Foundations of Mathematics
Section 2.3

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3(a) Let $\mathscr{A}$ be a family of sets and $B \in \mathscr{A}$, and $x \in \bigcup_{A \in \mathscr{A}} (A)$ then $x \in A$ for at least one $A \in \mathscr{A}$. As $B \in \mathscr{A}$ then $(\forall x \in B)\, x \in \bigcup_{A \in \mathscr{A}} (A)$, thus by definition of subset $B \subseteq \bigcup_{A \in \mathscr{A}} (A)$.

(b) Let $A \subseteq B$ for all $A \in \mathscr{A}$ where $\mathscr{A}$ is a family of sets. Then we know that for any $x \in A \in \mathscr{A}$ that $x \in B$ thus by the definition of subset the set $S = \{x : (\exists A \in \mathscr{A}) x \in A\} \subseteq B$ and by definition $S = \bigcup_{A \in \mathscr{A}} (A)$, thus $\bigcup_{A \in \mathscr{A}} (A) \subseteq B$.

11(a) The largest set X such that $X \subseteq A$ for all $A \in \mathscr{A}$ is $\bigcap_{B \in \mathscr{A}} (B)$. This is shown by proving that both $\bigcap_{B \in \mathscr{A}} (B) \subseteq A$ for all $A \in \mathscr{A}$ and that if $V \subseteq A$ for all $A \in \mathscr{A}$ then $V \subseteq \bigcap_{B \in \mathscr{A}} (B)$.

1. By theorem 2.3.1(a) we know that $\bigcap_{B \in \mathscr{A}} (B) \subseteq A$ for all $A \in \mathscr{A}$.
2. By theorem 2.3.2(a) we know that if $V \subseteq A$ for all $A \in \mathscr{A}$ then $V \subseteq \bigcap_{B \in \mathscr{A}} (B)$.

Thus we have shown that $\bigcap_{A \in \mathscr{A}} (A)$ is the largest set for which it is a subset of all $A \in \mathscr{A}$.

(b) The smallest set Y such that $A \subseteq Y$ for all $A \in \mathscr{A}$ is $\bigcup_{B \in \mathscr{A}} (B)$. This is shown by proving that both $A \subseteq \bigcup_{B \in \mathscr{A}} (B)$ for all $A \in \mathscr{A}$ and that if $A \subseteq W$ for all $A \in \mathscr{A}$ then $\bigcup_{B \in \mathscr{A}} (B) \subseteq W$.

1. By theorem 2.3.1(b) we know that $A \subseteq \bigcup_{B \in \mathscr{A}} (B)$ for all $A \in \mathscr{A}$.
2. By theorem 2.3.2(b) we know that if $A \subseteq W$ for all $A \in \mathscr{A}$ then $\bigcup_{B \in \mathscr{A}} (B) \subseteq W$.

Thus we have shwon that $\bigcup_{B \in \mathscr{A}} (B)$ is the smallest set for which it is a superset of all $A \in \mathscr{A}$.

12(a) Let $\mathscr{A} = \{\{1, 2, \ldots, 10\}, \{11, 12, \ldots, 20, 1\}\}$

(b) Let $\mathscr{B} = \{\{1, 2, \ldots, 17\}, \{18\}, \{19\}, set20\}$

(c) Let $\mathscr{C} = \{\{1\}, \{2\}, \ldots, \{20\}\}$

17(a) First we know that if $Q \subseteq W$, then $Q \cap W = Q$, and using this we will show by induction that for any $k \in \mathbb{N}$ that $\bigcap_{i=1}^{k} (A_i) = A_k$ given that for any pair $i \in \mathbb{N}$, $j \in \mathbb{N}$ where $i \leq j$ then $A_j \subseteq A_i$.

For our base case we let $k = 1$, and it is trivial to show that $\bigcap_{i=1}^{1} (A_i) = A_1$.

For our inductive case we assume that $\bigcap\limits_{i=1}^{k}(A_i) = A_k$ and we intend to show that $\bigcap\limits_{i=1}^{k+1}(A_i) = A_{k+1}$.

$$\begin{aligned}
\bigcap_{i=1}^{k+1}(A_i) &= \bigcap_{i=1}^{k}(A_i) \cap A_{k+1} \\
&= A_k \cap A_{k+1} \\
&= A_{k+1}
\end{aligned}$$

The last step, $A_k \cap A_{k+1} = A_{k+1}$ is true due to our knowledge that $A_{k+1} \subseteq A_k$.

Thus by PMI we know that for any $k \in \mathbb{N}$ that $\bigcap\limits_{i=1}^{k}(A_i) = A_k$ given that for any pair $i \in \mathbb{N}$, $j \in \mathbb{N}$ where $i \leq j$ then $A_j \subseteq A_i$.

(b) We again will let $A_j \subseteq A_i$ for any pair $i \in \mathbb{N}$, and $j \in \mathbb{N}$ where $i \leq j$, and again we will use mathematical induction, this time to show that $\bigcup\limits_{i=1}^{\infty}(A_i) = A_1$.

For our base case we show that $\bigcup\limits_{i=1}^{1}(A_i) = A_1$, which is trivial.

For our inductive case we assume $\bigcup\limits_{i=1}^{k}(A_i) = A_1$ and show that it holds for $\bigcup\limits_{i=1}^{k+1}(A_i) = A_1$.

$$\begin{aligned}
\bigcup_{i=1}^{k+1}(A_i) &= \bigcup_{i=1}^{k}(A_i) \cup A_{k+1} \\
&= A_1 \cup A_{k+1}
\end{aligned}$$

We know that $A_1 \cup A_{k+1} = A_1$ because $\forall(n \in \mathbb{N})A_n \subseteq A_1$, thus $A_{k+1} \subseteq A_1$, and because A_{k+1} is a subset of A_1 the union of the two is A_1 (there are no elements in A_{k+1}, not in A_1).

Thus by PMI we know that for any $k \in \mathbb{N}$, $\bigcup\limits_{i=1}^{k}(A_i) = A_1$, although this isn't what we were actually trying to prove. The question states the union up to ∞, although A_i is only defined for when $i \in \mathbb{N}$, thus it doesn't make sense to include infinity in this range.