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(i) Transforming a 2D Y using

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$y = \begin{bmatrix} y_x \\ y_y \end{bmatrix}$$

$$f_y(y) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \times e^{-\frac{1}{2}(y^T \Sigma^{-1} y)}$$

$$\Sigma = AA' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$|\Sigma| = 16.$$

$$\Sigma^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$\Rightarrow f_y(y) = \frac{1}{(2\pi)(4)} \times e^{-\frac{1}{2}[y_x \; y_y]\begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} y_x \\ y_y \end{bmatrix}}$$

$$\Rightarrow f_y(y) = \frac{1}{(2\pi) \times 4} \times e^{-\frac{1}{8}[y_x \; y_y]\begin{bmatrix} y_x \\ y_y \end{bmatrix}}.$$

$$\boxed{\Rightarrow f_y(y) = \frac{1}{4(2\pi)} \times e^{-\frac{1}{8}(y_x^2 + y_y^2)}}$$

Now,

$$\frac{\partial}{\partial y_x} f_y(y) = \frac{\partial}{\partial y_x}\left(\frac{1}{4(2\pi)} \times e^{-\frac{1}{8}[y_x^2 + y_y^2]}\right).$$

$$\frac{\partial}{\partial y_x} f_y(y) = \frac{1}{8\pi} \times e^{-\frac{1}{8}(y_x^2 + y_y^2)} \times \left(\frac{-2y_x}{8}\right)$$

$$\boxed{\Rightarrow \frac{\partial f_y(y)}{\partial y_x} = -\frac{1}{32\pi} e^{-\frac{1}{8}(y_x^2 + y_y^2)} \times y_x.}$$

$$\frac{\partial}{\partial y_y} f_y(y) = \frac{\partial}{\partial y_y}\left(\frac{1}{8\pi} \times e^{-\frac{1}{8}(y_x^2 + y_y^2)}\right)$$

$$= \frac{1}{8\pi} e^{-\frac{1}{8}(y_x^2 + y_y^2)} \times \left(-\frac{2y_y}{8}\right)$$

$$\boxed{\frac{\partial}{\partial y_y} f_y(y) = -\frac{1}{32\pi} e^{-\frac{1}{8}(y_x^2 + y_y^2)} \times y_y}$$

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Consider a vector,

$$\bar{U} = \frac{\partial}{\partial y_x} f_y(y)\,\hat{i} + \frac{\partial}{\partial y} f_y(y)\,\hat{j}$$

and unit vectors $\bar{U}_1$ & $\bar{U}_2$ along the directions $y = x$ and $y = -x$.

$$\Rightarrow \bar{U}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$\bar{U}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

Now consider,

$$\bar{U}\cdot\bar{U}_1 = \frac{\frac{\partial}{\partial y_x} f_y(y)}{\sqrt{2}} + \frac{\frac{\partial}{\partial y_y} f_y(y)}{\sqrt{2}}.$$

$$= \frac{-1}{32\pi} \times e^{-\frac{1}{8}(y_x^2 + y_y^2)} \times (y_x + y_y)$$

at (a, a).

$$\boxed{\Rightarrow \bar{U}\cdot\bar{U}_1 = -\frac{1}{32\pi} e^{-\frac{1}{4}(a^2)} (2a)}$$

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Now consider, $\bar{\nabla}.\bar{U}_2$

$$\bar{\nabla}.\bar{U}_2 = \frac{\frac{\partial}{\partial y_x} f_y(y)}{\sqrt{2}} - \frac{\frac{\partial}{\partial y_y} f_y(y)}{\sqrt{2}}$$

$$= -\frac{1}{32\pi} e^{-\frac{1}{8}(y_x^2+y_y^2)} (y_x - y_y)$$

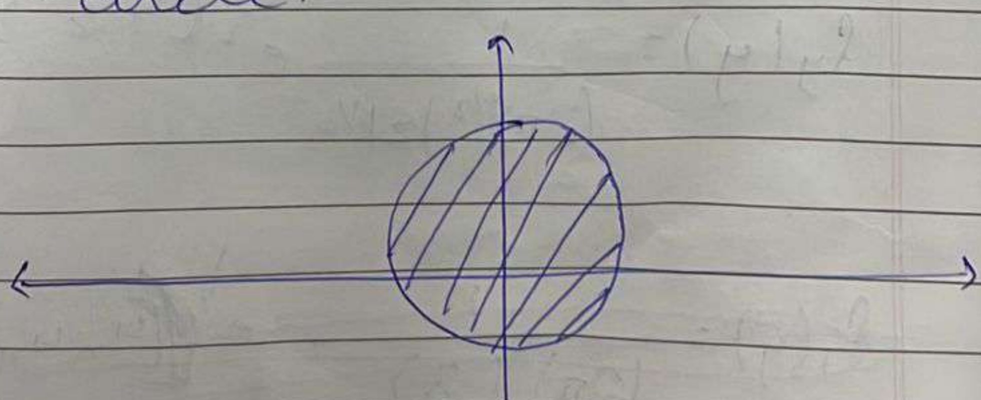
at $(a, -a)$

$$\bar{\nabla}.\bar{U}_2 = -\frac{1}{32\pi} \times e^{-\frac{1}{8}(2a^2)} (2a). \quad \text{--- (2)}$$

Since (1) = (2)

The rate of probability increase or probability decrease along $y = x$ & $y = -x$ is same.

Hence the cross section of the gaussian will be a circle.



(ii) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$y = \begin{bmatrix} y_x \\ y_y \end{bmatrix}$.

$$f_y(y) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}\left(y^T \Sigma^{-1} y\right)}$$

$$\Sigma = AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$|\Sigma| = 9$$

$$\Sigma^{-1} = \frac{1}{9} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$f_y(y) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}\left(y^T \Sigma^{-1} y\right)}$$

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$$= \frac{1}{(2\pi)(3)} e^{-\frac{1}{2}\times\frac{1}{9}\left([y_x\ y_y]\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}\begin{bmatrix}y_x\\ y_y\end{bmatrix}\right)}$$

$$= \frac{1}{6\pi} e^{-\frac{1}{18}\left([y_x\ y_y]\begin{bmatrix}5y_x + 4y_y\\ 4y_x + 5y_y\end{bmatrix}\right)}$$

$$= \frac{1}{6\pi} e^{-\frac{1}{18}\left(5y_x^2 + 4y_yy_x + 4y_xy_y + 5y_y^2\right)}$$

$$= \frac{1}{6\pi} e^{-\frac{1}{18}\left(5y_x^2 + 5y_y^2 + 8y_xy_y\right)}$$

Consider a vector,

$$\vec{v} = \frac{\partial}{\partial y_x} f_y(y)\,\hat{i} + \frac{\partial}{\partial y_y} f_y(y)\,\hat{j}$$

$$\frac{\partial}{\partial y_x} f_y(y) = \frac{1}{6\pi} e^{-\frac{1}{18}\left(5y_x^2 + 5y_y^2 + 8y_xy_y\right)} \times \left(-\frac{1}{18}\left(10y_x + 8y_y\right)\right)$$

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$$\frac{\partial}{\partial y_y} f_y(y) = \frac{1}{6\pi} e^{-\frac{1}{18}(5y_x^2 + 5y_y^2 + 8 \dots} \times \left(-\frac{10 y_y + 8y \dots}{18}\right)$$

Consider a vector, ②

$$\bar{v} = \frac{\partial}{\partial y_x} f_y(y)\,\hat{i} + \frac{\partial}{\partial y_y} f_y(y)\,\hat{j}$$

$$\& \ \bar{v}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j},$$

$$\bar{v}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$$

$$\bar{v}\cdot\bar{v}_1 = \frac{1}{6\pi} \times e^{-\frac{1}{18}(5y_x^2 + 5y_y^2 + 8y_x y_y)} \times \left(-\frac{1}{18}(10y_x + 8y \dots\right)$$

$$+ \frac{1}{6\pi} \times e^{-\frac{1}{18}(5y_x^2 + 5y_y^2 + 8y_x y_y)} \times \left(-\frac{1}{18}(10y_y + 8y \dots\right)$$

at (a, a)

$$\boxed{-\left(\frac{a}{6\pi} e^{-a^2} + \frac{a}{6\pi} e^{-a^2}\right)\frac{1}{\sqrt{2}}}$$

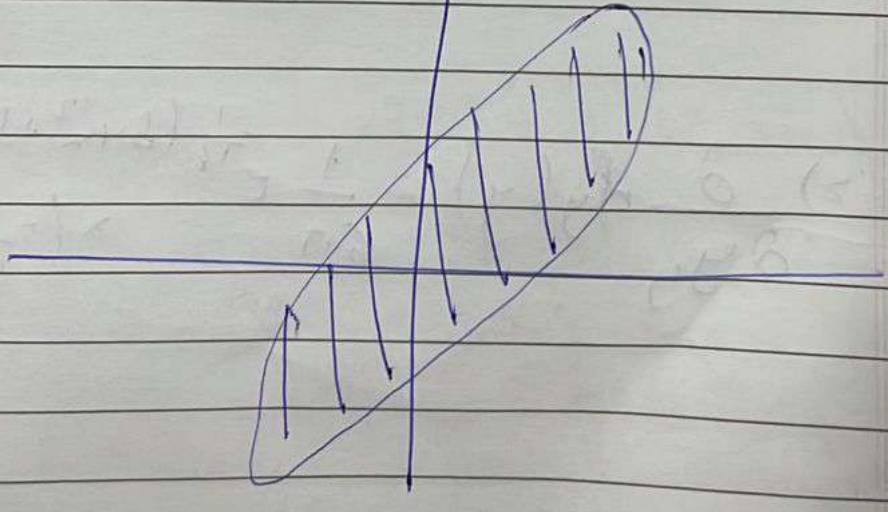
— ③

$$\bar{\upsilon}\cdot\bar{\upsilon}_2 = \frac{\frac{\partial}{\partial y_x} f_y(y)}{\sqrt{2}} - \frac{\frac{\partial}{\partial y_y} f_y(y)}{\sqrt{2}}$$

$\cancel{= \frac{1}{\sqrt{2}}\left(\frac{a}{6\pi} e^{-a^2}\right)}$ at $(a,-a)$.

$$= \frac{e^{-a^2}}{6\pi}\left(\frac{-a}{\sqrt{2}}\right)$$

Since $|\bar{\upsilon}\cdot\bar{\upsilon}_2| > |\bar{\upsilon}\cdot\bar{\upsilon}_1|$, the rate at which prob decreases along $y=x$ will be slower than along $y=-x$. Hence the cross section of the gaussian will be elliptical along $y=x$.



(iii) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow \Sigma = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$.

$|\Sigma| = 9$.

$\Rightarrow \Sigma^{-1} = \frac{1}{9}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$\Rightarrow f_y(y) = \frac{1}{6\pi} e^{-\frac{1}{18}(5y_x^2 + 5y_y^2 + 8y_x y_y)}$

$\Rightarrow \frac{\partial}{\partial y_x} f_y(y) = \frac{1}{6\pi} e^{-\frac{1}{18}(5y_x^2 + 5y_y^2 + 8y_x y_y)}$
$\times \left(-\frac{1}{18}(10y_x + 8y_y)\right)$

$\Rightarrow \frac{\partial}{\partial y_y} f_y(y) = \frac{1}{6\pi} e^{-\frac{1}{18}(5y_x^2 + 5y_y^2 + 8y_x y_y)}$
$\times \left(-\frac{1}{18}(10y_y + 8y_x)\right)$

Let $\bar{U} = \frac{\partial}{\partial y_x} f_y(y)\hat{i} + \frac{\partial}{\partial y_y} f_y(y)\hat{j}$

$\bar{U}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

$\bar{U}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$\bar{U}\cdot\bar{U}_1$ at (a, a)

$$= \frac{e^{-a^2}}{6\pi}\left(\frac{-2a}{\sqrt{2}}\right)$$

$\bar{U}\cdot\bar{U}_2$ at $(a, -a)$

$$= \frac{e^{-(a^2/a)}}{6\pi} \times \frac{-a}{9\sqrt{2}}$$

Here $|\bar{U}\cdot\bar{U}_1| > |\bar{U}\cdot\bar{U}_2|$

Hence the rate at which probability decreases along $y = x$ at a greater pace than along $y = -x$

Hence the cross section of the gaussian will be ~~along~~ elliptical along $y = -x$.