Evaluating logs

We already know how to evaluate simple logs like $\log_2 8$, because we understand that this is asking us the question "To what power do we have to raise 2 in order to get 8?" To answer it, we only need to solve the exponential equation

$$2^x = 8$$

We know that $2^3 = 8$, so it's easy to see that $x = 3$. Also, all the quantities in this problem are whole numbers, but log problems can be more complicated than this, and that's what we want to talk about here.

The base is greater than the argument

In all our examples so far, the argument has been greater than the base. In $\log_2(8)$, the base is 2 and the argument is 8, so argument > base. But what happens when the base is greater than the argument?

$$\log_{27}(3)$$

To evaluate this expression, we'll solve the exponential equation

$$27^x = 3$$

To solve this equation, it would help if we could rewrite it in such a way that we have the same base on both sides. We know that 27 is the same as 3^3, so we'll use that to rewrite the equation.

$$(3^3)^x = 3$$

Now remember the general rule of exponents which tells us that $(a^b)^c = a^{b \cdot c}$. Therefore, our equation becomes

$$3^{3x} = 3$$

$$3^{3x} = 3^1$$

If the bases are equal, then the exponents must also be equal in order for the equation to be true.

$$3x = 1$$

$$x = \frac{1}{3}$$

So we can see that

$$27^{\frac{1}{3}} = 3$$

In terms of logs, this translates to

$$\log_{27}(3) = \frac{1}{3}$$

Let's try another example.

Example

Find the value of the expression.

$$\log_{243}(3)$$

To evaluate this expression, we'll solve the exponential equation

$$243^x = 3$$

To do this, we'll express 243 as a power of 3.

$$243 = 3 \cdot 81 = 3 \cdot 9 \cdot 9 = 3^1 \cdot 3^2 \cdot 3^2 = 3^5$$

$$(3^5)^x = 3$$

$$3^{5x} = 3$$

$$3^{5x} = 3^1$$

Since the bases are equal, the only way to make this equation true is for the exponents to also be equal.

$$5x = 1$$

$$x = \frac{1}{5}$$

So we can see that

$$\log_{243}(3) = \frac{1}{5}$$

The argument is a fraction

Sometimes the argument will be a fraction, like this:

$$\log_2\left(\frac{1}{64}\right)$$

We'll evaluate this expression by solving the exponential equation

$$2^x = \frac{1}{64}$$

To do this, we'll express 64 as a power of 2.

$$2^x = \frac{1}{2^6}$$

$$2^x = 2^{-6}$$

The bases are equal, so the exponents must be equal. Therefore, $x = -6$ and

$$\log_2\left(\frac{1}{64}\right) = -6$$

Let's try another example.

Example

Find the value of the expression.

$$\log_5\left(\frac{1}{625}\right)$$

We'll evaluate this expression by solving the exponential equation

$$5^x = \frac{1}{625}$$

To do this, we'll express 625 as a power of 5.

$$625 = 5 \cdot 125 = 5 \cdot 5 \cdot 25 = 25 \cdot 25 = 5^2 \cdot 5^2 = 5^4$$

$$5^x = \frac{1}{5^4}$$

$$5^x = 5^{-4}$$

The bases are equal, so the exponents must be equal. Therefore, $x = -4$ and

$$\log_5\left(\frac{1}{625}\right) = -4$$

This method for evaluating logs will always work when we can solve an exponential equation that can be converted to a form in which the base is the same on both sides. Let's show a summary of the steps with one last example, in which we evaluate $\log_{32}(16)$.

$$\log_{32}(16)$$

$$32^x = 16$$

$$(2^5)^x = 2^4$$

$$2^{5x} = 2^4$$

$$5x = 4$$

$$x = \frac{4}{5}$$

So we can say

$$\log_{32}(16) = \frac{4}{5}$$

Let's try one more example.

Example

Find the value of the expression.

$$\log_{\frac{1}{625}}(5)$$

We'll evaluate this expression by solving the exponential equation

$$\left(\frac{1}{625}\right)^x = 5$$

$$\frac{1}{625} = \frac{1}{5^4} = 5^{-4}$$

$$(5^{-4})^x = 5$$

$$5^{-4x} = 5^1$$

Since the bases are equal, we can equate the exponents.

$$-4x = 1$$

$$x = -\frac{1}{4}$$

Therefore,

$$\log_{\frac{1}{625}}(5) = -\frac{1}{4}$$