4\pi|x-y|^5} \varphi(y)\, d\sigma(y)$$

is bounded. Analogously as above by the Rayleigh quotient estimate it can be shown that $\|(\tilde{\mathcal{K}}_D^R)^*\|_{\mathcal{L}(H^{-\frac{1}{2}},H^{-\frac{1}{2}})} \leq C\|\mathcal{K}_D^*\|_{\mathcal{L}(H^{-\frac{1}{2}},H^{-\frac{1}{2}})}$. $\square$

Remark 4.28 *By the Neumann series [22, Theorem 1.1] we have that $\mathcal{Q}_{D,0} + \delta\mathcal{Q}_{D,1}^R$ is invertible for δ sufficiently small enough and*

$$(\mathcal{Q}_{D,0} + \delta\mathcal{Q}_{D,1}^R)^{-1} = \sum_{n=0}^{\infty}(-1)^n\delta^n(\mathcal{Q}_{D,0}^{-1}\mathcal{Q}_{D,1}^R)^n\mathcal{Q}_{D,0}^{-1}. \tag{4.36}$$

Hence, we have an explicit formula for g_p in the asymptotic case.

4.4 Invertibility of $\mathcal{Q}_D^A$

In this section it is assumed that $D \Subset \mathbb{R}^3$ is a bounded simply connected domain of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$.

Definition 4.29 (Fredholm Operator) *[5, sec. 1.3.2] Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. $A \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is called Fredholm if $Im(A) \subseteq \mathcal{B}_2$ is closed, $Ker(A) \subseteq \mathcal{B}_1$ is finite-dimensional and $Im(A)$ is of finite co-dimension. We write $A \in Fred(\mathcal{B}_1, \mathcal{B}_2)$.*

For $A \in \text{Fred}(\mathcal{B}_1, \mathcal{B}_2)$ the index of A is given by

$$\begin{aligned}\text{ind}(A) &= \dim(Ker(A)) - \text{codim}(Im(A)) \\ &= \dim(Ker(A)) - \dim(\mathcal{B}_2/Im(A)).\end{aligned} \tag{4.37}$$

Lemma 4.30 $\mathcal{Q}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ *is Fredholm and of index zero.*

Proof As shown in the proof of [12, Theorem 5.1], we have

$$\mathcal{Q}_D^A = T + K,$$

for $T = \frac{1}{2}(\varepsilon_m I + (\mathcal{S}_D^A)^{-1}\mathcal{S}_D)$ an invertible and $K = \varepsilon_m\mathcal{K}_D^* - (\mathcal{K}_D^A)^*(\mathcal{S}_D^A)^{-1}\mathcal{S}_D$ a compact operator. Note that, since T is bijective, $T \in Fred(H^{-\frac{1}{2}}(\partial D), H^{-\frac{1}{2}}(\partial D))$ and $\text{ind}(T) = 0$. By [5, Proposition 1.4] $\mathcal{Q}_D^A$ is Fredholm and $\text{ind}(\mathcal{Q}_D^A) = \text{ind}(T)$. $\square$

Due to the closed range of Fredholm operators, it suffices to show injectivity of $\mathcal{Q}_D^A$ in order to show invertibility.

Theorem 4.31 $\mathcal{Q}_D^A : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ *is invertible.*

Proof This proof uses the technique found in the proof of [19, Theorem 2]. It is shown that $\mathcal{Q}_D^A$ is one to one.
Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$ such that $\mathcal{Q}_D^A \varphi = 0$, which writes as

$$\big(\varepsilon_m(\frac{1}{2}I + \mathcal{K}_D^*) - (-\frac{1}{2}I + (\mathcal{K}_D^A)^*)(\mathcal{S}_D^A)^{-1}\mathcal{S}_D\big)\varphi = 0.$$

Define $\psi(x) = (\mathcal{S}_D^A)^{-1}\mathcal{S}_D\varphi(x) \in H^{-\frac{1}{2}}(\partial D)$. By the jump relations (4.4b) and (4.21) of the normal derivatives it is obtained that φ and ψ satisfy

$$\begin{aligned} \varepsilon_m \frac{\partial \mathcal{S}_D}{\partial \nu}\varphi|_+ - \nu \cdot A\nabla\mathcal{S}_D^A\psi|_- &= 0 \quad \text{and} \\ \mathcal{S}_D^A\psi - \mathcal{S}_D\varphi &= 0 \quad \text{on} \quad \partial D. \end{aligned}$$

It is shown $\varphi = \psi = 0$. Let

$$u = \begin{cases} \mathcal{S}_D\varphi & \text{on} \quad \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{S}_D^A\psi & \text{on} \quad D. \end{cases}$$

Note that from $\mathcal{S}_D^A\psi - \mathcal{S}_D\varphi = 0$ on ∂D it follows that u is continuous across ∂D. Furthermore, $u \in H^{\frac{1}{2}}_{loc}(\mathbb{R}^3)$ and $u = O(|x|^{-1})$ for $|x| \to \infty$, due to the mapping properties of the single layer potential. Moreover, since $\mathcal{S}_D^A\psi$ is A-harmonic and $\mathcal{S}_D\varphi$ is harmonic on $\mathbb{R}^3 \setminus \partial D$ we know $\nabla \cdot ((\varepsilon_m\chi(\mathbb{R}^3 \setminus \overline{D}) + A\chi(D))\nabla u) = 0$ on $\mathbb{R}^3$.
Let $R \in \mathbb{R}_{>0}$ be sufficiently large, such that $D \subset B_R(0)$. Using Green's first identity it can be written

$$\begin{aligned} 0 &= \int_{B_R(0)} u\nabla \cdot \big((\varepsilon_m(\chi(\mathbb{R}^3 \setminus \overline{D}) + A\chi(D))\nabla u\big)\, dx \\ &= -\varepsilon_m \int_{B_R(0)\setminus\overline{D}} \nabla\mathcal{S}_D\varphi \ \cdot \nabla\mathcal{S}_D\varphi\, dx - \varepsilon_m \int_{\partial D} \mathcal{S}_D\varphi\ \nu \cdot \nabla\mathcal{S}_D\varphi|_+\, d\sigma \\ &\quad + \varepsilon_m \int_{\partial B_R(0)} \mathcal{S}_D\varphi\ \nu \cdot \nabla\mathcal{S}_D\varphi\, d\sigma - \int_D \nabla\mathcal{S}_D^A\psi \cdot A\nabla\mathcal{S}_D^A\psi\, dx \\ &\quad - \int_{\partial D} \mathcal{S}_D^A\psi\ \nu \cdot A\nabla\mathcal{S}_D^A\psi|_-\, d\sigma. \end{aligned}$$

By the equalities $\varepsilon_m \frac{\partial \mathcal{S}_D}{\partial \nu}\varphi|_+ = \nu \cdot A\nabla\mathcal{S}_D^A\psi|_-$ and $\mathcal{S}_D^A\psi = \mathcal{S}_D\varphi$ it is obtained that

$$\begin{aligned} &\varepsilon_m \int_{B_R(0)\setminus\overline{D}} \nabla\mathcal{S}_D\varphi \cdot \nabla\mathcal{S}_D\varphi\, dx + \int_D \nabla\mathcal{S}_D^A\psi \cdot A\nabla\mathcal{S}_D^A\psi\, dx \\ &\quad = \varepsilon_m \int_{\partial B_R(0)} \mathcal{S}_D\varphi\ \nu \cdot \nabla\mathcal{S}_D\varphi\, d\sigma. \end{aligned}$$

Clearly from $\mathcal{S}_D\varphi(x) = O(|x|^{-1})$ it follows $|\nabla\mathcal{S}_D\varphi(x)| = O(|x|^{-2})$ as $|x| \to \infty$. Moreover, applying the Cauchy-Schwarz inequality, there is a $C > 0$ such that

$$\lim_{R\to\infty} \left| \int_{\partial B_R(0)} \mathcal{S}_D\varphi \, \nu \cdot \nabla\mathcal{S}_D\varphi \, d\sigma \right| \le \lim_{R\to\infty} C \int_{\partial S^1} R^{-1} \, d\Omega = 0,$$

where S^1 denotes the unit sphere in $\mathbb{R}^3$ and $d\Omega$ its surface measure. Therefore, by taking the limit $R \to \infty$ of the above equations' left-hand side the identity

$$0 = \varepsilon_m \int_{\mathbb{R}^3\setminus\overline{D}} \nabla\mathcal{S}_D\varphi \cdot \nabla\mathcal{S}_D\varphi \, dx + \int_D \nabla\mathcal{S}_D^A\psi \cdot A\nabla\mathcal{S}_D^A\psi \, dx$$

is obtained. Because $\varepsilon_m > 0$ and A is a symmetric positive-definite it is concluded that u is constant on $\mathbb{R}^3 \setminus \partial D$. Recall that u is continuous across ∂D and $u = O(|x|^{-1})$ for $|x| \to \infty$, hence $u = 0$ on $\mathbb{R}^3$.
Moreover, for $v \in H_{loc}^{\frac{1}{2}}(\partial D)$ given by

$$v = \begin{cases} \mathcal{S}_D^A\psi & \text{on} \quad \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{S}_D\varphi & \text{on} \quad D, \end{cases}$$

we have $\nabla \cdot \big((\varepsilon_m\chi(D) + A\chi(\mathbb{R}^3 \setminus \overline{D}))\nabla v\big) = 0$. Analogously as for u it can be shown $v = 0$.
Hence, $\mathcal{S}_D^A\psi = \mathcal{S}_D\varphi = 0$ on $\mathbb{R}^3$, whereas $\varphi = \frac{\partial\mathcal{S}_D}{\partial\nu}\varphi|_+ - \frac{\partial\mathcal{S}_D}{\partial\nu}\varphi|_- = 0$ on ∂D.$\square$

Chapter 5

Translation, Rotation and Scaling Properties of Anisotropic Polarization Tensors

In the following chapter first the three-dimensional and second the two-dimension case is treated.

5.1 Translation, Rotation and Scaling of Anisotropic Polarization Tensors in Three Dimensions

The aim of this section is to study the behaviour of the APT under translation, rotation and scaling of the domain D. It is followed the approach given in [6, chp. 4.2]. Therein the behaviour of the GPT in two dimensions is studied.
Assume that $D \Subset \mathbb{R}^3$ is a simply connected bounded domain and ∂D of class $\mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$ then for $i, j \in \mathbb{N}^3$ $|i|, |j| > 0$, the APT writes as

$$M_{i,j}(\varepsilon_m I, A, D) = \int_{\partial D} x^j g_i(x)\, d\sigma(x). \tag{5.1a}$$

Recall equation (4.34). We have for $A = \varepsilon_c(I + \delta R)$, $\varepsilon_c > 0$, R symmetric and $\|R\| = O(1)$ as $\delta \to 0$

$$\begin{aligned} g_i = {} & (\mathcal{Q}_{D,0} + \delta \mathcal{Q}^R_{D,1})^{-1} \\ & \Big(- \varepsilon_m\, \nu \cdot \nabla x^i + \varepsilon_c \big(-\frac{1}{2} I + (\mathcal{K}^A_D)^* \big) \mathcal{S}_D^{-1} \\ & \quad + \delta \big(-\frac{1}{2} I + (\mathcal{K}^A_D)^* \big) \mathcal{B}^R_{D,1} \Big) [x^i] + o(\delta). \end{aligned} \tag{5.1b}$$

Moreover, for a general conductivity matrix A, we have by equation (4.23)

$$g_i = (\mathcal{Q}^A_D)^{-1} \big(- \varepsilon_m\, \nu \cdot \nabla x^i + \big(-\frac{1}{2} I + (\mathcal{K}^A_D)^* \big) (\mathcal{S}^A_D)^{-1} \big) [x^i]. \tag{5.1c}$$

5.1.1 Translation, Rotation and Scaling Properties of Perturbed First Order Anisotropic Polarization Tensors

First the behaviour of the APT using a homogeneous background medium and the formula for the asymptotic expansion of operators is considered. It is proven that $\left(\mathcal{Q}_{D,0} + \delta \mathcal{Q}^R_{D,1}\right)$ is invariant under translation and scaling and a rotation formula is derived. These findings are applicable in the transformation properties of the exact operator $\mathcal{Q}^A_D$.

Translation For a translated inclusion D^T invariance is shown.

Lemma 5.1 *Let $T \in \mathbb{R}^3$ and define $D^T = \{y + T|\ y \in D\}$, $x^T := x + T$ and $\varphi^T(x^T) := \varphi(x)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $y \in \partial D$ then $(\mathcal{Q}_{D^T,0} + \delta \mathcal{Q}^R_{D^T,1})[\varphi^T](x^T) = (\mathcal{Q}_{D,0} + \delta \mathcal{Q}^R_{D,1})[\varphi](x)$ for all $x \in \partial D$.*

Proof The claim is shown in two steps. Let φ, φ^T, x^T for $x \in \partial D$ and D^T be given as above.

i) By the definition of the operator $\mathcal{Q}_{D,0}$ eq. (4.31b) we have

$$\mathcal{Q}_{D^T,0}[\varphi^T](x^T) = \frac{\varepsilon_m + \varepsilon_c}{2} I[\varphi^T](x^T) + (\varepsilon_m - \varepsilon_c)\mathcal{K}^*_{D^T}[\varphi^T](x^T).$$

Note that $\mathcal{K}^*_D$ is translation invariant, i.e., $\mathcal{K}^*_{D^T}[\varphi^T](x^T) = \mathcal{K}^*_D[\varphi](x)$. The proof for the two-dimensional case can be found in [9, chp. 4.2.1]. The three-dimensional case can be shown analogously. Furthermore,

$$I[\varphi^T](x^T) = \varphi^T(x^T) = \varphi(x) = I[\varphi](x).$$

Therefore we have

$$\mathcal{Q}_{D^T,0}[\varphi^T](x^T) = \mathcal{Q}_{D,0}[\varphi](x).$$

ii) The translated operator $\mathcal{Q}_{D,1}$ as defined by eq. (4.31c) is

$$\mathcal{Q}^R_{D^T,1}[\varphi^T](x^T) = \varepsilon_c\Big(\big(\frac{1}{2}I - \mathcal{K}^*_{D^T}\big)\mathcal{B}^R_{D^T,1}\mathcal{S}_{D^T} - (\mathcal{K}^R_{D^T,1})^*\Big)[\varphi^T](x^T).$$

The single operators are acting under translation as follows:

$$\begin{aligned}(\mathcal{K}^R_{D^T,1})^*[\varphi^T](x^T) = &-\frac{1}{2}Tr(R)\mathcal{K}^*_{D^T}[\varphi^T](x^T)\\ &- \frac{3}{2}\int_{\partial D^T} \frac{\langle R(x^T - \tilde{y}), x^T - \tilde{y}\rangle\langle x^T - \tilde{y}, \nu_{x^T}\rangle}{4\pi|x^T - \tilde{y}|^5}\varphi^T(\tilde{y})\, d\sigma(\tilde{y}).\end{aligned}$$

Using the substitution of variables $\tilde{y} = y + T$, $y \in \partial D$, $x^T - \tilde{y} = x - y$, $d\sigma(\tilde{y}) = d\sigma(y)$ and $\varphi^T(x^T) = \varphi(x)$, it follows that

$$\begin{aligned}\int_{\partial D^T} & \frac{\langle R(x^T - \tilde{y}), x^T - \tilde{y}\rangle \langle x^T - \tilde{y}, \nu_{x^T}\rangle}{4\pi |x^T - \tilde{y}|^5} \varphi^T(\tilde{y})\, d\sigma(\tilde{y}) \\ &= \int_{\partial D} \frac{\langle R(x-y), x-y\rangle\langle x-y, \nu_x\rangle}{4\pi|x-y|^5}\varphi(y)\, d\sigma(y).\end{aligned}$$

Hence, $(\mathcal{K}^R_{D^T,1})^*[\varphi^T](x^T) = (\mathcal{K}^R_{D,1})^*[\varphi](x)$. By the same substitution of variables as above, the translated single layer potential equals to

$$\begin{aligned}\mathcal{S}_{D^T}[\varphi^T](x^T) &= \int_{\partial D^T} \Gamma(x^T - \tilde{y})\varphi^T(\tilde{y})\, d\sigma(\tilde{y}) \\ &= \int_{\partial D} \Gamma(x-y)\varphi^T(y^T)\, d\sigma(y) \\ &= \int_{\partial D} \Gamma(x-y)\varphi(y)\, d\sigma(y) = \mathcal{S}_D[\varphi](x).\end{aligned}$$

And by the definition of $\mathcal{S}^R_{D,1}$ eq. (4.30d) and the same argumentation it can be shown that

$$\begin{aligned}\mathcal{S}^R_{D,1}[\varphi^T](x^T) = &-\frac{1}{2}Tr(R)\mathcal{S}_{D^T}[\varphi^T](x^T) \\ &- \frac{1}{2}\int_{\partial D^T} \frac{\langle R(x^T - \tilde{y}), x^T - \tilde{y}\rangle}{4\pi|x^T - \tilde{y}|^3}\varphi^T(\tilde{y})\, d\sigma(\tilde{y}) \\ = &\, \mathcal{S}^R_{D,1}[\varphi](x).\end{aligned}$$

Let $\varphi(x) \in H^{-\frac{1}{2}}(\partial D)$ such that $\mathcal{S}_D[\varphi](x) = \psi(x)$. This is equivalent to

$$\begin{aligned}\psi(x) &= \int_{\partial D} \Gamma(x-y)\varphi(y)\, d\sigma(y) \\ &= \int_{\partial D^T} \Gamma(x^T - \tilde{y})\varphi^T(\tilde{y})\, d\sigma(\tilde{y}) = \mathcal{S}_{D^T}[\varphi^T](x^T)\end{aligned}$$

for $\varphi^T(y^T) := \varphi(y)$, where the substitution $\tilde{y} - T = y$ and $d\sigma(y) = d\sigma(\tilde{y})$ was used. Hence by applying the inverse operator we get $\mathcal{S}_D^{-1}[\psi](x) = \varphi(x) = \varphi^T(x^T) = \mathcal{S}_{D^T}^{-1}[\psi^T](x^T)$.
Furthermore, it follows that $\mathcal{B}^R_{D,1} = -\mathcal{S}_D^{-1}\mathcal{S}^R_{D,1}\mathcal{S}_D^{-1}$ is translation invariant, which shows the claim. $\square$

Lemma 5.2 *Let $T \in \mathbb{R}^3$, $x^T := x + T$, $D^T = \{x + T |\ x \in D\}$, $\varphi^T(x^T) := \varphi(x)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and $\psi^T(x^T) := \psi(x)$ for $\psi \in H^{\frac{1}{2}}(\partial D)$, then $\mathcal{K}^*_D[\varphi](x) = \mathcal{K}^*_{D^T}[\varphi^T](x^T)$, $\mathcal{S}_D[\varphi](x) = \mathcal{S}_{D^T}[\varphi^T](x^T)$, $\mathcal{S}_D^{-1}[\psi](x) = \mathcal{S}_{D^T}^{-1}[\psi^T](x^T)$ and $(\mathcal{K}^A_D)^*[\varphi](x) = (\mathcal{K}^A_{D^T})^*[\varphi^T](x^T)$ for all $x \in \partial D$.*

Proof It suffice to show the last claim. For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ it can be written by the same substitution as above

$$\begin{aligned}
(\mathcal{K}_D^A)^*[\varphi](x) &= \int_{\partial D} \frac{\langle y - x, \nu_x \rangle}{4\pi\sqrt{det(A)}|A_*(x-y)|^3} \varphi(y)\, d\sigma(y) \\
&= \int_{\partial D^T} \frac{\langle \tilde{y} - x^T, \nu_{x^T} \rangle}{4\pi\sqrt{det(A)}|A_*(x^T-\tilde{y})|^3} \varphi^T(\tilde{y})\, d\sigma(\tilde{y}) \\
&= (\mathcal{K}_{D^T}^A)^*[\varphi^T](x^T). \qquad \square
\end{aligned}$$

Following [9, chp. 4.2.1], which shows similar properties of the GPT, the translation behaviour of the APT is studied. Let $g_{D,i}$ be the solution of (3.10) for $i \in \mathbb{N}^3$ with $|i| > 0$. Then we have for $\delta \to 0$,

$$\begin{aligned}
\big(\mathcal{Q}_{D,0} + \delta\mathcal{Q}_{D,1}^R\big)[g_{D,i}] &= -\varepsilon_m \nu \cdot \nabla x^i \\
&+ \varepsilon_c\Big(\big(-\frac{1}{2}I + \big(\mathcal{K}_D^A\big)^*\big)\mathcal{S}_D^{-1} + \delta\big(-\frac{1}{2}I + \big(\mathcal{K}_D^A\big)^*\big)\mathcal{B}_{D,1}^R\Big)[x^i] + o(\delta). \quad (5.2)
\end{aligned}$$

For given j, $|j| > 0$, a multi-index, let the coefficients $c_{i,j}^T \in \mathbb{R}$ be given by

$$(x - T)^i = \sum_j c_{i,j}^T x^j \quad \text{for all} \quad x \in \mathbb{R}^3. \quad (5.3)$$

Observe that $c_{i,j}^T = 0$ if $|j| > |i|$. Note that from

$$x^i = (x + T - T)^i = (x^T - T)^i = \sum_j c_{i,j}^T (x^T)^j, \quad (5.4)$$

it follows, by the translation invariance of the derivative, that

$$\nu_x \cdot \nabla x^i = \sum_j c_{i,j}^T \nu_{x^T} \cdot \nabla (x^T)^j. \quad (5.5)$$

Furthermore, by the first part of Lemma 5.2, eq. (5.3), since $x^i \in H^{-\frac{1}{2}}(\partial D)$, and the linearity of $\mathcal{S}_{D^T}^{-1}$ we have

$$\mathcal{S}_D^{-1}[y^i](x) = \sum_j c_{i,j}^T \mathcal{S}_{D^T}^{-1}[(y^T)^j](x^T). \quad (5.6)$$

By applying the second part of Lemma 5.2 for $\psi(x) := \mathcal{S}_D^{-1}[y^i](x) = \psi^T(x^T) = \sum_j c_{i,j}^T \mathcal{S}_{D^T}^{-1}[(y^T)^j](x^T)$ and the linearity of $(\mathcal{K}_{D^T}^A)^*$, one obtains

$$(\mathcal{K}_D^A)^*\big(\mathcal{S}_D^{-1}[y^j]\big)(x) = \sum_j c_{i,j}^T (\mathcal{K}_{D^T}^A)^*\big(\mathcal{S}_{D^T}^{-1}[(y^T)^j]\big)(x^T). \quad (5.7)$$

Hence,

$$
\begin{aligned}
\Big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\Big)\mathcal{S}_D^{-1}[y^i](x) &= \sum_j c_{i,j}^T\Big(-\frac{1}{2}\mathcal{S}_{D^T}^{-1}[(y^T)^j](x^T) \\
&\qquad + (\mathcal{K}_{D^T}^A)^*\mathcal{S}_{D^T}^{-1}[(y^T)^j]\Big)(x^T) \\
&= \sum_j c_{i,j}^T\Big(-\frac{1}{2}I + (\mathcal{K}_{D^T}^A)^*\Big)\mathcal{S}_{D^T}^{-1}[(y^T)^j](x^T). \qquad (5.8)
\end{aligned}
$$

By translation invariance, i.e., Lemma 5.1, the operator $\mathcal{Q}_{D,0} + \delta\mathcal{Q}_{D,1}$ on a translated domain D^T is

$$
\begin{aligned}
&\big(\mathcal{Q}_{D^T,0} + \delta\mathcal{Q}_{D^T,1}^R\big)[g_{D,i}^T](x^T) = \big(\mathcal{Q}_{D,0} + \delta\mathcal{Q}_{D,1}^R\big)[g_{D,i}](x) \\
&\quad = -\varepsilon_m \nu \cdot \nabla x^i \qquad (5.9) \\
&\qquad + \varepsilon_c\Big(\Big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\Big)\mathcal{S}_D^{-1} + \delta\Big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\Big)\mathcal{B}_{D,1}^R\Big)[y^i](x) \\
&\qquad + o(\delta) \\
&\quad = -\varepsilon_m \sum_j c_{i,j}^T \nu_{x^T} \cdot \nabla (x^T)^j \qquad (5.10) \\
&\qquad + \sum_j c_{i,j}^T \varepsilon_c\Big(\Big(-\frac{1}{2}I - (\mathcal{K}_D^A)^*\Big)\mathcal{S}_D^{-1} \\
&\qquad\qquad + \delta\Big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\Big)\mathcal{B}_{D,1}^R\Big)[(y^T)^j](x) + o(\delta) \\
&= -\varepsilon_m \sum_j c_{i,j}^T \nu_{x^T} \cdot \nabla (x^T)^j \qquad (5.11) \\
&\qquad + \sum_j c_{i,j}^T \varepsilon_c\Big(\Big(\Big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\Big)\mathcal{S}_{D^T}^{-1} \\
&\qquad\qquad + \delta\Big(-\frac{1}{2}I + (\mathcal{K}_{D^T}^A)^*\Big)\mathcal{B}_{D^T,1}^R\Big)[(y^T)^j](x^T) + o(\delta). \\
&\quad = \sum_j c_{i,j}^T\big(\mathcal{Q}_{D^T,0} + \delta\mathcal{Q}_{D^T,1}^R\big)[g_{D^T,j}](x^T) + o(\delta). \qquad (5.12)
\end{aligned}
$$

Therefore, it is obtained for a sufficiently small $\delta > 0$ that by the injectivity of $\mathcal{Q}_{D^T,0} + \delta\mathcal{Q}_{D^T,1}^R$

$$
g_{D,i}^T = \sum_j c_{i,j}^T g_{D^T,j} + o(\delta) \quad \text{on} \quad \partial D^T. \qquad (5.13)
$$

As in [6, Proposition 4.6] it now can finally be shown:

Proposition 5.3 *Let $D^T := \{y + T \mid y \in D\}$. Then for $\delta \to 0$ the generalized anisotropic polarization tensor writes as*

$$
M_{i,j}(\varepsilon_m I, A, D) = \sum_{k,l} c_{j,k}^T c_{i,l}^T M_{l,k}(\varepsilon_m I, A, D^T) + o(\delta). \qquad (5.14)
$$

Proof By the substitution $x = \tilde{x} - T$ for $x \in \partial D$ and $g_i^T(x^T) = g_i(x)$ it follows

$$
\begin{aligned}
M_{i,j}(\varepsilon_m I, A, D) &= \int_{\partial D} x^j g_i(x)\, d\sigma(x)\\
&= \int_{\partial D^T} (\tilde{x} - T)^j g_{D,i}^T(\tilde{x})\, d\sigma(\tilde{x}).
\end{aligned}
$$

Using equations (5.3) and (5.13) it can be written

$$
\begin{aligned}
M_{i,j} &= \sum_{k,l} c_{j,k}^T c_{i,l}^T \int_{\partial D^T} \tilde{x}^k (g_{D^T,l}(\tilde{x}) + o(\delta))\, d\sigma(\tilde{x})\\
&= \sum_{k,l} c_{j,k}^T c_{i,l}^T M_{l,k}(\varepsilon_m I, A, D^T) + \sum_l c_{i,l}^T o(\delta) \int_{\partial D} x^j\, d\sigma(x)\\
&= \sum_{k,l} c_{j,k}^T c_{i,l}^T M_{l,k}(\varepsilon_m I, A, D^T) + o(\delta),
\end{aligned}
$$

where it was used that D is bounded and of class $\mathcal{C}^{1+\alpha}$, the coefficients $c_{i,l}^T$ are bounded and x^j is integrable. $\square$

Rotation For a rotated domain $D_{\alpha\beta\gamma}$ the associated conductivity matrix is similarity transformed by the associated rotation matrix.

For $y \in \mathbb{R}^3$ we denote the three rotation matrices around each coordinate axis [14, p. 189] and the rotated vector by

$$
B_\alpha := \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0\\ \sin(\alpha) & \cos(\alpha) & 0\\ 0 & 0 & 1 \end{pmatrix}, \tag{5.15a}
$$

$$
B_\beta := \begin{pmatrix} \cos(\beta) & 0 & -\sin(\beta)\\ 0 & 1 & 0\\ \sin(\beta) & 0 & \cos(\beta) \end{pmatrix}, \tag{5.15b}
$$

$$
B_\gamma := \begin{pmatrix} \cos(\gamma) & \sin(\gamma) & 0\\ -\sin(\gamma) & \cos(\gamma)) & 0\\ 0 & 0 & 1 \end{pmatrix} \tag{5.15c}
$$

$$
\text{for} \quad \alpha, \beta, \gamma \in [0, 2\pi) \text{ and}
$$

$$
y^{\alpha\beta\gamma} := B_{\alpha\beta\gamma} y \quad \text{for} \quad B_{\alpha\beta\gamma} := B_\alpha B_\beta B_\gamma. \tag{5.15d}
$$

Recall that B_α, B_β, B_γ and $B_{\alpha\beta\gamma} \in SO(3)$. The inverse of $B_{\alpha\beta\gamma}$ is denoted by $B_{-\alpha\beta\gamma} := B_{-\alpha-\beta-\gamma}$.
A rotated inclusion is denoted by $D_{\alpha\beta\gamma} = \{B_{\alpha\beta\gamma} y \mid y \in D\}$.

Lemma 5.4 *Let* $D_{\alpha\beta\gamma} = \{B_{\alpha\beta\gamma} y \mid y \in D\}$, $\varphi^{\alpha\beta\gamma}(y^{\alpha\beta\gamma}) := \varphi(y)$ *for* $\varphi \in H^{-\frac{1}{2}}(\partial D)$ *and* $y \in \partial D$, *then we have* $(\mathcal{Q}_{D_{\alpha\beta\gamma},0} + \delta \mathcal{Q}^R_{D_{\alpha\beta\gamma},1})[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = (\mathcal{Q}_{D,0} + \delta \mathcal{Q}_{D,1}^{-\alpha\beta\gamma R\alpha\beta\gamma})[\varphi](x)$ *for all* $x \in \partial D$, *where* ${}_{-\alpha\beta\gamma}R_{\alpha\beta\gamma} = B_{-\alpha\beta\gamma} R B_{\alpha\beta\gamma}$.

Lemma 5.5 *Let* $B_{\alpha\beta\gamma} \in SO(3)$, $x^{\alpha\beta\gamma} = B_{\alpha\beta\gamma}x$, $D_{\alpha\beta\gamma} = \{B_{\alpha\beta\gamma}y \mid y \in D\}$, $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\varphi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) := \varphi(x)$ *and* $\psi \in H^{\frac{1}{2}}(\partial D)$, $\psi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) := \psi(x)$ *then* $\mathcal{K}_D^*[\varphi](x) = \mathcal{K}^*_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$, $\mathcal{S}_D[\varphi](x) = \mathcal{S}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$, $\mathcal{S}_D^{-1}[\psi](x) = \mathcal{S}^{-1}_{D_{\alpha\beta\gamma}}[\psi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$ *and* $(\mathcal{K}_D^A)^*[\varphi](x) = (\mathcal{K}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}})^*[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$ *for all* $x \in \partial D$, *where* ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}$.

The proof of Lemma 5.4 and Lemma 5.5 can be found in appendix B.2 (see Lemma B.2 and B.3).

For calculating the rotation behaviour of the first order APT it can be proceed as in [6, chp. 4.2.2]. Therefore the crucial equations are:
Let $p \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$. Define the coefficients $r_{p,q}^{\alpha\beta\gamma} \in \mathbb{R}$ by

$$(B_{-\alpha\beta\gamma}x)_p = (x_{-\alpha\beta\gamma})_p = \sum_q r_{p,q}^{\alpha\beta\gamma} x_q \quad \text{for all} \quad x \in \mathbb{R}^3. \tag{5.16}$$

It can be written

$$x_p = (B_{-\alpha\beta\gamma}B_{\alpha\beta\gamma}x)_p = (B_{-\alpha\beta\gamma}x^{\alpha\beta\gamma})_p = \sum_q r_{p,q}^{\alpha\beta\gamma}(x^{\alpha\beta\gamma})_q. \tag{5.17}$$

From the transformation formulae $\nabla_x = B_{\alpha\beta\gamma}\nabla_{x^{\alpha\beta\gamma}}$ (see Appendix A, Lemma A.1), $B_{-\alpha\beta\gamma}\nu_{x^{\alpha\beta\gamma}} = \nu_x$ and using the substitution above it is derived that

$$\nu_x \cdot \nabla_x x_p = \sum_q r_{p,q}^{\alpha\beta\gamma} \nu_{x^{\alpha\beta\gamma}} \cdot \nabla_{x^{\alpha\beta\gamma}}(x^{\alpha\beta\gamma})_q. \tag{5.18}$$

Note that $\mathcal{Q}_{D,1}^R$ depends on the conductivity matrix A. Therefore using the same technique as for a translated for a rotated domain $D_{\alpha\beta\gamma}$, we have for δ sufficiently small

$$g_{D,p}^{\alpha\beta\gamma} = \sum_q r_{p,q}^{\alpha\beta\gamma} g_{D_{\alpha\beta\gamma},q}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}} + o(\delta) \quad \text{on} \quad \partial D_{\alpha\beta\gamma}, \tag{5.19}$$

where $g_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}$ denotes the solution of the anisotropic transmission problem with the conductivity matrix ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}$. In particular, this can be shown by making use of the ansatz

$$\begin{aligned}(\mathcal{Q}_{D_{\alpha\beta\gamma},0} + \delta\mathcal{Q}_{D_{\alpha\beta\gamma},1}^{\alpha\beta\gamma R_{-\alpha\beta\gamma}})[g_{D,p}^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})& \\ = (\mathcal{Q}_{D,0} + \delta\mathcal{Q}_{D,1}^R)[g_{D,p}](x),& \quad x \in \partial D.\end{aligned} \tag{5.20}$$

Proposition 5.6 *Let* $D_{\alpha\beta\gamma} := \{B_{\alpha\beta\gamma}y \mid y \in D\}$. *Then for* $\delta \to 0$ *the first order polarization tensor writes as*

$$m_{p,q}(\varepsilon_m I, A, D) \quad = \quad \sum_{s,t} r_{q,s}^{\alpha\beta\gamma} r_{p,t}^{\alpha\beta\gamma} m_{t,s}(\varepsilon_m I, {}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma}, D_{\alpha\beta\gamma}) \; + \; o(\delta), \tag{5.21}$$

where ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}$.

Scaling On a scaled inclusion sD also invariance is shown.

Lemma 5.7 *Let $s \in \mathbb{R}_{>0}$, define $sD := \{sy \mid y \in D\}$, $y^s = sy$ and $\varphi^s(y^s) := \varphi(y)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $y \in \partial D$ then $(\mathcal{Q}_{sD,0} + \delta\mathcal{Q}^R_{sD,1})[\varphi^s](x^s) = (\mathcal{Q}_{D,0} + \delta\mathcal{Q}^R_{D,1})[\varphi](x)$ for all $x \in \partial D$.*

Lemma 5.8 *Let $s \in \mathbb{R}_{>0}$, $x^s = sx$, $sD = \{sy \mid y \in D\}$, $\varphi^s(x^s) := \varphi(x)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and $\psi^s(x^s) := \psi(x)$ for $\psi \in H^{\frac{1}{2}}(\partial D)$ then $\mathcal{K}^*_D[\varphi](x) = \mathcal{K}^*_{sD}[\varphi^s](x^s)$, $\mathcal{S}_D[\varphi](x) = \frac{1}{s}\mathcal{S}_{sD}[\varphi](x^s)$, $\mathcal{S}^{-1}_D[\psi](x) = s\mathcal{S}^{-1}_{sD}[\psi^s](x^s)$ and $(\mathcal{K}^A_D)^*[\varphi](x)$ $= (\mathcal{K}^A_{sD})^*[\varphi^s](x^s)$ for all $x \in \partial D$.*

The reader interested in the proof of Lemmas 5.7 and 5.8 is referred to Appendix B.3 (Lemma B.4 and Lemma B.5).
The scaling behaviour of the perturbed first order APT is studied.
For $p \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$, and $s \in \mathbb{R}_{>0}$ we have for $sx = x^s$ that

$$x_p = (s^{-1}sx)_p = (s^{-1}x^s)_p = \frac{1}{s^{|p|}}(x^s)_p = \frac{1}{s}(x^s)_p. \tag{5.22}$$

Furthermore, we have by the transformation of the gradient under scaling (Appendix A, Lemma A.1) $\nabla_x = s\nabla_{x^s}$ and $\nu_{x^s}(x^s) = \nu_x(x)$, hence

$$\nu_x \cdot \nabla_x x_p = \nu_{x^s} \cdot \nabla_{x^s}(x^s)_p. \tag{5.23}$$

By defining $\psi(x) := x_p \in H^{\frac{1}{2}}(\partial D)$ and $\psi(x) = \psi^s(x^s) = 1/s(x^s)_p$, it is obtained, due to the linearity of $\mathcal{S}^{-1}_{sD}$, that

$$\mathcal{S}^{-1}_D[y_p](x) = s\mathcal{S}^{-1}_{sD}[\frac{1}{s}(y^s)_p](x^s) = \mathcal{S}^{-1}_{sD}[(y^{(s)})_p](x^s). \tag{5.24}$$

For $g_{D,p}$ the solution of (3.10) and $p \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$, as $\delta \to 0$ consequently we have

$$g^s_{D,p} = g_{sD,p} + o(\delta) \quad \text{on} \quad \partial sD. \tag{5.25}$$

By using the analogue technique as in the translation case, the main result of this section results:

Proposition 5.9 *Let $sD := \{sy \mid y \in D\}$. Then for $\delta \to 0$ the first order polarization tensor writes as*

$$m_{p,q}(\varepsilon_m I, A, D) = \frac{1}{s}m_{p,q}(\varepsilon_m I, A, sD) + o(\delta). \tag{5.26}$$

5.1.2 Translation, Rotation and Scaling Properties of Anisotropic Polarization Tensors

It is proven that $\mathcal{Q}^A_D$ is invariant under translation and scaling. Furthermore, a rotation formula is derived.

Translation

Lemma 5.10 *Let* $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\varphi^T(x^T) := \varphi(x)$, $x^T = x + T$, $T \in \mathbb{R}^3$, $D^T = \{y + T \mid y \in D\}$ *then* $\mathcal{S}^A_{D^T}[\varphi^T](x^T) = \mathcal{S}^A_D[\varphi](x)$.

Proof Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$. We have

$$\mathcal{S}^A_{D^T}[\varphi^T](x^T) = \int_{\partial D^T} \Gamma^A(x^T - \tilde{y})\varphi^T(\tilde{y})\, d\sigma(\tilde{y}).$$

Note that $\Gamma^A(x^T - \tilde{y}) = \Gamma^A(x - y)$ for $y = \tilde{y} - T$. Hence, by the substitution of variables $\tilde{y} = y + T$, $d\sigma(\tilde{y}) = d\sigma(y)$ and $\varphi^T(y^T) = \varphi(y)$ the identity

$$\mathcal{S}^A_{D^T}[\varphi^T](x^T) = \int_{\partial D} \Gamma^A(x - y)\varphi^T(y^T)\, d\sigma(y) = \mathcal{S}^A_D[\varphi](x),$$

is obtained. □

Lemma 5.11 *For* $\psi \in H^{\frac{1}{2}}(\partial D)$*, we have* $(\mathcal{S}^A_{D^T})^{-1}[\psi^T](x^T) = (\mathcal{S}^A_D)^{-1}[\psi](x)$.

Proof For $\psi \in H^{\frac{1}{2}}(\partial D)$, let $\varphi \in H^{-\frac{1}{2}}(\partial D)$ such that $\varphi(x) = (\mathcal{S}^A_D)^{-1}[\psi](x)$, which is equivalent to $\psi(x) = \mathcal{S}^A_D[\varphi](x) = \mathcal{S}^A_{D^T}[\varphi^T](x^T)$ by Lemma 5.10. Define $\psi^T(x^T) = \psi(x)$ for $x \in \partial D$ then, by applying $(\mathcal{S}^A_{D^T})^{-1}$ to $\psi(x)$ it can be written $\varphi^T(x^T) = (\mathcal{S}^A_{D^T})^{-1}[\psi^T](x^T)$, hence $\psi(x) = (\mathcal{S}^A_{D^T})^{-1}[\psi^T](x^T)$. □

Note that together with the translation invariance results found in section 5.1.1 it follows for $i, j \in \mathbb{N}^3$, $|i|, |j| > 0$, that

$$\mathcal{Q}^A_{D^T}[g^T_{D,i}](x^T) = \mathcal{Q}^A_D[g_{D,i}](x) = \sum_j c^T_{i,j} \mathcal{Q}^A_{D^T}[g_{D^T,j}](x^T), \tag{5.27a}$$

where the coefficients $c^T_{i,j} \in \mathbb{R}$ are given by

$$(x - T)^i = \sum_j c^T_{i,j} x^j \quad \text{for all} \quad x \in \mathbb{R}^3. \tag{5.27b}$$

Because $\mathcal{Q}^A_{D^T}$ is injective, it follows

$$g^T_{D,i} = \sum_j c^T_{i,j} g_{D^T,j} \quad \text{on} \quad \partial D^T. \tag{5.28}$$

The main result of this section, which can be proven like Proposition 5.3 with minor changes, is:

Proposition 5.12 *Let* $D^T := \{y + T \mid y \in D\}$ *then the general anisotropic polarization tensor writes as*

$$M_{i,j}(\varepsilon_m I, A, D) = \sum_{k,l} c^T_{j,k} c^T_{i,l} M_{l,k}(\varepsilon_m I, A, D^T). \tag{5.29}$$

Rotation In case of a rotated target a rotation formula is derived.

Lemma 5.13 *Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\varphi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) := \varphi(x)$, $x^{\alpha\beta\gamma} = B_{\alpha\beta\gamma}x$ for $B_{\alpha\beta\gamma} \in SO(3)$ and $D_{\alpha\beta\gamma} = \{B_{\alpha\beta\gamma}y \mid y \in D\}$ then $\mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = \mathcal{S}_D^A[\varphi](x)$, where ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}$.*

Proof Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$. We have

$$\mathcal{S}_D^A[\varphi](x) = \int_{\partial D} \Gamma^A(x-y)\varphi(y)\, d\sigma(y).$$

For $\tilde{y} = B_{\alpha\beta\gamma}y$ we have $d\sigma(y) = d\sigma(\tilde{y})$, because $B_{\alpha\beta\gamma} \in SO(3)$. Note that $\Gamma^A(x-y) = \Gamma^A(B_{-\alpha\beta\gamma}(x^{\alpha\beta\gamma} - \tilde{y})) = \Gamma^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}(x^{\alpha\beta\gamma} - \tilde{y})$ for $\tilde{y} = B_{\alpha\beta\gamma}y$ and define ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}$. Then we have $\det({}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma}) = \det(A)$, because $B_{\pm\alpha\beta\gamma} \in SO(3)$. Hence, for $\varphi^{\alpha\beta\gamma}(y^{\alpha\beta\gamma}) = \varphi(y)$ the single layer potential writes as

$$\begin{aligned}\mathcal{S}_D^A[\varphi](x) &= \int_{\partial D_{\alpha\beta\gamma}} \Gamma^A(B_{-\alpha\beta\gamma}(x^{\alpha\beta\gamma} - \tilde{y}))\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y}) \\ &= \int_{\partial D_{\alpha\beta\gamma}} \Gamma^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}(x^{\alpha\beta\gamma} - \tilde{y})\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y}) = \mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}),\end{aligned}$$

because for $B_{\alpha\beta\gamma} \in SO(3), d\sigma(y) = d\sigma(\tilde{y})$. □

Lemma 5.14 *For $\psi \in H^{\frac{1}{2}}(\partial D)$ we have $(\mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}})^{-1}[\psi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = (\mathcal{S}_D^A)^{-1}[\psi](x)$.*

Proof For $\psi \in H^{\frac{1}{2}}(\partial D)$, let $\varphi \in H^{-\frac{1}{2}}(\partial D)$ such that $\varphi(x) = (\mathcal{S}_D^A)^{-1}[\psi](x)$, which is equivalent to $\psi(x) = \mathcal{S}_D^A[\varphi](x) = \mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$. Define $\psi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) = \psi(x)$ for $x \in \partial D$ then, by applying $(\mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}})^{-1}$ to $\psi(x)$, it can be written $\varphi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) = (\mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}})^{-1}[\psi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$. Hence $\varphi(x) = (\mathcal{S}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}})^{-1}[\psi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$. □

Note that together with the rotation formulae found in section 5.1.1 it follows for $p, q \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$, that

$$\begin{aligned}\mathcal{Q}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}[g_{D,p}^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) &= \mathcal{Q}_D^A[g_{D,p}](x) \\ &= \sum_q r_{p,q}^{\alpha\beta\gamma}\mathcal{Q}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}[g_{D_{\alpha\beta\gamma},q}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}](x^{\alpha\beta\gamma}),\end{aligned} \tag{5.30a}$$

where the coefficients $r_{p,q}^{\alpha\beta\gamma} \in \mathbb{R}$ are given by

$$(x_{-\alpha\beta\gamma})_p = \sum_q r_{p,q}^{\alpha\beta\gamma}x_q \quad \text{for all} \quad x \in \mathbb{R}^3 \text{ and} \tag{5.30b}$$

$${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}. \tag{5.30c}$$

By the injectivity of $\mathcal{Q}_{D_{\alpha\beta\gamma}}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}}$, it follows that

$$g_{D,p}^{\alpha\beta\gamma} = \sum_q r_{p,q}^{\alpha\beta\gamma} g_{D_{\alpha\beta\gamma},q}^{\alpha\beta\gamma A_{-\alpha\beta\gamma}} \quad \text{on} \quad \partial D_{\alpha\beta\gamma}. \tag{5.31}$$

Hence, the first order polarization tensor rotation formula is given by:

Proposition 5.15 *Let $B_{\alpha\beta\gamma} \in SO(3)$, $D_{\alpha\beta\gamma} := \{B_{\alpha\beta\gamma} y \mid y \in D\}$ then the first order polarization tensor writes as*

$$m_{p,q}(\varepsilon_m I, A, D) = \sum_{r,s} r_{q,r}^{\alpha\beta\gamma} r_{p,s}^{\alpha\beta\gamma} m_{s,r}(\varepsilon_m I, {}_{\alpha\beta\gamma} A_{-\alpha\beta\gamma}, D_{\alpha\beta\gamma}). \tag{5.32}$$

Lemma 3.9 restricted to real symmetric matrices with unit determinate is consistent with Proposition 5.15. Because for $R = B_{-\alpha\beta\gamma} \in SO(3)$, $(B_{-\alpha\beta\gamma})_{p,q} = r_{p,q}^{\alpha\beta\gamma}$ for $p, q \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Scaling

Lemma 5.16 *Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\varphi^s(x^s) := \varphi(x)$, $x^s = sx$, $s \in \mathbb{R}_{>0}$, $sD = \{sy \mid y \in D\}$ then $\mathcal{S}_{sD}^A[\varphi^s](x^s) = s\mathcal{S}_D^A[\varphi](x)$.*

Proof Let $\varphi \in H^{-\frac{1}{2}}(\partial D)$. We have

$$\mathcal{S}_{sD}^A[\varphi^s](x^s) = \int_{\partial sD} \Gamma^A(x^s - \tilde{y})\varphi^s(\tilde{y})\, d\sigma(\tilde{y}).$$

Note that $\Gamma^A(x^s - \tilde{y}) = 1/s\Gamma^A(x - y)$ for $\tilde{y} = sy$. Hence, by the substitution of variables $\tilde{y} = sy$, $d\sigma(\tilde{y}) = s^2 d\sigma(y)$ and $\varphi^s(y^s) = \varphi(y)$ the identity

$$\mathcal{S}_{sD}^A[\varphi^s](x^s) = s\int_{\partial D} \Gamma^A(x - y)\varphi^s(y^s)\, d\sigma(y) = s\mathcal{S}_D^A[\varphi](x),$$

is obtained. $\square$

Clearly it follows:

Lemma 5.17 *For $\psi \in H^{\frac{1}{2}}(\partial D)$ we have $(\mathcal{S}_{sD}^A)^{-1}[\psi^s](x^s) = 1/s(\mathcal{S}_D^A)^{-1}[\psi](x)$.*

By applying the results found in section 5.1.1 it follows for $p \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the standard basis of $\mathbb{R}^3$, that

$$\mathcal{Q}_{sD}^A[g_{D,p}^s](x^s) = \mathcal{Q}_D^A[g_{D,p}](x) = \mathcal{Q}_{sD}^A[g_{sD,p}](x^s), \tag{5.33a}$$

$$\text{because} \quad x_p = 1/s(x^s)_p \quad \text{for all} \quad x \in \mathbb{R}^3. \tag{5.33b}$$

From the injectivity of $\mathcal{Q}_{sD}^A$ it is obtained that

$$g_{D,p}^s = g_{sD,p} \quad \text{on} \quad \partial sD. \tag{5.34}$$

The main result of this section is:

Proposition 5.18 *Let $s \in \mathbb{R}_{>0}$, $sD := \{sy \mid y \in D\}$ then the first order polarization tensor writes as*

$$m_{p,q}(\varepsilon_m I, A, D) = \frac{1}{s} m_{p,q}(\varepsilon_m I, A, sD). \tag{5.35}$$

5.2 Translation, Rotation and Scaling of $\mathcal{Q}_D^A$ in Two Dimensions

In the following the assumption is meat that $\mathcal{S}_D^A$ is invertible. The proofs in this section are left to the reader or can be found in [6, chp. 4.2].

5.2.1 Translation, Rotation and Scaling Properties

Translation Similarly as to the three-dimensional case the operator $\mathcal{Q}_D^A$ can be shown to be translation invariant.

Lemma 5.19 *Let $T \in \mathbb{R}^2$, $x^T := x + T$, $D^T = \{x + T \mid x \in D\}$, $\varphi^T(x^T) := \varphi(x)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and $\psi^T(x^T) := \psi(x)$ for $\psi \in H^{\frac{1}{2}}(\partial D)$, then $\mathcal{K}_D^*[\varphi](x) = \mathcal{K}_{D^T}^*[\varphi^T](x^T)$, $\mathcal{S}_D[\varphi](x) = \mathcal{S}_{D^T}[\varphi^T](x^T)$, $(\mathcal{K}_D^A)^*[\varphi](x) = (\mathcal{K}_{D^T}^A)^*[\varphi^T](x^T)$, $\mathcal{S}_D^A[\varphi](x) = \mathcal{S}_{D^T}^A[\varphi^T](x^T)$ and $(\mathcal{S}_D^A)^{-1}[\psi](x) = (\mathcal{S}_{D^T}^A)^{-1}[\psi^T](x^T)$ for all $x \in \partial D$.*

Proposition 5.20 *For all $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we have*

$$\mathcal{Q}_D^A[\varphi](x) = \mathcal{Q}_D^A[\varphi^T](x^T) \quad \text{for all} \quad x \in \partial D. \tag{5.36}$$

Rotation For a rotated domain D_θ the associated conductivity matrix is similarity transformed by the associated rotation matrix.

For $y \in \mathbb{R}^2$ let the around the origin rotated vector y^θ be given by [6, chp. 4.2]

$$y^\theta := \begin{pmatrix} cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := B_\theta y. \tag{5.37}$$

Recall that $B_\theta \in SO(2)$. The inverse of B_θ is denoted by $B_\theta^{-1} := B_{-\theta}$.
For a rotated inclusion it is written $D_\theta = \{B_\theta y \mid y \in D\}$.

Lemma 5.21 *For $B_\theta \in SO(2)$, $x^\theta = B_\theta x$, $D_\theta = \{B_\theta y \mid \ y \in D\}$, $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\varphi^\theta(x^\theta) := \varphi(x)$ and $\psi \in H^{\frac{1}{2}}(\partial D)$, $\psi^\theta(x^\theta) := \psi(x)$ then $\mathcal{S}_D[\varphi](x) = \mathcal{S}_{D^\theta}[\varphi^\theta](x^\theta)$ and $\mathcal{K}_{D_\theta}^*[\varphi^\theta](x^\theta) = \mathcal{K}_D^*[\varphi](x)$.*

Lemma 5.22 *For $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\psi \in H^{\frac{1}{2}}(\partial D)$, x^θ and D_θ as above we have $(\mathcal{K}_D^A)^*[\varphi](x) = (\mathcal{K}_{D_\theta}^{\theta A_{-\theta}})^*[\varphi^\theta](x^\theta)$ and $(\mathcal{S}_D^A)^{-1}[\psi](x) = (\mathcal{S}_{D_\theta}^{\theta A_{-\theta}})^{-1}[\psi^\theta](x^\theta)$ for all $x \in \partial D$, where ${}_\theta A_{-\theta} = B_\theta A B_{-\theta}$.*

Proposition 5.23 *For all $\varphi \in H^{-\frac{1}{2}}(\partial D)$ we have*

$$\mathcal{Q}_D^A[\varphi](x) = \mathcal{Q}_{D_\theta}^{\theta A_{-\theta}}[\varphi^\theta](x^\theta) \quad \text{for all} \quad x \in \partial D. \tag{5.38}$$

Scaling On a scaled inclusion sD also invariance is shown on the subspace of integral mean value zero functions.

Lemma 5.24 *Let $s \in \mathbb{R}_{>0}$, define $sD := \{sy \mid y \in D\}$, $y^s = sy$ and $\varphi^s(y^s) := \varphi(y)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $y \in \partial D$ then $\mathcal{K}_D^*[\varphi](x) = \mathcal{K}_{sD}^*[\varphi^s](x^s)$ and $(\mathcal{K}_D^A)^*[\varphi](x) = (\mathcal{K}_{sD}^A)^*[\varphi^s](x^s)$ for all $x \in \partial D$.*

Lemma 5.25 *For $\varphi \in H_0^{-\frac{1}{2}}(\partial D)$, $\psi \in H_0^{\frac{1}{2}}(\partial D)$, xs and sD as above we have $\mathcal{S}_D[\varphi](x) = \frac{1}{s}\mathcal{S}_{sD}[\varphi^s](x^s)$, $\mathcal{S}_D^{-1}[\varphi](x) = s(\mathcal{S}_{sD})^{-1}[\varphi^s](x^s)$ and $(\mathcal{S}_D^A)^{-1}[\psi](x) = s(\mathcal{S}_{sD}^A)^{-1}[\psi^s](x^s)$.*

Proposition 5.26 *For $\varphi \in H_0^{-\frac{1}{2}}(\partial D)$ we have*

$$\mathcal{Q}_D^A[\varphi](x) = \mathcal{Q}_{sD}^A[\varphi^s](x^s) \quad \text{for all} \quad x \in \partial D. \tag{5.39}$$

Chapter 6

Target Identification and Location

In this chapter a dictionary matching algorithm is designed. Under the assumption the targets are exact copies from a *dictionary* $\mathcal{D}$ up to rigid transformations and translations the derived invariants able to identify and locate targets.
The dictionary $\mathcal{D}$ is a collection of standard shapes centred at the origin and characteristic size one. The unknown shape B is transformed under an unknown rotation determined by the angles α, β and γ, scaling parameter s and translation T. Then again $D = T_T s R_{\alpha\beta\gamma} B$ for $B \in \mathcal{D}$, as well as A and ε_m, the anisotropic conductivity of the target and background, are given.

6.1 Shape Identification and Location in Three Dimensions

In this section properties of APTs derived in section 5.1.2 are applied to derive transform invariant parameters and a least squares problem. Through them, a target classification procedure and a method for determining its location is build.

As shown in Propositions 5.18 and 5.15 for any $B \in \mathcal{D}$ under rotation and scaling the first order APT matrix m writes as

$$m(\varepsilon_m I, {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, B) = R_{-\alpha\beta\gamma} m(\varepsilon_m I, A, R_{\alpha\beta\gamma} B) R_{\alpha\beta\gamma}, \tag{6.1}$$

$$\text{where} \quad {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma} = R_{-\alpha\beta\gamma} A R_{\alpha\beta\gamma} \text{ for } R_{\alpha\beta\gamma} \in SO(3) \quad \text{and}$$

$$m(\varepsilon_m I, A, B) = \frac{1}{s} m(\varepsilon_m I, A, sB) \text{ for } s \in \mathbb{R}_{>0}. \tag{6.2}$$

For a translated, scaled and rotated target D given by $D = T_T s R_{\alpha\beta\gamma} B$ with $B \in \mathcal{D}$ the first order APT writes as

$$m(\varepsilon_m I, {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D) = \frac{1}{s} R_{\alpha\beta\gamma} m(\varepsilon_m I, A, B) R_{-\alpha\beta\gamma}. \tag{6.3}$$

Moreover, let $i, j \in \mathbb{N}^3$ with $|i|, |j| > 0$ for the general APT by Proposition 5.12 the translation formula

$$M_{i,j}(\varepsilon_m I, A, B) = \sum_{k,l} c^T_{j,k} c^T_{i,l} M_{l,k}(\varepsilon_m I, A, T_T B) \quad \text{for} \quad (x-T)^i \quad = \sum_j c^T_{i,j} x^j$$

is obtained. Observe that for $|i| = 1$ and $|j| = 1$ the coefficients $c^T_{i,j} = \delta_{i,j}$. Hence, the first order polarization tensor is invariant under translation.

6.1.1 Shape Identification

First we define a ratio, which is invariant for any transformed domain:

$$\mathcal{I}(D) = \frac{tr(m(\varepsilon_m I, {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))^3}{\det(m(\varepsilon_m I, {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))} = \frac{tr(m(\varepsilon_m I, A, B))^3}{\det(m(\varepsilon_m I, A, B))}. \tag{6.4}$$

By the transformation invariance of the shape descriptor $\mathcal{I}$ the unknown shape can be measured by the simple procedure of Algorithm 1.

Algorithm 1 Shape identification

Require: Shape descriptor $\mathcal{I}(D)$ of an unknown target D;

for $B_n \in \mathcal{D}$ **do**
  $e_n \leftarrow |\mathcal{I}(D) - \mathcal{I}(\mathcal{B}_n)|$;
  $n \leftarrow n + 1$;
end for
return true dictionary element $n^* \leftarrow \arg\min_n e_n$.

6.1.2 Determination of Transformation Parameters

The scaling transformation parameter s can be uniquely determined by the formula:

$$s = \sqrt[3]{\frac{\det(m(\varepsilon_m I, A, B))}{\det(m(\varepsilon_m I, {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D))}}. \tag{6.5}$$

To determine the rotation parameters $\alpha, \beta, \gamma \in [0, 2\pi)$ the following non-linear least squares problem is imposed:

$$\begin{aligned}\hat{\alpha}\hat{\beta}\hat{\gamma} = \arg\min_{\alpha\beta\gamma} \| & m(\varepsilon_m I, {}_{-\alpha\beta\gamma} A_{\alpha\beta\gamma}, D) \\ & - \frac{1}{s} R_{\alpha\beta\gamma} m(\varepsilon_m I, A, D) R_{-\alpha\beta\gamma} \|^2_F,\end{aligned} \tag{6.6}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Because for i and j two fixed multi-indices with $|i|, |j| > 1$, we have

$$(x-T)^i = \sum_{j_1=0}^{i_1} \binom{i_1}{j_1} (-T_1)^{i_1-j_1} x_1^{j_1} \sum_{j_2=0}^{i_2} \binom{i_2}{j_2} (-T_2)^{i_2-j_2} x_2^{j_2} \sum_{j_3=0}^{i_3} \binom{i_3}{j_3} (-T_3)^{i_3-j_3} x_3^{j_3}. \quad (6.7)$$

The coefficients $c^T_{i,j}$ satisfy

$$\begin{aligned}\sum_j c^T_{i,j} x^j &= \sum_{j_1=1}^{i_1} \sum_{j_2=1}^{i_2} \sum_{j_3=1}^{i_3} \binom{i_1}{j_1} \binom{i_2}{j_2} \binom{i_3}{j_3} \quad (6.8)\\ &\quad (-T_1)^{i_1-j_1} (-T_2)^{i_2-j_2} (-T_3)^{i_3-j_3} x_1^{j_1} x_2^{j_2} x_3^{j_3} \\ &= \sum_j \binom{i}{j} (-T)^{i-j} x^j. \quad (6.9)\end{aligned}$$

And the general order APT under the translation operation equals to

$$\begin{aligned}& M_{i,j}(\varepsilon_m I_{,\alpha\beta\gamma} A_{-\alpha\beta\gamma}, D) \\ &= \sum_{k,l} c^T_{j,k} c^T_{i,l} M_{l,k}(\varepsilon_m I_{,\alpha\beta\gamma} A_{-\alpha\beta\gamma}, sR_{\alpha\beta\gamma} B) \\ &= \sum_{k,l} \binom{j}{k} \binom{i}{l} (-T)^{i+j-(k+l)} M_{l,k}(\varepsilon_m I_{,\alpha\beta\gamma} A_{-\alpha\beta\gamma}, sR_{\alpha\beta\gamma} B).\end{aligned}$$

We propose recovering T by the 2nd order APT for $|i| = |j| = 2$

$$\begin{aligned}\hat{T} = \arg\min_T |\sum_{k,l} \binom{j}{k} \binom{i}{l} (-T)^{i+j-(k+l)} M_{l,k}(\varepsilon_m I_{,\alpha\beta\gamma} A_{-\alpha\beta\gamma}, sR_{\alpha\beta\gamma} B) \\ - M_{l,k}(\varepsilon_m I_{,\alpha\beta\gamma} A_{-\alpha\beta\gamma}, B)|. \quad (6.10)\end{aligned}$$

6.2 Complex Contracted Anisotropic Polarization Tensors Under Rigid Motions and Scaling

In [4] complex contracted GPTs and their invariants with respect to rigid motions and scaling are derived. In this section complex contracted APTs are introduced and their properties with respect to rigid motions and scaling are derived.
Nice properties of $\mathcal{Q}^A_D$, such as translation and scaling invariance as well as relations with respect to rotations are used.
It is assumed that $\mathcal{Q}^A_D : H_0^{-\frac{1}{2}}(\partial D) \to H_0^{-\frac{1}{2}}(\partial D)$ for $D \subset \mathbb{R}^2$ is invertible.

Let $D \subset \mathbb{R}^2$ be a simply connected domain, $m, n \in \mathbb{N}_{>0}$ and $i, j \in \mathbb{N}^2$ with $|i|, |j| > 0$ be two multi-indices. Let the real coefficients a_i^m, a_j^m, b_i^n and b_j^n be given by the complex polynomial

$$P_m(x) = (x_1 + ix_2)^m = \sum_{|i|=m} a_i^m x^i + i \sum_{|j|=m} b_j^m x^j. \tag{6.11}$$

We define complex contracted APTs (complex CAPTs) by

$$\begin{aligned}\mathbb{N}_{mn}^{(1)}(\varepsilon_m I, A, D) &:= M_{mn}^{cc}(\varepsilon_m I, A, D) - M_{mn}^{ss}(\varepsilon_m I, A, D)\\ &\quad + i\big(M_{mn}^{cs}(\varepsilon_m I, A, D) + M_{mn}^{sc}(\varepsilon_m I, A, D)\big) \quad \text{and} \qquad (6.12a)\\ \mathbb{N}_{mn}^{(2)}(\varepsilon_m I, A, D) &:= M_{mn}^{cc}(\varepsilon_m I, A, D) + M_{mn}^{ss}(\varepsilon_m I, A, D)\\ &\quad + i\big(M_{mn}^{cs}(\varepsilon_m I, A, D) - M_{mn}^{sc}(\varepsilon_m I, A, D)\big), \qquad (6.12b)\end{aligned}$$

where the associated real CAPTs are

$$M_{mn}^{cc}(\varepsilon_m I, A, D) := \sum_{|i|=m} \sum_{|j|=n} a_i^m a_j^n M_{i,j}(\varepsilon_m I, A, D), \tag{6.13a}$$

$$M_{mn}^{cs}(\varepsilon_m I, A, D) := \sum_{|i|=m} \sum_{|j|=n} a_i^m b_j^n M_{i,j}(\varepsilon_m I, A, D), \tag{6.13b}$$

$$M_{mn}^{sc}(\varepsilon_m I, A, D) := \sum_{|i|=m} \sum_{|j|=n} b_i^m a_j^n M_{i,j}(\varepsilon_m I, A, D), \tag{6.13c}$$

$$M_{mn}^{ss}(\varepsilon_m I, A, D) := \sum_{|i|=m} \sum_{|j|=n} b_i^m b_j^n M_{i,j}(\varepsilon_m I, A, D). \tag{6.13d}$$

By the linearity of $\mathcal{Q}_D^A$ we have the identities

$$\begin{aligned}\mathbb{N}_{mn}^{(1)}(\varepsilon_m I, A, D) &= \int_{\partial D} P_n(y) (\mathcal{Q}_D^A)^{-1} [-\varepsilon_m \nu \cdot \nabla P_m\\ &\quad + \big(-\frac{1}{2} I + (\mathcal{K}_D^A)^*\big)(\mathcal{S}_D^A)^{-1} P_m](y)\, d\sigma(y), \qquad (6.14a)\\ \mathbb{N}_{mn}^{(2)}(\varepsilon_m I, A, D) &= \int_{\partial D} P_n(y) (\mathcal{Q}_D^A)^{-1} [-\varepsilon_m \nu \cdot \nabla \overline{P_m}\\ &\quad + \big(-\frac{1}{2} I + (\mathcal{K}_D^A)^*\big)(\mathcal{S}_D^A)^{-1} \overline{P_m}](y)\, d\sigma(y), \qquad (6.14b)\end{aligned}$$

where $\bar{z}$ denotes the conjugate of a complex number z. Note that $P_m(x)$ is holomorphic and therefore $P_m(x) \in L_0^2(\partial D)$.

6.3 Properties of Complex Conjugated Anisotropic Polarization Tensors

Complex CAPTs matrices are defined by $\mathbb{N}^{(1)} := (\mathbb{N}_{mn}^{(1)})_{mn}$ and $\mathbb{N}^{(2)} := (\mathbb{N}_{mn}^{(2)})_{mn}$. In this section it is shown that these matrices satisfy for a shifted, scaled and rotated domain $T_z sR_\theta D$

$$\mathbb{N}^{(1)}(T_z sR_\theta D) = C^z G^\omega \mathbb{N}^{(1)}(D, R_\theta AR_{-\theta}) G^\omega (C^z)^T, \tag{6.15a}$$

$$\mathbb{N}^{(2)}(T_z sR_\theta D) = \overline{C^z G^\omega} \mathbb{N}^{(2)}(D, R_\theta AR_{-\theta}) G^\omega (C^z)^T, \tag{6.15b}$$

$$\text{where} \quad G^\omega = diag(\omega^m) \quad \text{for } \omega^m = s^m e^{im\theta}$$

$$\text{and} \quad C^z_{mn} = \binom{m}{n} z^{m-n}.$$

By the symmetry of APTs due to Theorem 3.10 the matrices of the complex CAPTs are symmetric.

Proposition 6.1 *The matrix $\mathbb{N}^{(1)}$ of the complex CAPT is symmetric, i.e., $(\mathbb{N}^{(1)})^T = \mathbb{N}^{(1)}$ and $\mathbb{N}^{(2)}$ is Hermitian, i.e., $\overline{(\mathbb{N}^{(2)})}^T = \mathbb{N}^{(2)}$.*

6.3.1 Scaling, Translation and Dilatation of Complex Conjugated Anisotropic Polarization Tensors

The properties under scaling rotation and dilatation of the complex CAPTs are considered in this section. It is written $\mathbb{N}_{mn}(D)$ for simplification if the conductivity matrices are avoidable and

$$\mathbb{N}^{(1)}_{mn}(D) = \int_{\partial D} P_n(y) \varphi_{D,m}(y) \, d\sigma(y),$$

$$\mathbb{N}^{(2)}_{mn}(D) = \int_{\partial D} P_n(y) \overline{\varphi}_{D,m}(y) \, d\sigma(y),$$

$$\text{where} \quad \varphi_{D,m}(y) = (\mathcal{Q}^A_D)^{-1}[-\varepsilon_m \, \nu \cdot \nabla P_m + \big(-\frac{1}{2}I + (\mathcal{K}^A_D)^*\big)(\mathcal{S}^A_D)^{-1} P_m](y).$$

Scaling Let $s \in \mathbb{R}_{>0}$ and $sD = \{sy \mid y \in D\}$ be a scaled target. For all $m, n \in \mathbb{N}_{>0}$ we have the complex CAPTs scaling formula

$$\mathbb{N}^{(1)}_{mn}(sD) = s^{m+n} \mathbb{N}^{(1)}_{mn}(D) \quad \text{and} \tag{6.16a}$$

$$\mathbb{N}^{(2)}_{mn}(sD) = s^{m+n} \mathbb{N}^{(2)}_{mn}(D). \tag{6.16b}$$

Proof Because $P_m(sy) = s^m y^m$ and by the substitution $y^s = sy$, $d\sigma(y) = \det(\frac{1}{s}I)|s\nu(y)| d\sigma(y^s)$, it can be written

$$\mathbb{N}^{(1)}_{mn}(D) = \int_{\partial D} P_n(s^{-1}sy) \varphi^s_{D,m}(sy) \, d\sigma(y) = \frac{1}{s} s^{-n} \int_{\partial sD} P_n(y^s) \varphi^s_{sD,m}(y^s) \, d\sigma(y^s).$$

Whereas by the scaling invariance of $\mathcal{Q}^A_D$ we have

$$\begin{aligned} \mathcal{Q}^A_{sD}[\varphi^s_{D,m}](y^s) &= \mathcal{Q}^A_D[\varphi_{D,m}](y) \\ &= -\varepsilon_m \nu_y \cdot \nabla_y P_m(y) + (-\frac{1}{2}I + (\mathcal{K}^A_D)^*)(\mathcal{S}^A_D)^{-1} P_m(y). \end{aligned}$$

Due to the identity $\nu_s = \nu$ and the transformation of the derivative $\nabla_y = s\nabla_{y^s}$ (see Lemma A.1) it follows

$$\nu_y \cdot \nabla_y P_m(y) = s^{-m+1}\nu_s \cdot \nabla_{y^s} P_m(y^s),$$

hence by the substitution $P_m(y) = s^{-m}P_m(y^s)$ it is obtained that

$$\big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\big)(\mathcal{S}_D^A)^{-1}P_m(y) = s^{-m+1}\big(-\frac{1}{2}I + (\mathcal{K}_{sD}^A)^*\big)(\mathcal{S}_{sD}^A)^{-1}P_m(y^s).$$

It follows $\mathcal{Q}_{sD}^A[\varphi_{D,m}^s](y^s) = s^{-m+1}\mathcal{Q}_{sD}^A[\varphi_{sD,m}](y^s)$. Hence, by the injectivity of $\mathcal{Q}_D^A$ it can be written $\varphi_{D,m}^s(y^s) = s^{-m+1}\varphi_{sD,m}(y^s)$ and we have

$$\mathbb{N}_{mn}^{(1)}(D) = \frac{1}{s}s^{-n}\int_{\partial sD} P_n(y^s)s^{-m+1}\varphi_{sD,m}(y^s)d\sigma(y^s) = s^{-(m+n)}\mathbb{N}_{mn}^{(1)}(sD).$$

The second scaling equation can be shown analogously. □

Translation Define for $z \in \mathbb{R}^2$ the translated target $D^z = \{y + z \mid y \in D\}$. For all $m, n \in \mathbb{N}_{>0}$ we have the complex CAPT translation formula

$$\mathbb{N}_{mn}^{(1)}(D^z) = \sum_{l=1}^{m}\sum_{k=1}^{n} C_{ml}^z \mathbb{N}_{lk}^{(1)}(D) C_{nk}^z \quad \text{and} \tag{6.17a}$$

$$\mathbb{N}_{mn}^{(2)}(D^z) = \sum_{l=1}^{m}\sum_{k=1}^{n} \overline{C_{ml}^z} \mathbb{N}_{lk}^{(2)}(D) C_{nk}^z, \tag{6.17b}$$

$$\text{where } C_{mn}^z = \binom{m}{n} z^{m-n}.$$

Proof Note that we have for $y^z = y + z$

$$P_m(y^z) = (z+T)^m = \sum_{n=0}^{m}\binom{m}{n} y^n T^{m-n} = \sum_{n=0}^{m}\binom{m}{n} P_n(y) z^{m-n}.$$

Furthermore, it can be written

$$\mathcal{Q}_{D^z}^A[\varphi_{D^z,m}](y^z) = -\varepsilon_m \nu_y \cdot \nabla_{y^z} P_m(y^z) + \big(-\frac{1}{2}I + (\mathcal{K}_{D^z}^A)^*\big)(\mathcal{S}_{D^z}^A)^{-1}P_m(y^z).$$

By the translation invariance of $\big(-\frac{1}{2}I + (\mathcal{K}_{D^z}^A)^*\big)(\mathcal{S}_D^A)^{-1}$ it is obtained

$$\begin{aligned}
\mathcal{Q}_{D^z}^A[\varphi_{D^z,m}](y^z) &= \sum_{s=0}^{m}\binom{m}{s} z^{m-s}\big(-\varepsilon_m \nu_y \cdot \nabla_y P_s(y) + \big(-\frac{1}{2}I + (\mathcal{K}_D^A)^*\big)(\mathcal{S}_D^A)^{-1}P_s(y)\big) \\
&= \sum_{s=0}^{m}\binom{m}{s} z^{m-s}\mathcal{Q}_D^A[\varphi_{D,s}](y) \\
&= \sum_{s=0}^{m}\binom{m}{s} z^{m-s}\mathcal{Q}_{D^z}^A[\varphi_{D^z,s}^z](y^z).
\end{aligned}$$

Hence, we have $\varphi_{D^z,m}(y^z) = \sum_{s=0}^{m} \binom{m}{s} z^{m-s} \varphi^z_{D,s}(y^z)$. It follows that we have the complex CAPT equals to

$$\begin{aligned}
\mathbb{N}^{(1)}_{mn}(D^z) &= \int_{\partial D^z} P_n(y^z)\varphi_{D^z,m}(y^z)\, d\sigma(y^z) \\
&= \sum_{t=0}^{n} \binom{n}{t} z^{n-t} \int_{\partial D} P_t(y) \sum_{s=0}^{m} \binom{m}{s} T^{m-s} \varphi^z_{D,s}(y)\, d\sigma(y) \\
&= \sum_{s=1}^{m}\sum_{t=1}^{n} \binom{m}{s}\binom{n}{t} z^{m-s} z^{n-t} \mathbb{N}^{(1)}_{s,t}(D) = \sum_{s=1}^{m}\sum_{t=1}^{n} C^z_{ms}\mathbb{N}^{(1)}_{st}(D) C^z_{nt},
\end{aligned}$$

where it was used that $\mathbb{N}^{(1)}_{0t} = \mathbb{N}^{(1)}_{s0} = 0$ by the symmetry of $\mathbb{N}^{(1)}$ and $\int_{\partial D^z} \varphi^z_{D^z,s}\, d\sigma(y^z) = 0$. $\square$

Dilatation Let a rotated target be given by a dilatation in the complex plane $D_\theta = B_\theta D = \{e^{i\theta}x \mid x \in D\}$ for $\theta \in [0, 2\pi)$. For all $m, n \in \mathbb{N}_{>0}$ we have the complex CAPTs rotation formula

$$\mathbb{N}^{(1)}_{mn}(B_\theta D, A) = e^{i(m+n)\theta}\mathbb{N}^{(1)}_{mn}(D, B_\theta A B_{-\theta}) \quad \text{and} \tag{6.18a}$$

$$\mathbb{N}^{(2)}_{mn}(B_\theta D, A) = e^{i(n-m)\theta}\mathbb{N}^{(2)}_{mn}(D, B_\theta A B_{-\theta}). \tag{6.18b}$$

Proof Since $P_m(y) = e^{-im\theta}P_m(y_\theta)$ for $y_\theta = e^{i\theta}y$ and by the substitution $y_\theta = e^{i\theta}y$, $d\sigma(y) = d\sigma(y_\theta)$ for $\varphi^\theta_{D,m}(y^\theta) = \varphi_{D,m}(y)$, it can be shown that

$$\mathbb{N}^{(1)}_{mn}(D, A) = \int_{\partial D} P_n(y)\varphi_{D,m}(y)\, d\sigma(y) = e^{-in\theta}\int_{\partial D_\theta} P_n(y_\theta)\varphi^\theta_{D,m}(y_\theta)\, d\sigma(y_\theta).$$

By applying the rotation formula for $\mathcal{Q}^A_D$,

$$\begin{aligned}
\mathcal{Q}^{\theta A_{-\theta}}_{D_\theta}[\varphi^\theta_{D,m}](y_\theta) &= \mathcal{Q}^A_D[\varphi_{D,m}](y) \\
&= -\varepsilon_m \nu_y \cdot \nabla_y P_m(y) \\
&\quad + (-\frac{1}{2}I + (\mathcal{K}^A_D)^*)(\mathcal{S}^A_D)^{-1}P_m(y).
\end{aligned}$$

Due to the identity $\nu_{y_\theta} = e^{i\theta}\nu_y$ and the chain rule $\nabla_y = e^{i\theta}\nabla_{y_\theta}$ (see Lemma A.1), it follows that

$$\nu_y \cdot \nabla_y P_m(y) = e^{-im\theta}\nu_{y_\theta} \cdot \nabla_{y_\theta} P_m(y_\theta).$$

Hence, it is obtained that

$$\begin{aligned}
\mathcal{Q}^{\theta A_{-\theta}}_{D_\theta}[\varphi^\theta_{D,m}](y_\theta) &= -e^{-im\theta}\varepsilon_m \nu_{y_\theta} \cdot \nabla_{y_\theta} P_m(y_\theta) \\
&\quad + \big(-\frac{1}{2}I + (\mathcal{K}^{\theta A_{-\theta}}_{D_\theta})^*\big)(\mathcal{S}^{\theta A_{-\theta}}_{D_\theta})^{-1} e^{-im\theta} P_m(y_\theta) \\
&= e^{-im\theta}\varphi^{\theta A_{-\theta}}_{D_\theta, m}(y_\theta),
\end{aligned}$$

where $\varphi^{\theta A_{-\theta}}_{D_\theta,m}(y_\theta) = -\varepsilon_m \nu_{y_\theta} \cdot \nabla P_m(y_\theta) + \big(-\frac{1}{2}I + (\mathcal{K}^{\theta A_{-\theta}}_{D_\theta})^*\big)(\mathcal{S}^{\theta A_{-\theta}}_{D_\theta})^{-1} P_m(y_\theta)$.
Due to the injectivity of $\mathcal{Q}^A_D$ the complex contacted CAPT writes as

$$\begin{aligned}\mathbb{N}^{(1)}_{mn}(D,A) &= e^{-i(n+m)\theta} \int_{\partial D_\theta} P_n(\tilde{y}) \varphi^{\theta A_{-\theta}}_{D_\theta,m}(\tilde{y}) d\sigma(\tilde{y}) \\ &= e^{-i(n+m)\theta} \mathbb{N}^{(1)}_{mn}(D_\theta, B_\theta A B_{-\theta}).\end{aligned}$$

The second part of the proof is left to the reader. □

6.3.2 Shape Identification by Complex Contracted Anisotropic Polarization Tensors

In [4] two algorithms for shape classification are introduced using complex contracted GPTs. The equations, which are the heart of the approach, can also be stated for complex contracted APTs.

Let $\mathcal{D}$ be a given dictionary such that $B \in \mathcal{D}$ is centred at the origin.

Let $D = T_z s R_\theta B$ for $B \in \mathcal{D}$, we have

$$\mathbb{N}^{(1)}_{11}(D, R_{-\theta} A R_\theta) = \omega^2 \mathbb{N}^{(1)}_{11}(B, A), \tag{6.19a}$$

$$\mathbb{N}^{(1)}_{12}(D, R_{-\theta} A R_\theta) = 2\mathbb{N}^{(1)}_{11}(D, R_{-\theta} A R_\theta) z + \omega^3 \mathbb{N}^{(2)}_{12}(B, A), \tag{6.19b}$$

$$\mathbb{N}^{(2)}_{12}(D, R_{-\theta} A R_\theta) = 2\mathbb{N}^{(2)}_{11}(D, R_{-\theta} A R_\theta) z + s^2 \omega \mathbb{N}^{(2)}_{12}(B, A), \tag{6.19c}$$

where $\omega = s e^{i\theta}$.

Chapter 7

Conclusion

In this work an identification and location procedure for extended three-dimensional anisotropic targets was established. Moreover, it is shown that anisotropic and isotropic layer potentials and the Neumann-Poincaré operator satisfy common features.
We proofed that the anisotropic single and double layer potential have the same asymptotic behaviour as the isotropic operators and used the jump relations to show injectivity of the anisotropic single layer potential and state invertibility in two respectively three dimensions on the same Sobolev space as the isotropic. Thus, it is shown that the anisotropic Neumann-Poincaré operator is symmetrizable, satisfies Calderón's identity on $H^{-\frac{1}{2}}(\partial D)$ and a spectral decomposition on an equivalent Hilbert space is stated. The operator $\mathcal{Q}_D^A$ is introduced for bounded open targets $D \Subset \mathbb{R}^3$ of class $\mathcal{C}^{1+\alpha}$, that enables to solve the APT-defining integral equations. Applying Fredholm theory and layer potential techniques this operator is shown to be invertible in three dimension. Moreover, for a targets conductivity matrix A with anisotropy ratio close an isotropic medium there is an asymptotic expansion, which is shown to be invertible by the Neumann series. For the exact operator $\mathcal{Q}_D^A$ and the perturbed operator $\mathcal{Q}_D^A + \delta\mathcal{Q}_D^A$ a translation formula, a rotation formula and a scaling formula is deduced. Applying them consistent exact and asymptotic formulas for the APT follow. A translation formula of the general APT is shown, whereas the first order APT is translation invariant. For the first order polarization tensor the conductivity matrix is similarity transformed by the rotation matrix of the underlying target rotation and scaled by the scaling factors inverse. Also in two dimensions $\mathcal{Q}_D^A$ is translation and scaling invariant and under rotation A is similarity transformed. For target identification in three dimensions a first order APT based identification algorithm is given. An explicit formula for the scaling parameter, a non-linear least squares estimator for the rotation angles and, based on second order APTs, a minimization problem to extract the targets

location are imposed.

7.1 Discussion of Anisotropic Layer Potential Techniques

Because of the non-trivial kernel of the anisotropic singe layer potential in two dimensions (Theorem 4.19), invertibility of $\mathcal{S}_D^A$ could not be stated. In this case showing that $\mathcal{Q}_D^A$ is well-defined and invertible must be supplemented.

7.2 Discussion of Complex Contracted Anisotropic Polarization Tensors

It remains open if the APT is positive and similar results as [9, Theorem 4.11] can be established. As a consequence it would be possible to show positivity of the diagonal elements of the complex CAPT and to deteremine transform invarinat shape descriptors as in [4]. By positivity a scaling invariant quantitie as intoduced in [4] can be derived.

7.3 Outlook

An implementation of a numerical experiment using a dictionary of anisotropic target data from $\mathbb{R}^3$ is reasonable to approve the shape identification procedure by algorithm 1, the scaling transformation parameter formula given by eq. (6.4), the least squares estimators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ and the minimizer $\hat{T}$.

Appendix A

Properties of the Anisotropic Fundamental Solution of the Laplacian

In this appendix we use the Einstein summation convention with subscripts.

Lemma A.1 *Let $\tilde{x} = A_* x \in \mathbb{R}^d, d \geq 2$ and $A_*^2 = A^{-1}$ for $A \in \mathbb{R}^{d\times d}$ a constant symmetric positive-definite matrix, then*

$$\nabla_x = A_* \nabla_{\tilde{x}} \quad \textit{and} \quad \nabla_x \cdot A \nabla_x = \Delta_{\tilde{x}}. \tag{A.1}$$

Proof We have $\tilde{x}_i = A_{*ij} x_j$, therefore $\partial_{x_i} = \frac{\partial \tilde{x}_j}{\partial x_i} \partial_{\tilde{x}_j} = A_{*ij} \partial_{\tilde{x}_j}$, which shows $\nabla_x = A_* \nabla_{\tilde{x}}$.
It follows $A\nabla_x = AA_* \nabla_{\tilde{x}}$. Furthermore,

$$\begin{aligned}(\nabla_x \cdot A \nabla_x) &= \partial_{x_i}(A\nabla_x)_i = \partial_{x_i}(AA_*)_{il}\partial_{\tilde{x}_l} = A_{*i,j}\partial_{\tilde{x}_j}(AA_*)_{il}\partial_{\tilde{x}_l} \\ &= \partial_{\tilde{x}_j}(A_*AA_*)_{jl}\partial_{\tilde{x}_l} = \partial_{\tilde{x}_j} I_{j,l}\partial_{\tilde{x}_l} = \partial_{\tilde{x}_j}\delta_{j,l}\partial_{\tilde{x}_l} = \partial_{\tilde{x}_j}\partial_{\tilde{x}_j} \\ &= \nabla_{\tilde{x}} \cdot \nabla_{\tilde{x}} = \Delta_{\tilde{x}},\end{aligned}$$

where the symmetry of A_* was used at the end of the last line and $A_*^2 = A^{-1}$ in the following. □

Lemma A.2 *For $a \in \mathbb{N}$, $a \geq 3$ and $A_*^2 = A^{-1}$, A a symmetric positive-definite matrix we have for $x, y \in \mathbb{R}^d$, $x \neq y$ and $d \geq 2$,*

i) $\nabla_y \frac{1}{|A_*(x-y)|^{a-2}} = \frac{-a+2}{|A_*(x-y)|^a} A^{-1}(x-y)$,

ii) $\nu_y \cdot A \nabla_y \dfrac{1}{|A_*(x-y)|^{a-2}} = (-a+2)\dfrac{\langle x-y, \nu_y\rangle}{|A_*(x-y)|^a}$,

iii) $\nabla_y |A_*(x-y)|^3 = -3|A_*(x-y)|A^{-1}(x-y)$ *and* *iv)* $\nu_y \cdot A\nabla_y \log(|A_*(x-y)|) = -\dfrac{\langle x-y, \nu_y\rangle}{|A_*(x-y)|^2}.$

This can be proven similarly as Lemma A.1, using transformation rules.

Lemma A.3 *Let $A \in \mathbb{R}^{d\times d}, d \geq 2$ be a symmetric positive-definite matrix, then*

$$\nabla \cdot A\nabla\Gamma^A(x-y) = \delta(x-y), \tag{A.2}$$

where δ denotes the Dirac delta.

Proof Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. We have

$$\int_{\mathbb{R}^d} \nabla_y \cdot A\nabla_y \Gamma^A(x-y)\varphi(y)\, dy = \int_{\mathbb{R}^d} \nabla_y \cdot A\nabla_y \frac{1}{\sqrt{\det(A)}}\Gamma(A_*(x-y))\varphi(y)\, dy.$$

Using the integration by substitution formula with $\tilde{y} = A_* y$, $dy = \frac{1}{\det(A_*)}d\tilde{y}$ and $\det(A_*) = \frac{1}{\sqrt{\det(A)}}$, since $A_*^2 = A^{-1}$ and by Lemma A.1, the above writes as

$$\int_{\mathbb{R}^d} \Delta_{\tilde{y}}\Gamma(\tilde{x}-\tilde{y})\varphi(A_*^{-1}\tilde{y})\, d\tilde{y} = \int_{\mathbb{R}^d} \delta(\tilde{x}-\tilde{y})\varphi(A_*^{-1}\tilde{y})\, d\tilde{y}.$$

By another substitution, $y = A_*^{-1}\tilde{y}$, $d\tilde{y} = \det(A_*)dy$, it follows that

$$\int_{\mathbb{R}^d} \nabla_y \cdot A\nabla_y \Gamma^A(x-y)\varphi(y)\, dy = \det(A_*)\int_{\mathbb{R}^d} \delta(A_*(x-y))\varphi(y)\, dy.$$

Note that, by substitution it follows for $B \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \delta(Bx)\, dx = \int_{\mathbb{R}^d} \frac{1}{\det(B)}\delta(\tilde{x})\, d\tilde{x} = \frac{1}{\det(B)},$$

i.e., $\delta(Bx) = \frac{1}{\det(B)}\delta(x)$. Hence, the above equals to

$$\det(A_*)\int_{\mathbb{R}^d} \delta(A_*(x-y))\varphi(y)\, d\sigma(y) = \int_{\mathbb{R}^d} \delta(x-y)\varphi(y)\, dy = \varphi(x). \quad \square$$

Lemma A.4 *For $d \geq 2$ let $y_0 \in D$ and $y \in \partial D$, then*

$$|\Gamma^A(x-y) - \Gamma^A(x-y_0)| = O(|x|^{1-d}) \quad \text{as } |x| \to \infty. \tag{A.3}$$

Proof The Taylor expansion of $\frac{1}{|A_*(x-y)|}$ is applied in order to show the asymptotic behaviour. It writes as

$$\frac{1}{|A_*(x-y)|} = \sum_{n=0}^{\infty} \frac{1}{n!}(y\cdot\nabla_{\tilde{y}})^n \frac{1}{|A_*(x-\tilde{y})|}\Big|_{\tilde{y}=0}.$$

Let $y \in \partial D$. Since $x \neq y$ for $|x| \to \infty$ from Lemma A.2 ii) we have $y \cdot \nabla_{\tilde{y}} \frac{1}{|A_*(x-\tilde{y})|} = \frac{y \cdot A^{-1}(x-\tilde{y})}{|A_*(x-\tilde{y})|^3}$, therefore $y \cdot \nabla_{\tilde{y}} \frac{1}{|A_*(x-\tilde{y}|}\Big|_{\tilde{y}=0} = \frac{y \cdot A^{-1}x}{|A_*x|^3}$. Moreover, it can be written

$$\begin{aligned}
\partial_{\tilde{y}_j} \frac{y \cdot A^{-1}(x-\tilde{y})}{|A_*(x-y)|^3} &= \partial_{\tilde{y}_j} \frac{y_i A^{-1}_{i,k}(x-\tilde{y})_k}{|A_*(x-\tilde{y})|^3} \\
&= \frac{y_i A^{-1}_{i,k}(-\delta_{k,j})|A_*(x-\tilde{y})|^3 - y_i A^{-1}_{i,k}(y-\tilde{y})_k \partial_{\tilde{y}_j}|A_*(x-\tilde{y})|^3}{|A_*(x-\tilde{y})|^6} \\
&= \frac{-y_i A^{-1}_{i,j}|A_*(x-\tilde{y})|^3 - y_i A^{-1}_{i,k}(y-\tilde{y})_k\big(-3A^{-1}(x-\tilde{y})\big)_j|A_*(x-\tilde{y})|}{|A_*(x-\tilde{y})|^6} \\
&= -\frac{y_i A^{-1}_{i,j}}{|A_*(x-\tilde{y})|^3} + \frac{3y_i A^{-1}_{i,k}(y-\tilde{y})_k A^{-1}(x-\tilde{y})_j}{|A_*(x-\tilde{y})|^5} \\
&= -\frac{(A^{-1}y)_j}{|A_*(x-\tilde{y})|^3} + \frac{3\langle y, A^{-1}(y-\tilde{y}\rangle\big(A^{-1}(x-\tilde{y})\big)_j}{|A_*(x-\tilde{y})|^5},
\end{aligned}$$

where Lemma A.2 iii) was applied from the second to third line. It follows that

$$(y \cdot \nabla_{\tilde{y}})^2 \frac{1}{|A_*(x-\tilde{y})|} = -\frac{\langle y, A^{-1}y\rangle}{|A_*(x-\tilde{y})|^3} + \frac{3\langle y, A^{-1}(y-\tilde{y})\rangle\langle y, A^{-1}(x-\tilde{y})\rangle}{|A_*(x-\tilde{y})|^5}.$$

Therefore, the second term in the Taylor series is

$$(y \cdot \nabla_{\tilde{y}})^2 \frac{1}{|A_*(x-\tilde{y})|}\Big|_{\tilde{y}=0} = -\frac{\langle y, A^{-1}y\rangle}{|A_*(x-\tilde{y})|^3} + \frac{3\langle y, A^{-1}y\rangle\langle y, A^{-1}x\rangle}{|A_*x|^5}.$$

Hence, using the Taylor series it is obtained

$$\frac{1}{|A_*(x-y)|} = \frac{1}{|A_*x|} + \frac{\langle y, A^{-1}x\rangle}{|A_*x|^3} + O(\frac{1}{|A_*x|^3}) \quad \text{as} \quad |x| \to \infty.$$

Inserting above it follows that for $d \geq 3$

$$\begin{aligned}
|\Gamma^A(x-y) - \Gamma^A(x-y_0)| &= C\frac{\langle (y-y_0), A^{-1}x\rangle}{|A_*x|^3} + O(\frac{1}{|A_*x|^3}) \\
&\leq \frac{|A_*(y-y_0)||A_*x|}{|A_*x|^3} + O(\frac{1}{|A_*x|^3}) \\
&= \frac{|A_*(y-y_0)|}{|A_*x|^2} + O(\frac{1}{|A_*x|^3}) = O(\frac{1}{|A_*x|^2}),
\end{aligned}$$

where the Cauchy-Schwarz inequality was used. Recall $|A_*x|$ equals to the norm induced by the inner product $\langle x, y\rangle_{A^{-1}} = \langle x, A^{-1}y\rangle$ for $x, y \in \mathbb{R}^d$. By

the equivalence of norms on finite dimensional vector spaces it is obtained $|\Gamma^A(x-y) - \Gamma^A(x-y_0)| = O(\frac{1}{|x|^2})$ for $|x| \to \infty$.
Now assume $d = 2$, we have for C some positive constant

$$\begin{aligned}|\Gamma^A(x-y) - \Gamma^A(x-y_0)| &= C|\log(|A_*(x-y)| - \log(|A_*(x-y_0)|)| \\ &= C\log(\frac{|A_*(x-y)|}{|A_*(x-y_0)|}) \leq C\big(\frac{|A_*(x-y)|}{|A_*(x-y_0)|} - 1\big).\end{aligned}$$

Analogously to the case as for $d = 3$, by inserting the Taylor expansion of $\frac{1}{|A_*(x-y)|}$ into above estimate the claim can be proven. $\square$

Appendix B

Translation, Rotation and Scaling Properties of Perturbed Anisotropic Polarization Tensors

The proofs from section 5.2.1 are stated in this appendix.

B.1 Translation

Lemma B.1 *Let $T \in \mathbb{R}^3$, $x^T := x + T$, $D^T = \{x + T |\ x \in D\}$, $\varphi^T(x^T) := \varphi(x)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and $\psi^T(x^T) := \psi(x)$ for $\psi \in H^{\frac{1}{2}}(\partial D)$, then*

i) $\mathcal{S}_D^{-1}[\psi](x) = \mathcal{S}_{D^T}^{-1}[\psi^T](x^T)$ *for all $x \in \partial D$ and*

ii) $(\mathcal{K}_D^A)^*[\varphi](x) = (\mathcal{K}_{D^T}^A)^*[\varphi^T](x^T)$ *for all $x \in \partial D$.*

Proof i) Let $\varphi(x) \in H^{-\frac{1}{2}}(\partial D)$ such that $\mathcal{S}_D[\varphi](x) = \psi(x)$. This is equivalent to

$$\begin{aligned}\psi(x) &= \int_{\partial D} \Gamma(x-y)\varphi(y)\, d\sigma(y) \\ &= \int_{\partial D^T} \Gamma(x^T - \tilde{y})\varphi^T(\tilde{y})\, d\sigma(\tilde{y}) = \mathcal{S}_{D^T}[\varphi^T](x^T),\end{aligned}$$

for $\varphi^T(y^T) := \varphi(y)$, where the substitution $\tilde{y} - T = y$ and $d\sigma(y) = d\sigma(\tilde{y})$ was used. Hence by applying the inverse operator we get $\mathcal{S}_D^{-1}[\psi](x) = \varphi(x) = \varphi^T(x^T) = \mathcal{S}_{D^T}^{-1}[\psi^T](x^T)$.

ii) For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ it can be written by the same substitution as above

$$\begin{aligned}(\mathcal{K}_D^A)^*[\varphi](x) &= \int_{\partial D} \frac{\langle y-x, \nu_x\rangle}{4\pi\sqrt{det(A)}|A_*(x-y)|^3}\varphi(y)\, d\sigma(y)\\ &= \int_{\partial D^T} \frac{\langle \tilde{y}-x^T, \nu_{x^T}\rangle}{4\pi\sqrt{det(A)}|A_*(x^T-\tilde{y})|^3}\varphi^T(\tilde{y})\, d\sigma(\tilde{y})\\ &= (\mathcal{K}_{D^T}^A)^*[\varphi^T](x^T).\end{aligned}$$ □

B.2 Rotation

Lemma B.2 *Let $D_{\alpha\beta\gamma} := \{B_{\alpha\beta\gamma}y \mid y \in D\}$ and $\varphi^{\alpha\beta\gamma}(y^{\alpha\beta\gamma}) := \varphi(y)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and $y \in \partial D$, then we have*

i) $\mathcal{Q}_{D_{\alpha\beta\gamma},0}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = \mathcal{Q}_{D,0}[\varphi](x)$ *and*

ii) $\mathcal{Q}^R_{D_{\alpha\beta\gamma},1}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = \mathcal{Q}_{D,1}^{-\alpha\beta\gamma R\alpha\beta\gamma}[\varphi](x)$.

Proof This proof is analogously build up as the proof of Lemma **??**. Let φ, $\varphi^{\alpha\beta\gamma}$, $D_{\alpha\beta\gamma}$, $x^{\alpha\beta\gamma}$ for $x \in \partial D$ be given as above.

i) We have $\mathcal{K}^*_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = \mathcal{K}^*_D[\varphi](x)$. A proof for the case $d = 2$ can be found in [6, chp. 4.2.2]. The three-dimensional case can be shown analogously.

ii) Recall that the definition of $\mathcal{Q}^R_D$ on a rotated domain is given by

$$\mathcal{Q}^R_{D_{\alpha\beta\gamma},1}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = \varepsilon_c\big((\frac{1}{2}I - \mathcal{K}^*_{D_{\alpha\beta\gamma}})\mathcal{B}^R_{D_{\alpha\beta\gamma},1}\mathcal{S}_{D_{\alpha\beta\gamma}} - (\mathcal{K}^R_{D_{\alpha\beta\gamma},1})^*\big)[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}).$$

Therefore first it is shown that

$$(\mathcal{K}^R_{D_{\alpha\beta\gamma},1})^*[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = (\mathcal{K}_{D,1}^{-\alpha\beta\gamma R_{\alpha\beta\gamma}})^*[\varphi](x) \quad \text{for} \quad {}_{-\alpha\beta\gamma}R_{\alpha\beta\gamma} = B_{-\alpha\beta\gamma}RB_{\alpha\beta\gamma}.$$

Since, due to $\mathcal{K}^R_{D,1}$ on a rotated domain writes as

$$\begin{aligned}(\mathcal{K}^R_{D_{\alpha\beta\gamma},1})^*[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = &-\frac{1}{2}Tr(R)\mathcal{K}^*_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})\\ &-\frac{3}{2}\int_{\partial D_{\alpha\beta\gamma}} \frac{\langle R(x^{\alpha\beta\gamma}-\tilde{y}), x^{\alpha\beta\gamma}-\tilde{y}\rangle\langle x^{\alpha\beta\gamma}-\tilde{y}, \nu_{x^{\alpha\beta\gamma}}\rangle}{4\pi|x^{\alpha\beta\gamma}-\tilde{y}|^5}\\ &\quad \varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y}),\end{aligned}$$

just the integral term is calculated. Applying the substitution of variables formula $\tilde{y} = B_{\alpha\beta\gamma}y$ and $d\sigma(\tilde{y}) = det(B_{\alpha\beta\gamma})d\sigma(y) = d\sigma(y)$, it is obtained this equals to

$$\int_{\partial D} \frac{\langle R(x^{\alpha\beta\gamma} - B_{\alpha\beta\gamma}y), x^{\alpha\beta\gamma} - B_{\alpha\beta\gamma}y\rangle\langle x^{\alpha\beta\gamma} - B_{\alpha\beta\gamma}y, \nu_{x^{\alpha\beta\gamma}}\rangle}{4\pi|x^{\alpha\beta\gamma} - B_{\alpha\beta\gamma}y|^5}\varphi^{\alpha\beta\gamma}(B_{\alpha\beta\gamma}y)\, d\sigma(y)$$
$$= \int_{\partial D} \frac{\langle RB_{\alpha\beta\gamma}(x-y), B_{\alpha\beta\gamma}(x-y)\rangle\langle B_{\alpha\beta\gamma}(x-y), B_{\alpha\beta\gamma}\nu_x\rangle}{4\pi|B_{\alpha\beta\gamma}(x-y)|^5}\varphi(y)\, d\sigma(y).$$

We have $\langle B_{\alpha\beta\gamma}B_{-\alpha\beta\gamma}RB_{\alpha\beta\gamma}z, B_{\alpha\beta\gamma}z\rangle = \langle B_{-\alpha\beta\gamma}RB_{\alpha\beta\gamma}z, z\rangle$ and $|B_{\alpha\beta\gamma}z| = |z|$ for $z \in \mathbb{R}^3$, because $B_{\alpha\beta\gamma} \in SO(3)$. It follows the above can be written as

$$\int_{\partial D} \frac{\langle B_{-\alpha\beta\gamma}RB_{\alpha\beta\gamma}(x-y), x-y\rangle\langle x-y, \nu_x\rangle}{4\pi|x-y|^5}\varphi(y)\, d\sigma(y),$$

which shows $(\mathcal{K}^R_{D_{\alpha\beta\gamma},1})^*[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = (\mathcal{K}^{-\alpha\beta\gamma R_{\alpha\beta\gamma}}_{D,1})^*[\varphi](x)$.

Next, we show that the single layer potential is rotation invariant. Note that the three-dimensional fundamental solution of the Laplacian satisfies

$$\Gamma(B_{\alpha\beta\gamma}(x-y)) = -\frac{1}{4\pi|B_{\alpha\beta\gamma}(x-y)|} = -\frac{1}{4\pi|x-y|} = \Gamma(x-y).$$

Hence, applying the substitution of variables formula as above the single layer potential satisfies

$$\begin{aligned}
\mathcal{S}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) &= \int_{D_{\alpha\beta\gamma}} \Gamma(x^{\alpha\beta\gamma} - \tilde{y})\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y}) \\
&= \int_{\partial D} \Gamma(x^{\alpha\beta\gamma} - B_{\alpha\beta\gamma}y)\varphi^{\alpha\beta\gamma}(B_{\alpha\beta\gamma}y) d\sigma(y) \\
&= \int_{\partial D} \Gamma(B^{\alpha\beta\gamma}(x-y))\varphi(y)\, d\sigma(y) \\
&= \int_{\partial D} \Gamma(x-y)\varphi(y)\, d\sigma(y) = \mathcal{S}_D[\varphi](x).
\end{aligned}$$

And by using equation (4.30d) and the same calculation it can be shown that

$$\begin{aligned}
\mathcal{S}^R_{D_{\alpha\beta\gamma},1}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) &= -\frac{1}{2}Tr(R)\mathcal{S}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) \\
&\quad - \frac{1}{2}\int_{\partial D_{\alpha\beta\gamma}} \frac{\langle R(x^{\alpha\beta\gamma} - \tilde{y}), x^{\alpha\beta\gamma} - \tilde{y}\rangle}{4\pi|x^{\alpha\beta\gamma} - \tilde{y}|^3}\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y}) \\
&= -\frac{1}{2}Tr(R)\mathcal{S}_D[\varphi](x) \\
&\quad - \frac{1}{2}\int_{\partial D} \frac{\langle B_{\alpha\beta\gamma}B_{-\alpha\beta\gamma}RB_{\alpha\beta\gamma}(x-y), B_{\alpha\beta\gamma}(x-y)\rangle}{4\pi|B_{\alpha\beta\gamma}(x-y)|^3} \\
&\qquad \varphi^{\alpha\beta\gamma}(y)\, d\sigma(y) \\
&= \mathcal{S}^{-\alpha\beta\gamma R\alpha\beta\gamma}_{D,1}[\varphi](x),
\end{aligned}$$

where ${}_{-\alpha\beta\gamma}R_{\alpha\beta\gamma} := B_{-\alpha\beta\gamma}RB_{\alpha\beta\gamma}$.
Let $\varphi(x) \in H^{-\frac{1}{2}}(\partial D)$ such that $\mathcal{S}_D[\varphi](x) = \psi(x)$. This is equivalent to

$$\begin{aligned}\psi(x) &= \int_{\partial D} \Gamma(x-y)\varphi(y)\, d\sigma(y)\\ &= \int_{\partial D_{\alpha\beta\gamma}} \Gamma(x^{\alpha\beta\gamma} - \tilde{y})\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y}) = \mathcal{S}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})\end{aligned}$$

for $\varphi^{\alpha\beta\gamma}(y^{\alpha\beta\gamma}) := \varphi(y)$, where the substitution $\tilde{y} = B_{\alpha\beta\gamma}y$ and $d\sigma(y) = d\sigma(\tilde{y})$ was used. Hence $\psi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) = \mathcal{S}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$ and by invertibility of $\mathcal{S}_D$ we have $\mathcal{S}^{-1}_{D_{\alpha\beta\gamma}}[\psi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = \varphi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) = \varphi(x) = \mathcal{S}_D^{-1}[\psi](x)$.
Furthermore, it follows $\mathcal{B}^R_{D_{\alpha\beta\gamma},1}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}) = -\mathcal{S}_D^{-1}\mathcal{S}_{D,1}^{{}_{-\alpha\beta\gamma}R_{\alpha\beta\gamma}}\mathcal{S}_D^{-1}[\varphi](x) = \mathcal{B}_{D,1}^{{}_{-\alpha\beta\gamma}R_{\alpha\beta\gamma}}[\varphi](x)$, which shows the claim. □

Lemma B.3 *Let $B_{\alpha\beta\gamma} \in SO(3)$, $x^{\alpha\beta\gamma} = B_{\alpha\beta\gamma}x$, $D_{\alpha\beta\gamma} = \{B_{\alpha\beta\gamma}y \mid y \in D\}$, $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $\varphi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) := \varphi(x)$ and $\psi \in H^{\frac{1}{2}}(\partial D)$, $\psi^{\alpha\beta\gamma}(x^{\alpha\beta\gamma}) := \psi(x)$, then $\mathcal{S}_D[\varphi](x) = \mathcal{S}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$, $\mathcal{S}_D^{-1}[\varphi](x) = \mathcal{S}^{-1}_{D_{\alpha\beta\gamma}}[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$ and $(\mathcal{K}_D^A)^*[\varphi](x) = (\mathcal{K}_{D_{\alpha\beta\gamma}}^{{}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma}})^*[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma})$ for all $x \in \partial D$, where ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma} = B_{\alpha\beta\gamma}AB_{-\alpha\beta\gamma}$.*

Proof It suffice to show the last claim.
For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ it can be written by the substitution $\tilde{y} = B_{\alpha\beta\gamma}y$ and $d\sigma(\tilde{y}) = d\sigma(y)$ that

$$\begin{aligned}(\mathcal{K}_D^A)^*[\varphi](x) &= \int_{\partial D} \frac{\langle y-x, \nu_x\rangle}{4\pi\sqrt{det(A)}|A_*(x-y)|^3}\varphi(y)\, d\sigma(y)\\ &= \int_{\partial D_{\alpha\beta\gamma}} \frac{\langle \tilde{y} - x^{\alpha\beta\gamma}, \nu_{x^{\alpha\beta\gamma}}\rangle}{4\pi\sqrt{det(A)}|A_*B_{-\alpha\beta\gamma}(x^{\alpha\beta\gamma} - \tilde{y})|^3}\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y})\\ &= \int_{\partial D_{\alpha\beta\gamma}} \frac{\langle \tilde{y} - x^{\alpha\beta\gamma}, \nu_{x^{\alpha\beta\gamma}}\rangle}{4\pi\sqrt{det(A)}|B_{\alpha\beta\gamma}A_*B_{-\alpha\beta\gamma}(x^{\alpha\beta\gamma} - \tilde{y})|^3}\varphi^{\alpha\beta\gamma}(\tilde{y})\, d\sigma(\tilde{y})\\ &= (\mathcal{K}_{D_{\alpha\beta\gamma}}^{{}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma}})^*[\varphi^{\alpha\beta\gamma}](x^{\alpha\beta\gamma}),\end{aligned}$$

where it was used that $({}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma})^{-1} = (B_{\alpha\beta\gamma}A_*B_{-\alpha\beta\gamma})^2$, $\det({}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma}) = \det(A)$ and ${}_{\alpha\beta\gamma}A_{-\alpha\beta\gamma}$ is symmetric, because $A_*^2 = A^{-1}$ and $B_{\alpha\beta\gamma} \in SO(3)$. □

B.3 Scaling

Lemma B.4 *Let $s \in \mathbb{R}_{>0}$, define $sD := \{sy \mid y \in D\}$, $y^s = sy$ and $\varphi^s(y^s) := \varphi(y)$ for $\varphi \in H^{-\frac{1}{2}}(\partial D)$, $y \in \partial D$ then*

i) $\mathcal{Q}_{sD,0}[\varphi^s](x^s) = \mathcal{Q}_{D,0}[\varphi](x)$ *for all* $x \in \partial D$ *and* *ii)* $\mathcal{Q}^R_{sD,1}[\varphi^s](x^s) = \mathcal{Q}^R_{D,1}[\varphi](x)$ *for all* $x \in \partial D$.

Proof This proof is analogously structured as the proof of Lemma 5.1.
Let φ, φ^s, x^s for $x \in \partial D$ and sD be given as above.

i) By the definition of the operator $\mathcal{Q}_{sD,0}$, eq. (4.31b), we have

$$\mathcal{Q}_{sD,0}[\varphi^s](x^s) = \frac{\varepsilon_m + \varepsilon_c}{2} I[\varphi^s](x^s) + (\varepsilon_m - \varepsilon_c)\mathcal{K}^*_{sD}[\varphi^s](x^s).$$

Note that $\mathcal{K}^*_D$ is translation invariant, i.e., $\mathcal{K}^*_{sD}[\varphi^s](x^s) = \mathcal{K}^*_D[\varphi](x)$. The proof for the two-dimensional case can be found in [6, chp. 4.2.3]. The three-dimensional case can be shown analogously. Furthermore, the identity is scaling invariant, which shows the claim.

ii) Due to eq. (4.31c) the operator on a scaled domain $\mathcal{Q}^R_{sD,1}$ is given by

$$\mathcal{Q}^R_{sD,1}[\varphi^s](x^s) = \varepsilon_c\big((\frac{1}{2}I - \mathcal{K}^*_{sD})\mathcal{B}^R_{sD,1}\mathcal{S}_{sD} - (\mathcal{K}^R_{sD,1})^*\big)[\varphi^s](x^s).$$

The single operators are acting under scaling like in the following:

$$\begin{aligned}(\mathcal{K}^R_{sD,1})^*[\varphi^s](x^s) = &-\frac{1}{2}Tr(R)\mathcal{K}^*_{sD}[\varphi^s](x^s)\\ &-\frac{3}{2}\int_{\partial sD}\frac{\langle R(x^s-\tilde{y}), x^s-\tilde{y}\rangle\langle x^s-\tilde{y}, \nu_{x^s}\rangle}{4\pi|x^s-\tilde{y}|^5}\varphi^s(\tilde{y})\, d\sigma(\tilde{y}).\end{aligned}$$

Using the substitution of variables formula $\tilde{y} = sy$ and $d\sigma(\tilde{y}) = det(sI)$ $|(1/sI)\nu(y)|d\sigma(y)$ the above integral can be written as

$$\begin{aligned}&\int_{\partial D} s^2\frac{\langle R(x^s-sy), x^s-sy\rangle\langle x^s-sy, \nu_x\rangle}{4\pi|x^s-y^s|^5}\varphi^s(sy)\, d\sigma(y)\\ &= \int_{\partial D} s^2\frac{s^2\langle R(x-y), x-y\rangle s\langle x-y, \nu_x\rangle}{4\pi s^5|x-y|^5}\varphi(y)\, d\sigma(y)\\ &= \int_{\partial D}\frac{\langle R(x-y), x-y\rangle\langle x-y, \nu_x\rangle}{4\pi|x-y|^5}\varphi(y)\, d\sigma(y).\end{aligned}$$

Hence, $(\mathcal{K}^R_{sD,1})^*[\varphi^s](x^s) = (\mathcal{K}^R_{D,1})^*[\varphi](x)$.
Note that $\Gamma(sx) = \frac{1}{s}\Gamma(x)$, which follows directly form its definition. Therefore it can be written, using the substitution $\tilde{y} = sy$ and $d\sigma(\tilde{y}) = \det(sI)|(1/sI)\nu_y|\, d\sigma(y)$,

$$\mathcal{S}_{sD}[\varphi^s](x^s) = \int_{\partial sD}\Gamma(x^s-\tilde{y})\varphi^s(\tilde{y})\, d\sigma(\tilde{y}) = s\mathcal{S}_D[\varphi](x).$$

As a direct consequence it follows for any $\psi \in H^{\frac{1}{2}}(\partial D)$ that

$$\mathcal{S}^{-1}_{sD}[\psi^s](x^s) = \frac{1}{s}\mathcal{S}^{-1}_D[\psi](x).$$

This can be proven in the same manner as Lemma 5.2, i) with some minor changes. Furthermore, by eq. (4.30d) and repeating the same calculation as above it can be written

$$\mathcal{S}^R_{sD,1}[\varphi^s](x^s) = -\frac{1}{2}Tr(R)\mathcal{S}_{sD}[\varphi^s](x^s) - \frac{1}{2}\int_{\partial sD} \frac{\langle R(x^s-\tilde{y}), x^s-\tilde{y}\rangle}{4\pi|x^s-\tilde{y}|^3}\varphi^s(\tilde{y})d\sigma(\tilde{y}).$$

By the substitution of variables formula as above the integral in above term writes as

$$\begin{aligned}&\int_{\partial D} s^2 \frac{\langle R(x^s-sy), x^s-sy\rangle}{4\pi|x^s-sy|^3}\varphi^s(sy)\ d\sigma(y)\\ &= \int_{\partial D} s^2 \frac{s^2\langle R(x-y), x-y\rangle}{4\pi s^3|x-y|^3}\varphi(y)\ d\sigma(y)\\ &= s\int_{\partial D} \frac{\langle R(x-y), x-y\rangle}{4\pi|x-y|^3}\varphi(y)\ d\sigma(y).\end{aligned}$$

It follows $\mathcal{S}^R_{sD,1}[\varphi^s](x^s) = s\mathcal{S}^R_{D,1}[\varphi](x)$.
Moreover, $\mathcal{B}^R_{D,1}\mathcal{S}_D$ is scaling invariant, because $\mathcal{B}^R_{sD,1}\mathcal{S}_{sD}[\varphi^s](x^s)$ $= -\mathcal{S}^{-1}_{sD}\mathcal{S}^R_{sD,1}[\varphi^s](x^s) = -\mathcal{S}^{-1}_{D}\mathcal{S}^R_{D,1}[\varphi](x) = \mathcal{B}^R_{D,1}\mathcal{S}_D[\varphi](x)$. □

Lemma B.5 *Let $s \in \mathbb{R}_{>0}$, $x^s = sx$, $sD = \{sy|\ y \in D\}$, $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and $\varphi^s(x^s) := \varphi(x)$ then $\mathcal{S}_D[\varphi](x) = \frac{1}{s}\mathcal{S}_{sD}[\varphi^s](x^s)$, $\mathcal{S}^{-1}_D[\varphi](x) = s\mathcal{S}^{-1}_{sD}[\varphi^s](x^s)$ and $(\mathcal{K}^A_D)^*[\varphi](x) = (\mathcal{K}^A_{sD})^*[\varphi^s](x^s)$ for all $x \in \partial D$.*

Proof It suffice to show the last claim. For $\varphi \in H^{-\frac{1}{2}}(\partial D)$ it can be written by the same substitution $\tilde{y} = sy$, $d\sigma(y) = det(1/sI)|s\nu(y)|\ d\sigma(y)$ that

$$\begin{aligned}(\mathcal{K}^A_D)^*[\varphi](x) &= \int_{\partial D} \frac{\langle y-x, \nu_x\rangle}{4\pi\sqrt{det(A)}|A_*(x-y)|^3}\varphi(y)\ d\sigma(y)\\ &= \int_{\partial sD} \frac{1}{s^2}\frac{\frac{1}{s}\langle \tilde{y}-x^s, \nu_{x^s}\rangle}{\frac{1}{s^3}4\pi\sqrt{det(A)}|A_*(x^s-\tilde{y})|^3}\varphi^s(\tilde{y})\ d\sigma(\tilde{y})\\ &= (\mathcal{K}^A_{sD})^*[\varphi^s](x^s).\end{aligned}$$

□

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