

# Topological Graph Theory

Benji Altman

April 13, 2019

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Overview</b>	<b>1</b>
2.1	Graphs . . . . .	1
2.2	Königsberg, and it's seven bridges . . . . .	2
2.3	History . . . . .	3
<b>3</b>	<b>Graph Theory Background</b>	<b>3</b>
3.1	Planer Graphs . . . . .	3
3.2	Graph Drawing . . . . .	4
3.3	Face . . . . .	4
3.4	Path . . . . .	5
3.5	Connected . . . . .	6
3.6	Contraction . . . . .	7
3.7	Subdivision . . . . .	7
<b>4</b>	<b>Kuratowski's Theorem</b>	<b>9</b>
4.1	$K_{3,3}$ and $K_5$ . . . . .	9
	<b>Glossary</b>	<b>9</b>

## 1 Introduction

Topological graph theory is an entire field within topology and as such this paper is by no means meant to cover all of topological graph theory in any depth. This paper instead will first cover a rather shallow overview of the field, followed by a more in depth study of graphs and their genus. The overview will mainly be focused on giving a thorough understanding of what topological graph theory is as well as to briefly cover the history of the field. In giving an overview of the field we will cover some of the basic concepts and definitions needed for the more rigorous part of the paper. After the overview we will dive into Kuratowski's Theorem, We will go through and attempt to have an intuitive understanding of a Kuratowski's Theorem and it's proof. After proving Kuratowski's Theorem, we will continue onto talking about generalizations of the theorem and map colorings, however their coverage will be rather shallow and lacking proofs.

## 2 Overview

### 2.1 Graphs

Before we talk about topological graph theory with any level of understanding we must first understand what a graph is.

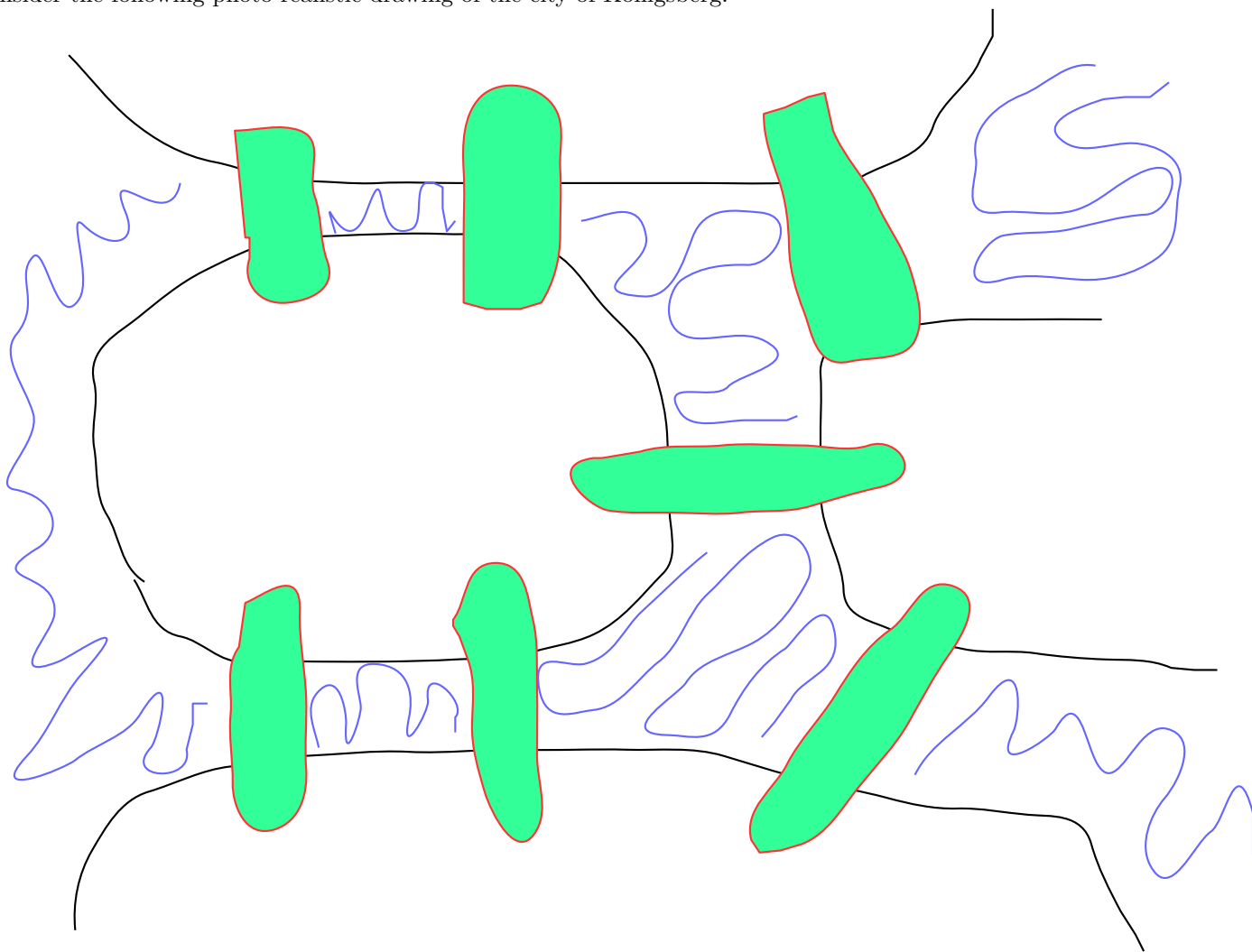
A graph is generally defined as a set of vertices combined with a set of edges between vertices, however here it may be more useful to think about them visually with a simple representation.

Consider first a set of points, this may be thought of as just drawing dots on a sheet of paper. Each of these points will be called a vertex. Now we may start drawing lines between vertices. Lines may cross over each other and need not be straight. There is no requirement that all vertices have a line going to it. Each of these lines are called an edge. We will simply insist that no edge connects two vertices and that we do not have multiple edges between the same pair of vertices.

Once we have drawn this we have a representation of a graph. If we were to move the vertices around on the paper but leave them having the same edges (the same vertices are connected to the point as they were before). we would be left with the same graph. That is to say, it doesn't mater where we put a vertex on our sheet, the graph exists independently of the representation we draw.

## 2.2 Königsberg, and it's seven bridges

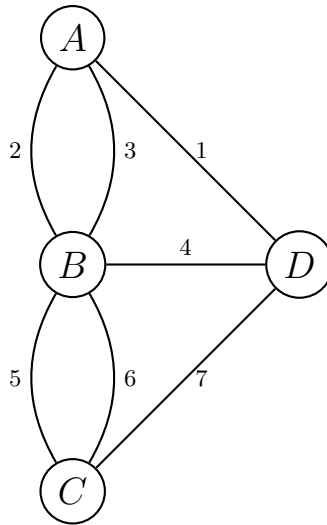
Consider the following photo-realistic drawing of the city of Königsberg.



Now the question is, if we get to choose where we start, can we go for a stroll and cross every bridge exactly once?

I first came across this question in the 8<sup>th</sup> grade, and it was presented to us during geometry class. While undoubtedly an interesting problem, it is quite misleading to try and think of this as a geometry problem. Instead we will try and reduce it to a graph problem.

Let us start by thinking of every island as a vertex and every bridge as an edge. We find the following graph.



It is worth noting two things about the above diagram. First that the labels on the vertices and edges are unrelated to the problem, but have been added simply to make referring to parts of the graph much easier. Second that whatever the above image depicts, does not fit our definition of a graph.

Notice that edge 2 and edge 3 both connect vertex  $A$  to  $B$ , as well edges 5 and 6 do for  $B$  and  $C$ . This is a strict violation of our definition for a graph. The issue of course then comes to what would one call such a beast as this where, presumably, one is able to have as many connections between any pair of vertices and could even have connections from a vertex to itself.

I am particularly glad that you're paying enough attention to notice that the diagram does not depict a graph. This is what we will refer to as a multigraph. It is worth noting that graphs are a type of multigraph, so anything we show to be true for all multigraphs, is also true for all graphs.

Now to solve this problem we need to make one simple observation about how we walk. If we are to go to island (or vertex) we must also leave that island, unless it is the last island we arrive on. This means, that with the exception of the island we start on and end on, each island must have an even number of bridges connected to it. On the multigraph we would say that we need all vertices but a start and end vertex to have an even number of edges. If we look at the multigraph above, we have four vertices that have an odd number of edges connected to them.

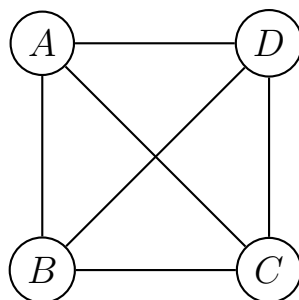
## 2.3 History

The seven bridges of Königsberg problem was solved by Euler in 1736. In mathematics this problem is of great historical significance as it is considered to be the beginning of graph theory as well as a sort of precursor to topology.

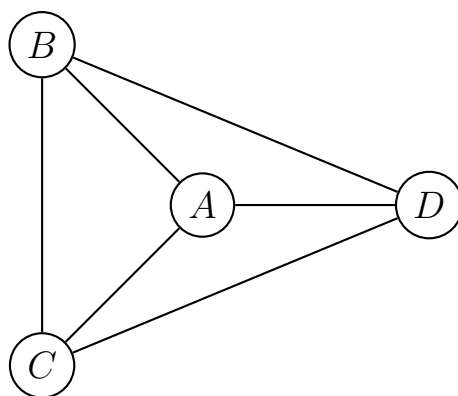
# 3 Graph Theory Background

## 3.1 Planer Graphs

Consider the following graph.



This graph is called a complete graph as every vertex is connected to every other vertex by an edge; in fact this graph in particular is called  $K_4$  as it is the complete four vertex graph. We would like to find out if we can draw this above graph without having any lines crossing. We can in fact draw this graph without any intersections and for any skeptics who may be reading this, the below is  $K_4$  without any edges intersecting.



So if this graph can be drawn without intersection, can any graph be drawn without intersections? If some can and some can't how do we tell which can be drawn and which can not? The answer to this comes in Kuratowski's theorem, however before we can even state this theorem we need to build up a bit of terminology for graph theory.

## 3.2 Graph Drawing

A graph drawing is exactly what it sounds like. We've already seen drawings of graphs like the one for Königsberg and two drawings of  $K_4$ , this means there may be multiple distinct drawings for the same graph. Now we don't need to be very worried about what defines a drawing, that won't be important to us. Simply think of it as the drawing.

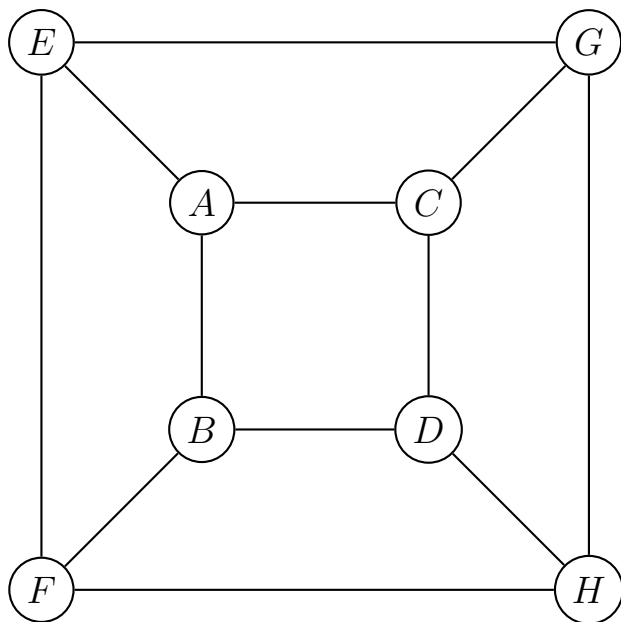
If a drawing has no edges intersecting it is said to be a plane drawing. If there exists such a drawing for a particular graph, then that graph is a planar graph.

## 3.3 Face

Faces are a bit of an odd property here as they fundamentally are actually properties only of plane drawings and not graphs themselves; however the number of faces, as we will see stays consistent between any plane drawing of a planar graph and as such the number of faces is a property that planar graphs have.

A face is defined as a connected space that contains no edges or vertices and itself is bounded by edges and vertices.

If you think back to high-school geometry and cubes you may recall that at each of the corners is a vertex, the lines connecting vertices are edges and the area between the edges are faces. In a graph we have vertices connected by edges and when we draw them there are empty areas enclosed by edges and vertices. For example the following would correspond with a cube



Now we can imagine that if we look at the cube straight on maybe  $ABDC$  would be the face we are looking at. So too we see the area enclosed in  $ABDC$  is a face by the definition we gave. The same is true for  $ABFE$ ,  $ACGE$ ,  $DBFH$ , and  $DCGH$ , however that leaves us with only five faces on this cube. The last face must logically come from  $EFHG$ , however the area enclosed by that contains all the other edges and vertices, so it doesn't fit our definition. However we may notice that the infinite space outside  $EFHG$  does not contain any vertices, and therefore we get a sixth face.

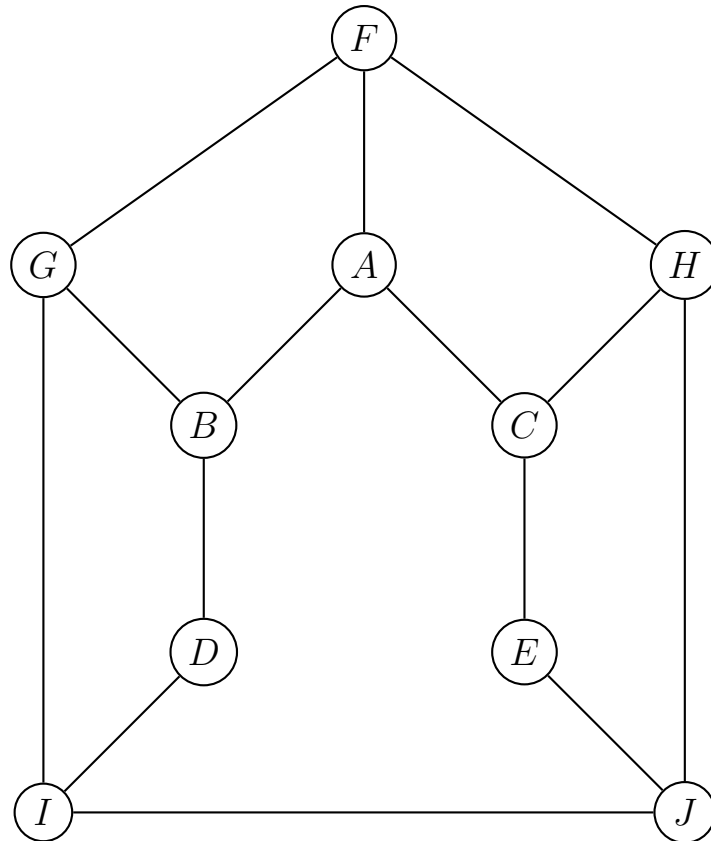
Consider then the following very simple graph:



Now we still only have one face, however it doesn't have as nice a boundary as they did in the cube. However if we look at all of space excluding  $A$  and  $B$  then we still get a valid space by our definition.

### 3.4 Path

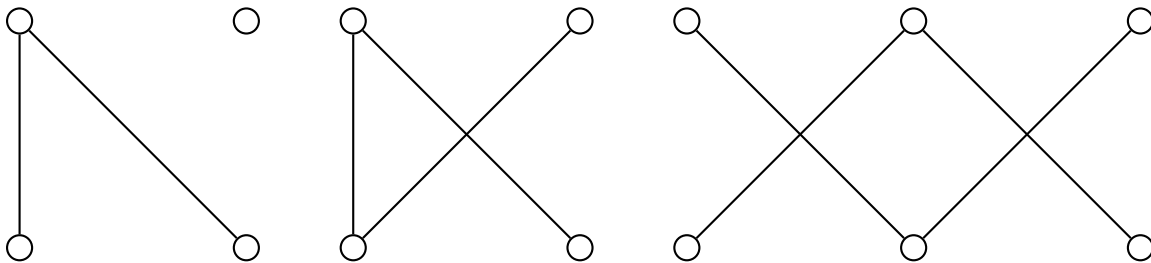
Now moving onto a more traditional graph property we have the concept of a path. A path is defined as a finite sequence of vertices with the property that each element of the sequence (excluding the last one) has an edge from it to the next element. For example consider the following graph.



Now the sequence  $BGF$  is a path as  $B$  connects to  $G$  and  $G$  connects to  $F$ . The sequence  $ABGFACEJHCABA$  is also a path, however  $JEDI$  is not a path as  $E$  has no edge to  $D$ . Notice that how we draw the graph has nothing to do with what is and is not a path.

### 3.5 Connected

A graph is said to be connected if for any pair of vertices,  $(a, b)$  there is a path from  $a$  to  $b$ . So if we consider the following graphs

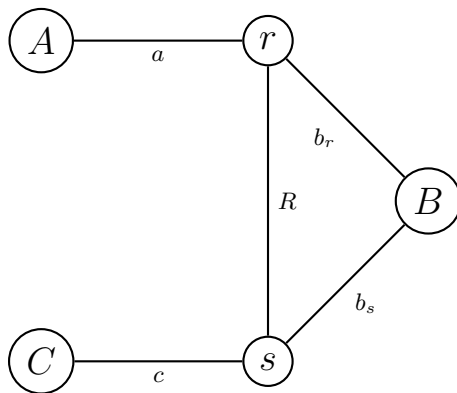


We find that only the graph in the middle is connected. For the graph on the right consider any vertex on the bottom, there is no path to the vertex above it. The graph on the left has the upper right vertex isolated from the rest of the graph.

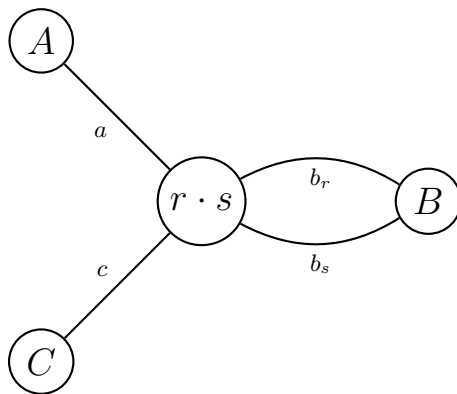
Any graph, connected or not, can be broken into connected components. To do this we simply take a vertex and every other vertex connected to it and call that one component, and then repeat with a vertex not in that component. This sort of breaking apart is nice as often we will prove things about connected graphs that are true about all graphs. For example if all components of a graph are planar then the entire graph must be planar, this will be proven below and it allows us to only deal with connected graphs.

### 3.6 Contraction

This is not a property, but rather an operation or action that we perform on a graph. Let us consider the following graph.



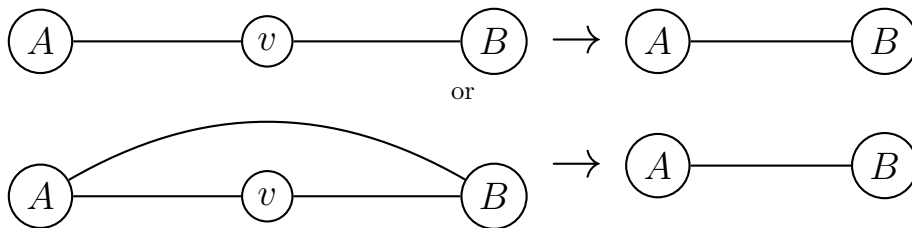
We wish to perform a contraction on edge  $R$ . To be clear all graph contractions are on edges. So we will make a new vertex,  $r \cdot s$  which has all the edges of  $r$  and all the edges of  $s$  except the edge we are contracting across. In this case the edge we are contracting across is  $R$  and thus we get the following multigraph.



This can then be reduced into a graph again by simply treating  $b_r$  and  $b_s$  as the same edge. This operation is particularly important as we will prove that if a graph is planar, then so is any graph or multigraph obtained by contracting an edge.

### 3.7 Subdivision

To define this we first describe, we first have to describe an operation where we take a vertex  $v$  that has degree 2, and replace with an edge as follows.



Here  $A$  and  $B$  may be connected in any way to a larger graph, however only this subgraph will change.

Now we are going to call this operation vertex replacement, and using it we define a graph subdivision as follows.

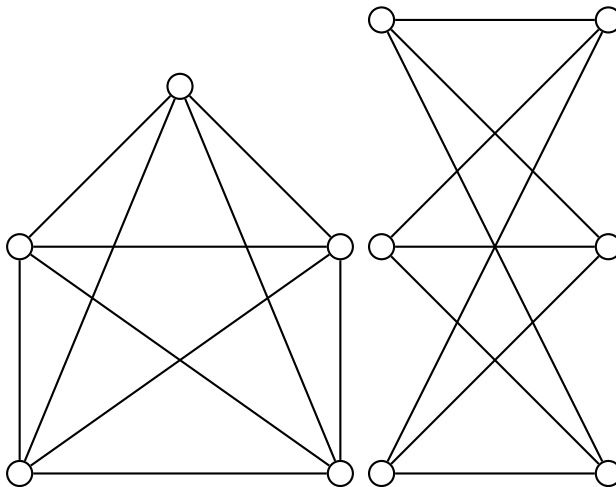
**Definition 3.1** (graph subdivision). A graph  $G$  is a subdivision of  $H$  if  $H$  may be produced by some series of vertex replacements on  $G$ .



## 4 Kuratowski's Theorem

**Theorem 4.1** (Kuratowski's Theorem). *A graph  $G$  is nonplanar if and only if it contains a subgraph that is a subdivision of  $K_{3,3}$  or  $K_5$*

Here  $K_5$  refers to the complete graph on 5 vertices, and  $K_{3,3}$  is the complete bipartite graph with three vertices in both partitions. Drawings of both these graphs are below with ( $K_5$  on left and  $K_{3,3}$  on right).



### 4.1 $K_{3,3}$ and $K_5$

Our first step is proving that  $K_{3,3}$  and  $K_5$  are nonplanar. To do this we are going to use the following theorem.

**Theorem 4.2** (Euler's formula on planar graphs). *For any plane drawing of a graph, we have*

$$v - e + f = 2$$

, where  $v$  is the number of vertices in the graph,  $e$  is the number of edges in the graph, and  $f$  is the number of faces in the plane drawing of the graph.

Before we can prove this we need to define the dual of a planar graph.

**Definition 4.1** (graph dual). For any plane drawing  $G$  of a graph, we may construct its dual  $G^*$  by putting a single vertex in each face, and putting an edge between any two adjacent faces (faces that share an edge). Note that this dual is not necessarily a graph, but will fit our definition of a multigraph.

For example consider the following graph drawing xunit=.5pt,yunit=.5pt,runit=.5pt (333.3333333,237.3333333)  
curcolor0 0 1 [linestyle=None,fillstyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linestyle=None,fillstyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linestyle=None,fillstyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linestyle=None,fillstyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linestyle=None,fillstyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor0 0 1 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor]  
curcolor1 0 0 [linestyle=None,fillstyle=solid,fillcolor=curcolor] curcolor1 0 0 [linestyle=None,fillstyle=solid,fillcolor=curcolor]  
curcolor1 0 0 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor1 0 0 [linestyle=None,fillstyle=solid,fillcolor=curcolor]  
curcolor1 0 0 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor1 0 0 [linestyle=None,fillstyle=solid,fillcolor=curcolor]  
curcolor1 0 0 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor1 0 0 [linestyle=None,fillstyle=solid,fillcolor=curcolor]  
curcolor1 0 0 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor] curcolor1 0 0 [linewidth=1.33333334,linestyle=solid,fillcolor=curcolor,linestyle=da

1.99255] curcolor1 0 0 [linewidth=1.3333334,linicolor=curcolor,linestyle=dashed,dash=2.98883 1.99255]  
curcolor1 0 0 [linewidth=1.3333334,linicolor=curcolor,linestyle=dashed,dash=2.98883 1.99255] curcolor1 0  
0 [linewidth=1.3333334,linicolor=curcolor,linestyle=dashed,dash=2.98883 1.99255] curcolor1 0 0 [linewidth=1.3333334,lin  
1.99255] curcolor1 0 0 [linewidth=1.3333334,linicolor=curcolor,linestyle=dashed,dash=2.98883 1.99255]  
curcolor1 0 0 [linewidth=1.3333334,linicolor=curcolor,linestyle=dashed,dash=2.98883 1.99255]

*Proof.*

□

## Glossary

**complete graph** A graph with all possible edges included, the notation  $K_n$  is used to denote the complete with  $n$  vertices. 4

**contraction** A graph operation where one removes an edge by fusing two vertices together. 7

**edge** A connection or line between vertices in a graph. 1, 2, 3, 4, 7

**graph** A set of vertices and edges. 1, 2, 3, 4, 7

**multigraph** Like a graph, but multiple edges may connect the same vertex pair, and an edge may connect a vertex to itself. 3, 7

**set** A collection of objects. 1, 2

**vertex** A point or node in a graph. 1, 2, 3, 4, 7

## References