Exercise - b

$$E[\tau] = \sum_{m=0}^{\infty} m \cdot P(\tau = m)$$

$\Rightarrow E[\tau] = P(\tau=1) + 2 \cdot P(\tau=2) + \cdots$

$\Rightarrow E[\tau] = p + 2p(1-p) + 3p(1-p^2) + \cdots$ — ①

Multiplying ~~by~~ ~~d~~ on both sides. by $(1-p)$

$\Rightarrow E[\tau](1-p) = p(1-p) + 2p(1-p)^2 + 3p(1-p)^3 + \cdots$ — ②

② − ①

$$E[\tau](1-p) = p(1-p) + 2p(1-p)^2 + 3p(1-p)^3 + \cdots$$

$$E[\tau] = p + 2p(1-p) + 3p(1-p)^2 + \cdots$$

$$-pE[\tau] = -p - p(1-p) - p(1-p)^2 - p(1-p)^3 - \cdots$$

$\Rightarrow pE[\tau] = p + p(1-p) + p(1-p)^2 + p(1-p)^3 + \cdots$

$\Rightarrow E[\tau] = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots$ — ③

We know that,
$0 < p < 1$

$\Rightarrow 1-p < 1$

$\Rightarrow$ the G.P in ③ converges and the sum is given as.

$$E[\tau] = \frac{1}{1-(1-p)}$$

$$\boxed{\Rightarrow E[\tau] = \frac{1}{p}.}$$