Topological Graph Theory

Benji Altman

May 11, 2019

Contents

1 Introduction

Topological graph theory is a field within topology, and therefore this paper is by no means meant to cover all of topological graph theory in any depth. This paper instead will first be concerned with a rather shallow overview of the field (section 2), followed by a more in-depth study of graph theory (section 3). The overview will mainly be focused on conveying a thorough understanding of what topological graph theory is, as well as briefly covering the history of the field. After the overview, section 3 of this paper will build up definitions from graph theory that will needed for the later, more rigorous parts of this paper. After that this paper will dive into Kuratowski's Theorem. It will start by defining K_5 and $K_{3,3}$ and showing that they are both nonplanar. Up to this point the paper is written to be accessible by anyone, no matter their level of math background. After discussing the nonplanarity of K_5 and $K_{3,3}$ this paper continues by thoroughly proving Kuratowski's theorem. The audience for this part is expected to have at least an undergraduate understanding of topology, and some exposure to graph theory. After Kuratowski's theorem, this paper finishes with a brief discussion of what lies beyond Kuratowski's theorem.

2 Overview

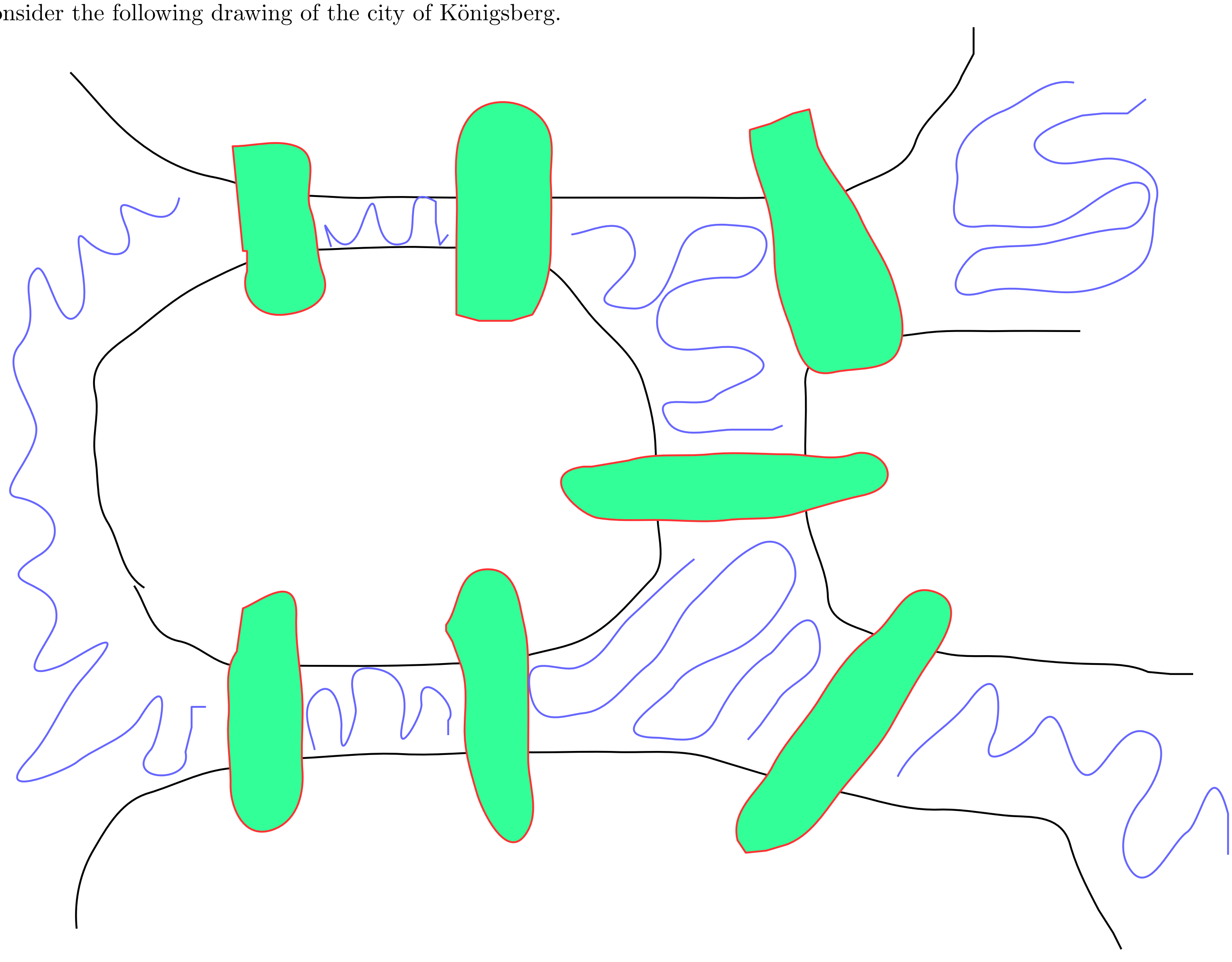
2.1 Graphs

Before we talk about topological graph theory one must first understand what a graph is. A graph is generally defined as a set of vertices combined with a set of edges between vertices. However, here it may be more useful to think about of a graph visually with a simple representation. Consider first a set of points. This may be thought of as just dots on a sheet of paper. Each of these points will be called a vertex. Then we may start drawing lines between vertices. Lines may cross over each other, and need not be straight. There is no requirement that all vertices have a line going to them. These lines will be called edges. We will simply insist that no edge connect a vertex to itself and that we do not have multiple edges between the same pair of vertices.

Once we have drawn this, we have a representation of a graph. If we were to move the vertices around on the paper but leave them with the same edges, we would be left with the same graph. That is to say, it doesn't matter where we put a vertex, the graph exists independently of the representation.

2.2 Königsberg, and its seven bridges

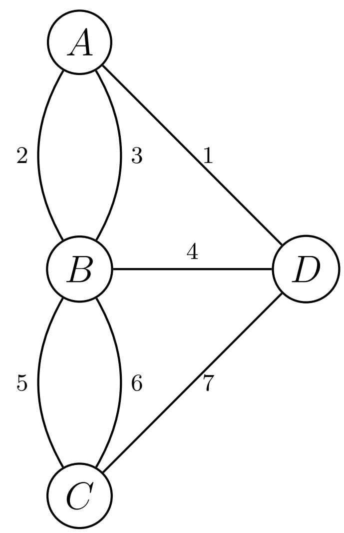
Consider the following drawing of the city of Königsberg.



Now the question is, if we get to choose where we start, can we go for a stroll and cross every bridge exactly once?

I first came across this question in an 8$^{\text{th}}$ grade geometry class. While undoubtedly an interesting problem, it is quite misleading to think of this as a geometry problem. Instead we will reduce it to a graph problem.

Let us start by thinking of every island as a vertex and every bridge as an edge. We find the following graph:



It is worth noting two things about the above diagram. First, that the labels on the vertices and edges are unrelated to the problem, but have been added simply to make referring to parts of the graph much easier. Second, that whatever the above image depicts, does not fit our definition of a graph.

Notice that edge 2 and edge 3 both connect vertex A to B, just as 5 and 6 do for vertices B and C. This is a strict violation of our definition for a graph. The issue, of course, then comes to what would one call this graph-like object where one is able to have any number of connections between any pair of vertices.

This is what we will call a multigraph. A multigraph is like a graph, but can have an edge that connects a vertex to itself and can have two vertices that share multiple edges. It is worth noting that any graph is a multigraph, so anything shown to be true for all multigraphs must be true for all graphs.

Now to solve the Königsberg problem we need to make one simple observation about how we walk. If we are to go to an island (or vertex) we must also leave that island unless it is the last island we go to. This means, that with the exception of the island we start on and end on, each island must have an even number of bridges connected to it.

We now translate the Königsberg problem into the language of multigraphs. The question becomes "If we pick the right vertex and start going across edges to other vertices, is it possible to cross every edge exactly once?" We can translate what we've discovered about islands needing an even number of bridges to say that we can only make this sort of journey if all but two vertices (a start and end vertex) have an even number of edges. Looking at our multigraph representation of the city of Königsberg we find that all four of the vertices have four edges. This means it is impossible to take a single walk and cross every bridge exactly once.

The seven bridges of Königsberg problem was solved by Euler in 1736.[1] In the history of mathematics this problem is of some significance as it is considered to be the beginning of graph theory as well as a sort of precursor to topology.[2]

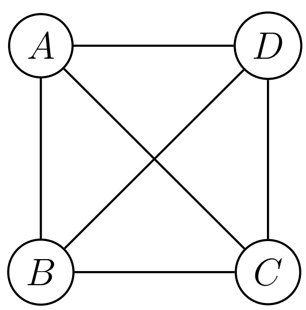
2.3 What is topological graph theory?

Topology is the area of math in which we deal with different spaces and their fundamental properties. Much of topological graph theory has to do with the relationship between graphs and spaces. For example this paper focuses on Kuratowski's theorem, which deals with which graphs are able to be drawn in a plane. This paper uses Kuratowski's theorem as an introduction to topological graph theory.

3 Graph Theory Background

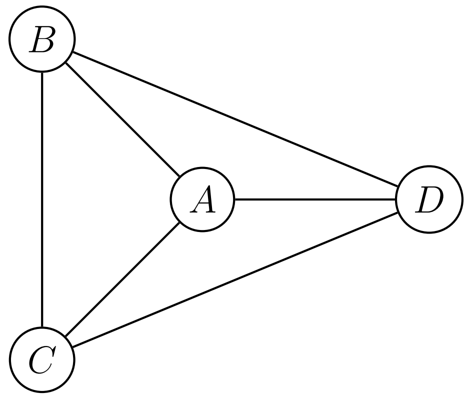
3.1 Planar Graphs

Consider the following graph.



This graph is called a complete graph as every vertex is connected to every other vertex by an edge. In fact this graph in particular is called K_4 as it is the complete graph on four vertices. We would like to find out if we can draw this graph without having any lines crossing. This is in fact possible, and for any skeptics who may be reading this the below is K_4 without any edges intersecting.

[1] https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg
[2] https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg



So if this graph can be drawn without any intersection, can any graph be drawn without intersections? If some can and some can't, how do we tell which can be drawn and which can not? This is the question at the center of Kuratowski's theorem. The rest of this section will serve two purposes. The first purpose of this section is to build up familiarity with graph theory for those without any previous exposure. The second purpose of this section is to familiarize the reader with definitions and terms from graph theory that will be needed later in our proof of Kuratowski's theorem.

3.2 Graph Drawing

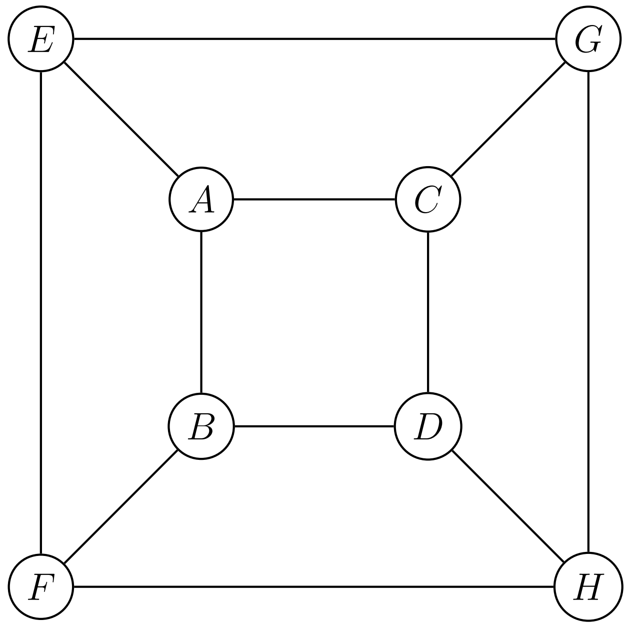
A graph drawing is exactly what it sounds like. We've already seen drawings of graphs like the one for Königsberg and two drawings of K_4. The fact that there can be two distinct drawings of K_4 shows that there can be multiple distinct drawings for the same graph. Simply thinking of a drawing as a drawing on a sheet of paper will be good enough until section 4.2 where we formally define a graph drawing.

If a drawing has no edges intersecting, it is said to be a plane drawing. If there exists such a drawing for a particular graph, then that graph is a planar graph.

3.3 Face

Faces are a bit of an odd property here. Fundamentally faces are a property of plane drawings of a graph, not of the graph itself. A face is defined as a connected space that contains no edges or vertices and itself is bounded by edges and vertices.

If you think back to high-school geometry and cubes you may recall that a each of the corners is a vertex, the lines connecting vertices are edges, and the area between the edges are faces. In a graph we have vertices connected by edges, and when we draw them there are empty areas enclosed by edges and vertices. For example the following would correspond with a cube:



We can start going through the faces of the above graph drawing. We clearly see that $ABDC$ bounds an area that contains to vertices or edges. This means that this area is a face. We can see that $EABF$, $EACG$, $GCDH$, and $FBDH$ all have the same property on their interior. There is however one final remaining face. This face is bordered by $EFHG$, and is called the outer face. This is all the infinite space outside the bounds of the graph. Every plane drawing of a graph will have an outer face.

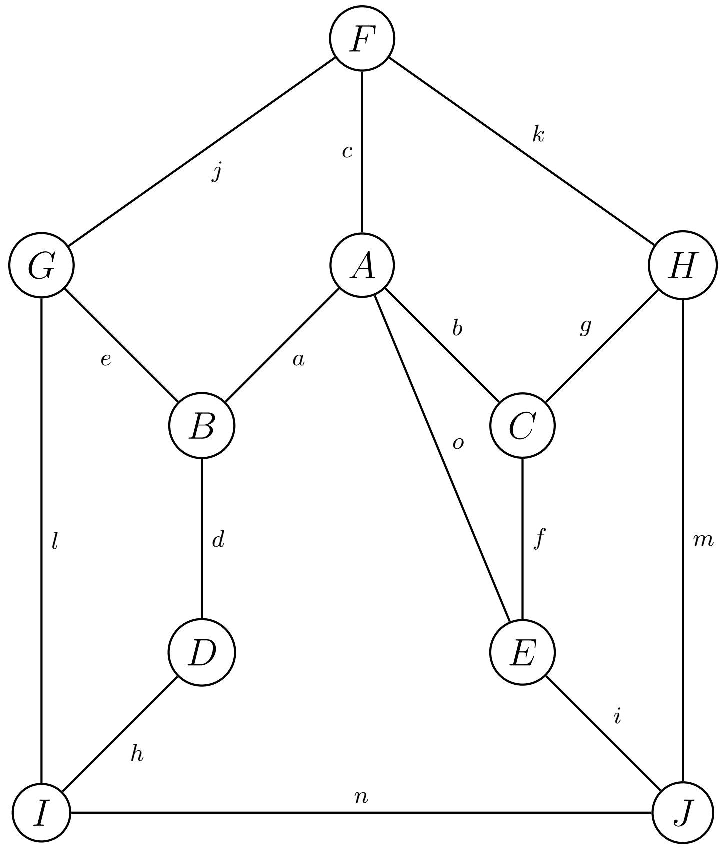
3.4 Walks

Now moving onto a more traditional graph property we have the concept of a walk. A walk can be thought of as if you were standing on some vertex v_0 and just kept following edges to neighboring vertices some number of times. We define it as follows.

Definition 3.1 (walk)**.** A walk is a finite sequence of vertices and edges with the following properties.

- Every odd element in the sequence is a vertex.
- Every even element in the sequence is an edge that connects the vertices before and after it.
- A walk always begins and ends with a vertex.

For example consider the following graph.



Now the sequence $BeGjF$ is a walk, as we can think of starting at vertex B, walking along edge e to vertex G, then along edge j to vertex F and finishing there. The sequence $AaBaA$ is also a walk as we could start at vertex A, go along edge a to vertex B, then back along edge a to vertex A where we finish. The sequence $CfEi$ however isn't a walk, as it doesn't end with a vertex.

3.4.1 Path

Definition 3.2 (path)**.** A path is a walk in which no vertex or edge can appear twice.

For example with the above graph $BeGjF$ is not only a walk, but also a path. However, the sequence $AaBaA$, while it is a walk, is not a path as both edge a and vertex A are used twice.

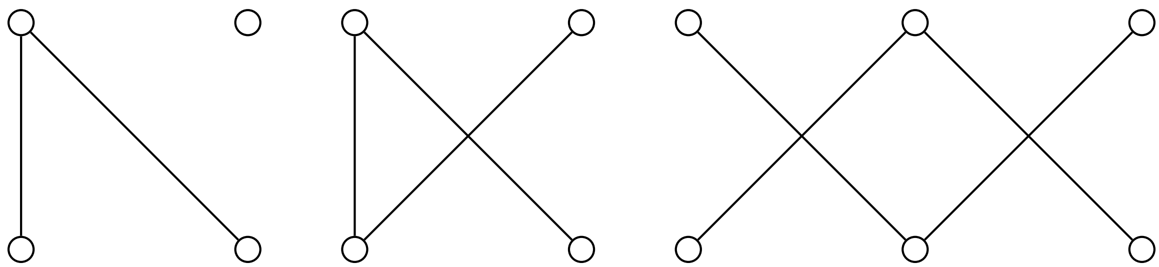
3.4.2 Cycle

Definition 3.3 (cycle)**.** A cycle is a walk, of length greater than one, in which no vertex or edge can appear twice, with the exception of the first and last vertex which must be the same.

The definition of a cycle is quite similar to that of a path. It effectively is a path that loops back and ends where it started. The stipulation that a cycle must be "of length greater than one" simply means that the walk A (just the vertex A and nothing else) is not a cycle. Examples of a cycle would include $AbCgHkFcA$, $AbCfEoA$, and $FkHmJnIlGjF$. It is important to note that something like $FcAcF$ is not a cycle as it includes the edge c twice.

3.5 Connected

A graph is said to be connected if for any pair of vertices, (a, b) there is a walk from a to b. So if we consider the following graphs



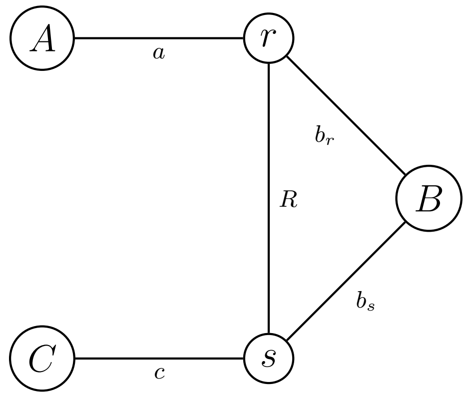
We find that only the graph in the middle is connected. For the graph on the right consider any vertex on the bottom, there is no path to the vertex above it. The graph on the left is has the upper right vertex isolated from the rest of the graph.

Any graph, connected or not, can be broken into connected components. To do this we simply take a vertex and every other vertex connected to it and call that one component, and then repeat with a vertex not in that component. This sort of breaking apart is nice as often we will prove things about connected graphs that are true about all graphs. For example if all components of a graph are planar then the entire graph must be planar, this will be proven in section 4.3 and it allows us to only deal with connected graphs.

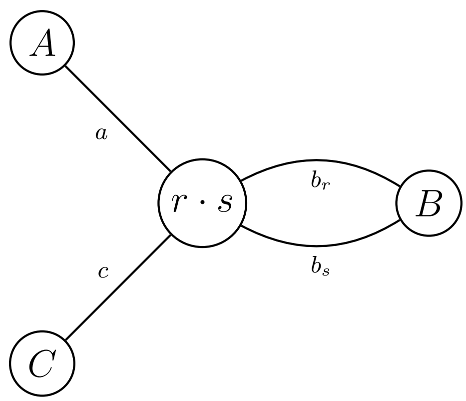
3.6 Contraction

3.6.1 Edge contraction

This is not a property, but rather an operation or action that we perform on a graph. Let us consider the following graph.



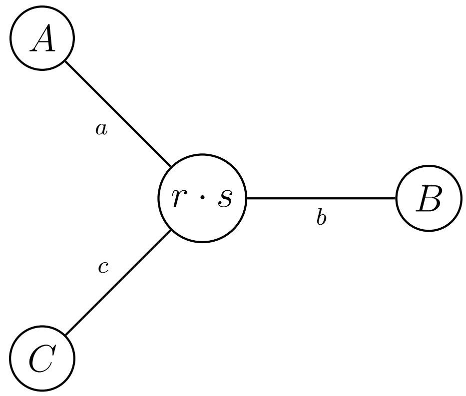
We wish to perform a contraction on edge R. To be clear all graph contractions are on edges. So we will make a new vertex, $r \cdot s$ which has all the edges of r and all the edges of s except the edge we are contracting across. In this case the edge we are contracting across is R and thus we get the following multigraph.



Often we may prefer to have the end result be a graph rather than a multigraph, which brings us to our next kind of contraction

3.6.2 Simple edge contraction

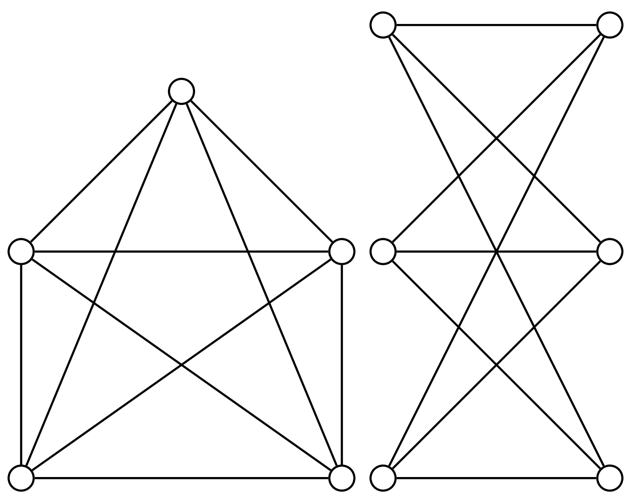
We can take any multigraph produced by an edge contraction and for any edge pairs that go between the same vertices, simply replace it with a single edge. This will reduce the multigraph back to a graph, and from the previous example the result of a simple edge contraction on R will give the below graph.



4 Kuratowski's Theorem

Theorem 4.1 (Kuratowski)**.** *A graph G is nonplanar if and only if it contains a subgraph that is a subdivision of $K_{3,3}$ or K_5*

Here K_5 refers to the complete graph on 5 vertices, and $K_{3,3}$ is the complete bipartite graph with three vertices in both partitions. Drawings of both theses graphs are below with (K_5 on left and $K_{3,3}$ on right).



4.1 K_5 and $K_{3,3}$

Our first step is proving that $K_{3,3}$ and K_5 are nonplanar. To do this we are going to use the following theorem.

Theorem 4.2 (Euler's formula on planar multigraphs)**.** *For any plane drawing of a multigraph (with the exception of a multigraph with no vertices), we have*

$$v - e + f = 2$$

, where v is the number of vertices in the multigraph, e is the number of edges in the multigraph, and f is the number of faces in the plane drawing of the multigraph.

Proof. [3] First consider a connected multigraph with no edges, as this is connected we may only have a single vertex. This produces a single and no edges so we find

$$v - e + f = 1 - 0 + 1 = 2$$

Now that we know that for 0 edges this rule fits, then either the rule $(v - e + f - 2)$ is true no matter the number of edges, or there is some number at which point this rule breaks, and $v - e + f \neq 2$. If there is a number that breaks this rule, then there must be a lowest number (let's call it k) that breaks this rule. For any multigraph with k edges, choose any edge e in the graph.

- If e connects a vertex to itself it is a loop and its removal will result in the loss of one edge and one face (each loop creates a face[4]). The resulting multigraph has less than k edges and thus we know that for it

$$\begin{aligned} 2 &= v - e + f \\ &= v - (e + 1) + (f + 1) \end{aligned}$$

  and as the original graph had one more edge and one more face it too would fit this rule.

- If e is not a loop we may perform an edge contraction on it and this will reduce both the number of edges and the number of vertices by one. Again this gives us a multigraph with less than k edges so we know that for it

$$\begin{aligned} 2 &= v - e + f \\ &= (v + 1) - (e + 1) + f \end{aligned}$$

  and as the original graph had one more edge and one more vertex it to would fit this rule.

From this we know that there can not possibly be some k, and thus all multigraphs, regardless of the number of edges must fit this rule. □

This leads to a nice corollary, that will be helpful when talking about graphs.

[3] This proof comes from: `https://www.ics.uci.edu/~eppstein/junkyard/euler/iedge.html`.
[4] this is due to the Jordan curve theorem

Corallary 4.3. Any plane drawing of a specific multigraph G has the same number of faces.

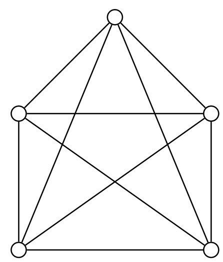
Proof. Any planar multigraph can be broken into connected components. Each component of a planar multigraph is a connected planar multigraph and thus any planar drawing of it fits the rule $v - e + f = 2$. Now any plane drawing we make of a planar multigraph will be made of parts that already have a constant number of faces.[5]

□

This means that given any planar multigraph G, we can talk about the number of faces G has without referring to any drawing of G, as all plane drawings will have the same number of faces.

Now using this we can prove that $K_{3,3}$ and K_5 are nonplanar.

Theorem 4.4. *K_5 is nonplanar.*



K_5

Proof. [6] K_5 is a graph[7], and as such an edge can not be a loop and two vertices can share at most one edge. This means that in any plane drawing of a graph, there must be at least 3 edges bordering every face.

- To only have one edge bordering a face would require that the edge be a loop (which we can't have in a graph).
- To only have only two edges bordering a face would require that some pair of vertices share more than one edge (which we can't have in a graph).

Additionally every edge borders no more than two faces, so from this we know that in any graph $2e \geq 3f$ or $\frac{2}{3}e \geq f$. Now K_5 has 10 edges and 5 vertices, so we know that

$$\frac{2}{3}10 = \frac{20}{3} = 6.\bar{6} \geq f$$

[5]This is not a complete proof, to complete it you simply make an induction on the number of components drawn and realize each time you add a component you add exactly $f_i - 1$ new faces (if f_i is the number of faces in the i^{th} component drawn).

[6]This proof comes from `http://www.math.cmu.edu/~mradclif/teaching/228F16/Kuratowski.pdf`.

[7]Recall that all graphs are multigraphs, but not all multigraphs are graphs.

We know that K_5 is a connected graph (and thus also a connected multigraph) so if K_5 were planar then we would have

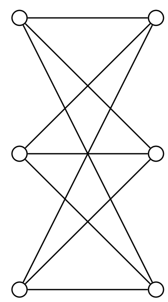
$$2 = v - e + f = 5 - 10 + f \leq 5 - 10 + 6.\bar{6} = 1.\bar{6}$$

which is clearly false as

$$2 \not\leq 1.\bar{6}$$

Therefore K_5 must not be planar. □

Theorem 4.5. *$K_{3,3}$ is nonplanar.*



$K_{3,3}$

Proof. [8] $K_{3,3}$ is a bipartite graph, meaning that there is some way to break the graph into two collections of vertices, where there are no edges that stay within one of these collections. If we look at the drawing of $K_{3,3}$ above we see that the right and left works as these collections for us. There is no edge that goes between two vertices on the right or two vertices on the left. This means that if we can make a plane drawing of $K_{3,3}$ each face must have at least four edges on its boundary. We know from our proof of K_5's nonplanarity that each face in a graph must have at least three edges on its boundary. If any face were to have three edges then the cycle bounding the face would have exactly three vertices as well. Each pair of vertices within these three vertices would share an edge and therefore there is no way to split them up into two groups such that neither group has an edge within it. Additionally every edge borders no more than two faces, so from this we know that in any bipartite graph $2e \geq 4f$ or $e \geq 2f$.

Now $K_{3,3}$ has 6 vertices and 9 edges. $K_{3,3}$ is bipartite so the number of faces, $f \leq \frac{e}{2} = \frac{9}{2} = 4.5$. $K_{3,3}$ is connected, so if it were planar we would have Euler's formula giving us

$$2 = v - e + f = 6 - 9 + f \leq 6 - 9 + 4.5 = 1.5$$

[8] This proof comes from `http://www.math.cmu.edu/~mradclif/teaching/228F16/Kuratowski.pdf`.

and this is false as

$$2 \not\leq 1.5$$

Therefore $K_{3,3}$ is nonplanar. □

4.2 Drawings

Up to this point the paper has been writing such that anyone with a basic mathematical competency and interest could understand and work through the content. From this point on the paper moves a bit faster and assumes knowledge of basic mathematical notation and more experience with math.

The purpose of the next couple sections is to build up the mathematical machinery that we need for our proof of Kuratowski's theorem (theorem 4.1). In this section we will be proving a slew of lemmas and theorems, as well as provide a couple definitions, all of which will be needed in our final proof.

We start by going back and formalizing our idea of a graph drawing. All this is meant to do is use mathematical language to define what we have already been thinking of as a graph drawing.

Definition 4.1 (Drawing)**.** A drawing is a subset, D, of some topological space S. For plane drawings this space will be $\mathbb{R}^2$ with its standard topology.

Definition 4.2 (Graph Drawing)**.** Given a graph G, with vertex set V and edge set E, a drawing D is said to be a drawing of G if and only if the following criteria are met:

- There exists an injective function $v : V \to D$.
- If there is an edge $e \in E$ between vertices a and b then there is a continuous injective function $\hat{e} : [0, 1] \to S$ with $\hat{e}(0) = v(a)$ and $\hat{e}(1) = v(b)$ and the image of e is contained within D.[9]
- For any vertex $a \in V$ and edge $e \in E$ and any real number $x \in (0, 1)$, $\hat{e}(x) \neq v(a)$.
- For any two edges $e \neq f$, $\hat{e}(x) \neq \hat{f}(y)$ for all $x, y \in (0, 1)$.
- For all $d \in D$ there exists either a vertex $a \in V$ such that $v(a) = d$ or there exists an edge $e \in E$ such that d is in the image of $\hat{e}$.

4.3 Subgraph lemmas

Lemma 4.6. *If a graph G can be drawn in a space, S, then so can any subgraph of G.*

[9] In a multigraph drawing we may have $a = b$ and thus $\hat{e}(0) = \hat{e}(1)$ making $\hat{e}$ not injective, however it must still be injective in $[0, 1)$.

Proof. Let D be a graph drawing of G in S. Simply remove all edges and vertices from D that H does not have and we have a new drawing $D' \subset D$ that is a graph drawing of H in S. $\square$

Definition 4.3 (contactable)**.** A graph G is said to be contactable to a graph H if there is some series of simple edge contractions that when performed on G produce H.

Lemma 4.7. *If a graph G with edge e is planar then so is the graph produced by performing a simple edge contraction on e.*

Proof. Let G be a planar graph with edge e between x and y. et G' be the graph produced by an edge contraction on e. We are going to prove this using induction on the number of edges of y.

Statement: G' is planar and there is a plane drawing of G' where all vertices, v, that border a face that y borders in G also a face that $x \cdot y$ borders in G' ($x \cdot y$ is the vertex produced by contracting across e).

Base case: Suppose y only has edge e. Clearly $G' \subset G$ so by lemma 4.6 G' must be planar. Additionally as y has only one edge it can not possibly lie upon any cycles in G, this means that y can only border at most 1 face, and as it has an edge to x it must share this face with x. We may draw G' simply be removing e and y from any drawing of G and relabeling x as $x \cdot y$. Clearly then the statement above must be true when y has only one edge.

Inductive step: Let $z \neq x$ be a vertex sharing edge $\bar{e}$ with y. Assume that the statement is true in $G \setminus \{\bar{e}\}$. Now perform a contraction on e in $G \setminus \{\bar{e}\}$ to produce H. $x \cdot y$ in H must border a face $\bar{f}$ that z borders so we may add in edge $\bar{e}$ to H in face $\bar{f}$ and produce a plane drawing of G'. Further let v be a vertex bordering a face f in G that y also borders. We then know that there is a face f' in H that borders both v and $x \cdot y$. If $f' \neq \bar{f}$ then clearly $x \cdot y$ and v both boarder f' in G' as well as H. If $f' = \bar{f}$ then f' is split into two parts by $\bar{e}$ in G', both of which are bordered by $x \cdot y$. Therefore our statement must be true for G.

Now by induction we have proven the statement is true on all graphs and therefore our lemma is proved. $\square$

Corallary 4.8. If a graph G is planar, and if G is contactable to a graph H, then H must also be planar.

Proof. Let $c_1, c_2, \ldots, c_n$ be the sequence of contractions on G that produce H. Let H_i be the graph produced after the first i contractions on G, so $H_i = G$ and $H_n = H$. If G is planar then clearly H_0 is planar. If H_i is planar, then by the above lemma clearly H_{i+1} is planar. Now by induction, if G is planar H must be planar. $\square$

Lemma 4.9. *If a graph G has a subgraph that is contactable to H as a subgraph, then graph G' that is contactable to G also contains a subgraph that is contactable to H.*

Proof. Let H be a graph. Let H' be contactable to H. Let G be a graph that contains H' as a subgraph. Let G' be a graph that is contactable to G. There is some sequence of simple edge contractions $h_1, h_2, \dots, h_\ell$ that produce H from H' as is there some sequence of simple edge contractions $g_1, g_2, \dots, g_m$ that produce G from G'. If we reverse these sequences we get sequences of inverse simple edge contractions that take produce H' from H and G' from G. Each inverse simple edge contraction will be done on a single vertex (and turn it into two vertices that share an edge). Let $f_i = h_{\ell-i+1}$ and $e_i = g_{i-m+1}$ so that they are the reversed sequences of g and h. Take the subsequence, $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ that consists of all the inverse simple edge contractions that act on a vertex within the subgraph H' of G. Now we can produce H'' as the graph produced by performing the following inverse simple edge contractions on H

$$f_1, f_2, \dots, f_\ell, e_{i_1}, e_{i_2}, \dots, e_{i_n}$$

Clearly now H'' is contactable to H and H'' must be a subgraph of G'. □

Lemma 4.10. *If a graph G is nonplanar than at least one of its connected components is nonplanar.*

Proof. Suppose there exists a nonplanar graph G for which all of its components are planar. If each component is planar we may also make a drawing of it in any space homeomorphic to $\mathbb{R}^2$. Simply embed each component in a separate open epsilon ball about an integer point, with $\varepsilon \leq \frac{1}{2}$, and we find a contradiction with a planar embedding of G. □

4.4 The Jordan Curve Theorem

Theorem 4.11 (Jordan curve theorem)**.** *Let C be a Jordan curve in the plane $\mathbb{R}^2$. Then its complement, $\mathbb{R}^2 \setminus C$, consists of exactly two connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior), and the curve C is the boundary of each component.*[10]

Definition 4.4 (Jordan curve)**.** A Jordan curve is a curve, $f : [0,1] \to \mathbb{R}^2$, with the following properties:

- f is continuous
- f is injective on the interval $[0,1)$
- $f(0) = f(1)$

Lemma 4.12. *For any plane drawing of a graph, G, containing cycle C, C is a Jordan curve in the drawing.*

[10] This statement of the Jordan curve theorem comes from `https://en.wikipedia.org/wiki/Jordan_curve_theorem#Definitions_and_the_statement_of_the_Jordan_theorem`.

Proof. Consider a plane drawing of a graph G with cycle C. We may write out C as $v_1e_1v_2e_2 \dots v_ke_kv_1$. We know that each of these e_is have a corresponding curve $\hat{e}_i$ that starts at the end of the previous ($\hat{e}_{i-1}$) curve and ends at the beginning of the next ($\hat{e}_{i+1}$) curve. This means we can construct a new continuous curve simply by stringing each of these together and then resize the domain. What we end with will then be a curve $f : [0,1] \to \mathbb{R}^2$ that is continuous. We will also have $f(0) = f(1)$ as v_1 as e_1 starts at v_1 and e_k ends at v_1. We will also have f injective on $[0, 1)$ as we know that edges do not overlap each other (this is formalized in the fourth bullet point of our definition for a drawing). This then means that our cycle creates a Jordan curve. □

4.5 Connectivity lemmas

Lemma 4.13. *For any face, F, in a plane drawing, D, of a graph G, there is a corresponding drawing plane drawing of G, D', where all edges and vertices that border F in D, now border the outer face.*

Proof. Let G be a graph with face F in plane drawing D. Now let f^{-1} be a stereographic projection from the punctured sphere onto the plane. Therefore $f(D)$ is a drawing of G on the punctured sphere. If $f(D)$ is a drawing of G on the punctured sphere then it must also be a drawing of G on the sphere. Now we pick a point on the interior of F and puncture the sphere there rather than where our original puncture was.[11] We may now use a sterographic projection, g to map $f(D)$ from the new punctured sphere back to $\mathbb{R}^2$ creating $D' = g(f(D))$. We know that in D' the face that corresponds to F is the outer face as it contained the punctured point in $g^{-1}(D')$. □

Definition 4.5 (cut set)**.** In a graph G, any set of vertices, whose removal would yield a disconnected graph is called a cut set.

Definition 4.6 (k connected)**.** A graph, G, is k connected if its minimal cut set is of size k. The complete graph on n vertices is defined to be $n - 1$ connected.

Lemma 4.14. *For all non-empty graphs, G, there exists exactly one number $k \in \mathbb{N}$ such that G is k connected.*

Proof. If a graph is a complete graph then two no vertices a and b can ever be separated by a cut set as they will always share an edge. Therefore there are no cut sets in a complete graph.

[11]There is a problem in this proof. How do we know that F has an interior? Simply put if we have a finite number of vertices and never use any space filling curves for edges we will never fill space. We know that we are never forced to use a space filling curve as for any space filling curve c we may create a new curve c' that connects the same end points, stays within c's image, and is not space filling.

If a graph is not a complete graph, then there exists some vertex pair a, b such that there is no edge between a and b. Let S be the set of all vertices in our graph except a and b. It is clear that S must be a cut set of G. Therefore G must also have a minimal cut set. □

Definition 4.7 (neighborhood)**.** The neighborhood of a vertex v in a graph G is the set of vertices that are adjaacent to v. This is denoted by $N(v)$. In a graph (not multigraph) we will never have $v \in N(v)$ as v can not have an edge to itself.

Definition 4.8 (disjoint paths)**.** A set of paths, S, is disjoint if no two paths share a vertex. A path $p \in S$ is said to be disjoint from the the other paths in S if S is disjoint.

Definition 4.9 (internally disjoint paths)**.** A set of paths, S, is internally disjoint if no to paths share an internal vertex. That is no two paths, $p_1 \neq p_2 \in S$ share a vertex other than their first or last vertex. A path $p \in S$ is said to be internally disjoint from the other paths in S if S is internally disjoint.

Theorem 4.15 (Menger's theorem)**.** *A non-empty graph is n connected if and only if among all pairs of vertices $a \neq b$ in G the least number of internally disjoint paths from a to b is n.*

Proof. [12] This is trivially true on complete graphs, so for the rest of this proof we will only be considering graphs that are not complete.

Let A and B be vertex sets in a finite graph G. Define an AB path as any path that starts in A and has no other vertices in A, and ends in B and has no other vertices in B. Notice that if $v \in A \cap B$ then the path v is an AB path. Define an AB separator as any vertex set, S, for which all AB paths contain at least one member of S. Define an AB connector a set of disjoint AB paths.

We will show by induction that for any pair of vertex sets A, B in a graph G, the maximum size of an AB connector is also the minimum size of an AB separator. First however note that the size of the minimum AB separator always forms an upper bound for the size of an AB connector, as every path in an AB connector must pass through one element of every AB separator. Likewise the maximum size of an AB connector forms a lower bound for the size of an AB separator.

Base case: Suppose G has no edges, therefore all paths in G are single vertex paths. For all $v \in A \cap B$ the path v is an AB path, and any other path will either not start in A or not end in B. This means that any AB separator contains $A \cap B$ and that our maximal AB connector is composed of single vertex paths for each vertex in $A \cap B$. Therefore our statement must be true when G has no edges.

Inductive case: Suppose H is a graph with $k > 1$ edges. Further suppose that for any sets pair of vertex sets A, B in a graph with $k - 1$ edges, the size of the maximum AB connector is also the size of the minimum

[12] This proof comes from `https://en.wikipedia.org/wiki/Menger%27s_theorem#Other_proofs`.

AB separator. Choose vertex sets A and B within H and suppose the minimum size of an AB separator is n. Choose an edge $e \in H$ and consider $H \setminus \{e\}$. Any AB separator in H is also an AB separator in $H \setminus \{e\}$, so the size of the minimum AB separator in $H \setminus \{e\}$ is at most n.

If $H \setminus \{e\}$ has a minimum AB separator of size n then $H \setminus \{e\}$ must have a maximum AB connector of size n. This AB connector is also a AB connector of size n in H. Therefore H has a maximum AB connector of size n.

If $H \setminus \{e\}$ has a minimum AB separator of size less than n, then let S be such an AB separator. Any AB path in H must contain a vertex in S or the edge e. Now we know that $|S| = n - 1$ as S combined with either vertex of e is an AB separator in H. We know there exists some AB path in H that contains edge e and no vertices in S as otherwise S would be an AB separator in H. Along this path let us call the earlier vertex of e, e_a and the later vertex e_b. In $(H \setminus \{e\}) \setminus S$ we have a path from A to e_a and a path from e_b to B. This also means there is no path from e_a to B or from A to e_b in $(H \setminus \{e\}) \setminus S$ as otherwise S would not be an AB separator in $H \setminus \{e\}$. Now we let $S_a = S \cup \{e_a\}$ and $S_b = S \cup \{e_b\}$. S_a and S_b are both vertex sets within $H \setminus \{e\}$ so the minimal AS_a separators is the same size as the maximal AS_a connector and the minimal S_bB separator is the same size as the maximal S_bB connector by our inductive hypothesis. Now let T_a be a minimal AS_a separator in $H \setminus \{e\}$ and T_b be a minimal S_bB separator in $H \setminus \{e\}$. We know that T_a is a minimal AS_a separator in H as any path through e from A must have already passed through S_a. Likewise we know that T_b is a minimal S_bB separator in H for the same reason. As T_a is a AS_a separator in H and S_a is a AB separator in H then any AB path must contain an element of S_1 and also contains an element of T_a. Likewise as T_b is a S_bB separator in H and S_b is a AB separator in H then any AB contains an element of S_2 and therefore also contains an element of T_b. From this we know that both T_a and T_b must be at least size n. This then means that the minimal AS_a connector in $H \setminus \{e\}$ is at least size n, so it must be exactly n as there are only n elements in S_a. Likewise the maximal S_bB connector must be of size n. Finally we can construct an AB connector in H of size n by taking an AS_a path starting at A going to S_a, if we are at e_a then take edge e to e_b then we are in S_b and we can continue down a S_bB path. There can not be a larger AB connector as we have an AB separator of size n and this forms an upper bound like before.

Now by induction we know that for any graph G, and any pair of vertex sets A, B within G, the maximal AB connector is the same size as the minimal AB separator.

Now let G be a graph with vertices $a \neq b$. Further let $A = N(a)$ and $B = N(b)$.

$$(\implies)$$

First let us suppose G is n connected, meaning the smallest cut set in G is n. If a is not adjacent to b

then any AB separator serves as a cut set in G, separating a from b. It follows that the size of the smallest AB separator is at least n, therefore there is an AB connector with at least n paths. Therefore there are at least n internally disjoint paths from a to b.

Now let us suppose a is adjacent to b and further let us suppose that there can be an AB separator of size $k < n$. First off we know that $|A| \geq n$ and $|B| \geq n$ as otherwise we would have one of the following:

1. A or B would serve as a cut set of size less than n separating a or b from the rest of G. This clearly is a contradiction.

2. A and B both contain all vertices except a and b respectively. If G has m vertices then G can be at most $m-1$ connected and therefore we still have $|A| = m-1 = |B|$.

If S is a minimal AB separator of size $k < n$ then there are no AB paths that do not contain a vertex in S. Let $\bar{a} \in A$ and $\bar{b} \in B$ and let p be a path from $\bar{a}$ to $\bar{b}$. p must contain a final vertex, p_i, that is within A. p must contain an earliest vertex, p_j, that is within B and after p_i. If we consider the subpath of p simply going from p_i to p_j we have an AB path. This means that any path from $\bar{a}$ to $\bar{b}$ must pass through S, this means that if $\bar{a}$ is not adjacent to $\bar{b}$ and $\bar{a} \neq \bar{b}$ S is a cut set separating $\bar{a}$ from $\bar{b}$. Therefore as $|S| = k < n$ we know that all for all $\bar{a} \in A$ and all $\bar{b} \in B$, either $\bar{a} = \bar{b}$ or $\bar{a}$ is adjacent to $\bar{b}$. Without loss of generality we can suppose that $|A| \leq |B|$. Now every vertex in $A \setminus B$ has an AB path to every vertex in $B \setminus A$ as they are adjacent. Now for every vertex $a' \in A$ we have a mutually distinct AB path from a' to some vertex in B. This is because either $a' \in A \cap B$ and it is already an AB path or a' is adjacent to all $b' \in B$ and simply select one that isn't used by any other AB path. We know that $|A| \geq n$, so we have a contradiction and the minimal AB separator must still be at least size n. We then have at least one AB connector of at least size n, and therefore there are at least n internally disjoint paths from a to b.

Further we know there is a cut set of size n in G. Therefore there must be some vertex pair α, β that are separated by such a cut set of size n. This means that there is a minimal $N(\alpha)N(\beta)$ separator of size n. Therefore the maximal $N(\alpha)N(\beta)$ connector is size n. Therefore if G is n connected, then among all pairs of vertices $a \neq b$ in G the least number of internally disjoint paths from a to b is n.

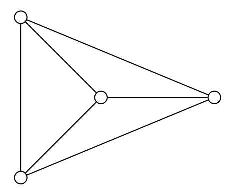
$$(\Longleftarrow)$$

Now let us suppose G is not n connected, then the graph must be $k \neq n$ connected. By what we have shown above we know that among all the vertex pairs $a \neq b$ in G the least number of internally disjoint paths from a to b is $k \neq n$. $\square$

Definition 4.10 (minimal nonplanar graph)**.** A nonplanar graph G is said to be a minimal nonplanar graph if every proper subgraph of G is planar.

Lemma 4.16. *Any connected minimal nonplanar graph is at least 2 connected.*

Proof. [13] First consider the complete graph. We know that K_4 is planar (see below), therefore by lemma 4.6 all subsets of K_4, are planar. Any complete graph K_i with $i < 4$ is a subgraph of K_4 (simply remove any $4 - i$ vertices from K_4) so a nonplanar complete graph must be at least 4 connected.



Now let G be a minimal nonplanar graph that is not a complete graph. By lemma 4.10 G contains connected component, G', that is nonplanar. We know $G' \subset G$ and that all proper subsets of G are planar, therefore $G' = G$. This means that G' is the only component in G so G must be at least 1 connected.

If G is 1 connected then there exists some vertex v for which $G \setminus \{v\}$ into at least two components. Let H be one such component and H' be the graph produced by adding v back into H (H' is v with all of H and all edges between H and v). Now H' is a proper subgraph of G so it must be planar. We also know that $G \setminus H$ is a proper subgraph of G and therefore must also be planar. By lemma 4.13 we may draw both $G \setminus H$ and H' such that v borders the outer face. We may split $\mathbb{R}^2$ into two disjoint subsets

$$R^- = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x < 0\right\}$$
$$R^+ = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x > 0\right\}$$

then draw H' in R^- with the exception of putting v at $(0, 0)$ and then draw $G \setminus H$ in R^+ with the exception of putting v at $(0, 0)$. This then yields a plane drawing of G, so G must be at least 2 connected. $\square$

Definition 4.11 (lobe)**.** Let G be a graph with cut set S. Further let $C_1, C_2, \ldots C_n$ be the connected components of $G \setminus S$. A lobe L_i of S is then what we would get if we took the subset of G containing all of C_i and S. This can more formally be written as

$$L_i = G \setminus \left(\bigcup_{j \neq i} C_j \right)$$

[13] This proof comes from `http://math.uchicago.edu/~may/REU2017/REUPapers/Xu,Yifan.pdf`

Lemma 4.17. *Let G be a nonplanar 2 connected graph with minimal cut set $S = \{x, y\}$. Further if L is a lobe of S let L^+ be the graph of L with an edge added between x and y. There exists some lobe L of S for which L^+ is nonplanar.*

Proof. [14] Let G be a graph with lobes $L_1, L_2, \ldots, L_n$ of separating set $S = \{x, y\}$. Suppose that for all i, L_i^+ is planar. We may then draw L_1^+ on the plane. If we have a planar drawing of $\bigcup_{j=1}^{i-1} L_j^+$ we can then embed a drawing of L_i^+ where the edge between x and y borders the outer edge into a face that is bordered by the edge between x and y in our planar drawing of $\bigcup_{j=1}^{i-1} L_j^+$. We then can redraw this embedding such that x and y in L_i^+ are at the same points as x and y in $\bigcup_{j=1}^{i-1} L_j^+$ yielding a planar drawing of $\bigcup_{j=1}^{i} L_j^+$. By induction then $\bigcup_{i=1}^{n} L_i^+$ is planar. Clearly $G \subset \bigcup_{i=1}^{n} L_i^+$ so by lemma 4.6 we know G must be planar. □

Lemma 4.18. *If G is a graph with the properties listed below, then G is at least 3-connected.*

1. *G does not contain any subgraph that is contactable to of $K_{3,3}$ or K_5.*
2. *G is nonplanar.*
3. *Of all graphs satisfying properties (1) and (2), G has the least number of edges.*
4. *Of all graphs satisfying properties (1), (2) and (3), G has the least number of vertices.*

Proof. [15] Let G be a graph fitting the above properties. Any proper subgraph, H, of G will be missing at least one vertex or one edge from G. If H is missing a vertex from G, then H must break one of the first 3 properties. H can not contain a subgraph that is contactable to $K_{3,3}$ or K_5 as then G would also contain it and H can not have more vertices than G. This leaves only that H must be planar. If H is missing an edge from G, then H must break one of the first 2 properties. Again H can not contain a subgraph that is contactable to $K_{3,3}$ or k_5 as then G would also contain it. This again leaves only that H must be planar. This means that all proper subgraphs H of G are planar so G is minimally nonplanar. Additionally we know that G is not a complete graph as all complete graphs either contain K_5 or are planar.

By lemma 4.16 we know that G is at least 2 connected. Suppose that G is 2 connected. Let $\{x\}\, y = S$ be a cut set in G. We know by lemma 4.17 that there is a lobe L for which adding an edge between x and y would yield a nonplanar graph L^+. Clearly L^+ has less edges then G, so L^+ must contain a subgraph that is contractible to $K_{3,3}$ or K_5. Consider another lobe M of S. There must be a path, p, from x to y in M as otherwise G would not be either 0 or 1 connected. Clearly $L \cup p$ is a subgraph of G and $L \cup p$ is contractible to L^+. By lemma 4.9 we then know that $L \cup p$ contains a subgraph that is contractible to $K_{3,3}$ or K_5. This gives us a contradiction so G can not be 2-connected and therefore must be at least 3 connected. □

[14]This proof comes from `http://www.cs.rpi.edu/~goldberg/14-GT/19-kurat.pdf`.
[15]This proof comes from `http://www.cs.rpi.edu/~goldberg/14-GT/19-kurat.pdf`.

4.6 Bridges

Let G be a connected graph with cycle C. Let G' be the graph of G when all edges of C are removed.

Definition 4.12 (partial bridge)**.** A partial bridge, B, of C in G is any connected subgraph of G' that contains at lest one edge and for any two edges a and b in B, there is a walk in B that contains both a and b and does not pass through any vertex of C (it may begin or end on with a vertex of C, but no internal vertex may be in C).

Definition 4.13 (bridge)**.** A partial bridge, B, is a bridge of C in G if there does not exist a partial bridge $B' \neq B$ of C in G such that $B \subset B'$.

Alternatively a bridge is a maximal partial bridge.

Bridges are the final tool we use to prove Kuratowski's theorem. We will now build up some lemmas, that we will need and might help us to build up a conception of a bridge. For these let G be a connected graph with cycle C and let G' be the subgraph. All bridges and partial bridges will be in reference to C.

Lemma 4.19. *Let W be a walk in G' that does not pass through C (it may start and end in C) and has at least one edge. Then the graph, H, composed of the vertices and edges in W is a partial bridge.*

Proof. First H is a subgraph of G' as W is a walk in G'. Second H contains at least one edge as W contains at least one edge. Third H is connected trivially. Finally W is a walk containing all edges of H and not passing through C, so H must be a partial bridge. $\square$

Lemma 4.20. *Let A and B be partial bridges sharing a vertex, $v \notin C$, then $A \cup B$ is a partial bridge.*

Proof. First $A \cup B$ is trivially a subgraph of G' and contains at least one edge as both A and B contain at least one edge. Additionally $A \cup B$ is connected as for any pair of vertices $a, b \in A \cup B$ we have the following possibilities:

- If both a and b are in A or both a and b are in B, then there is a walk from a to b within A or B respectively as both A and B are connected.
- Otherwise one of the vertices is in A and the other is in B. Without loss of generality let $a \in A$ and $b \in B$, then we can construct a walk from a to v within A as both are in A and a walk from v to b as within B both are in B. We then combine these two walks and have a walk from a to b that stays within $A \cup B$.

Therefore $A \cup B$ is connected. Finally we can make a similar argument that for any edges in $a, b \in A \cup B$ there is a walk within $A \cup B$ that doesn't pass through C that contains both a and b.

- If both a and b are in A or both a and b are in B then there is such a walk as A and B are both partial bridges.
- If a and b are not both in A or both in B then without loss of generality we can let $a \in A$ and $b \in B$. First as A and B are both connected and contain at least one edge, then v must have at least one edge in each A and B. Now we know there exists a walk in A containing both a and an edge of v and there is a walk in B containing both b and an edge of v, and both these walks do not pass through C. Any walk passing though an edge of v must also contain v, so both these paths must contain v. Without loss of generality we will assume the walk in A contains v after a and the walk in B contains b after v. We now construct a walk that starts as the first walk up until has passed through a and continued on to v, then we start following the walk in B as it does after v until reaching b. This walk's internal vertices are composed only of internal vertices from the walk in A, the walk in B and of v. All of these are known not to be in C.

Now we have shown that $A \cup B$ is a partial bridge. □

Corallary 4.21. No two bridges of the same cycle will share any edges and may only share a vertex if the vertex is in the cycle.

Proof. Let A and B be bridges of C. If they share a vertex that is not in C, then by lemma 4.20 we know that $A \cup B$ is a partial bridge, and it would contain both A and B, so then $A = A \cup B = B$. If A and B share an edge, e, then they must also share the vertices of e. If either one of e's vertices are not in C then we already know that $A = B$. If both of e's vertices are in C then e is the only edge in both A and B as for any other edge e' a walk that contains both e and e' must through one of e's vertices to get between e and e'. Additionally neither A nor B can contain any other vertices than e's vertices as they are connected. Therefore $A = B$ again. □

Lemma 4.22. *Every bridge B of cycle C contains at least one vertex in C.*

Proof. Let B be a partial bridge that does not contain any vertex in C. We know C and B are contained within a larger graph G that is connected, so there must be a walk from any vertex in B to any vertex in C. Consider then a walk W that starts at vertex $b \in B$ and ends at vertex $c \in C$. There must be some first vertex in W that is in C, so let us call W' the subwalk of W from b to that first vertex in C. Now we know that W''s first vertex in C is also its last vertex, so it does not pass through C. Therefore by lemma 4.19 the graph H composed of W''s vertices and edges is a partial bridge. We know also that $B \cup H$ is a partial bridge by lemma 4.20. Finally $B \neq B \cup H$ as $B \cup H$ contains a vertex in C and B does not. Now we have a partial bridge, $B \cup H$ that contains B, so we know B is not a bridge. □

Lemma 4.23. *In a plane drawing of a graph G, given some cycle C, every bridge, B, of C will either entirely be in the interior or exterior of C with the exception of the vertices of B that are in C.*

Proof. We know that C splits the plane into the interior and exterior. We know every vertex and edge that are not in C, must be entirely in the interior or exterior. As B is a subgraph G' which contains no edges of C then every edge in B is entirely in the interior or exterior.

Now assume for the sake of contradiction there is an edge, $e \in B$ that is in the exterior and an edge $i \in B$ that is in the interior. Then any walk containing both e and i must pass over C which can only be done at vertices of C. This then means that every walk contain e and i passes through C and thus we have a contradiction.

Now for any vertex $v \in B$ there is some edge $e \in B$ that connects to v. If v is in the interior of C then e must also be entirely in the interior, and if v is in the exterior then e must be entirely in the exterior. This means that if any part of B is in the interior, then B must have at least one edge in the interior and if any part of B is in the exterior then B must have at least one edge in the exterior. We then can conclude that all of B must be solely in the interior of C or the exterior of C. □

4.6.1 Some more useful definitions about bridges

Definition 4.14 (vertex of attachment)**.** If B is a bridge of cycle C, then any vertex in both B and C is a vertex of attachment

Definition 4.15 (degree)**.** The degree of a bridge is the number of vertices of attachment it has.

Definition 4.16 (Skew)**.** Let A and B be bridges of cycle C. A and B are skew one another if there exists vertices of attachment $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with ordering a_1, b_1, a_2, b_2 in C and $\{a_1, a_2\} \cap \{b_1, b_2\} = \varnothing$.

Lemma 4.24. *If A and B are bridges of cycle C in a planar graph G, and A and B are skew one another then A and B can not both be drawn in interior or both in the exterior of C.*

Proof. In C the vertices a_1 and a_2 separate C into two parts. One part contains b_1 and the other contains b_2. If A is drawn on the interior of C then there must be a path from a_1 to a_2 already drawn on the interior of C. This means that any path drawn from b_1 to b_2 on the interior of C, must pass over this path. The same argument works for the exterior of C. □

4.7 The home stretch

We can finally provide the proof for Kuratowski's theorem.

Theorem 4.1 (Kuratowski)**.** *A graph G is nonplanar if and only if it contains a subgraph that is a subdivision of $K_{3,3}$ or K_5*

Proof. [16] Suppose that G is a graph with the following properties.

1. G does not contain any subgraph that is contactable to of $K_{3,3}$ or K_5.
2. G is nonplanar.
3. Of all graphs satisfying properties (1) and (2), G has the least number of edges.
4. Of all graphs satisfying properties (1), (2) and (3), G has the least number of vertices.

We will try and show that G does not exist.

First off we have lemma 4.18 which says that G must be at least 3 connected. Additionally we know that G must be minimally nonplanar.[17] Now choose an edge $e \in G$ between x and y and let $G' = G \setminus \{e\}$. Clearly G' is planar. Because G is 3 connected, by Menger's theorem (theorem 4.15) we know that between any two vertices $a \neq b$ there are at least 3 internally disjoint paths in G. From this we can see that in G' there must be at least 2 internally disjoint paths between a and b. By Menger's theorem (theorem 4.15) we know that G' is 2 connected. This also means that every pair of paths in G' lie upon a cycle, start at a go down one path to b, then follow the other internally distinct path back to a. Let $\mathcal{C}$ be the set of all cycles that x and y lie on in G'. Let $\mathcal{D}$ be the set of all plane drawings of G'. By lemma 4.12 we know that every drawing $D \in \mathcal{D}$ we know that all cycles $C \in \mathcal{C}$ form a Jordan curve. Now choose a pair $(C, D) \in \mathcal{C} \times \mathcal{D}$ such the curve created by C in D has the maximum possible number of edges in its interior. Now lets take a look at the bridges of C in D.

C has no bridges of degree 1 as their vertex of attachment would be a single vertex cut set and we know G' is at least 2 connected. Now we can break C into two parts. If we start at x and travel along C to y we get a path we are calling p. If we continue along C from y back to x we get another path q.

Claim: There are no external bridges of C that have more than one vertex of attachment in either p or q.

Proof: Assume there is an external bridge B of C with vertices of attachment b_1, b_2 with either both in p or both in q. Let r be the path (either p or q) that contains b_1 and b_2. Further assume without loss of generality that b_1 comes before b_2 in r. As B is connected, there must be a path s in B from b_1 to b_2. Now construct a path r' that starts where r starts, goes down r to b_1 then follows s to b_2 where it continues down

[16] This proof comes from `http://math.uchicago.edu/~may/REU2017/REUPapers/Xu,Yifan.pdf`. This paper also is where most of the structure of everything that has been written from the beginning of this section up to this point comes from.

[17] This was proved in the beginning of the proof for lemma 4.18.

r until r's end. If $b_2 = r_j$ this would look like

$$r' = r_1 r_2 \dots b_1 s_1 s_2 \dots b_2 r_{j+1} r_{j+2} \dots r_{|r|}$$

Now we can construct C' to be C but replace r with r'. C' clearly contains both x and y, and it also will have more edges on it's interior than C did. This is a contradiction as C is the cycle containing x and y with the most edges on the interior.

Now by the pigeonhole principle there can not be any external bridges of C with degree greater than 2. Further no external bridge may have x or y as a vertex of attachment as x and y are in both p and q. This means that all external bridges of C are degree two with one vertex of attachment in p and one vertex of attachment in q and neither in $\{x, y\}$. We also know there is at least one external bridge of C as otherwise x and y would both be bordering the outer face so the edge e can be drawn within the outer face yielding G to be a planar graph. By similar logic there must be some internal bridge I that has at least one vertex of attachment in $p \setminus \{x, y\}$ and at least one vertex of attachment in $q \setminus \{x, y\}$ (This is because if there was no such bridge x and y would share a face and we could draw in e making G planar).

Let E be an external bridge of C with vertices of attachment e_p and e_q with e_p in $p \setminus \{x, y\}$ and $e_q \setminus \{x, y\}$ in q. If we consider $C \setminus \{e_p, e_q\}$ we get two connected components. One of these components will have x and the other will have y. Let us define $X(E)$ to be the component containing x and $Y(E)$ to be the component containing y. Additionally there must be some internal bridge of C with a vertex of attachment in both $X(E)$ and $Y(E)$ as otherwise E could be redrawn on the inside of C.

Claim: There must exist some external bridge E of C for which there is an internal bridge B of C with at least one vertex of attachment in all of the following:

1. $X(E)$
2. $Y(E)$
3. $p \setminus \{x, y\}$
4. $q \setminus \{x, y\}$.

Proof: Assume there is no such bridge external bridge E. We will show a contradiction as we will be able to redraw G' such that we can add e in the interior of C.

For any bridge B on the interior of C that does not have a vertex of attachment in $p \setminus \{x, y\}$, B does not prevent the edge e from being added in the interior of C. The same is true for any bridge on the interior of C with no vertex of attachment in $p \setminus \{x, y\}$ this means that if we can redraw G' such that all bridges on the interior of C have no vertices of attachment in $p \setminus \{x, y\}$ or no vertices of attachment in $q \setminus \{x, y\}$ then we will have shown a contradiction.

Let B be a bridge on the interior of C that has a vertex of attachment in both $q \setminus \{x, y\}$ and in $p \setminus \{x, y\}$. If there is any bridge, E, that is skew with B then E must be on the exterior and it's existence would contradict our initial assumption. As there may not be any bridges that are skew with B we may simply redraw B on the exterior of C. This then yields a contradiction, so the claim as been proved.

Now that we know there is some external bridge E of C for which there is an internal bridge B of C with at least one vertex of attachment in each of $X(E), Y(E), q \setminus \{x, y\}, p \setminus \{x, y\}$. This means that the set $B \cap C$ (the set of all vertices of attachment of B) must intersect each of the following sets $X(E), Y(E), q \setminus \{x, y\}, p \setminus \{x, y\}$. Choose $A \subset B \cap C$ such that A intersects with each of $X(E), Y(E), q \setminus \{x, y\}, p \setminus \{x, y\}$ and there is no $A' \subset B \cap C$ for which $|A'| < A$ and A' intersects with each of $X(E), Y(E), q \setminus \{x, y\}, p \setminus \{x, y\}$.

We know that $|A| \neq 0$ as then it cant intersect anything. We know that $|A| \neq 1$ as then there exists only one $a \in A$ and we can not have both $a \in X(E)$ and $a \in Y(E)$ as $X(E) \cap Y(E) = \varnothing$. We know that $|A| < 5$ by the pigeonhole principle. This means that we only have three possibilities left: $|A| = 2$, $|A| = 3$, or $|A| = 4$. Lets go through them.

1. If $|A| = 2$ then we may let $A = \{u, v\}$ with $u \in X(E)$ and $v \in Y(E)$. Now either $u \in q \setminus \{x, y\}$ and $v \in p \setminus \{x, y\}$, or $v \in q \setminus \{x, y\}$ and $u \in p \setminus \{x, y\}$. Either way this will result in G containing a subgraph that contracts to $K_{3,3}$.

2. If $|A| = 3$. There must be some $a \in A$ for which $a \in \{x, y, e_p, e_q\}$ as otherwise A would contain a subset that fit our previous case. Whatever a's value, it will be in exactly one of the four sets $X(E), Y(E), q \setminus \{x, y\}, p \setminus \{x, y\}$. This means no other element can also be in whichever of these sets a is in as then we could simply remove a from A.

   Suppose there is another element $b \in A$ with $b \in \{x, y, e_p, e_q\}$. We know that $\{a, b\} \neq \{x, y\}$ and $\{a, b\} \neq \{e_p, e_q\}$ as then we would have only one vertex left that would need to be in two disjoint sets. This means that we can say without loss of generality let $a \in \{x, y\}$ and $b \in \{e_p, e_q\}$. This leaves one last element $c \in A$ with c along the path in C between the element of $\{x, y\} \setminus \{a\}$ and the element of $\{e_p, e_q\} \setminus \{b\}$. By simply removing the path between a and b in C we would get a graph that looks very similar to what we got when $|A| = 2$. This will result again in G containing a subgraph that contracts to $K_{3,3}$.

   Suppose there is no other element in A that is also in $\{x, y, e_p, e_q\}$. We know then that if $a \in \{x, y\}$ then the other two elements are on the paths in C between the element in $\{x, y\} \setminus \{x\}$ and e_p or e_q respectively. Let $b \in A$ be on the path in C between e_p and the element of $\{x, y\} \setminus \{x\}$. Simply contract all the edges on C between e_p and b and we have contracted to a graph that would be in our previous case. Then by lemma 4.9 we know that G must contain a subgraph that is a subdivision of $K_{3,3}$. If $a \in \{e_p, e_q\}$ then we can make the same argument that both the other elements of A will be on paths between the element of $\{e_p, e_q\} \setminus a$ and x or y respectively. Then we may let $b \in A$ be on the path in C between x and the element of $\{e_p, e_q\} \setminus a$, and then contract all the edges on the path along C between b and x. This again leaves us at the previous case so G will have a subgraph that contracts to $K_{3,3}$.

3. If $|A| = 4$ then $A = \{x, y, e_q, e_p\}$ as otherwise one of the previous cases would be a subset of A. In this, when we add back in edge e we find that G contains a subgraph that is contactable to K_5.

Now we have shown a contradiction with our original statement, so there is no graph G that is nonplanar does not contain a subgraph that is contactable to K_5 or $K_{3,3}$. Further suppose G is a graph that contains a subgraph H that is contactable to either K_5 or $K_{3,3}$. We know that K_5 is nonplanar by theorem 4.4 and that K_3 is nonplanar by theorem 4.5. It follows then from lemma 4.8 that H is nonplanar. Finally G must also be nonplanar by lemma 4.6. $\square$

5 Beyond Kuratowski's theorem

Kuratowski's theorem is an unexpectedly elegant statement that says there are only two base case nonplanar graphs, and that all other nonplanar graphs derive their nonplanarity from at least one of these two graphs. A logical next step might be to ask about different surfaces. We've found which graphs can be drawn on a plane, but what about a sphere or a torus? It turns out that the set of graphs that can be drawn on a sphere is the same set of graphs that can be drawn on the plane,[18] however this is not true for a torus and certainly not true in general. For example $K_{3,3}$ can be drawn on both a torus and a Möbius strip. So the question naturally arises, do all surfaces have a set of base forbidden graphs like $\{K_{3,3}, K_5\}$ is for the plane.

The graph minor theorem answers this question, at least in part. It states that any countable set $\mathcal{G}$ of finite graphs, contains a finite subset $\mathcal{H} \subset \mathcal{G}$ such that for all $G \in \mathcal{G}$ there is some $H \in \mathcal{H}$ such that G contains a subset that is contactable to H.[19] This means that if a surface S has the following property

- For any graph G that can be drawn in S, any graph produced by preforming an edge contraction on G is able to be drawn in S.

then S has a finite set of forbidden graphs.

[18]We actually showed most of what is needed to prove this in our proof for lemma 4.13.

[19]This comes from `https://arxiv.org/pdf/1608.04066.pdf`.

Another reasonable question is, now that we've talked so much about what can be drawn in $\mathbb{R}^2$, what about in $\mathbb{R}^k$. This turns out to be a rather disappointing line of inquiry. When $k = 1$ we can only draw line graphs and unions thereof.[20] When $k \geq 3$ we can draw any graph whatsoever pretty easily. Simply make a drawing that allows for edges to cross in $\mathbb{R}^2$ and wherever there is a crossing simply lift "up" one edge.[21]

5.1 Genus of a graph

We can also flip the question we are asking and rather ask which surfaces can a given graph can be drawn on. The genus of a graph is one such way to characterize this. If we let S_0 be the sphere and S_n be the sphere with n handles attached to it. This would mean S_1 is a torus and S_n is an n holed torus. Now every graph G can be drawn on some space S_n. To see this simply draw G on a sphere and allow crossings. Now wherever there is a cross simply add a handle and draw one of the edges over the handle and the other going through the handle. Armed with this knowledge we can define the genus of a graph, G, to be the least possible n such that G can be drawn on G_n. There is also a nonorientable analog of this using a sequence of nonorientable spaces.

6 Conclusion

We have built up a basic understanding of graphs and graph theory. We've thoroughly proven Kuratowski's theorem, and hopefully in a way that is approachable. We then spent a bit of time talking about possible next steps. The graph minor theorem for example is a spectacular theorem that deserves far more attention then we are giving it. This however is where we end the paper. Thank you for reading.

7 Acknowledgments

I would like to thank Professor Shaw for his extreme patients while I wrote this paper. I would like to thank my family (especially my mother) who pushed me to continue writing the paper. Finally I would like to thank anyone who took time to help revise the paper. Thank you.

[20] A line graph is any graph G such that there is some path p in G that contains every vertex and every edge in G.

[21] Up here refers to any direction perpendicular to the plane we initially drew the graph in.