Foundations of Mathematics
Section 1.5 Problems

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$\mathbb{R}$

3(c) We are trying to prove $4 \not| x^2 \implies x \in \{\text{Odds}\}$, where $x \in \mathbb{Z}$. By contrapositive, $4 \not| x^2 \implies x \in \{\text{Odds}\}$ is equivalent to trying to prove $4|x^2 \impliedby x \in 2\mathbb{Z}$.

Let x be an even number. Thus it can be written as $2k$, where $k \in \mathbb{Z}$. Now

$$x^2 = 4k^2$$

and $4k^2$ is divisible by 4 by definition of divisibility. Thus $4 \not| x^2 \implies x \in \{\text{Odds}\}$.

5 Let $\mathfrak{C}$ be a circle with center at $(2, 4)$ and radius length of r, this will be referred to throughout problem 5.

(a) The point $(-1, 5)$ is distance d_1 from the center of $\mathfrak{C}$, and the point $(5, 1)$ is distance d_2 from the center of $\mathfrak{C}$.

$$\begin{aligned} d_1 &= \sqrt{(2-(-1))^2 + (4-5)^2} \\ &= \sqrt{3^2 + (-1)^2} \\ &= \sqrt{9+1} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned} d_2 &= \sqrt{(5-2)^2 + (1-4)^2} \\ &= \sqrt{3^2 + (-3)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

$$\sqrt{10} = d_1 \neq d_2 = 3\sqrt{2}$$

By the definition of circle, two points of of different distance from the centerpoint may not both lie upon it's edge.

(b) The distance from the line $y = x - 6$, which will herein be referred to as $\mathfrak{L}$, to the center point of $\mathfrak{C}$ at x is given by the function:

$$\begin{aligned} \mathfrak{D}(x) &= \sqrt{(2-x)^2 + (4-(x-6))^2} \\ &= \sqrt{(2-x)^2 + (10-x)^2} \\ &= \left((2-x)^2 + (10-x)^2\right)^{\frac{1}{2}} \end{aligned}$$

We can find $\mathfrak{D}(x)$'s minima by using calculus.

$$\begin{aligned}\mathfrak{D}'(x) &= \frac{1}{2}\cdot\left((2-x)^2+(10-x)^2\right)^{-\frac{1}{2}}\cdot(2\cdot(2-x)\cdot(-1)+2\cdot(10-x)\cdot(-1)) \\ &= \frac{2(x-6)}{\sqrt{(2-x)^2+(10-x)^2}}\end{aligned}$$

$\mathfrak{D}'(x)=0$ when $(2(x-6)=0)\wedge\left(\sqrt{(2-x)^2+(10-x)^2}\neq 0\right)$

$$\begin{aligned}2(x-6)&=0\\ x-6&=0\\ x&=6\end{aligned}$$

$$\begin{aligned}\sqrt{(2-6)^2+(10-6)^2} &= \sqrt{(-4)^2+4^2}\\ &= 4\sqrt{2}\\ &\neq 0\end{aligned}$$

Thus $\mathfrak{D}$'s only critical point is when $x=6$. We can tell $x=6$ is a minima for $\mathfrak{D}(x)$ if $\mathfrak{D}''(6)>0$.

$$\begin{aligned}\mathfrak{D}'(x) &= 2(x-6)\left((2-x)^2+(10-x)^2\right)^{-1/2}\\ \mathfrak{D}''(x) &= 2\left((2-x)^2+(10-x)^2\right)^{-1/2}-(x-6)\left((2-x)^2+(10-x)^2\right)^{-3/2}\cdot 2\,(x-12)\\ \mathfrak{D}''(6) &= 2\left((2-6)^2+(10-6)^2\right)^{-1/2}-(6-6)\left((2-6)^2+(10-6)^2\right)^{-3/2}\cdot 2\,(6-12)\\ &= 2\left((-4)^2+(4)^2\right)^{-1/2}-(0)\left((-4)^2+4^2\right)^{-3/2}\cdot 2\,(-6)\\ &= \frac{\sqrt{2}}{4}>0\end{aligned}$$

Thus the distance between $\mathfrak{C}$'s center point and $\mathfrak{L}$ is $\mathfrak{D}(6)$, which is equal to

$$\begin{aligned}\mathfrak{D}(6) &= \sqrt{(2-6)^2+(4-(6-6))^2}\\ &= \sqrt{(-4)^2+4^2}\\ &= \sqrt{16\cdot 2}\\ &= 4\sqrt{2}\end{aligned}$$

If $4\sqrt{2}>r$ then $\mathfrak{L}$ does not cross over $\mathfrak{C}$ because $\mathfrak{L}$ will never have a point as close or closer then r to $\mathfrak{C}$'s center point.

We can say that $\sqrt{2}>\frac{4}{3}$ because $\sqrt{2}^2=2>\frac{4}{3}^2=\frac{16}{9}$, thus $4\sqrt{2}>4\cdot\frac{4}{3}=\frac{16}{3}>\frac{15}{3}=5$. Thus $4\sqrt{2}>5$ so if $r=5$, $\mathfrak{L}$ will not intersect $\mathfrak{C}$.

(c) Claim: $(0,3)$ is not inside $\mathfrak{C}\implies(3,1)$ is not inside $\mathfrak{C}$. We proove this by contrapostive, so we are going to show that if $(3,1)$ is inside $\mathfrak{C}\implies(0,3)$ is inside $\mathfrak{C}$.

Proof: The point $(0,3)$ is of distance $\sqrt{(2-0)^2+(4-3)^2}$ from the center of $\mathfrak{C}$.

$$\begin{aligned}\sqrt{(2-0)^2+(4-3)^2} &= \sqrt{2^2+1^2}\\ &= \sqrt{4+1}\\ &= \sqrt{5}\end{aligned}$$

The point $(3,1)$ is of distance $\sqrt{(3-2)^2+(1-4)^2}$ from the center of $\mathfrak{C}$.

$$\begin{aligned}\sqrt{(3-2)^2+(1-4)^2} &= \sqrt{1^2+(-3)^2}\\ &= \sqrt{1+9}\\ &= \sqrt{10}\end{aligned}$$

For the point $(3,1)$ to be in $\mathfrak{C}$, $r > \sqrt{10}$, thus $(0,3)$ will always be contained within that circle as it is only $\sqrt{5}$ from the center of $\mathfrak{C}$.

10 Prove that $\sqrt{5} \notin \mathbb{Q}$. This will be shown by contradiction so we start by assuming $\sqrt{5} \in \mathbb{Q}$, which by definition of $\mathbb{Q}$ means that it can be written as $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_+$ and there are no common factors between a and b. [1] We also know, by definition of square root, that $\sqrt{5}^2 = 5$, thus $\frac{a}{b}^2 = 5$ and $a^2 = 5b^2$.

We also will show that for any integer x, if it is not divisible by some prime number d, then x^2 is also not divisible by d. We show this by saying that any value x can be written as $(-1)^b p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots p_v^{e_v}$, where

11 $b \in \{0,1\}, \forall_{p_k, k \in \mathbb{N}} (p_k \in \mathbb{P})$

[1] $\mathbb{Z}_+$ is the set of positive integers, ie: $\{1, 2, 3, \ldots\}$.