Graphing parabolas

In this lesson we'll learn how to identify the characteristics of a parabola and go back and forth between the equation of a parabola and its graph.

A parabola is the solution to an equation $y = f(x)$ where $f(x)$ is a quadratic polynomial. Therefore, a parabola is the solution to a non-linear equation. There are two forms of the equation of a parabola that are especially helpful when we want to know something about a parabola.

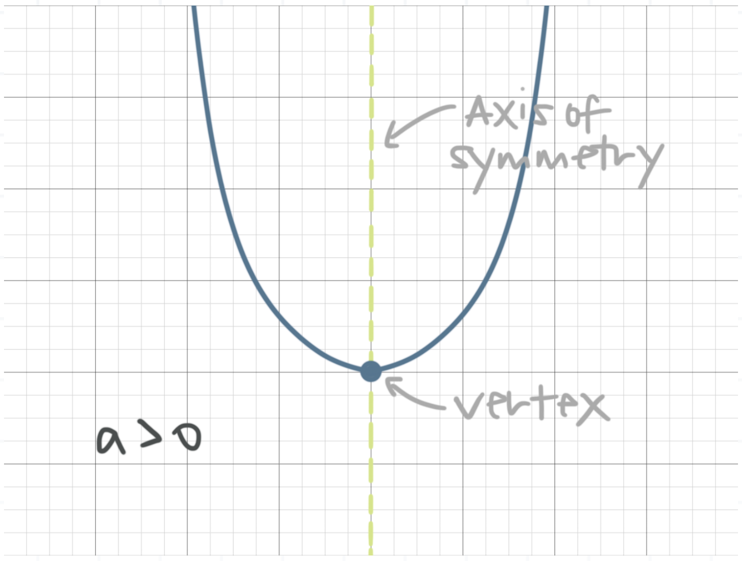
	Standard form	**Vertex form**
Equation	$y = ax^2 + bx + c$	$y = a(x - h)^2 + k$
Axis of symmetry	$x = -b/2a$	$x = h$
Vertex	$(-b/2a, f(-b/2a))$	(h, k)

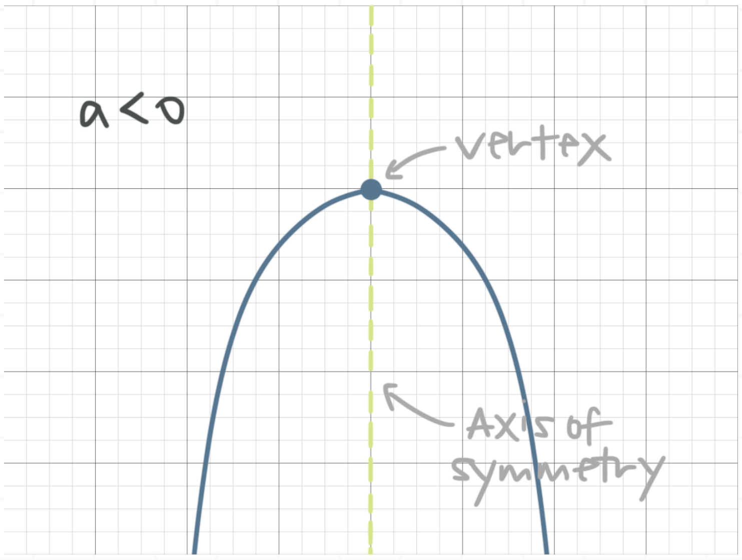
To convert from standard form to vertex form we complete the square, but to convert from vertex form to standard form we expand the square and then distribute and simplify.

Let's talk about the different parts of a parabola.

In both forms (standard and vertex), if $a > 0$ the parabola opens upwards and the vertex is the point at the bottom of the parabola (the point with the minimum value of y).

In both forms (standard and vertex), if $a < 0$ the parabola opens downwards and the vertex is the point at the top of the parabola (the point with the maximum value of y).





Let's do a few problems.

Example

Write the equation in vertex form.

$$y = 2x^2 + 36x + 170$$

To convert the standard equation of the parabola to vertex form from standard form, we'll need to complete the square.

Before we complete the square, we'll factor the coefficient of the x^2 term, which is 2, out of the first two terms on the right-hand side of the given equation.

$$y = 2x^2 + 36x + 170$$

$$y = 2(x^2 + 18x) + 170$$

To complete the square, we need to find the number d that satisfies the equation

$$x^2 + 18x + d^2 = (x + d)^2$$

That is, we need to find the number d for which

$$x^2 + 18x + d^2 = x^2 + 2dx + d^2$$

This means that the coefficient of the x term of the expression inside the parentheses must be equal to $2d$. That coefficient is 18, so we'll set $2d$ to 18 and solve for d.

$$2d = 18 \quad \to \quad d = 9$$

To keep our equation balanced, we need to add and subtract d^2 (81) inside the parentheses, and then distribute, regroup, and simplify.

$$y = 2(x^2 + 18x) + 170$$

$$y = 2(x^2 + 18x + 81 - 81) + 170$$

$$y = 2(x^2 + 18x + 81) + 2(-81) + 170$$

$$y = 2(x^2 + 18x + 81) - 162 + 170$$

$$y = 2(x^2 + 18x + 81) + 8$$

Finally, we'll factor the expression that's now inside the parentheses $(x^2 + 18x + 81)$. By construction ("completing the square"), that expression factors as $(x + d)^2$.

$$x^2 + 18x + 81 = (x + d)^2$$

$$x^2 + 18x + 81 = (x + 9)^2$$

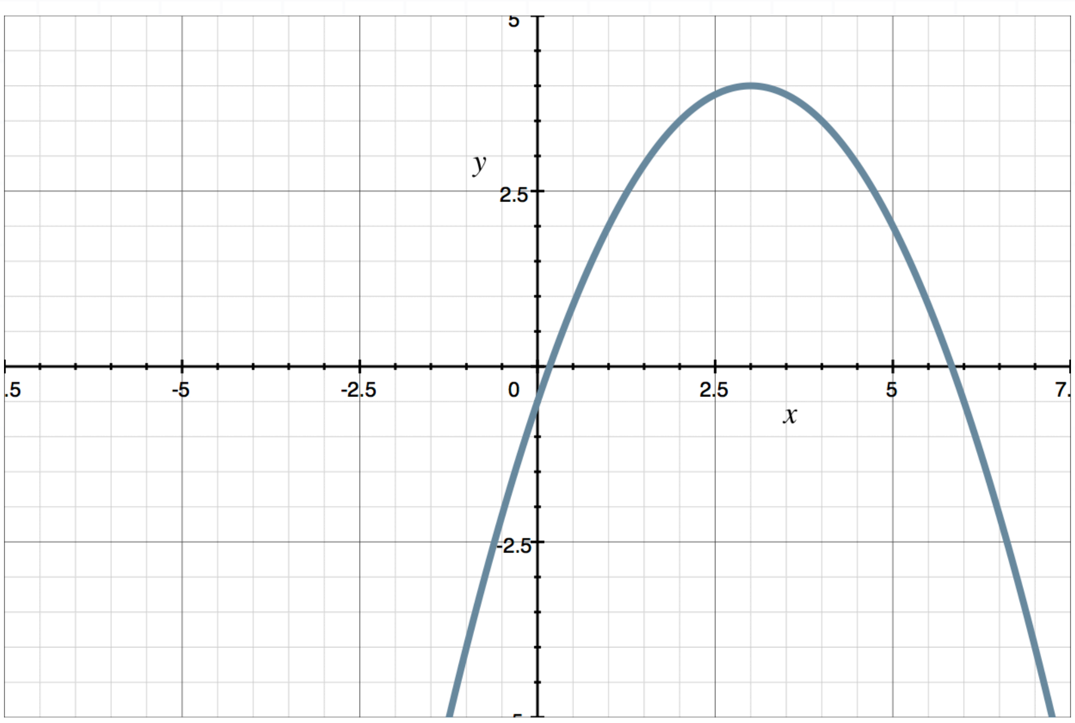
The vertex form of the equation is

$$y = 2(x+9)^2 + 8$$

Let's try one where we need to interpret a graph.

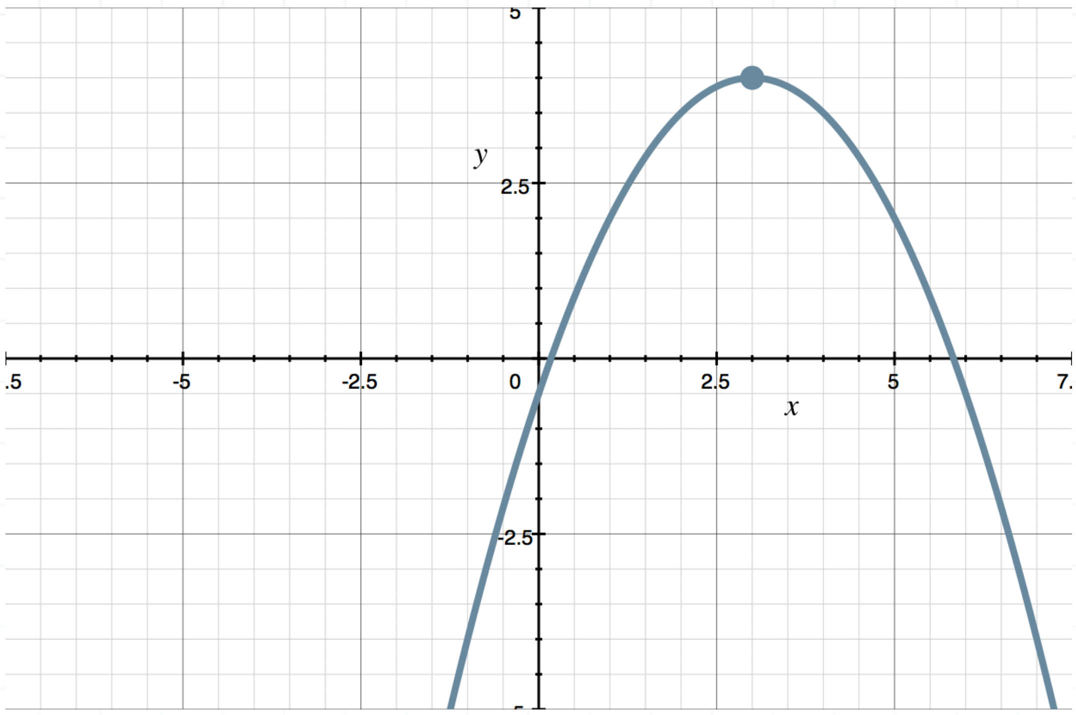
Example

The coefficient of the x^2 term in the equation of the parabola shown in the graph is $-1/2$. What is the equation of the parabola in standard form?



Remember, the vertex form of the equation of a parabola is $y = a(x-h)^2 + k$, where (h, k) are the coordinates of the vertex.

We know that $a = -1/2$, and we can read the coordinates of the vertex from the graph: $(3,4)$.



So we know that $h = 3$ and $k = 4$. Let's put what we know into the vertex form of the equation of a parabola.

$$y = a(x - h)^2 + k$$

$$y = -\frac{1}{2}(x - 3)^2 + 4$$

Now we want to go from vertex form to standard form, so we'll expand the square:

$$y = -\frac{1}{2}(x - 3)(x - 3) + 4$$

$$y = -\frac{1}{2}(x^2 - 6x + 9) + 4$$

Distribute the $-1/2$ over all the terms inside the parentheses.

$$y = -\frac{1}{2}(x^2) - \frac{1}{2}(-6x) - \frac{1}{2}(9) + 4$$

$$y = -\frac{1}{2}x^2 + 3x - \frac{9}{2} + 4$$

$$y = -\frac{1}{2}x^2 + 3x - \frac{9}{2} + \frac{8}{2}$$

$$y = -\frac{1}{2}x^2 + 3x - \frac{1}{2}$$

Sketching parabolas

Let's outline the general steps for sketching parabolas.

1. Find the coordinates of the vertex.

2. Find the value of the y-intercept.

3. Find the x-coordinates of the x-intercepts by solving the equation $f(x) = 0$.

4. Make sure we've found at least one point to either side of the vertex. This helps make a better sketch. If we already have two x -intercepts from the previous step, we can use those. Otherwise, if we have zero or just one x-intercept, we can then use another x value or use the axis of symmetry and the y-intercept to get the second point.

5. Sketch the graph.

Let's do an example.

Example

Graph the parabola.

$$y = -x^2 + 2x + 8$$

Comparing this equation to the standard form of a parabola, we can identify $a = -1$, $b = 2$, and $c = 8$. Since $a < 0$, the parabola opens downwards and the vertex is the point at the top of the parabola.

The vertex lies on the axis of symmetry, so its x-coordinate is $-b/2a$.

$$x = -\frac{2}{2(-1)} = 1$$

$$y = -1^2 + 2(1) + 8 = -1 + 2 + 8 = 9$$

So the vertex is $(1,9)$. The equation of the axis of symmetry is $x = 1$. Now we need to find x-intercepts, which we can do by substituting $y = 0$ into the equation and solving for x.

$$y = -x^2 + 2x + 8$$

$$0 = -x^2 + 2x + 8$$

$$0 = (x + 2)(x - 4)$$

$x = -2$ and $x = 4$

So the x-intercepts are $(-2,0)$ and $(4,0)$. To find the y-intercept, we need to find $y(0)$.

$$y(0) = -0^2 + 2(0) + 8 = 8$$

So the y-intercept is $(0,8)$. We have all the information we need to sketch the graph, so a sketch of the parabola is