

The maximum mean discrepancy and Generative Adversarial Networks

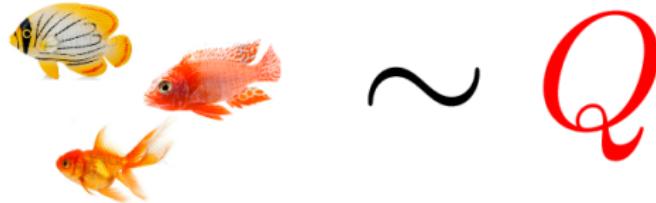
Arthur Gretton

Gatsby Computational Neuroscience Unit,
University College London

MLSS Moscow, 2019

A motivation: comparing two samples

- Given: Samples from unknown distributions P and Q .
- Goal: do P and Q differ?



A real-life example: two-sample tests

- Have: Two collections of samples X , Y from unknown distributions P and Q .
- Goal: do P and Q differ?



MNIST samples

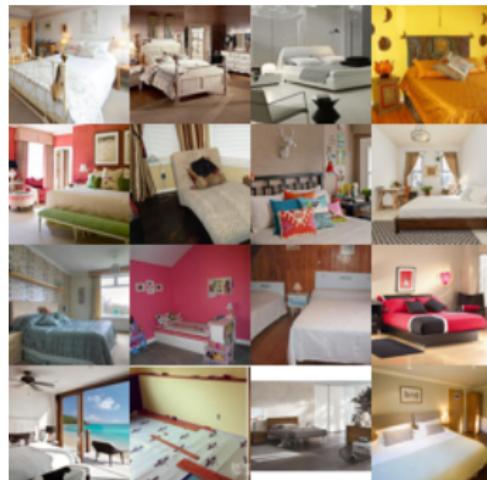


Samples from a GAN

Significant difference in GAN and MNIST?

Training implicit generative models

- Have: One collection of samples X from unknown distribution P .
- Goal: generate samples Q that look like P



LSUN bedroom samples P



Generated Q , MMD GAN

Using a critic $D(P, Q)$ to train a GAN

(Binkowski, Sutherland, Arbel, G., ICLR 2018),
(Arbel, Sutherland, Binkowski, G., NeurIPS 2018)

Training generative models

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**The
Guardian**

A portrait created by AI just sold for \$432,000. But is it really art?

An image of Edmond de Belamy, created by a computer, has just been sold at Christie's. But no algorithm can capture our complex human consciousness



▲ Portrait of Edmond Bellamy at Christie's in New York. Photograph: Timothy A Clary/AFP/Getty Images

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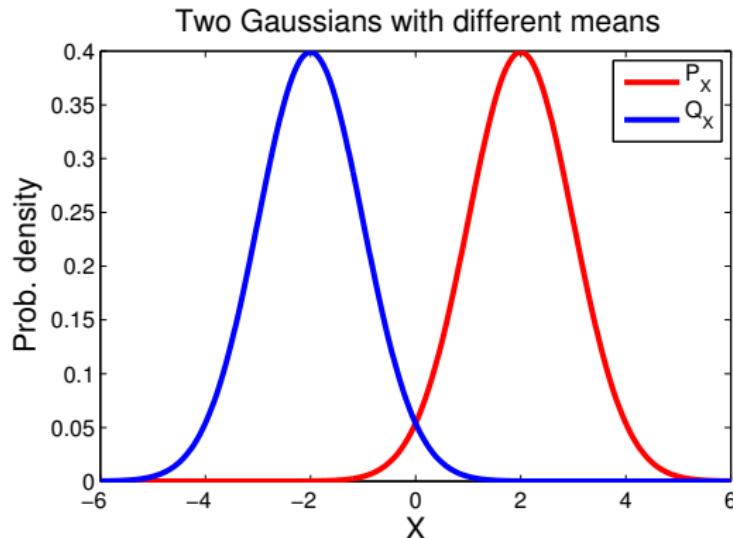
Outline

- Measures of distance between distributions...
 - Difference in feature means
 - Integral probability metrics (not just a technicality!)
- Statistical testing for evaluating GAN quality
- GAN critic design
 - Gradient regularisation and data adaptivity
 - Evaluating GAN performance? Problems with Inception and FID.

Differences in distributions

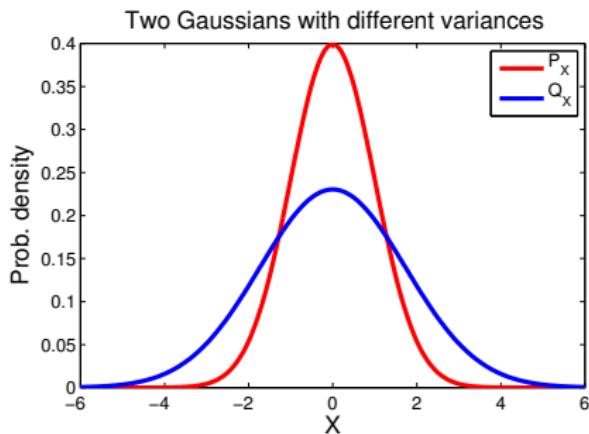
Feature mean difference

- Simple example: 2 Gaussians with different means
- Answer: t-test



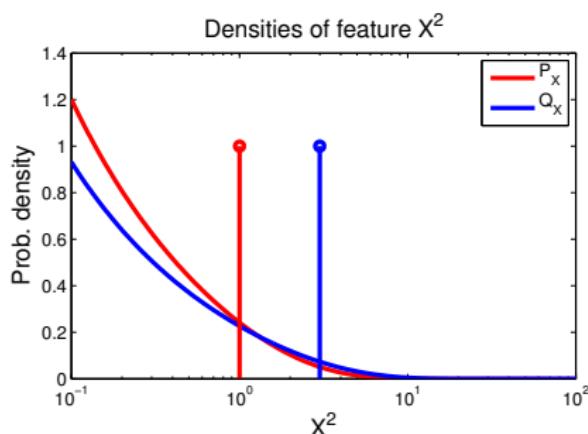
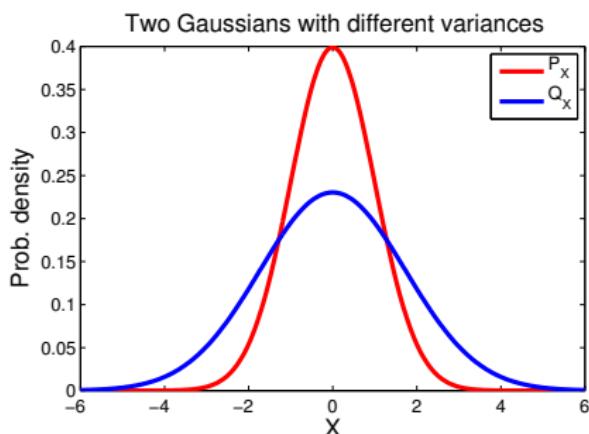
Feature mean difference

- Two Gaussians with same means, different variance
- Idea: look at difference in **means of features** of the RVs
- In Gaussian case: second order features of form $\varphi(x) = x^2$



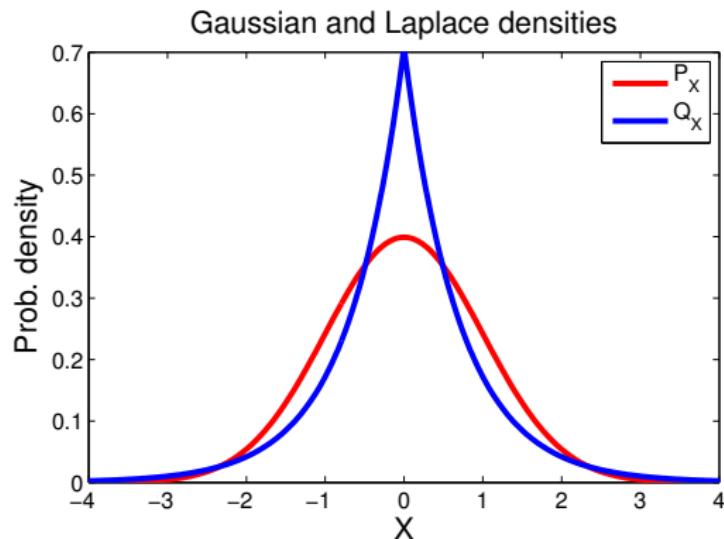
Feature mean difference

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Feature mean difference

- Gaussian and Laplace distributions
- Same mean *and* same variance
- Difference in means using **higher order features**...RKHS



Infinitely many features using kernels

Kernels: dot products
of features

Feature map $\varphi(x) \in \mathcal{F}$,

$$\varphi(x) = [\dots \varphi_i(x) \dots] \in \ell_2$$

For positive definite k ,

$$k(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{F}}$$

Infinitely many features
 $\varphi(x)$, dot product in
closed form!

Infinitely many features using kernels

Kernels: dot products
of features

Exponentiated quadratic kernel

$$k(x, x') = \exp(-\gamma \|x - x'\|^2)$$

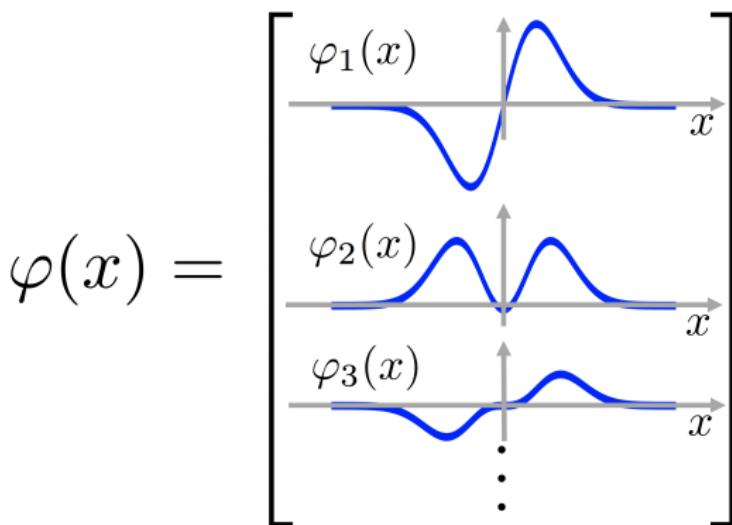
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Infinitely many features
 $\varphi(x)$, dot product in
closed form!



Infinitely many features of *distributions*

Given P a Borel **probability measure** on \mathcal{X} , define feature map of probability P ,

$$\mu_P = [\dots \mathbf{E}_P [\varphi_i(X)] \dots]$$

For positive definite $k(x, x')$,

$$\langle \mu_P, \mu_Q \rangle_{\mathcal{F}} = \mathbf{E}_{P,Q} k(\textcolor{blue}{x}, \textcolor{red}{y})$$

for $x \sim P$ and $y \sim Q$.

Fine print: feature map $\varphi(x)$ must be Bochner integrable for all probability measures considered.
Always true if kernel bounded.

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Fine print: feature map $\varphi(x)$ must be Bochner integrable for all probability measures considered.
Always true if kernel bounded.

The maximum mean discrepancy

The maximum mean discrepancy is the distance between feature means:

$$\begin{aligned} MMD^2(P, Q) &= \|\mu_P - \mu_Q\|_{\mathcal{F}}^2 \\ &= \langle \mu_P, \mu_P \rangle_{\mathcal{F}} + \langle \mu_Q, \mu_Q \rangle_{\mathcal{F}} - 2 \langle \mu_P, \mu_Q \rangle_{\mathcal{F}} \\ &= \underbrace{\mathbf{E}_P k(X, X')}_{(a)} + \underbrace{\mathbf{E}_Q k(Y, Y')}_{(a)} - 2 \underbrace{\mathbf{E}_{P, Q} k(X, Y)}_{(b)} \end{aligned}$$

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(a)= within distrib. similarity, (b)= cross-distrib. similarity.

Illustration of MMD

- Dogs ($= P$) and fish ($= Q$) example revisited
- Each entry is one of $k(\text{dog}_i, \text{dog}_j)$, $k(\text{dog}_i, \text{fish}_j)$, or $k(\text{fish}_i, \text{fish}_j)$

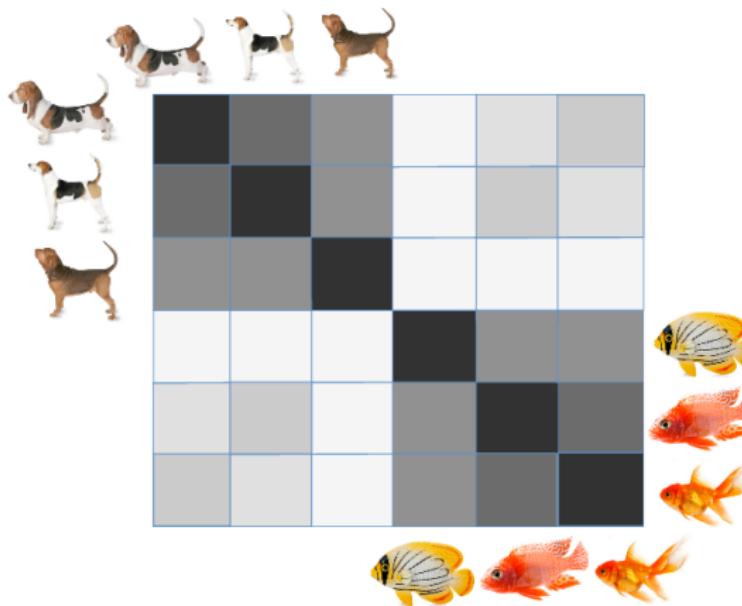
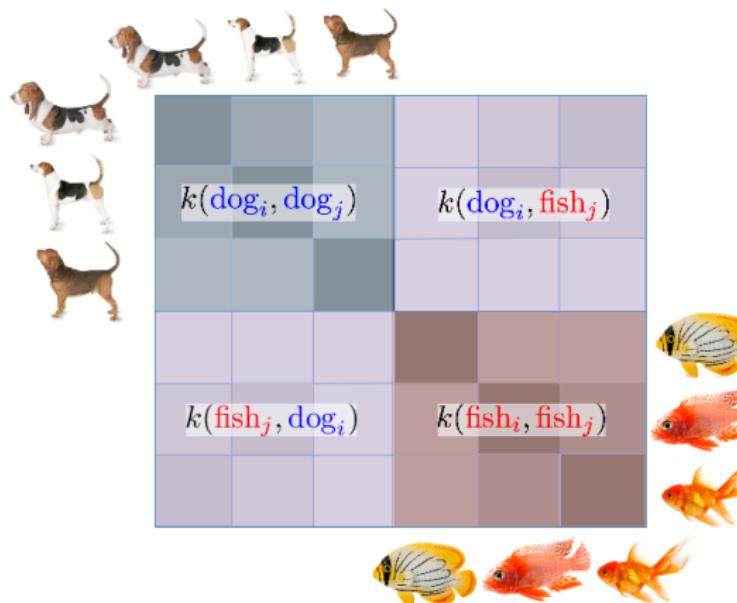


Illustration of MMD

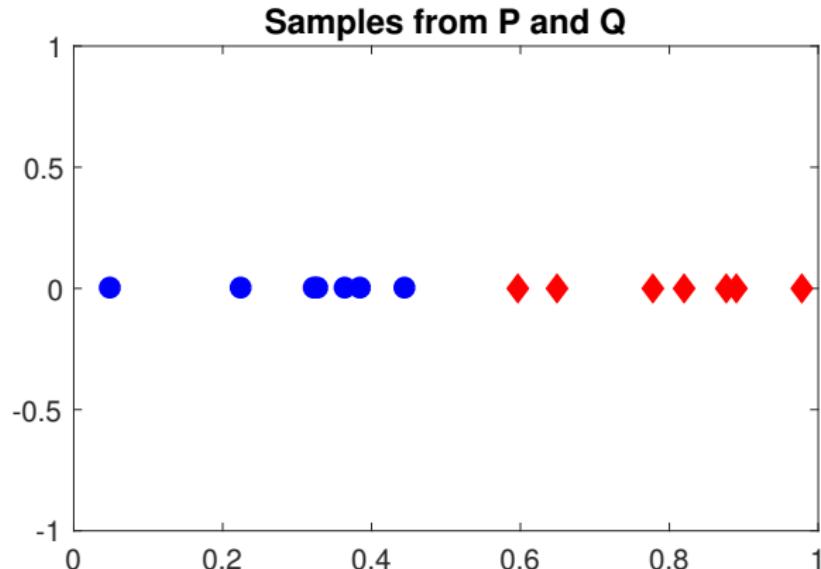
The maximum mean discrepancy:

$$\widehat{MMD}^2 = \frac{1}{n(n-1)} \sum_{i \neq j} k(\text{dog}_i, \text{dog}_j) + \frac{1}{n(n-1)} \sum_{i \neq j} k(\text{fish}_i, \text{fish}_j) - \frac{2}{n^2} \sum_{i,j} k(\text{dog}_i, \text{fish}_j)$$



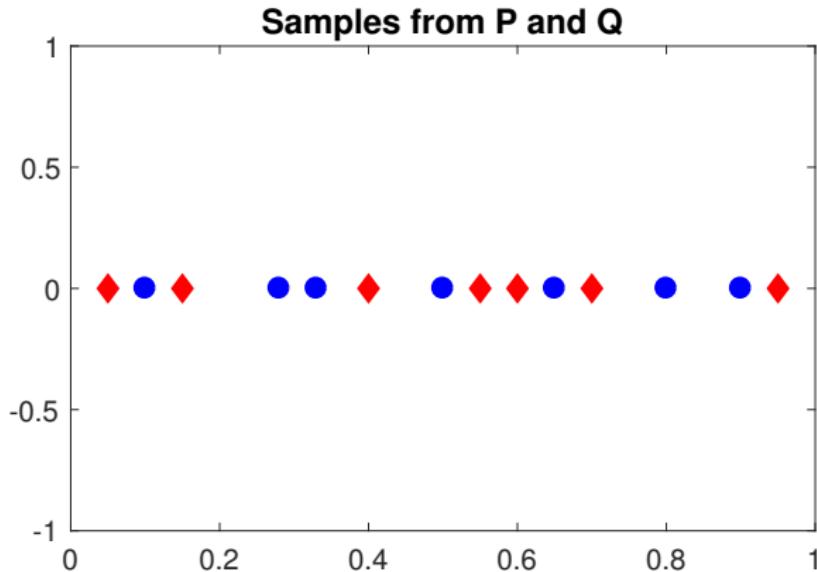
Integral probability metrics

Are P and Q different?



Integral probability metrics

Are P and Q different?

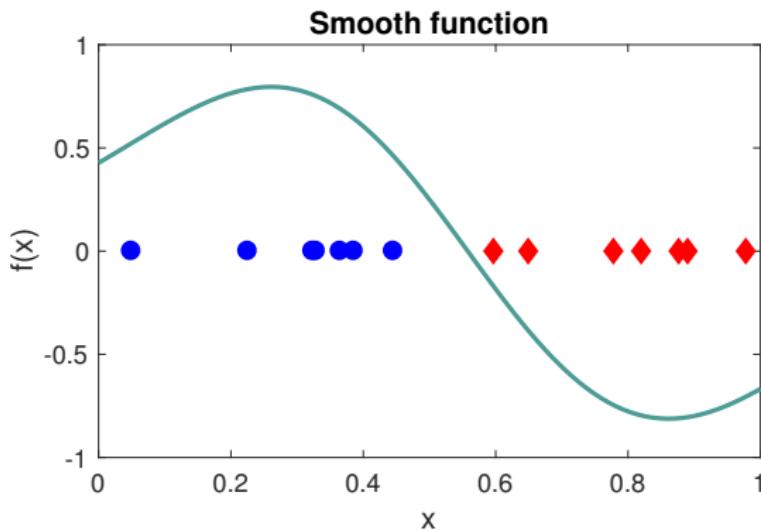


Integral probability metrics

Integral probability metric:

Find a "well behaved function" $f(x)$ to maximize

$$\mathbf{E}_P f(X) - \mathbf{E}_Q f(Y)$$

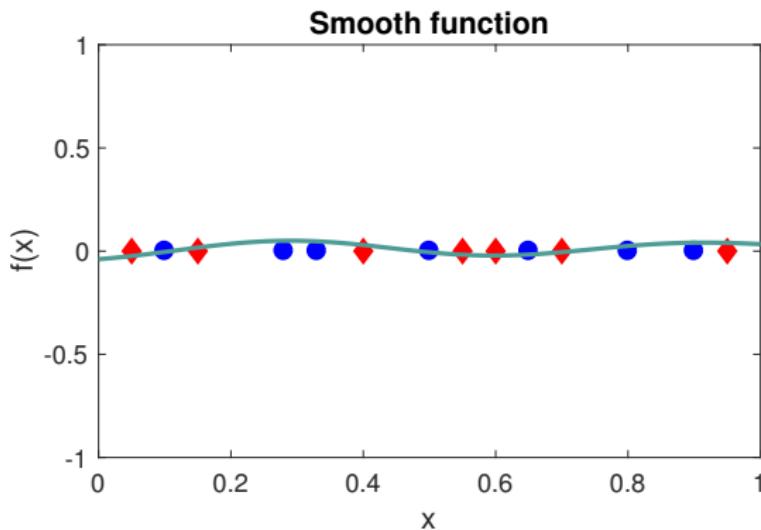


MMD as an integral probability metric

Integral probability metric:

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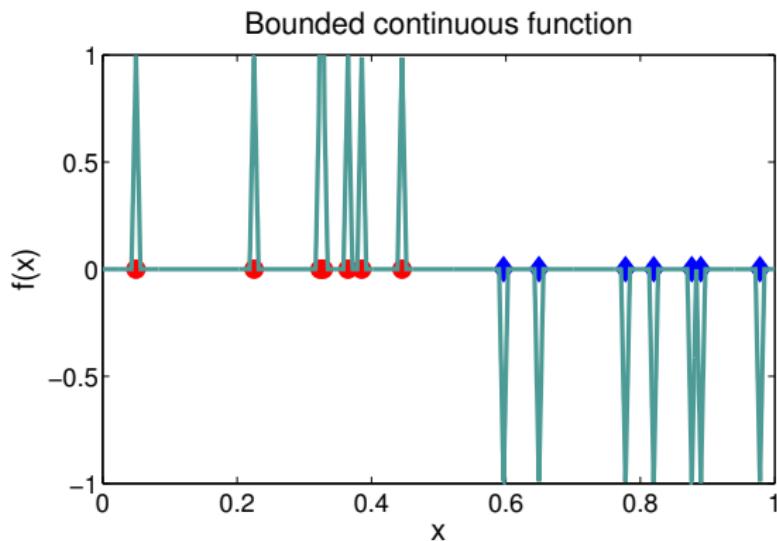
$$\mathbf{E}_P f(\mathcal{X}) - \mathbf{E}_Q f(\mathcal{Y})$$



MMD as an integral probability metric

What if the function is **not well behaved?**

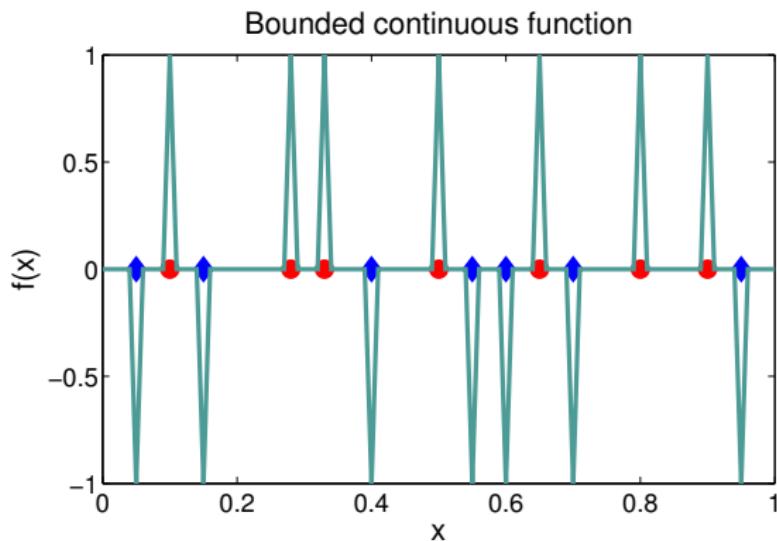
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MMD as an integral probability metric

What if the function is **not well behaved?**

$$\mathbf{E}_P f(X) - \mathbf{E}_Q f(Y)$$



MMD as an integral probability metric

Maximum mean discrepancy: smooth function for P vs Q

$$MMD(P, Q; \mathcal{F}) := \sup_{\|f\| \leq 1} [\mathbf{E}_{Pf}(X) - \mathbf{E}_{Qf}(Y)]$$

$(\mathcal{F} = \text{unit ball in RKHS } \mathcal{F})$

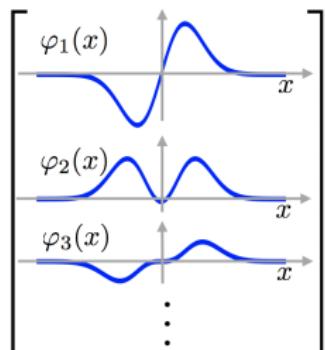
MMD as an integral probability metric

Maximum mean discrepancy: smooth function for P vs Q

$$MMD(P, Q; \mathcal{F}) := \sup_{\|\mathbf{f}\| \leq 1} [\mathbf{E}_{P\mathbf{f}}(X) - \mathbf{E}_{Q\mathbf{f}}(Y)]$$

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Functions are linear combinations of features:

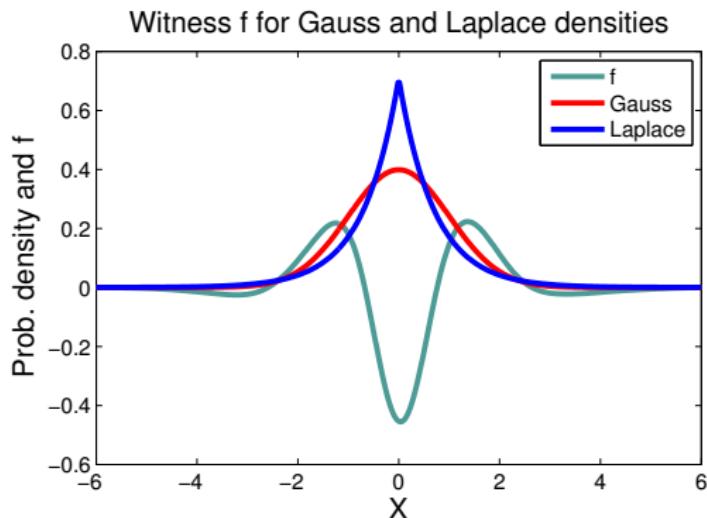
$$\mathbf{f}(x) = \langle \mathbf{f}, \varphi(x) \rangle_{\mathcal{F}} = \sum_{\ell=1}^{\infty} f_{\ell} \varphi_{\ell}(x) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix}^{\top}$$

$$\|\mathbf{f}\|_{\mathcal{F}}^2 := \sum_{i=1}^{\infty} f_i^2 \leq 1$$

MMD as an integral probability metric

Maximum mean discrepancy: smooth function for P vs Q

$$MMD(P, Q; \mathcal{F}) := \sup_{\|f\| \leq 1} [\mathbf{E}_P f(X) - \mathbf{E}_Q f(Y)]$$

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MMD as an integral probability metric

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Expectations of functions are linear combinations
of expected features

$$\mathbf{E}_P(f(X)) = \langle f, \mathbf{E}_P \varphi(X) \rangle_{\mathcal{F}} = \langle f, \mu_P \rangle_{\mathcal{F}}$$

(always true if kernel is bounded)

MMD as an integral probability metric

Maximum mean discrepancy: smooth function for P vs Q

$$MMD(P, Q; \mathcal{F}) := \sup_{\|f\| \leq 1} [\mathbf{E}_{Pf}(X) - \mathbf{E}_{Qf}(Y)]$$

$(\mathcal{F} = \text{unit ball in RKHS } \mathcal{F})$

For characteristic RKHS \mathcal{F} , $MMD(P, Q; \mathcal{F}) = 0$ iff $P = Q$

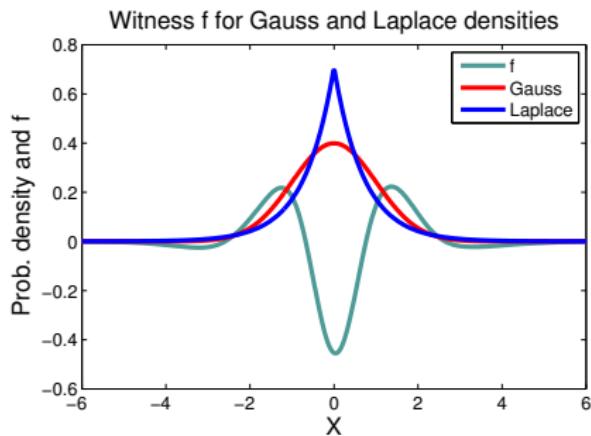
Other choices for witness function class:

- Bounded continuous [Dudley, 2002]
- Bounded variation 1 (Kolmogorov metric) [Müller, 1997]
- Lipschitz (Wasserstein distances) [Dudley, 2002]

Integral prob. metric vs feature difference

The MMD:

$$\begin{aligned} MMD(P, Q; F) \\ = \sup_{f \in F} [E_P f(X) - E_Q f(Y)] \end{aligned}$$



Integral prob. metric vs feature difference

The MMD:

use

$$\begin{aligned} MMD(P, Q; \mathcal{F}) &= \sup_{f \in \mathcal{F}} [\mathbf{E}_{Pf}(X) - \mathbf{E}_{Qf}(Y)] \\ &= \sup_{f \in \mathcal{F}} \langle f, \mu_P - \mu_Q \rangle_{\mathcal{F}} \end{aligned}$$
$$\mathbf{E}_{Pf}(X) = \langle \mu_P, f \rangle_{\mathcal{F}}$$

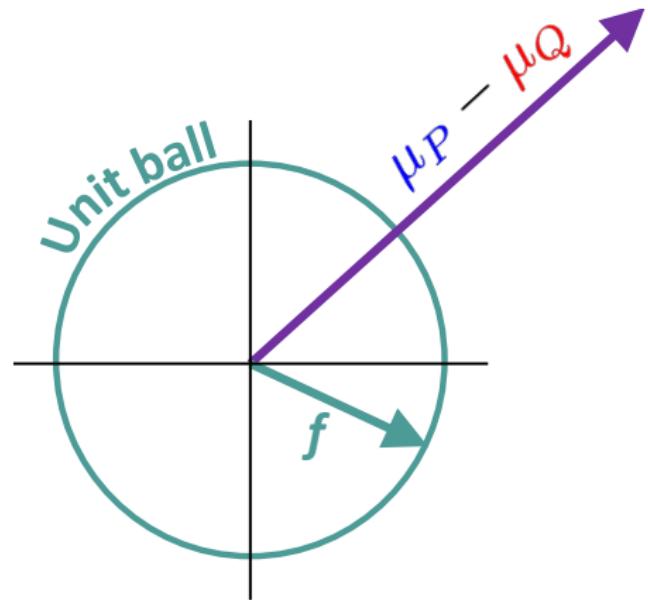
Integral prob. metric vs feature difference

The MMD:

$$MMD(P, Q; F)$$

$$= \sup_{f \in F} [\mathbf{E}_P f(X) - \mathbf{E}_Q f(Y)]$$

$$= \sup_{f \in F} \langle f, \mu_P - \mu_Q \rangle_{\mathcal{F}}$$



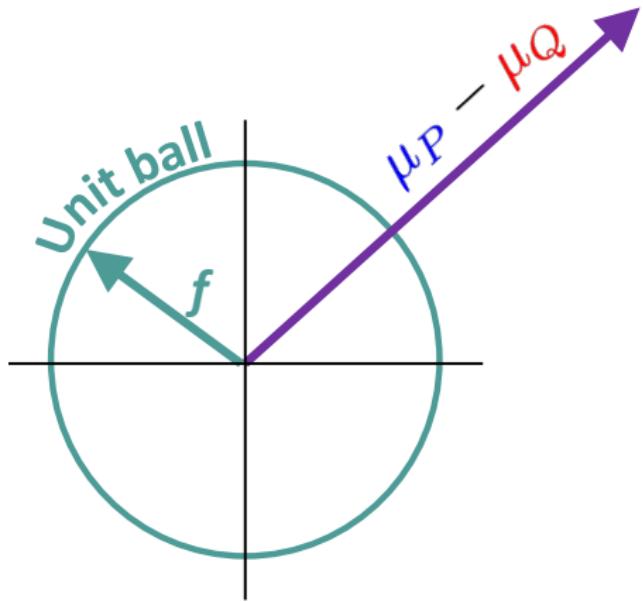
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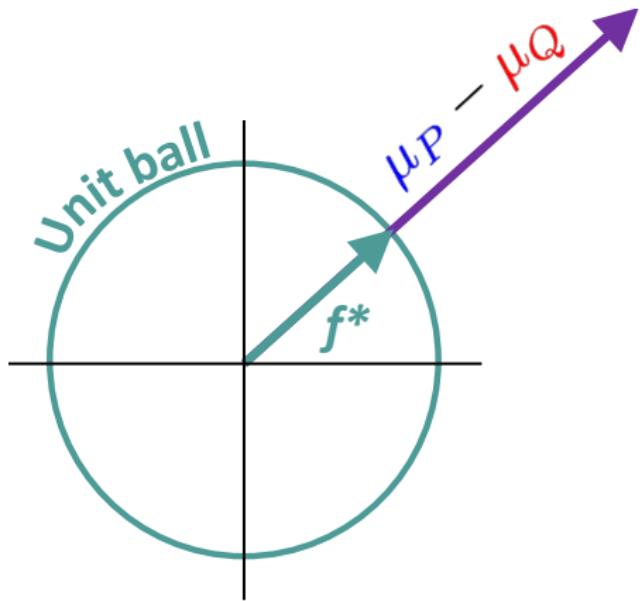
Integral prob. metric vs feature difference

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$$f^* = \frac{\mu_P - \mu_Q}{\|\mu_P - \mu_Q\|}$$

Integral prob. metric vs feature difference

The MMD:

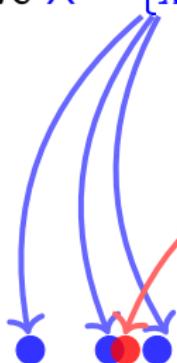
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Function view and feature view equivalent
(kernel case only)

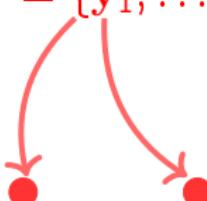
Construction of MMD witness

Construction of empirical **witness function** (proof: next slide!)

Observe $X = \{x_1, \dots, x_n\} \sim P$

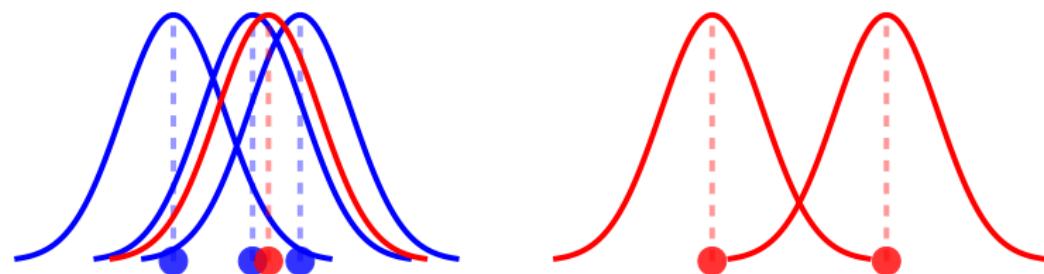


Observe $Y = \{y_1, \dots, y_n\} \sim Q$



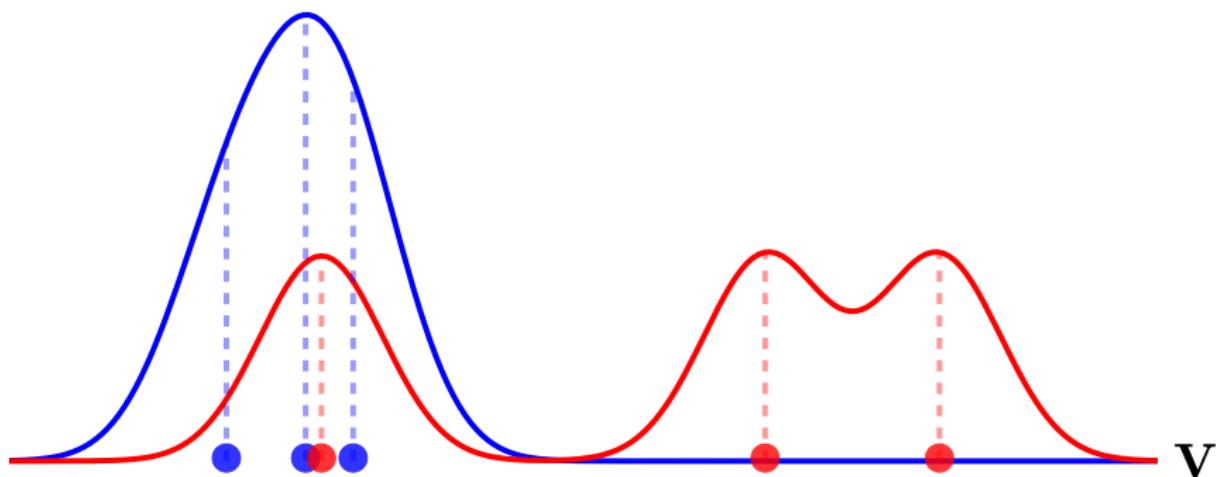
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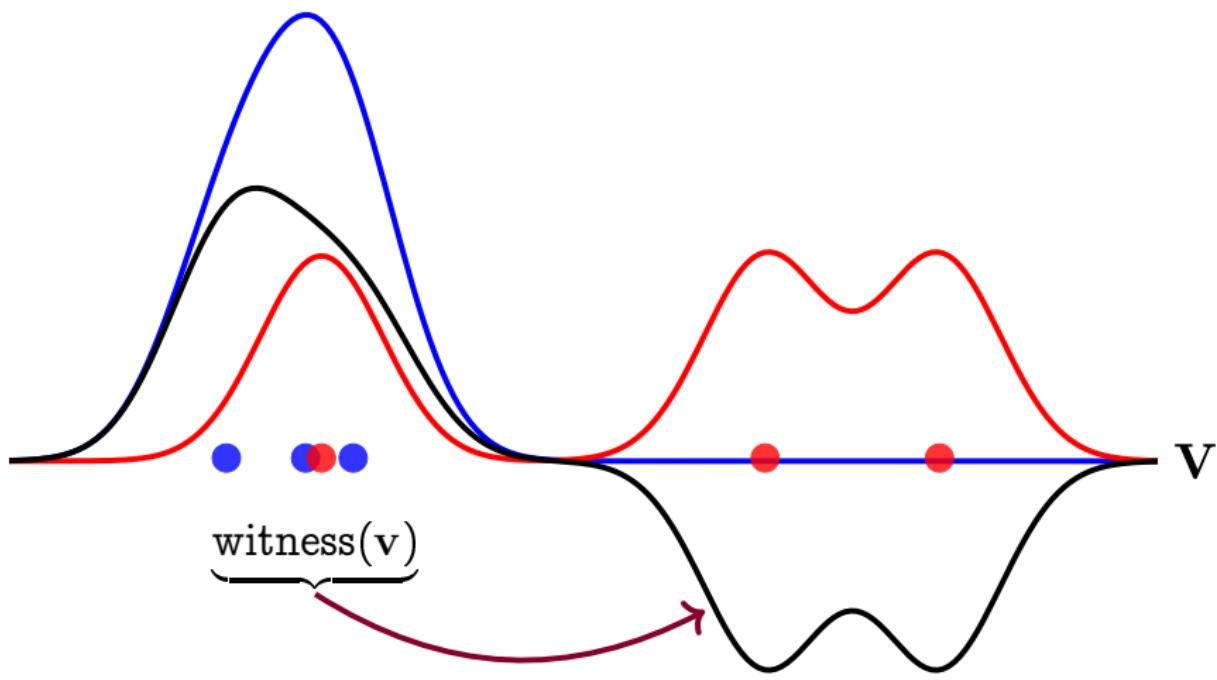
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Construction of MMD witness

Construction of empirical **witness function** (proof: next slide!)



Derivation of empirical witness function

Recall the witness function expression

$$f^* \propto \mu_P - \mu_Q$$

Derivation of empirical witness function

Recall the **witness function** expression

$$f^* \propto \mu_P - \mu_Q$$

The empirical feature mean for P

$$\hat{\mu}_P := \frac{1}{n} \sum_{i=1}^n \varphi(x_i)$$

Derivation of empirical witness function

Recall the **witness function** expression

$$\textcolor{teal}{f}^* \propto \mu_P - \mu_Q$$

The empirical feature mean for P

$$\widehat{\mu}_P := \frac{1}{n} \sum_{i=1}^n \varphi(x_i)$$

The empirical witness function at v

$$\textcolor{teal}{f}^*(v) = \langle \textcolor{teal}{f}^*, \varphi(v) \rangle_{\mathcal{F}}$$

Derivation of empirical witness function

Recall the **witness function** expression

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Derivation of empirical witness function

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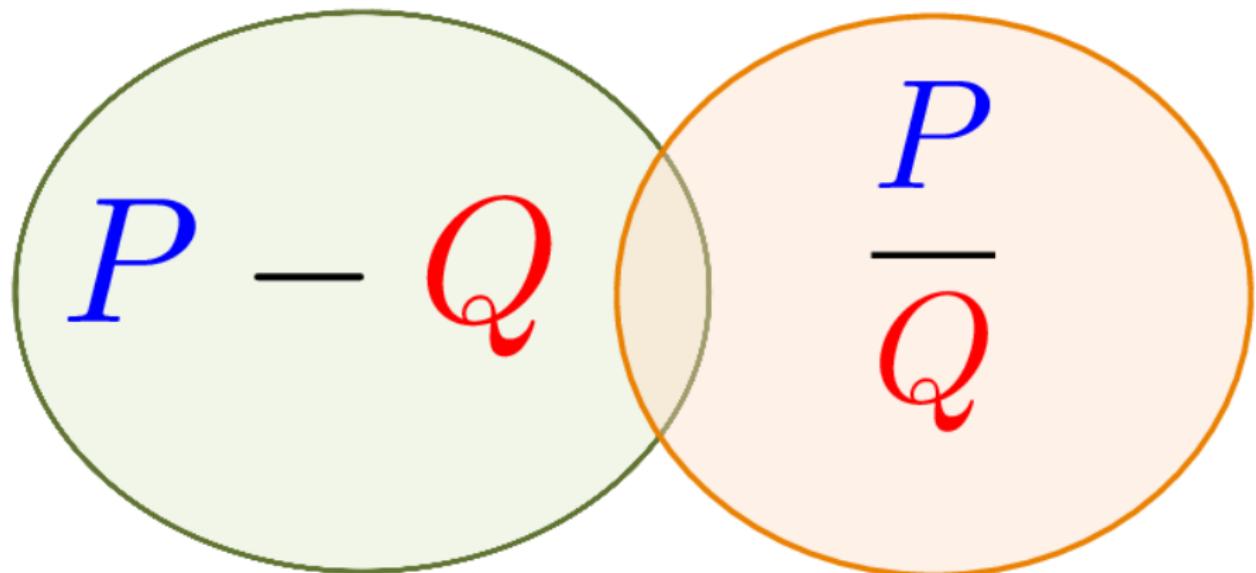
The empirical witness function at v

$$\begin{aligned} f^*(v) &= \langle f^*, \varphi(v) \rangle_{\mathcal{F}} \\ &\propto \langle \hat{\mu}_P - \hat{\mu}_Q, \varphi(v) \rangle_{\mathcal{F}} \\ &= \frac{1}{n} \sum_{i=1}^n k(\textcolor{blue}{x}_i, v) - \frac{1}{n} \sum_{i=1}^n k(\textcolor{red}{y}_i, v) \end{aligned}$$

Don't need explicit feature coefficients $f^* := [\ f_1^* \ f_2^* \ \dots \]$

Interlude: divergence measures

Divergences



Divergences

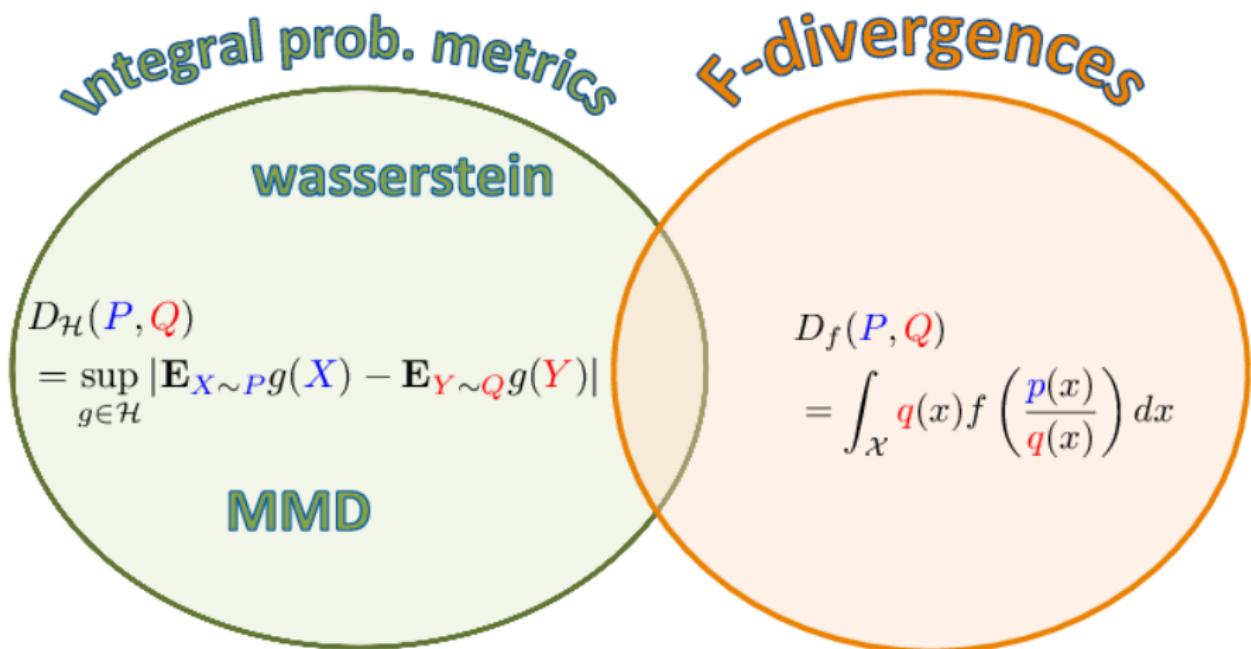
Integral prob. metrics

F-divergences

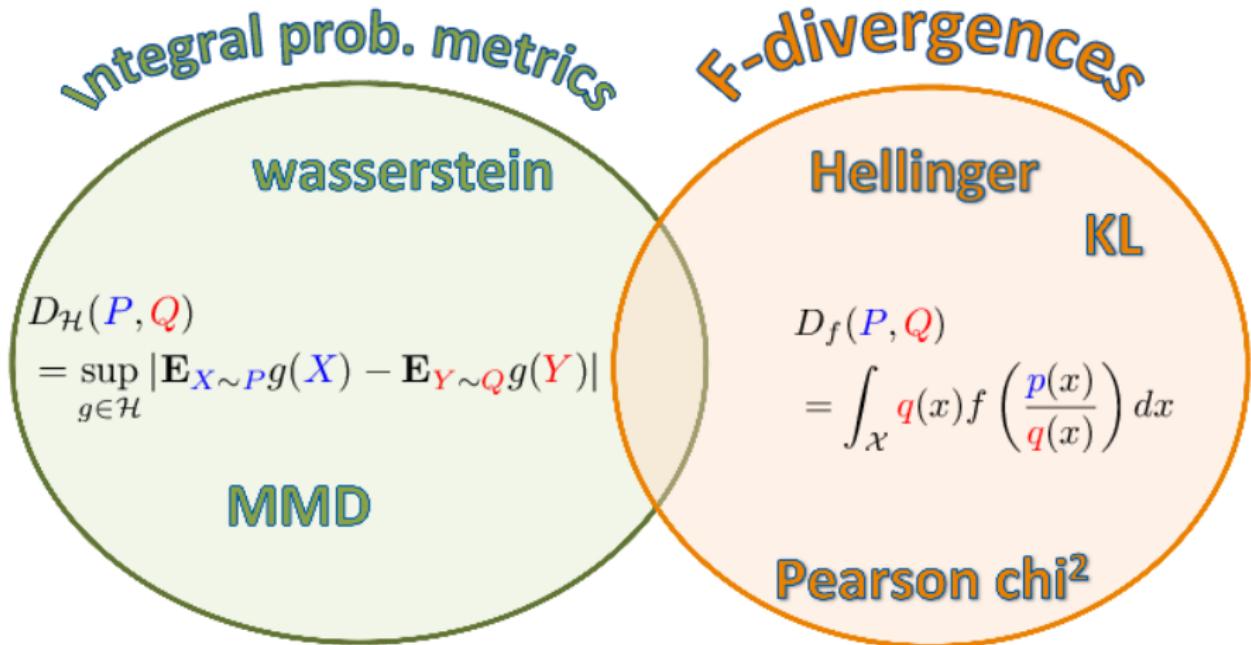
$$D_{\mathcal{H}}(\mathbf{P}, \mathbf{Q}) = \sup_{g \in \mathcal{H}} |\mathbf{E}_{X \sim \mathbf{P}} g(X) - \mathbf{E}_{Y \sim \mathbf{Q}} g(Y)|$$

$$D_f(\mathbf{P}, \mathbf{Q}) = \int_{\mathcal{X}} q(x) f\left(\frac{p(x)}{q(x)}\right) dx$$

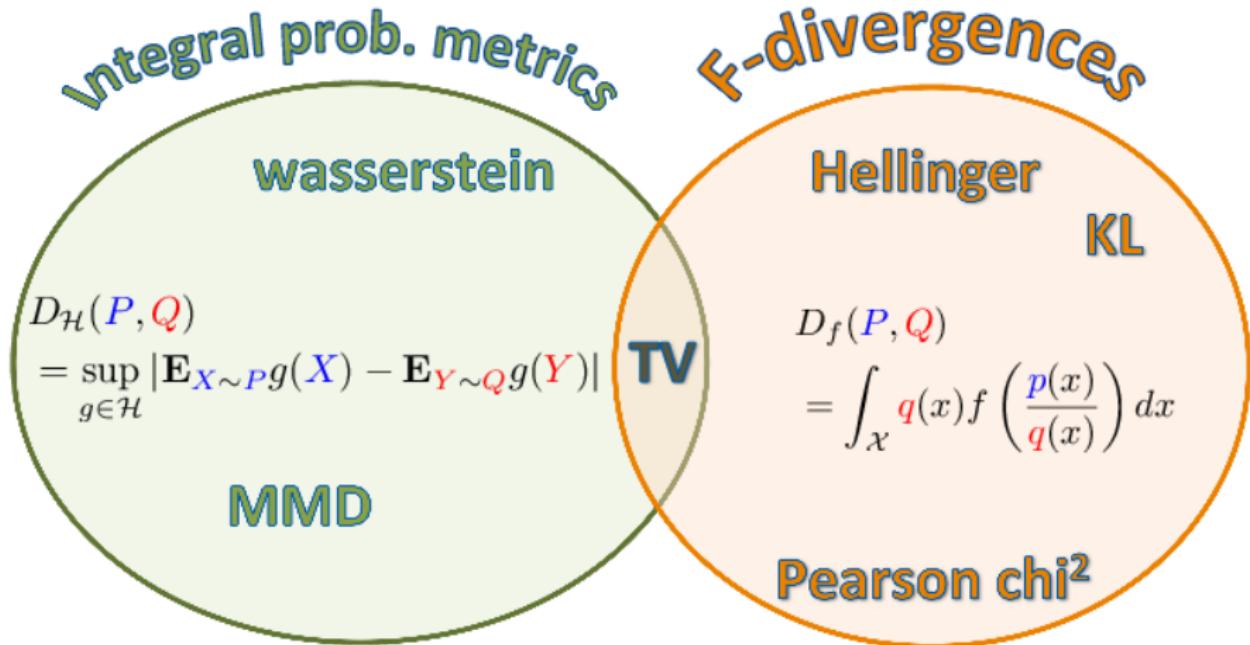
Divergences



Divergences



Divergences



Sriperumbudur, Fukumizu, G, Schoelkopf, Lanckriet (2012)

Two-Sample Testing with MMD

A statistical test using MMD

The empirical MMD:

$$\widehat{MMD}^2 = \frac{1}{n(n-1)} \sum_{i \neq j} k(\mathbf{x}_i, \mathbf{x}_j) + \frac{1}{n(n-1)} \sum_{i \neq j} k(\mathbf{y}_i, \mathbf{y}_j) - \frac{2}{n^2} \sum_{i,j} k(\mathbf{x}_i, \mathbf{y}_j)$$

How does this help decide whether $P = Q$?

A statistical test using MMD

The empirical MMD:

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Perspective from [statistical hypothesis testing](#):

- Null hypothesis \mathcal{H}_0 when $P = Q$
 - should see \widehat{MMD}^2 “close to zero”.
- Alternative hypothesis \mathcal{H}_1 when $P \neq Q$
 - should see \widehat{MMD}^2 “far from zero”

A statistical test using MMD

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Perspective from [statistical hypothesis testing](#):

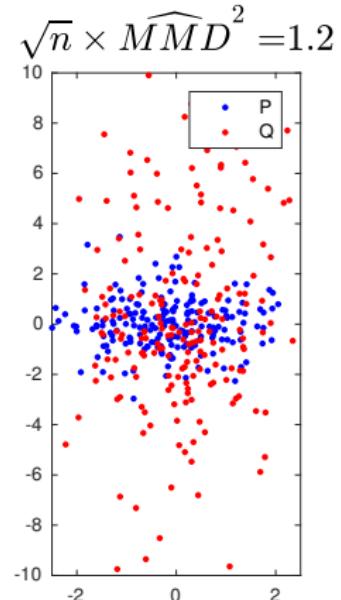
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 - should see \widehat{MMD}^2 “far from zero”

Want [Threshold](#) c_α for \widehat{MMD}^2 to get [false positive rate](#) α

Behaviour of \widehat{MMD}^2 when $P \neq Q$

Draw $n = 200$ i.i.d samples from P and Q

- Laplace with different y-variance.
- $\sqrt{n} \times \widehat{MMD}^2 = 1.2$

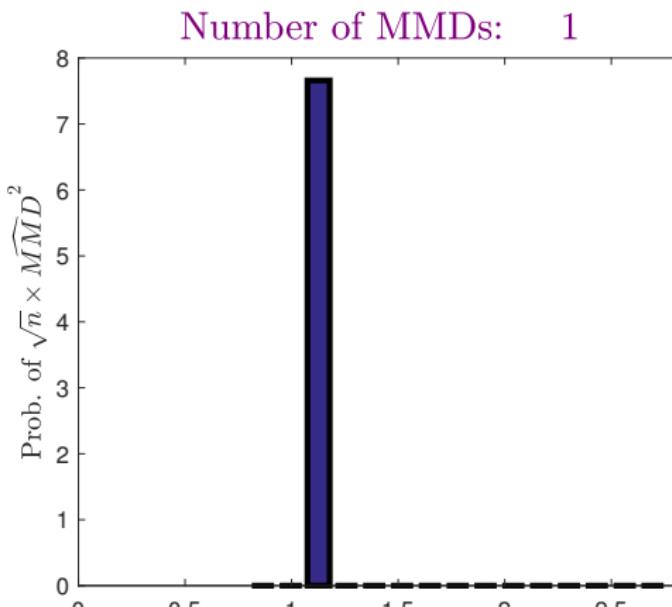


Draw $n = 200$ i.i.d samples from P and Q

Behaviour of \widehat{MMD}^2 when $P \neq Q$

Laplace with different y-variance.

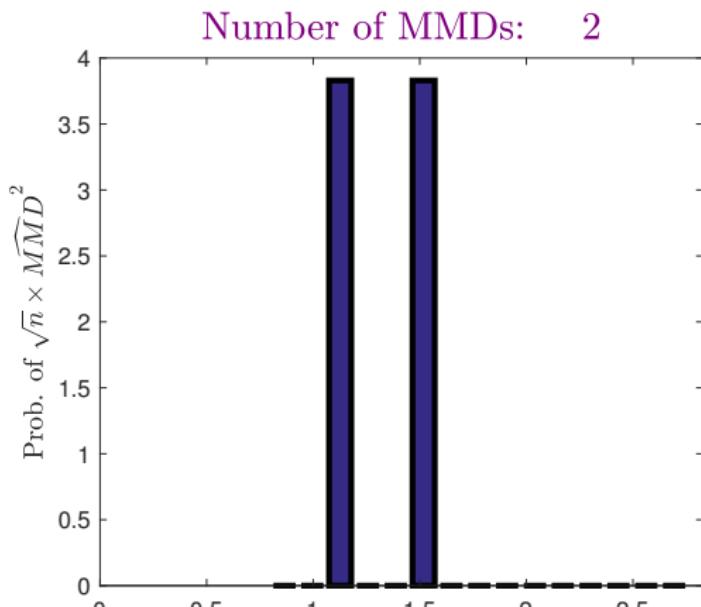
$$\sqrt{n} \times \widehat{MMD}^2 = 1.2$$



Draw $n = 200$ **new** samples from P and Q
Behaviour of \widehat{MMD}^2 when $P \neq Q$

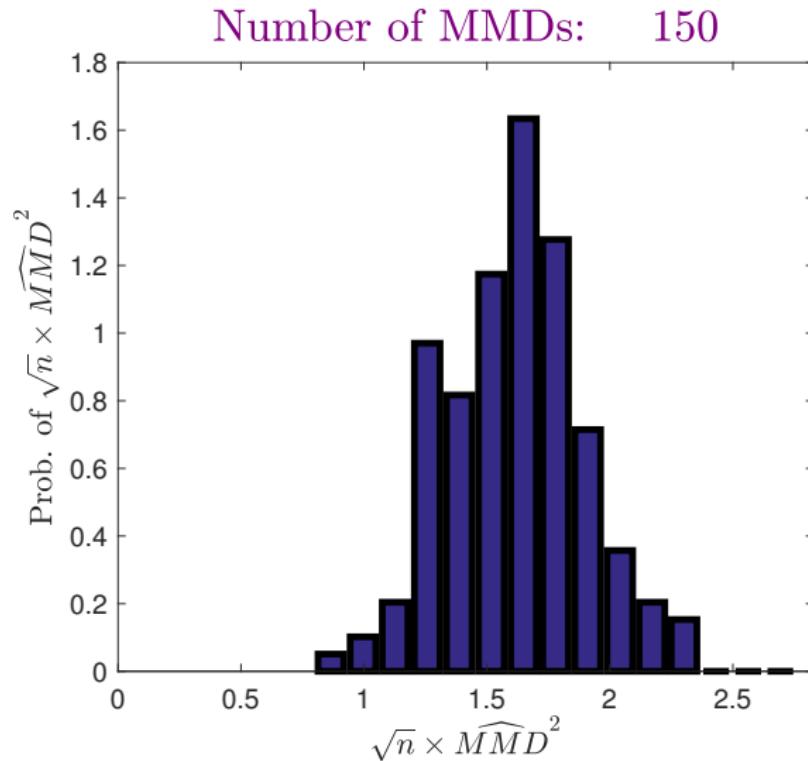
■ Laplace with different y-variance.

■ $\sqrt{n} \times \widehat{MMD}^2 = 1.5$



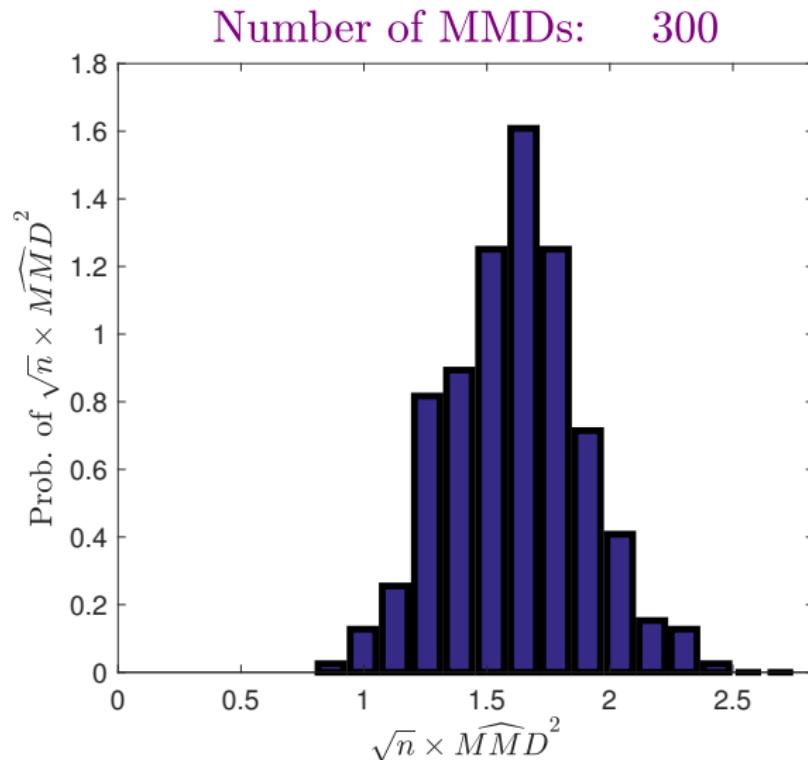
Behaviour of \widehat{MMD}^2 when $P \neq Q$

Repeat this 150 times ...



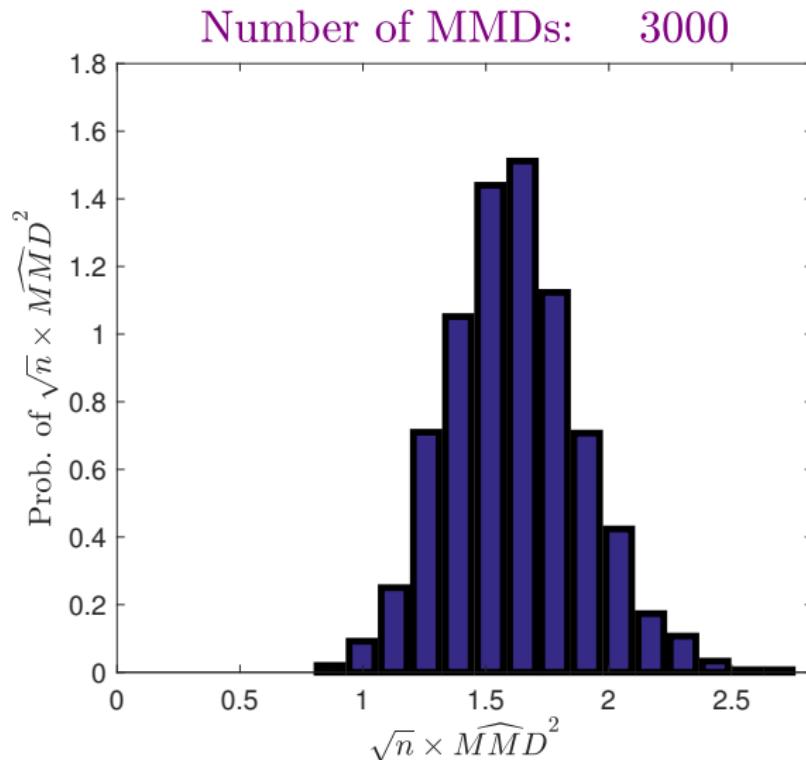
Behaviour of \widehat{MMD}^2 when $P \neq Q$

Repeat this 300 times ...



Behaviour of \widehat{MMD}^2 when $P \neq Q$

Repeat this 3000 times . . .

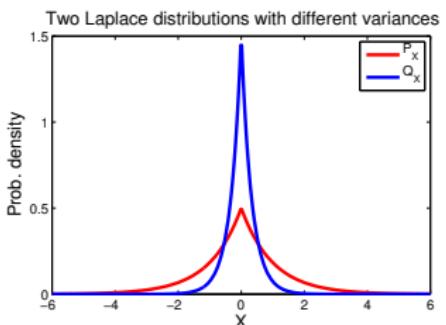
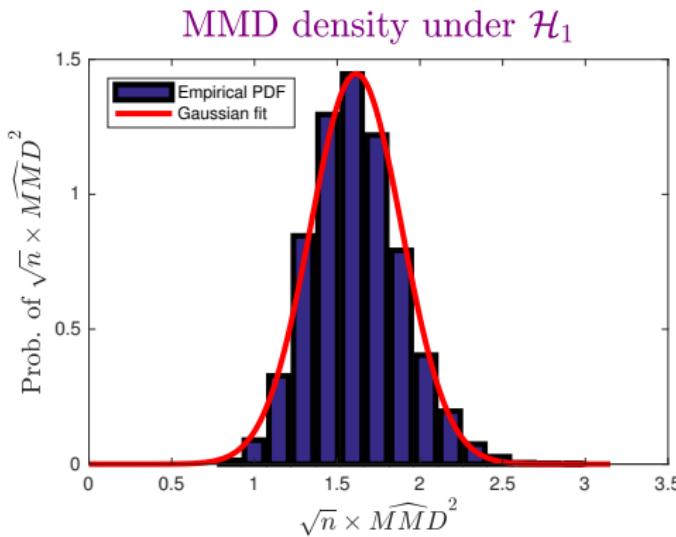


Asymptotics of \widehat{MMD}^2 when $P \neq Q$

When $P \neq Q$, statistic is asymptotically normal,

$$\frac{\widehat{MMD}^2 - \text{MMD}(P, Q)}{\sqrt{V_n(P, Q)}} \xrightarrow{D} \mathcal{N}(0, 1),$$

where variance $V_n(P, Q) = O(n^{-1})$.

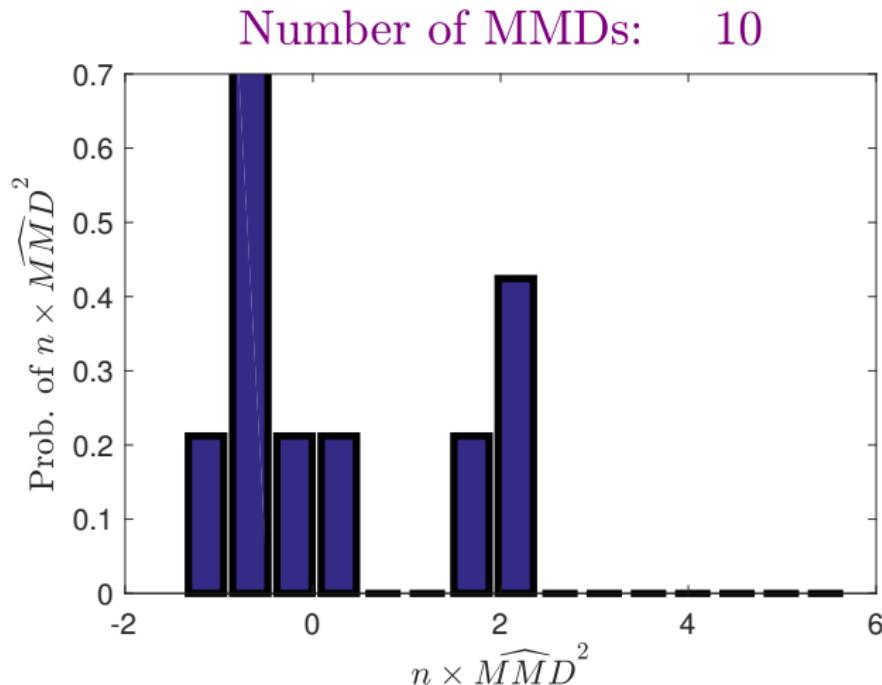


Behaviour of \widehat{MMD}^2 when $P = Q$

What happens when P and Q are the same?

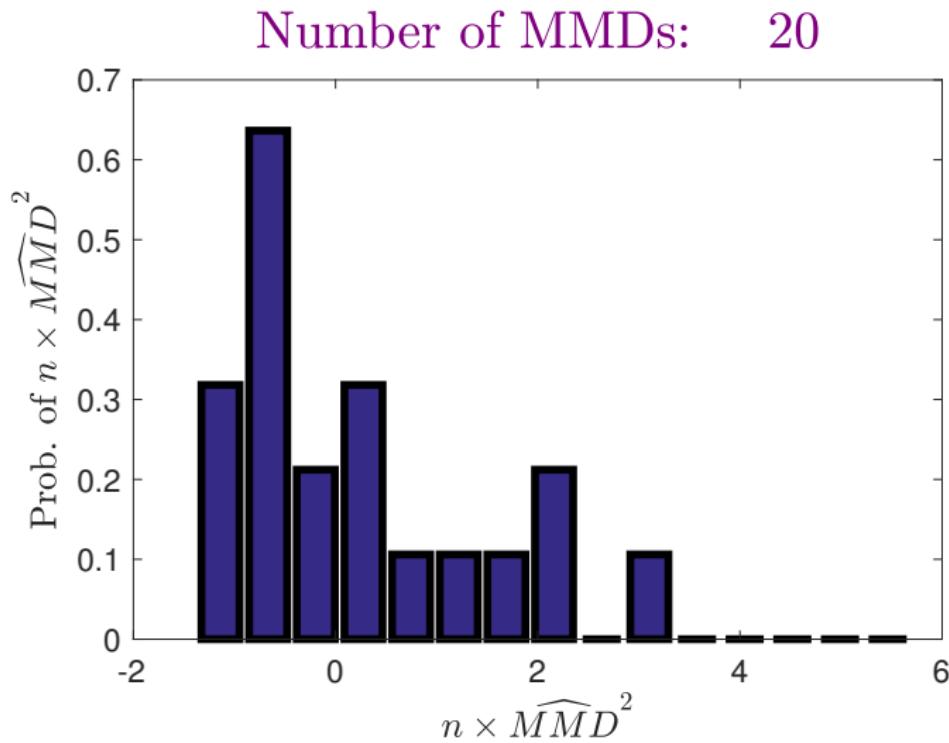
Behaviour of \widehat{MMD}^2 when $P = Q$

- Case of $P = Q = \mathcal{N}(0, 1)$



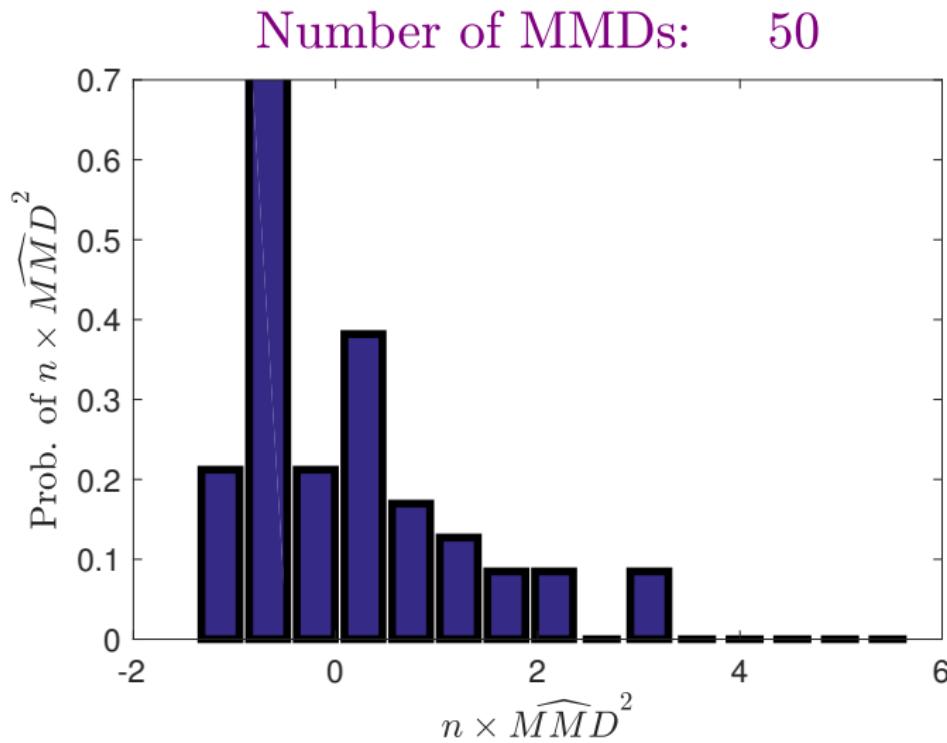
Behaviour of \widehat{MMD}^2 when $P = Q$

- Case of $P = Q = \mathcal{N}(0, 1)$



Behaviour of \widehat{MMD}^2 when $P = Q$

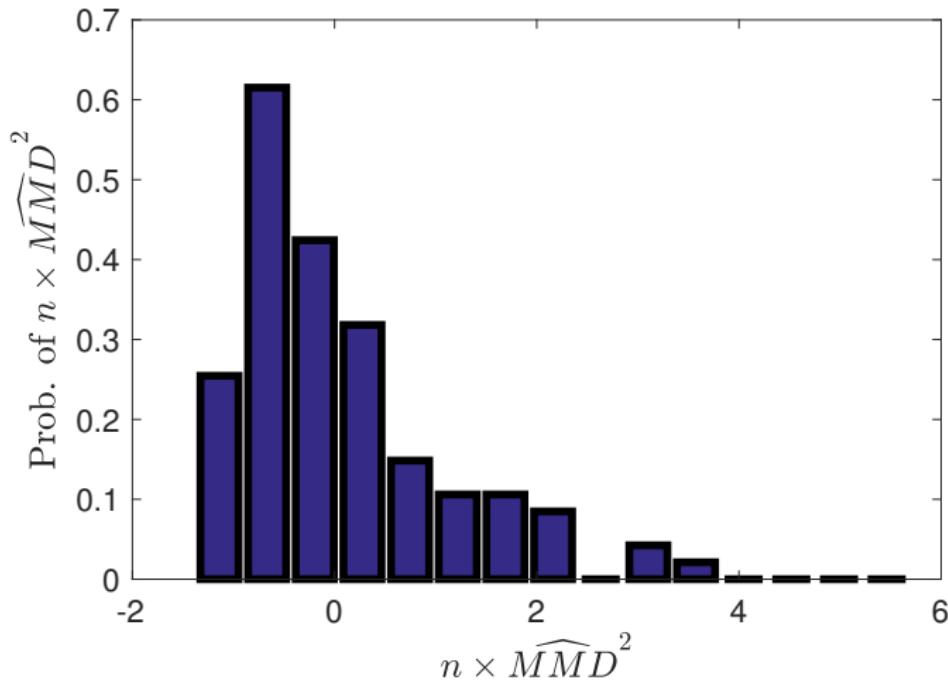
- Case of $P = Q = \mathcal{N}(0, 1)$



Behaviour of \widehat{MMD}^2 when $P = Q$

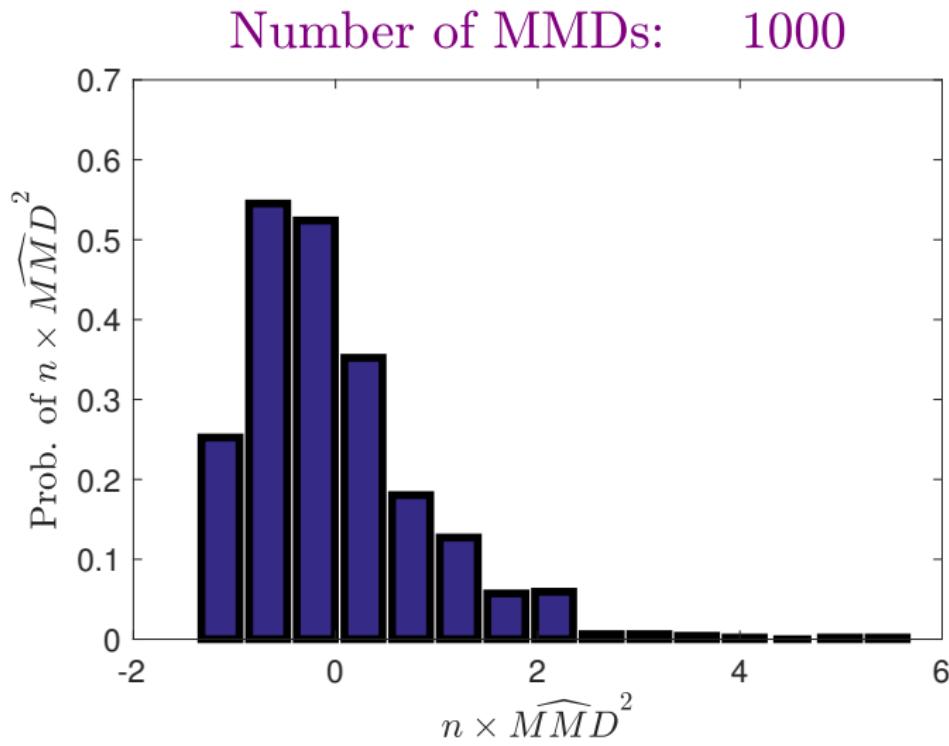
- Case of $P = Q = \mathcal{N}(0, 1)$

Number of MMDs: 100



Behaviour of \widehat{MMD}^2 when $P = Q$

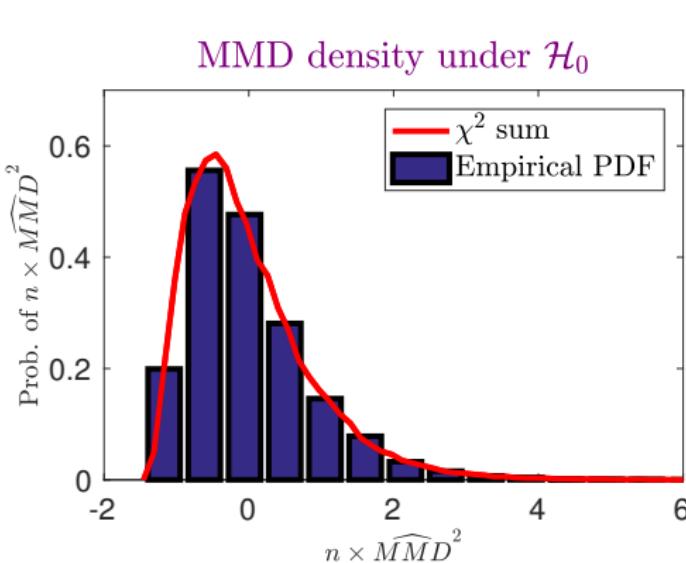
- Case of $P = Q = \mathcal{N}(0, 1)$



Asymptotics of \widehat{MMD}^2 when $P = Q$

Where $P = Q$, statistic has asymptotic distribution

$$n\widehat{MMD}^2 \sim \sum_{l=1}^{\infty} \lambda_l [z_l^2 - 2]$$



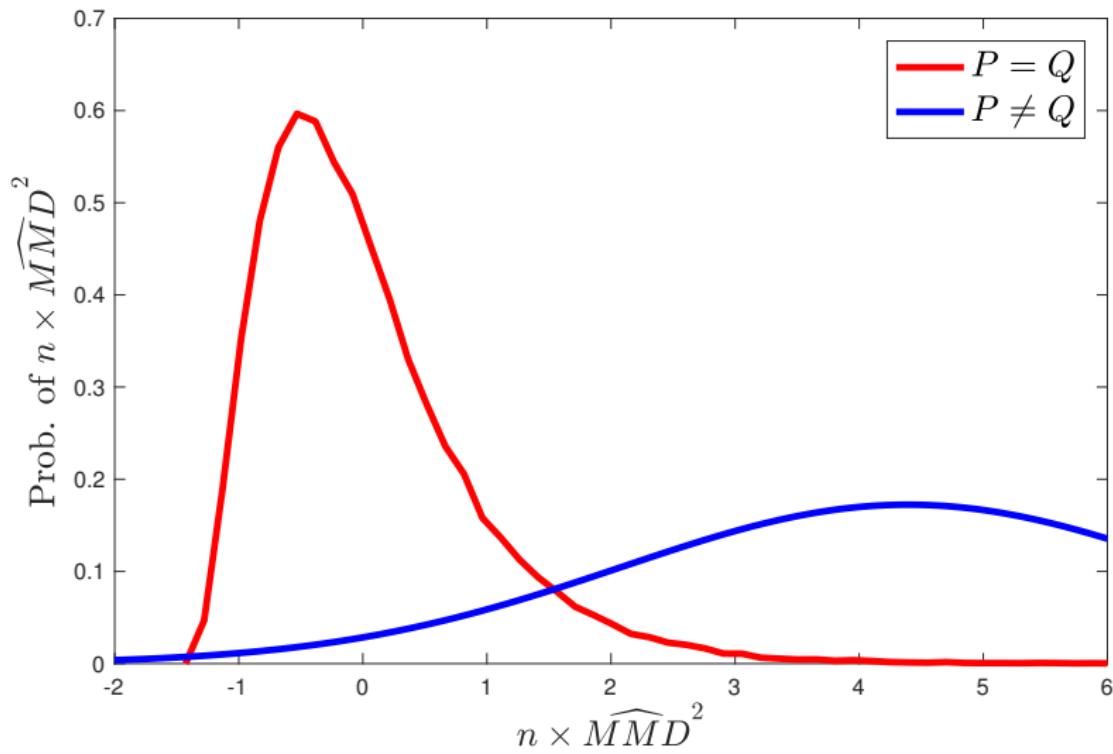
where

$$\lambda_i \psi_i(x') = \underbrace{\int_{\mathcal{X}} \tilde{k}(x, x') \psi_i(x) dP(x)}_{\text{centred}}$$

$$z_l \sim \mathcal{N}(0, 2) \quad \text{i.i.d.}$$

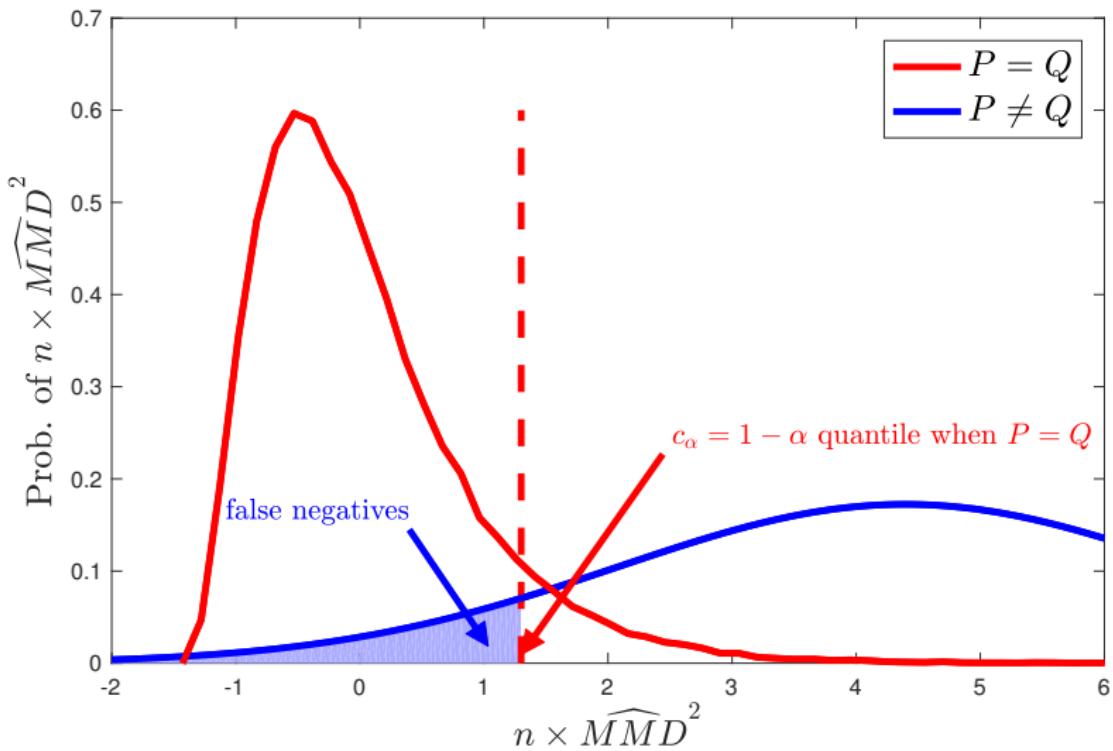
A statistical test

A summary of the asymptotics:



A statistical test

Test construction: (G., Borgwardt, Rasch, Schoelkopf, and Smola, JMLR 2012)



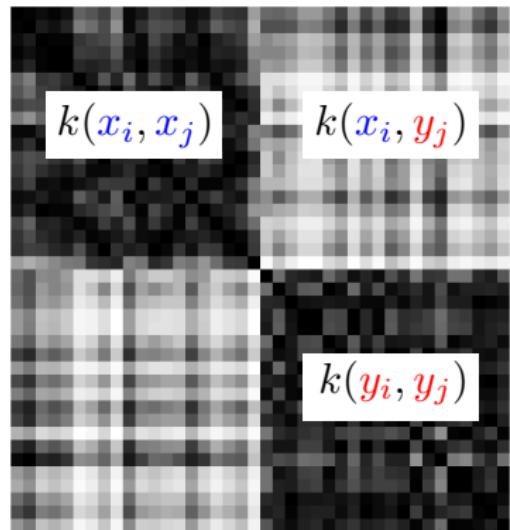
How do we get test threshold c_α ?

Original empirical MMD for dogs and fish:

$$X = \begin{bmatrix} \text{Basset Hound} & \text{Beagle} & \text{Basset Hound} & \dots \end{bmatrix}$$

$$Y = \begin{bmatrix} \text{Butterfly Fish} & \text{Coral Fish} & \text{Goldfish} & \dots \end{bmatrix}$$

$$\widehat{\text{MMD}}^2 = \frac{1}{n(n-1)} \sum_{i \neq j} k(\mathbf{x}_i, \mathbf{x}_j) + \frac{1}{n(n-1)} \sum_{i \neq j} k(\mathbf{y}_i, \mathbf{y}_j) - \frac{2}{n^2} \sum_{i,j} k(\mathbf{x}_i, \mathbf{y}_j)$$



How do we get test threshold c_α ?

Permuted **dog** and **fish** samples (**merdogs**):

$$\tilde{X} = \begin{bmatrix} \text{fish emoji} & \text{dog emoji} & \text{fish emoji} & \dots \end{bmatrix}$$

$$\tilde{Y} = \begin{bmatrix} \text{dog emoji} & \text{fish emoji} & \text{dog emoji} & \dots \end{bmatrix}$$

How do we get test threshold c_α ?

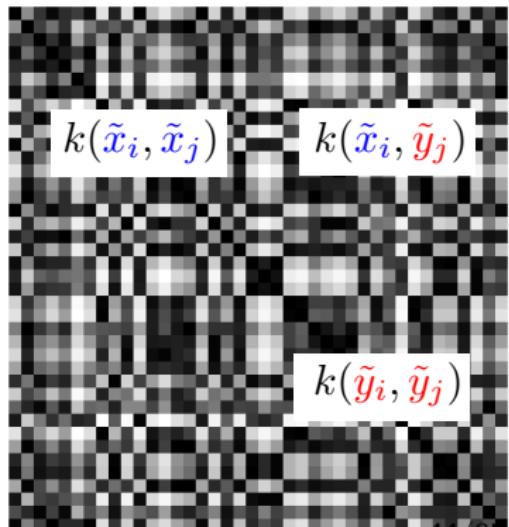
Permuted dog and fish samples (**merdogs**):

$$\tilde{X} = [\text{fish emoji} \quad \text{dog emoji} \quad \text{fish emoji} \quad \dots]$$

$$\tilde{Y} = [\text{dog emoji} \quad \text{fish emoji} \quad \text{dog emoji} \quad \dots]$$

$$\begin{aligned}\widehat{MMD}^2 &= \frac{1}{n(n-1)} \sum_{i \neq j} k(\tilde{x}_i, \tilde{x}_j) \\ &\quad + \frac{1}{n(n-1)} \sum_{i \neq j} k(\tilde{y}_i, \tilde{y}_j) \\ &\quad - \frac{2}{n^2} \sum_{i,j} k(\tilde{x}_i, \tilde{y}_j)\end{aligned}$$

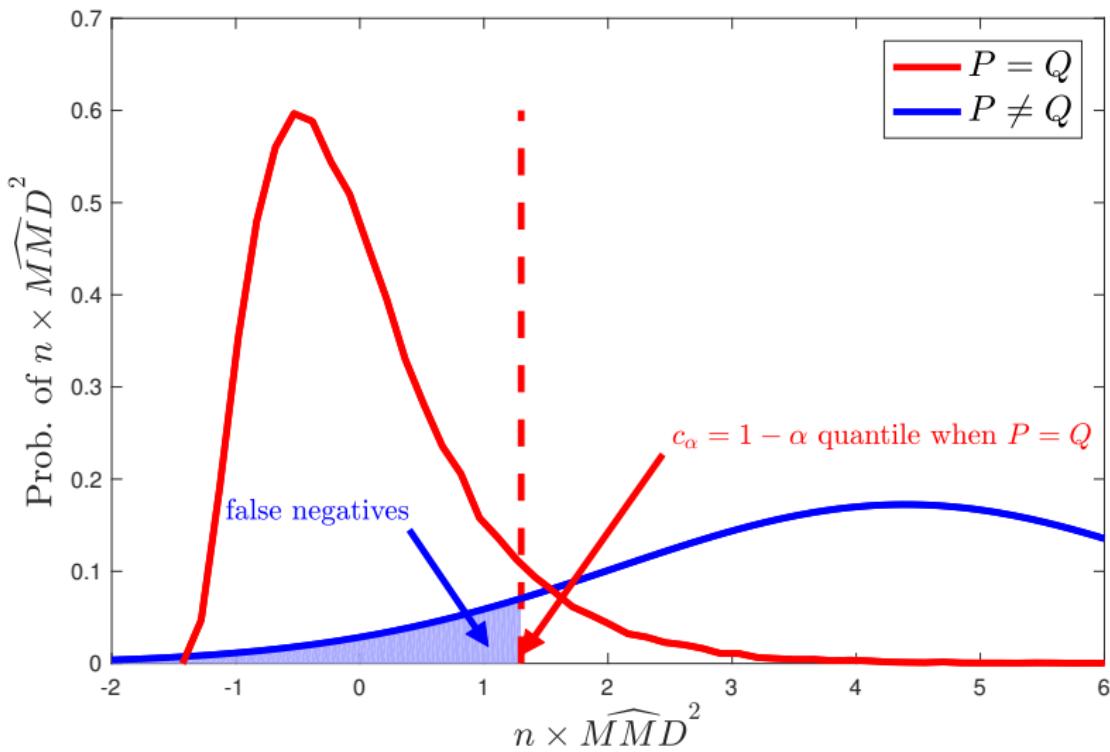
Permutation simulates
 $P = Q$



How to choose the best kernel:
optimising the kernel parameters

Graphical illustration

- Maximising test power same as minimizing false negatives



Optimizing kernel for test power

The power of our test:

$$\Pr_{P \neq Q} \left(n \widehat{\text{MMD}}^2 > \hat{c}_\alpha \right)$$

Optimizing kernel for test power

The power of our test:

$$\begin{aligned} & \Pr_{P \neq Q} \left(n \widehat{\text{MMD}}^2 > \hat{c}_\alpha \right) \\ & \rightarrow \Phi \left(\frac{\text{MMD}^2(P, Q)}{\sqrt{V_n(P, Q)}} - \frac{c_\alpha}{n \sqrt{V_n(P, Q)}} \right) \end{aligned}$$

where

- Φ is the CDF of the standard normal distribution.
- \hat{c}_α is an estimate of c_α test threshold.

Optimizing kernel for test power

The power of our test:

$$\begin{aligned} & \Pr_{P \neq Q} \left(n \widehat{\text{MMD}}^2 > \hat{c}_\alpha \right) \\ & \rightarrow \Phi \left(\underbrace{\frac{\text{MMD}^2(P, Q)}{\sqrt{V_n(P, Q)}}}_{O(n^{1/2})} - \underbrace{\frac{c_\alpha}{n \sqrt{V_n(P, Q)}}}_{O(n^{-1/2})} \right) \end{aligned}$$

Variance under \mathcal{H}_1 decreases as $\sqrt{V_n(P, Q)} \sim O(n^{-1/2})$

For large n , second term negligible!

Optimizing kernel for test power

The power of our test:

$$\Pr_{P \neq Q} \left(n \widehat{\text{MMD}}^2 > \hat{c}_\alpha \right)$$
$$\rightarrow \Phi \left(\frac{\text{MMD}^2(P, Q)}{\sqrt{V_n(P, Q)}} - \frac{c_\alpha}{n \sqrt{V_n(P, Q)}} \right)$$

To maximize test power, maximize

$$\frac{\text{MMD}^2(P, Q)}{\sqrt{V_n(P, Q)}}$$

(Sutherland, Tung, Strathmann, De, Ramdas, Smola, G., ICLR 2017)

Code: github.com/dougal-sutherland/opt-mmd

Troubleshooting for generative adversarial networks



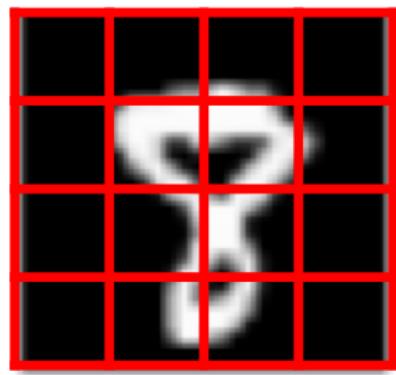
MNIST samples



Samples from a GAN

The ARD kernel

σ_1	σ_2	σ_3
σ_i	σ_{i+1}	σ_{i+2}

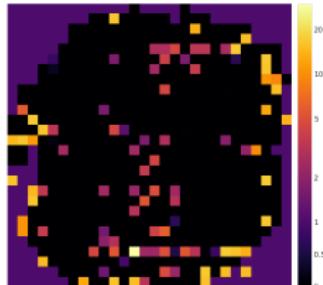


$$k(\boldsymbol{\gamma}, \boldsymbol{z}) = \prod_{i=1}^D \exp \left(\frac{-(\boldsymbol{\gamma}[i] - \boldsymbol{z}[i])^2}{\sigma_i^2} \right)$$

Troubleshooting for generative adversarial networks



MNIST samples



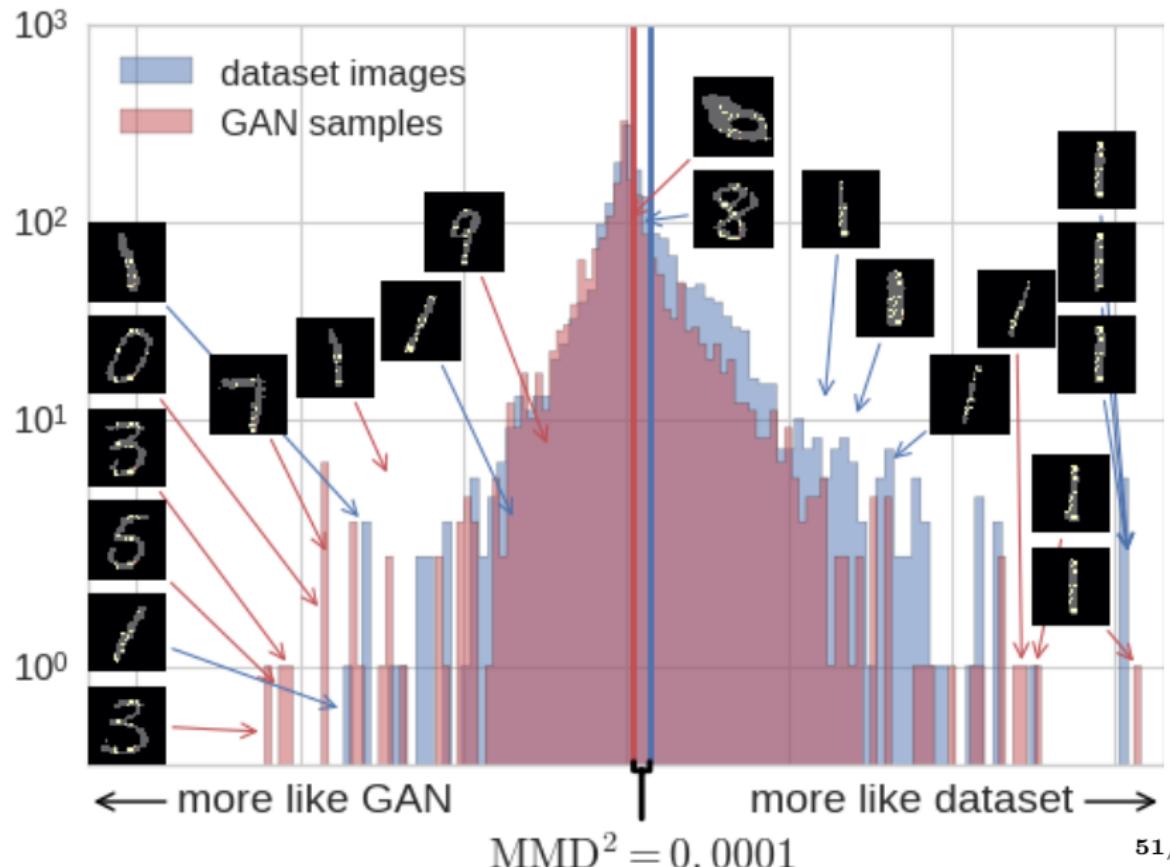
ARD map



Samples from a GAN

- Power for **optimized ARD kernel**: 1.00 at $\alpha = 0.01$
- Power for optimized RBF kernel: 0.57 at $\alpha = 0.01$

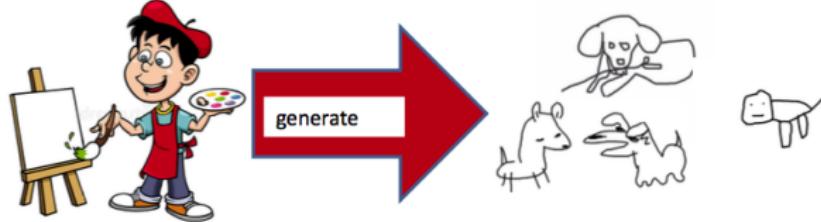
Troubleshooting generative adversarial networks



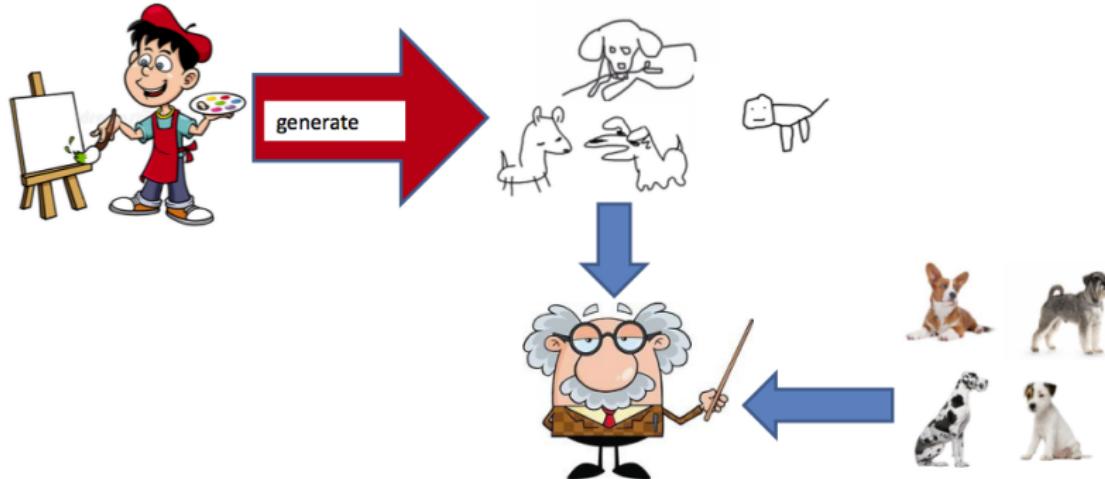
Training Generative Adversarial Networks with MMD Critic

Training Generative Adversarial Networks: Myths and Conjectures

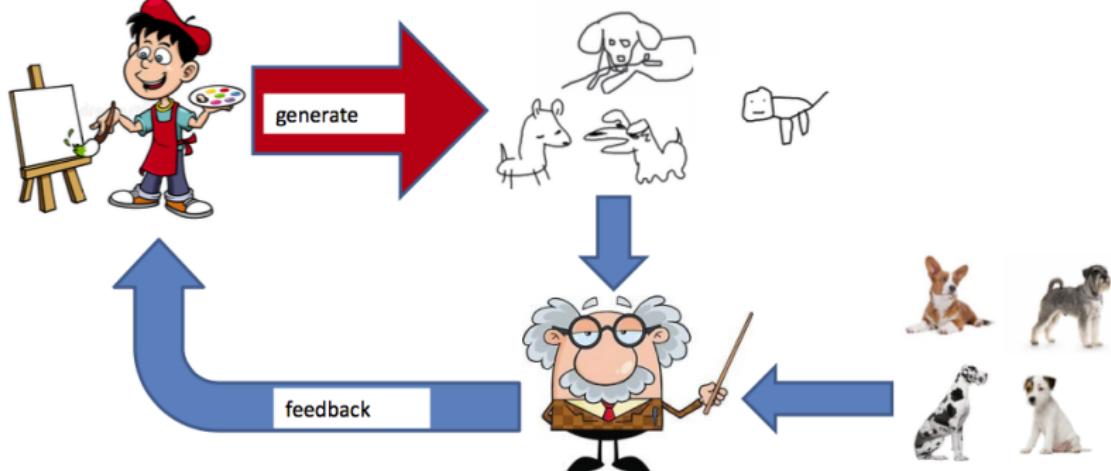
Reminder: GAN setting



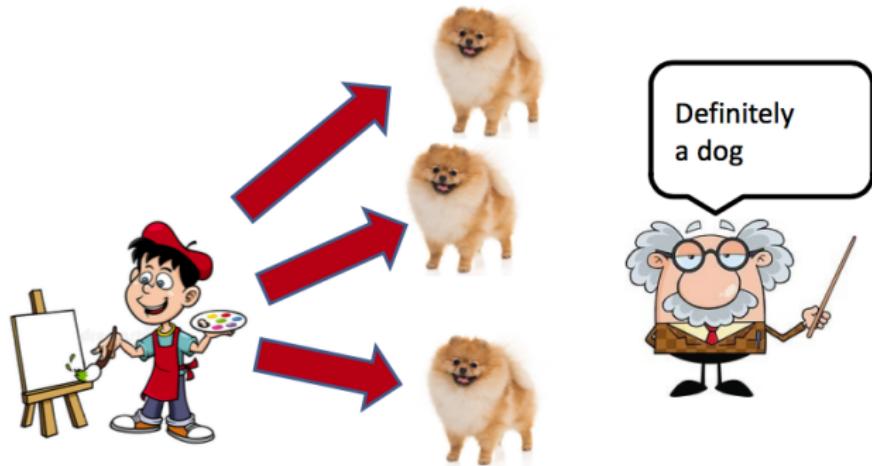
Reminder: GAN setting



Reminder: GAN setting



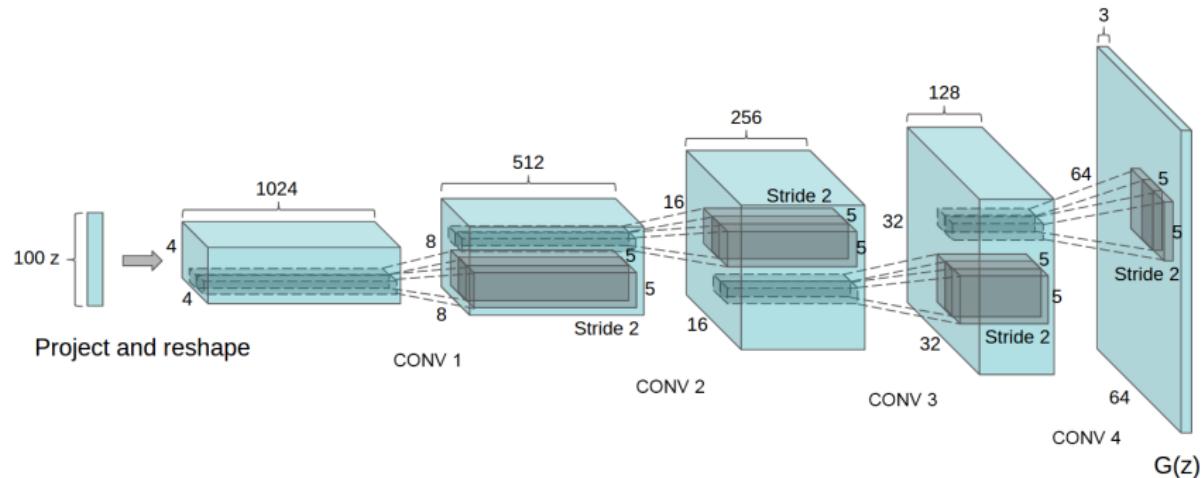
Why is classification not enough?



Classification **not** enough!
Need to compare **sets**

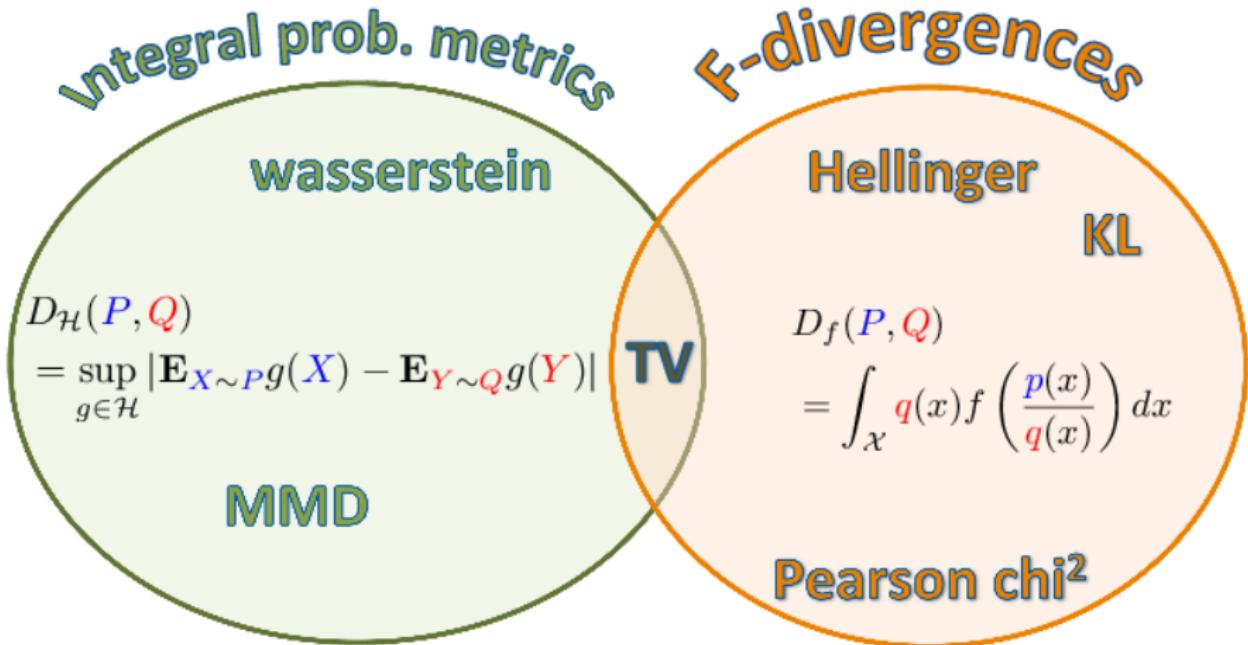
(otherwise student can just produce the **same dog** over and over)

What I won't cover: the generator



Radford, Metz, Chintala, ICLR 2016

Choices of critic



Sriperumbudur, Fukumizu, G, Schoelkopf, Lanckriet (2012)

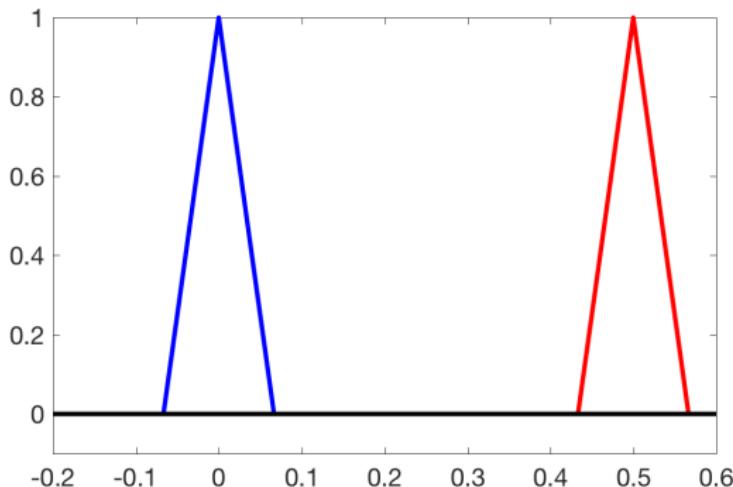
F-divergence as critic



An **unhelpful** critic: Jensen-Shannon, Arjovsky and Bottou [ICLR 2017]

$$D_{JS}(\textcolor{blue}{P}, \textcolor{red}{Q}) = \frac{1}{2}D_{KL}\left(\textcolor{blue}{p}, \frac{\textcolor{blue}{p}+\textcolor{red}{q}}{2}\right) + \frac{1}{2}D_{KL}\left(\textcolor{red}{q}, \frac{\textcolor{blue}{p}+\textcolor{red}{q}}{2}\right)$$

$$D_{JS}(\textcolor{blue}{P}, \textcolor{red}{Q}) = \log 2$$



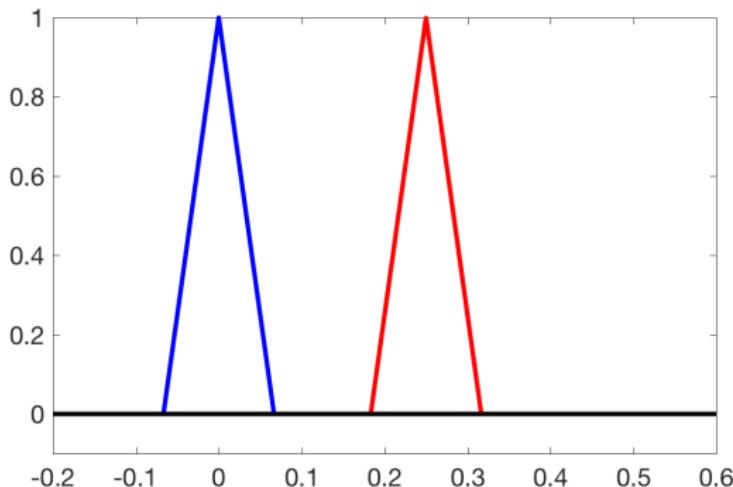
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F-divergence as critic



An **unhelpful** critic: Jensen-Shannon, Arjovsky and Bottou [ICLR 2017]

$$D_{JS}(\textcolor{blue}{P}, \textcolor{red}{Q}) = \frac{1}{2} D_{KL}\left(\textcolor{blue}{p}, \frac{\textcolor{blue}{p} + \textcolor{red}{q}}{2}\right) + \frac{1}{2} D_{KL}\left(\textcolor{red}{q}, \frac{\textcolor{blue}{p} + \textcolor{red}{q}}{2}\right)$$

What is done in practice?

F-divergence as critic



An **unhelpful** critic: Jensen-Shannon, Arjovsky and Bottou [ICLR 2017]

$$D_{JS}(P, Q) = \frac{1}{2} D_{KL}\left(P, \frac{P+Q}{2}\right) + \frac{1}{2} D_{KL}\left(Q, \frac{P+Q}{2}\right)$$

What is done in practice?

- Use a **variational approximation** to the critic, alternate generator and critic training (we will return to this!) Goodfellow et al. [NeurIPS 2014], Nowozin et al. [NeurIPS 2016]

F-divergence as critic



An **unhelpful** critic: Jensen-Shannon, Arjovsky and Bottou [ICLR 2017]

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F-divergence as critic



An **unhelpful** critic: Jensen-Shannon, Arjovsky and Bottou [ICLR 2017]

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What is done in practice?

- Use a **variational approximation** to the critic, alternate generator and critic training (we will return to this!) Goodfellow et al. [NeurIPS 2014], Nowozin et al. [NeurIPS 2016]
- Add “**instance noise**” to the reference and generator observations Sonderby et al. [arXiv 2016], Arjovsky and Bottou [ICLR 2017]
 - ...or (approximately equivalently) a **gradient penalty** for the variational critic (we will return to this!) Roth et al [NeurIPS 2017], Nagarajan and Kolter [NeurIPS 2017], Mescheder et al. [ICML 2018]

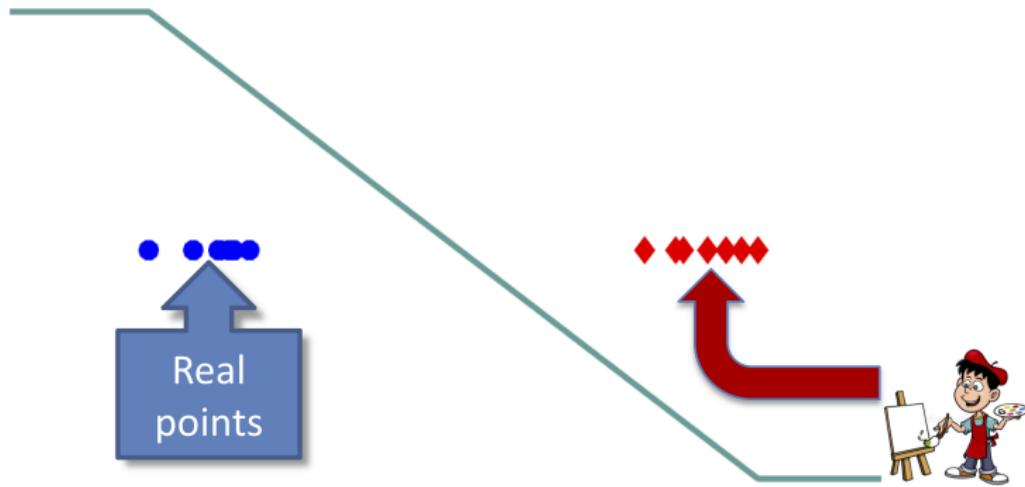
Wasserstein distance as critic



A **helpful** critic witness:

$$W_1(P, Q) = \sup_{\|\textcolor{teal}{f}\|_L \leq 1} E_P f(X) - E_Q f(Y).$$
$$\|\textcolor{teal}{f}\|_L := \sup_{x \neq y} |f(x) - f(y)| / \|x - y\|$$

$$W_1=0.88$$



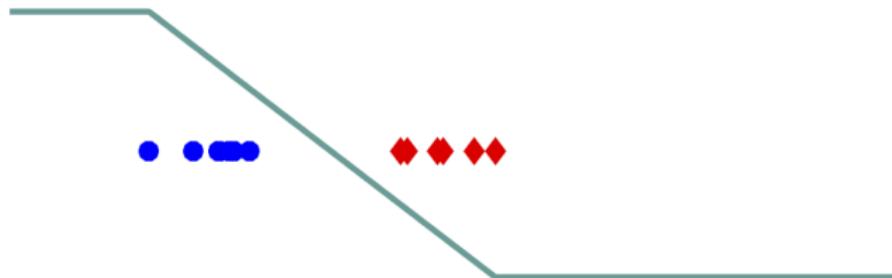
Wasserstein distance as critic



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$$W_1=0.65$$



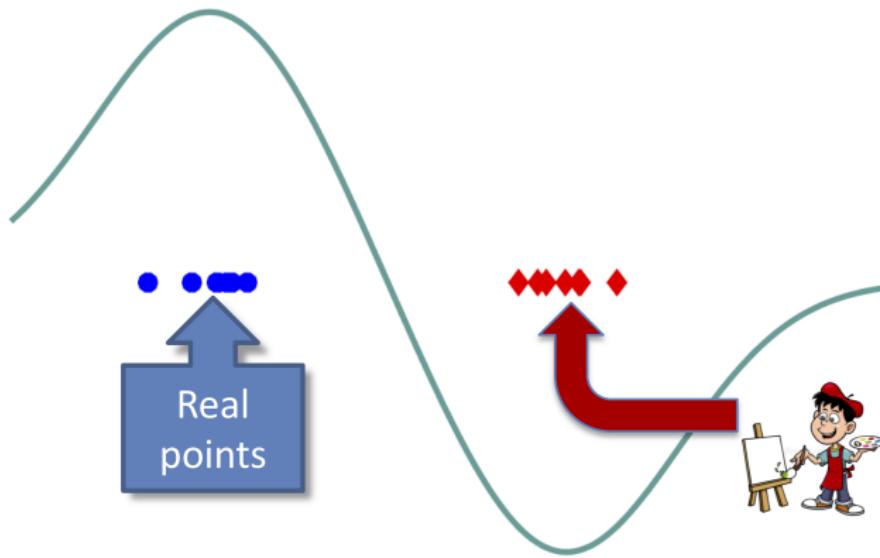
MMD as critic



A **helpful** critic witness:

$$MMD(P, Q) = \sup_{\|f\|_{\mathcal{F}} \leq 1} E_P f(X) - E_Q f(Y).$$

MMD=1.8



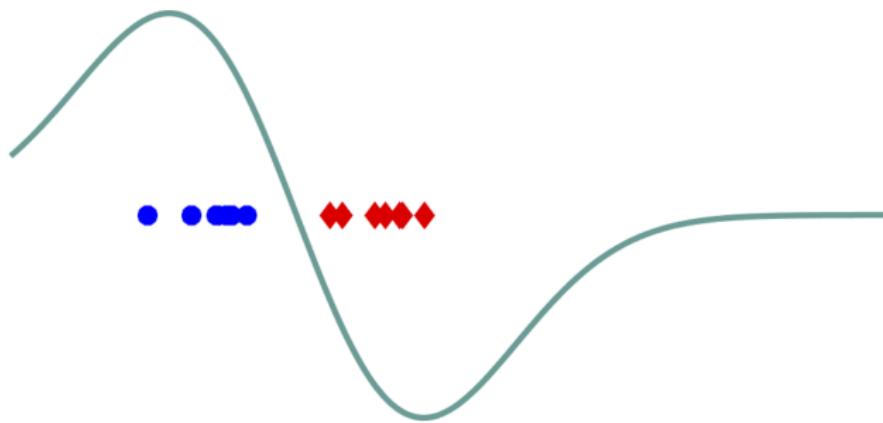
MMD as critic



A **helpful** critic witness:

$$MMD(P, Q) = \sup_{\|f\|_{\mathcal{F}} \leq 1} E_P f(X) - E_Q f(Y)$$

MMD=1.1

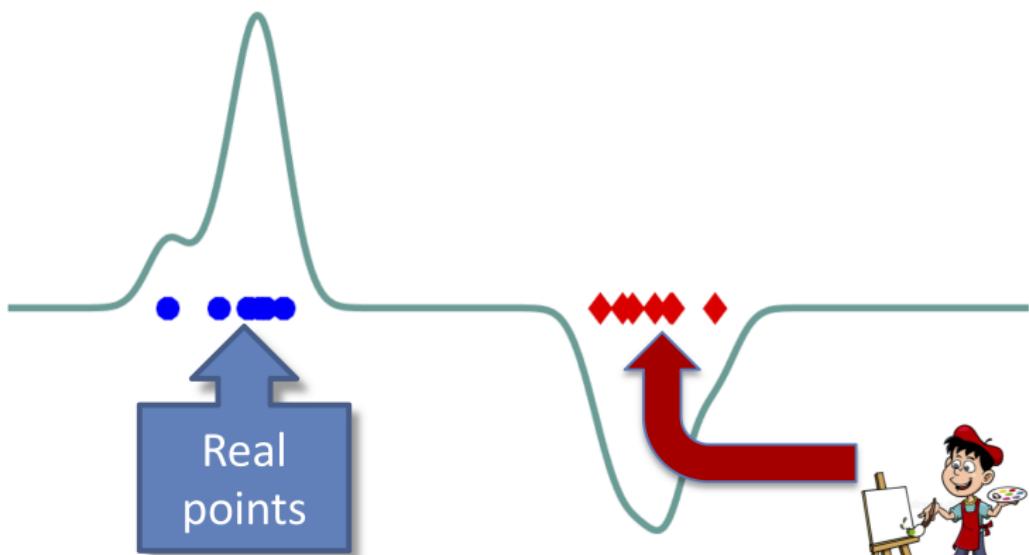


MMD as critic



An **unhelpful** critic witness:
 $MMD(P, Q)$ with a narrow kernel.

$MMD=0.64$

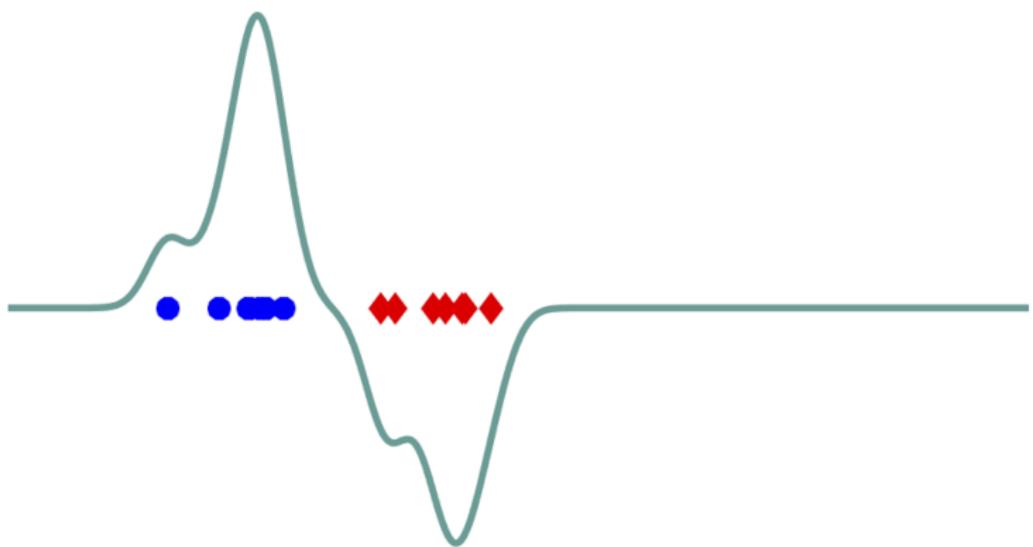


MMD as critic



An **unhelpful** critic witness:
 $MMD(P, Q)$ with a narrow kernel.

$MMD=0.64$



MMD for GAN critic

Can you use MMD as a critic to train GANs?

From ICML 2015:

Generative Moment Matching Networks

Yujia Li¹

Kevin Swersky¹

Richard Zemel^{1,2}

YUJIALI@CS.TORONTO.EDU

KSWERSKY@CS.TORONTO.EDU

ZEMEL@CS.TORONTO.EDU

¹Department of Computer Science, University of Toronto, Toronto, ON, CANADA

²Canadian Institute for Advanced Research, Toronto, ON, CANADA

From UAI 2015:

Training generative neural networks via Maximum Mean Discrepancy optimization

Gintare Karolina Dziugaite
University of Cambridge

Daniel M. Roy
University of Toronto

Zoubin Ghahramani
University of Cambridge

MMD for GAN critic

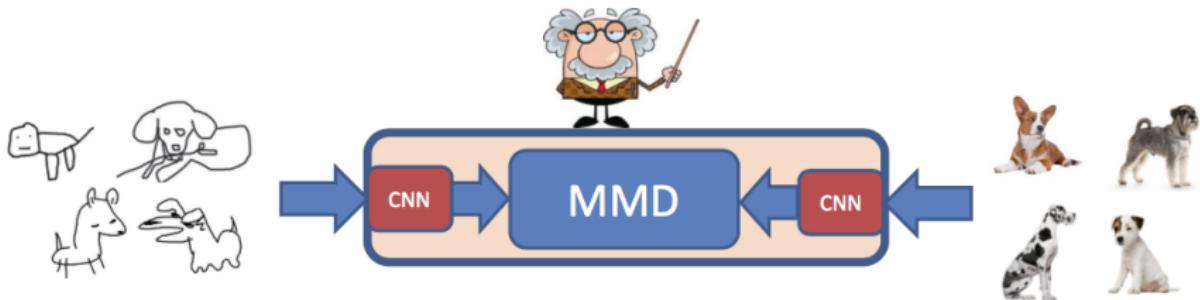
Can you use MMD as a critic to train GANs?



Need better image features.

CNN features for an MMD witness

- Add convolutional features!
- The **critic** (teacher) also needs to be trained.



$$\hat{\kappa}(x, y) = h_\psi^\top(x) h_\psi(y)$$

where $h_\psi(x)$ is a CNN map:

- **Wasserstein GAN** Arjovsky et al.
[ICML 2017]
- **WGAN-GP** Gulrajani et al.
[NeurIPS 2017]

$$\hat{\kappa}(x, y) = k(h_\psi(x), h_\psi(y))$$

where $h_\psi(x)$ is a CNN map,

k is e.g. an exponentiated quadratic kernel

MMD Li et al., [NeurIPS 2017]

Cramer Bellemare et al. [2017]

Coulomb Unterthiner et al., [ICLR 2018]

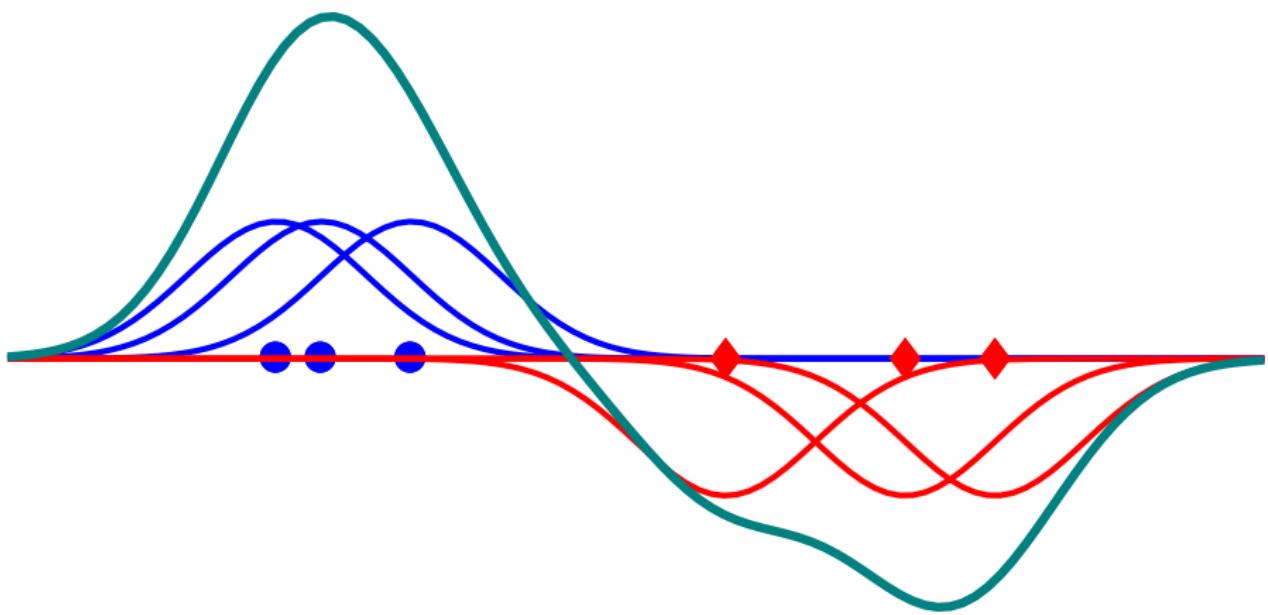
Demystifying MMD GANs Bink 61/85

Sutherland, Arbel, G., [ICLR 2018]

Witness function, kernels on deep features

Reminder: witness function,

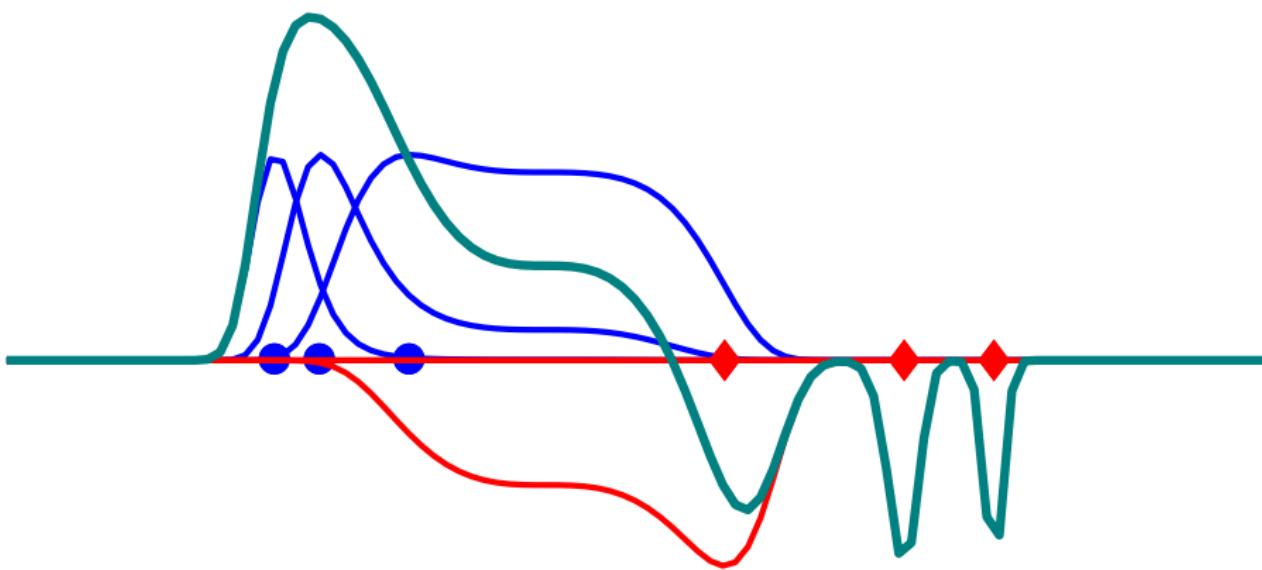
$k(x, y)$ is exponentiated quadratic



Witness function, kernels on deep features

Reminder: witness function,

$k(h_\psi(x), h_\psi(y))$ with nonlinear h_ψ and exp. quadratic k



Challenges for learned critic features

Learned critic features:

MMD with kernel $k(h_\psi(x), h_\psi(y))$ must give useful gradient to generator.

Challenges for learned critic features

Learned critic features:

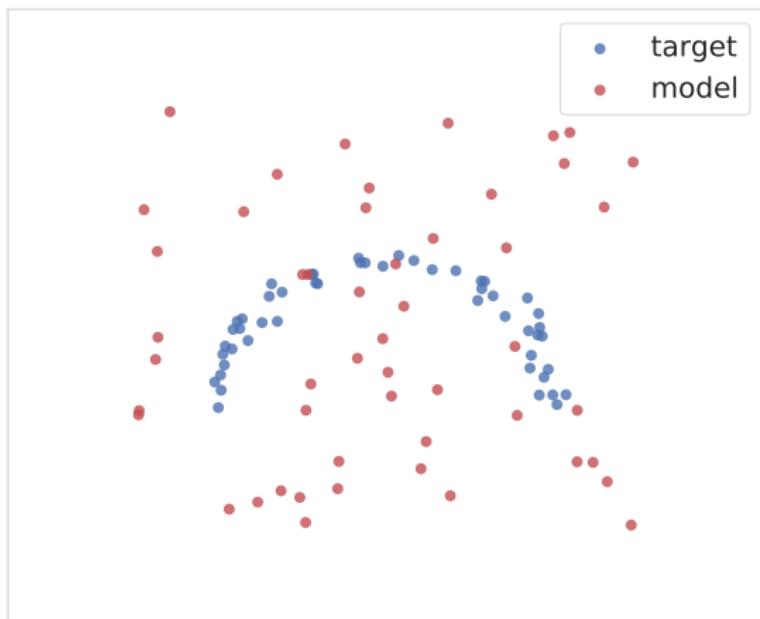
MMD with kernel $k(h_\psi(x), h_\psi(y))$ must give useful gradient to generator.

Relation with test power?

If the MMD with kernel $k(h_\psi(x), h_\psi(y))$ gives a powerful test, will it be a good critic?

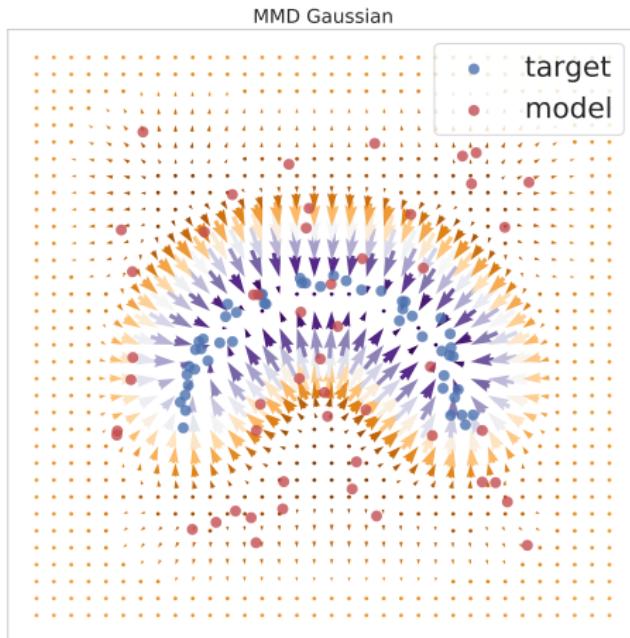
A simple 2-D example

Samples from target P and model Q



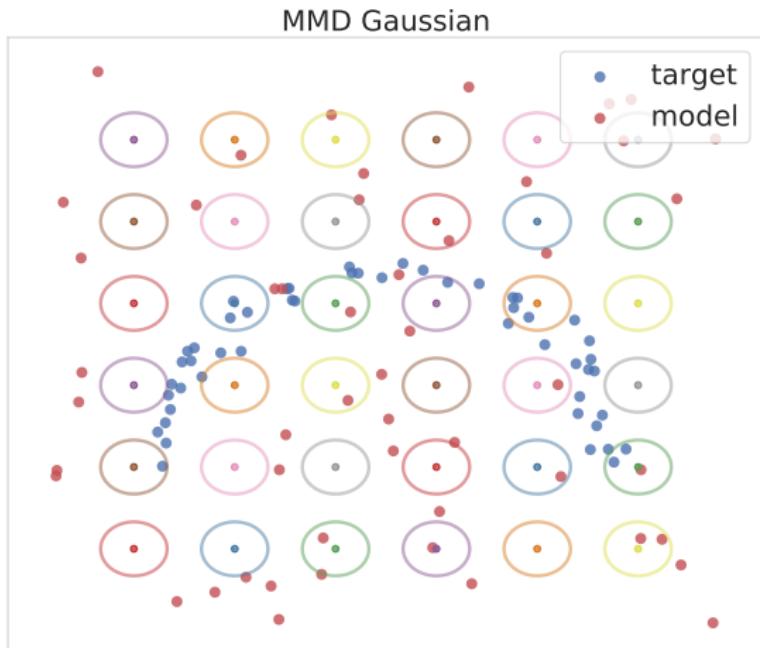
A simple 2-D example

Witness gradient, MMD with exp. quad. kernel $k(x, y)$



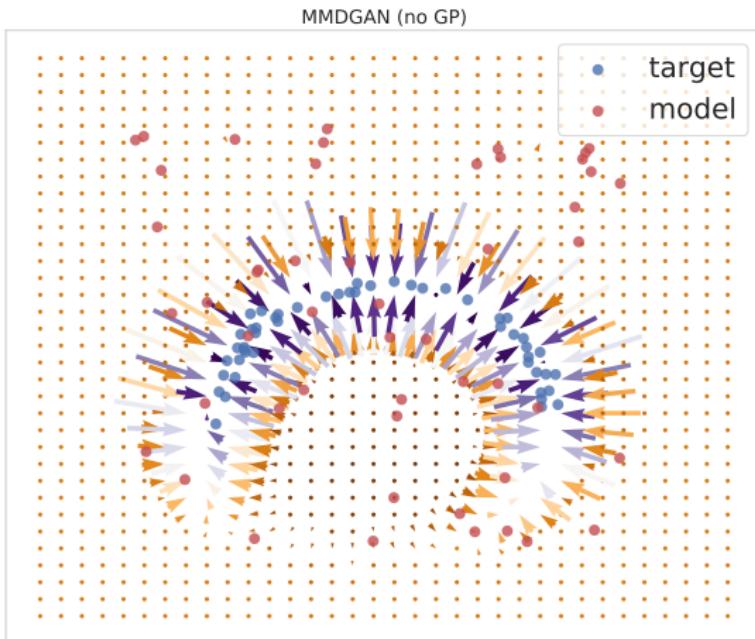
A simple 2-D example

What the kernels $k(x, y)$ look like



A simple 2-D example

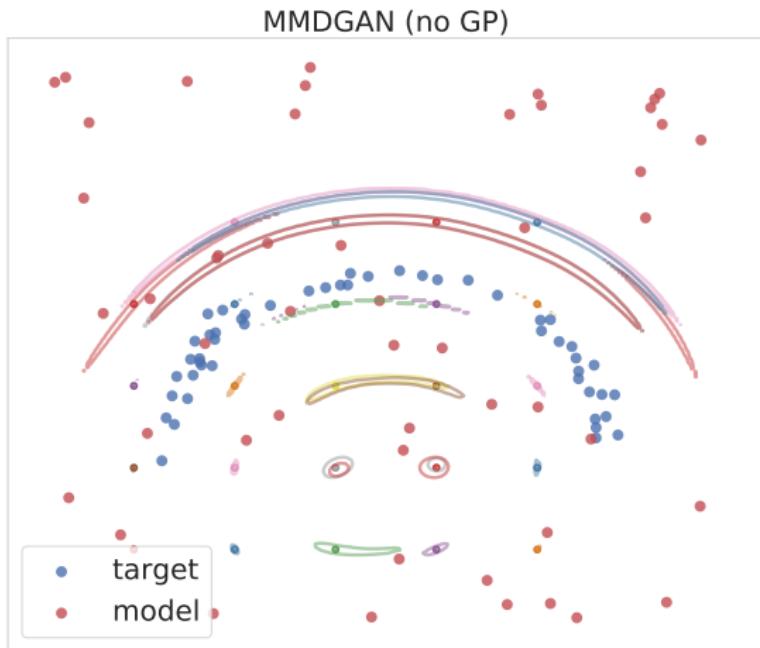
Witness gradient, maximise MMD to learn $h_\psi(x)$ for $k(h_\psi(x), h_\psi(y))$



(4 layer, fully connected, RELU, skipthrough 1-4, early stage)
64/85

A simple 2-D example

What the kernels $k(h_\psi(x), h_\psi(y))$ look like



(4 layer, fully connected, RELU, skipthrough 1-4, **early stage**)_{64/85}

A simple 2-D example



A data-adaptive gradient penalty

- New gradient regulariser Arbel, Sutherland, Binkowski, G. [NeurIPS 2018]
 - Also related to Sobolev GAN Mroueh et al. [ICLR 2018]
-

On gradient regularizers for MMD GANs

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A data-adaptive gradient penalty

- New gradient regulariser Arbel, Sutherland, Binkowski, G. [NeurIPS 2018]
- Also related to Sobolev GAN Mroueh et al. [ICLR 2018]

Modified witness constraint:

$$\widetilde{MMD} := \sup_{\|\mathbf{f}\|_S \leq 1} [\mathbb{E}_{\mathbf{P}} f(\mathbf{X}) - \mathbb{E}_{\mathbf{Q}} f(\mathbf{Y})]$$

where

$$\|\mathbf{f}\|_S^2 = \|\mathbf{f}\|_{L_2(\mathbf{P})}^2 + \|\nabla \mathbf{f}\|_{L_2(\mathbf{P})}^2 + \lambda \|\mathbf{f}\|_k^2$$

The equation is accompanied by three orange boxes with blue arrows pointing upwards, each representing a component of the norm:
1. L₂ norm control (orange box)
2. Gradient control (purple box)
3. RKHS smoothness (blue box)

Maximise \widetilde{MMD} wrt critic features

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Problem: not computationally feasible: $\mathcal{O}(n^3)$ per iteration.

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Maximise scaled MMD over critic features:

$$SMMD(P, \lambda) = \sigma_{P, \lambda} MMD$$

where

$$\sigma_{P, \lambda}^2 = \lambda + \int k(h_\psi(\mathbf{x}), h_\psi(\mathbf{x})) dP(x) + \sum_{i=1}^d \int \partial_i \partial_{i+d} k(h_\psi(\mathbf{x}), h_\psi(\mathbf{x})) dP(x)$$

Replace expensive constraint with cheap upper bound:

$$\|\mathbf{f}\|_S^2 \leq \sigma_{P, \lambda}^{-1} \|\mathbf{f}\|_k^2$$

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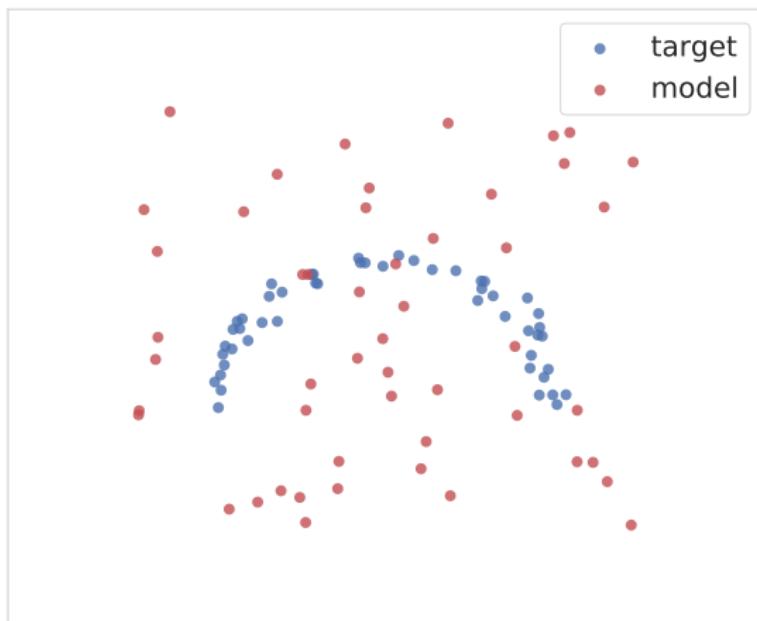
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Idea: rather than regularise the critic or witness function, regularise features directly

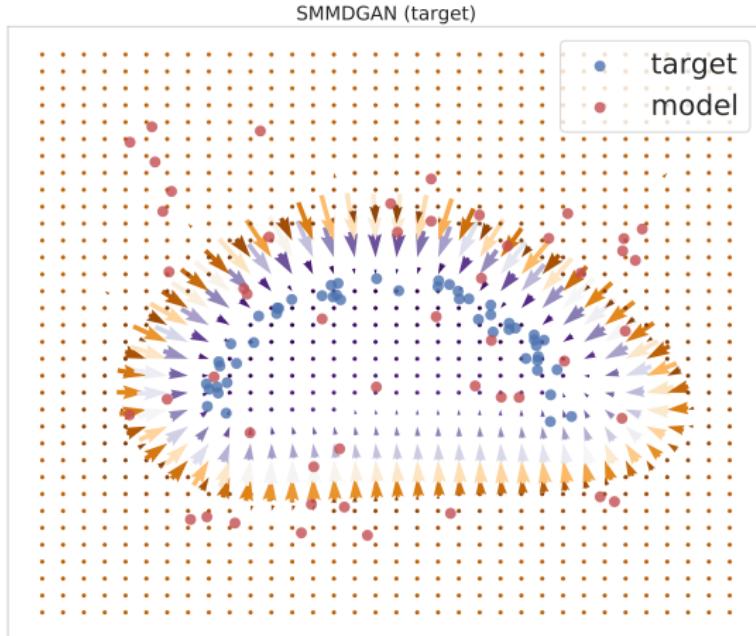
Simple 2-D example revisited

Samples from target P and model Q



Simple 2-D example revisited

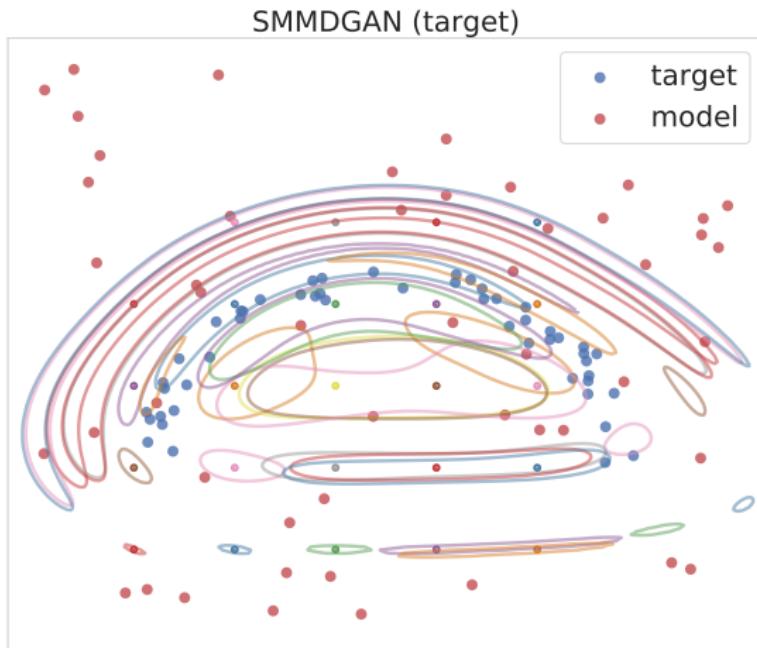
Witness gradient, **maximise** $SMMD(P, \lambda)$
to learn $h_\psi(x)$ for $k(h_\psi(x), h_\psi(y))$



(**early** stage of critic optimisation)

Simple 2-D example revisited

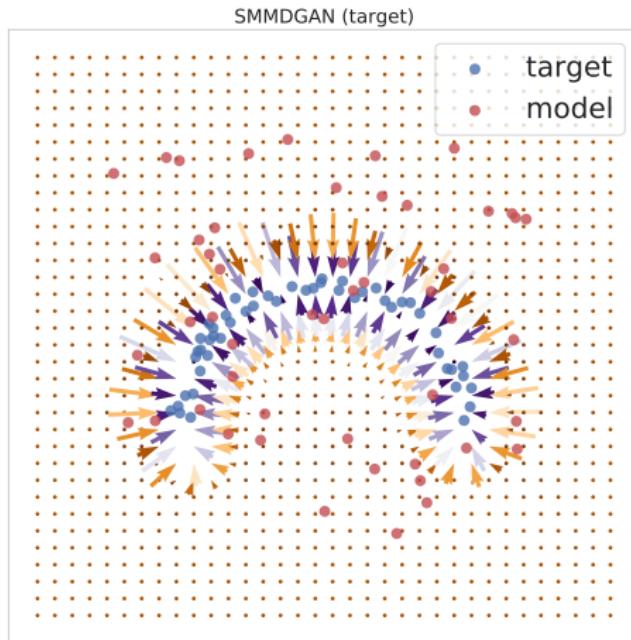
What the kernels $k(h_\psi(x), h_\psi(y))$ look like



(**early** stage of critic optimisation)

Simple 2-D example revisited

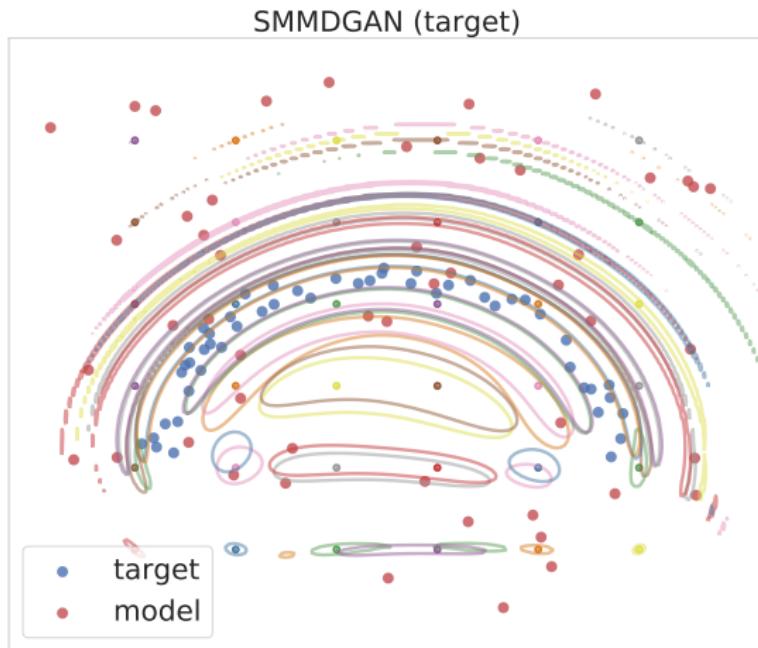
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(**late** stage of critic optimisation)

Simple 2-D example revisited

What the kernels $k(h_\psi(x), h_\psi(y))$ look like



(late stage of critic optimisation)

Our empirical observations

Data-adaptive critic loss:

- Witness function class for $SMMD(P, \lambda)$ depends on P .
 - Without data-dependent regularisation, maximising MMD over features h_ψ of kernel $k(h_\psi(x), h_\psi(y))$ is **unhelpful**.
- Data-dependent regularisation also applies to variational form in f-GANs Roth et al [NeurIPS 2017, eq. 19 and 20]

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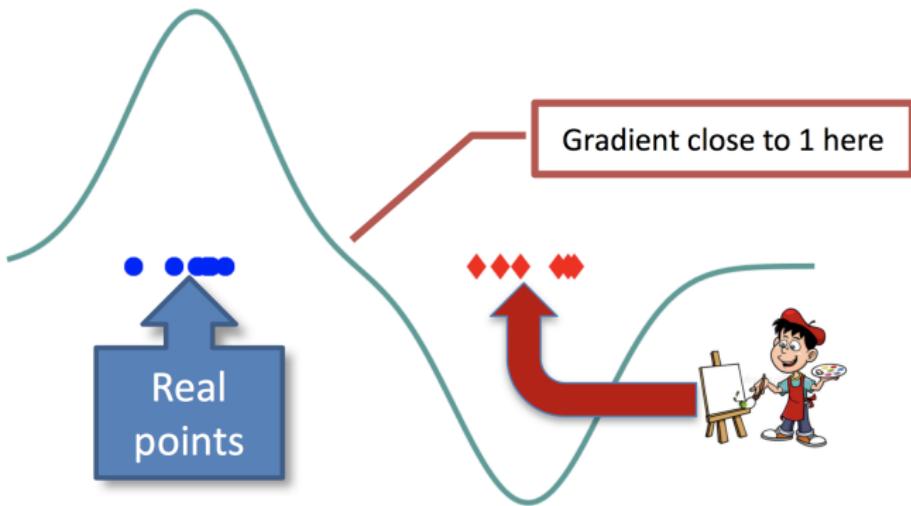
Alternate critic and generator training:

- Weaker critics can give better signals to poor (early stage) generators.

WGAN-GP

Wasserstein GAN Arjovsky et al. [ICML 2017]

WGAN-GP Gukrajani et al. [NeurIPS 2017]



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- Given a generator G_θ with parameters θ to be trained.
Samples $Y \sim G_\theta(Z)$ where $Z \sim R$



- Given critic features h_ψ with parameters ψ to be trained. f_ψ a linear function, $\mathfrak{K}(x, y) = h_\psi^\top(x)h_\psi(y)$.

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WGAN-GP gradient penalty:

$$\max_{\psi} \mathbf{E}_{X \sim P} f_\psi(\mathbf{X}) - \mathbf{E}_{Z \sim R} f_\psi(G_\theta(\mathbf{Z})) + \lambda \mathbf{E}_{\widetilde{\mathbf{X}}} \left(\|\nabla_{\widetilde{\mathbf{X}}} f_\psi(\widetilde{\mathbf{X}})\| - 1 \right)^2$$

where

$$\widetilde{\mathbf{X}} = \gamma \mathbf{x}_i + (1 - \gamma) G_\theta(\mathbf{z}_j)$$

$$\gamma \sim \mathcal{U}([0, 1]) \quad x_i \in \{\mathbf{x}_\ell\}_{\ell=1}^m \quad z_j \in \{\mathbf{z}_\ell\}_{\ell=1}^n$$

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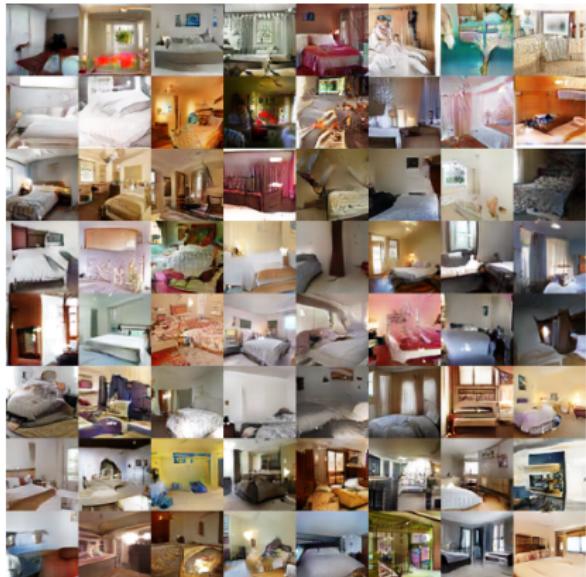
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Again: data-dependent gradient regularisation on witness class

Linear vs nonlinear kernels

- Critic features from DCGAN: an f -filter critic has f , $2f$, $4f$ and $8f$ convolutional filters in layers 1-4. LSUN 64×64 .

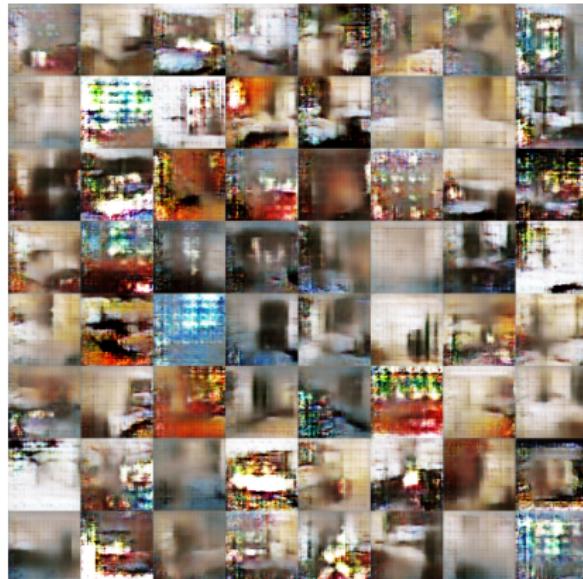
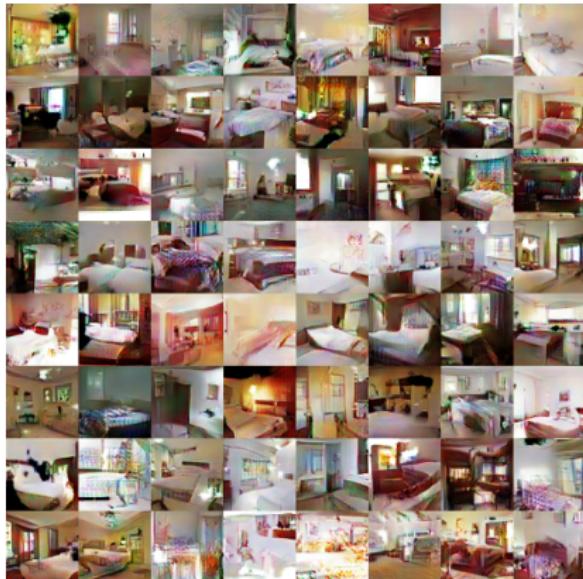


$k(\mathbf{h}_\psi(\mathbf{x}), \mathbf{h}_\psi(\mathbf{y}))$, $f = 64$,
KID=3

$\mathbf{h}_\psi^\top(\mathbf{x})\mathbf{h}_\psi(\mathbf{y})$, $f = 64$, KID=4

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$k(h_\psi(x), h_\psi(y))$, $f = 16$,
KID=9

$h_\psi^\top(x)h_\psi(y)$, $f = 16$, KID=37

Evaluation and experiments

Evaluation of GANs

The inception score? Salimans et al. [NeurIPS 2016]

Based on the classification output $p(y|x)$ of the inception model Szegedy et al. [ICLR 2014],

$$E_X \exp KL(P(y|X) \| P(y)).$$

High when:

- predictive label distribution $P(y|x)$ has low entropy (good quality images)
- label entropy $P(y)$ is high (good variety).

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Problem: relies on a trained classifier! Can't be used on new categories (celeb, bedroom...)

Evaluation of GANs

The Frechet inception distance? Heusel et al. [NeurIPS 2017]

Fits Gaussians to features in the inception architecture (pool3 layer):

$$FID(\mathcal{P}, \mathcal{Q}) = \|\mu_{\mathcal{P}} - \mu_{\mathcal{Q}}\|^2 + \text{tr}(\Sigma_{\mathcal{P}}) + \text{tr}(\Sigma_{\mathcal{Q}}) - 2\text{tr}\left((\Sigma_{\mathcal{P}}\Sigma_{\mathcal{Q}})^{\frac{1}{2}}\right)$$

where $\mu_{\mathcal{P}}$ and $\Sigma_{\mathcal{P}}$ are the feature mean and covariance of \mathcal{P}

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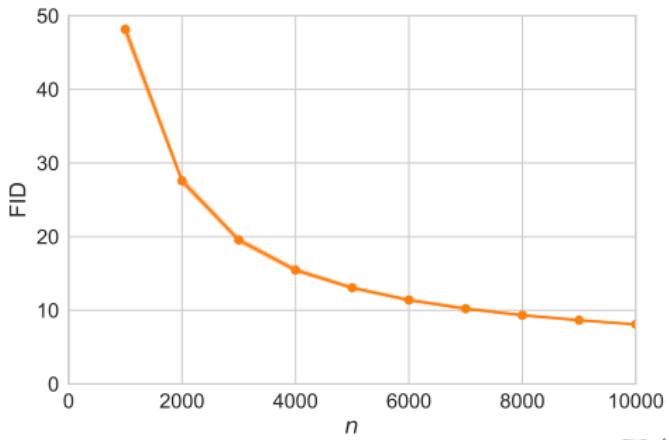
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where μ_P and Σ_P are the feature mean and covariance of P

Problem: bias. For finite samples can consistently give incorrect answer.

- Bias demo,
CIFAR-10 train vs
test



Evaluation of GANs

The FID can give the wrong answer in theory.

Assume m samples from P and $n \rightarrow \infty$ samples from Q .

Given two alternatives:

$$P_1 \sim \mathcal{N}(0, (1 - m^{-1})^2) \quad P_2 \sim \mathcal{N}(0, 1) \quad Q \sim \mathcal{N}(0, 1).$$

Clearly,

$$FID(P_1, Q) = \frac{1}{m^2} > FID(P_2, Q) = 0$$

Given m samples from P_1 and P_2 ,

$$FID(\widehat{P}_1, Q) < FID(\widehat{P}_2, Q).$$

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Evaluation of GANs

The FID can give the wrong answer in practice.

Let $d = 2048$, and define

$$P_1 = \text{relu}(\mathcal{N}(0, I_d)) \quad P_2 = \text{relu}(\mathcal{N}(1, .8\Sigma + .2I_d)) \quad Q = \text{relu}(\mathcal{N}(1, I_d))$$

where $\Sigma = \frac{4}{d}CC^T$, with C a $d \times d$ matrix with iid standard normal entries.

For a random draw of C :

$$FID(P_1, Q) \approx 1123.0 > 1114.8 \approx FID(P_2, Q)$$

With $m = 50\,000$ samples,

$$FID(\widehat{P}_1, Q) \approx 1133.7 < 1136.2 \approx FID(\widehat{P}_2, Q)$$

At $m = 100\,000$ samples, the ordering of the estimates is correct.

This behavior is similar for other random draws of C .

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The kernel inception distance (KID)

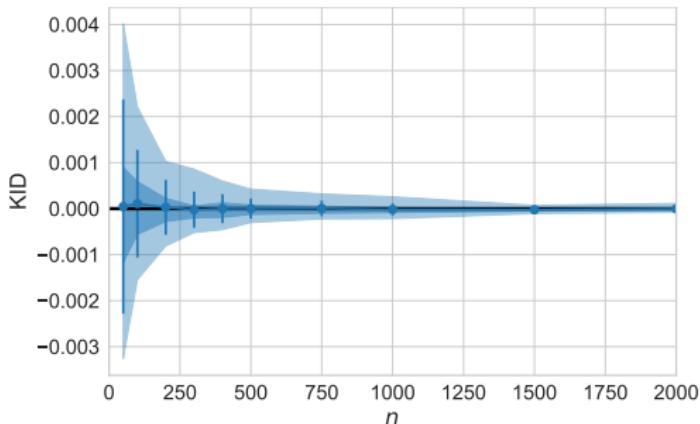
The Kernel inception distance Binkowski, Sutherland, Arbel, G. [ICLR 2018]

Measures similarity of the samples' representations in the inception architecture (pool3 layer)

MMD with kernel

$$k(x, y) = \left(\frac{1}{d} x^\top y + 1 \right)^3.$$

- Checks match for feature means, variances, skewness
- **Unbiased** : eg CIFAR-10 train/test



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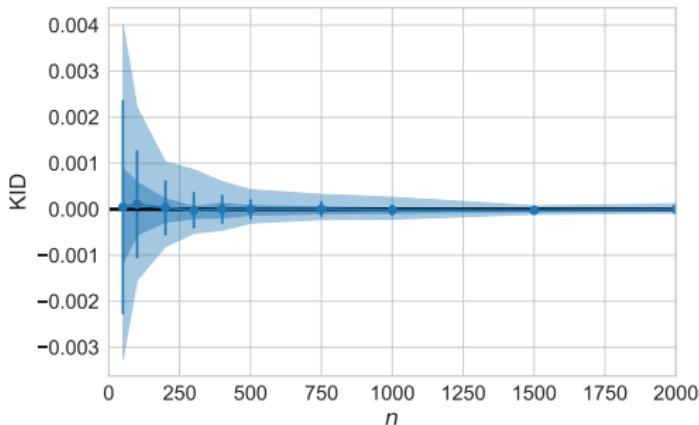
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...“but isn't KID computationally costly?”

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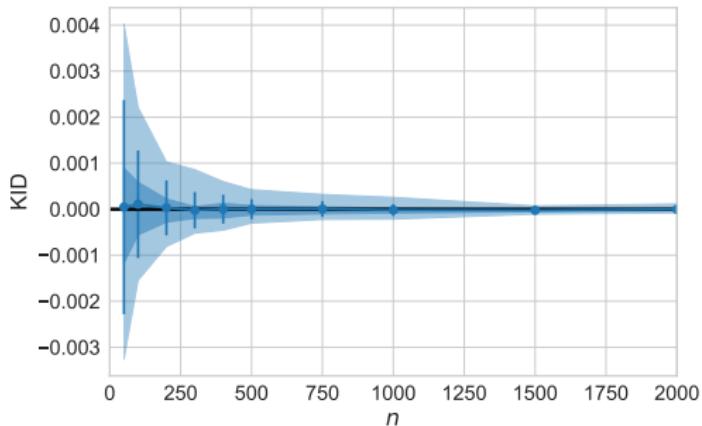
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...“but isn't KID is computationally costly?”

“Block” KID implementation is cheaper than FID: see paper
(or use Tensorflow implementation)!

The kernel inception distance (KID)

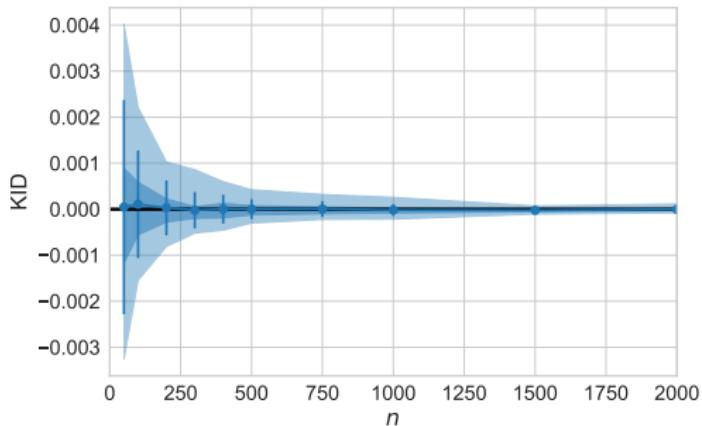
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Also used for automatic learning rate adjustment: if $KID(\hat{P}_{t+1}, Q)$ not significantly better than $KID(\hat{P}_t, Q)$ then reduce learning rate.

[Bounliphone et al. ICLR 2016]

Benchmarks for comparison (all from ICLR 2018)

SPECTRAL NORMALIZATION FOR GENERATIVE ADVERSARIAL NETWORKS

Takeru Miyato¹, Toshiki Kataoka¹, Masanori Koyama², Yuichi Yoshida³

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¹Preferred Networks, Inc. ²Ritsumeikan University ³National Institute of Informatics

We combine with scaled MMD

DEMYSTIFYING MMD GANS

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Our ICLR
2018
paper

SOBOLEV GAN

Youssef Mroueh¹, Chun-Liang Li^{2,*}, Tom Sercombe^{1,*}, Anant Raj^{3,*} & Yu Cheng¹

† IBM Research AI

◦ Carnegie Mellon University

◊ Max Planck Institute for Intelligent Systems

* denotes Equal Contribution

{mroueh, chengyu}@us.ibm.com, chunli@cs.cmu.edu,

tom.sercombe@ibm.com, anant.raj@tuebingen.mpg.de

BOUNDARY-SEEKING GENERATIVE ADVERSARIAL NETWORKS

R Devon Hjelm*

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Kyunghyun Cho

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CIFAR Azrieli Global Scholar

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Yoshua Bengio

MILA, University of Montréal, CIFAR, IVADO

yoshua.bengio@umontreal.ca

Results: celebrity faces 160×160

KID scores:

- Sobolev GAN:
14
- SN-GAN:
18
- Old MMD
GAN:
13
- SMMD GAN:
6

202 599 face images, re-sized and cropped to 160 × 160

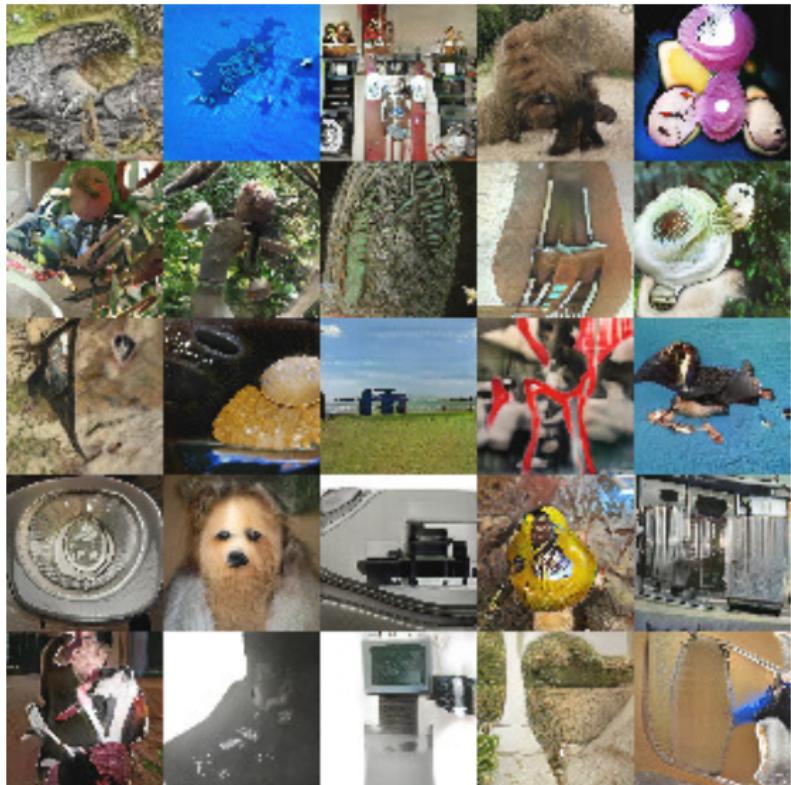


Results: unconditional imagenet 64×64

KID scores:

- BGAN:
47
- SN-GAN:
44
- SMMD GAN:
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ILSVRC2012 (ImageNet)
dataset, 1 281 167 images,
resized to 64×64 . 1000
classes.

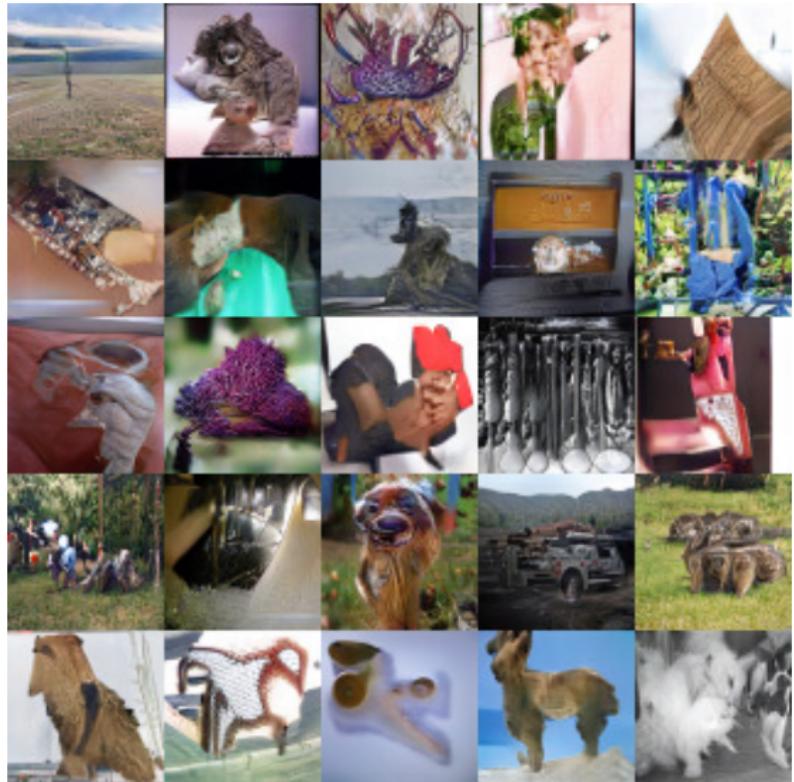


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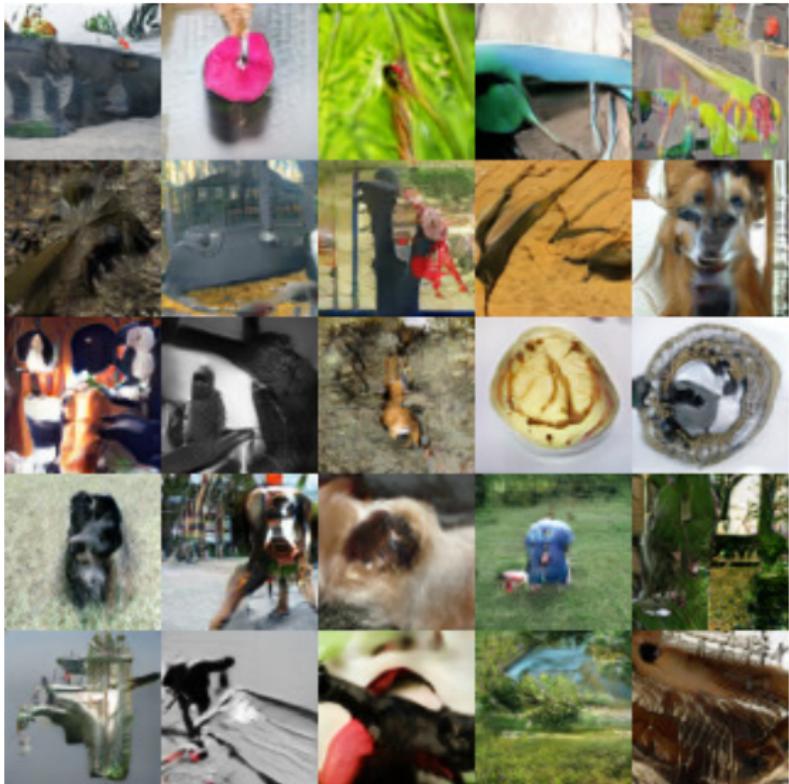


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Summary

- MMD critic gives state-of-the-art performance for GAN training (FID and KID)
 - use convolutional input features
 - train with new gradient regulariser
- Faster training, simpler critic network
- Reasons for good performance:
 - Unlike WGAN-GP, MMD loss still a valid critic when features not optimal
 - Kernel features do some of the “work”, so simpler h_ψ features possible.
 - Better gradient/feature regulariser gives better critic

“Demystifying MMD GANs,” including KID score, ICLR 2018:

<https://github.com/mbinkowski/MMD-GAN>

Gradient regularised MMD, NeurIPS 2018:

<https://github.com/MichaelArbel/Scaled-MMD-GAN>

Co-authors

From Gatsby:

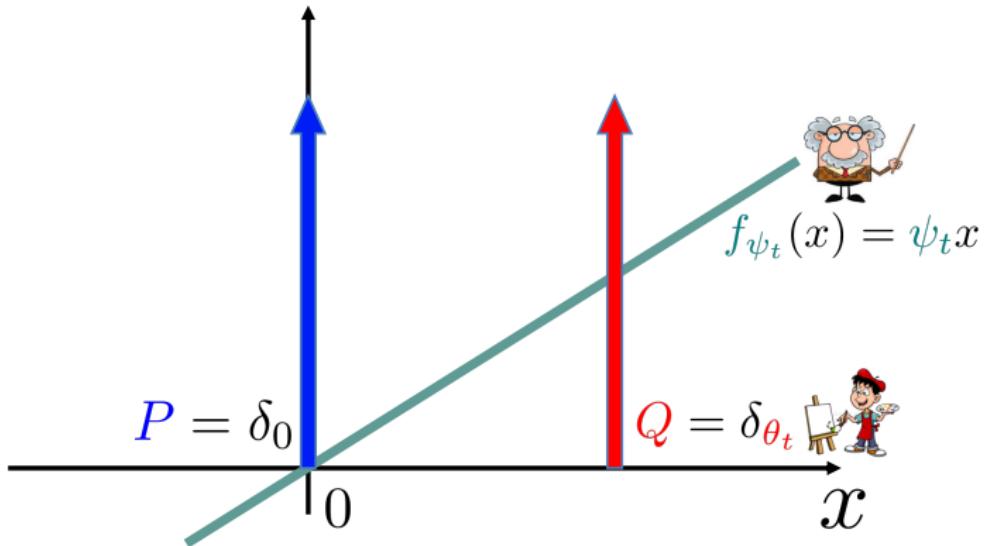
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External collaborators:

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Questions?



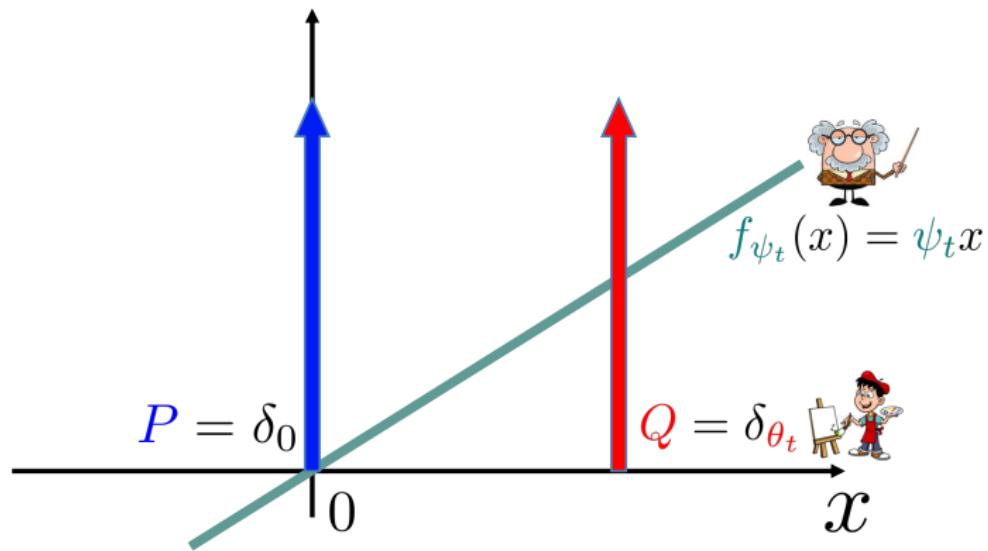


$$\begin{aligned}
 D(P, Q; \psi_t) &= \mathbf{E}_Q f_{\psi_t}(Y) - \mathbf{E}_P f_{\psi_t}(X) \\
 &= \psi_t \theta_t
 \end{aligned}$$

Mescheder et al. [ICML 2018]

Optimization: simple example

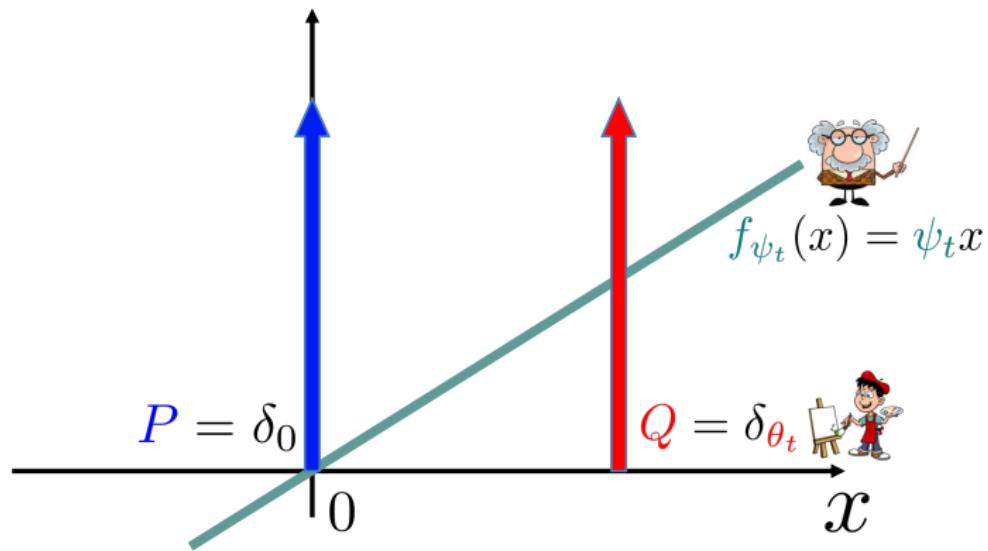
Gradient descent on generator:



$$\frac{\partial}{\partial \theta} D(P, Q; \psi_t) = \frac{\partial}{\partial \theta} \psi_t \theta_t = \psi_t$$

Optimization: simple example

Gradient descent on generator:

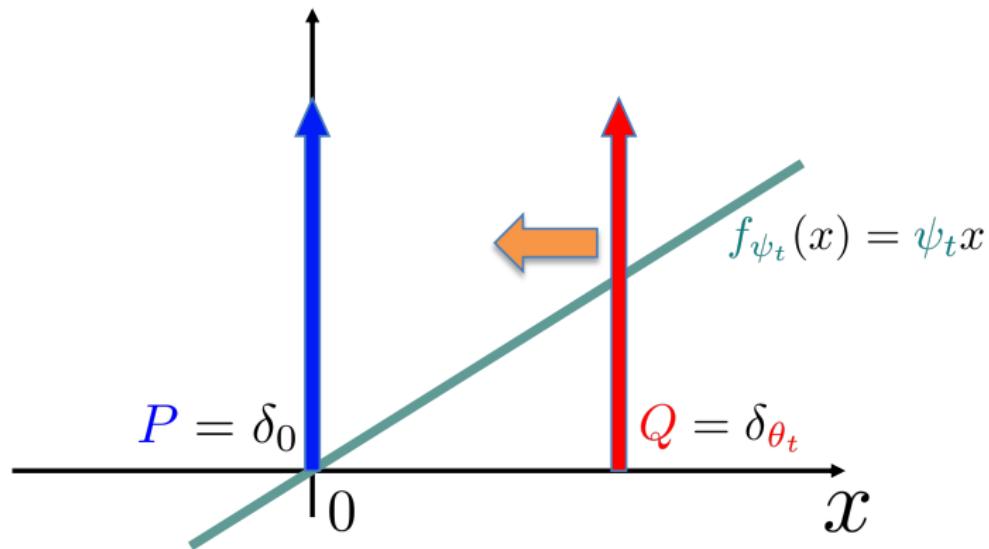


$$\frac{\partial}{\partial \theta} D(P, Q; \psi_t) = \frac{\partial}{\partial \theta} \psi_t \theta_t = \psi_t$$

$$\theta_{t+1} = \theta_t - \gamma \frac{\partial}{\partial \theta} D(P, Q; \psi_t) = \theta_t - \gamma \psi_t$$

Optimization: simple example

Gradient descent on generator:

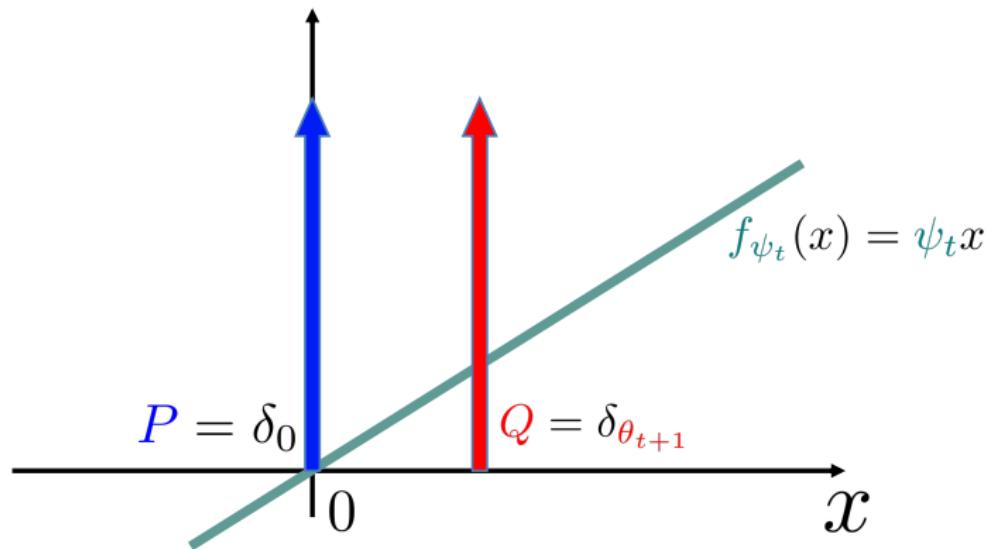


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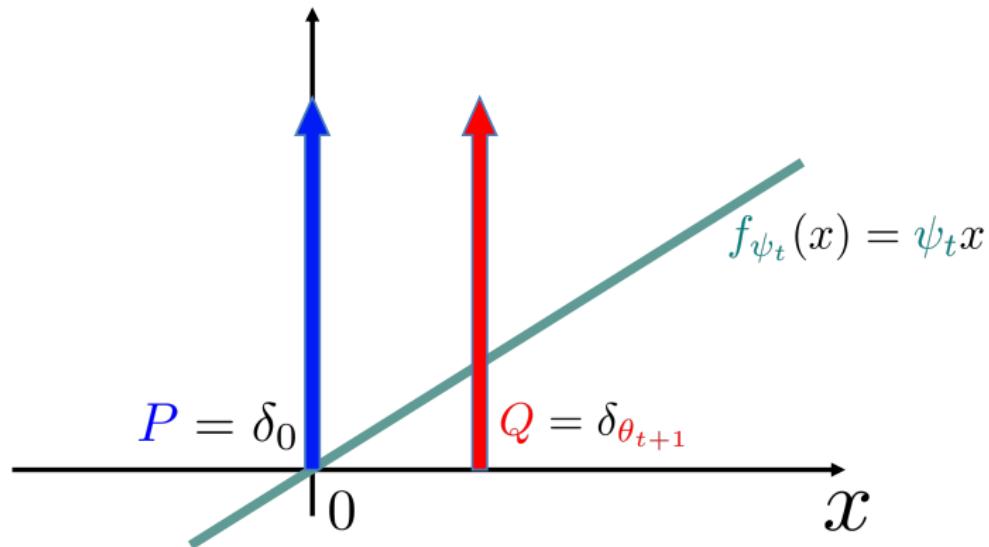


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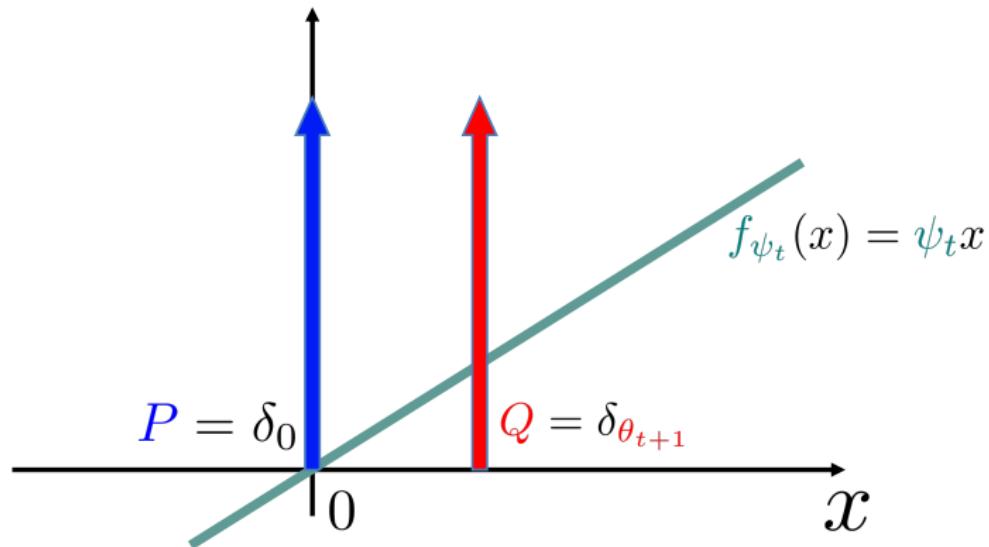
Gradient **ascent** on critic:



$$\frac{\partial}{\partial \psi} D(P, Q; \psi_t) = \theta_{t+1}$$

Optimization: simple example

Gradient **ascent** on critic:

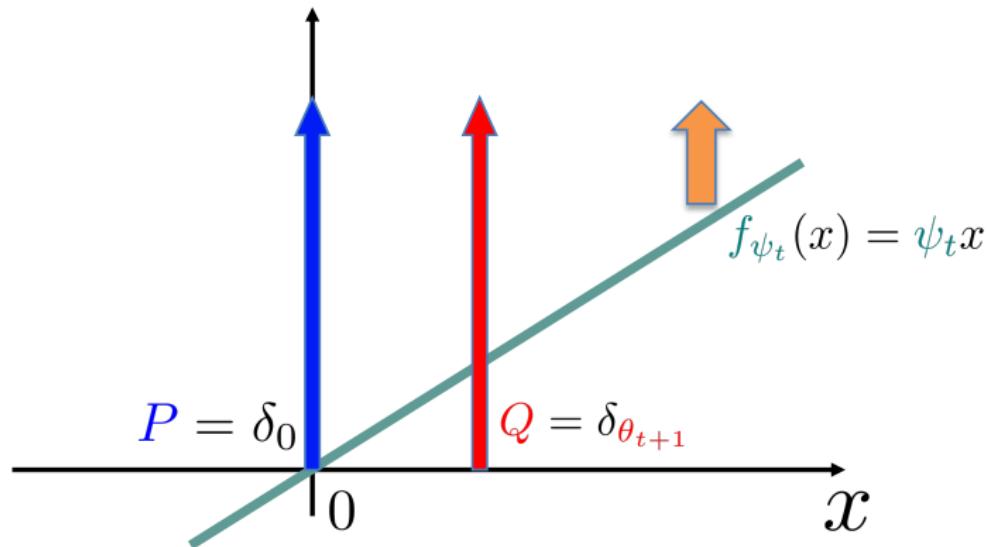


$$\frac{\partial}{\partial \psi} D(P, Q; \psi_t) = \theta_{t+1}$$

$$\psi_{t+1} = \psi_t + \lambda \frac{\partial}{\partial \psi} D(P, Q; \psi_t) = \psi_t + \lambda \theta_{t+1}$$

Optimization: simple example

Gradient **ascent** on critic:

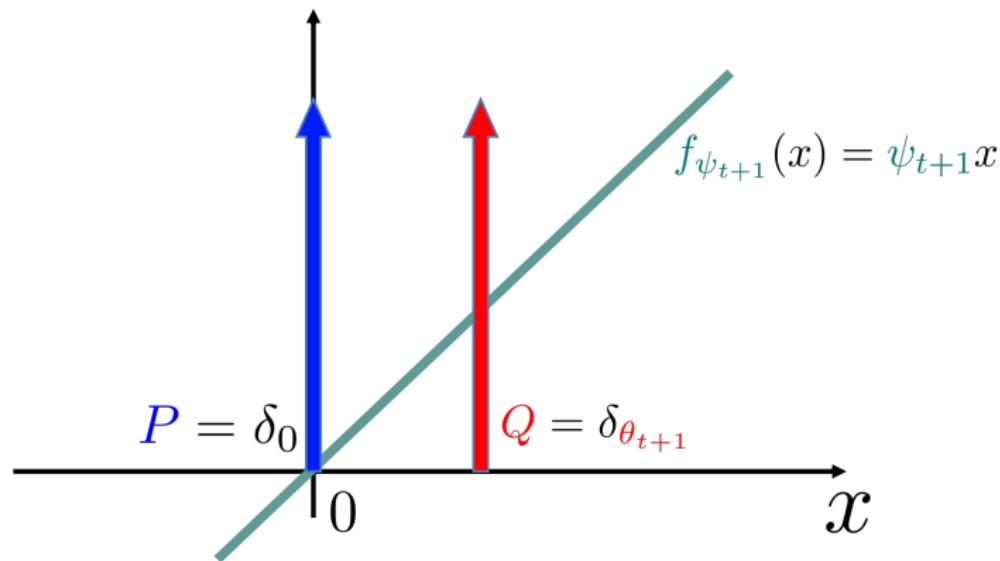


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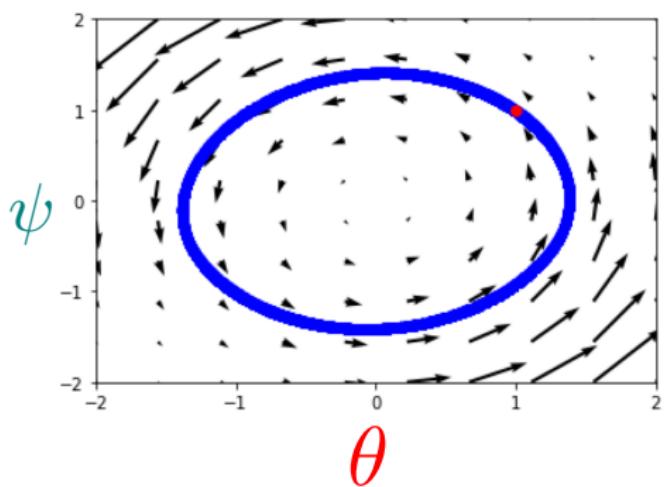
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Optimization: simple example

Idealised continuous system (infinitely small learning rate)

$$\begin{bmatrix} \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\nabla_{\psi} D(P, Q; \psi) \\ \nabla_{\theta} D(P, Q; \psi) \end{bmatrix}$$

Every integral curve $(\psi(t), \theta(t))$ of the gradient vector field satisfies $\psi^2(t) + \theta^2(t) = c$ for all $t \in [0, \infty)$.



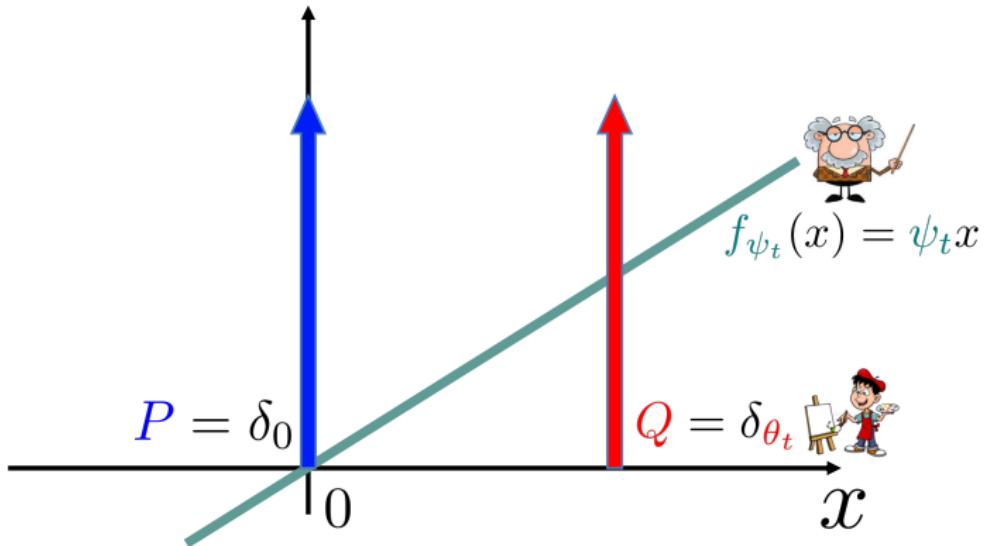
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A solution: control witness gradient



$$\begin{aligned}
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Mescheder et al. [ICML 2018]

Convergence issues for WGAN-GP penalty

WGAN-GP style gradient penalty may not converge near solution

Nagarajan and Kolter [NeurIPS 2017], Mescheder et al. [ICML 2018], Balduzzi et al. [ICML 2018]

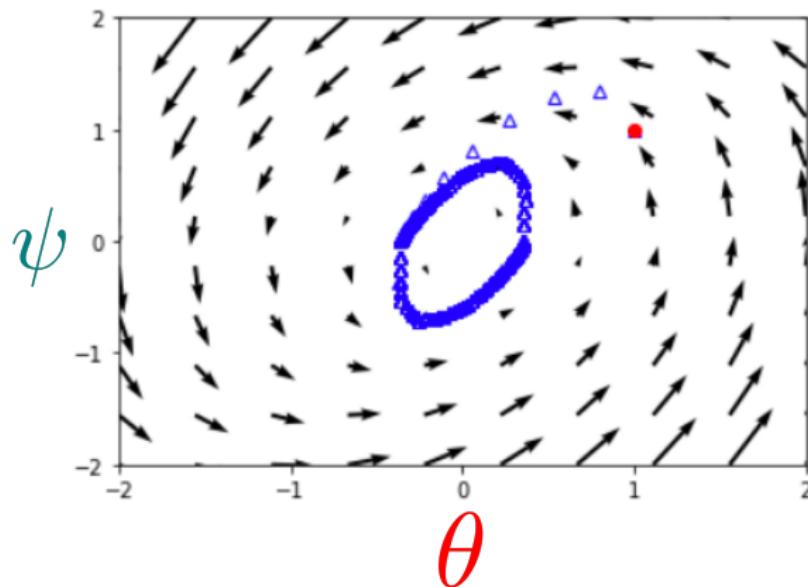


Figure from Mescheder et al. [ICML 2018]