

Topological Graph Theory

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1 Introduction

Topological graph theory is an entire field within topology and as such this paper is by no means meant to cover all of topological graph theory in any depth. This paper instead will first cover a rather shallow overview of the field, followed by a more in depth study of graphs and their genus. The overview will mainly be focused on giving a thorough understanding of what topological graph theory is as well as to briefly cover the history of the field. In giving an overview of the field we will cover some of the basic concepts and definitions needed for the more rigorous part of the paper. After the overview we will dive into Kuratowski's Theorem, We will go through and attempt to have an intuitive understanding of a Kuratowski's Theorem and its proof. After proving Kuratowski's Theorem, we will continue onto talking about generalizations of the theorem and map colorings, however their coverage will be rather shallow and lacking proofs.

2 Overview

2.1 Graphs

Before we talk about topological graph theory with any level of understanding we must first understand what a graph is.

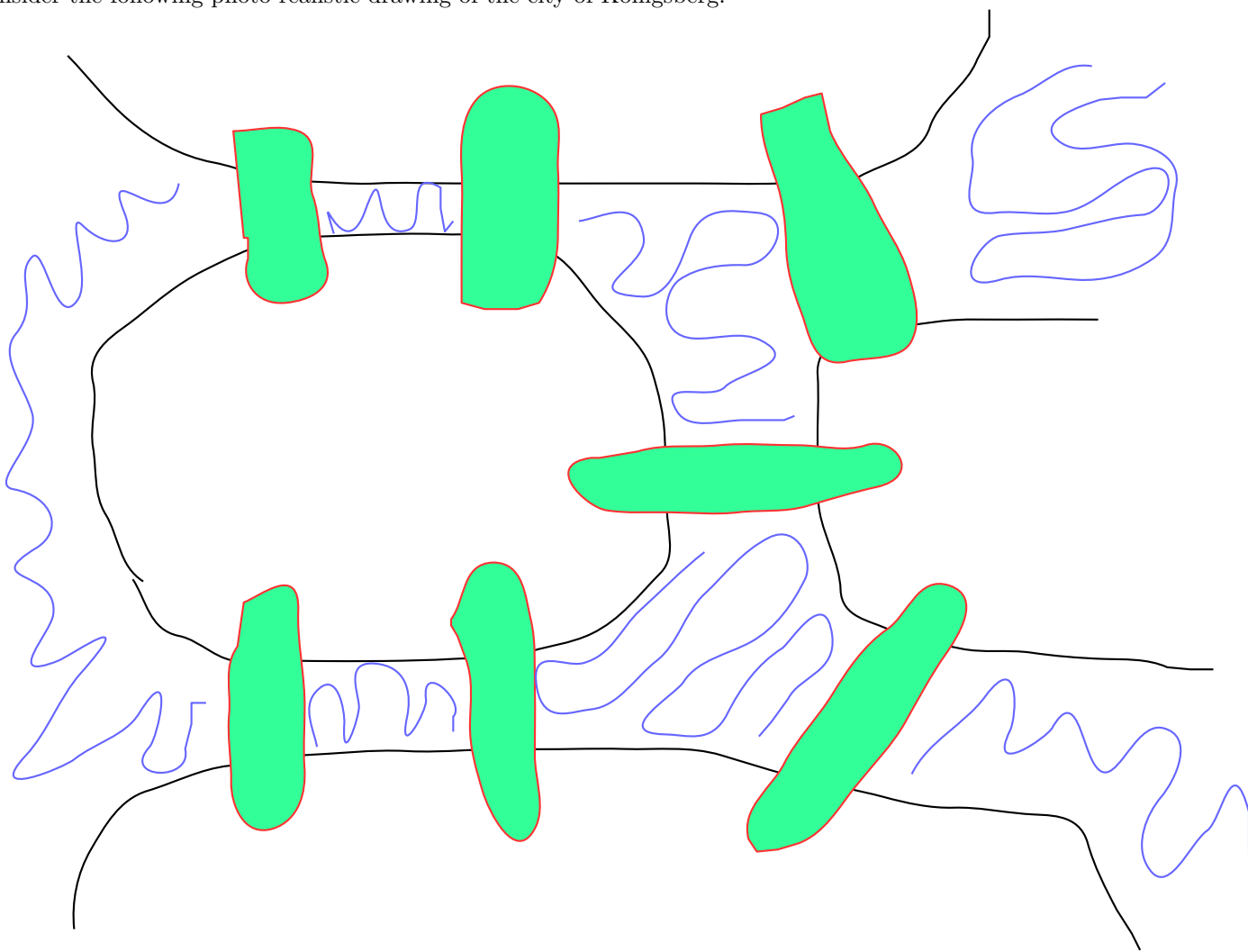
A graph is generally defined as a set of vertices combined with a set of edges between vertices, however here it may be more useful to think about them visually with a simple representation.

Consider first a set of points, this may be thought of as just drawing dots on a sheet of paper. Each of these points will be called a vertex. Now we may start drawing lines between vertices. Lines may cross over each other and need not be straight. There is no requirement that all vertices have a line going to it. Each of these lines are called an edge. We will simply insist that no edge connects two vertices and that we do not have multiple edges between the same pair of vertices.

Once we have drawn this we have a representation of a graph. If we were to move the vertices around on the paper but leave them having the same edges (the same vertices are connected to the point as they were before). we would be left with the same graph. That is to say, it doesn't matter where we put a vertex on our sheet, the graph exists independently of the representation we draw.

2.2 Königsberg, and its seven bridges

Consider the following photo-realistic drawing of the city of Königsberg.

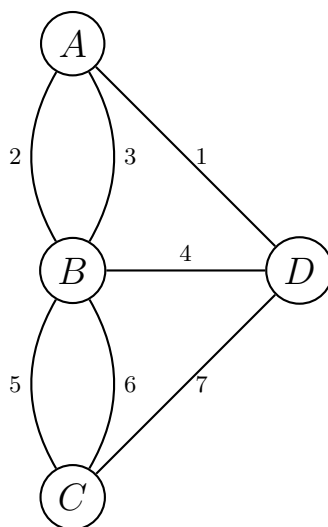


Now the question is, if we get to choose where we start, can we go for a stroll and cross every bridge exactly once?

I first came across this question in the 8th grade, and it was presented to us during geometry class. While undoubtedly an interesting problem, it is quite misleading to try and think of this as a geometry problem. Instead we will try and reduce it to a graph problem.

Let us start by thinking of every island as a vertex and every bridge as an edge. We find the following

graph.



It is worth noting two things about the above diagram. First that the labels on the vertices and edges are unrelated to the problem, but have been added simply to make referring to parts of the graph much easier. Second that whatever the above image depicts, does not fit our definition of a graph.

Notice that edge 2 and edge 3 both connect vertex A to B , as well edges 5 and 6 do for B and C . This is a strict violation of our definition for a graph. The issue of course then comes to what would one call such a beast as this where, presumably, one is able to have as many connections between any pair of vertices and could even have connections from a vertex to itself.

I am particularly glad that you're paying enough attention to notice that the diagram does not depict a graph. This is what we will refer to as a multigraph. It is worth noting that graphs are a type of multigraph, so anything we show to be true for all multigraphs, is also true for all graphs.

Now to solve this problem we need to make one simple observation about how we walk. If we are to go to island (or vertex) we must also leave that island, unless it is the last island we arrive on. This means, that with the exception of the island we start on and end on, each island must have an even number of bridges connected to it. On the multigraph we would say that we need all vertices but a start and end vertex to have an even number of edges. If we look at the multigraph above, we have four vertices that have an odd number of edges connected to them.

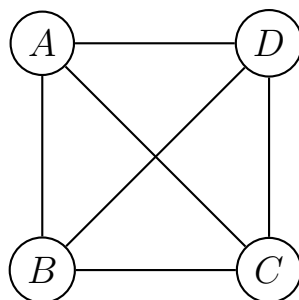
2.3 History

The seven bridges of Königsberg problem was solved by Euler in 1736. In mathematics this problem is of great historical significance as it is considered to be the beginning of graph theory as well as a sort of precursor to topology.

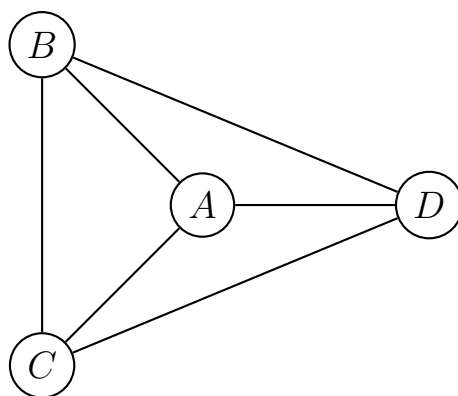
3 Graph Theory Background

3.1 Planer Graphs

Consider the following graph.



This graph is called a complete graph as every vertex is connected to every other vertex by an edge; in fact this graph in particular is called K_4 as it is the complete four vertex graph. We would like to find out if we can draw this above graph without having any lines crossing. We can in fact draw this graph without any intersections and for any skeptics who may be reading this, the below is K_4 without any edges intersecting.



So if this graph can be drawn without intersection, can any graph be drawn without intersections? If some can and some can't how do we tell which can be drawn and which can not? The answer to this comes in Kuratowski's theorem, however before we can even state this theorem we need to build up a bit of terminology for graph theory.

3.2 Graph Drawing

A graph drawing is exactly what it sounds like. We've already seen drawings of graphs like the one for Königsberg and two drawings of K_4 , this means there may be multiple distinct drawings for the same graph. Now we don't need to be very worried about what defines a drawing, that won't be important to us. Simply think of it as the drawing.

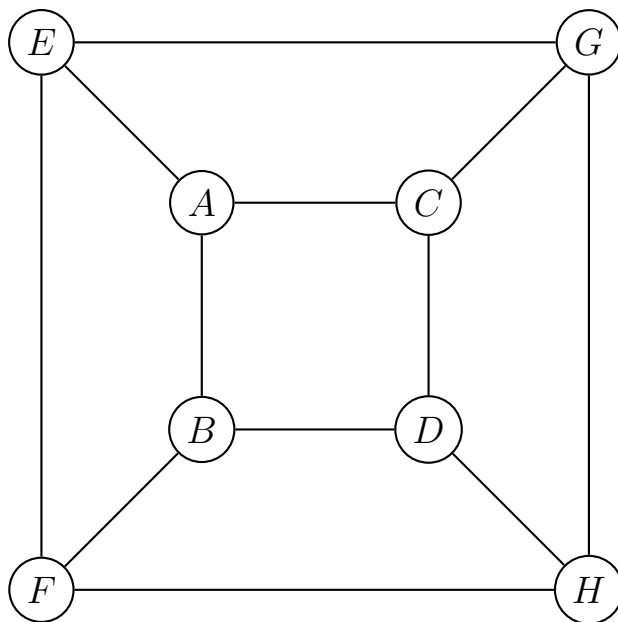
If a drawing has no edges intersecting it is said to be a plane drawing. If there exists such a drawing for a particular graph, then that graph is a planar graph.

3.3 Face

Faces are a bit of an odd property here as they fundamentally are actually properties only of plane drawings and not graphs themselves; however the number of faces, as we will see stays consistent between any plane drawing of a planar graph and as such the number of faces is a property that planar graphs have.

A face is defined as a connected space that contains no edges or vertices and itself is bounded by edges and vertices.

If you think back to high-school geometry and cubes you may recall that at each of the corners is a vertex, the lines connecting vertices are edges and the area between the edges are faces. In a graph we have vertices connected by edges and when we draw them there are empty areas enclosed by edges and vertices. For example the following would correspond with a cube



Now we can imagine that if we look at the cube straight on maybe $ABDC$ would be the face we are looking at. So too we see the area enclosed in $ABDC$ is a face by the definition we gave. The same is true for $ABFE$, $ACGE$, $DBFH$, and $DCGH$, however that leaves us with only five faces on this cube. The last face must logically come from $EFHG$, however the area enclosed by that contains all the other edges and vertices, so it doesn't fit our definition. However we may notice that the infinite space outside $EFHG$ does not contain any vertices, and therefore we get a sixth face.

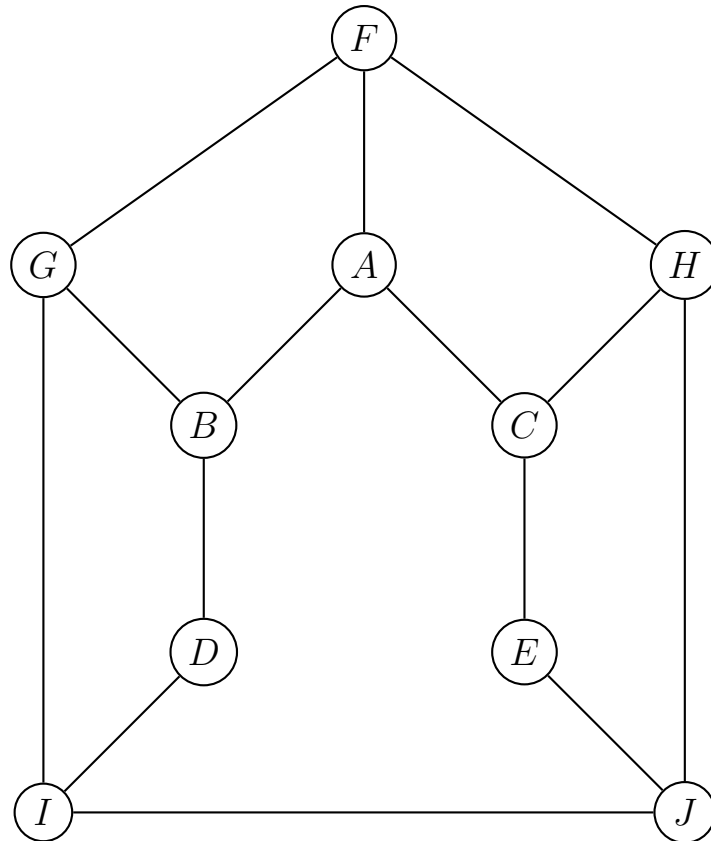
Consider then the following very simple graph:



Now we still only have one face, however it doesn't have as nice a boundary as they did in the cube. However if we look at all of space excluding A and B then we still get a valid space by our definition.

3.4 Path

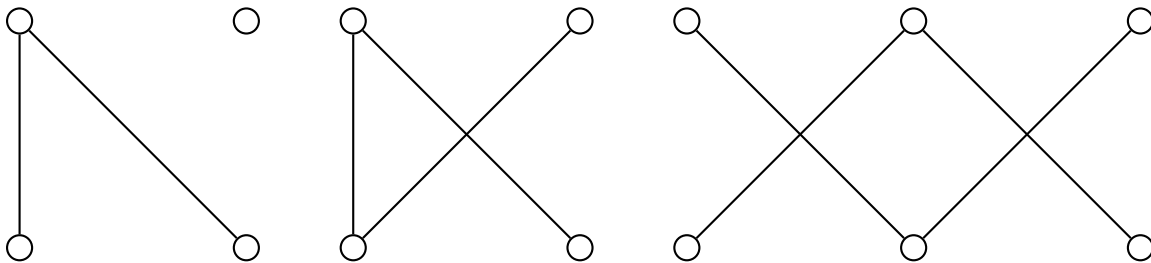
Now moving onto a more traditional graph property we have the concept of a path. A path is defined as a finite sequence of vertices with the property that each element of the sequence (excluding the last one) has an edge from it to the next element. For example consider the following graph.



Now the sequence BGF is a path as B connects to G and G connects to F . The sequence $ABGFACEJHCABA$ is also a path, however $JEDI$ is not a path as E has no edge to D . Notice that how we draw the graph has nothing to do with what is and is not a path.

3.5 Connected

A graph is said to be connected if for any pair of vertices, (a, b) there is a path from a to b . So if we consider the following graphs

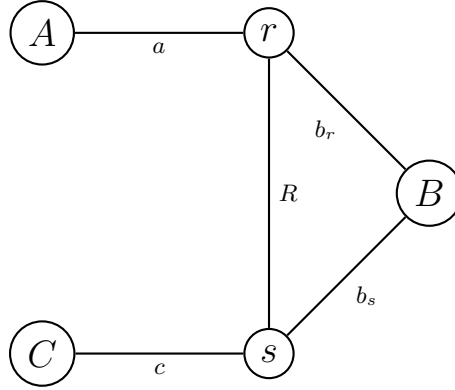


We find that only the graph in the middle is connected. For the graph on the right consider any vertex on the bottom, there is no path to the vertex above it. The graph on the left has the upper right vertex isolated from the rest of the graph.

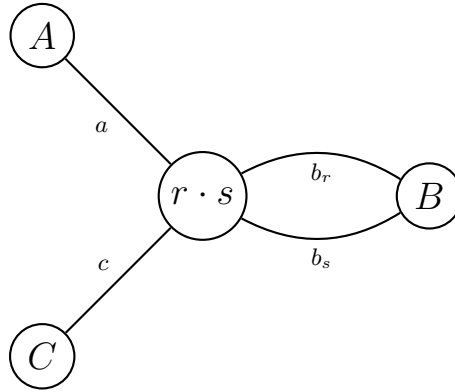
Any graph, connected or not, can be broken into connected components. To do this we simply take a vertex and every other vertex connected to it and call that one component, and then repeat with a vertex not in that component. This sort of breaking apart is nice as often we will prove things about connected graphs that are true about all graphs. For example if all components of a graph are planar then the entire graph must be planar, this will be proven below and it allows us to only deal with connected graphs.

3.6 Contraction

This is not a property, but rather an operation or action that we perform on a graph. Let us consider the following graph.



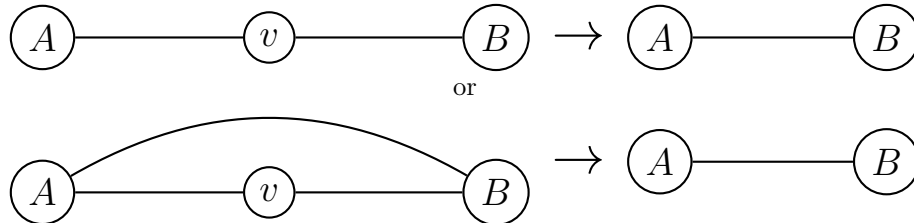
We wish to perform a contraction on edge R . To be clear all graph contractions are on edges. So we will make a new vertex, $r \cdot s$ which has all the edges of r and all the edges of s except the edge we are contracting across. In this case the edge we are contracting across is R and thus we get the following multigraph.



This can then be reduced into a graph again by simply treating b_r and b_s as the same edge. This operation is particularly important as we will prove that if a graph is planar, then so is any graph or multigraph obtained by contracting an edge.

3.7 Subdivision

To define this we first describe, we first have to describe an operation where we take a vertex v that has degree 2, and replace with an edge as follows.



Here A and B may be connected in any way to a larger graph, however only this subgraph will change.

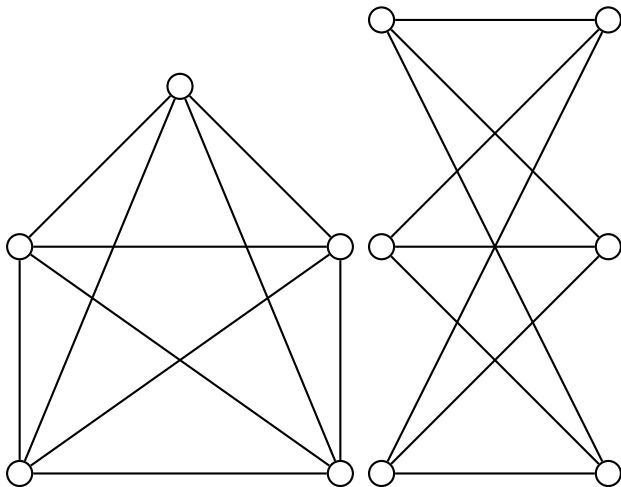
Now we are going to call this operation vertex replacement, and using it we define a graph subdivision as follows.

Definition 3.1 (graph subdivision). A graph G is a subdivision of H if H may be produced by some series of vertex replacements on G .

4 Kuratowski's Theorem

Theorem 4.1 (Kuratowski's Theorem). *A graph G is nonplanar if and only if it contains a subgraph that is a subdivision of $K_{3,3}$ or K_5*

Here K_5 refers to the complete graph on 5 vertices, and $K_{3,3}$ is the complete bipartite graph with three vertices in both partitions. Drawings of both these graphs are below with (K_5 on left and $K_{3,3}$ on right).



4.1 K_5 and $K_{3,3}$

Our first step is proving that $K_{3,3}$ and K_5 are nonplanar. To do this we are going to use the following theorem.

Theorem 4.2 (Euler's formula on planar multigraphs). *For any plane drawing of a multigraph (with the exception of a multigraph with no vertices), we have*

$$v - e + f = 2$$

, where v is the number of vertices in the multigraph, e is the number of edges in the multigraph, and f is the number of faces in the plane drawing of the multigraph.

Proof. First consider a connected multigraph with no edges, as this is connected we may only have a single vertex. This produces a single and no edges so we find

$$v - e + f = 1 - 0 + 1 = 2$$

Now that we know that for 0 edges this rule fits, then either the rule ($v - e + f = 2$) is true no matter the number of edges, or there is some number at which point this rule breaks, and $v - e + f \neq 2$. If there is a number that breaks this rule, then there must be a lowest number (let's call it k) that breaks this rule. For any multigraph with k edges, choose any edge e in the graph.

- If e connects a vertex to itself it is a loop and its removal will result in the loss of one edge and one face (each loop creates a face¹). The resulting multigraph has less than k edges and thus we know that for it

$$\begin{aligned} 2 &= v - e + f \\ &= v - (e + 1) + (f + 1) \end{aligned}$$

and as the original graph had one more edge and one more face it too would fit this rule.

¹Jordan curve theorem

- If e is not a loop we may perform an edge contraction on it and this will reduce both the number of edges and the number of vertices by one. Again this gives us a multigraph with less than k edges so we know that for it

$$\begin{aligned} 2 &= v - e + f \\ &= (v + 1) - (e + 1) + f \end{aligned}$$

and as the original graph had one more edge and one more vertex it would fit this rule.

From this we know that there can not possibly be some k , and thus all multigraphs, regardless of the number of edges must fit this rule. \square

This leads to a nice corollary, that will be helpful when talking about graphs.

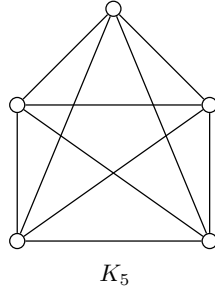
Corollary 4.3. Any plane drawing of any multigraph has the same number of faces.

Proof. Any planar multigraph can be broken into connected components. Each component of a planar multigraph is a connected planar multigraph and thus any planar drawing of it fits the rule $v - e + f = 2$. Now any plane drawing we make of a planar multigraph will be made of parts that already have a constant number of faces.² \square

This means that given any planar multigraph G , we can talk about the number of faces G has without referring to any drawing of G , as all plane drawings will have the same number of faces.

Now using this we can prove that $K_{3,3}$ and K_5 are nonplanar.

Theorem 4.4. K_5 is nonplanar.



Proof. K_5 is a graph³, and as such an edge can not be a loop and two vertices can share at most one edge. This means that in any plane drawing of a graph, there must be at least 3 edges bordering every face.

- To only have one edge bordering a face would require that the edge be a loop (which we can't have in a graph).
- To only have only two edges bordering a face would require that some pair of vertices share more than one edge (which we can't have in a graph).

Additionally every edge borders no more than two faces, so from this we know that in any graph $2e \geq 3f$ or $\frac{2}{3}e \geq f$. Now K_5 has 10 edges and 5 vertices, so we know that

$$\frac{2}{3}10 = \frac{20}{3} = 6.\bar{6} \geq f$$

We know that K_5 is a connected graph (and thus also a connected multigraph) so if K_5 were planar then we would have

$$2 = v - e + f = 5 - 10 + f \leq 5 - 10 + 6.\bar{6} = 1.\bar{6}$$

²This is not a complete proof, to complete it you simply make an induction on the number of components drawn and realize each time you add a component you add exactly $f_i - 1$ new faces (if f_i is the number of faces in the i^{th} component drawn).

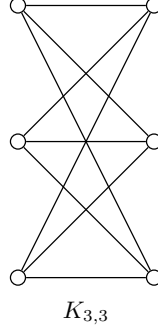
³Recall that all graphs are multigraphs, but not all multigraphs are graphs.

which is clearly false as

$$2 \not\leq 1.\bar{6}$$

Therefore K_5 must not be planar. □

Theorem 4.5. $K_{3,3}$ is nonplanar.



Proof. $K_{3,3}$ is a bipartite graph, meaning that there is some way to break the graph into two collections of vertices, where there are no edges that stay within one of these collections. If we look at the drawing of $K_{3,3}$ above we see that the right and left works as these collections for us. There is no edge that goes between two vertices on the right or two vertices on the left. This means that if we can make a plane drawing of $K_{3,3}$ each face must have at least four edges on its boundary. We know from our proof of K_5 's nonplanarity that each face in a graph must have at least three edges on its boundary. If any face were to have three edges then the cycle bounding the face would have exactly three vertices as well. Each pair of vertices within these three vertices would share an edge and therefore there is no way to split them up into two groups such that neither group has an edge within it. Additionally every edge borders no more than two faces, so from this we know that in any bipartite graph $2e \geq 4f$ or $e \geq 2f$.

Now $K_{3,3}$ has 6 vertices and 9 edges. $K_{3,3}$ is bipartite so the number of faces, $f \leq \frac{e}{2} = \frac{9}{2} = 4.5$. $K_{3,3}$ is connected, so if it were planar we would have Euler's formula giving us

$$2 = v - e + f = 6 - 9 + f \leq 6 - 9 + 4.5 = 1.5$$

and this is false as

$$2 \not\leq 1.5$$

Therefore $K_{3,3}$ is nonplanar. □

4.2 Some basic lemmas

This is the point in the paper where it is no longer written to be understood by everyone.

Lemma 4.6. *If a graph G has a nonplanar subgraph, then G is nonplanar.*

Lemma 4.7. *If a graph G is a subdivision of a nonplanar graph, then G is nonplanar.*

Definition 4.1 (minimal nonplanar graph). A nonplanar graph G is said to be a minimal nonplanar graph if for all edges $e \in G$ the removal of e would yield a graph.

Lemma 4.8. *For any nonplanar graph G there exists a minimal nonplanar subgraph.*

Proof. Let G be a nonplanar graph with n edges. The removal of all edges from G would yield a graph with only vertices which is trivially planar. This means there is some least number k such that there exists a nonplanar subgraph of G with k edges. Any such nonplanar subgraph of G , must be a minimal nonplanar graph as the removal of any edge from it would yield a subgraph of G with less than k edges. □

Lemma 4.9. *Every minimal nonplanar graph contains a connected minimal nonplanar subgraph.*

Proof. Let G be a minimal nonplanar graph. If G is connected then G contains itself as a minimal nonplanar subgraph. If G is not connected then it is composed of connected components. When embedding G within the plane, if every component was planar we could embed them into any space homeomorphic to \mathbb{R}^2 . For example each component could be embedded in an open disk with radius $1/2$ about a separate point in \mathbb{Z}^2 , and this would then be a plane drawing. This means that at least one component in G must be nonplanar. Now every other component in this graph may not have any edges as their removal would still yield a nonplanar graph (which breaks the definition of minimal nonplanar). This means that there is only one nonplanar component, and further if any edge from this component is to be removed then G would be planar. Now by . From this we get that this component is itself a connected minimal nonplanar subgraph of G . \square

Glossary

complete graph A graph with all possible edges included, the notation K_n is used to denote the complete with n vertices. 4

contraction A graph operation where one removes an edge by fusing two vertices together. 7

edge A connection or line between vertices in a graph. 2, 3, 4, 7

graph A set of vertices and edges. 1, 2, 3, 4, 7

multigraph Like a graph, but multiple edges may connect the same vertex pair, and an edge may connect a vertex to itself. 3, 7

set A collection of objects. 2

vertex A point or node in a graph. 2, 3, 4, 7

References