COM1009
Introduction to Algorithms and Data Structures

Lecture #9

▶AVL Trees: a class of self-balancing trees

Dr Dirk Sudholt
Department of Computer Science
University of Sheffield

d.sudholt@sheffield.ac.uk

https://staffwww.dcs.shef.ac.uk/people/D.Sudholt/campus_only/com1009.htm

Reading: Lecture notes on MOLE

Aims of this lecture

- To see a class of **self-balancing trees** guaranteeing operations in time $O(\log n)$.
- To show that the depth of AVL trees is $O(\log n)$.
- To show how to perform insertions and deletions, **rebalancing the tree** through **rotations** whenever it becomes unbalanced.

Self-balancing trees

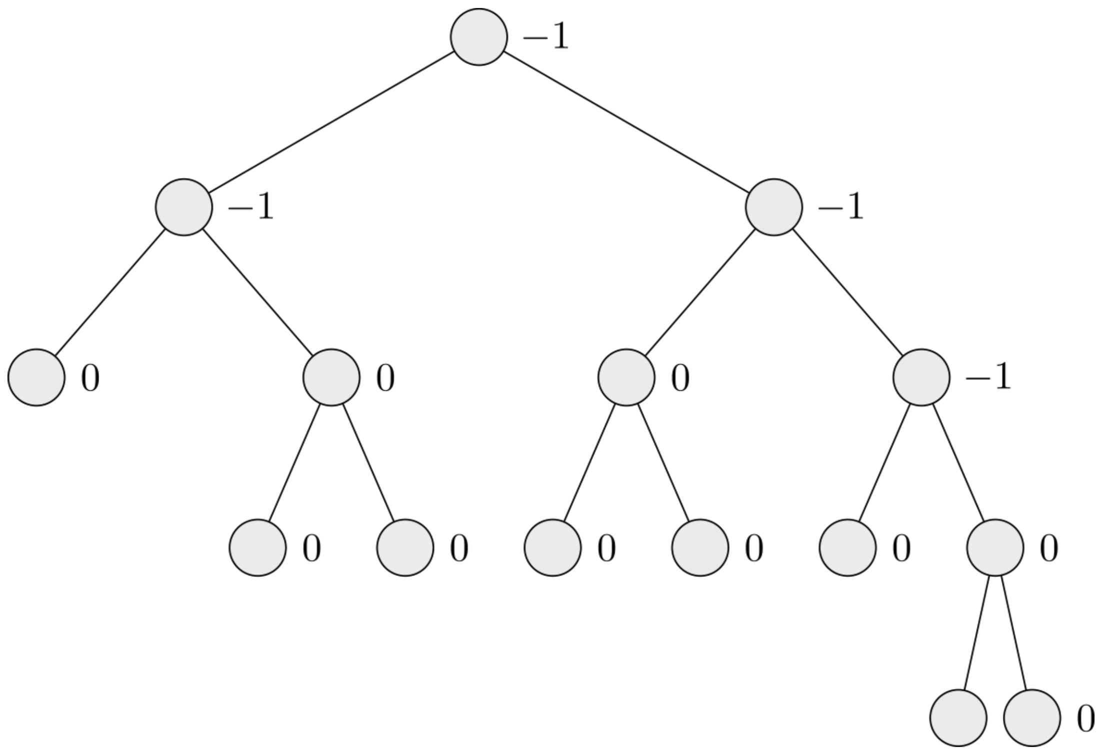
- There are various types of binary search trees that are guaranteed to have depth $O(\log n)$.
  - AVL Trees
  - 2-3 Trees
  - B-Trees
  - Red-black Trees
  - Splay Trees
  - Van Emde Boas Trees
  - ...

AVL Trees

- Invented by and named after Adelson-Velskii and Landis.
- **Invariant**: all nodes are **locally balanced**.
- A binary tree is called AVL tree if for every node the following holds: the height of the left subtree and the height of the right subtree only differ by at most 1.
- Let *v* be a node and T_l, T_r be its left and right subtrees, respectively. Then $bal(v) \coloneqq h(T_l) - h(T_r)$ is the **balance factor** of *v*, $h(\,)$ denoting the height of a tree.
- In an AVL tree hence for every node *v* we have $bal(v) \in \{-1, 0, +1\}$.

▶Balance properties

- The local property does **not** mean that all leaves are on two levels. AVL trees can be lopsided, see this example:



- However, overall the tree is still pretty balanced.

▶Estimating the depth of an AVL tree

Theorem: the height of an AVL tree with n nodes is at most

$$h \leq \frac{1}{\log((\sqrt{5}+1)/2)} \log n \approx 1.44 \log n.$$

Proof outline:

- Consider the minimum number of nodes in any AVL tree of height h and call it $A(h)$.
  - This means that any AVL tree of height h will have $n \geq A(h)$ nodes.
- Show that $A(h)$ (and thus n) is exponentially large in h.
  - Will show that $A(h)$ is similar to Fibonacci numbers.
- Take logarithms (+maths) to get the claimed bound.

Minimum number of nodes in an AVL tree

- Let $A(h)$ be the minimum number of nodes in any AVL tree of height h.
  - An AVL tree with height 0 consists of the root only, hence $A(0) = 1$.
  - The smallest AVL tree of height 1 has two nodes, hence $A(1) = 2$.
  - An AVL tree of height h has to have a root with one subtree of height $h - 1$, and the other subtree of height at least $h - 2$. Hence $A(h) = 1 + A(h-1) + A(h-2)$.
- This is similar to the Fibonacci numbers (bar the "$1 +$"):
  - $Fib(0) = Fib(1) = 1$ and
  - $Fib(h) = Fib(h-1) + Fib(h-2)$.
  - Handy closed form: $$\mathrm{Fib}(k) \geq \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{k+1} - 1\right]$$

Link to Fibonacci numbers

- We prove by induction that $A(h) = Fib(h+2) - 1$.
- Base case: $A(0) = 1 = 2 - 1 = Fib(2) - 1$
  and $A(1) = 2 = 3 - 1 = Fib(3) - 1$.
- Assume that the claim holds for $\mathrm{A}(h-1)$ and $\mathrm{A}(h-2)$, then

$$\begin{aligned} A(h) &= 1 + A(h-1) + A(h-2) && \text{(by recurrence)} \\ &= 1 + \mathrm{Fib}(h+1) - 1 + \mathrm{Fib}(h) - 1 && \text{(2x induction hypothesis)} \\ &= \mathrm{Fib}(h+1) + \mathrm{Fib}(h) - 1 \\ &= \mathrm{Fib}(h+2) - 1 && \text{(by definition of } \mathrm{Fib}(h+2)\text{).} \end{aligned}$$

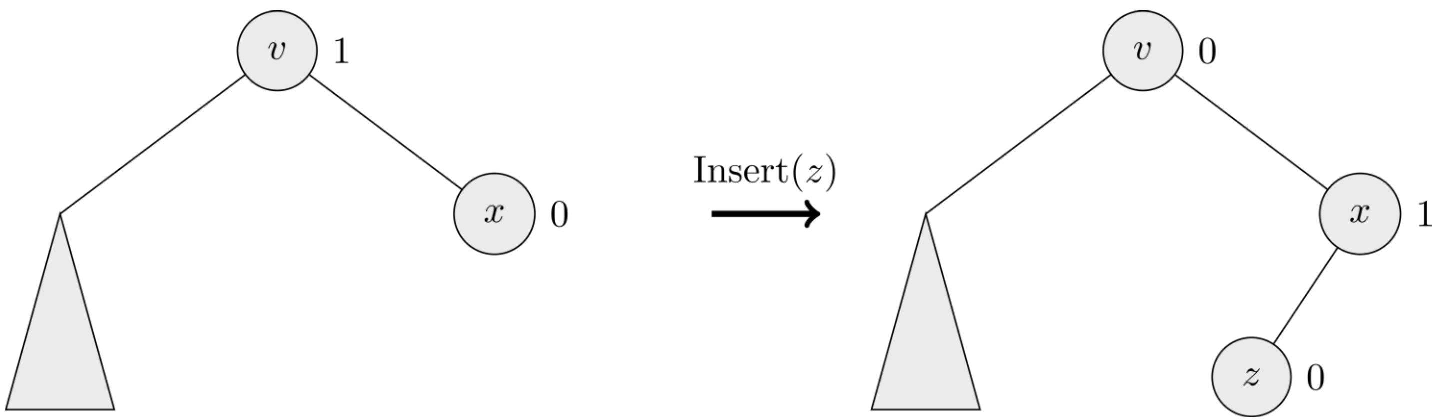
- Every AVL tree with n nodes and height h has
  $n \geq A(h) \geq Fib(h+2) - 1$.
- Plugging in closed form for Fib gives $\left(\frac{\sqrt{5}+1}{2}\right)^{h+3} \leq \sqrt{5}n + 1$
- Taking logarithm of base $\frac{\sqrt{5}+1}{2}$: $h + 3 \leq \log_{(\sqrt{5}+1)/2}(\sqrt{5}n + 1)$
  Converting to $\log_2$ completes proof. $\Rightarrow h \leq \log_{(\sqrt{5}+1)/2}(n)$

Search in an AVL Tree

- Works like in an ordinary binary search tree.

Inserting in an AVL Tree

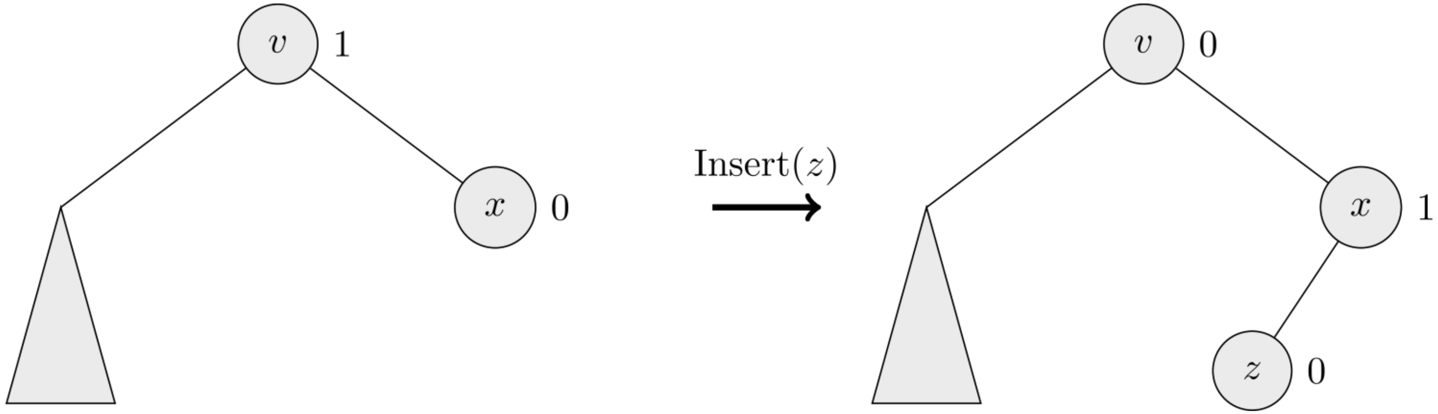
- Works like in an ordinary binary search tree.
- But the tree may become unbalanced, hence we need to **rebalance**. (We focus on ideas here; writing code is easy.)
- We record the **search path** to new element, and then **work back up the search path** to rebalance so long as the height of the current subtree has increased.
- Let *v* be the current node and its right child *x* be on the search path (left child is symmetric)



▶Insert (1)

Case 1: $bal(v) = 1$.

- Left subtree of *v* was higher than right subtree before insertion.
- After inserting *z*, the right subtree has increased its height, hence the subtree at *v* is now balanced.
- The height of *v* has not changed, hence rebalancing is done.



Insert (2)

Case 2: $bal(v) = 0$.

- Both subtrees of v were balanced before insertion.
- After inserting *z*, the right subtree has increased its height, hence now $bal(v) = -1$.
- The height of the subtree at *v* has increased, hence we need to continue rebalancing at v's parent to check for imbalances further up the tree.
- If *v* was the root, we stop.

Insert (3)

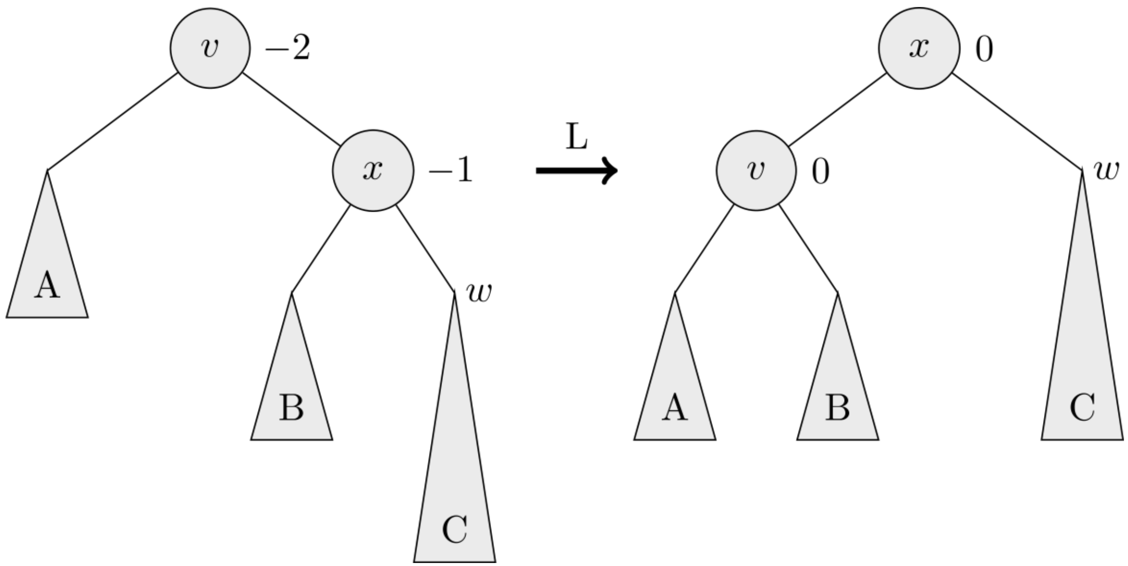
Case 3: $bal(v) = -1$.

- After insertion, the tree has become unbalanced: $bal(v) = -2$.
- Search path contains nodes *v, x, w* whose subtrees increased in height.
- We distinguish two sub-cases, depending on whether *w* is the right child or the left child of x.

▶Insert (4)

Sub-case 3-1: w is the right child of x.

- The tree is lopsided because of an “outside” problem.
- Now **rotate** the tree to the left: x becomes the parent of *v*, and x's left subtree B becomes a subtree of v.

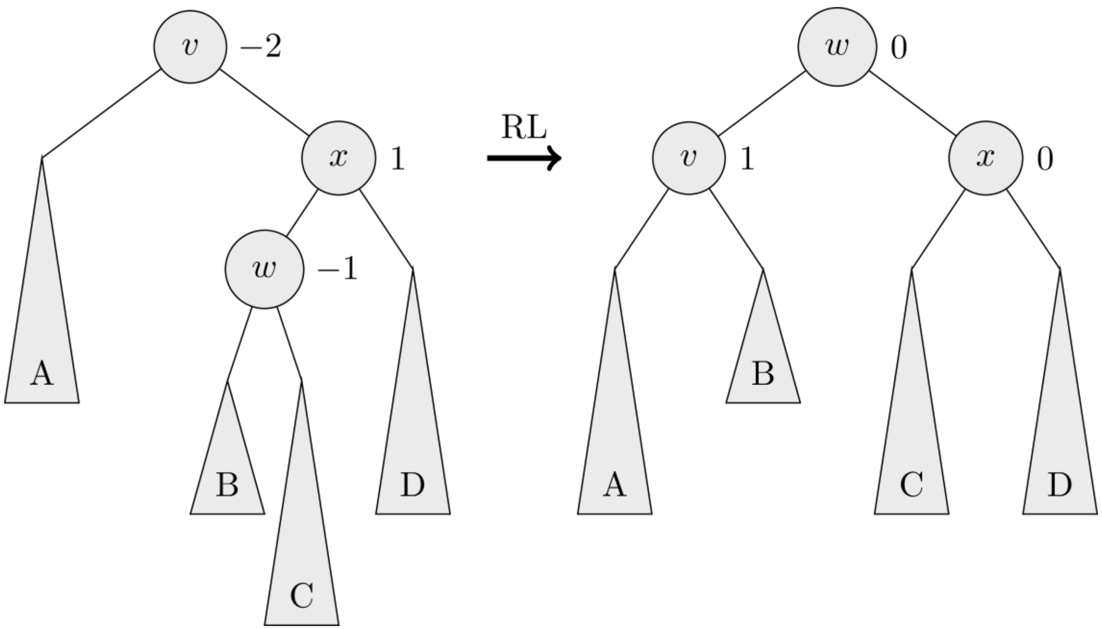


- Height of whole subtree is the same as before insert. Done.

▶Insert (5)

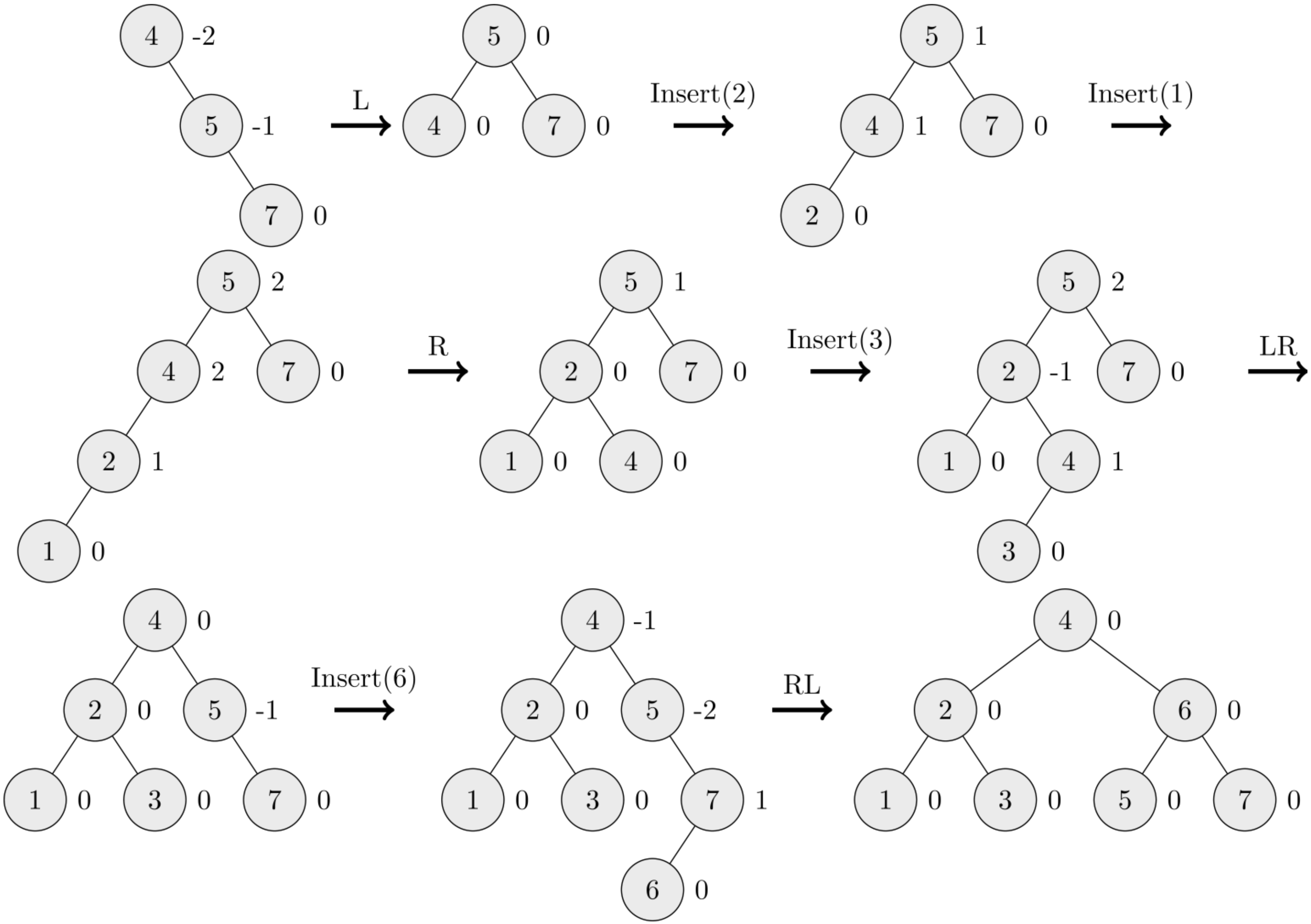
Sub-case 3-2: w is the left child of x.

- The tree is lopsided because of an “inside” problem.
- Now need a **double rotation** to rebalance the tree: a right rotation at x, followed by an immediate left rotation at v.



NB: heights of B and C could be the other way round.

Insert: Example



Runtime of Insert

- Inserting an element takes time $O(h)$.
- Rebalancing:
  - Insert finishes with the first rotation/double rotation.
  - All rotations (L/R/LR/RL) take time $O(1)$.
  - Backing up the search path takes time $O(1)$ for each node on the search path, hence time $O(h)$ overall.
  - This includes the time to update balance factors.
- Total runtime of Insert: $O(h) = O(\log n)$.

▶Deleting in an AVL Tree

- Like for Insert, we work backwards up the search path to rebalance so long as the height of the current subtree has decreased.
- Assume without loss of generality that delete decreased the height of the *left* subtree.
- **Case 1**: $bal(v) = 1$. Here deletion decreased the height of the higher subtree, leading to $bal(v) = 0$.
  However, the height of *v* has decreased, so we need to iterate the rebalance procedure with *v*'s parent.
- **Case 2**: $bal(v) = 0$. Then we update $bal(v) = -1$ and note that the height of *v*'s subtree has not decreased, so the rebalancing is complete.

Delete (2)

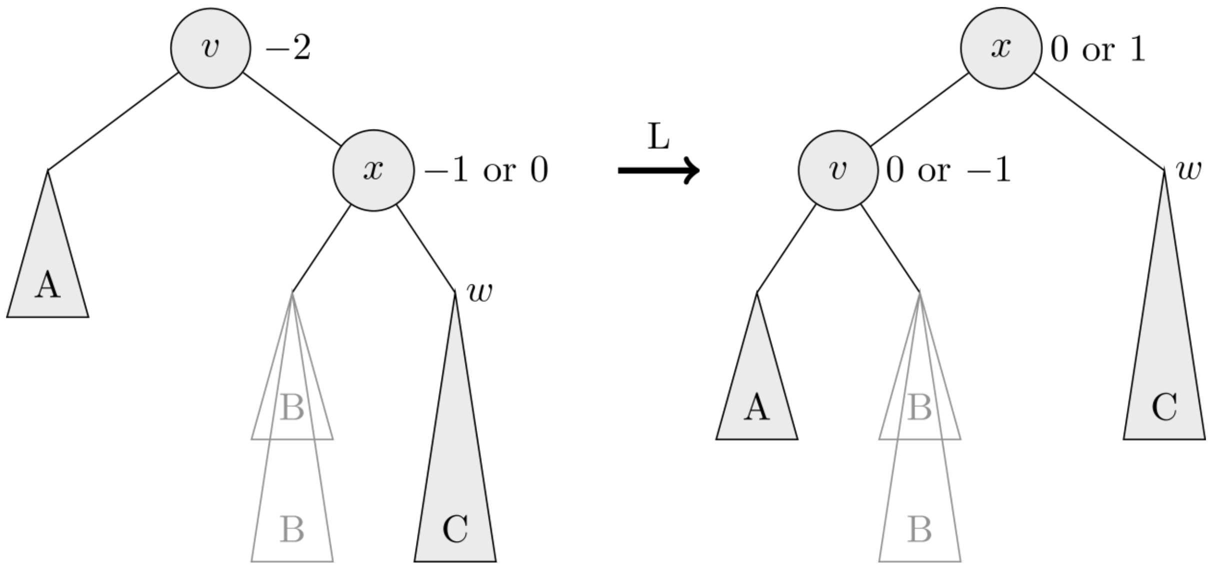
Case 3: $bal(v) = -1$.

- After deletion, the shallower subtree has become even more shallow: $bal(v) = -2$.
- Consider path of nodes *v, x, w* whose subtrees are now too high.
- We distinguish two sub-cases, depending on whether *w* is the right child or the left child of *x*.

▶Delete (3)

Sub-case 3-1: $\mathrm{bal}(x) \in \{-1, 0\}$.

- The tree is lopsided because of an "outside" problem.
- Now **rotate** the tree to the left.
- Two possibilities for the height of B.

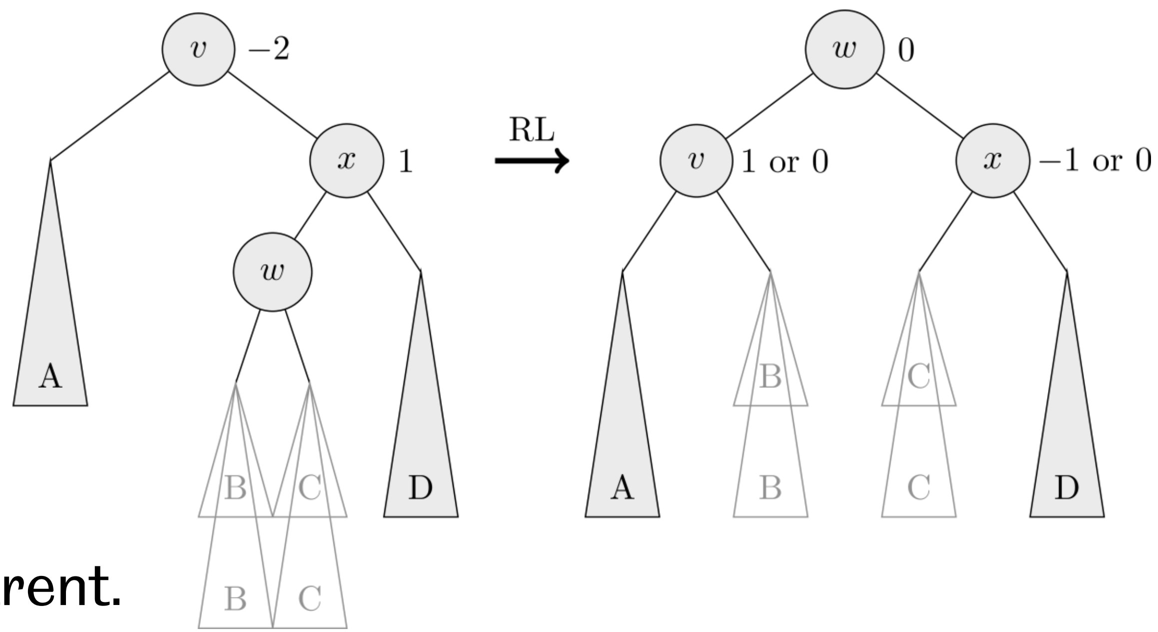


- If B was high, we're done. Otherwise, iterated at x's parent.

Delete (4)

Sub-case 3-2: $bal(x) = 1$.

- The tree is lopsided because of an "inside" problem.
- Again need a **double rotation.**
- B and C can have one of two heights; one must be high.



- Continue at parent.

Runtime of Delete

- Delete may not finish with the first rotation/double rotation.
- Still, the time spent at each node on the search path is $O(1)$, so we still get a time of $O(h) = O(\log n)$.

Summary

- AVL trees with n elements have height $O(\log n)$.
- AVL trees with n nodes execute the following operations in time $O(\log n)$
  - **Searching, Minimum, Maximum, Successor**
    - Follows since AVL trees are binary search trees whose height is always $h = O(\log n)$.
  - **Insertion**
  - **Deletion**
- Greater efficiency from a simple idea: rotating nodes.