Combinatorics Homework Chapter 1

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February 20, 2017

Prob 1.2 We can refer to each square on any chess board uniquely with notation (x, y) where x is the number of squares to the right, and y is the number of squares to the left.[1] The problem states that the upper right hand corner or $(0, 0)$ is white, thus we know that the white squares can be defined as "The set of all squares (x, y) where $x + y$ is even."[2] Thus if X is on an odd row

[1]Thus $(0, 0)$ would be the upper right hand corner, $(0, 1)$ would be one square down from that, and $(1, 0)$ would be one square left of the upper right hand corner.

[2]This can be proved by defining black and white squares as the following

1. A black square is any square adjacent to a white square.
2. A white square is any square, either adjacent to a black square, or is $(0, 0)$.
3. A square, (a_x, a_y), is adjacent to another square, (b_x, b_y), iff $(b_x = a_x \pm 1 \wedge b_y = a_y) \vee (b_x = a_x \wedge b_y = a_y \pm 1)$

Let us also define a function $f(s) = s_x + s_y$, where $s = (s_x, s_y)$. Notice that for any square q, where $f(q) = n$, for any adjacent square to it, r, $f(r) = n \pm 1$. That means that for any q where $f(q)$ is odd, all $f(r)$ is even, where r is any adjacent square, and if $f(q)$ is even, then all $f(r)$ would have been odd.

Now to prove that $\forall_{x \in \{\text{white squares}\}} (f(x) \text{ is even})$ and $\forall_{x \in \{\text{black squares}\}} (f(x) \text{ is odd})$ we start by showing

$$f((0, 0)) = 0 \in \{\text{Evens}\} \tag{1}$$

and we know that (0,0) is a white square by definition. Now let $f((a, b)) = k$, where a and b are natural numbers (ie: in the set $\{0, 1, 2, \ldots\}$), thus

$$\begin{aligned} f((a+1, b)) &= a + 1 + b = k + 1 \\ f((a-1, b)) &= a - 1 + b = k - 1 \\ f((a, b+1)) &= a + b + 1 = k + 1 \\ f((a, b-1)) &= a + b - 1 = k - 1 \end{aligned} \tag{2}$$

so for adjacent square to (a, b), $(a, b)'$, $f((a, b)')$ can be written as $a + b \pm 1$ or $f((a, b)) \pm 1$. Any even number ± 1 is an odd number, and any odd number ± 1 is an even number. Now we can say that for any square X, any adjacent square to it X', is odd iff X is even, and is even iff X is odd. We also know by definitions 1 and 2 that X' is black if X is white and X' is white if X is black. We can now show by induction that if $f(X)$ is even, then X is a white square, and if $f(X)$ is odd, then X is a black square.