Foundations of Mathematics
Section 3.1

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Let $R = \{(1,5), (2,2), (3,4), (5,2)\}$, $S = \{(2,4), (3,4), (3,1), (5,5)\}$, and $T = (1,4), (3,5), (4,1)$. Find

(a)

$R \circ S$

$$R \circ S = \{(3,5), (5,2)\}$$

(b)

$R \circ T$

$$R \circ T = \{(3,2), (4,5)\}$$

(f)

$T \circ T$

$$T \circ T = \{(1,1)\}$$

(g)

$R \circ (S \circ T)$

$$\begin{aligned} S \circ T &= \{(3,5)\} \\ R \circ (S \circ T) &= \{(3,2)\} \end{aligned}$$

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Let R be a relation from A to B and S be a relation from B to C.

(a)

Prove that $\boldsymbol{Rng}\left(R^{-1}\right) = \boldsymbol{Dom}(R)$

$$\begin{aligned} R^{-1} &= \{(y,x) : (x,y) \in R\} \\ \boldsymbol{Rng}\left(R^{-1}\right) &= \left\{x : \exists_y \left((y,x) \in R^{-1}\right)\right\} \\ \boldsymbol{Dom}(R) &= \{x : \exists_y ((x,y) \in R)\} \end{aligned}$$

Let $v \in \boldsymbol{Rng}\left(R^{-1}\right)$, thus there exists some y such that $(y,v) \in R^{-1}$, now let w be one possible value for y. Now we know that $(w,v) \in R^{-1}$ so we also know that $(v,w) \in R$. We now can say $v \in \boldsymbol{Dom}(R)$, and as that would be true for any $v \in \boldsymbol{Rng}\left(R^{-1}\right)$, we then know $\boldsymbol{Rng}\left(R^{-1}\right) \subseteq \boldsymbol{Dom}(R)$.

Now let $v \in \boldsymbol{Dom}(R)$, there must now be some y such that $(v,y) \in R$ and we will now let w represent one such value of y. Now we can say $(v,w) \in R$ which also means that $(w,v) \in R^{-1}$ and that $v \in \boldsymbol{Rng}\left(R^{-1}\right)$, and because this is true for any $v \in \boldsymbol{Dom}(R)$, then $\boldsymbol{Dom}(R) \subseteq \boldsymbol{Rng}\left(R^{-1}\right)$

Now we can finally say $\boldsymbol{Dom}(R) = \boldsymbol{Rng}\left(R^{-1}\right)$.

(b)

Prove that $\boldsymbol{Dom}(S \circ R) \subseteq \boldsymbol{Dom}(R)$.

$$
\begin{aligned}
S \circ R &= \{(a, c) : \exists_b\,((a, b) \in R \textit{ and } (b, c) \in S)\} \\
\boldsymbol{Dom}(S \circ R) &= \{a : \exists_c\,((a, c) \in S \circ R)\} \\
\boldsymbol{Dom}(R) &= \{a : \exists_b\,((a, b) \in R)\}
\end{aligned}
$$

First let $x \in \boldsymbol{Dom}(S \circ R)$, that means that there is some c such that $(a, c) \in S \circ R$, we will let y be one such value of c. Now that we know $(x, y) \in S \circ R$ we can say there is some b such that $(x, b) \in R$, we will let z be one such value of b. Now we notice that $(x, z) \in R$, which means that $x \in \boldsymbol{Dom}(R)$, and because that could be any $x \in \boldsymbol{Dom}(S \circ R)$ we know that $\boldsymbol{Dom}(S \circ R) \subseteq \boldsymbol{Dom}(R)$.

(c)

Show by example that $\boldsymbol{Dom}(S \circ R) = \boldsymbol{Dom}(R)$ may be false.

$$
\begin{aligned}
R &= \{(0, 0)\} \\
S &= \varnothing \\
S \circ R &= \varnothing \\
\boldsymbol{Dom}(R) &= \{0\} \\
\boldsymbol{Dom}(S \circ R) &= \varnothing \\
\boldsymbol{Dom}(S \circ R) &= \varnothing \neq \{0\} = \boldsymbol{Dom}(R)
\end{aligned}
$$

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Complete the proof of Theorem 3.1.2 by proving that if R is a relation from A to B and S is a relation from B to C, then

(c)

$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

$$
\begin{aligned}
S \circ R &= \{(a, c) : \exists_b\,((a, b) \in R \textit{ and } (b, c) \in S)\} \\
(S \circ R)^{-1} &= \{(c, a) : (a, c) \in S \circ R\} \\
R^{-1} &= \{(b, a) : (a, b) \in R\} \\
S^{-1} &= \{(c, b) : (b, c) \in S\} \\
R^{-1} \circ S^{-1} &= \left\{(c, a) : \exists_b \left((c, b) \in S^{-1} \textit{ and } (b, a) \in R^{-1}\right)\right\}
\end{aligned}
$$

First we will start by showing $(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$ and then we will show $(S \circ R)^{-1} \supseteq R^{-1} \circ S^{-1}$.

For our first step we let $(x, y) \in (S \circ R)^{-1}$, which means that $(y, x) \in S \circ R$. Now we can say there is some b such that $(y, b) \in R$ and $(b, x) \in S$, we will let z be a possible value for b, which means $(y, z) \in R$ and $(z, x) \in S$. Now we can say that $(z, y) \in R^{-1}$ and that $(x, z) \in S^{-1}$. Now we can show that $(x, y) \in R^{-1} \circ S^{-1}$, which means that $(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$

For our second step we will start by letting $(x, y) \in R^{-1} \circ S^{-1}$, using this we can say that there exists a b such that $(x, b) \in S^{-1}$ and $(b, y) \in R^{-1}$, we will let z be one such possible value of b. Now we have $(x, z) \in S^{-1}$ and $(z, x) \in R^{-1}$

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Prove that if A has m elements and B has n elements, then there are 2^{mn} different relations from A to B.

First let us notice that the maximal relation, one with every possible pair from A to B is $A \times B$. Now we want to find how many possible subsets of $A \times B$ there are, as any subset is a unique relation from A to B, there may be no other possible relations that are not subsets of $A \times B$ as $A \times B$ has all ordered pairs (a, b) where $a \in A$ and $b \in B$, and thus all relations must be subsets of $A \times B$. Now to count how many subsets there are we simply must realize that any element may either be in a subset or not in a subset, then we are looking at two possibilities for each element in $A \times B$. This translates to $2^{\overline{\overline{A \times B}}}$ different relations (this could also be looked at as $\overline{\overline{\mathscr{P}(A \times B)}}$.) We now simply need to find how many elements there are in $A \times B$ and here we use the multiplication rule and find $\overline{\overline{A \times B}} = \overline{\overline{A}} \times \overline{\overline{B}}$ which we know is $m \cdot n$. Thus the total number of possible relations from A to B is 2^{mn}.

15(a)

Let R be a relation from A to B. For $a \in A$, define the **vertical section of** R **at** a to be $V_a = \{y \in B : (a, y) \in R\}$. Prove that $\bigcup_{a \in A} (V_a) = \boldsymbol{Rng}(R)$.

Let

$$y \in \bigcup_{a \in A} (V_a)$$

then we can say

$$\begin{aligned} \exists_x (y \in V_x) &\implies \exists_x ((x, y) \in R) \\ &\implies y \in \boldsymbol{Rng}(R) \\ &\implies \bigcup_{a \in A} (V_a) \subseteq \boldsymbol{Rng}(R) \end{aligned}$$

Now let

$$y \in \boldsymbol{Rng}(R)$$

then we can say

$$\begin{aligned} \exists_x ((x, y)) \in R &\implies \exists_x (y \in V_x) \\ &\implies y \in \bigcup_{a \in A} (V_a) \\ &\implies \boldsymbol{Rng}(R) \subseteq \bigcup_{a \in A} (V_a) \end{aligned}$$

Now we have shown

$$\boldsymbol{Rng}(R) = \bigcup_{a \in A} (V_a)$$

(b)

Let R be a relation from A to B. For $b \in B$, define the **horizontal section of** R **at** b to be $H_b = \{x \in A : (x, b) \in R\}$. Prove that $\bigcup_{b \in B} (H_b) = \boldsymbol{Dom}(R)$.