Hypothesis testing for the population proportion

Up to now we've been focused mostly on hypothesis testing for the mean, but we can also perform hypothesis testing for a proportion. In order for the test to work, we'll need $np \geq 5$ and $n(1-p) \geq 5$, where p is the population proportion (or we can substitute the sample proportion $\hat{p}$ when we don't know the population proportion p).

We want the number of successes and failures to be at least 5, because that threshold means that the probability distribution will approximate the normal curve. Keep in mind that we have to be careful about distinguishing between the sample proportion $\hat{p}$ and the p-value of the test.

One-tailed test

Just like for the mean, a one-tailed test for the proportion indicates directionality, so our hypothesis statements will either be

$$H_0: p \leq k$$

$$H_a: p > k$$

if we suspect that $p > k$, or

$$H_0: p \geq k$$

$$H_a: p < k$$

when we suspect that $p < k$. Once we set the hypothesis tests, we'll pick a significance level, calculate the test statistic, and state the conclusion.

Example

We want to test the hypothesis that more than 32% of Americans watch the Super Bowl, so we collect a random sample of $1{,}000$ Americans and find that 350 of them watched the game. What can we conclude at a significance level of $\alpha = 0.05$?

First build the hypothesis statements.

H_0: At most 32% of Americans watched the Super Bowl, $p \leq 0.32$

H_a: More than 32% of Americans watched the Super Bowl, $p > 0.32$

The sample proportion is

$$\hat{p} = \frac{x}{n} = \frac{350}{1{,}000} = 0.35$$

Then find the standard error of the proportion.

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.32(1-0.32)}{1{,}000}} = \sqrt{\frac{0.2176}{1{,}000}} \approx 0.0148$$

Now we have enough to find the test statistic.

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{0.35 - 0.32}{0.0148} \approx 2.03$$

The critical value for 95% confidence with an upper-tailed test is $z = 1.65$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633

Using the critical value approach, we can say that our z-value exceeds $z = 1.65$,

$$z \approx 2.03 > z_\alpha = 1.65$$

and therefore falls in the region of rejection, which means we'll reject the null hypothesis and conclude that more than $32\,\%$ of Americans watch the Super Bowl.

We know our findings are significant at $\alpha = 0.05$, but we can find the p-value to state a higher level of significance that corresponds to $z \approx 2.03$ and not just $z = 1.65$. The test statistic $z \approx 2.03$ gives a value of 0.9788 in the z-table.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857

Which means the conclusion isn't only significant at $\alpha = 0.05$, but it's actually significant at

$$1 - 0.9788 = 0.0212$$

The result is significant at the 0.0212 level, so as long as $\alpha \geq 0.0212$, we'll be able to reject H_0.

Two-tailed test

The two-tailed test for the proportion follows the same steps as the one-tailed test, other than the fact that we split the alpha value into both tails.

Let's continue with the same Super Bowl example we were using, but this time we'll say that we don't have a guess about directionality, and instead will simply hypothesize that the proportion of Americans who watch the Super Bowl is some proportion other than 32%.

Example (cont'd)

We want to test the hypothesis that 32% of Americans watch the Super Bowl, so we collect a random sample of $1{,}000$ Americans and find that 350 of them watched the game. What can we conclude at a significance level of $\alpha = 0.05$?

First build the hypothesis statements.

H_0: 32% of Americans watched the Super Bowl, $p = 0.32$

H_a: The proportion of Americans who watched the Super Bowl was not 32%, $p \neq 0.32$

We already calculated that the standard error of the proportion is $\sigma_{\hat{p}} \approx 0.0148$, the sample proportion is $\hat{p} = 0.35$, and the z-value of the test-statistic is $z \approx 2.03$.

Because we're doing a two-tailed test, $\alpha = 0.05$ needs to be split as 0.025 in the lower tail and 0.025 in the upper tail. Which means we're looking for the value in the z-table that corresponds to $1 - 0.025 = 0.9750$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	**.9750**	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

So $z = \pm 1.96$ will be the critical values. Our z-value exceeds $z = 1.96$ and therefore still falls in the region of rejection (even though we've switched from a one-tailed test to a two-tailed test), which means we'll again reject the null hypothesis and conclude that the proportion of Americans who watch the Super Bowl is some proportion other than 32%.