Rationalizing with conjugate method

Remember that “rationalize the denominator” just means “get the radicals out of the denominator.”

We already know how to rationalize the denominator if the denominator has just one term and it consists of a square root and nothing else.

Example

Rationalize the denominator.

$$\frac{6}{\sqrt{13}}$$

If we multiply the denominator by $\sqrt{13}$, we’ll get rid of the square root there. But in order to keep the value of the fraction the same, we have to multiply both the numerator and the denominator by $\sqrt{13}$.

$$\frac{6}{\sqrt{13}}\left(\frac{\sqrt{13}}{\sqrt{13}}\right)$$

$$\frac{6\sqrt{13}}{13}$$

Now that the square root is out of the denominator, we’ve rationalized the denominator.

But how do we rationalize the denominator when it has two terms, such as the denominator in the expression

$$\frac{3}{5-\sqrt{3}}$$

In a case like this one, where the denominator is the sum or difference of two terms, at least one of which contains a square root (and neither term contains a higher root - a cube root, a fourth root, ...), we can use the *conjugate method* to rationalize the denominator.

The conjugate of a binomial has the same two terms, but with the opposite sign in between. For $5-\sqrt{3}$, we keep the same two terms, 5 and $\sqrt{3}$, but we change the sign in the middle. Since the sign in this case is negative, we'll change it to a positive sign.

Original binomial	$5-\sqrt{3}$
Its conjugate	$5+\sqrt{3}$

To rationalize the denominator of a fraction where the denominator is a binomial, we'll multiply both the numerator and denominator by the conjugate.

As we're doing these problems, let's also remember these facts:

Fact 1: We can multiply any number by 1 without changing its value.

$$\frac{1}{\sqrt{2}-5}\cdot 1 \text{ is the same as } \frac{1}{\sqrt{2}-5}$$

Fact 2: We can write 1 as any nonzero number divided by itself, for example,

$$\frac{\sqrt{2}+5}{\sqrt{2}+5} = 1$$

Example

Rationalize the denominator.

$$\frac{3-\sqrt{2}}{\sqrt{2}-5}$$

We want to use the conjugate method to get the radical out of the denominator. Remember that the conjugate of a binomial has the same two terms but with the opposite sign between them. So the conjugate of $\sqrt{2}-5$ is $\sqrt{2}+5$. This is the binomial that both the numerator and denominator have to be multiplied by.

$$\frac{3-\sqrt{2}}{\sqrt{2}-5}$$

$$\frac{3-\sqrt{2}}{\sqrt{2}-5} \cdot \frac{\sqrt{2}+5}{\sqrt{2}+5}$$

Now this becomes a binomial multiplication problem. We need to make sure to multiply our first terms, outer terms, inner terms, and last terms (FOIL).

$$\frac{\left(3-\sqrt{2}\right)\left(\sqrt{2}+5\right)}{\left(\sqrt{2}-5\right)\left(\sqrt{2}+5\right)}$$

$$\frac{3\sqrt{2}+15-2-5\sqrt{2}}{2+5\sqrt{2}-5\sqrt{2}-25}$$

Group together the terms that contain $\sqrt{2}$, and group together the terms that have no radical. Then combine (add) the terms within each group. Do this for the numerator and denominator separately.

$$\frac{\left(3\sqrt{2}-5\sqrt{2}\right)+(15-2)}{\left(5\sqrt{2}-5\sqrt{2}\right)+(2-25)}$$

$$\frac{-2\sqrt{2}+13}{-23}$$

Multiply by $(-1)/(-1)$ to remove the negative sign from the denominator.

$$\frac{-2\sqrt{2}+13}{-23}\cdot\frac{-1}{-1}$$

$$\frac{2\sqrt{2}-13}{23}$$

Let's do another example.

Example

Rationalize the denominator.

$$\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$

We want to use the conjugate method to get the radical out of the denominator. Remember that the conjugate of a binomial has the same two terms but with the opposite sign between them. So the conjugate of $\sqrt{5}+\sqrt{7}$ is $\sqrt{5}-\sqrt{7}$. This is the binomial that both the numerator and denominator have to be multiplied by.

$$\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$

$$\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}} \cdot \frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}-\sqrt{7}}$$

Now this becomes a binomial multiplication problem. We need to make sure to multiply our first terms, outer terms, inner terms, and last terms.

$$\frac{\left(\sqrt{5}-\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)}{\left(\sqrt{5}+\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)}$$

$$\frac{5-\sqrt{35}-\sqrt{35}+7}{5-\sqrt{35}+\sqrt{35}-7}$$

$$\frac{5-2\sqrt{35}+7}{5-7}$$

$$\frac{12-2\sqrt{35}}{-2}$$

Factor out a 2 in the numerator.

$$\frac{2\left(6-\sqrt{35}\right)}{-2}$$

Now cancel the 2 in the numerator against the 2 in the denominator.

$$\frac{6-\sqrt{35}}{-1}$$

$$-6+\sqrt{35}$$

$$\sqrt{35}-6$$

Let's try one more with numbers.

Example

Rationalize the denominator.

$$\frac{3}{5-\sqrt{3}}$$

Since the denominator is a binomial in which one of the terms is a square root, we need to multiply the numerator and denominator by the conjugate of the binomial in order to rationalize the denominator.

$$\frac{3}{5-\sqrt{3}}\left(\frac{5+\sqrt{3}}{5+\sqrt{3}}\right)$$

$$\frac{15+3\sqrt{3}}{25+5\sqrt{3}-5\sqrt{3}-3}$$

Using the conjugate method, the two terms in the middle of the denominator will always cancel with each other.

$$\frac{15+3\sqrt{3}}{25-3}$$

$$\frac{15+3\sqrt{3}}{22}$$

Now that the square root is out of the denominator, we've rationalized the denominator.

We can even use the conjugate method with variables, and with square roots in both terms in the denominator.

Example

Rationalize the denominator.

$$\frac{a+\sqrt{3}}{-3\sqrt{a}+\sqrt{3}}$$

First, we'll switch the order of the terms in the denominator so that we lead with a positive term instead of a negative term, just to make things a little simpler.

$$\frac{a+\sqrt{3}}{\sqrt{3}-3\sqrt{a}}$$

Then we'll multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{a+\sqrt{3}}{\sqrt{3}-3\sqrt{a}}\left(\frac{\sqrt{3}+3\sqrt{a}}{\sqrt{3}+3\sqrt{a}}\right)$$

$$\frac{a\sqrt{3}+3a\sqrt{a}+3+3\sqrt{3a}}{3+3\sqrt{3a}-3\sqrt{3a}-9a}$$

Using the conjugate method, the two terms in the middle of the denominator will always cancel with each other.

$$\frac{a\sqrt{3}+3a\sqrt{a}+3+3\sqrt{3a}}{3-9a}$$