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$$\text{Entropy} = -\sum p(r) \log(P(r))$$

① Uniform.

Let r be a ~~rand~~ uniform distribution in $[0, n]$. Then $p(r)\ \forall r \in [0,n]$

$$= \frac{1}{n}$$

$$\Rightarrow \text{Entropy} = -\sum_n \frac{1}{n} \log\left(\frac{1}{n}\right)$$

$$\boxed{\Rightarrow \text{Entropy} = \log n}$$

② Gaussian

$p(x)$ can be given by

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\Rightarrow \text{Entropy} = -\int_{-\infty}^{\infty} p(x) \ln p(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} \times \ln\left(\frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}\right) dx.$$

$$\Rightarrow \text{Entropy} = \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma} \times \log\left(\frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left[\ln\left(\sqrt{2\pi}\,\sigma\right) + \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

Let $\frac{x-\mu}{\sqrt{2}\sigma} = t$

$$\boxed{\Rightarrow dx = \sqrt{2}\sigma\, dt}$$

$$\Rightarrow \text{Entropy} = \frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{\infty} e^{-t^2} \ln\left(\sqrt{2\pi}\,\sigma\right) dt + \int_{-\infty}^{\infty} t^2 e^{-t^2}\, dt\right]$$

$$\boxed{\Rightarrow \text{Entropy} = \frac{\sqrt{\pi}}{2}}$$

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(3) Exponential

$p(x) = \lambda e^{-\lambda x}$ & $p(x)$ is not defined for $x < 0$

$$\text{Entropy} = -\int_{-\infty}^{\infty} p(x) \ln(p(x))\, dx.$$

$$\Rightarrow \text{Entropy} = -\int_{0}^{\infty} \lambda e^{-\lambda x} \ln(\lambda e^{-\lambda x})\, dx.$$

$$= \lambda \int_{0}^{\infty} e^{-\lambda x} [\ln \lambda + \ln e^{-\lambda x}]\, dx.$$

$$= \lambda \int_{0}^{\infty} e^{-\lambda x} \ln \lambda\, dx + \lambda \int_{0}^{\infty} e^{-\lambda x} \ln(e^{-\lambda x})\, dx$$

$$= \left[\ln \lambda \cdot \cancel{\lambda} \left(\frac{e^{-\lambda x}}{-\cancel{\lambda}} \right) \Big|_0^{\infty} \right] + \lambda \left[\int_0^{\infty} e^{-\lambda x} (-\lambda x)\, dx \right] \quad \text{--- (1)}.$$

Consider,

$$I_1 = \lambda \int_0^{\infty} e^{-\lambda x} (-\lambda x)\, dx.$$

Let $e^{-\lambda x} = t$. $\Rightarrow x = \frac{-\ln t}{\lambda}$

$\Rightarrow e^{-\lambda x} (-\lambda)\, dx = dt.$

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$$\Rightarrow I_1 = \lambda \int_0^\infty -\frac{\ln t}{\lambda} \, dt.$$

$$= -\int_0^\infty \ln t \, dt.$$

$$= (t \ln t - t)\Big|_0^\infty \rightarrow ②$$

Substituting ② in ①:

$$\text{Entropy} = \left[\left(e^{-\lambda x} \ln\left(\frac{1}{\lambda}\right) \right) - \left(e^{-\lambda x} (-\lambda x) - e^{-\lambda x} \right) \right]_0^\infty$$

$$\boxed{= 1 - \ln \lambda}$$