Topology Midterm

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April 7, 2017

17.5 In order to show that, for any order topology, $\overline{(a,b)} \subset [a,b]$ we first notice that $[a,b] \supset (a,b)$ and that $[a,b]$ is closed. By definition we know $\overline{(a,b)} = \bigcap$ all closed supersets of (a,b). We now notice that $[a,b]$ is one such closed superset of (a,b), thus $\overline{(a,b)} \subset [a,b]$.

We now will look to see when $\overline{(a,b)} = [a,b]$. We already know that $\overline{(a,b)} \subset [a,b]$, and to have equality we only need $[a,b] \subset \overline{(a,b)}$. Let us start by noticing that $[a,b]$ is the union of disjoint sets (a,b) and $\{a,b\}$. Now if $[a,b]$ is to be a subset of $\overline{(a,b)}$ then that would be the same as saying $(a,b) \cup \{a,b\} \subset \overline{(a,b)}$ thus both (a,b) and $\{a,b\}$ must be subsets of $\overline{(a,b)}$. We know that $(a,b) \subset \overline{(a,b)}$ as $\overline{(a,b)} = (a,b) \cup (a,b)'$, and because we know that $\{a,b\}$ is disjoint from (a,b) we can then say $[a,b] \subset \overline{(a,b)} \implies \{a,b\} \subset (a,b)'$. We also can say

$$\begin{aligned}
\{a,b\} \subset (a,b)' &\implies \{a,b\} \subset \overline{(a,b)} \\
&\implies \{a,b\} \cup (a,b) \subset \overline{(a,b)} \\
&\implies [a,b] \subset \overline{(a,b)}
\end{aligned}$$

and thus, iff a and b are limit points for the interval (a,b), then our equality $([a,b] = \overline{(a,b)})$ holds.

17.17 Consider the lower limit topology on $\mathbb{R}$, and the topology given by the basis $\mathcal{C}$ of Exercise 8 §13. Determine the closures of the intervals $A = \left(0, \sqrt{2}\right)$ and $B = \left(\sqrt{2}, 3\right)$ in these two topologies.

Basis $\mathcal{C}$ of Exercise 8 §13:

$$\mathcal{C} = \{[a,b) \,|a < b \ and \ a,b \in \mathbb{Q}\}$$

First we will consider our topology to be $\mathbb{R}_\ell$:

Let C be an interval in the form (a,b), where $a,b \in \mathbb{R}$. By definition we know that $\overline{C}$ is the intersection of all closed sets that contain C. We know that $[a,b) \in \mathbb{R}_\ell$ and that $[a,b) \supset A$, thus $\overline{C} \subset [a,b)$.

Now if we can show that $[a,b) \subset \overline{C}$ then we will know that $[a,b) = \overline{C}$.

First we note that by theorem 17.6 $\overline{C} = C \cup C'$, now because we know $C \subset \overline{C}$ then we can say if $[a,b) \setminus C \subset \overline{C} \setminus C$ then $[a,b) = \overline{C}$. We also know that $\overline{C} \setminus C \subset C'$, thus we can say that if $[a,b) \setminus C \subset C'$ then $[a,b) = \overline{C}$. Next we find that $[a,b) \setminus C = \{a\}$ so if $a \in C'$ then $[a,b) = \overline{C}$. We will show $a \in C'$ by contradiction.

Let us assume $a \notin C'$ then there is an interval $[x,y)$, where $x,y \in \mathbb{R}$, that contains a but no elements in C. By definition $[x,y) = \{k | x \leq k < y\}$, so if $a \in [x,y)$ then $x \leq a < y$. Now we can construct an interval $(a,y) \subset [x,y)$ which is not empty as $y > a$ and thus it will contain some elements of C. We now have a contradiction, thus $a \in C'$, thus

$$[a,b) = \overline{C}$$

Now if we let $a = 0$ and $b = \sqrt{2}$ then we know $\overline{\left(0, \sqrt{2}\right)} = \overline{A} = \left[0, \sqrt{2}\right)$.

Now if we let $a = \sqrt{2}$ and $b = 3$ then we know $\overline{\left(\sqrt{2}, 3\right)} = \overline{B} = \left[\sqrt{2}, 3\right)$.

Now we will to continue on to the topology $\mathscr{C}$, which is given by basis $\mathcal{C}$.

Let us first attempt to find $\overline{(0,\sqrt{2})}$. We will consider the set $\left[0,\sqrt{2}\right]$, and attempt to show that it is closed by showing it's compliment is open.

$$\begin{aligned}\left[0,\sqrt{2}\right]^{\mathsf{c}} &= (-\infty,0)\cup\left(\sqrt{2},\infty\right)\\ &= \left(\bigcup_{a<b<0\ and\ a,b\in\mathbb{Q}}([a,b))\cup\bigcup_{\sqrt{2}<a<b\ and\ a,b\in\mathbb{Q}}([a,b))\right)\in\mathscr{C}\end{aligned}$$

Thus $\left[0,\sqrt{2}\right]$ is closed, and thus $\overline{(0,\sqrt{2})}\subset\left[0,\sqrt{2}\right]=\left(0,\sqrt{2}\right)\cup\left\{0,\sqrt{2}\right\}$. Now to find $\overline{(0,\sqrt{2})}$ we simply must determine if 0 is a limit point of $\left(0,\sqrt{2}\right)$ and if $\sqrt{2}$ is a limit point of $\left(0,\sqrt{2}\right)$. If an open set contains 0 then it must contain an interval $[\alpha,\beta)$, where $\alpha\leq 0$ and is rational, and $\beta>0$ and is rational. Because $\beta>0$ then there must be some number between 0 and β that is in $\left(0,\sqrt{2}\right)$, thus 0 is a limit point of $\left(0,\sqrt{2}\right)$. If an open set contains $\sqrt{2}$ then it must contain an interval $[\alpha,\beta)$, where $\alpha\leq\sqrt{2}$ and is rational, and $\beta>\sqrt{2}$ and is rational. Because α is rational $\alpha\neq\sqrt{2}$, thus there must be a number between α and $\sqrt{2}$ that is in $\left(0,\sqrt{2}\right)$, thus $\sqrt{2}$ is a limit point of $\left(0,\sqrt{2}\right)$. Thus we may now say that $\overline{(0,\sqrt{2})}=\left[0,\sqrt{2}\right]$.

Now let us attempt to find $\overline{(\sqrt{2},3)}$. We will consider the set $\left[\sqrt{2},3\right)$, and attempt to show that it is closed by showing it's compliment is open.

$$\begin{aligned}\left[\sqrt{2},3\right)^{\mathsf{c}} &= \left(-\infty,\sqrt{2}\right)\cup[3,\infty)\\ &= \left(\bigcup_{a<b<\sqrt{2}\ and\ a,b\in\mathbb{Q}}([a,b))\cup\bigcup_{3\leq a<b\ and\ a,b\in\mathbb{Q}}([a,b))\right)\in\mathscr{C}\end{aligned}$$

Thus $\left[\sqrt{2},3\right)$ is closed, and thust $\overline{(\sqrt{2},3)}\subset\left[\sqrt{2},3\right)=\left(\sqrt{2},3\right)\cup\left\{\sqrt{2}\right\}$. To to find $\overline{(\sqrt{2},3)}$ we must simply determine if $\sqrt{2}$ is a limit point of $\left(\sqrt{2},3\right)$. If an open set contains $\sqrt{2}$ then it must contain an interval $[\alpha,\beta)$, where $\alpha\leq\sqrt{2}$ and is rational, and $\beta>\sqrt{2}$ and is rational. Because $\beta>\sqrt{2}$ then there must be a number between $\sqrt{2}$ and β that is in $\left(\sqrt{2},3\right)$, thus $\sqrt{2}$ is a limit point of $\left(\sqrt{2},3\right)$. Thus we may now say that $\overline{(\sqrt{2},3)}=\left[\sqrt{2},3\right)$.

18.5 Consider the linear function $f:\mathbb{R}\to\mathbb{R}$ defined as $f(x)=\frac{x-a}{b-a}$. We know all linear functions are continious, we have the homeomorphisms

$$\begin{aligned}f((a,b)) &= \left(\frac{a-a}{b-a},\frac{b-a}{b-a}\right)\\ &= (0,1)\end{aligned}$$

$$\begin{aligned}f([a,b]) &= \left[\frac{a-a}{b-a},\frac{b-a}{b-a}\right]\\ &= [0,1]\end{aligned}$$

thus we have shown homeomophism between (a,b) and $(0,1)$, and between $[a,b]$ and $[0,1]$.

18.8(a)

(b)

19.7

20.4