Finding a function from its inverse

The nice thing about functions and their inverses is that if we know two points, say (a_1, b_1) and (a_2, b_2), of the inverse of a function $f(x)$, then we also know that two of the points of $f(x)$ are (b_1, a_1), and (b_2, a_2). This works out very nicely if we know two points of the inverse of a linear function and we want to find that linear function.

Now we may be wondering if the inverse of a linear function is also a linear function, and the answer to this question is Yes.

To find $f^{-1}(x)$, we can first replace $f(x)$ with y, then switch x with y,

$$y = mx + b$$

$$x = my + b$$

solve for y,

$$x - b = my$$

$$\frac{x - b}{m} = y$$

$$\frac{1}{m} \cdot x - \frac{b}{m} = y$$

and finally replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{1}{m} \cdot x - \frac{b}{m}$$

Let's look at an example.

Example

Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(3) = 4$$

$$f^{-1}(-1) = 5$$

This means that $(3,4)$ and $(-1,5)$ are points of the function $f^{-1}(x)$, which is the inverse of $f(x)$. Therefore, $(4,3)$ and $(5, -1)$ are points of $f(x)$. Now we can use these points on the line that represents $f(x)$ to find the equation of the line. Let's begin by finding the slope m.

$$m = \frac{3-(-1)}{4-5} = \frac{4}{-1} = -4$$

Let's find the y-intercept. We can use the slope we just found ($m = -4$) and the slope-intercept form of the equation of a line ($y = mx + b$), together with the coordinates of one point on the line, to solve for b. Let's use the point $(4,3)$.

$$3 = -4(4) + b$$

$$3 = -16 + b$$

$$3 + 16 = b$$

$$19 = b$$

The equation of the line that represents $f(x)$ is then

$$f(x) = -4x + 19$$

If we like, we can also use the points of the inverse function to find the equation of the line that represents $f^{-1}(x)$ first, and then use that to find $f(x)$.

Example

Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-2) = 8$$

$$f^{-1}(-5) = 14$$

Let's begin by finding the equation of the line that represents $f^{-1}(x)$.

Use the points $(-2,8)$ and $(-5,14)$ to find the slope of that line.

$$m = \frac{14 - 8}{-5 - (-2)} = \frac{6}{-3} = -2$$

Let's use the point-slope form of the equation of a line ($y - y_1 = m(x - x_1)$) to solve for the y-intercept this time (although we could still use the slope-intercept form to solve for the y-intercept). To get the point-slope form, we need the slope and the coordinates of one point. We know that $m = -2$, and we can use the point $(-2,8)$.

$$y - y_1 = m(x - x_1)$$

$$y - 8 = -2(x - (-2))$$

$$y - 8 = -2(x + 2)$$

$$y - 8 = -2x - 4$$

$$y = -2x + 4$$

Remember, this is the equation of the line that represents $f^{-1}(x)$. To get $f(x)$, we'll switch x with y, then solve for y, and finally replace y with $f(x)$.

$$x = -2y + 4$$

$$x - 4 = -2y$$

$$-\frac{1}{2}(x - 4) = -\frac{1}{2}(-2y)$$

$$-\frac{1}{2}x + 2 = y$$

$$f(x) = -\frac{1}{2}x + 2$$

As we can see, there's more than one way to solve these types of problems, so we should just use whichever method we're most comfortable with.