# Algebra 1 Notes

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# Variables

Algebra is when we first start to heavily use **variables**, which are symbols for numbers we don't know yet. We usually use a letter like $x$ or $y$ to indicate a variable.

Before Algebra, we focus almost exclusively on **constants**, which are numbers on their own. For example, take the simple statement,

$$3 + 2 = 5$$

In this equation, $3$, $2$, and $5$ are all regular numbers on their own, so we would call them "constants."

But if we replace the $3$ in the equation with $x$, we could write the equation as

$$x + 2 = 5$$

In this equation, $x$ is a variable, while $2$ and $5$ are still constants. We call $x$ a "variable" because it represents a number we don't know. However, just by looking at the equation $x + 2 = 5$, and also given that we know we started with $3 + 2 = 5$, we realize that we can solve for the value of the variable. Since this equation statement is so simple, we can see right away that the only value of the variable that makes the equation true is $x = 3$.

Later in this Algebra course, we'll spend a lot of time talking about how to solve equations like these for the values of the variables they contain.

## A little more vocabulary

Let's look at another equation, like

$$3x - 10 = 8$$

We know already that $x$ is a variable and that $10$ and $8$ are constants. When we have a number attached to and in front of a variable, like the $3$ in this equation, we call it a **coefficient**. When we have a coefficient in front of a variable, it means that we're supposed to multiply the coefficient by the variable. So $3x$ means "three multiplied by x," or "three times x."

We can also have coefficients that aren't numbers, but are actually letters instead. If we typically use letters like $x$ and $y$ to represent variables, we typically use letters like $a$ and $b$ to represent constant coefficients. So in an expression like,

$$ax^2 + bx + c$$

we can say that $x$ is the variable, $a$ and $b$ are both constant coefficients, and $c$ is a constant.

We called $ax^2 + bx + c$ an expression, and we did that to distinguish this from an equation. **Equations** include an equals sign, and tell us that whatever we have on the left side of the equals sign is equivalent/equal to/has the same value as whatever we have on the right side of the equals sign.

On the other hand, **expressions** don't include an equals sign, but instead are just groups of terms, where a **term** is a single number or a variable, or numbers and variables multiplied together. So $ax^2 + bx + c$ is an expression because it doesn't include an equals sign, and instead a collection of three terms, $ax^2$, $bx$, and $c$.

Let's look at an example so that we can get a little practice identifying these things.

**Example**

Use the vocabulary we've learned to describe as many parts of the statement as possible.

$$3x - 10 = 8$$

Let's start by talking about equations and expressions. We can say that $3x - 10 = 8$ is an equation, because it includes an equals sign that tells us that the left and right sides must have the same value. We could also say that $3x - 10$ is an expression and $8$ is an expression, and therefore that the equation sets those two expressions equal to one another.

The terms in the equation are $3x$, $10$, and $8$, where $x$ is a variable, and $10$ and $8$ are constants. The $x$ variable has a constant coefficient of $3$.

# Identifying multiplication

When we first learned about multiplication, we most likely learned about the times symbol, $\times$. Now that we're getting into more advanced math, the traditional $\times$ symbol can often be confused with the variable $x$.

Instead, we'll start using parentheses or a dot to indicate multiplication. All of these are different ways of indicating multiplication,

| | |
|---|---|
| Times | $a \times b = c$ |
| Dot | $a \cdot b = c$ |
| Parentheses | $(a)(b) = c$ |
| Variables next to each other | $ab = c$ |

but we'll stop using the times symbol and start using the other three options.

The last way to indicate multiplication (without any multiplication symbol) is used only for multiplication of two or more variables (like $ab$ or $xyz$) or for multiplication of a number by one or more variables (like $3x$ or $4ac$).

If we're multiplying two numbers, like $3 \cdot 4$, we have to use a multiplication symbol to avoid confusing $3 \cdot 4$ with $34$.

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**Example**

Write $5$ times $2$ in three different ways.

When we multiply $5$ and $2$ together, we can write the product as

$$5 \times 2 = 10$$

$$5 \cdot 2 = 10$$

$$(5)(2) = 10$$

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Let's try another example to identify multiplication.

**Example**

Simplify the expression.

$$2 \cdot 4 \times 3(5)(2 \cdot 2)$$

All of these symbols represent multiplication. Multiplication can be done in any order, so if we multiply these values together, we get

$$8 \times 3(5)(2 \cdot 2)$$

$$8 \times 15(4)$$

$$8 \times 60$$

$$480$$

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# Associative Property

When we add or multiply real numbers, it doesn't matter how those numbers are grouped; the result of the multiplication will always be the same.

## Associative property

Intuitively we already know this, but now we want to formally say that this is the **Associative Property** of multiplication.

| | |
|---|---|
| Associative Property of Addition | $(a+b)+c=a+(b+c)$ |
| Associative Property of Multiplication | $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ |

In these formulas, $a$, $b$, and $c$ represent numbers. The Associative Property of Addition tells us that it doesn't matter if we first add $b$ to $a$, and then add $c$ to the result, or if we first add $c$ to $b$, and then add the result to $a$. We'll get the same answer both ways.

Similarly, the Associative Property of Multiplication tells us that it doesn't matter if we first multiply $a$ by $b$, and then multiply the result by $c$, or if we first multiply $b$ by $c$, and then multiply $a$ by the result. We'll get the same answer both ways.

"Associative" comes from the word "associate," so we want to remember that "associate," in terms of math, refers to grouping with parentheses. In other words, in an example of the Associative Property, the numbers will stay in the same order but the parentheses will move.

**Example**

Use the Associative Property to write the expression a different way, without performing the addition.

$$3 + (6 + 7)$$

We know that when we apply the Associative Property of Addition, the parentheses move but the numbers don't. So we could keep the numbers where they are, but move the parentheses to rewrite $3 + (6 + 7)$ as

$$(3 + 6) + 7$$

We didn't have to perform the addition to solve this problem, but we can also see that the two expressions are equal.

$$3 + (6 + 7) = 3 + (13) = 16$$

$$(3 + 6) + 7 = (9) + 7 = 16$$

We get $16$ as the answer, regardless of the order in which we perform the addition.

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Let's try another example, this time with the Associative Property of Multiplication.

**Example**

Is the equation true or false?

$$(2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5)$$

The equation is true, because of the Associative Property of Multiplication. The order of the numbers stayed the same but the parentheses moved.

We can see that the expressions on both sides of the equation each simplify to $30$.

$$(2 \cdot 3) \cdot 5 = (6) \cdot 5 = 30$$

$$2 \cdot (3 \cdot 5) = 2 \cdot (15) = 30$$

# Commutative Property

The Commutative Property is similar to the Associative Property, but the Commutative Property tells us specifically that we can add or multiply numbers in any order. For example, $2 + 5 = 5 + 2$.

## Commutative Property

Like the Associative Property, the **Commutative Property** applies to both addition and to multiplication.

| | |
|---|---|
| Commutative Property of Addition | $a + b = b + a$ |
| Commutative Property of Multiplication | $a \cdot b = b \cdot a$ |

"Commutative" comes from the word "commute," and since commute means to move, we can remember that when we use the commutative property, the numbers will move around.

Let's do an example where we use the Commutative Property to rewrite the expression.

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**Example**

Use the Commutative Property to write the expression a different way, without performing the multiplication.

$$5 \cdot 3 \cdot 2$$

We know that when we apply the Commutative Property of Multiplication, the numbers change places. So we could use the Commutative Property to rewrite $5 \cdot 3 \cdot 2$ as any of these:

$$5 \cdot 2 \cdot 3$$

$$3 \cdot 5 \cdot 2 \qquad 3 \cdot 2 \cdot 5$$

$$2 \cdot 5 \cdot 3 \qquad 2 \cdot 3 \cdot 5$$

What we see is that, regardless of the order in which we perform the multiplication, we always get the same value.

$$5 \cdot 3 \cdot 2 = 15 \cdot 2 = 30 \qquad 5 \cdot 2 \cdot 3 = 10 \cdot 3 = 30$$

$$3 \cdot 5 \cdot 2 = 15 \cdot 2 = 30 \qquad 3 \cdot 2 \cdot 5 = 6 \cdot 5 = 30$$

$$2 \cdot 5 \cdot 3 = 10 \cdot 3 = 30 \qquad 2 \cdot 3 \cdot 5 = 6 \cdot 5 = 30$$

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Let’s try another example, this time with the Commutative Property of Addition.

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**Example**

Are the expressions $4 + 12 + 7$ and $7 + 4 + 12$ equivalent?

The equation is true by the Commutative Property of Addition, which tells us that we can change the order of the terms, and the sum of the terms will still be the same.

Since we've just moved the $7$ from its position as the last term to become the first term, we've only changed the order of the addition, and that won't change the value of the sum.

If we simplify both expressions, we get the same value.

$$4 + 12 + 7 = 16 + 7 = 23$$

$$7 + 4 + 12 = 11 + 12 = 23$$

# Transitive Property

The **Transitive Property** is another logic property of Algebra, which simply tells us that,

if $a = b$ and $b = c$, then $a = c$

Of course, this makes sense if we think about it in terms of real numbers. All we're saying here with the Transitive Property is that, if we know that $5 = 4 + 1$, and we also know that $4 + 1 = 2 + 3$, then we can conclude that $5 = 2 + 3$.

$$5 = 4+1$$
$$4+1 = 2+3$$

"Transitive" comes from the word "transit" which means to move from one place to another. In this case we can "jump" over the middle and link the ends together, since the ends are both equal to the middle.

Let's do an example where we apply the Transitive Property.

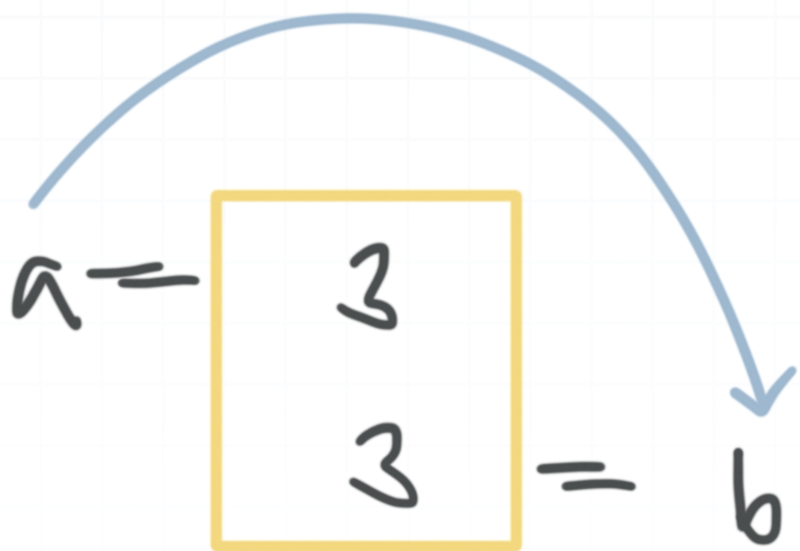
### Example

For numbers $a$ and $b$, what can we say about the relationship between $a$ and $b$ if both of the following statements are true?

$a = 3$

$b = 3$

We can take the equation $b = 3$ and "turn it around" (switch the left-hand side with the right-hand side), which gives $3 = b$. Now we have the following:




Since $a = 3$ and $3 = b$, the transitive property tells us that $a = b$.

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Let's try another example of the Transitive Property.

**Example**

Consider the variables $x$, $y$, and $z$. Use the Transitive Property to write an equation that relates the variable $x$ to the variable $z$, with no mention of the variable $y$.

$x = y$

$$y = 2z + 4$$

These expressions seem more complicated, but it doesn't matter what the expressions are. If we know that $x = y$ and that $y = 2z + 4$, then Transitive Property tells us that $x = 2z + 4$.

$$x = y$$
$$y = 2z + 4$$

# Understood 1

We talked earlier about $x$ as a variable, and about how, in a term like $3x$, the $3$ is a constant coefficient on $x$ that multiplies it. So $3x$ is "$3$ multiplied by $x$." What we want to say now is that, while $x$ by itself looks like it has no coefficient, it actually has a coefficient of $1$.

## The "Understood 1"

In other words, whenever anything has a coefficient of $1$, we just don't write it, because we agree that it's implied. So when we see $x$, we know automatically that it means $1x$. So think about $3x + x$ as $3x + 1x$, which simplifies to $4x$. We'll talk more later about how to add like terms like $3x + x$.

The same goes for subtraction. We should know that $3x - x$ is means $3x - 1x$, which simplifies to $2x$.

And if we think about it more, we realize that essentially everything has an understood coefficient of $1$. To pick a random example, even the number $7$ we could think of as $1(7)$. There's can always be an implied coefficient of $1$, because we know that multiplying something by $1$ doesn't change its value (because $1$ is the identity number for multiplication, which we learned about in Pre-Algebra).

This idea of the "understood $1$" also extends to denominators. Every value has an implied denominator of $1$, since dividing something by $1$ doesn't change its value. So $x$ is understood to be $x/1$, in the same way that $7$ is understood to be $7/1$.

And finally, we can extend this idea to exponents, so $x$ is understood to be $x^1$, and $7$ is understood to be $7^1$, since raising any value to the power of $1$ doesn't change its value.

It's a little absurd, but technically we could combine this "understood $1$" rule for coefficients, denominators, and exponents, and rewrite $x$ and $7$ as

$$x = \frac{1x^1}{1} \qquad\qquad 7 = \frac{1(7)^1}{1}$$

## Why does this matter?

Let's be honest, it might seem a little odd to make this weird point that $x = 1x^1/1$. So why are we even bothering?

We've already seen that this concept helps us simplify something like $3x + x$ or $3x - x$, but let's look at some more examples to see when this "understood $1$" comes into play.

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**Example**

Rewrite the expression as one fraction.

$$\frac{2}{3} + x$$

If we want to rewrite this sum as only one fraction, our first step is to realize that $x$ by itself is actually a fraction, too. If we rewrite $x$ as $x/1$, now we have the sum of two fractions. Start by simplifying the expression in parentheses, using the fact that $x = 1x$.

$$\frac{2}{3} + \frac{x}{1}$$

Remember from Pre-Algebra that, in order to add fractions, we need a common denominator, meaning that the denominators have to the be same. Right now, one denominator is $3$ and the other is $1$. If we multiply the second fraction by $3/3$ (which is like multiplying by $1$, and therefore doesn't change the value of the original fraction), then we'll make the denominators the same.

$$\frac{2}{3} + \frac{x \cdot 3}{1 \cdot 3}$$

$$\frac{2}{3} + \frac{3x}{3}$$

Now that the denominators are equivalent, we can add the fractions. When we do, the denominator stays the same, and the numerators get added.

$$\frac{2 + 3x}{3}$$

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This time let's go in the opposite direction, by simplifying an expression that contains some "understood $1$s" that we can eliminate.

**Example**

Simplify the expression.

$$\frac{1}{1(1x^1)}$$

We know that $x^1$ is the same as $x$ by itself, so we can remove the exponent without changing the value of the expression.

$$\frac{1}{1(1x)}$$

We know that $1x$ is the same as $x$ by itself, so we remove the coefficient without changing the value of the expression.

$$\frac{1}{1(x)}$$

Within the denominator, the parentheses indicate multiplication, so we have $1$ multiplied by $x$. We can simplify the denominator to $1x$, and then remove the "understood $1$" coefficient, and the expression becomes just

$$\frac{1}{x}$$

This is as far as we can simplify. The "understood $1$" concept allows us to simplify $x/1$ to just $x$, but we can't simplify $1/x$ to just $x$. Dividing something by $1$ doesn't change its value, which is why $x/1$ simplifies to $x$. But dividing $1$ by $x$ is the opposite operation and doesn't simplify the same way.

# Adding and subtracting like terms

Now that we understand that $x = 1x^1$ (this is the concept of the "understood $1$"), we can start talking about how to combine like terms. We'll begin here by looking at how to add and subtract like terms, and then later we'll talk about how to multiply and divide like terms.

## What is a "like term"?

We define like terms differently depending on whether we're adding and subtracting, or multiplying and dividing. When we're adding and subtracting, **like terms** are terms with equivalent bases and equivalent exponents.

For example, $x^2$ and $3x^2$ are like terms because they both have base $x$ and an exponent of $2$. The coefficients are $1$ and $3$, but the coefficients don't have to match.

So when we add $x^2 + 3x^2$, we get $4x^2$. One trick for adding terms like these is to identify the like term, and then replace that like term with something easier to think about. It's silly, but let's pretend that, instead of $x^2$, we have "apples." It's can also be helpful to write in any "understood $1$." So we can think about $x^2 + 3x^2$ as

$$1x^2 + 3x^2$$

$$1(\text{apples}) + 3(\text{apples})$$

$4(\text{apples})$

$4x^2$

Let's look at a much more complicated example, so that we can show that it's actually not that much more complicated at all.

**Example**

Simplify the expression.

$$2(x^2+1)^3+(x^2+1)^3-3(x^2+1)^3+5(x^2+1)^3$$

This expression includes a lot more than we've learned how to deal with yet. But let's just focus in on the "like terms" concept.

All of these terms have the same base, $(x^2+1)$, and the same exponent, $3$. The coefficients are $2$, $1$, $3$, and $5$. So let's rewrite the expression as

$2$ of these $+\ 1$ of these $-\ 3$ of these $+\ 5$ of these

If we think about just $2+1-3+5$, we get $5$. So we can simplify the expression as

$5$ of these

$$5(x^2+1)^3$$

Alternatively, if we don't like using text, we can always make a substitution with a different variable. For instance, if we substitute $A=(x^2+1)$, the expression is

$$2A + A - 3A + 5A$$

$$2A + 1A - 3A + 5A$$

$$(2 + 1 - 3 + 5)A$$

$$5A$$

$$5(x^2 + 1)^3$$

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Let's try one more example of adding and subtracting like terms.

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**Example**

Simplify the expression.

$$x - 3x^2 + 4x + 7x^2$$

We need to realize that the four terms in this expression aren't all like terms. The $x$ and $4x$ terms are alike because the base is $x$ and the exponent is the "understood 1." And the $-3x^2$ and $7x^2$ terms are alike because the base is $x$ and the exponent is 2.

So we'll start by reordering terms in the expression to group like terms together.

$$x + 4x - 3x^2 + 7x^2$$

Combine the $x$ terms.

$$(1 + 4)x - 3x^2 + 7x^2$$

$$5x - 3x^2 + 7x^2$$

Combine the $x^2$ terms.

$$5x + (-3 + 7)x^2$$

$$5x + 4x^2$$

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# Multiplying and dividing like terms

Remember that, when adding and subtracting, **like terms** are terms with the same base and same exponent. For instance, $-4x^2$ and $3x^2$ are like terms because they have the same base, $x$, and the same exponent, $2$. The coefficients are different, $-4$ and $3$, but the coefficients can be different and the terms are still alike.

On the other hand, $3x^2$ and $-4x^5$ are not like terms for addition and subtraction because, while they have the same base, the exponents $2$ and $5$ are different.

## Like terms for multiplication and division

When we're multiplying and dividing, the base still has to be the same (just like when were adding and subtracting), but the exponents can be different.

So while $3x^2$ and $-4x^5$ *can't* be added or subtracted, the sum is just $3x^2 - 4x^5$ and the sum can't be simplified any further. But they *can* be multiplied or divided. When we multiply terms with the same base, the base stays the same and the exponents get added. The coefficients get multiplied together to become the new coefficient.

$$4x^2 \cdot 3x^5$$

$$(4 \cdot 3)x^{2+5}$$

$$12x^7$$

Similarly, when we divide terms with the same base, the base stays the same and the exponents get subtracted (we subtract the exponent in the denominator from the exponent in the numerator). The coefficients get divided to become the new coefficient.

$$\frac{4x^3}{2x}$$

$$\frac{4}{2}x^{3-1}$$

$$2x^2$$

We won't be able to multiply or divide terms when the bases aren't the same. For example, the bases of $x^2$ and $y^3$ are different, so we can multiply the terms, we just can't combine or simplify the product in any way.

$$x^2 \cdot y^3 = x^2y^3$$

We can also divide these terms, we just can't combine or simplify the quotient in any way.

$$x^2 \div y^3 = \frac{x^2}{y^3}$$

Let's work through an example to see this in action.

**Example**

Simplify the expression.

$$\frac{x^3}{x} - (2x)(3x)$$

In the fraction, the terms in the numerator and denominator are alike, because they have the same base, $x$. So to simplify the fraction, the base will stay the same and we'll subtract the exponents.

$$x^{3-1} - (2x)(3x)$$

$$x^2 - (2x)(3x)$$

To simplify the product of the terms $(2x)(3x)$, we can see that the terms are alike because they have the same base, $x$. To simplify the product, the coefficients will be multiplied to get the new coefficient, the base will stay the same, and the exponents will get added.

$$x^2 - (2 \cdot 3)x^{1+1}$$

$$x^2 - 6x^2$$

Now we realize that we're left with subtraction. Terms are only alike for the purpose of addition and subtraction when the base and exponent are both equivalent. For both of these terms, the base is $x$ and the exponent is $2$. So we have like terms for subtraction, and we're subtracting six $x^2$ terms from one $x^2$ terms, so the result will be

$$(1 - 6)x^2$$

$$-5x^2$$

# Distributive Property

Let's imagine that we're shopping for fruit at the grocery store, and we decide to buy $3$ apples and $8$ oranges. We notice that the price for apples and oranges today is actually equivalent; they're all $\$0.75$ per piece of fruit.

To calculate the total cost of the fruit, we want to realize that, because the cost of both apples and oranges is equivalent, we can compute our total cost two ways.

1. Calculate the total cost of the apples, calculate the total cost of the oranges, and add those totals together to get the grand total.

   $\$0.75(A) + \$0.75(O)$

   $\$0.75(3) + \$0.75(8)$

   $\$2.25 + \$6.00$

   $\$8.25$

2. Find the total pieces of fruit, then calculate the cost of the total to get the grand total.

   $\$0.75(A + O)$

   $\$0.75(3 + 8)$

   $\$0.75(11)$

   $\$8.25$

## The Distributive Property

The **Distributive Property** is the rule that tells us we're able to make this calculation both ways. Put simply it tells us that $a(b+c)=ab+ac$.

In other words, if we start at this point in our apples and oranges example,

$$\$0.75(3+8)$$

we can "distribute" the $\$0.75$ across the terms inside the parentheses to rewrite the expression as

$$\$0.75(3)+\$0.75(8)$$

Mathematically, we're saying that we can take the coefficient that's sitting in front of the parentheses, and multiply it across all the terms inside the parentheses. And the Distributive Property doesn't apply just to sums, we could also use the property to say

$$a(b+c)=ab+ac \qquad (a+b)c=ac+bc$$

$$a(b-c)=ab-ac \qquad (a-b)c=ac-bc$$

In other words, distributing removes the parentheses from the expression. When we're first learning to distribute, it's useful to write out every multiplication that results from the distribution, and then simplify them in a separate step.

Let's do an example where we introduce a variable into the expression.

### Example

Use the Distributive Property to expand the expression.

$$4(3 - x)$$

Multiply the coefficient outside the parentheses, $4$, by each term inside the parentheses.

$$4(3) - 4(x)$$

$$12 - 4x$$

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Let's do another example that's a little more complicated.

### Example

Use the Distributive Property to expand the expression.

$$-x(x^2 - 2x + 1)$$

It doesn't matter that there are now more than two terms inside the parentheses. We'll still just multiply the coefficient, $-x$, by each term inside the parentheses.

We'll pay special attention to the second term, $-x(-2x)$. If we ignore the negative signs, we know that $x(2x)$ is $2x^2$. The two negatives will cancel with one another to become a positive.

$$-x(x^2) - x(-2x) - x(1)$$

$$-x^3 + 2x^2 - x$$

# Distributive Property with fractions

Remember that the Distributive Property is a method we can use to simplify an expression by multiplying the coefficient outside the parentheses by each term inside the parentheses.

$$a(b+c)=ab+ac$$

All we want to say now is that the Distributive Property applies to fractions in exactly the same way. Introducing fractions just means we have to remember rules for fraction multiplication.

When we multiply two fractions, we multiply the numerators to get the new numerator, and we multiply the denominators to get the new denominator.

$$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{ac}{bd}+\frac{ae}{bf}$$

## Understood $1$ and cancelling common factors

There are two other important points to keep in mind when we're applying the Distributive Property and there are fractions involved.

First, if we're multiplying a fraction by a term that isn't a fraction, remember that the idea of the "understood $1$" tells us that we can rewrite any value with a denominator of $1$. So given any term that isn't a fraction, we can turn it into a fraction by giving it a denominator of $1$.

Second, once we've applied the Distributive Property, we need to decide whether we can cancel any common factors from the numerator and denominator of any of the fractions in the result.

When we first learned about fractions in Pre-Algebra, we talked about how to cancel common factors to simplify something like $2/4$ to $1/2$, or to simplify $10/30$ to $1/3$. Now that we have variables involved, we'll need to be able to cancel common factors to turn something like $3x/x^2$ into $3/x$, or something like $y^4/y$ into $y^3/1 = y^3$.

Let's do some examples.

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**Example**

Use the Distributive Property to rewrite the expression.

$$-\frac{1}{3}(3y + 15)$$

Multiply the coefficient outside the parentheses, $-1/3$, by each term inside the parentheses.

$$-\frac{1}{3}(3y) - \frac{1}{3}(15)$$

When we multiply a fraction by a non-fraction, the non-fraction just multiplies the numerator,

$$-\frac{1(3y)}{3} - \frac{1(15)}{3}$$

$$-\frac{3y}{3}-\frac{15}{3}$$

but we could also change the non-fractions into fractions by giving them a denominator of $1$, and we'll arrive at the same place.

$$-\frac{1}{3}\left(\frac{3y}{1}\right)-\frac{1}{3}\left(\frac{15}{1}\right)$$

$$-\frac{1(3y)}{3(1)}-\frac{1(15)}{3(1)}$$

$$-\frac{3y}{3}-\frac{15}{3}$$

Either way, now that we've applied the Distributive Property, we need to consider both fractions to see if we can cancel any common factors. In the first fraction, we see a common factor of $3$ in the numerator and denominator, so we can cancel it. Remember that this is because we're cancelling out the $3/3$, and $3/3=1$, and cancelling $1$ out of the fraction won't change the value of the fraction.

$$-y-\frac{15}{3}$$

In the second fraction, we see a common factor of $3$ in the numerator and denominator, so we can cancel it.

$$-y-\frac{3(5)}{3}$$

$$-y-5$$

Let's do another example that's a little more complicated.

**Example**

Use the Distributive Property to expand the expression.

$$\frac{3a}{b^2}\left(\frac{4c}{5b}+\frac{a^3}{3b^2}\right)$$

Multiply the coefficient $3a/b^2$ by each term inside the parentheses.

$$\frac{3a}{b^2}\left(\frac{4c}{5b}\right)+\frac{3a}{b^2}\left(\frac{a^3}{3b^2}\right)$$

In each term, we'll use fraction multiplication, multiplying the numerators together to get the new numerator, and multiplying the denominators together to get the new denominator.

$$\frac{3a(4c)}{b^2(5b)}+\frac{3a(a^3)}{b^2(3b^2)}$$

$$\frac{12ac}{5b^3}+\frac{3a^4}{3b^4}$$

We need to consider whether we can cancel any common factors within each of the remaining fractions. Within the first fraction, there's no common factors between $12$ and $5$, so we can't simplify the coefficients.

The numerator includes $a$ and $c$ only, and the denominator includes $b$ only, so there are no common factors that we can see among those constants.

Within the second fraction, the only common factor is a common factor of $3$ in the numerator and denominator, so we'll cancel that, and the resulting expression will be

$$\frac{12ac}{5b^3}+\frac{a^4}{b^4}$$

---

Let's try one more example of the Distributive Property with fractions.

**Example**

Use the distributive property to expand the expression.

$$\left(xz^2-\frac{x^2y^3}{z^2}\right)\frac{xy^2}{z}$$

Multiply each term inside the parentheses by the coefficient outside the parentheses, $xy^2/z$.

$$xz^2\left(\frac{xy^2}{z}\right)-\frac{x^2y^3}{z^2}\left(\frac{xy^2}{z}\right)$$

Multiply the numerators and the denominators separately (remember that if there's no denominator, then the denominator is $1$).

$$\frac{xz^2(xy^2)}{1(z)} - \frac{x^2y^3(xy^2)}{z^2(z)}$$

$$\frac{x^2y^2z^2}{z} - \frac{x^3y^5}{z^3}$$

Within the first fraction, we can cancel one factor of $z$.

$$\frac{x^2y^2z(z)}{z(1)} - \frac{x^3y^5}{z^3}$$

$$\frac{x^2y^2z}{1} - \frac{x^3y^5}{z^3}$$

$$x^2y^2z - \frac{x^3y^5}{z^3}$$

There are no common factors in the second fraction, since the numerator contains only $x$ and $y$, while the denominator contains only $z$.

# PEMDAS and order of operations

In any math expression, grouping symbols are used to group factors or terms together. When we evaluate a math expression, we have to perform the operations that are enclosed in grouping symbols before we perform other kinds of operations in that expression.

We're most familiar with parentheses as one kind of grouping symbol, but there are others.

| | |
|---|---|
| Parentheses | ( ) |
| Brackets (square brackets) | [ ] |
| Braces (curly braces) | { } |
| Absolute Value | \| \| |

In some expressions, a division symbol (either the ÷ symbol or the "fraction line" that separates the numerator of a fraction from the denominator) is used as a grouping symbol. In the fraction

$$\frac{a}{b+c}$$

for example, the division symbol tells us that we have to first perform the addition in the denominator $(b+c)$, and then second divide the numerator $a$ by the result.

Similarly, the absolute value in $a|b-c|$ tells us that we have to first perform the subtraction $b-c$, then second take the absolute value of the result, then third multiply by $a$.

## PEMDAS and order of operations

When we're given an expression and we want to evaluate it, we have to perform the indicated operations in the correct order; that order has come to be known as the **order of operations**, or **PEMDAS**.

The first letter P tells us that the first thing we have to do is perform operations that are enclosed in grouping symbols; the reason why P is used for this is that **P**arentheses are the most commonly used grouping symbol. The order of operations for PEMDAS is

| | |
|---|---|
| **P**arentheses | (all grouping symbols) |
| **E**xponents | (powers and roots) |
| **M**ultiplication/**D**ivision | (From left to right and top to bottom, performing each multiplication/division as we come to it) |
| **A**ddition/**S**ubtraction | (From left to right and top to bottom, performing each addition/subtraction as we come to it) |

All grouping symbols other than the division symbol actually consist of a pair of symbols:

- an opening (left) parenthesis and a closing (right) parenthesis
- an opening (left) bracket and a closing (right) bracket

- an opening (left) brace and a closing (right) brace
- an opening absolute value line and a closing absolute value line

But keep in mind that when we refer in general to a “pair” of grouping symbols, in some cases we could be referring to just a division symbol.

Sometimes one pair of grouping symbols is inside another pair. When that happens, we have to start by performing the operation that’s enclosed in the innermost pair of grouping symbols, then work our way outwards.

Let’s do an example of how to apply PEMDAS.

---

**Example**

Simplify the expression.

$$[15 - (2 + 4)] \cdot 5 - 3$$

<u>**P**</u> Start by performing the operation that’s enclosed in the innermost pair of grouping symbols (the addition in the parentheses).

$$[15 - (6)] \cdot 5 - 3$$

$$[15 - 6] \cdot 5 - 3$$

Now perform the operation that’s enclosed in the remaining pair of grouping symbols (the subtraction in the brackets).

$$[9] \cdot 5 - 3$$

$$9 \cdot 5 - 3$$

**E** There are no exponents, **MD** so we'll perform all the multiplication and division together, working from left to right in the expression.

$$45 - 3$$

**AS** Finally, we'll do all the addition and subtraction together, working from left to right in the expression.

$$42$$

---

Let's try another example of applying PEMDAS.

---

**Example**

Simplify the expression.

$$3[(4 - 1) + 7] - (8 + 2)$$

**P** Start by performing the operation that's enclosed in the innermost pair of grouping symbols (the subtraction $4 - 1$ in the parentheses).

$$3[(3) + 7] - (8 + 2)$$

$$3[3 + 7] - (8 + 2)$$

Neither of the two remaining pairs of grouping symbols is inside the other pair, so perform the operations enclosed in those two pairs of grouping symbols (the addition $3+7$ in the square brackets, and the addition $8+2$ in the parentheses) separately.

$$3[10]-(10)$$

$$3[10]-10$$

**E** There are no exponents, **MD** so we'll perform all the multiplication and division together, working from left to right in the expression.

$$30-10$$

**AS** Finally, we'll do all the addition and subtraction together, working from left to right in the expression.

$$20$$

---

We also want to make sure we know how to deal with grouping symbols in the denominator of a fraction.

**Example**

Simplify the expression.

$$\frac{3}{(4-1)+7}+\frac{(8+2)-4}{(12+2)-4}$$

**P** When we have grouping symbols within a fraction, we want to first simplify any grouping in the numerator,

$$\frac{3}{(4-1)+7}+\frac{10-4}{(12+2)-4}$$

then simplify any grouping in the denominator.

$$\frac{3}{3+7}+\frac{10-4}{14-4}$$

Before we can address the division represented by the fraction lines, we still need to simplify each numerator individually and each denominator individually. As we work inside each numerator and denominator, we still stick to PEMDAS. In the numerators and denominators we have left, we only have addition and subtraction, so we get

$$\frac{3}{10}+\frac{6}{10}$$

**E** There are no exponents, **MD** and there's no real value here to performing the division, since actually performing the division would just turn each fraction into a decimal number. **AS** So we'll perform all the addition and subtraction together, working from left to right in the expression.

$$\frac{3}{10}+\frac{6}{10}$$

$$\frac{9}{10}$$

---

Let's look at an example that includes exponents.

**Example**

Apply order of operations to simplify the expression.

$$3^3 + 9 \div (5 - 2) \cdot (4)^2$$

Parentheses

$$3^3 + 9 \div (5 - 2) \cdot (4)^2$$

$$3^3 + 9 \div (3) \cdot (4)^2$$

Exponents

$$27 + 9 \div (3) \cdot (16)$$

Multiplication and Division together, from left to right

$$27 + 3 \cdot 16$$

$$27 + 48$$

Addition and Subtraction together, from left to right

$$75$$

---

Let's try one more example using the order of operations.

**Example**

Apply order of operations to simplify the expression.

$$6 + 2(3x + 1)$$

Parentheses

$3x$ and $1$ are not like terms, so we can't simplify the expression inside the parentheses

Exponents

There are no exponents in the expression

Multiplication and Division together, from left to right

$$6 + 2(3x + 1)$$

$$6 + 2(3x) + 2(1)$$

$$6 + 6x + 2$$

Addition and Subtraction together, from left to right

$$6x + 8$$

# Evaluating expressions

Now that we have a background on important Algebra operations, we want to start evaluating expressions and solving equations.

Remember that an expression is a collection of terms, like $3x - 8$, $x - y$, or $abc + c^3 + ab$, while equations include an equals sign and set two expressions equal to one another, like $x + 5 = 12$ or $-2(3x + 1) = 3(-5x + 11) + 1$.

We'll start here with evaluating expressions, which will prepare us later to solve equations.

## How to evaluate expressions

When we evaluate an expression, all we're doing is replacing variables with numbers, or substituting numbers for variables, and then simplifying the expression using order of operations.

Let's do an example.

---

**Example**

Given $x = 8$ and $y = 3$, evaluate the expression.

$$x - y$$

Plug in $8$ for $x$, and $3$ for $y$, and then simplify.

$$x - y$$

$$8 - 3$$

$$5$$

---

Let's try another example.

---

**Example**

If $a = 2$, $b = -1$, and $c = 3$, find the value of the expression.

$$abc + c^3 + ab$$

Plug in $2$ for $a$, $-1$ for $b$, and $3$ for $c$. When we substitute $a = 2$, $b = -1$, and $c = 3$, we get

$$abc + c^3 + ab$$

$$(2)(-1)(3) + 3^3 + (2)(-1)$$

Now simplify using order of operations. All the values within parentheses are simplified as much as they can be, and the parentheses that remain are there just to indicate multiplication, so we'll deal with them later in the order of operations.

We'll address the exponent first.

$$(2)(-1)(3) + 27 + (2)(-1)$$

Perform the multiplication from left to right.

$$(-2)(3) + 27 + (2)(-1)$$

$$-6 + 27 + (2)(-1)$$

$$-6 + 27 - 2$$

Perform the addition and subtraction, from left to right.

$$21 - 2$$

$$19$$

# Inverse operations

In order to prepare to solve equations, we need to understand **inverse operations**, which are opposite operations that undo each other. For example, addition undoes subtraction and vice versa, and division undoes multiplication and vice versa.

## Defining inverse operations

For instance, because addition and subtraction are inverse operations,

- Addition is the inverse of subtraction
- Subtraction is the inverse of addition

adding and subtracting the same value from something won't change the value we started with. So let's say we start with $10$. If we both add and subtract $3$,

$$10 + 3 - 3$$

the addition and subtraction undo each other, and we're left with $10$, the same value we started with.

Similarly, because multiplication and division are inverse operations,

- Multiplication is the inverse of division
- Division in the inverse of multiplication

multiplying and dividing by the same value won't change the value we started with. If we start with $-4$ and both multiply and divide by $2$,

$$\frac{-4(2)}{2}$$

the multiplication and division undo each other, and we're left with $-4$, the same value we started with.

Finally, because exponents and roots are inverse operations,

- Exponents are the inverse of roots
- Roots are the inverse of exponents

raising something to a power and taking the root won't change the value we started with. If we start with $3$ and both raise it to the power of $2$ and take the square root,

$$\sqrt{3^2}$$

the exponent and root undo each other, and we're left with $3$, the same value we started with.

- An exponent of $2$ and a square root are inverses, $\sqrt{x^2}$
- An exponent of $3$ and a third root are inverses, $\sqrt[3]{x^3}$
- An exponent of $4$ and a fourth root are inverses, $\sqrt[4]{x^4}$
- ...

Let's do an example.

**Example**

What should replace the question mark to make the equation true?

$$2+7 \quad ? \quad = 2$$

Reading this equation from left to right, we can see that we're starting with $2$ on the left, and ending with $2$ on the right. Since the final value isn't different from the original value, we need to use an inverse operation that will undo the addition of $7$ that we see in the middle of the equation.

$$2+7-7=2$$

In other words, because we added $7$, we needed to subtract $7$. Those inverse operations undo each other, leaving us with a result that's unchanged from the value we started with.

The question mark should be replaced with $-7$.

---

Let's try another example with inverse operations.

**Example**

What should replace the question mark to make the equation true?

$$4 \cdot 3 \quad ? \quad = 4$$

Because we're multiplying by $3$, we have to undo that operation by dividing by $3$. Multiplication and division are inverse operations, so dividing by $3$ will undo the multiplication by $3$.

$$4 \cdot 3 \div 3 = 4$$

The question mark should be replaced with $\div 3$.

---

# Simple equations

To solve simple equations, start by thinking about what's happening to the variable. For instance, let's think about the equation $x+5=12$. We're adding $5$ to $x$ in order to get $12$, which means that if we want to get $x$ by itself, we have to get rid of the extra $5$.

Well what's the inverse operation of addition? How do we "undo" the addition of $5$? Since the inverse operation of addition is subtraction, we'll subtract $5$ to undo the addition of $5$.

Whatever we do to one side of the equation, we have to do to the other, so if we want to subtract $5$ from one side, we have to make sure to subtract $5$ from both sides.

## How to solve simple equations

In other words, solving simple equations is really just undoing everything that's happening to the variable in order to get the variable by itself.

We'll solve equations by working the order of operations in reverse. So we'll undo all the addition and subtraction first, then we'll undo all the multiplication and division, etc.

We'll keep applying inverse operations until the variable is alone, always remembering to do the same thing to both sides of the equation so that it stays balanced.

**Example**

Solve the equation for $x$.

$$x - 3 = 10$$

In this example, $3$ is being subtracted from $x$. To undo that subtraction, we need to add $3$ to both sides of the equation.

$$x - 3 + 3 = 10 + 3$$

$$x + 0 = 13$$

Remember that $0$ can be added or subtracted from anything and it won't change the value of that thing, which means that $x + 0$ simplifies to just $x$, and the solution to the equation is

$$x = 13$$

If we plug $x = 13$ back into the original equation, we see that substituting this value makes the equation true.

$$13 - 3 = 10$$

$$10 = 10$$

---

Let's try another example where we solve a simple equation.

**Example**

Solve for the variable.

$$3x + 5 = 11$$

In this example, $x$ is being multiplied by $3$ and then $5$ is being added to the result. To solve for $x$, we work backwards from the order of operations, so we need to first undo the addition by subtracting $5$ from both sides of the equation.

$$3x + 5 - 5 = 11 - 5$$

$$3x + 0 = 6$$

$$3x = 6$$

Now we need to undo the multiplication by $3$. Division is the inverse operation of multiplication, so we'll divide both sides by $3$.

$$\frac{3x}{3} = \frac{6}{3}$$

$$x = 2$$

If we plug $x = 2$ back into the original equation, we see that substituting this value makes the equation true.

$$3(2) + 5 = 11$$

$$6 + 5 = 11$$

$$11 = 11$$

# Balancing equations

Remember that equations have an equals sign and expressions do not. We balance or solve equations, but simplify or evaluate expressions. And we've seen now how to solve simple equations by using inverse operations to isolate the variable.

Now we'll look in more detail about how to solve equations by keeping them balanced.

## The equation scale

Think of an equation as a two-sided scale that we always have to keep in balance.

What we do to one side of an equation we have to do to the other, otherwise the scale won't stay balanced. For example, if we add something to one side, we have to add the same value to the other side.

Or if we take something away from one side, we also have to take it away from the other side.

But don't confuse changing the amount of weight on the scale with moving weight around on the same side of the scale. We can still rewrite one side of the equation without rewriting the other side, we just can't change the value of one side of the equation without changing the value of the other side.

So given an equation like,

$$3x(x+2) = 6x - 2$$

if we just want to distribute the $3x$ across the $(x+2)$ on the left, then we can rewrite the equation as

$$3x(x) + 3x(2) = 6x - 2$$

$$3x^2 + 6x = 6x - 2$$

All we did here was rewrite the left side; we didn't actually change its value. In other words, it's like we changed the left side from $3+1$ to $2+2$. It looks different, but the value isn't different.

In contrast, if we want to add $4$ to one side of the original equation, then we have to add $4$ to both sides,

$$3x(x+2) + 4 = 6x - 2 + 4$$

because adding $4$ actually does affect the value. So if we want to make any change to one side of the equation that'll affect the "weight" on that side of the "scale," then we need to make sure we equally change the weight on the other side of the scale (equally change the amount on the other side of the equation).

In general, to solve equations, we'll

1. Simplify both sides of the equation as much as possible using the order of operations.

2. If the variable we're trying to solve for appears on both sides of the equation, combine those terms on one side using inverse operations.

3. Move all constant values to the other side of the equation (opposite the variable) using inverse operations.

We'll remember to always keep the equation balanced as we go. Let's do an example.

**Example**

Solve for the variable by keeping the equation balanced.

$$-2(3x+1)=3(-5x+11)+1$$

Simplify both sides of the equation by distributing and then combining like terms.

$$-2(3x)-2(1)=3(-5x)+3(11)+1$$

$$-6x-2=-15x+33+1$$

$$-6x - 2 = -15x + 34$$

Use inverse operations to move all the $x$ terms to one side.

$$-6x + 15x - 2 = -15x + 15x + 34$$

$$9x - 2 = 34$$

Use inverse operations to move all the constants to the other side of the equation.

$$9x - 2 + 2 = 34 + 2$$

$$9x = 36$$

Use inverse operations to solve for $x$.

$$\frac{9x}{9} = \frac{36}{9}$$

$$x = 4$$

---

Let's try another example of balancing equations.

---

**Example**

Solve for the variable.

$$5(6a - 3) = -(1 - 9a) + 7$$

Simplify both sides of the equation by distributing and then combining like terms.

$$5(6a) + 5(-3) = -(1) - (-9a) + 7$$

$$30a - 15 = -1 + 9a + 7$$

$$30a - 15 = 9a + 6$$

Use inverse operations to move all the $a$ terms to one side.

$$30a - 9a - 15 = 9a - 9a + 6$$

$$21a - 15 = 6$$

Use inverse operations to move all the constants to the other side of the equation.

$$21a - 15 + 15 = 6 + 15$$

$$21a = 21$$

Use inverse operations to solve for $a$.

$$\frac{21a}{21} = \frac{21}{21}$$

$$a = 1$$

---

Let's look at another example.

**Example**

Solve for the variable.

$$2x - 3 = 3x + 1$$

Both sides are already as simplified as they can be, so we'll start by moving all the $x$ terms to one side.

$$2x - 2x - 3 = 3x - 2x + 1$$

$$-3 = x + 1$$

Use inverse operations to move all the constants to the other side.

$$-3 - 1 = x + 1 - 1$$

$$-4 = x$$

$$x = -4$$

---

Let's try another example of solving equations with variables on both sides.

**Example**

Solve for the variable.

$$10x - 13 = 4x + x - 6$$

Start by combining like terms to simplify the right side of the equation.

$$10x - 13 = 5x - 6$$

Move all the $x$ terms to one side.

$$10x - 5x - 13 = 5x - 5x - 6$$

$$5x - 13 = -6$$

Move all the constants to the other side.

$$5x - 13 + 13 = -6 + 13$$

$$5x = 7$$

Solve for $x$.

$$\frac{5x}{5} = \frac{7}{5}$$

$$x = \frac{7}{5}$$

---

# Equations with subscripts

Sometimes we'll encounter subscripted variables. A subscripted variable is just a variable that has a subscript attached to it, and a **subscript** is a small number that comes just after, and at a lower level than the variable.

## When to use subscripts

It's really common to see subscripts in the sciences, like chemistry or physics. Think about the chemical formula for water, $H_2O$. The $2$ that we see on the H is a subscript.

In mathematics, we'll often use subscripts to represent the same kind of value for different subjects.

For instance, let's say we're working a problem where two people, Abigail and Phoebe, are covering some distance at a particular rate of speed. We essentially have four variables in this situation: Abigail's distance, Phoebe's distance, Abigail's rate, and Phoebe's rate. Instead of using four variables like $v$, $x$, $y$, and $z$, we can use more descriptive variables, along with subscripts, to represent the four unknowns. Specifically, $A_D$, $P_D$, $A_R$, and $P_R$ would make a lot of sense.

## Solving equations with subscripts

Luckily, solving equations with subscripted variables isn't any different than solving equations with normal variables. In other words, we need to think about a subscripted variable just like we would a normal variable.

Let's do an example to get some practice with these.

**Example**

The pressure and volume of a gas are related according to the equation $P_1V_1 - P_2V_2 = 0$, where $P_1$ and $V_1$ are the original pressure and volume, and $P_2$ and $V_2$ are the new pressure and volume. If the original pressure is $1.4$ the original volume is $210$, and the new pressure is $28$, what is the new volume?

We know that $P_1 = 1.4$, $V_1 = 210$, and $P_2 = 28$, so we can plug these values into the equation relating these variables, and we get

$$P_1V_1 - P_2V_2 = 0$$

$$(1.4)(210) - (28)V_2 = 0$$

Solve for new volume, $V_2$, by simplifying the left side and then using inverse operations to isolate $V_2$.

$$294 - 28V_2 = 0$$

$$294 - 28V_2 + 28V_2 = 0 + 28V_2$$

$$294 = 28V_2$$

$$\frac{294}{28} = \frac{28V_2}{28}$$

$$10.5 = V_2$$

$$V_2 = 10.5$$

---

Let's try another example with subscripted variables.

**Example**

A car travels at a $50$ mph for $125$ miles, then speeds up and travels at a new constant speed for another $153$ miles. If the total time for the trip is $4.75$ hours, how fast does the car go during the second part of the trip?

We'll use an equation that relates distance, rate, and time for an object in motion. The equation is

$$\text{distance} = \text{rate} \times \text{time}$$

and tells us that multiplying how fast something is moving and the amount of time it's been moving is equal to the distance that it's moved. We can manipulate this equation to solve for any of the three values. For example, we can divide both sides by rate to get an equation for time.

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

The total time for the trip (which we'll call $t$) is the sum of the times for the two parts. Let $d_1$ and $r_1$ be the distance and rate on the first part of the trip, and let $d_2$ and $r_2$ be the distance and rate on the second part. If we divide $d_1$ by $r_1$, we get the time for the first part; similarly if we divide $d_2$ by $r_2$, we get the time for the second part. Therefore,

$$\frac{d_1}{r_1} + \frac{d_2}{r_2} = t$$

Start by plugging in what we know, which is $d_1 = 125$, $r_1 = 50$, $d_2 = 153$, and $t = 4.75$.

$$\frac{125}{50} + \frac{153}{r_2} = 4.75$$

$$2.5 + \frac{153}{r_2} = 4.75$$

Use inverse operations to isolate $r_2$.

$$2.5 - 2.5 + \frac{153}{r_2} = 4.75 - 2.5$$

$$\frac{153}{r_2} = 2.25$$

$$\frac{153}{r_2} \cdot r_2 = 2.25 \cdot r_2$$

$$153 = 2.25r_2$$

$$\frac{153}{2.25} = \frac{2.25r_2}{2.25}$$

$$68 = r_2$$

$$r_2 = 68$$

The car travels at $68$ mph for the second part of the trip.

# Word problems into equations

Word problems can be tricky at first, because it can be challenging to translate the words into actual mathematical equations that we can solve.

## Translating word problems

Since this translation is usually the hardest part, we want to outline exactly how certain words and phrases are translated into expressions or equations. The table shows some common words and phrases.

| | Words | Phrases | Expressions |
|---|---|---|---|
| **Addition** | sum, total, more than, added, increased, plus | 3 more than a number, the sum of 5 and a number | 3+x<br>5+n |
| **Subtraction** | less, minus, decreased by, difference, less than | 12 decreased by a number, the difference of 7 and a number | 12-n<br>7-x |
| **Multiplication** | product, times, multiplied, of | the product of a number and 2, 2/3 of a number | 2x<br>(2/3)n |
| **Division** | quotient, divided by, divided into | 15 divided by a number, the quotient of a number and 4 | 15/n<br>x/4 |

Let's work through an example where we translate a simple phrase into a mathematical expression.

### Example

Write the phrase as an algebraic expression.

"four less than twice $x$"

The phrase "twice $x$" means "$2$ times $x$," which we know means to multiply, and so we can write it as $2x$. Now we have

"four less than $2x$"

"Less" means subtraction, so we'll subtract $4$ from $2x$.

$$2x - 4$$

We might be tempted to write the $4$ first and express "four less than $2x$" as $4 - 2x$, but that would be incorrect, and we can use specific numbers to visualize this.

If we were asked what number is four less than $10$, we'd know it's $6$, and that we'd have to subtract $4$ from $10$, which is written as $10 - 4$. So "four less than $2x$" will be $2x - 4$.

Once we've translated a phrase into a math expression, we can work with the math in different ways. This next example asks us to translate a phase into math, but then simplify the mathematical expression.

**Example**

Find the value of the expression.

$$\frac{1}{4} \text{ of } 120$$

In math, the word "of" (immediately after a fraction) tells us to multiply. Therefore, the mathematical expression of the phrase will be

$$\frac{1}{4} \cdot 120$$

Because we were asked to actually find the value of the expression, we'll perform the multiplication to get the simplified value.

$$\frac{1(120)}{4}$$

$$30$$

---

Not only can we translate phrases into expressions, but we can write equations from some phrases as well.

**Example**

John's age is four less than twice Mary's age. If Mary is $18$, how old is John?

The first step in solving a word problem like this is to define the variables, meaning that we want to state the particular quantity that each variable stands for.

In this problem, we have two quantities: Mary's age and John's age, so let's let $M$ be Mary's age, and let $J$ be John's age.

Now we need to translate each word or phrase into mathematical symbols. Here, "John's age" is translated as "$2M - 4$."

How about the word "is" (in "John's age <u>is</u> four less than twice Mary's age")? Well, "is" is translated as an equals sign. To see this, it may help to think of the word "is" as having the same meaning (in math) as "is equal to." Combining all of these, we get the equation

$$J = 2M - 4$$

We're given Mary's age as $18$, so we substitute $18$ for $M$ and then solve for $J$.

$$J = 2(18) - 4$$

$$J = 36 - 4$$

$$J = 32$$

The final step is to answer the question that was asked. Here, we're asked for John's age. Since we defined $J$ as John's age, the answer is $32$.

---

If instead of the last example, we'd been asked,

> "Currently, John's age is four less than twice Mary's age. If Mary is now $18$, how old will John be seven years from now?"

then it would be convenient to define $M$ as Mary's age now, and $J$ as John's age now, because we're given a relationship between Mary's current age and John's current age.

Then in the last step (answering the question that was asked), we'd have to evaluate $J + 7$ (to get John's age seven years from now), and our answer would be $32 + 7 = 39$.

# Consecutive integers

Remember that **integers** are “whole numbers” that are either positive, negative, or $0$, which means we’re not including fractions or decimals.

## Consecutive integer word problems

**Consecutive integers** are integers that are one unit apart from each other.

$$\dots\ -5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5\ \dots$$

So, for example, the three consecutive integers that immediately follow $-2$ are $-1,\ 0,\ 1$, and the three consecutive integers that immediately precede $-1$ are $-4,\ -3,\ -2$.

Solving for a set of consecutive integers is a word problem we commonly see in Algebra, so let’s do some examples.

### Example

Find three consecutive integers that sum to $39$.

First we’ll define a variable, $n$, as the smallest of these three consecutive integers. Since consecutive integers are one unit apart from each other, the other two consecutive integers must be $n+1$ and $n+1+1$, or $n+2$. Since the sum of the three consecutive integers is $39$, we can say

$$n + (n + 1) + (n + 2) = 39$$

Removing the parentheses, and then grouping and combining like terms, we get

$$n + n + 1 + n + 2 = 39$$

$$(n + n + n) + (1 + 2) = 39$$

$$3n + 3 = 39$$

With both sides of the equation simplified, we'll use inverse operations to solve for $n$.

$$3n + 3 - 3 = 39 - 3$$

$$3n = 36$$

$$\frac{3n}{3} = \frac{36}{3}$$

$$n = 12$$

It's really important to remember here that $n = 12$ isn't automatically the correct answer. We have to look back at the specific question that was asked, and give the answer to it. We were asked to find three consecutive integers that have a sum of $39$.

We found $n = 12$ to be the smallest of the three consecutive integers, so the other two integers are $n + 1 = 12 + 1 = 13$ and $n + 2 = 12 + 2 = 14$. We can double-check that the three consecutive integers are $12$, $13$, and $14$ by plugging them back into the sum equation.

$$n + (n + 1) + (n + 2)$$

$$12 + 13 + 14$$

$$25 + 14$$

$$39$$

---

Let's look at another common integer problem. This time, we'll be looking for consecutive even integers.

**Example**

Find three consecutive even integers that have a sum of $54$.

First we'll define a variable, $n$, as the smallest of the three consecutive even integers. The next two consecutive even integers are $n + 2$ and $n + 2 + 2$, or $n + 4$. Since the sum of the three consecutive even integers is $54$, we have

$$n + (n + 2) + (n + 4) = 54$$

Removing the parentheses, and then grouping and combining like terms, we get

$$n + n + 2 + n + 4 = 54$$

$$(n + n + n) + (2 + 4) = 54$$

$$3n + 6 = 54$$

With both sides of the equation simplified, we'll use inverse operations to solve for $n$.

$$3n + 6 - 6 = 54 - 6$$

$$3n = 48$$

$$\frac{3n}{3} = \frac{48}{3}$$

$$n = 16$$

Again, we have to look back at the question that was asked, and give the answer to it. We were asked to find three consecutive even integers with a sum of $54$.

What we found is that the smallest of the three consecutive even integers is $n = 16$. The other two consecutive integers are $n + 2 = 16 + 2 = 18$ and $n + 4 = 16 + 4 = 20$. We can double-check that the three consecutive integers are $16$, $18$, and $20$ by plugging them back into the sum equation.

$$n + (n + 2) + (n + 4)$$

$$16 + 18 + 20$$

$$34 + 20$$

$$54$$

# Adding and subtracting polynomials

We've already defined expressions and equations, so we know that an expression is a collection of terms.

Now we want to define a **polynomial** as an expression that's the sum and/or difference of a finite number of terms, where the terms include only constants, variables, and positive integer exponents. A polynomial can't include division by a variable (we can't have a variable in any denominator).

The word "polynomial" is the combination of "poly" meaning "many" and "nomial" meaning "term." So we can think broadly about polynomials as "many terms." That being said, polynomials can still include only one term; they don't necessarily have to include multiple terms.

These are all examples of polynomials:

$$3x^2+6x-8 \qquad 2x^3+5x^2-7x+1$$

$$-12x^2y^2+y \qquad 3x$$

$$-7 \qquad x^3+3xy-6x^2-4$$

$$3x^2-\frac{1}{2}x+1$$

But these are not polynomials,

$3x^{-2}-y^2$ because of the negative exponent

$x^{\frac{1}{2}}$ because of the fractional exponent

$$-\frac{1}{2x} + 1 - y^3 + y^2 \qquad \text{because of the variable in the denominator}$$

It's most common to write the terms of a polynomial in descending order of their exponents, so the first term of the polynomial will usually be the term with the largest exponent.

The largest exponent is the **degree** of the polynomial. So, for example, if the largest exponent in the polynomial is $3$, then the polynomial is a third-degree polynomial.

## The sum or difference of polynomials

In this lesson we want to learn to add and subtract polynomials, which turns out is really just a matter of adding and subtracting like terms, something we already know how to do.

Remember that for the purposes of addition and subtraction, like terms are terms that have the same base and the same exponent. We combine like terms by adding or subtracting the coefficients while keeping the base and the exponent the same.

The sum or difference of polynomials will always itself be a polynomial. Let's do an example where we add two polynomials.

---

**Example**

Simplify the expression.

$$(3x^2 + 6x - 8) + (-12x^2 + 1)$$

First, remove the parentheses.

$$3x^2 + 6x - 8 - 12x^2 + 1$$

Group like terms together in descending order of their exponents.

$$(3x^2 - 12x^2) + 6x + (-8 + 1)$$

Combine like terms by performing the addition and/or subtraction of their coefficients.

$$(3 - 12)x^2 + 6x + (-8 + 1)$$

$$-9x^2 + 6x - 7$$

---

Let's try another example, this time we'll subtract one polynomial from the other.

**Example**

Simplify the expression.

$$(2x^3 + 5x^2 - 7x + 1) - (x^3 + 3x - 6x^2 - 4)$$

First, remove the parentheses. Because we're subtracting the second polynomial, we need to distribute the negative sign across each term in the second polynomial when we remove the parentheses.

$$2x^3 + 5x^2 - 7x + 1 - x^3 - 3x + 6x^2 + 4$$

Group like terms together in descending order of their exponents.

$$(2x^3 - x^3) + (5x^2 + 6x^2) + (-7x - 3x) + (1 + 4)$$

$$(2 - 1)x^3 + (5 + 6)x^2 + (-7 - 3)x + (1 + 4)$$

$$x^3 + 11x^2 - 10x + 5$$

---

# Multiplying polynomials

We've learned to add and subtract polynomials (which was really just about adding and subtracting like terms), and now we want to learn to multiply and divide polynomials, starting with multiplication of polynomials.

In the same way that the sum of two polynomials is always itself a polynomial, the product of two polynomials will also always be a polynomial.

## Binomial multiplication and FOIL

To multiply two polynomials, we'll always need to apply the Distributive Property. From what we already learned about the Distributive Property, we know how to distribute a single coefficient across parentheses, like this:

$$2(x+3)$$

$$2(x)+2(3)$$

$$2x+6$$

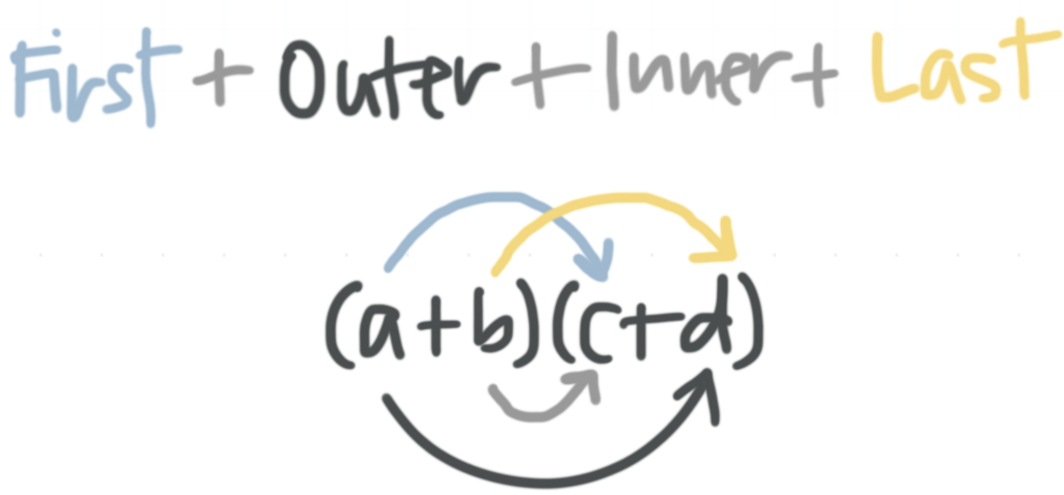
But now we want to replace the single coefficient with a polynomial coefficient. So in place of the single coefficient of $2$ in the expression above, we could instead use a polynomial like $x+4$, in which case we'd have the product of two polynomials,

$$(x+4)(x+3)$$

When we're talking about polynomials, a single term is a **monomial**, two terms is a **binomial**, and three terms is a **trinomial**. We don't usually assign a special name to polynomials with four or more terms.

So the product above is the multiplication of two binomials, and whenever we multiply two binomials, we can use an acronym, FOIL, to help us with the multiplication.

FOIL is a way to help us remember to multiply each term in the first binomial by each term in the second binomial. FOIL stands for First, Outer, Inner, Last, which is the order of the four terms in the result of the multiplication. It also indicates which terms in the given binomials are multiplied to produce each term in the result.




The firsts are the first term in the first set of parentheses (which is $a$) and the first term in the second set of parentheses (which is $c$), so the first term in the result (first) is the product of $a$ and $c$, or $ac$.

The outers are the outside term in the first set of parentheses (which is $a$) and the outside term in the second set of parentheses (which is $d$), so the second term in the result (outer) is the product of $a$ and $d$, or $ad$.

Similarly, the inners are $b$ and $c$, and the lasts are $b$ and $d$, so the third term in the result (inner) is $bc$, and the fourth term in the result (last) is $bd$.

$$(a+b)(c+d) = ac + ad + bc + bd$$

$$(a+b)(c-d) = ac - ad + bc - bd$$

$$(a-b)(c+d) = ac + ad - bc - bd$$

$$(a-b)(c-d) = ac - ad - bc + bd$$

Really though, all we're doing with binomial multiplication is multiplying every term in the first polynomial by every term in the second polynomial. Looking at the example $(x+4)(x+3)$, we need to multiply both $x$ and $4$ from the first polynomial by the second polynomial.

$$(x+4)(x+3)$$

$$x(x+3) + 4(x+3)$$

$$x(x) + x(3) + 4(x) + 4(3)$$

$$x^2 + 3x + 4x + 12$$

$$x^2 + 7x + 12$$

We can also make a chart for the binomial multiplication $(a+b)(c+d)$ in which the terms $a$ and $b$ from the first set of parentheses go along the left side, and the terms $c$ and $d$ from the second set of parentheses go across the top. Then we multiply each term along the left side by both terms across the top, and write the individual results in the chart. The four results all get added together to make the expanded polynomial.

For the binomial multiplication, $(a+b)(c+d)$, we get $ac+ad+bc+bd$.

| | c | d |
|---|---|---|
| **a** | ac | ad |
| **b** | bc | bd |

For the binomial multiplication, $(a-b)(c-d)$, we get $ac-ad-bc+bd$.

| | c | -d |
|---|---|---|
| **a** | ac | -ad |
| **-b** | -bc | bd |

Let's do an example with binomial multiplication.

**Example**

Expand the expression.

$$(x+2)(x-7)$$

If we multiply the firsts $x$ and $x$, the outers $x$ and $-7$, the inners $2$ and $x$, and the lasts $2$ and $-7$, and add all the results together, we get

$$(x)(x)+(x)(-7)+(2)(x)+(2)(-7)$$

$$x^2-7x+2x-14$$

$$x^2-5x-14$$

Let's try another example of multiplying binomials.

**Example**

Expand the expression.

$$(x+3)^2$$

This is "$(x+3)$ squared," which means that the binomial $(x+3)$ is multiplied by itself.

$$(x+3)(x+3)$$

$$(x)(x)+(x)(3)+(3)(x)+(3)(3)$$

$$x^2+3x+3x+9$$

$$x^2+6x+9$$

## Multiplying more than two binomials

We now know how to multiply two binomials together, but we're not limited to just two binomials, nor are we limited to only binomials. We can multiply together as many binomials as we like, and we can multiply polynomials that have more than two terms (trinomials, etc.).

No matter the number and length of the polynomials, the key here is just to make sure we multiply every term in each polynomial by every term in every other polynomial.

For instance, let's say we want to multiply $(x+3)(x+3)(x+3)$. We take two polynomials at a time, so we'll multiply the first and second $(x+3)$ binomials together.

$$[(x+3)(x+3)](x+3)$$

$$[(x)(x)+(x)(3)+(3)(x)+(3)(3)](x+3)$$

$$[x^2+3x+3x+9](x+3)$$

$$(x^2+6x+9)(x+3)$$

Now we'll multiply each term in the trinomial by each term in the binomial.

$$x^2(x+3)+6x(x+3)+9(x+3)$$

$$x^2(x)+x^2(3)+6x(x)+6x(3)+9(x)+9(3)$$

$$x^3+3x^2+6x^2+18x+9x+27$$

$$x^3+9x^2+27x+27$$

Let's do an example, and this time we'll use tables to organize our results.

---

**Example**

Use the Distributive Property to expand the expression.

$$3x(x+4)(x+1)(x-2)$$

We can do the multiplication in any order, but let's start by distributing the $3x$ across the $x+4$.

$$[3x(x+4)](x+1)(x-2)$$

$$[3x(x)+3x(4)](x+1)(x-2)$$

$$(3x^2+12x)(x+1)(x-2)$$

Now let's use a chart to distribute the $3x^2+12x$ across the $x+1$.

| | **x** | **1** |
|---|---|---|
| **3x²** | 3x³ | 3x² |
| **12x** | 12x² | 12x |

When we add all the results in the chart together and then combine like terms, we get

$$3x^3+3x^2+12x^2+12x$$

$$3x^3+15x^2+12x$$

So the remaining expression is now

$$(3x^3+15x^2+12x)(x-2)$$

Finally, we'll distribute the $3x^3+15x^2+12x$ across the $x-2$. We'll use another chart.

| | $x$ | $-2$ |
|---|---|---|
| $3x^3$ | $3x^4$ | $-6x^3$ |
| $15x^2$ | $15x^3$ | $-30x^2$ |
| $12x$ | $12x^2$ | $-24x$ |

When we add all the results in the chart together and then combine like terms, we get

$$3x^4 - 6x^3 + 15x^3 - 30x^2 + 12x^2 - 24x$$

$$3x^4 + 9x^3 - 18x^2 - 24x$$

# Dividing polynomials

We've looked at adding, subtracting, and multiplying polynomials, and now we finally want to jump into how to divide polynomials. Think about dividing polynomials as long division, but with variables.

## Review of long division

In lower-level math, when we first learned how to divide real numbers, we learned to use long division. Let's review long division by working through an example where we divide $146$ by $13$.

### Example

Use long division to divide $146$ by $13$.

The long division will look like,

$$\begin{array}{rl} & 11 \quad R3 \\ 13 & \overline{)\,146} \\ & \underline{-13\downarrow} \\ & 16 \\ & \underline{-13} \\ & 3 \end{array}$$

but to work through this step-by-step, we need to start by thinking “How many times does $13$ go into $14$?” It goes in $1$ time, so we write a $1$ above the long division sign and line it up with the $4$.

Then we multiply $13 \times 1$ and get $13$, which means we subtract $13$ from $14$ and get $1$. We bring down the $6$ so that the remaining $1$ becomes $16$.

How many times does $13$ go into $16$? It goes in $1$ time, so we write another $1$ above the long division sign, this time lined up with the $6$.

We multiply $13 \times 1 = 13$, which means we subtract $13$ from $16$ and get $3$. Since $13$ doesn’t go into $3$, and there’s nothing left to bring down, we have a remainder of $3$.

Our answer to $146 \div 13$ is $11$ with a remainder of $3$, or

$$\frac{146}{13} = 11 + \frac{3}{13}$$

---

As a reminder, the value being divided, $146$, is the **dividend**. The value we divide by, $13$, is the **divisor**. The result of the division, $11 + 3/13$, is the **quotient**, and if the divisor doesn’t divide evenly into the dividend, then the quotient will include a **remainder**. In this example, the remainder is $3$.

Quotient $\qquad$ Remainder

$$
\begin{array}{rrl}
 & 11 & R3 \\
\text{Divisor} \rightarrow 13 & \overline{)146} & \leftarrow \text{Dividend} \\
 & -13\phantom{6} & \\
\hline
 & 16 & \\
 & -13 & \\
\hline
 & 3 & \leftarrow \text{Remainder}
\end{array}
$$

## Polynomial long division

Now let's look at this same problem again, but this time using polynomial long division. This time we'll divide $x^2+4x+6$ (notice that the coefficients of this polynomial are $1$, $4$, and $6$) by $x+3$ (notice that $x$ has a coefficient of $1$, and we follow that by a constant $3$).

$$
\begin{array}{rl}
 & x+1 \qquad R3 \\
x+3 & \overline{)x^2+4x+6} \\
 & -(x^2+3x) \\
\hline
 & \qquad\quad x+6 \\
 & \quad\ \ -(x+3) \\
\hline
 & \qquad\qquad 3
\end{array}
$$

The leading term in the dividend $x^2 + 4x + 6$ is $x^2$, and the leading term in the divisor $x + 3$ is $x$. So we start by thinking, "What do we need to multiply by $x$ to get $x^2$?" The answer is $x$, so we write $x$ above the long division sign and line it up with the $x^2$.

Then we multiply $x + 3$ by $x$ and get $x^2 + 3x$, which means we subtract $x^2 + 3x$ from $x^2 + 4x$ and get $x$. Bring down the $+6$.

What do we need to multiply $x$ by to get $x$? We need to multiply by $1$, so we write $+1$ next to the $x$ above the long division sign.

We multiply $(x + 3) \cdot 1 = x + 3$, so we subtract $x + 3$ from $x + 6$ and get $3$.

Our answer is the quotient $x + 1$ with a remainder of $3$. When we do polynomial long division, we should write the remainder as a fraction, with the remainder in the numerator and the divisor in the denominator, so we should write this answer as

$$\frac{x^2 + 4x + 6}{x + 3} = x + 1 + \frac{3}{x + 3}$$

Let's do another example where we divide a trinomial by a binomial.

---

**Example**

Simplify the expression using polynomial long division.

$$(x^2 + 3x - 5) \div (x - 2)$$

Use polynomial long division to simplify.

$$
\begin{array}{r l}
 & x+5 \quad R5 \\
x-2 & \overline{\big)\, x^2 + 3x - 5} \\
 & -(x^2 - 2x) \\
\hline
 & 5x - 5 \\
 & -(5x - 10) \\
\hline
 & 5
\end{array}
$$

We get a quotient of $x+5$ with a remainder of $5$, so

$$\frac{x^2+3x-5}{x-2} = x + 5 + \frac{5}{x-2}$$

---

Let's try another example of dividing polynomials.

### Example

Use polynomial long division to simplify the expression.

$$(2x^3 + x^2 + 4) \div (x+1)$$

Use polynomial long division to simplify.

$$
\begin{array}{rl}
 & 2x^2 - x + 1 \qquad R3 \\
x+1 & \overline{)\, 2x^3 + x^2 + 0x + 4} \\
 & \underline{-(2x^3 + 2x^2)} \\
 & \qquad -x^2 + 0x \\
 & \qquad \underline{-(-x^2 - x)} \\
 & \qquad\qquad x + 4 \\
 & \qquad\qquad \underline{-(x + 1)} \\
 & \qquad\qquad\qquad 3
\end{array}
$$

When we multiplied the second term in the quotient $-x$, by the divisor $x+1$, we found $-x^2 - x$. There was no $x$ term in the dividend that we could bring down, so we added a "placeholder" $0x$ into the dividend. That way, we could bring down the $0x$ to pair with the $-x^2$, and then subtract $-x^2 - x$ from $x^2 + 0x$.

We get a quotient of $2x^2 - x + 1$ with a remainder of $3$, so

$$\frac{2x^3 + x^2 + 4}{x+1} = 2x^2 - x + 1 + \frac{3}{x+1}$$

# Multiplying multivariable polynomials

We define a **multivariable polynomial** as a polynomial that includes two or more variables. For instance, $2x^3 - 3xy - y^2$ is a multivariable polynomial because it includes multiple variables, $x$ and $y$.

We already know how to multiply like terms, and we already know how to multiply single variable polynomials, and we'll be able to multiply multivariable polynomials if we combine these two concepts.

Remember that, if we multiply terms with like bases, then the base stays the same and we add the exponents. So $3x^2(x^4) = 3x^{2+4} = 3x^6$. If we're multiplying multivariable terms, then we still add exponents for like bases. So $3x^2y(x^4y^3) = 3x^{2+4}y^{1+3} = 3x^6y^4$.

With multiple variables involved, and as the polynomials get longer, it can be especially useful here to use a chart to organize the polynomial multiplication so that we don't get lost as we're working through each term.

Let's look at an example where we multiply multivariable polynomials.

---

**Example**

Expand the expression.

$$(x - 2y)(2x^3 - 3xy - y^2)$$

Let's use a chart to make sure we distribute every term in the first polynomial by every term in the second polynomial.

| | $2x^3$ | $-3xy$ | $-y^2$ |
|---|---|---|---|
| $x$ | $2x^4$ | $-3x^2y$ | $-xy^2$ |
| $-2y$ | $-4x^3y$ | $6xy^2$ | $2y^3$ |

If we add every term in the body of the chart, we get

$$2x^4 - 3x^2y - xy^2 - 4x^3y + 6xy^2 + 2y^3$$

Next, we'll rearrange the terms in descending order of powers of $x$. We'll list any terms that don't include $x$ at the end, in ascending order of powers of $y$.

$$2x^4 - 4x^3y - 3x^2y - xy^2 + 6xy^2 + 2y^3$$

$$2x^4 - 4x^3y - 3x^2y + (-xy^2 + 6xy^2) + 2y^3$$

$$2x^4 - 4x^3y - 3x^2y + 5xy^2 + 2y^3$$

---

Let's try another example of multiplying multivariable polynomials.

---

**Example**

Expand the expression.

$$(2x + 3y)(x - y) + (x + y)(4x - 2y)$$

We'll multiply the first pair of binomials.

$$(2x+3y)(x-y)+(x+y)(4x-2y)$$

$$2x(x-y)+3y(x-y)+(x+y)(4x-2y)$$

$$2x(x)-2x(y)+3y(x)-3y(y)+(x+y)(4x-2y)$$

$$2x^2-2xy+3xy-3y^2+(x+y)(4x-2y)$$

$$2x^2+xy-3y^2+(x+y)(4x-2y)$$

Multiply the second pair of binomials.

$$2x^2+xy-3y^2+x(4x-2y)+y(4x-2y)$$

$$2x^2+xy-3y^2+x(4x)-x(2y)+y(4x)-y(2y)$$

$$2x^2+xy-3y^2+4x^2-2xy+4xy-2y^2$$

$$(2x^2+4x^2)+(xy-2xy+4xy)+(-3y^2-2y^2)$$

$$(2+4)x^2+(1-2+4)xy+(-3-2)y^2$$

$$6x^2+3xy-5y^2$$

# Dividing multivariable polynomials

Dividing multivariable polynomials is very similar to dividing single-variable polynomials. We'll still use long division, but now we'll have more than one variable.

## Filling in for missing terms in the dividend

When we were working with only one variable, we could easily fill in missing terms in advance. For example, if the divided contained an $x^3$ term and an $x$ term, we can fill in the missing $x^2$ term in advance before we start the division, because we can easily see the progression of $x^3$, $x^2$, $x$, ....

But as soon as the divided includes more than one variable, filling in these missing terms can be difficult. For that reason, we'll usually begin the division, and then fill in any missing terms that we need as we go.

For instance, in this long division problem,

$$
\begin{array}{r l}
 & x^2 \qquad + xy \quad - y^2 \\
x+y & \overline{)\, x^3 + 2x^2y + 0xy^2 - y^3} \\
 & -(x^3 + x^2y) \\
\hline
 & \qquad x^2y + 0xy^2 \\
 & \qquad -(x^2y + xy^2) \\
\hline
 & \qquad\qquad -xy^2 - y^3 \\
 & \qquad\qquad -(-xy^2 - y^3) \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

we'll only add the $0xy^2$ term into the dividend once we get the $x^2y + xy^2$ expression on the fourth line. When we arrive at $x^2y + xy^2$, we realize that we have to subtract the $xy^2$ term, and that we don't have a like term in the dividend. So it's at that point that we add the $0xy^2$ term into the dividend.

## Ordering the dividend's terms

Again, when we were working with only one variable, it was simple to order the terms in the dividend by listing them from highest power to lowest power. The $x^3$ term would come first, then the $x^2$ term, then the $x$ term, etc.

But when we have two variables, we should put the terms in order of descending power, based on the power of the variable that leads the divisor.

So if the divisor is $x + y$, we should order the terms in the dividend by descending powers of $x$. Let's do an example.

---

**Example**

Find the quotient.

$$\frac{2x^3 + 12x^2y + 15xy^2 - 9y^3}{x + 3y}$$

If we order the divisor as $x+3y$, then $x$ is the leading variable in the divisor, and we therefore want to order the terms in the dividend by decreasing powers of $x$.

$$
\begin{array}{r l}
 & \quad 2x^2 + 6xy - 3y^2 \\
x+3y & \overline{\big)\, 2x^3 + 12x^2y + 15xy^2 - 9y^3} \\
 & \underline{-(2x^3 + 6x^2y)} \\
 & \qquad\quad 6x^2y + 15xy^2 \\
 & \qquad \underline{-(6x^2y + 18xy^2)} \\
 & \qquad\qquad\quad -3xy^2 - 9y^3 \\
 & \qquad\qquad \underline{-(-3xy^2 - 9y^3)} \\
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

But if we order the divisor as $3y+x$ instead, then $y$ is the leading variable in the divisor, and we therefore want to order the terms in the dividend by decreasing powers of $y$.

$$
\begin{array}{r l}
 & \quad -3y^2 + 6xy + 2x^2 \\
3y+x & \overline{\big)\, -9y^3 + 15xy^2 + 12x^2y + 2x^3} \\
 & \underline{-(-9y^3 - 3xy^2)} \\
 & \qquad\quad 18xy^2 + 12x^2y \\
 & \qquad \underline{-(18xy^2 + 6x^2y)} \\
 & \qquad\qquad\quad 6x^2y + 2x^3 \\
 & \qquad\qquad \underline{-(6x^2y + 2x^3)} \\
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

In both cases, the result is the same, we find the same quotient, but the terms in the quotient show up in the opposite order.

# Greatest common factor

We've learned to apply the Distributive Property to distribute coefficients across polynomials, and even to multiply polynomials. But now we want to work that same process backwards.

In other words, if before we used the Distributive Property to change $3(x+2)$ to $3x+6$, now we want to learn how to change $3x+6$ into $3(x+2)$. This reverse process is called **factoring**, and we say that, from $3x+6$, we "factor out" a $3$ to get $3(x+2)$. Factoring is like "un-distributing."

## What to "factor out"

In order to factor some value out of a polynomial, all the terms in that polynomial need to contain the factor being taken out. Any factor that's shared by all the terms is a **common factor**, and the factor that consists of everything that's shared by all the terms is the **greatest common factor (GCF)**.

So $3$ is a common factor in $6x^3+9x^2+12x$, because $3$ can be factored out to get $3(2x^3+3x^2+4x)$. But $3x$ is the greatest common factor, because $3x$ is the most we can factor out of $6x^3+9x^2+12x$. When we do, we get $3x(2x^2+3x+4)$. We know $3x$ is the greatest common factor because there's no remaining common factor among $2x^2$, $3x$, and $4$.

The first thing we want to do here is learn to factor the greatest common factor out of a polynomial. If we're having trouble identifying the greatest common factor, we can always write out all the factors of each term. For example, we could write $2x^3y + 4x^2y^2 + 8xy$ as

$$2x^3y \quad + \quad 4x^2y^2 \quad + \quad 8xy$$

$$2 \cdot x \cdot x \cdot x \cdot y \quad + \quad 2 \cdot 2 \cdot x \cdot x \cdot y \cdot y \quad + \quad 2 \cdot 2 \cdot 2 \cdot x \cdot y$$

Now, to identify the greatest common factor, we just need to collect all the factors that are shared by each term. All three terms share one factor of $2$,

$$2 \quad (x \cdot x \cdot x \cdot y \quad + \quad 2 \cdot x \cdot x \cdot y \cdot y \quad + \quad 2 \cdot 2 \cdot x \cdot y)$$

one factor of $x$,

$$2x \quad (x \cdot x \cdot y \quad + \quad 2 \cdot x \cdot y \cdot y \quad + \quad 2 \cdot 2 \cdot y)$$

and one factor of $y$.

$$2xy \quad (x \cdot x \quad + \quad 2 \cdot x \cdot y \quad + \quad 2 \cdot 2)$$

$$2xy(x^2 + 2xy + 4)$$

We can see that $2xy$ is the greatest common factor, because there are no remaining common factors between $x^2$, $2xy$, and $4$. Remember, even though $x^2$ and $2xy$ still have a common factor of $x$, the $4$ doesn't share that common factor, so it can't be factored out. Similarly, even though $2xy$ and $4$ still have a common factor of $2$, the $x^2$ doesn't share that common factor, so it can't be factored out either.

Let's do another example.

---

**Example**

Identify the greatest common factor and factor it out of the polynomial.

$$3x^2 - 15xy$$

Write out all the factors of each term.

$$3 \cdot x \cdot x - 3 \cdot 5 \cdot x \cdot y$$

Both terms share one factor of $3$ and one factor of $x$. Factoring a $3x$ out of each term leaves just $x$ in the first term, and just $5y$ in the second term.

$$3x(x - 5y)$$

---

Let's try another example with the greatest common factor.

---

**Example**

Factor out the greatest common factor.

$$4x^2y - 6x^4y^2 + 8x^3y^4$$

Write out all the factors of each term.

$$2 \cdot 2 \cdot x \cdot x \cdot y - 2 \cdot 3 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y + 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$$

The only factors that are shared by all three terms are a $2$, an $x \cdot x$, and a $y$, so the greatest common factor is $2x^2y$. When we factor out the $2x^2y$, we have to divide each term by $2x^2y$.

$$2x^2y(2 - 3x^2y + 4xy^3)$$

---

Let's do one more.

---

**Example**

Factor the trinomial by pulling out the greatest common factor.

$$4s^4t^3 + 16s^2t^2 - 24st^4$$

Write out all the factors of each term.

$$2 \cdot 2 \cdot s \cdot s \cdot s \cdot s \cdot t \cdot t \cdot t + 2 \cdot 2 \cdot 2 \cdot 2 \cdot s \cdot s \cdot t \cdot t - 2 \cdot 2 \cdot 2 \cdot 3 \cdot s \cdot t \cdot t \cdot t \cdot t$$

The only factors that are shared by all three terms are a $2 \cdot 2$, an $s$, and a $t \cdot t$. Therefore, the greatest common factor is $4st^2$. Factoring out $4st^2$ gives

$$4st^2(s^3t + 4s - 6t^2)$$

---

# Quadratic polynomials

Remember that the degree of a polynomial is given by its largest exponent. So, for example, if the largest exponent in the polynomial is $3$, then the polynomial is a third-degree polynomial.

**Quadratic polynomials**, or **quadratics**, are second-degree polynomials. Our goal now is to learn to factor quadratics. In the same way that we said factoring out the greatest common factor was the inverse process to the Distributive Property, factoring a quadratic is the inverse process of "FOILing" two binomials.

In other words, we'll be starting with the quadratic and reversing the FOIL process to find the two binomial factors that were originally multiplied to get to the quadratic.

To factor a quadratic given in standard form, $ax^2 + bx + c$, where $a = 1$ and $c \neq 0$, we need to look for a pair of factors of $c$ that multiply to $c$ and sum to $b$.

Let's look at an example.

**Example**

Factor the quadratic polynomial.

$$x^2 - x - 20$$

This quadratic is given in standard form $ax^2 + bx + c$, with $a = 1$ and $c = -20 \neq 0$. Which means we can try to factor it.

We'll start by listing all pairs of factors of the constant term, $-20$, and their sums. We're looking for the pair of factors that have a sum of $b = -1$.

| Factors of -20 | Sum |
|---|---|
| -1 and 20 | 19 |
| 1 and -20 | -19 |
| -2 and 10 | 8 |
| 2 and -10 | -8 |
| -4 and 5 | 1 |
| 4 and -5 | -1 |

Since $4$ and $-5$ are the only factor pair with sum of $-1$, they must be the factors we need. If the factors are $x_1$ and $x_2$, then the quadratic will always factor as $(x + x_1)(x + x_2)$. So the quadratic $x^2 - x - 20$ factor as

$$(x + 4)(x - 5)$$

To check that we factored the quadratic correctly, we'll use the Distributive Property to multiply the binomials $(x + 4)(x - 5)$ to get

$$(x)(x) + (x)(-5) + (4)(x) + (4)(-5)$$

$$x^2 - 5x + 4x - 20$$

$$x^2 - x - 20$$

Let's try another example of factoring a quadratic polynomial.

### Example

Factor the quadratic as the product of two binomials.

$$x^2 - 8x + 15$$

This quadratic is given in standard form $ax^2 + bx + c$, with $a = 1$ and $c = 15 \neq 0$. Which means we can try to factor it.

We'll start by listing all pairs of factors of the constant term, $15$, and their sums. We're looking for the pair of factors that have a sum of $b = -8$.

| Factors of 15 | Sum |
|---|---|
| 1 and 15 | 16 |
| -1 and -15 | -16 |
| 3 and 5 | 8 |
| -3 and -5 | -8 |

The factors $-3$ and $-5$ have a sum of $-8$, so they're the correct factors, and the quadratic must factor as

$$(x-3)(x-5)$$

To check the factoring answer, we'll FOIL the product of the binomials.

$$(x)(x) + (x)(-5) + (-3)(x) + (-3)(-5)$$

$$x^2 - 5x - 3x + 15$$

$$x^2 - 8x + 15$$

---

If the coefficient $a$ on the $x^2$ term in a quadratic is either $-1$ or the greatest common factor of the polynomial, we can first factor that out and then factor the remaining quadratic.

---

**Example**

Factor the quadratic polynomial.

$$4x^2 - 20x + 24$$

The greatest common factor of this polynomial is $4$, so we first factor out a $4$.

$$4(x^2 - 5x + 6)$$

The remaining quadratic is $x^2 - 5x + 6$, which means we're looking for a pair of factors of $6$ that sum to $-5$.

Since $(-3)(-2) = 6$ and $(-3) + (-2) = -5$, we see that $x^2 - 5x + 6$ can be factored as $(x-3)(x-2)$, which means the given quadratic can be factored as

$$4(x-3)(x-2)$$

---

In later lessons, we'll learn how to factor more complicated quadratic polynomials, including quadratics in which

- the coefficient $a$ on the $x^2$ term is neither $1$ nor $-1$,
- the coefficient $a$ on the $x^2$ term isn't the greatest common factor of the quadratic, and when
- the constant $b$ is zero.

# Difference of squares

Factoring the difference of squares is a special case of factoring a quadratic.

We know we have a difference of squares when the quadratic is in standard form $ax^2 + bx + c$, when $b = 0$, and when $ax^2$ and $c$ are both perfect squares, such that the difference of squares is given as

$$ax^2 - c$$

Whenever we have a quadratic that we can define as the difference of squares, it will always factor as $(\sqrt{a}x + \sqrt{c})(\sqrt{a}x - \sqrt{c})$.

All the roots make this binomial look complicated, but all we need to do to factor the difference of squares is to take the square root of each term, then one binomial is the sum of the roots, and the other binomial is the difference of the roots.

Let's look at an example so that we can see how to factor the difference of squares.

---

**Example**

Factor the difference of squares.

$$9x^2 - 16$$

We can see that both $9x^2$ and $16$ are perfect squares. The term $9x^2$ is the perfect square of $3x$ because $(3x)^2 = 9x^2$, and the term $16$ is the perfect square of $4$ because $4^2 = 16$. In other words, we could rewrite the difference of squares as

$$9x^2 - 16$$

$$(3x)^2 - 4^2$$

To factor this difference of squares, we'll split the quadratic into the product of two binomials, which will be the sum and difference of $3x$ and $4$.

$$(3x + 4)(3x - 4)$$

We can double check our factoring by multiplying out the binomials.

$$(3x)(3x) + (3x)(-4) + (4)(3x) + (4)(-4)$$

$$9x^2 - 12x + 12x - 16$$

$$9x^2 - 16$$

---

Let's do another example.

---

**Example**

Factor the difference of squares.

$$x^2 - 25$$

Since $x^2$ and $25$ are both perfect squares (the squares of $x$ and $5$, respectively), $x^2 - 25$ is factored as

$$(x+5)(x-5)$$

---

Let's try another example, this time where we factor a multivariable difference of squares.

---

**Example**

Factor the binomial.

$$64x^4y^2 - 9z^6$$

Notice that $64x^4y^2 = (8x^2y)^2$, and that $9z^6 = (3z^3)^2$. Therefore, $64x^4y^2 - 9z^6$ a difference of squares that we can factor as

$$(8x^2y + 3z^3)(8x^2y - 3z^3)$$

---

# Zero Theorem

So far, we've been learning to factor quadratic expressions, but now we want to switch to solving quadratic equations. In other words, instead of factoring just $x^2 - x - 20$, we want to learn to solve $x^2 - x - 20 = 0$.

## The Zero Theorem

We already know that $x^2 - x - 20$ will factor as $(x+4)(x-5)$. Now that the quadratic is part of an equation, the factoring is no different. We can still factor the **quadratic equation** to rewrite it as

$$x^2 - x - 20 = 0$$

$$(x+4)(x-5) = 0$$

Once we've factored the quadratic equation, let's imagine that $A = x + 4$ and $B = x - 5$. Then this quadratic equation can be written as $AB = 0$.

What we can say about the values of $A$ and $B$ in $AB = 0$? Well, the only way to make the product $AB$ equal to $0$ is either for $A$ to be $0$, for $B$ to be $0$. This is the **Zero Theorem**, which tells us that, given $AB = 0$, we know

$A = 0$ or $B = 0$

Which means that, using the example $(x+4)(x-5) = 0$, it must be true that $x + 4 = 0$ or $x - 5 = 0$, which means that $x = -4$ or $x = 5$. These are the values of $x$ that make the equation true, and we were able to find them by applying the Zero Theorem.

When we solve for the solutions of a quadratic, we can call them the "**solutions**," the "**roots**," or the "**zeros**" of the quadratic.

Keep in mind that this Theorem only works when one side of the equation is $0$. So given something like $(x+4)(x-5)=3$, we can't break that down as $x+4=3$ or $x-5=3$.

Let's look at another example where we have to factor the quadratic and then apply the Zero Theorem to find the roots.

---

**Example**

Find the roots of the equation.

$$x^2-13x+36=0$$

The roots of this equation are the values of $x$ at which the polynomial on the left side is equal to $0$. If we factor the left side,

$$x^2-13x+36=0$$

$$(x-4)(x-9)=0$$

then the Zero Theorem tells us that

$$x-4=0 \quad \to \quad x=4$$

or

$$x-9=0 \quad \to \quad x=9$$

So the roots are $x = 4$ and $x = 9$.

---

Let’s do another example.

---

**Example**

Find the zeros of the equation.

$$x^2 - 8x + 7 = 0$$

Factor the left side.

$$x^2 - 8x + 7 = 0$$

$$(x - 7)(x - 1) = 0$$

The Zero Theorem tells us that

$$x - 7 = 0 \quad \rightarrow \quad x = 7$$

or

$$x - 1 = 0 \quad \rightarrow \quad x = 1$$

The zeros are $x = 1$ and $x = 7$.

---

# Completing the square

We just saw how to use the Zero Theorem to find the roots of a quadratic equation. All we had to do was factor the quadratic, set each factor equal to $0$ individually, and then solve each of those equations to find the roots.

But how do we find the roots of a quadratic when it won't easily factor? If the quadratic won't factor, or even if we just can't figure out how to factor it, we can always complete the square in order to find the roots.

## How to complete the square

**Completing the square** is one method we can use when we can't find the roots of a quadratic by factoring.

In standard form, we write a quadratic equation as $ax^2 + bx + c = 0$. But in order to complete the square, we really need the coefficient on $x^2$ to be $a = 1$. So if the quadratic we're working on has $a \neq 1$, then our first step has to be to divide through both sides of the equation by $a$, so that the coefficient on $x^2$ becomes $1$.

Once that's done, we could define the quadratic with new coefficients as $x^2 + bx + c = 0$. From here, we'll follow the same set of steps every time in order to complete the square.

1. Calculate $b/2$, then square the result to get $(b/2)^2$.

2. Add $(b/2)^2$ to both sides of the equation to get

$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c = \left(\frac{b}{2}\right)^2$$

3. Subtract $c$ from both sides.

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$$

4. Factor the left side. It will always factor as a perfect square.

$$\left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right) = \left(\frac{b}{2}\right)^2 - c$$

$$\left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 - c$$

5. Take the square root of both sides, remembering to include a ± sign on the right side, then subtract $b/2$ from both sides.

$$x + \frac{b}{2} = \pm\sqrt{\left(\frac{b}{2}\right)^2 - c}$$

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

## The number of solutions

Using the method of completing the square, we now know that, given a quadratic equation in the form $x^2 + bx + c = 0$, the root(s) of the quadratic will always take the form

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Based on the form of this solution, we can see that the quadratic can have either zero real roots, one real root, or two real roots, depending on the value of the square root in this solution equation.

- If $\left(\frac{b}{2}\right)^2 - c < 0$, then the quadratic has zero real roots (the roots are complex)
- If $\left(\frac{b}{2}\right)^2 - c = 0$, then the quadratic has one root, $x = -\frac{b}{2}$
- If $\left(\frac{b}{2}\right)^2 - c > 0$, then the quadratic has two roots, $x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$

Let's do an example so that we can see these steps in action.

---

**Example**

Solve for $x$ by completing the square.

$$x^2 + 6x + 4 = 0$$

There is no pair of factors of $4$ that have a sum of $6$, so we'll need to solve by completing the square. In this quadratic, $b = 6$, so $b/2 = 6/2 = 3$, and therefore $(b/2)^2 = 3^2 = 9$.

Add $9$ to both sides of the equation, then subtract $4$ from both sides.

$$x^2 + 6x + 9 + 4 = 0 + 9$$

$$x^2 + 6x + 9 = 5$$

Factor what remains on the left side as a perfect square.

$$(x + 3)(x + 3) = 5$$

$$(x + 3)^2 = 5$$

Take the square root of both sides, then subtract $3$.

$$x + 3 = \pm\sqrt{5}$$

$$x = -3 \pm \sqrt{5}$$

These values of $x$, $x = -3 - \sqrt{5}$ and $x = -3 + \sqrt{5}$, are the roots of the quadratic. Because the value under the square root was positive, $5 > 0$, we found two roots for the quadratic equation.

---

## Complex roots

We said earlier that the quadratic would have no real roots when $(b/2)^2 - c < 0$. That's because we can't take the square root of negative number. At least, we can't do it with real numbers alone.

But there are actually another set of numbers that we can work with, complex numbers. **Complex numbers** are numbers that include both real and imaginary numbers. An **imaginary number** is any number that includes the imaginary number $i$, where $i = \sqrt{-1}$.

Let's try an example where the roots of the quadratic are complex.

---

**Example**

Solve for $x$ by completing the square.

$$x^2 - 4x + 7 = 0$$

There is no pair of factors of $7$ that have a sum of $-4$, so we'll need to solve by completing the square. In this quadratic, $b = -4$, so $b/2 = -4/2 = -2$, and therefore $(b/2)^2 = (-2)^2 = 4$.

Add $4$ to both sides of the equation, then subtract $7$ from both sides.

$$x^2 - 4x + 4 + 7 = 0 + 4$$

$$x^2 - 4x + 4 = -3$$

Factor what remains on the left side as a perfect square.

$$(x - 2)(x - 2) = -3$$

$$(x-2)^2 = -3$$

Take the square root of both sides, then subtract $2$.

$$x - 2 = \pm\sqrt{-3}$$

$$x = 2 \pm \sqrt{-3}$$

This is where imaginary numbers come in. We can rewrite $\sqrt{-3}$ as $\sqrt{3}\sqrt{-1}$. Then, because $i = \sqrt{-1}$, we can rewrite the solutions as

$$x = 2 \pm \sqrt{3}\sqrt{-1}$$

$$x = 2 \pm \sqrt{3}i$$

So, while the quadratic has no solutions in terms of real numbers only, it does have the two complex roots $x = 2 - \sqrt{3}i$ and $x = 2 + \sqrt{3}i$.

# Quadratic formula

Now that we know how to complete the square, we can introduce the quadratic formula, which is certainly one of the most famous formulas from Algebra.

## The quadratic formula

The quadratic formula gives us any solutions to a quadratic equation $ax^2 + bx + c = 0$ as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Just as we saw when we learned to complete the square, the number of solutions to the quadratic can be determined by the value under the square root, **the discriminant**, $b^2 - 4ac$.

- When $b^2 - 4ac = 0$, the solution is one real number
- When $b^2 - 4ac > 0$, the solutions are two real numbers
- When $b^2 - 4ac < 0$, the solutions are two real complex numbers

## Building the quadratic formula

If we complete the square for a quadratic in standard form, $ax^2 + bx + c = 0$, we would start by dividing through the equation by $a$, since we said before that we can only complete the square when the coefficient on $x^2$ is $1$.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

We'll divide $b/a$ by $2$ to get $b/2a$, then we'll square that result to get $(b/2a)^2 = b^2/4a^2$. Adding this value to both sides and then subtracting $c/a$ gives

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} = 0 + \frac{b^2}{4a^2}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

The left side will always factor as a perfect square,

$$\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

and we can find a common denominator to combine the fractions on the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}\left(\frac{4a}{4a}\right)$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Take the square root of both sides.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Subtract $b/2a$ from both sides to solve for $x$, then combine the fractions.

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

Just like the method of completing the square, we can always use the quadratic formula to find the roots of a quadratic. Let's do an example with a positive discriminant.

**Example**

Solve for $x$ using the quadratic formula.

$$x^2 + 2x - 8 = 0$$

For this particular quadratic equation, $a = 1$, $b = 2$, and $c = -8$. Plugging these values into the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

$$x = \frac{-2 \pm \sqrt{36}}{2}$$

$$x = \frac{-2 \pm 6}{2}$$

We can factor a $2$ out of the numerator,

$$x = \frac{2(-1 \pm 3)}{2}$$

and then cancel the common factor of $2$ from the numerator and denominator.

$$x = -1 \pm 3$$

$x = -1 - 3$ or $x = -1 + 3$

$x = -4$ or $x = 2$

There are two real roots. That makes sense, since the value of the discriminant is positive, $b^2 - 4ac = 2^2 - 4(1)(-8) = 4 + 32 = 36 > 0$.

---

Let's try another example, this time where $a \neq 1$.

---

**Example**

Solve for the roots of the quadratic using the quadratic formula.

$$3x^2 + 6x + 2 = 0$$

We can identify $a = 3$, $b = 6$, and $c = 2$. Plugging these values into the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{-6 \pm \sqrt{12}}{6}$$

$$x = \frac{-6 \pm \sqrt{4}\sqrt{3}}{6}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{6}$$

We can factor a $2$ out of the numerator and denominator,

$$x = \frac{2(-3 \pm \sqrt{3})}{2(3)}$$

and then cancel the common factor of $2$ from the numerator and denominator.

$$x = \frac{-3 \pm \sqrt{3}}{3}$$

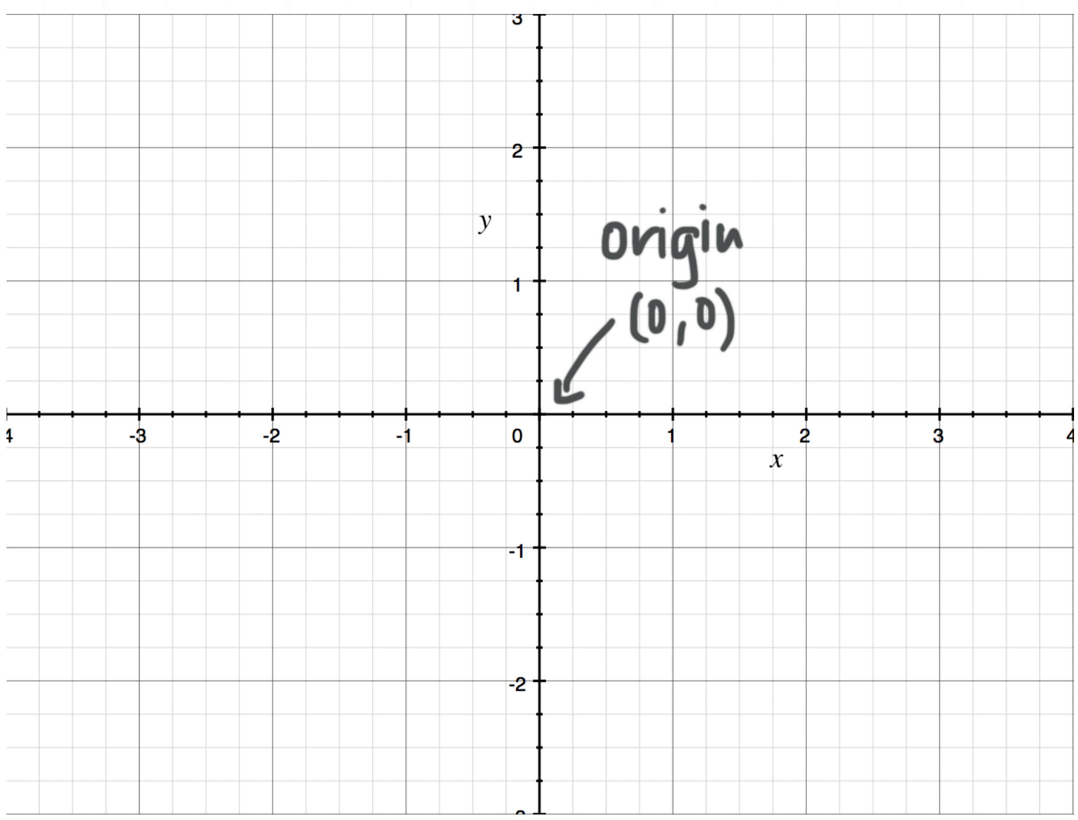
There are two real roots. That makes sense, since the value of the discriminant is positive, $b^2 - 4ac = 6^2 - 4(3)(2) = 36 - 24 = 12 > 0$.

# Cartesian coordinate system

The Cartesian coordinate system is the two-dimensional plane in which we graph points and equations. We can think about this plane as a surface, like a piece of paper, that extends forever in all directions.

The plane is defined by the **coordinate axes**, a pair of perpendicular number lines, one horizontal and one vertical. The horizontal axis is the $x$-axis and the vertical axis is the $y$-axis, and they meet at the **origin**, which is the point $(0,0)$.

We draw arrows at the ends of the axes to indicate that they extend forever. The horizontal axis is negative on the left and positive on the right, while the vertical axis is negative at the bottom and positive on the top.




We represent every point in the plane by a pair of numbers $(x, y)$, called its **coordinates**, where $x$ (the horizontal coordinate or the $x$-coordinate) is the horizontal (left-right) location of the point, and $y$ (the vertical coordinate or the $y$-coordinate) is the vertical (up-down) location of the point.

So the $x$-coordinate of a point in the plane is negative to the left of the $y$-axis, positive to the right of the $y$-axis, and $0$ if it's on the $y$-axis itself. Similarly, the $y$-coordinate of a point in the plane is negative below the $x$-axis, positive above the $x$-axis, and $0$ if it's on the $x$-axis.

The origin is the center of the coordinate system, so its coordinates are $(x, y) = (0,0)$.

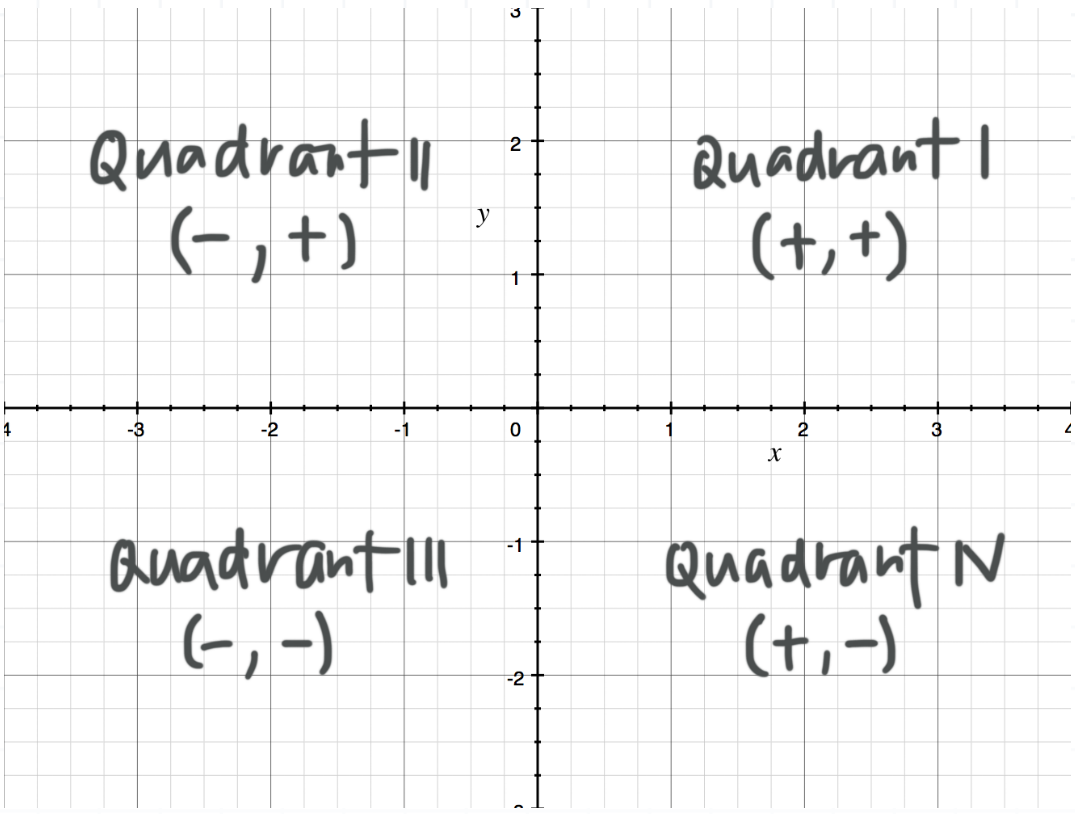
The axes divide the coordinate plane into four **quadrants**. Quadrant I is where $x$ and $y$ are both positive. The other three quadrants are named in counterclockwise order.

Quadrant I: both $x$ and $y$ are positive $(x, y) = ( + , + )$

Quadrant II: $x$ is negative and $y$ is positive $(x, y) = ( - , + )$

Quadrant III: both $x$ and $y$ are negative $(x, y) = ( - , - )$

Quadrant IV: $x$ is positive and $y$ is negative $(x, y) = ( + , - )$




Quadrants I, II, III, and IV are also called the first, second, third, and fourth quadrants, respectively.

We graph a point in the plane by placing a dot at its location in the Cartesian coordinate system. We sometimes say that we "plot a point," which means the same thing.

Let's do an example where we plot a point in the plane.

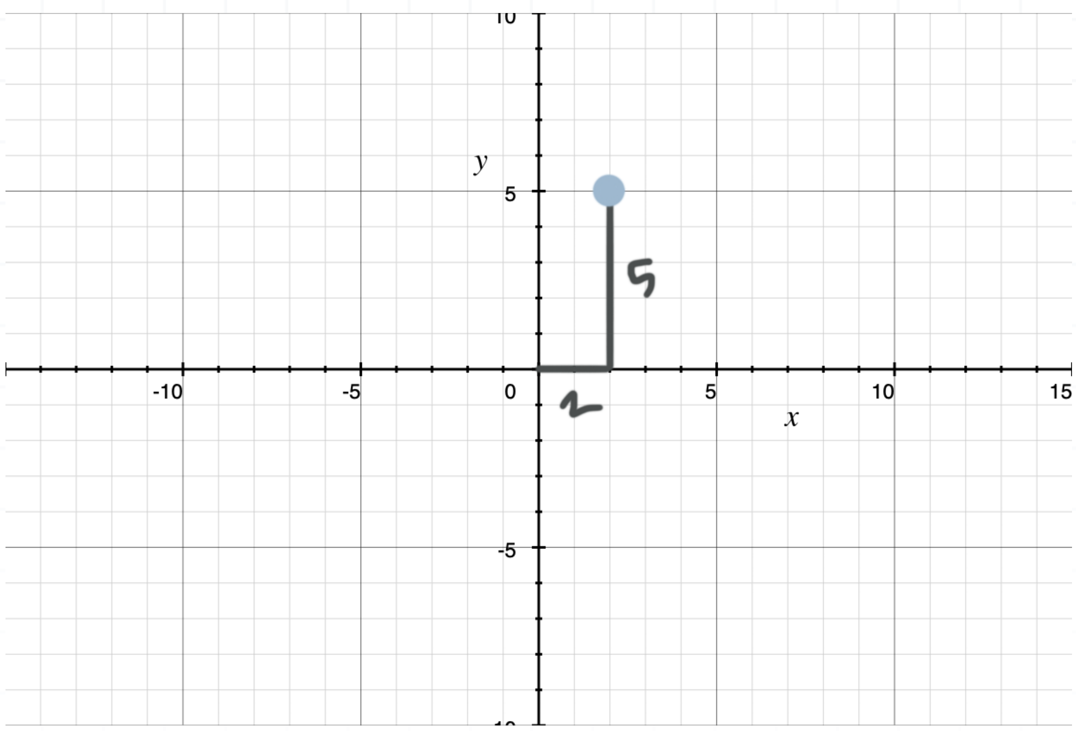
---

**Example**

Graph the point in the Cartesian coordinate system.

$$(2,5)$$

Remember that points are in the form $(x, y)$, so the $2$ tells us how to move on the $x$-axis (left or right) and the $5$ tells us how to move on the $y$-axis (up or down). Since the $x$-coordinate $2$ is positive, we move $2$ units from the origin in the direction of the positive $x$-axis (to the right). And since the $y$-coordinate $5$ is positive, we move $5$ units up from there in the direction of the positive $y$-axis (up).



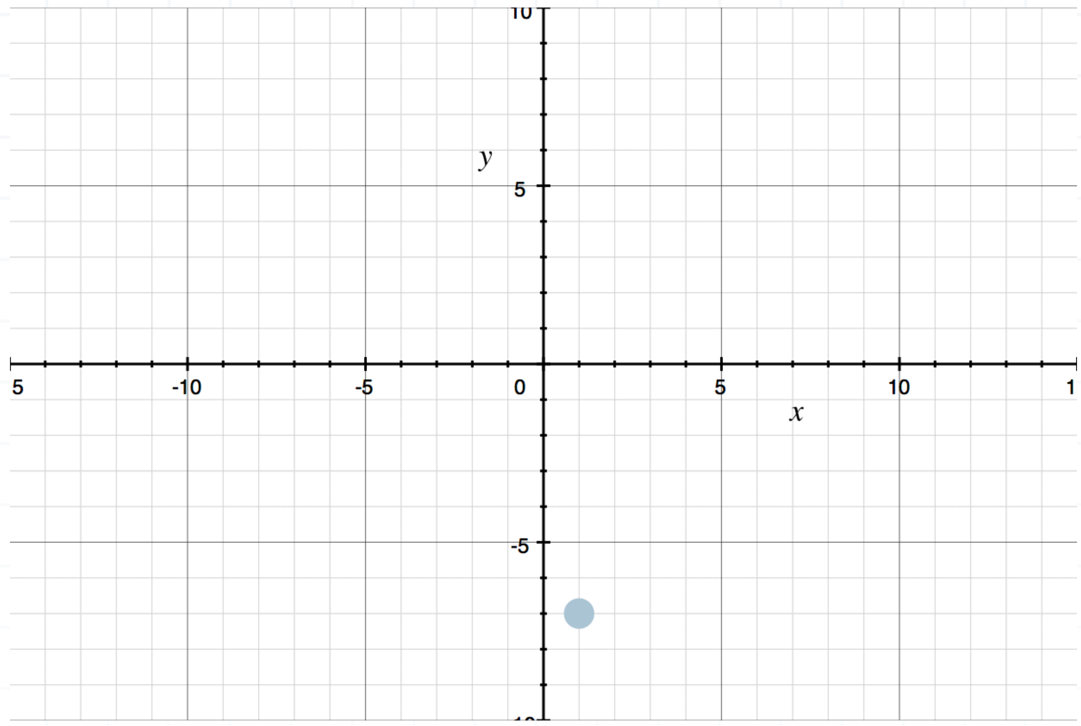

Let's try another example of plotting a point in the plane.

**Example**

In which quadrant should we plot the point?

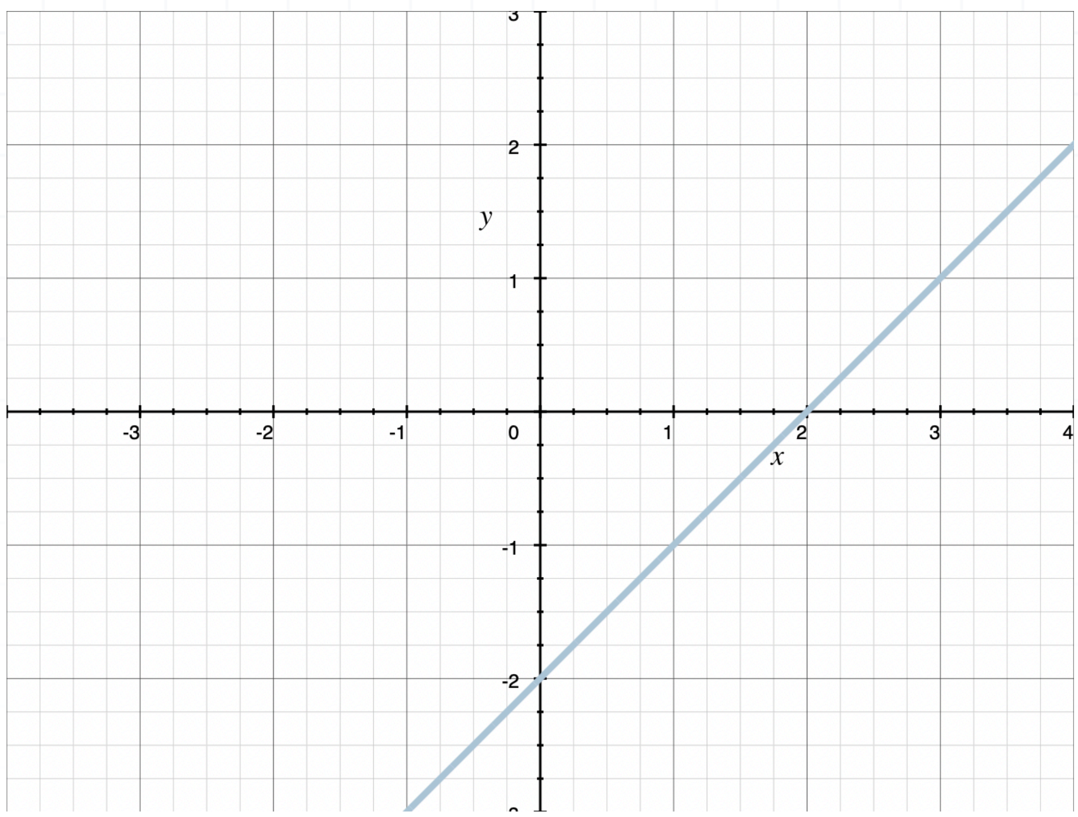
$$(1, -7)$$

Since the $x$-coordinate is positive and the $y$-coordinate is negative, the point should be plotted in the fourth quadrant, Quadrant IV.


10
y
5
5
-10
-5
0
5
10
1
x
-5

# Slope

We've learned to plot points in the Cartesian coordinate system, and now we want to be able to plot lines in the same plane. For example, the line $y = x - 2$ is sketched in the plane as




## How to find the slope

Every line has its own slope, where the **slope** is the "steepness" of the line, or the rate of change of the $y$-coordinates of the points on the graph as we move horizontally from left to right. If we think about the vertical change as "rise" and the horizontal change as "run," then we can write the slope formula as

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

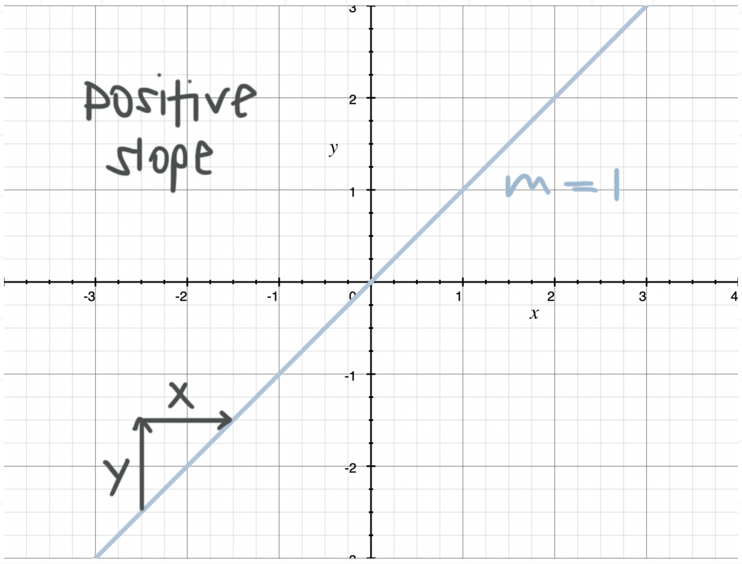
where $(x_1, y_1)$ is one point on the line and $(x_2, y_2)$ is another point on the line. This is the algebraic way of finding the slope, which we can use when we can't look at the graph, or when it's difficult to figure out rise/run by looking at the graph.

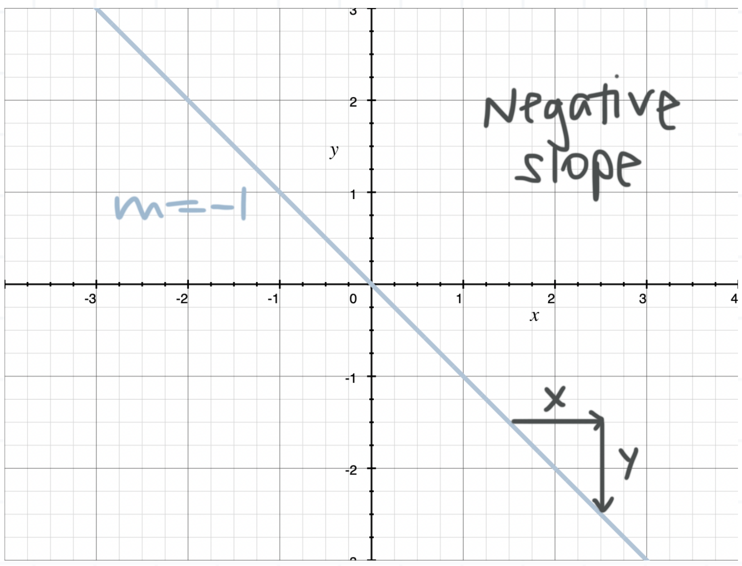
It never matters which point we call $(x_1, y_1)$ and which one we call $(x_2, y_2)$. The slope will be the same either way. Also, it doesn't matter which pair of points on the line we use to find the slope. The slope will always be the same.

## Sign of the slope

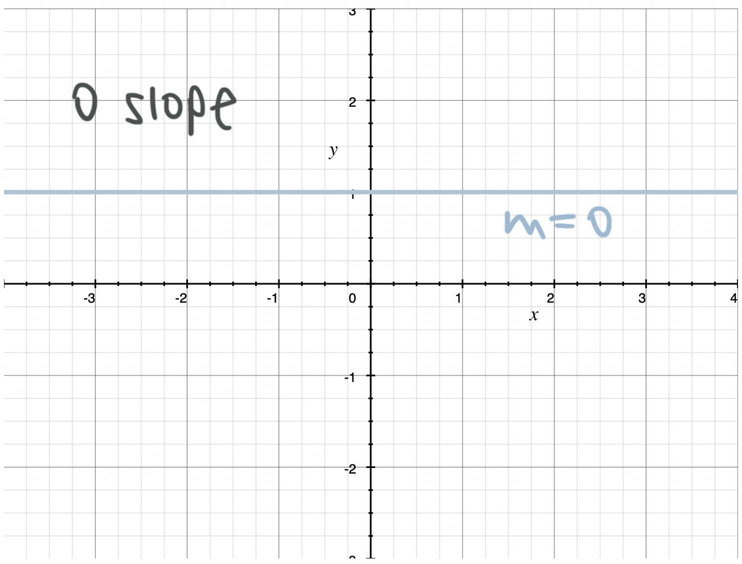
The slope is given by the change in $y$ compared to the change in $x$, which means slopes can be positive, negative, zero (horizontal), and undefined (vertical).

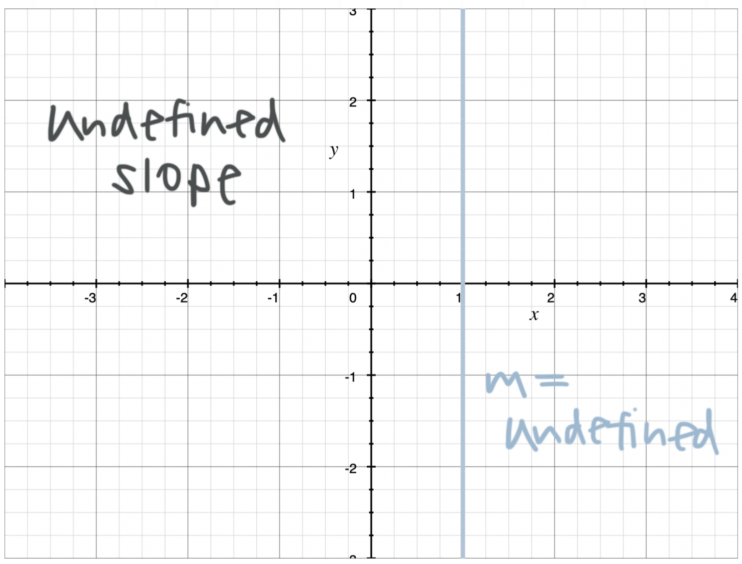
Lines with a positive slope move up as we move to the right, so the value of $y$ increases as $x$ increases. On the other hand, lines with a negative slope move down as we move to the right, so the value of $y$ decreases as $x$ increases.




Any perfectly horizontal line has a slope of $0$, because there's no change in $y$ for any change in $x$, so $m=0/(x_2-x_1)=0$. On the other hand, any perfectly vertical line has a slope that's undefined, because there's no change in $x$ for any change in $y$, so $m=(y_2-y_1)/0$, but we can't divide by $0$.



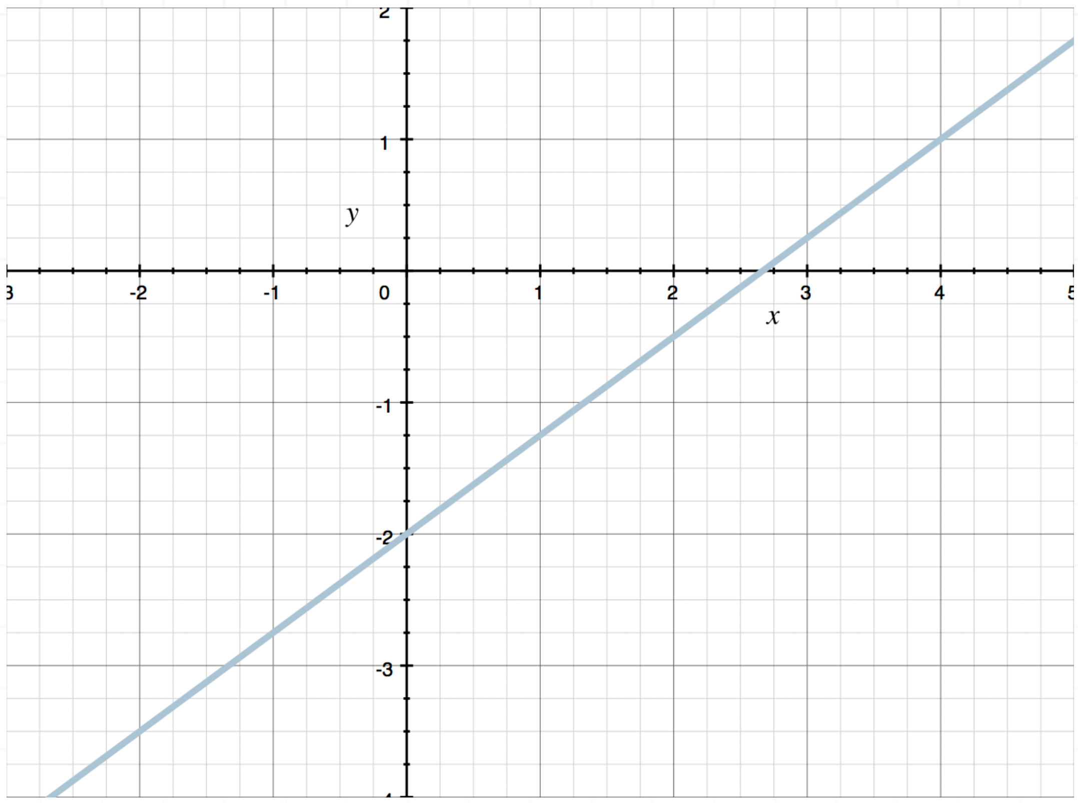




Let's do an example where we determine the slope of the line by looking at the graph.

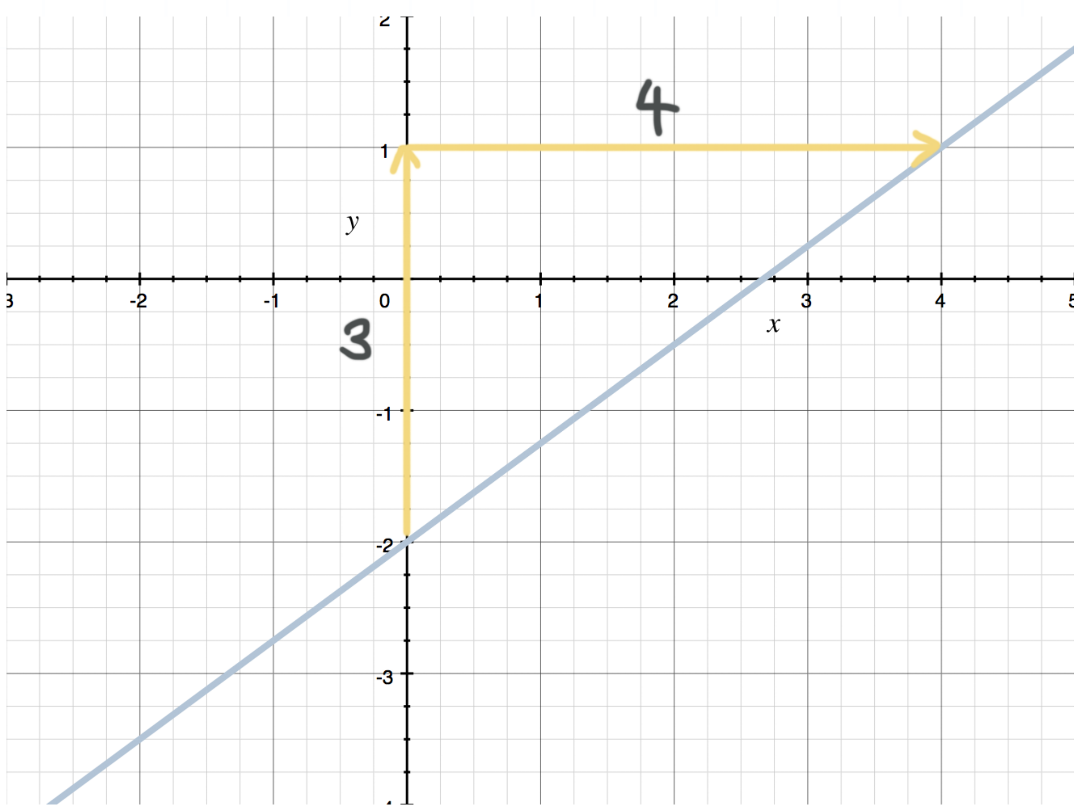
---

**Example**

What is the slope of the line?




Graphically, we can find two points on the graph and count the rise (up and down) and run (to the right).



The rise is $3$ and the run is $4$, so the slope is $3/4$. We could have also used two points on the line, like $(0,-2)$ and $(4,1)$, to calculate the slope algebraically.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-2)}{4-0}=\frac{3}{4}$$

---

Let's try another example.

### Example

What is the slope of the line that passes through $(-1,5)$ and $(3,-3)$?

It doesn't matter which point we label as $(x_1,y_1)$ and which one we label as $(x_2,y_2)$, we'll get the same slope either way. But let's say $(x_1,y_1)=(-1,5)$ and $(x_2,y_2)=(3,-3)$.

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$m=\frac{-3-5}{3-(-1)}$$

$$m=\frac{-8}{4}$$

$$m=-2$$

---

# Point-slope and slope-intercept forms of a line

We have two options for writing the equation of a line: point-slope form and slope-intercept form.

| | |
|---|---|
| Point-slope form | $y - y_1 = m(x - x_1)$ |
| Slope-intercept form | $y = mx + b$ |

Both forms require that we know at least two of the following pieces of information about the line:

1. One point, $(x_1, y_1)$
2. A second point, $(x_2, y_2)$
3. The slope, $m$
4. The $y$-intercept, $b$ (the $y$-value where the line crosses the $y$-axis)

If we know any two of these values, we can find the equation of the line in both forms.

## Point-slope form

The equation of a line in point-slope form is

$$y - y_1 = m(x - x_1)$$

In this form, $(x_1, y_1)$ is a point on the line, and $m$ is the slope. To use this form when we know two points on the line but we don't know the slope, we'll find $m$ as

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Then we'll plug the slope $m$ and the point $(x_1, y_1)$ into point-slope form. Let's do an example where we know the slope and one point on the line.

---

**Example**

Write the equation of the line in point-slope form.

$$m = -\frac{1}{4}$$

$$(-6,1)$$

Since we've been given the slope of the line and a point on the line, we can use the point-slope form to find the equation of the line. We'll plug $m = -1/4$ and the point $(-6,1)$ into the point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{4}(x - (-6))$$

$$y - 1 = -\frac{1}{4}(x + 6)$$

Let's try an example where we know two points on the line.

**Example**

Find the point-slope form of the equation of the line that passes through the points $(-2,-4)$ and $(3,5)$.

We'll start by finding the slope of the line. It never matters which point we use for $(x_1, y_1)$ and which one we use for $(x_2, y_2)$, as long as we stay consistent. Let's set

$$(x_1, y_1) = (-2,-4)$$

$$(x_2, y_2) = (3,5)$$

Plug these into the formula for slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{5 - (-4)}{3 - (-2)}$$

$$m = \frac{9}{5}$$

Next, substitute $m = 9/5$ and the point $(-2,-4)$ into point-slope form.

$$y - (-4) = \frac{9}{5}(x - (-2))$$

$$y+4=\frac{9}{5}(x+2)$$

---

## Slope-intercept form

The equation of a line in slope-intercept form is

$$y=mx+b$$

where $m$ is the slope and $b$ is the $y$-intercept. Since the $y$-intercept, $b$, is the $y$-coordinate of the point at which the graph crosses the $y$-axis, and since the $x$-coordinate of every point on the $y$-axis is $0$, the point where the graph crosses the $y$-axis is $(0,b)$.

Let's try an example where we know the slope and a point on the line.

---

**Example**

Find the equation of the line in slope-intercept form.

$$m=-3$$

$$(0,1)$$

If we recognize that the point $(0,1)$ lies on the $y$-axis (since its $x$-coordinate is $0$), then we can recognize that the $y$-intercept is $y=b=1$, and we can substitute $m=-3$ and $b=1$ into slope-intercept form.

$$y = mx + b$$

$$y = -3x + 1$$

Even if we were unsure about whether $(0,1)$ gave us the $y$-intercept, we could still plug the slope $m = -3$ and the point $(0,1)$ into the point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -3(x - 0)$$

$$y - 1 = -3x$$

$$y = -3x + 1$$

---

Let's try an example where we know two points on the line.

**Example**

Find the slope-intercept form of the equation of the line that passes through $(-1, -2)$ and $(3, -4)$.

First we need to find the slope, so let's set

$$(x_1, y_1) = (-1, -2)$$

$$(x_2, y_2) = (3, -4)$$

and then plug these points into the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-4 - (-2)}{3 - (-1)}$$

$$m = \frac{-2}{4}$$

$$m = -\frac{1}{2}$$

Next, substitute $m = -1/2$ and $(x_1, y_1) = (-1, -2)$ into the point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = -\frac{1}{2}(x - (-1))$$

$$y + 2 = -\frac{1}{2}(x + 1)$$

$$y + 2 = -\frac{1}{2}x - \frac{1}{2}$$

$$y = -\frac{1}{2}x - \frac{5}{2}$$

## Converting between the forms

Regardless of which form we start with, point-slope or slope-intercept, we can easily convert back and forth between the two.

We've worked four examples so far in this lesson. In this first two, we found equations in point-slope form,

$$y-1=-\frac{1}{4}(x+6) \qquad y+4=\frac{9}{5}(x+2)$$

To convert equations in point-slope form to equations in slope-intercept form, we simplify the right side of the equation and then solve for $y$.

$$y-1=-\frac{1}{4}x-\frac{6}{4} \qquad y+4=\frac{9}{5}x+\frac{18}{5}$$

$$y=-\frac{1}{4}x-\frac{6}{4}+1 \qquad y=\frac{9}{5}x+\frac{18}{5}-4$$

$$y=-\frac{1}{4}x-\frac{6}{4}+\frac{4}{4} \qquad y=\frac{9}{5}x+\frac{18}{5}-\frac{20}{5}$$

$$y=-\frac{1}{4}x-\frac{1}{2} \qquad y=\frac{9}{5}x-\frac{2}{5}$$

Now the equations are in slope-intercept form, $y=mx+b$.

In the third and fourth examples, we found equations in slope-intercept form,

$$y=-3x+1 \qquad y=-\frac{1}{2}x-\frac{5}{2}$$

To convert equations in slope-intercept form to equations in point-slope form, we factor out the coefficient on $x$,

$$y = -3\left(x - \frac{1}{3}\right) \qquad\qquad y = -\frac{1}{2}(x + 5)$$

Now the equations are in slope-intercept form, $y - y_1 = m(x - x_1)$. When we convert this way, we'll always have $y_1 = 0$.

## When lines can't be written in either form

Perfectly vertical lines can't be written in either point-slope or slope-intercept forms, because the slope of a vertical line is undefined.

Given two points on a vertical line $(x_1, y_1)$ and $(x_2, y_2)$, we know that $x_1 = x_2$, which means $x_2 - x_1 = 0$ and the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{0}$$

which is undefined because division by $0$ is undefined. So instead of writing vertical lines in point-slope or slope-intercept form, we'll write them as $x = c$. For example, $x = 3$ is the equation of the vertical line that passes through the point $(3,0)$ on the horizontal axis.

# Graphing linear equations

A **linear equation** is the equation of a line, so when we graph linear equations, it means that we're graphing lines. To sketch the graph of a line, we want to start by putting the equation into point-slope form or slope-intercept form.

| | |
|---|---|
| Point-slope form | $y - y_1 = m(x - x_1)$ |
| Slope-intercept form | $y = mx + b$ |

Remember that, in these equations, $m$ is the slope of the line and $b$ is the $y$-intercept (the $y$-coordinate of the point where the line crosses the $y$-axis).

When we're graphing linear equations, we can also use the **intercepts** of the line to help us, which are the points where the line crosses the major axes. We already know that $b$ is the $y$-intercept, and we can find the $x$-intercept by setting $y = 0$.

Let's do an example with a line in slope-intercept form.

---

### Example

What is the $y$-intercept of the line?

$$y = -\frac{2}{3}x$$

This equation is in slope-intercept form, but the $y$-intercept is missing. However, we could actually rewrite the equation of the line as

$$y = -\frac{2}{3}x + 0$$

Written this way, we haven't changed the value of either side of the equation at all, but we can see that the $y$-intercept is $0$, which means the line passes through the origin.

---

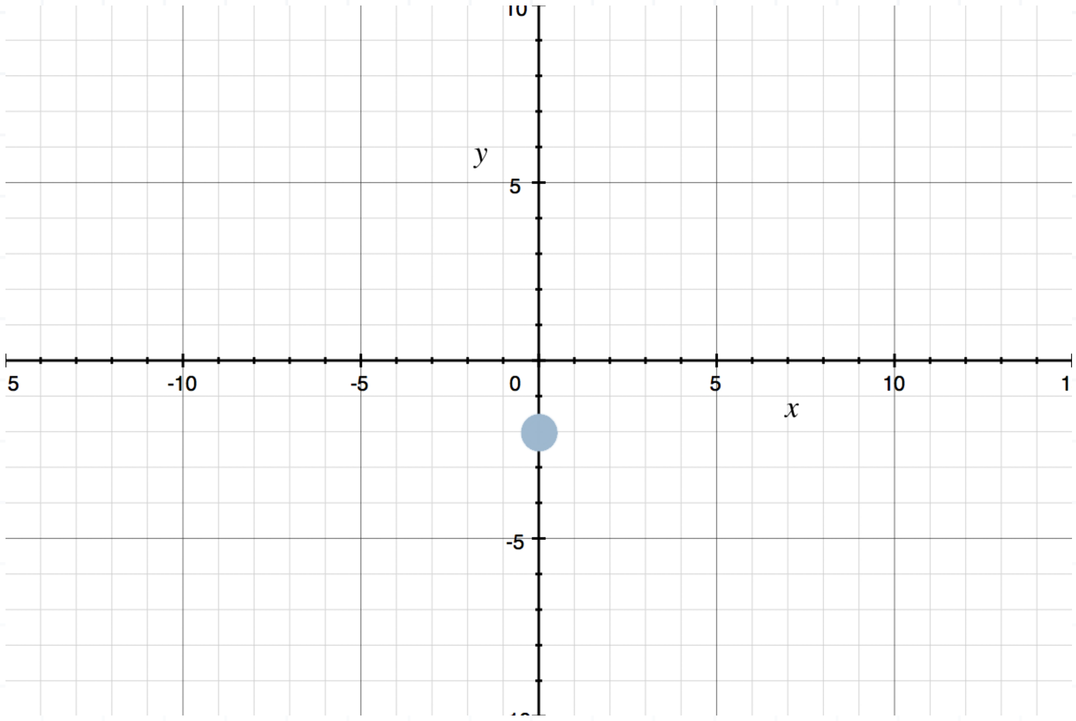
Let's look at an example where we graph a line from slope-intercept form.

**Example**

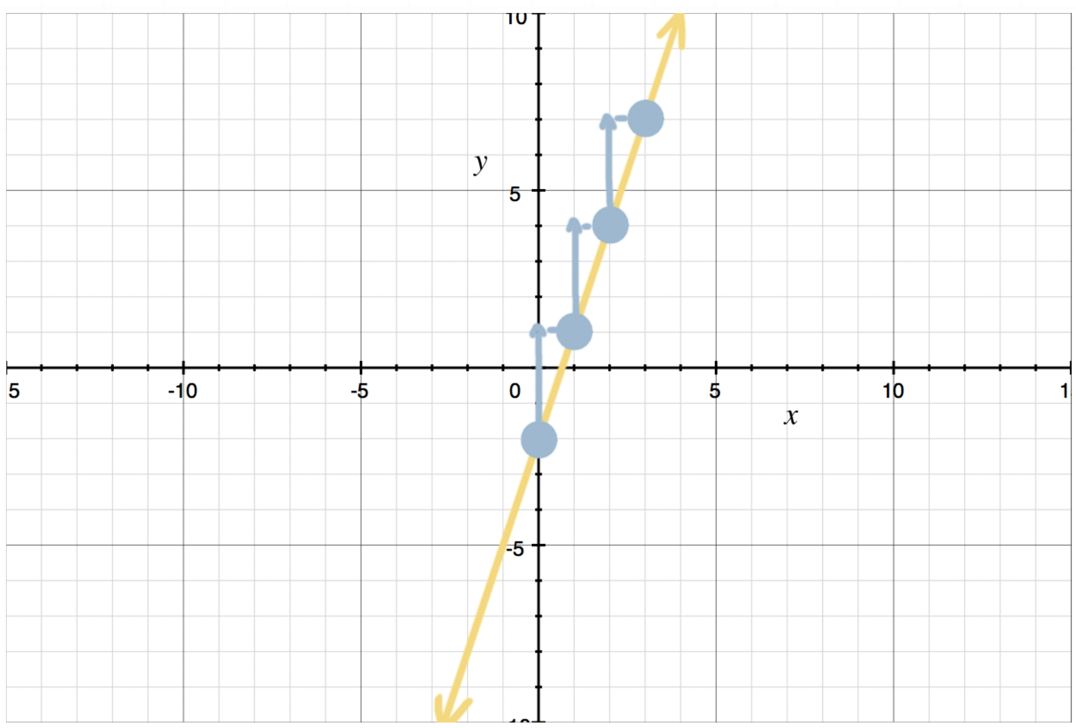
Graph the line.

$$y = 3x - 2$$

This equation is in slope-intercept form, so it's ready to be graphed. We'll plot the $y$-intercept $b = -2$ by placing a point at $(0, -2)$, or two units down from the origin on the $y$-axis.



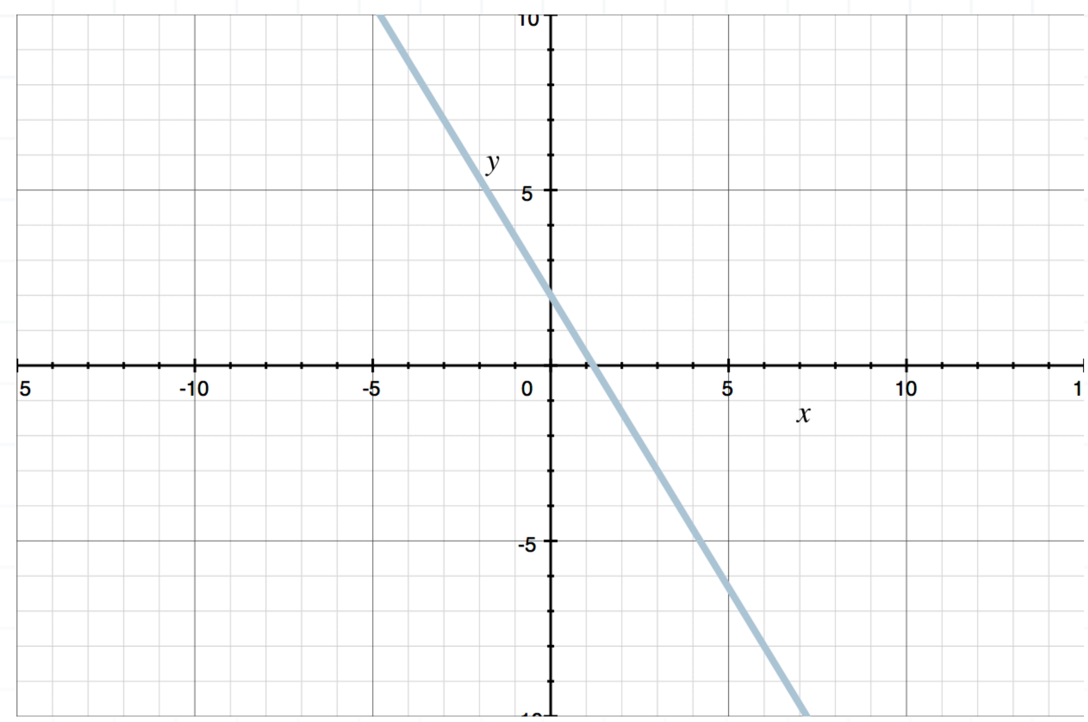

Since the slope of this line is $m = 3$, or $m = 3/1$, we'll move up $3$ units and right $1$ unit, and then plot a new point. So a sketch of the line is




Let's try another example, this time where we work backwards from the graph of the line to find the equation.

## Example

Write the equation of the line shown in the graph.




First, identify the $y$-intercept. In this case the graph of the line crosses the $y$-axis at $b = 2$. Next, we'll find the slope by identifying another clear point on the graph, like $(3, -3)$. To get from the $y$-intercept to the point $(3, -3)$, we'll go $5$ units down and then $3$ units to the right, so the slope is $m = -5/3$ and the equation of the line is

$$y = mx + b$$

$$y = -\frac{5}{3}x + 2$$

---

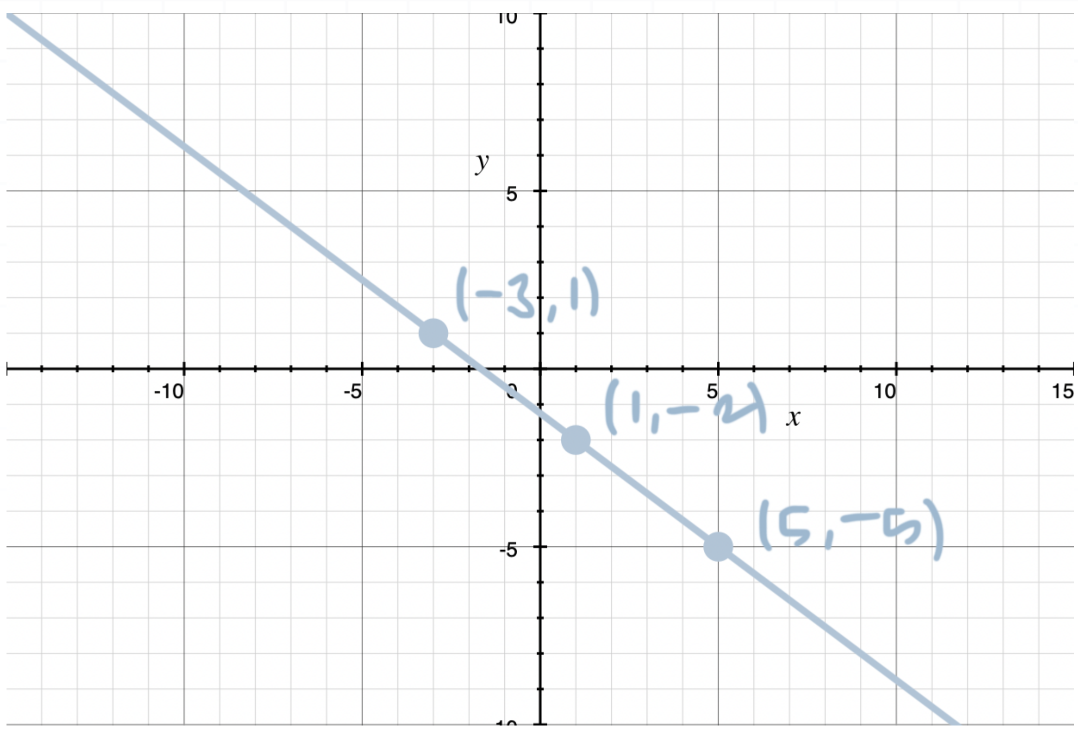
Let's try one more example where we have to interpret the slope of the line.

**Example**

How can we use the slope to find another point on the graph if the slope is $m = -3/4$ and the line passes through $(x_1, y_1) = (1, -2)$?

Starting at the point $(1, -2)$, we can find a second point in two ways.

We can either move up $3$ and to the left $4$ to plot the second point at $(1 - 4, -2 + 3) = (-3, 1)$, or we can move down $3$ and to the right $4$ to plot the second point at $(1 + 4, -2 - 3) = (5, -5)$.




And we could keep going, moving up and left or down and right, plotting more points along the line.

# Function notation

We already know that the math expression $y = 10 - 2x$ is an equation, because the equals sign tells us that the two expressions $y$ and $10 - 2x$ are equivalent.

What we want to say now is that some, but not all, equations are functions. Specifically, a **function** is an equation that only gives one output of the dependent variable for each input of the independent variable.

In an equation defined by $x$ and $y$, we typically say that the **independent variable** is $x$ and that the **dependent variable** is $y$, because the output value we get for $y$ "depends on" the input value we choose for $x$.

So we could say that $y = 10 - 2x$ is "a function $y$ of $x$" if we never get multiple output values of $y$ for any input value of $x$. The table below shows some input-output pairs for the equation $y = 10 - 2x$.

| x Input | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y Output | 14 | 12 | 10 | 8 | 6 |

We substitute the value of $x$ in the equation to find the associated value of $y$. For instance, substituting the input $x = 0$ gives an output $y = 10 - 2(0) = 10$.

Because we get a single output $y$ for every input $x$, and because this will always be the case for any input we choose, we can say that $y = 10 - 2x$ is a function.

## Equations that aren't functions

Any equation that gives multiple outputs for a single input is not a function. For example, $y^2 = x$ is not a function, because when we choose an input, like $x = 4$, we get $y = \pm 2$ as a pair of outputs, because both $(-2)^2 = 4$ and $2^2 = 4$.

Keep in mind that it's possible to find some inputs that only give one output, and other inputs that give multiple outputs. In $y^2 = x$, the input $x = 4$ gave two outputs $y = \pm 2$, but the input $x = 0$ gives only one output $y = 0$.

It doesn't matter if there are some inputs that only give one output. An equation is only a function if *all* inputs each only give one output. So if an equation gives one output at every input for all inputs, except for one single input that gives multiple outputs, that single input value is enough to disqualify the entire equation, and that equation is not a function.

The equations we've been dealing with so far in this course have all been equations of lines, quadratics, or other higher-degree polynomials. Every polynomial is always a function, with the exception of perfectly vertical lines. So any vertical line in the form $x = a$ is not a function, but all other lines and polynomials are always functions.

## How to express functions

When the equation *is* a function, we have the option to write it in function notation. To express $y = 10 - 2x$ as a function, we'd usually write it as $y(x) = 10 - 2x$ or as $f(x) = 10 - 2x$. The notation $y(x)$ we read as "$y$ of $x$," and it tells us that we're identifying the function with $y$, and the function $y$ is defined in terms of the variable $x$. So when we see $y(x)$, the parentheses don't indicate multiplication; they tell us that we have a function for $y$ in terms of $x$.

It's actually most common to see functions named with $f$, where the $f$ stands for "function," but technically we can use any letter that we want as the name of the function. So the function $f(x)$ is the function $f$ defined in terms of the variable $x$. We can also call $x$ the **argument** of the function.

## Input-output pairs satisfy the function

We showed earlier how to find input-output pairs for the function $f(x) = 10 - 2x$, and we listed some of those pairs in a table.

| x Input | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y Output | 14 | 12 | 10 | 8 | 6 |

In function notation, we replace $x$ with the desired input, which, for the input-output pairs in this table, looks like

$$f(-2) = 10 - 2(-2) = 10 + 4 = 14$$

$$f(-1) = 10 - 2(-1) = 10 + 2 = 12$$

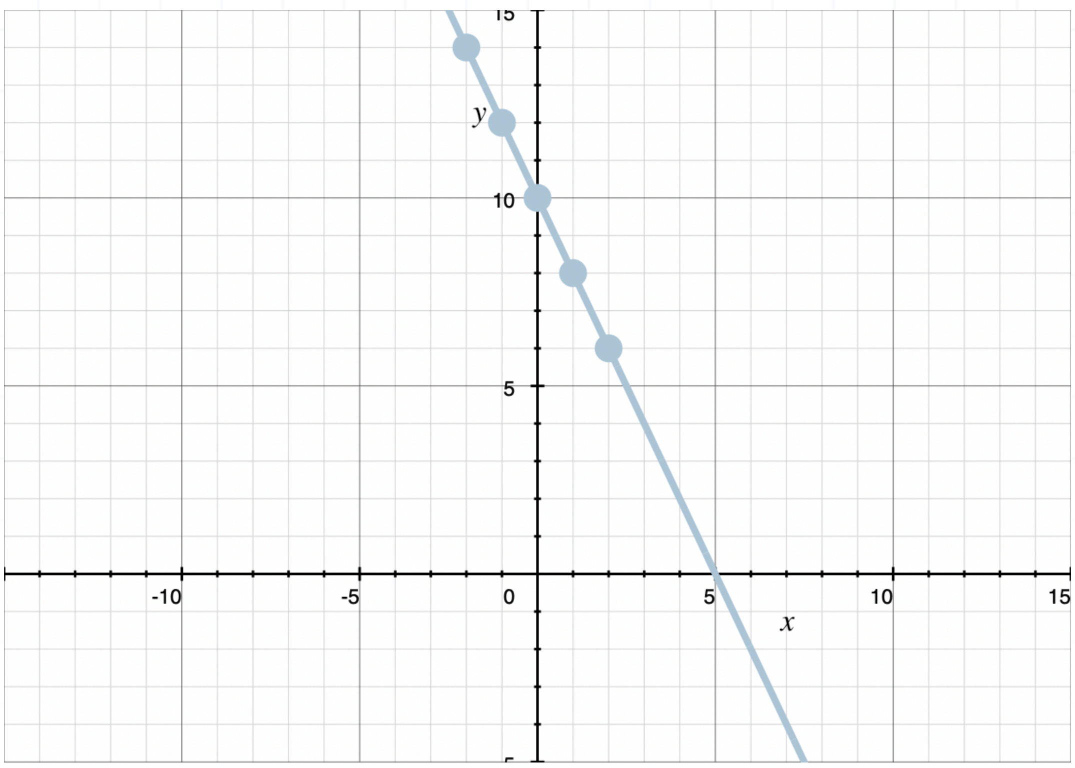
$$f(0) = 10 - 2(0) = 10 - 0 = 10$$

$$f(1) = 10 - 2(1) = 10 - 2 = 8$$

$$f(2) = 10 - 2(2) = 10 - 4 = 6$$

And keep in mind that these input-output pairs, because they satisfy the function, will exist as points on the graph of the function.

So if we sketch $f(x)=10-2x$ in the Cartesian coordinate system, we can see that the graph passes through $(-2,14)$, $(-1,12)$, $(0,10)$, $(1,8)$, and $(2,6)$.




Let’s do an example where we evaluate the function at a specific input.

---

**Example**

If $f(x)=10-2x$, find $f(-6)$.

Here, $-6$ is a specific input (a specific value of the variable $x$), so we’ll substitute $-6$ for the $x$ in the function $f(x)$ (for the $x$ in $10-2x$). Remember to use parentheses when plugging in numbers.

$$f(-6)=10-2(-6)$$

$$f(-6) = 10 + 12$$

$$f(-6) = 22$$

---

Let's try another example with function notation, this time with a quadratic polynomial.

### Example

If $f(x) = x^2 - 7x + 12$, find $f(4)$.

Plug in $4$ for every $x$ in the expression for $f(x)$ (for every $x$ in $x^2 - 7x + 12$).

$$f(4) = 4^2 - 7(4) + 12$$

$$f(4) = 16 - 28 + 12$$

$$f(4) = -12 + 12$$

$$f(4) = 0$$

---

# Domain and range

Now that we understand when an equation is a function, we want to be able to define the domain and range of the function.

## The domain

The **domain** of a function is all the values we can input into the function that don't cause it to be undefined. So the domain of a linear function (the equation of a line), is just all real numbers, because there are no values we can plug into a linear equation that make it undefined.

But if the function is a fraction, then any values that make the denominator $0$ will be excluded from the domain, since the fraction is undefined wherever its denominator is $0$. So these are some functions and their domains.

$$f(x) = \frac{1}{x} \qquad x \neq 0 \qquad\qquad g(x) = \frac{1}{x-3} \qquad x \neq 3$$

The function $f$ has a domain that includes all numbers except $0$ ($x \neq 0$ means $x$ "not equal to" $0$), and the function $g$ has a domain that includes all numbers except $3$ ($x \neq 3$ means $x$ "not equal to" $3$). The numbers excluded from the domain are the values that make the function undefined.

Let's also think about functions with roots. Any values that make the radicand (the value under the radical) negative will be excluded from the domain, since a radical is undefined wherever its radicand is negative. These are some more functions and their domains.

$$f(x) = \sqrt{x} \quad x \geq 0 \qquad\qquad g(x) = \sqrt{x-3} \quad x \geq 3$$

The notation $x \geq 0$ tells us that $x$ has to be “greater than or equal to” $0$, and the notation $x \geq 3$ tells us that $x$ has to be “greater than or equal to” $3$.

## The range

Once we know the domain of the function, then we can identify its **range**, which is the entire set of output values that can result from all the inputs in the domain. So if the domain is all of the allowable $x$ values, the range is all the possible $y$ values.

For instance, these are the ranges of the fractional functions we’ve been looking at:

$$f(x) = \frac{1}{x} \quad f(x) \neq 0 \qquad\qquad g(x) = \frac{1}{x-3} \quad g(x) \neq 0$$

With both numerators equal to $1$, there’s no way to make $f(x) = 0$ or $g(x) = 0$, so $0$ is not part of either function’s range.

Similarly, for the radical functions, we’ve set the domains so that the radicands will never be $0$, which means we’ll be taking the square root of a value that’s either $0$ or positive. The square root of $0$ is $0$, and the square root of a positive number is a positive number, which means the range of each function is all values that aren’t $0$.

$$f(x) = \sqrt{x} \quad f(x) \geq 0 \qquad\qquad g(x) = \sqrt{x-3} \quad g(x) \geq 0$$

The notation $f(x) \geq 0$ tells us that the function $f(x)$ has to be "greater than or equal to" $0$.

Let's do an example where we find the domain of another function.

---

### Example

What are the domain and range of the function?

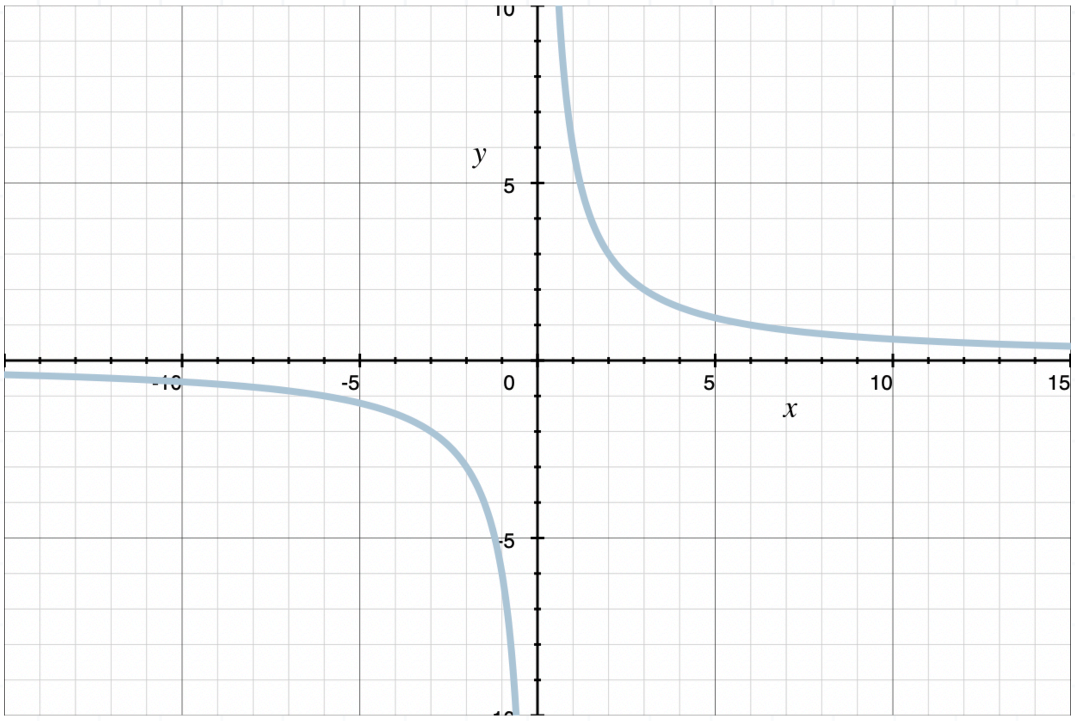
$$f(x) = \frac{6}{x}$$

An input of $x = 0$ would make the denominator of the function $0$, which would make the function undefined, so the domain includes all numbers except $x = 0$.

Then the range will include all values except $0$, because there's no value we can input for $x$ that will cause the value of $f(x)$ to be $0$.

| | |
|---|---|
| Domain | $x \neq 0$ |
| Range | $f(x) \neq 0$ |

If we graph this function, we see that it's not defined at $x = 0$ (along the vertical axis), and not defined at $f(x) = y = 0$ (along the horizontal axis).




We can also define a function by a set of $(x, y)$ coordinate points. When a function is defined this way, the domain is the set of $x$-values in the coordinate points, and the range is the set of $y$-values in the coordinate points.

Let's look at an example where we find the domain and range of a function that's defined by a set of coordinate points.

### Example

What are the domain and range of the function defined by the set of points?

$$(-2,4),\ (1,3),\ (2,5),\ (4,3)$$

If we collect all the $x$-values from the points, they make up the domain. And if we collect all the $y$-values from the points, they make up the range.

Domain: $-2, 1, 2, 4$

Range: $4, 3, 5, 3$

We don't need to list the same number more than once, and we'd prefer to arrange the values in ascending order, so we can actually give the domain and range as

Domain: $-2, 1, 2, 4$

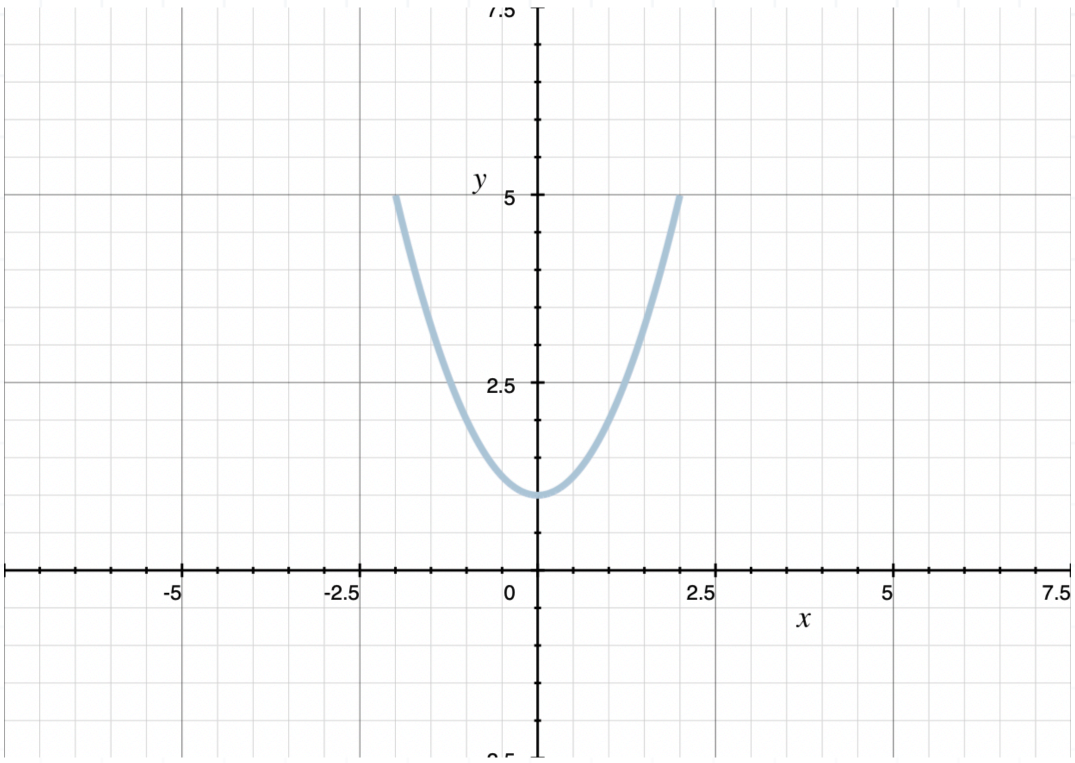
Range: $3, 4, 5$

---

We can also find the domain and range of a graph, just by looking at its sketch.

**Example**

What are the domain and range of the graph?




The domain is all of the $x$-values defined by the sketch, so we need to look at where the function is defined horizontally in the sketch, from its left-most point to its right-most point.

The range is all of the $y$-values defined by the sketch, so we need to look at where the function is defined vertically in the sketch, from its lowest point to its highest point.

Domain: $-2 \leq x \leq 2$

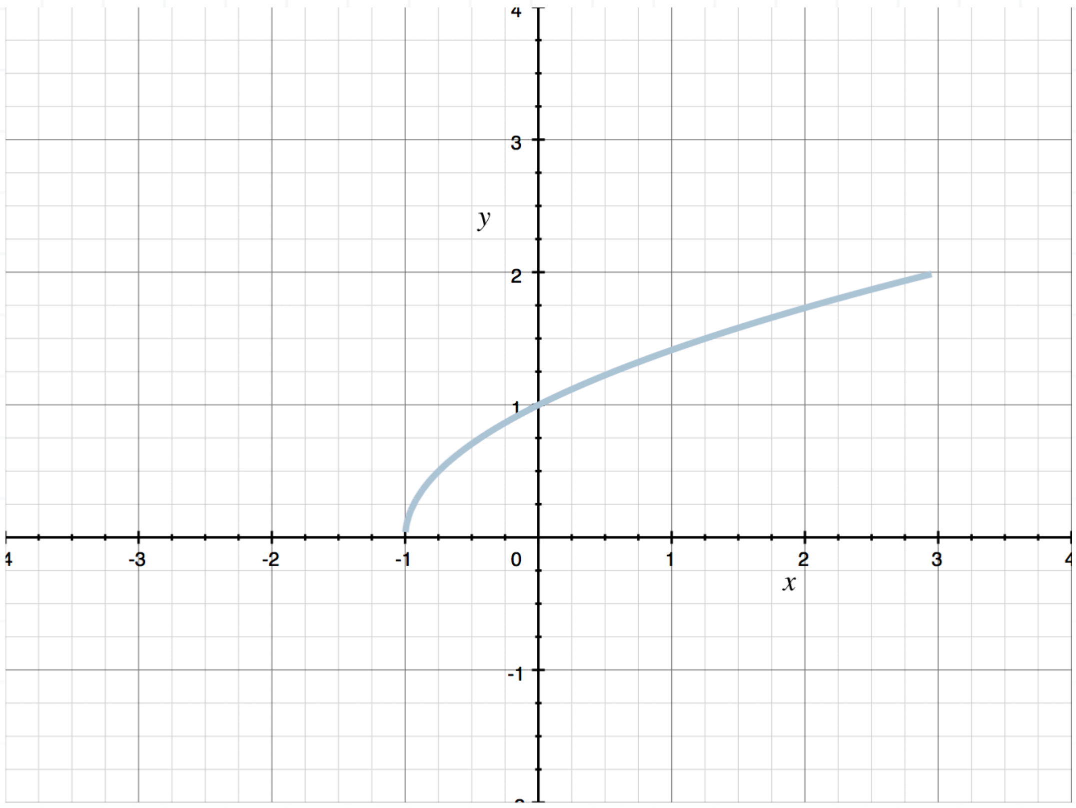
Range: $1 \leq y \leq 5$

---

Let's try another example of finding domain and range from a graph.

**Example**

What are the domain and range of the function?




The domain is all of the $x$-values defined by the sketch, so we need to look at where the function is defined horizontally in the sketch, from its left-most point to its right-most point.

The range is all of the $y$-values defined by the sketch, so we need to look at where the function is defined vertically in the sketch, from its lowest point to its highest point.

Domain: $-1 \leq x \leq 3$

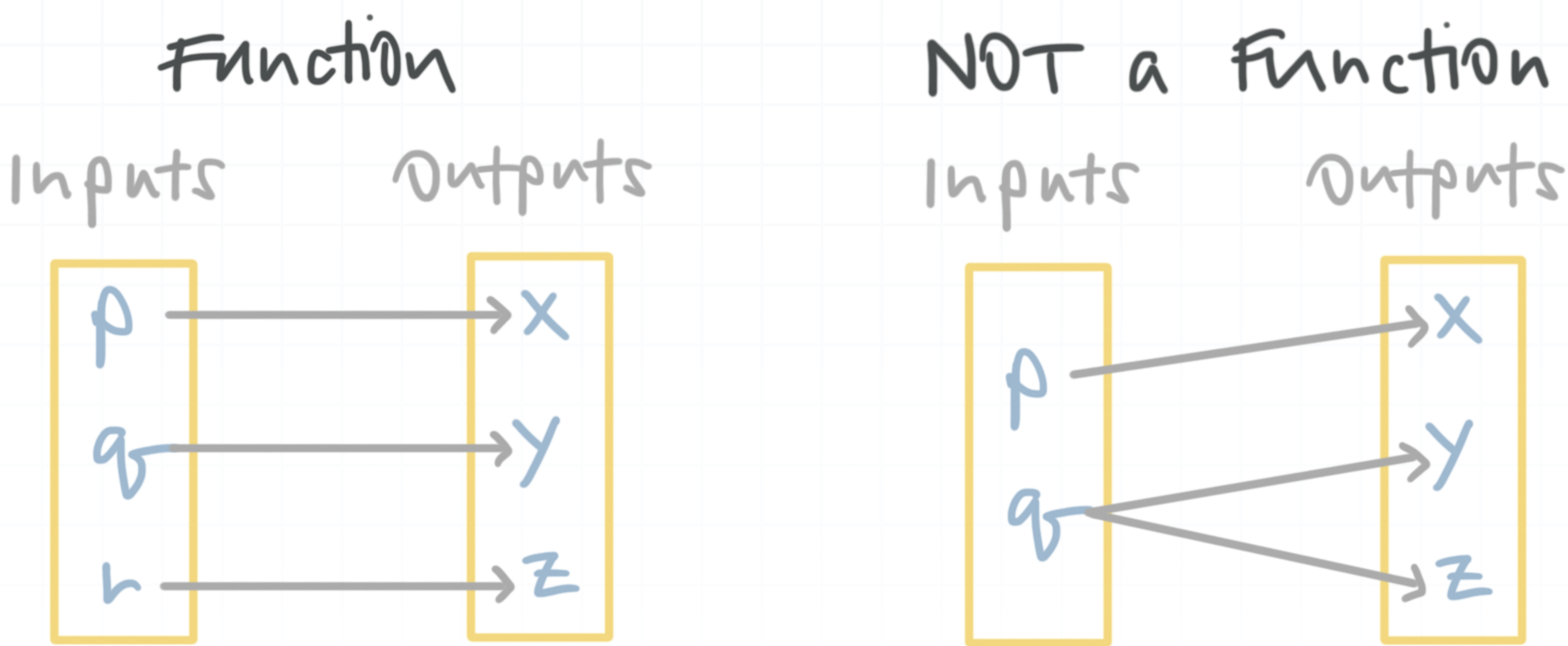
Range: $0 \leq y \leq 2$

---

# Testing for functions

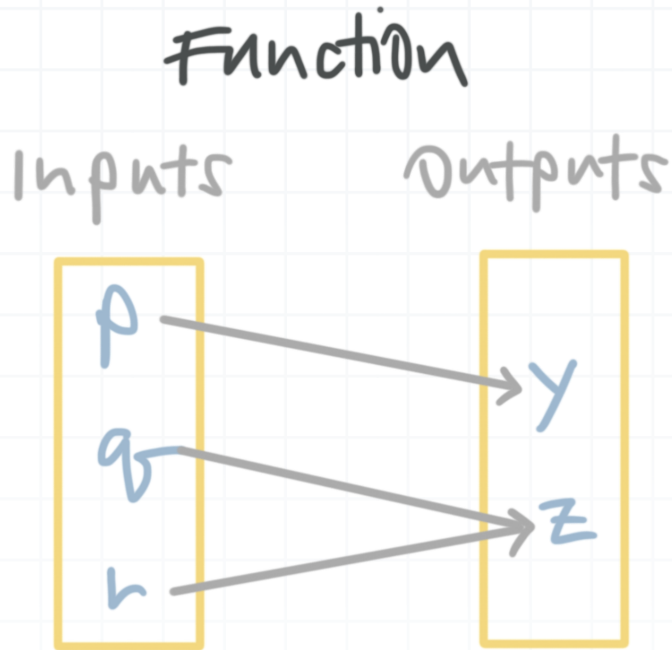
We already know that an equation is a function if every input is associated with only one output.

If we create a visual representation of this idea, it might look like this:




The image on the left could define a function because each of the inputs connects to only one output. In contrast, the image on the right is definitely not a function because there's an input that produces more than one output.

Keep in mind that there's a third scenario, which we can illustrate as




This scenario is still a function, because each input is still only associated with one output. The fact that two different inputs happen to have the same output value doesn't matter, because the one-input-one-output relationship is still preserved, so the image still represents s function.

That being said, if we're only given a set of coordinate points that represent the relation, we can still test that point set to determine whether or not it represents a function. We just need to make sure that there's only one $y$-value for every $x$-value.

Let's do an example.

## Example

Which point set, $A$ or $B$, represents a function?

$$A: (1,2), (2,4), (2,3)$$

$$B: (1, -3), (2, -4), (3, -5)$$

Set $B$ represents a function because each input has only one output. Set $A$ doesn't represent a function because the input $2$ has both an output of $3$ and an output of $4$, which means there are multiple outputs associated with the single input.

---

We can also try to determine algebraically whether an equation represents a function. Let's do an example with an equation.

**Example**

Determine algebraically whether the equation represents a function.

$$x^2 + y^2 = 1$$

If we can show that there are multiple outputs $y$ associated with a single input $x$, then we'll prove that the equation doesn't represent a function.

Let's see what's happening with the equation at $x = 0$ by plugging in $x = 0$.

$$(0)^2 + y^2 = 1$$

$$y^2 = 1$$

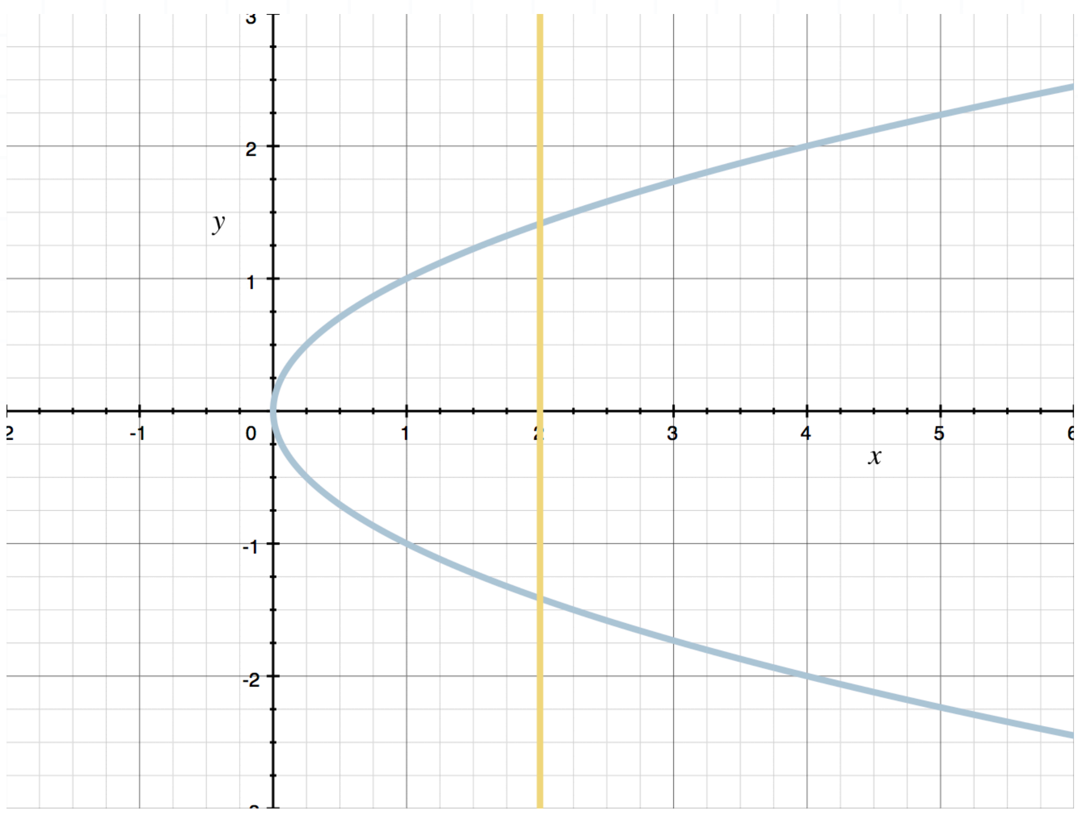
$$y = \pm 1$$

From this result, we can conclude that, at $x = 0$, $y$ can be both $1$ and $-1$. Since a function can have only one output $y$ for any input $x$, this equation can't represent a function.

# Vertical Line Test

If an equation is a function, then its graph will pass the Vertical Line Test. The **Vertical Line Test** (VLT) says that a graph represents a function if no perfectly vertical line crosses the graph more than once.

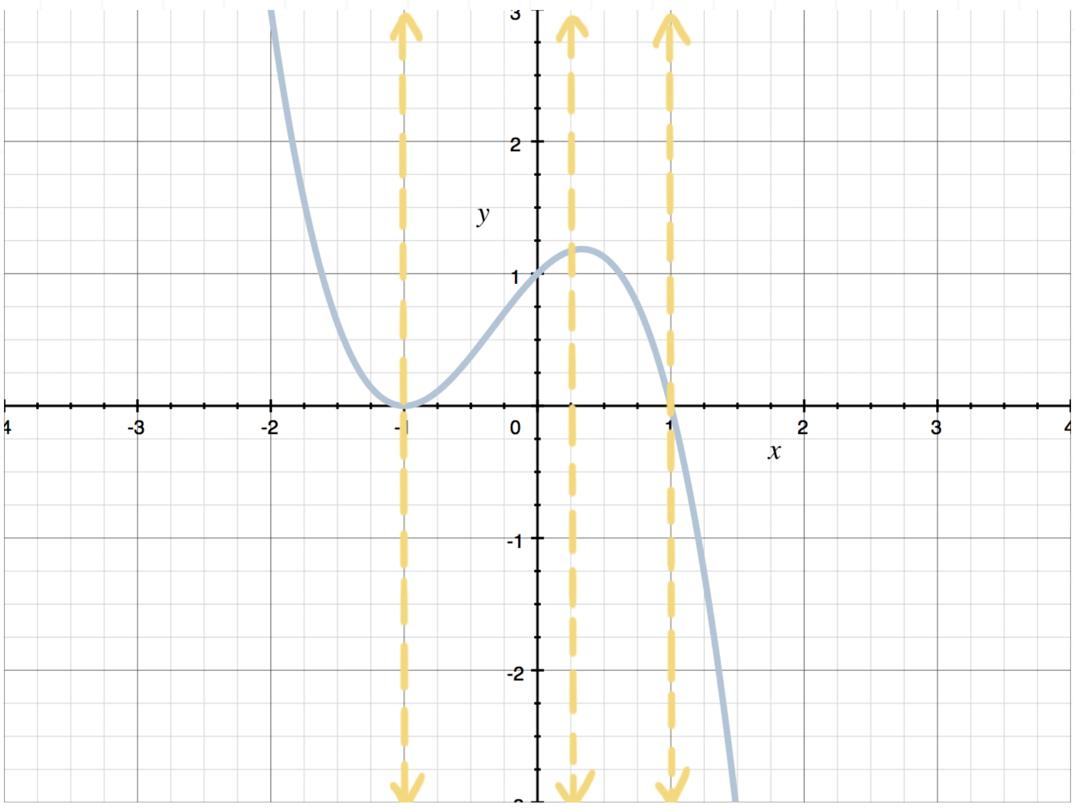
The graph below doesn't pass the Vertical Line Test because we can draw a vertical line that intersects it more than once. It takes only one vertical line intersecting the graph more than once for it to fail the Vertical Line Test.



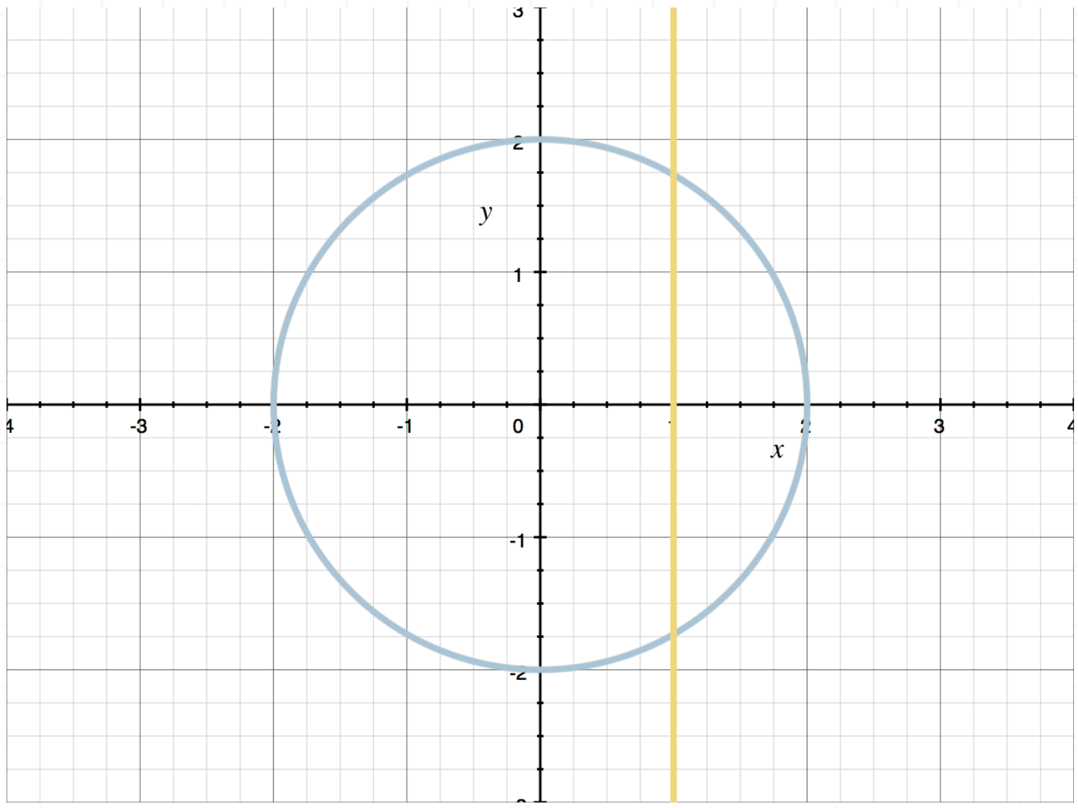

So, even if we can sketch a line or lines that only intersect the graph in one place, we only need to find one vertical line that intersects the graph multiple times in order for the graph to fail the VLT.

The reasoning behind the Vertical Line Test comes from the one-input-one-output rules for functions that we've already learned. If we can find a vertical line that crosses the graph more than once, it means there are multiple output values for the single input value, and therefore we know that the graph doesn't represent a function.

This graph passes the Vertical Line Test, so it represents a function. Any vertical line we can draw will cross the graph no more than once.




We know that the circle below doesn't represent a function, because any vertical line we draw at some $x$ that's strictly between $-2$ and $2$ (not "right at" $-2$ or $2$) will cross the graph twice, which causes the graph to fail the Vertical Line Test. In fact, circles can never represent functions, because they never pass the Vertical Line Test.




Let's do an example.

**Example**

How many times can a vertical line touch a graph in order for it to pass the Vertical Line Test?

Every vertical line can touch a graph at most once in order for the graph to pass the Vertical Line Test. If a graph passes the Vertical Line Test, it's the graph of a function.

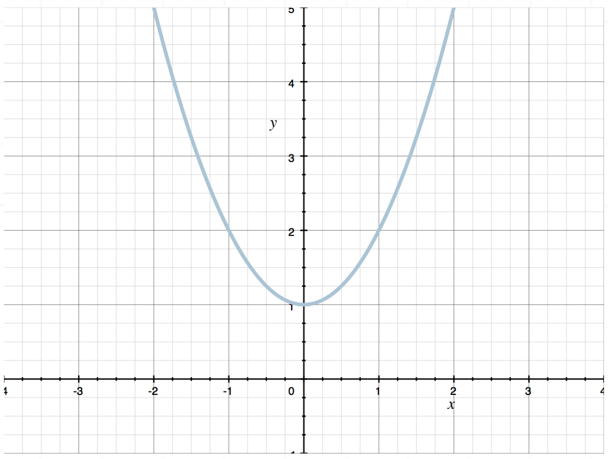
---

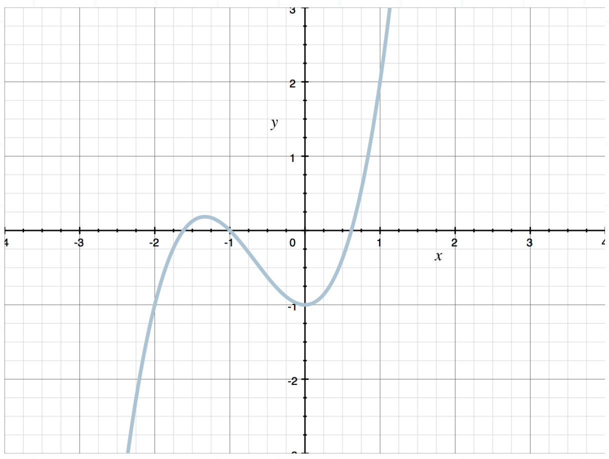
Let's do an example where we sketch a few functions that pass the Vertical Line Test.

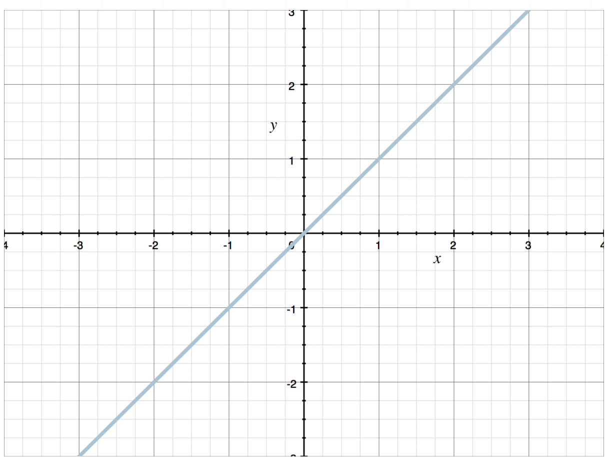
**Example**

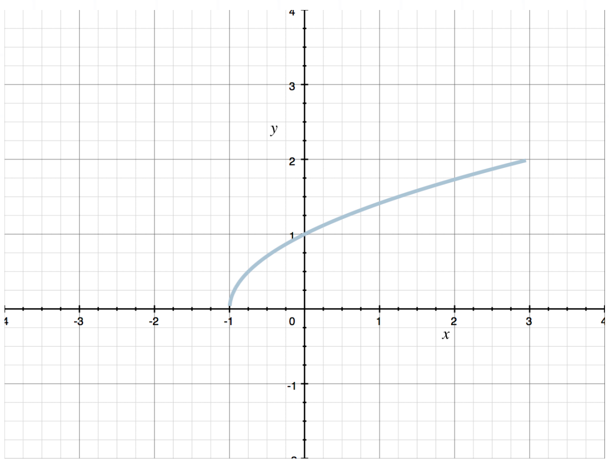
Create a graph that represents a function and explain why it's a function.

There are an infinite number of functions we could sketch, so we just need to sketch one that passes the Vertical Line Test. Any vertical line can touch the graph at most once. Below are some examples of graphs of functions.









# Sum of functions

We can actually do arithmetic operations with functions, like addition and multiplication, in the same way we might do operations with polynomials.

Let's start by looking at how to add add functions, and in the next lesson we'll look at function multiplication.

## Adding functions

To add functions, we simply add the equations together, combining terms only when they're alike. So the sum of two functions $f$ and $g$ is

$$(f+g)(x)=f(x)+g(x)$$

We'll only be able to combine terms in $f$ with terms in $g$ if we have like terms. Once we've added the functions together, we can evaluate the sum at a specific value of $x$.

Alternatively, if we're ultimately trying to find the value of the sum at a particular value of $x$, we could evaluate each function individually at that value, and then add the results.

Let's do an example so that we can see both of these routes to the solution.

---

**Example**

Find $(f+g)(3)$ if $f(x)=x^2-x+4$ and $g(x)=x-2$.

We need to find $(f+g)(3)$, which we could rewrite as $f(3)+g(3)$. So we can input $x=3$ into each function and then add the outputs.

First, let's find $f(3)$.

$$f(x)=x^2-x+4$$

$$f(3)=3^2-3+4$$

$$f(3)=9-3+4$$

$$f(3)=10$$

Now let's find $g(3)$.

$$g(x)=x-2$$

$$g(3)=3-2$$

$$g(3)=1$$

Now we can add the outputs to find the sum.

$$(f+g)(3)=f(3)+g(3)$$

$$(f+g)(3)=10+1$$

$$(f+g)(3)=11$$

We could also have added the functions first, and then plugged in $x=3$ to get the answer.

$$(f+g)(x) = (x^2 - x + 4) + (x - 2)$$

$$(f+g)(x) = x^2 - x + 4 + x - 2$$

$$(f+g)(x) = x^2 + 2$$

Evaluate the sum at $x = 3$.

$$(f+g)(3) = 3^2 + 2$$

$$(f+g)(3) = 9 + 2$$

$$(f+g)(3) = 11$$

---

Let's try another example of a sum of functions.

**Example**

Find $(g+h)(-2)$ if $g(x) = x^2 + 5x$ and $h(x) = 3 - x$.

We need to find $(g+h)(-2)$, which we could rewrite as $g(-2) + h(-2)$. So we can input $x = -2$ into the expression for each function and then add the outputs.

First, let's find $g(-2)$.

$$g(x) = x^2 + 5x$$

$$g(-2) = (-2)^2 + 5(-2)$$

$$g(-2) = 4 - 10$$

$$g(-2) = -6$$

Now let's find $h(-2)$.

$$h(x) = 3 - x$$

$$h(-2) = 3 - (-2)$$

$$h(-2) = 5$$

Now we'll find the sum of the functions at $x = -2$.

$$(g + h)(-2) = g(-2) + h(-2)$$

$$(g + h)(-2) = -6 + 5$$

$$(g + h)(-2) = -1$$

We could also have added the functions first, and then plugged in $x = -2$ to get the answer.

$$(g + h)(x) = (x^2 + 5x) + (3 - x)$$

$$(g + h)(x) = x^2 + 5x + 3 - x$$

$$(g + h)(x) = x^2 + 4x + 3$$

Evaluate the sum at $x = -2$.

$$(g + h)(-2) = (-2)^2 + 4(-2) + 3$$

$$(g + h)(-2) = 4 - 8 + 3$$

$$(g+h)(-2) = -4+3$$

$$(g+h)(-2) = -1$$

---

If the only information we have are pairs of inputs and outputs for each function, instead of their equations, there's only one way to add them, and that's to add the outputs. Suppose, for example, that we have functions $f$ and $g$ given as point sets.

$$f:(1,-9),\ (-2,8),\ (-5,16),\ (3,2)$$

$$g:(1,12),\ (-2,10),\ (-5,9),\ (3,-4)$$

Then the only way to find $(f+g)(-5)$ is to add the values of $f(-5)$ and $g(-5)$.

$$(f+g)(-5) = f(-5)+g(-5)$$

$$(f+g)(-5) = 16+9$$

$$(f+g)(-5) = 25$$

# Product of functions

Similarly to the way we can add functions to get their sum, we can also multiply functions to get their product.

## Multiplying functions

To find the product of functions, we multiply them, making sure to enclose both functions in parentheses and apply the Distributive Property appropriately.

$$(fg)(x) = f(x) \cdot g(x)$$

To find the value of the product at a specific value of $x$, we can either multiply the functions first and then evaluate the product at that value of $x$, or we can evaluate each function individually at that value, and then multiply those results.

Let's do an example so that we can see both methods.

---

**Example**

Find $(fg)(-4)$ if $f(x) = x + 2$ and $g(x) = x - 5$.

We need to find $(fg)(-4)$, which we could rewrite as $f(-4) \cdot g(-4)$. So we can substitute $x = -4$ into each function and then multiply the results.

First, let's find $f(-4)$.

$$f(x) = x + 2$$

$$f(-4) = -4 + 2$$

$$f(-4) = -2$$

Now let's find $g(-4)$.

$$g(x) = x - 5$$

$$g(-4) = -4 - 5$$

$$g(-4) = -9$$

Then the product $(fg)(-4)$ is

$$(fg)(-4) = f(-4) \cdot g(-4)$$

$$(fg)(-4) = -2 \cdot -9$$

$$(fg)(-4) = 18$$

We also could have multiplied the expressions for the functions, and then substituted $x = -4$ to get the answer.

$$(fg)(x) = (x + 2)(x - 5)$$

$$(fg)(x) = x^2 - 5x + 2x - 10$$

$$(fg)(x) = x^2 - 3x - 10$$

Evaluate the product at $x = -4$.

$$(fg)(-4) = (-4)^2 - 3(-4) - 10$$

$$(fg)(-4) = 16 + 12 - 10$$

$$(fg)(-4) = 28 - 10$$

$$(fg)(-4) = 18$$

---

Let's try another example where we find the product of functions, but don't evaluate at any specific value.

**Example**

Find $(gh)(x)$ if $g(x) = x + 6$ and $h(x) = x - 8$.

We need to find $(gh)(x)$ by multiplying the expressions for the functions. Our answer will be a new function instead of a single number, since there's no specific value of $x$ at which we're evaluating.

$$(gh)(x) = (x + 6)(x - 8)$$

$$(gh)(x) = x^2 - 8x + 6x - 48$$

$$(gh)(x) = x^2 - 2x - 48$$

---

# Even, odd, or neither

We can classify functions as even, odd, or neither even nor odd. Each of these classifications corresponds to a particular type of symmetry of the graph of the function.

In fact, it's often easiest to tell whether a function is even, odd, or neither by looking at its graph. Sometimes it's difficult or impossible to graph a function, so there is an algebraic way to check as well.

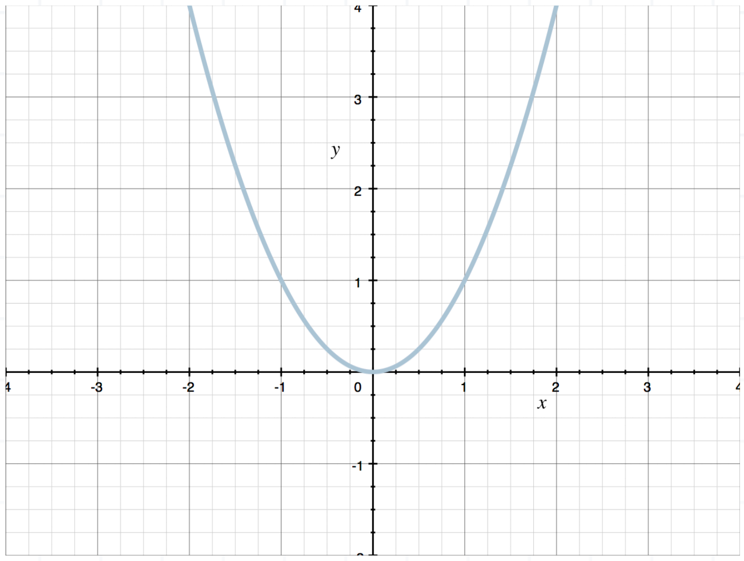
## Even functions

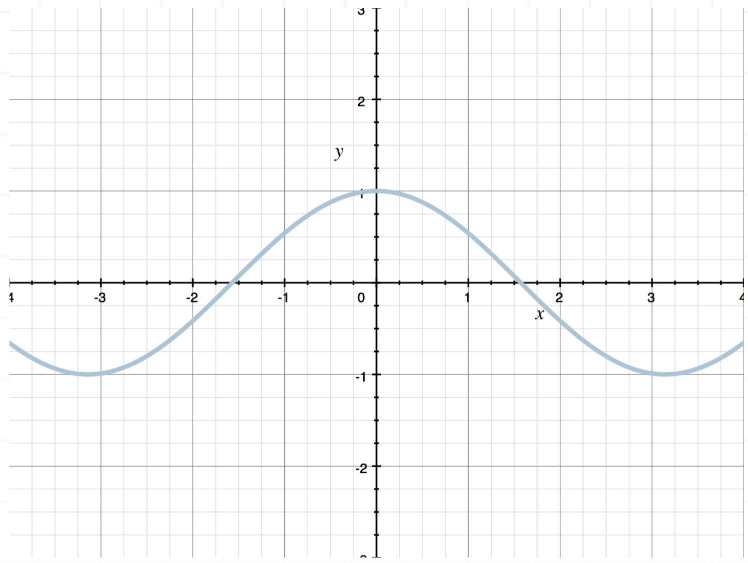
Functions that are even are symmetric with respect to the $y$-axis. When we plug $-x$ into the expression for an even function, it will simplify to the expression for the original function. This means that it doesn't matter whether we plug in $x$ or $-x$, our output will be the same.

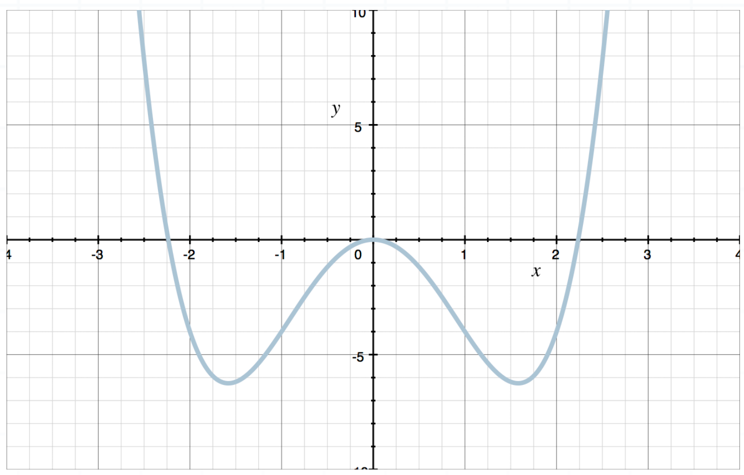
$$f(-x) = f(x)$$

What this means in terms of the graph of an even function is that the part that's to the left of the $y$-axis is a mirror image of the part that's to the right of the $y$-axis.

Below are graphs that are symmetric with respect to the $y$-axis and therefore represent even functions.







We can also identify even functions given points in a table. If a function is even, then opposite values of $x$ will have equivalent values of $y$. For instance, $x = 1$ and $x = -1$ will give the same value of $y$, $x = 2$ and $x = -2$ will give the same value of $y$, $x = 3$ and $x = -3$ will give the same value of $y$, etc.

As an example, this table of values could represent an even function.

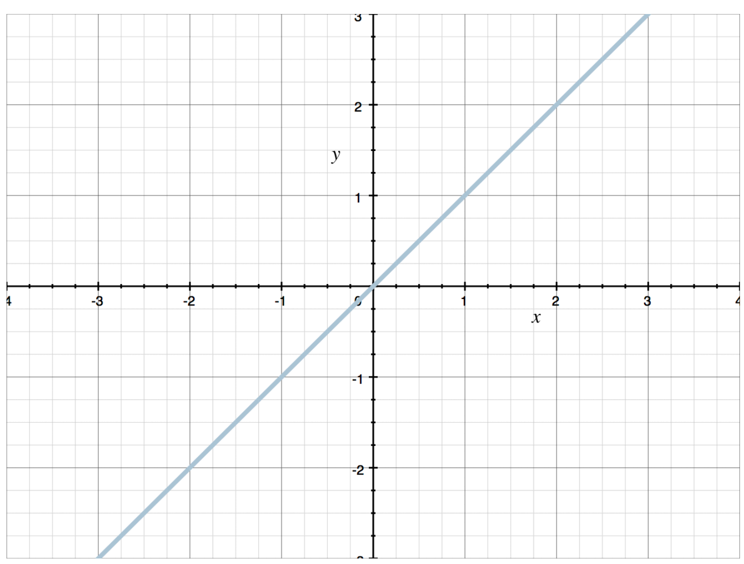
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| f(x) | 1 | -1 | 2 | 4 | 2 | -1 | 1 |

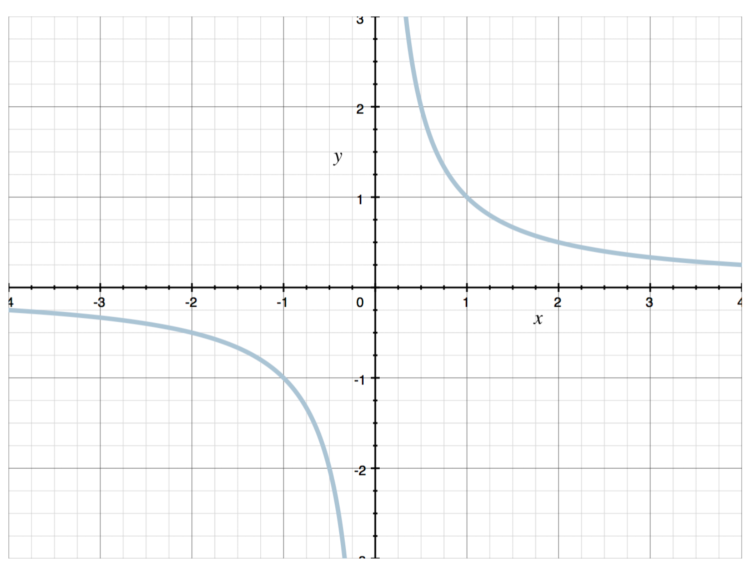
## Odd functions

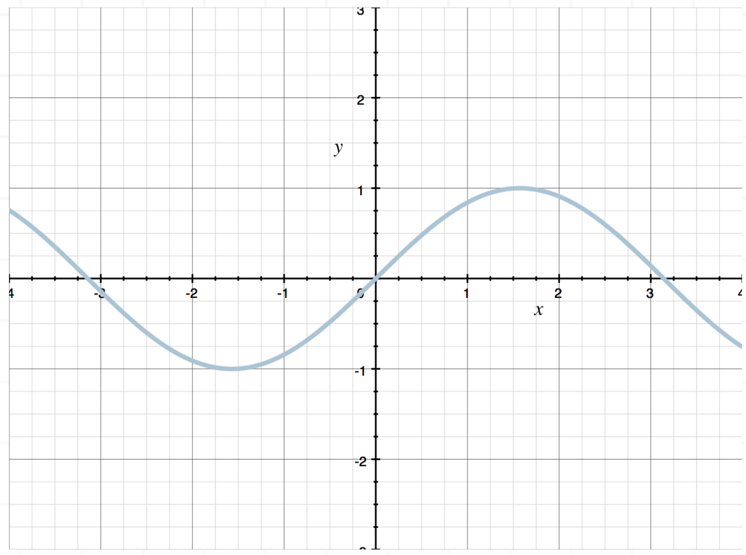
Functions that are odd are symmetric with respect to the origin. When we plug $-x$ into the expression for an odd function, it will simplify to the negative of the expression for the original function, or the expression for the original function multiplied by $-1$. This means that when we plug in $-x$, we'll get essentially the same output that we get when you plug in $x$, the only difference being that its sign will be opposite the sign of the original output.

$$f(-x) = -f(x)$$

Below are graphs that are symmetric with respect to the origin and therefore represent odd functions. Be sure to visually compare quadrants that are diagonal from each other (quadrants I and III, and quadrants II and IV). For every first-quadrant point $(x, y)$ in the graph of an odd function, there's a third-quadrant point on the graph with coordinates $(-x, -y)$. Similarly, for every second-quadrant point $(x, y)$ in the graph of an odd function, there's a fourth-quadrant point on the graph with coordinates $(-x, -y)$.





We can also identify odd functions given points in a table. If a function is odd, then opposite values of $x$ will have opposite values of $y$. For instance, $x = 1$ and $x = -1$ might give $y = -2$ and $y = 2$, $x = 2$ and $x = -2$ might give $y = 5$ and $y = -5$, $x = 3$ and $x = -3$ might give $y = -1$ and $y = 1$, etc.

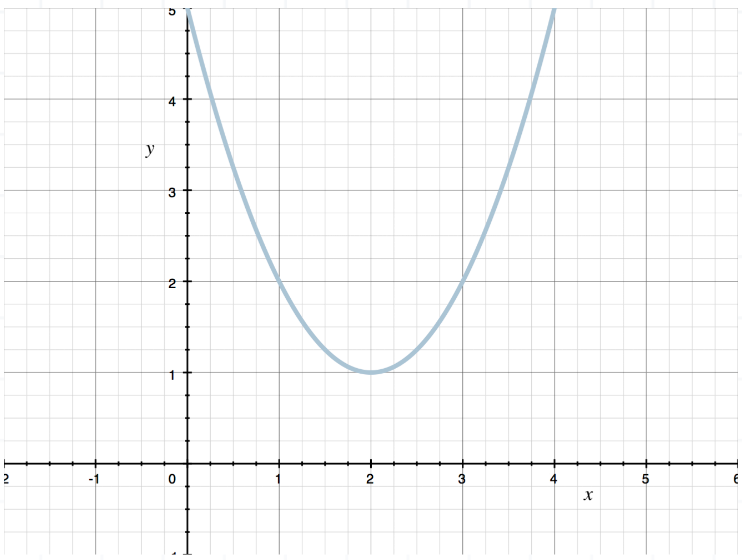
As an example, this table of values could represent an odd function.

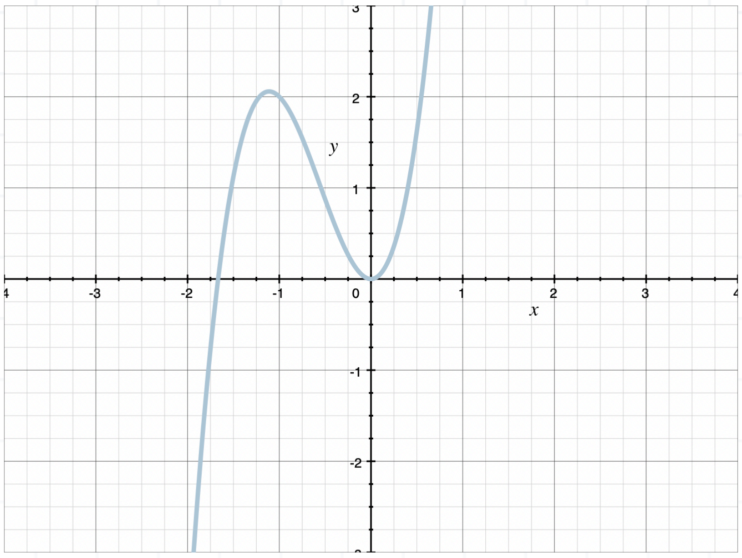
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| f(x) | -1 | -2 | 1 | 0 | -1 | 2 | 1 |

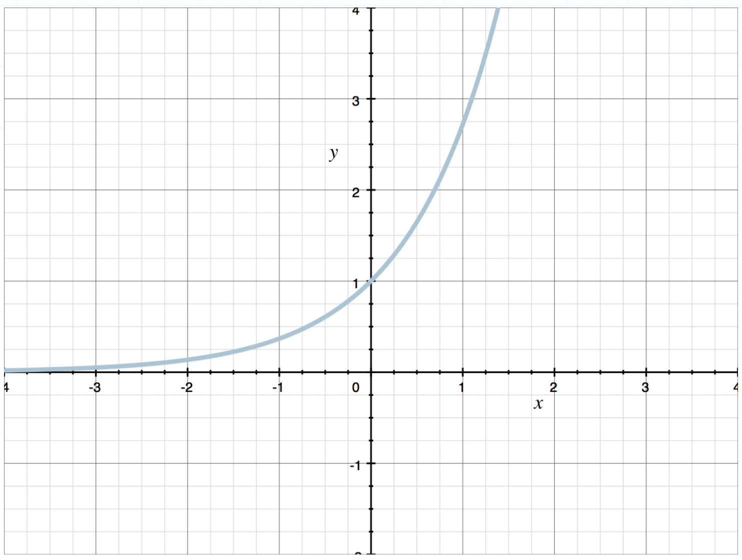
## Neither even nor odd

Functions that aren't even and aren't odd are not symmetric with respect to the $y$-axis, and also not symmetric with respect to the origin

It's possible that a graph could be symmetric with respect to the $x$-axis, but then it wouldn't pass the Vertical Line Test and therefore wouldn't represent a function.







We can also identify functions that aren't even or odd given points in a table. If a function is neither even nor odd, then opposite values of $x$ won't consistently correspond to equivalent values of $y$ or opposite values of $y$. For instance, $x = 1$ and $x = -1$ might give $y = 2$ and $y = -1$, while $x = 2$ and $x = -2$ might give $y = 3$ and $y = 2$, etc.

As an example, this table of values could represent a function that's neither even nor odd.

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| f(x) | 1 | -1 | 1 | -3 | 2 | 0 | 5 |

Let's do an example where we determine whether a function is even, odd, or neither.

### Example

Is the function even, odd, or neither?

$$f(x) = x^5 - 3x^3$$

To use algebra to classify the function, we need to find the expression for $f(-x)$, so we'll replace every $x$ (in the expression for $f(x)$) with $-x$.

$$f(-x) = (-x)^5 - 3(-x)^3$$

Remember that

$$(-x)^5 = (-1x)^5 = (-1)^5x^5$$

and

$$(-x)^3 = (-1x)^3 = (-1)^3x^3$$

Raising $-1$ to an odd power gives $-1$, so

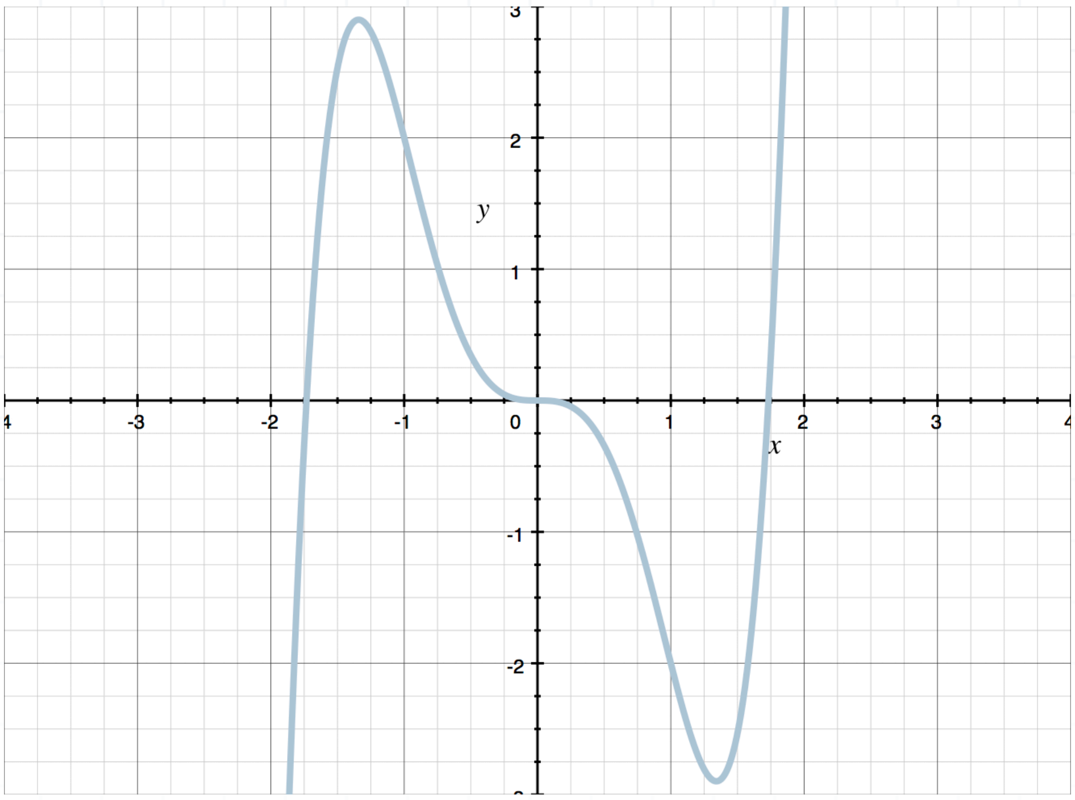
$$f(-x) = (-1)x^5 - 3(-1)x^3$$

Factor out a $-1$, and then simplify.

$$f(-x) = -1(x^5 - 3x^3)$$

$$f(-x) = -(x^5 - 3x^3)$$

Since $f(-x) = -f(x)$, the function is odd. We can see that the graph is symmetric with respect to the origin.

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Let's try another example of even, odd, or neither.

**Example**

Is the function even, odd, or neither?

$$f(x) = 5x^2 - x^4$$

To use algebra to classify the function, we need to find the expression for $f(-x)$, so we'll replace every $x$ (in the expression for $f(x)$) with $-x$.

$$f(-x) = 5(-x)^2 - (-x)^4$$

Remember that

$$(-x)^2 = (-1x)^2 = (-1)^2x^2$$

and

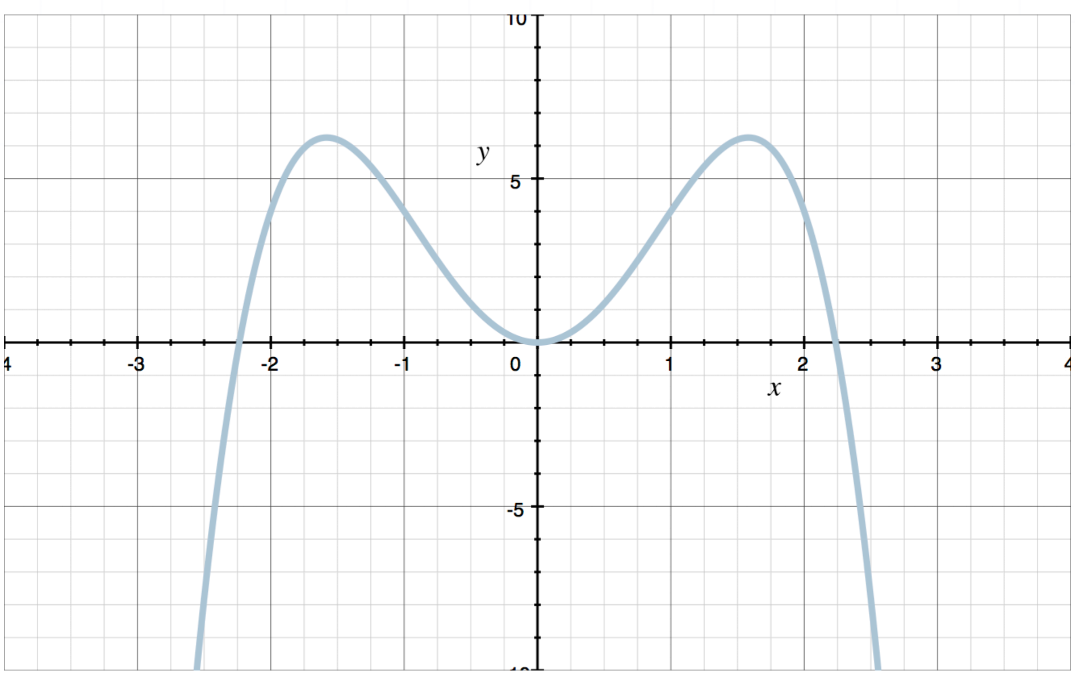
$$(-x)^4 = (-1x)^4 = (-1)^4x^4$$

Raising $-1$ to an even power gives $1$, so

$$f(-x) = 5(1)x^2 - (1)x^4$$

$$f(-x) = 5x^2 - x^4$$

Since $f(-x) = f(x)$, the function is even. We can see that the graph is symmetric with respect to the $y$-axis.



# Trichotomy

Up to now we've been dealing almost exclusively with equations, which tell us that the expressions on the left and right sides of the equals sign are equivalent.

But there are other ways to describe the relationship between two numbers or two expressions. Instead of defining them as equal, we can say that one is greater than the other, or that one is less than the other.

## The Law of Trichotomy

In fact, the **Law of Trichotomy** tells us that two numbers (or expressions) can have exactly one of three possible relationships:

- The first number is smaller than the second number, $a < b$
- The first number is greater than the second number, $a > b$
- The first number is equal to the second number, $a = b$

Given two numbers, we know that they must have exactly one of these three relationships. It's impossible for them to have more than one of these relationships at the same time, and it's also impossible that they aren't related in one of these three ways.

So, no matter which two numbers (or expressions) we choose, the Law of Trichotomy tells us that they'll take on exactly one of these three relationships.

Because of this fact, we can also make the following three statements:

- If $a$ is not greater than $b$ and also not equal to $b$, then $a$ must be less than $b$. If $a \ngeq b$, then $a < b$.

- If $a$ is not less than $b$ and also not equal to $b$, then $a$ must be greater than $b$. If $a \nleq b$, then $a > b$.

- If $a$ is not greater than $b$ and also not less than $b$, then $a$ must be equal to $b$. If $a \nless b$ and $a \ngtr b$, then $a = b$.

Let's do an example where we use the Law of Trichotomy to describe the relationship between two numbers.

**Example**

Describe the relationship between $2$ and $-7$.

Using the Law of Trichotomy, we know we can describe this relationship in three says. First, we know that $2$ and $-7$ are not equal to one another. But we can also say that $-7$ is less than $2$, or that $2$ is greater than $-7$.

$$2 \neq -7$$

$$2 > -7$$

$$-7 < 2$$

Let's do another example, but this time we'll have to make a slightly different conclusion.

**Example**

If $4x + 5 \nleq 2x + 7$, how can we rewrite the inequality?

The inequality symbol tells us that $4x + 5$ is not less than $2x + 7$, and also that $4x + 5$ is not equal to $2x + 7$.

But the Law of Trichotomy tells us that two expressions must either be equal, or that one has to be greater than the other or less than the other.

So if $4x + 5$ is not less than $2x + 7$ and also not equal to $2x + 7$, then the only possibility is that $4x + 5$ is greater than $2x + 7$. So we can rewrite the inequality statement as

$$4x + 5 > 2x + 7$$

# Inequalities and negative numbers

Now that we understand the Law of Trichotomy and the idea behind inequalities and inequality statements, let's turn toward learning to actually solve these inequalities.

## Solving inequalities

We'll solve inequalities using all the same methods we used to solve equations. In other words, just like equations, we want to follow order of operations and keep the inequality balanced as we do. We can essentially treat the inequality sign as an equals sign, and solve it as if it were an equation.

The only difference between equations and inequalities is what we have to do with the multiplication or division of negative values. When we multiply or divide both sides of an inequality by a negative value, we have to reverse the direction of the inequality.

So if we started with a less than sign ($<$) and we multiply or divide through the inequality by negative number, we need to change the less than sign $<$ to a greater than sign $>$. Likewise, if we start with $>$ and multiply or divide through by a negative number, we need to flip the inequality to $<$.

If we have a greater than or equal to sign ($\geq$), or a less than or equal to sign ($\leq$), the "equals" part of the inequality sign is unaffected when we multiply or divide by a negative number, but the less than or greater than part still flips. In other words, multiplying or dividing by a negative number changes $\geq$ to $\leq$, and changes $\leq$ to $\geq$.

Let's do an example so that we can solve an inequality, including flipping the sign.

**Example**

Solve the inequality.

$$-x + 3 > 12$$

First, subtract $3$ from both sides, just like we would if this inequality was an equation instead.

$$-x + 3 - 3 > 12 - 3$$

$$-x > 9$$

Now, to solve for $x$, we have to multiply both sides by $-1$, which means we have to change the direction of the inequality sign at the same time that we do the multiplication.

$$(-1)(-x) < 9(-1)$$

$$x < -9$$

---

Let's try another example of solving inequalities with negatives.

**Example**

Solve the inequality.

$$-2x + 4 \geq -6$$

First, subtract $4$ from both sides, just like we would if this inequality was an equation instead.

$$-2x + 4 - 4 \geq -6 - 4$$

$$-2x \geq -10$$

Now we have to divide both sides by $-2$, so we have to change the direction of the inequality sign at the same time that we do the division.

$$\frac{-2x}{-2} \leq \frac{-10}{-2}$$

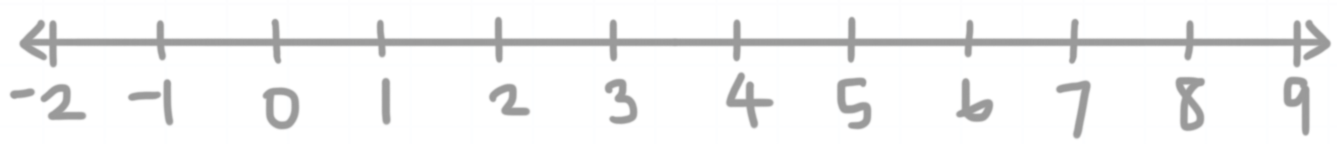
$$x \leq 5$$

# Graphing inequalities on a number line

In previous lessons we looked at how to graph points and lines in the Cartesian coordinate system, which was defined by the horizontal $x$-axis and the vertical $y$-axis.

## Number lines

In this lesson we're talking about graphing inequalities, which will be done on a number line. We can think of the number line as simply the horizontal $x$-axis from the Cartesian coordinate system, with the vertical $y$-axis stripped away. So a number line might look like this,




and we can see that it's one-dimensional, where the Cartesian coordinate system was two-dimensional.

In other words, the number line is simply a horizontal line, extending infinitely in both directions. We'll see negative numbers to the left of $0$ and positive numbers to the right of $0$, and we'll be graphing inequalities on these kinds of number lines.

## Solving inequalities

Before we can graph an inequality, we have to solve it. To make the inequality easier to graph, it's helpful to always write the solution where the variable is by itself on the left side of the inequality statement.

In other words, we'd like to get the solution into a form resembling one of these:

$x > a$ $\qquad$ $x \neq a$ $\qquad$ $x \geq a$

$x < a$ $\qquad$ $x \leq a$

If we solve an inequality and the result has the variable on the right side instead of the left side, we just need to swap the sides and reverse the sign of the inequality. For example, if we solve an inequality and get $6 > x$, we can turn it around and write it as $x < 6$.

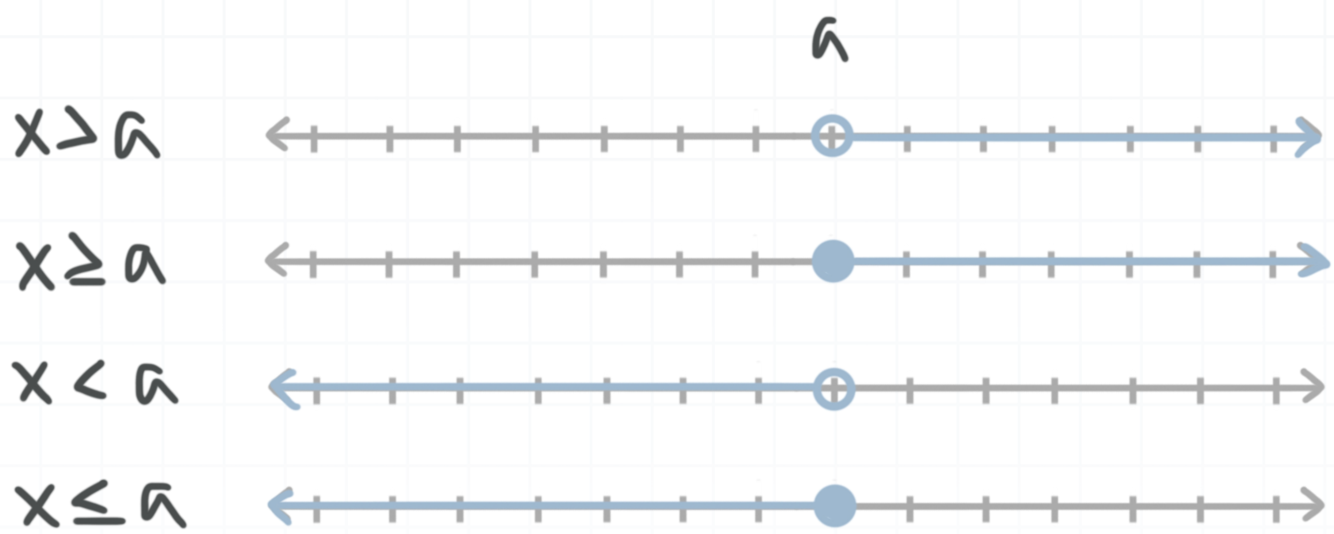
## Graphing inequalities

Once we've solve the inequality and put the variable on the left side, we can graph it on a number line.

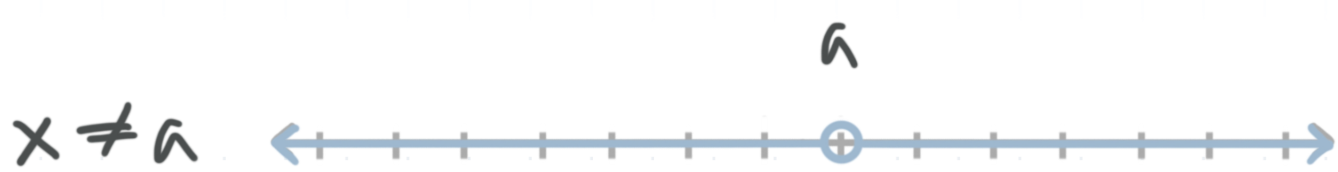
If the solution is either $x \geq a$ or $x \leq a$, then $a$ is included in the solution and we draw a solid circle at $x = a$ on the number line, and then a ray that extends from $a$ to the right or left.

If the solution is either $x > a$ or $x < a$, then $a$ is excluded from the solution and we draw an open circle at $x = a$ on the number line, and then a ray that extends from $a$ to the right or left.

To summarize, given $>$ or $\geq$, the arrow will go to the right, but if we have $<$ or $\leq$, the arrow will go to the left. If we have $>$ or $<$, the circle will be open, and if we have $\geq$ or $\leq$, the circle will be solid.




If the inequality is $x \neq a$ ("$x$ not equal to $a$"), the solution consists of all the numbers other than $a$, so we draw an open circle at $a$ and then sketch two rays, one that extends out to the left from $a$, and the other that extends out to the right from $a$.




Let's do an example with a "less than" inequality.

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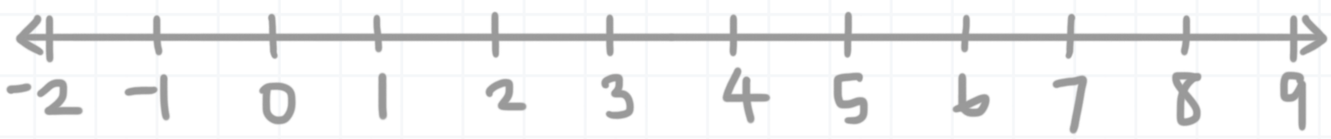
**Example**

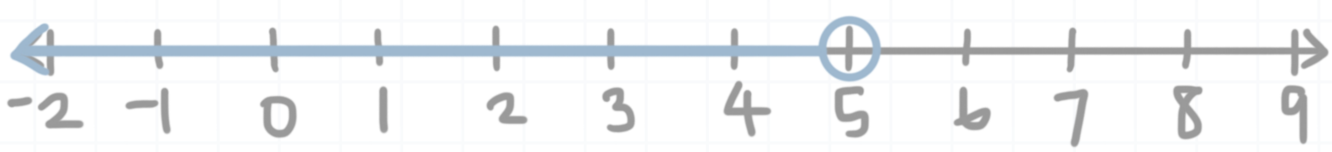
Graph the inequality on a number line.

$$x < 5$$

First, we'll draw a number line that includes $5$.




Next, we'll draw an open circle at $5$ because $5$ isn't part of the solution. Finally, we'll sketch a ray that points left from $5$, since the solution consists of all the numbers less than $5$.

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Let's try another example, this time with a "greater than or equal to" inequality.

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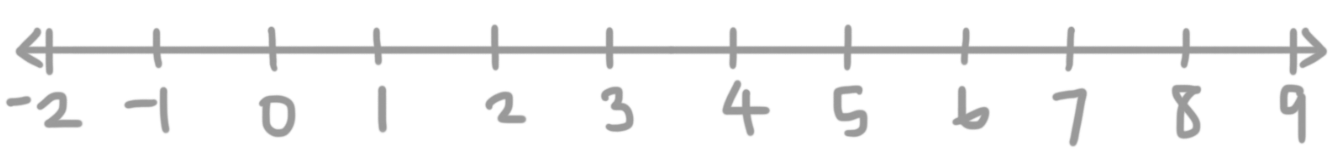
**Example**

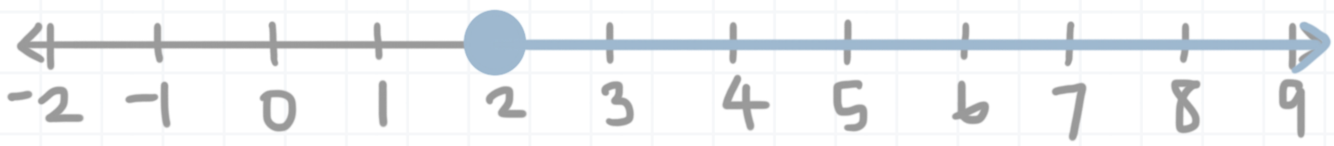
Graph the inequality on a number line.

$$x \geq 2$$

First, we'll draw a number line that includes $2$.




Next, we'll draw a solid circle at $2$ because $2$ is part of the solution. Finally, we'll sketch a ray that points right from $2$, since the solution consists of all the numbers greater than or equal to $2$.




Let’s try another example of graphing inequalities, this time where we have to simplify the inequality first.

**Example**

Graph $3(x+2)+1 \leq x-5$ on a number line.

Use the Distributive Property to simplify the parentheses.

$$3(x+2)+1 \leq x-5$$

$$3x+6+1 \leq x-5$$

$$3x+7 \leq x-5$$

Subtract $7$ from both sides.

$$3x+7-7 \leq x-5-7$$

$$3x \leq x-12$$

Subtract $x$ from both sides.

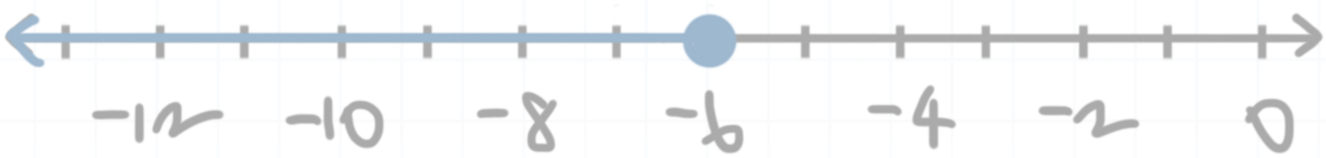
$$3x-x \leq x-x-12$$

$$2x \leq -12$$

Divide both sides by $2$. We don't have to flip the direction of the inequality because we're dividing by a positive number, not a negative number.

$$x \leq -6$$

Because the solution is a "less than or equal to" inequality, we'll draw a solid circle at $x = -6$, and then a ray that extends out to the left from that point.



# Graphing disjunctions on a number line

Now that we know how to graph inequalities, we want to say that there are two other types of inequality statements we can sketch on a number line: disjunctions and conjunctions. In this lesson, we'll focus on disjunctions.

Think about disjunctions as "or statements," like

"$x > 5$ or $x < 3$"

If we sketch this inequality statement on a number line, we get


What we see is that there are two separate pieces of the inequality statement, the $x < 3$ portion on the left, then a gap, then the $x > 5$ piece on the right.

This inequality statement tells us that a value of $x$ will satisfy the inequality (the disjunction), as long as it's *either* less than $3$ *or* greater than $5$. So $x = 2$ would satisfy the disjunction because it satisfies the $x < 3$ piece (even though it doesn't satisfy the $x > 5$ piece), and $x = 6$ would satisfy the disjunction because it satisfies the $x > 5$ piece (even though it doesn't satisfy the $x < 3$ piece).

The only values that don't satisfy this particular disjunction are the values of $x$ that put is in the "gap" we see on the number line between the pieces of the disjunction, specifically values of $x$ between $3$ and $5$, including $3$ and $5$ themselves.

Let's look at an example of how to graph a disjunction by splitting the inequality statements into two separate inequalities, graphing each one, and then graphing the overlap.

**Example**

Graph the disjunction.

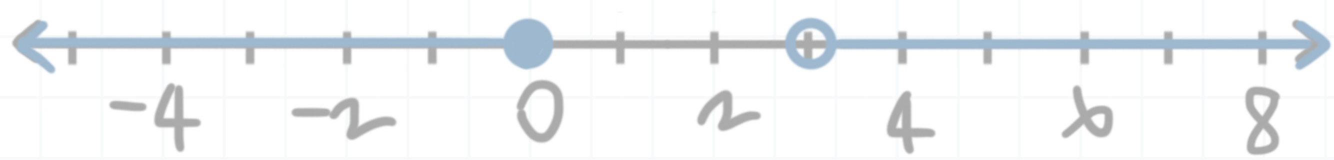
$x > 3$ or $x \leq 0$

We need to graph the disjunction of the inequalities $x > 3$ and $x \leq 0$, but first let's see how they can be graphed separately.

The graph of the inequality $x > 3$ has an open circle at $3$ (because we have a "greater than" sign), and because of the "greater than" part of the inequality, the arrow goes to the right.


The graph of the inequality $x \leq 0$ has a solid circle at $0$ (because we have a "less than or equal to" sign), and because of the "less than" part of the inequality, the arrow goes to the left.

A number is a solution to the compound inequality if the number is a solution to at least one of the inequalities. So the solution, and a sketch of the disjunction on the number line, is

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Let's try another example of graphing disjunctions.

---

**Example**

Graph the disjunction of the inequalities.

$2x - 1 > 5$ or $x + 3 < 2$

First, we'll solve and graph the two inequalities separately. To begin solving $2x - 1 > 5$, add $1$ to both sides.

$$2x - 1 > 5$$

$$2x - 1 + 1 > 5 + 1$$

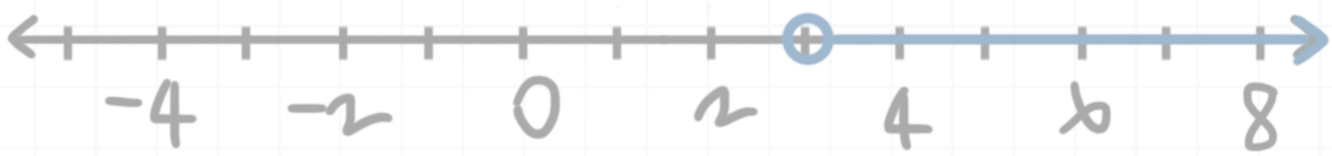
$$2x > 6$$

Now divide both sides by $2$.

$$\frac{2x}{2} > \frac{6}{2}$$

$$x > 3$$

The graph of the inequality $x > 3$ has an open circle at $3$ (because we have just a “greater than” sign), and because it’s a “greater than “inequality, the arrow goes to the right.




To begin solving the inequality $x + 3 < 2$, subtract $3$ from both sides.

$$x + 3 < 2$$

$$x + 3 - 3 < 2 - 3$$

$$x < -1$$

The graph of the inequality $x < -1$ has an open circle at $-1$ (because we have a “less than” sign), and because of the “less than” part of the inequality, the arrow goes to the left.


So the graph of the disjunction will be

# Graphing conjunctions on a number line

If disjunctions are "or statements," then conjunctions are "and statements."

Given a disjunction like "$x > 5$ or $x < 3$," values of $x$ satisfy the disjunction when they either satisfies $x > 5$ or when it satisfies $x < 3$. But given a conjunction like "$1 \leq x$ and $x \leq 8$," values of $x$ satisfy the conjunction only when they satisfy both $1 \leq x$ and $x \leq 8$.

## Conjunctions as one inequality statement

The four "simple" forms for a conjunction of two inequalities are

$$a \leq x \leq b$$

$$a \leq x < b$$

$$a < x \leq b$$

$$a < x < b$$

Each of these may look like just a single inequality, but each of them is actually two inequalities. If we break each of these apart into an "and statement," we'd rewrite them this way:

$a \leq x \leq b$ $\quad\quad$ $a \leq x$ and $x \leq b$

$a \leq x < b$ $\quad\quad$ $a \leq x$ and $x < b$

$a < x \le b$ $\quad$ $a < x$ and $x \le b$

$a < x < b$ $\quad$ $a < x$ and $x < b$

As an example, $1 \le x \le 8$ is actually the pair of inequalities $1 \le x$ and $x \le 8$.

## Graphing conjunctions

To graph a conjunction of two inequalities, we can first graph the inequalities separately and see where they overlap. Then we graph their conjunction (the overlap) on a separate number line, by including all the points that are on the graphs of both (not just one) of the inequalities (and no other points).

Let's look at an example of how to graph conjunctions by splitting the inequality into two separate inequalities, graphing each one, and then graphing the overlap.

---

**Example**

Graph the conjunction.

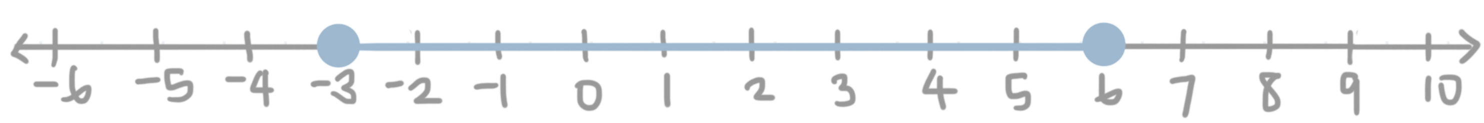
$$-3 \le x \le 6$$

We need to graph the conjunction of the inequalities $-3 \le x$ and $x \le 6$, but first let's see how they graph separately.

The inequality $-3 \leq x$ can be turned around and written as $x \geq -3$. So its graph has a solid circle at $-3$ (because we have a “greater than or equal to” sign), and because of the greater than part of the inequality, the arrow goes to the right.


The graph of the inequality $x \leq 6$ has a solid circle at $6$ (because we have a “less than or equal to” sign), and because of the less than part of the inequality, the arrow goes to the left.


To graph the conjunction of the two inequalities, we’ll graph only the overlap: a solid circle at $-3$ (because $-3$ is on the graphs of both of the inequalities), a solid circle at $6$ (because $6$ is on the graphs of both of the inequalities), and everything between $-3$ and $6$.

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Sometimes we’ll have two separate inequalities and will need to graph their conjunction. To do that, we may need to solve one or both of the inequalities before we can write their conjunction in one of the four “simple” forms.

### Example

Graph the conjunction of the inequalities $3x + 1 > -5$ and $2x - 4 \leq 6$.

First, we'll solve and graph the two inequalities separately. To begin solving $3x + 1 > -5$, subtract $1$ from both sides.

$$3x + 1 > -5$$

$$3x + 1 - 1 > -5 - 1$$

$$3x > -6$$

Now divide both sides by $3$.

$$\frac{3x}{3} > \frac{-6}{3}$$

$$x > -2$$

The graph of the inequality $x > -2$ has an open circle at $-2$ (because we have just a "greater than" sign), and because it's a greater than inequality, the arrow goes to the right.


To begin solving the inequality $2x - 4 \leq 6$, add $4$ to both sides.

$$2x - 4 \leq 6$$

$$2x - 4 + 4 \leq 6 + 4$$

$$2x \leq 10$$

Now divide both sides by $2$.

$$\frac{2x}{2} \leq \frac{10}{2}$$

$$x \leq 5$$

The graph of the inequality $x \leq 5$ has a solid circle at $5$ (because we have a "less than or equal to" sign), and because of the less than part of the inequality, the arrow goes to the left.


We can combine the solutions we found ($x > -2$ and $x \leq 5$) and write their conjunction in one of the four "simple" forms. If we take the first inequality ($x > -2$) and turn it around, we get $-2 < x$.

So the conjunction of the inequalities $x > -2$ and $x \leq 5$ can be written as

$$-2 < x \leq 5$$

The graph of the conjunction will be the overlap of the graphs of the two separate inequalities: an open circle at $-2$ (because $-2$ isn't on the graph of $-2 < x$), a solid circle at $5$ (because $5$ is on the graphs of both of the separate inequalities), and everything between $-2$ and $5$.


---

Let's try another example of graphing conjunctions.

**Example**

Graph the values of $x$ that satisfy the following inequalities.

$x \geq 5$ and $x \neq 7$

We'll draw a solid circle at $5$, because $5$ is part of the solution, then we'll sketch the solution extending out to the right from that point, since the solution includes all the numbers greater than $5$.


But we need to exclude $x = 7$, which means the arrow will go to the right, but it'll be interrupted by an open circle at $x = 7$.


---

One thing we should be aware of is that there are pairs of inequalities whose solutions have nothing in common.

For example, suppose we were given two inequalities, $x > 8$ and $x < -1$. In that case, the solution of their conjunction consists of all the numbers that are (simultaneously) greater than $8$, and at the same time less than $-1$. Well, no such number exists (no number that's greater than $8$ is also less than $-1$), so the graph of the conjunction of the inequalities $x > 8$ and $x < -1$ is just an empty number line, with nothing drawn on it.

# Graphing inequalities in the plane

In this lesson we'll move away from the number line and look at how to graph linear inequalities in a coordinate plane.

## Two steps for graphing an inequality

To graph a linear inequality, we start by drawing the boundary line.

- The boundary line will be dashed if the inequality is $<$ or $>$, which indicates that the boundary line isn't part of the graph of the inequality.

- The boundary line will be solid if the inequality is $\leq$ or $\geq$, which indicates that the boundary line is part of the graph of the inequality.

After we draw the boundary line, we'll shade in the side of the line that satisfies the inequality.

- Shade above the line if we have a $>$ or $\geq$ inequality.

- Shade below the line if we have a $<$ or $\leq$ inequality.

Sometimes trying to determine which side of the line to shade can be a little confusing. One technique we can use to make it a little easier is to pick a point that's not on the line and substitute it into the inequality. If the inequality is true, then we shade part of the graph that contains that point.

If the inequality is false, then we shade part of the graph that doesn't contain that point.

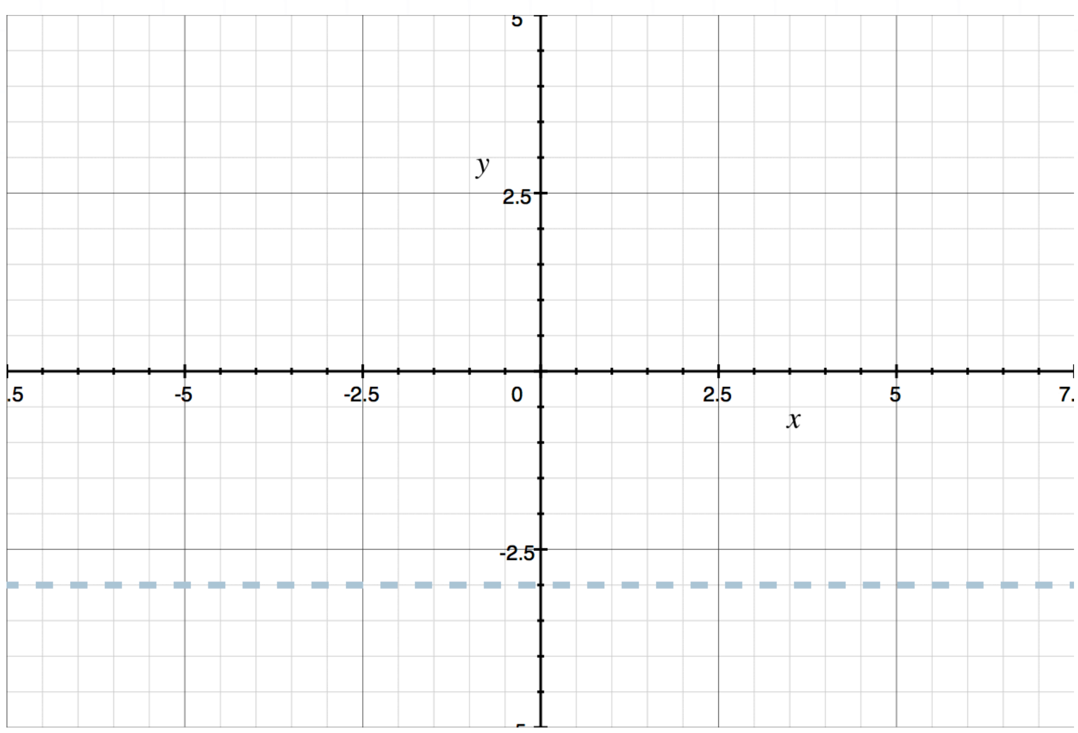
Let's do some examples so that we can get the idea.

**Example**

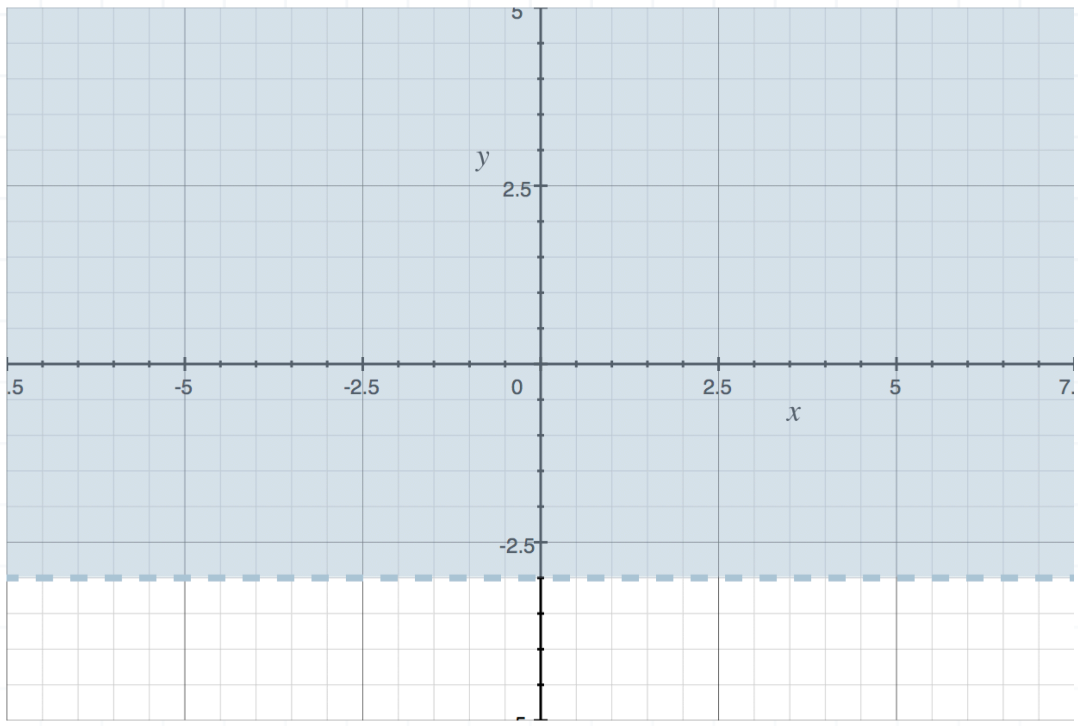
Graph the inequality.

$$y > -3$$

Let's begin by drawing the boundary line $y = -3$ with a dashed line since the sign is $>$.




Now because we have the $>$ symbol, we need to shade above the line.



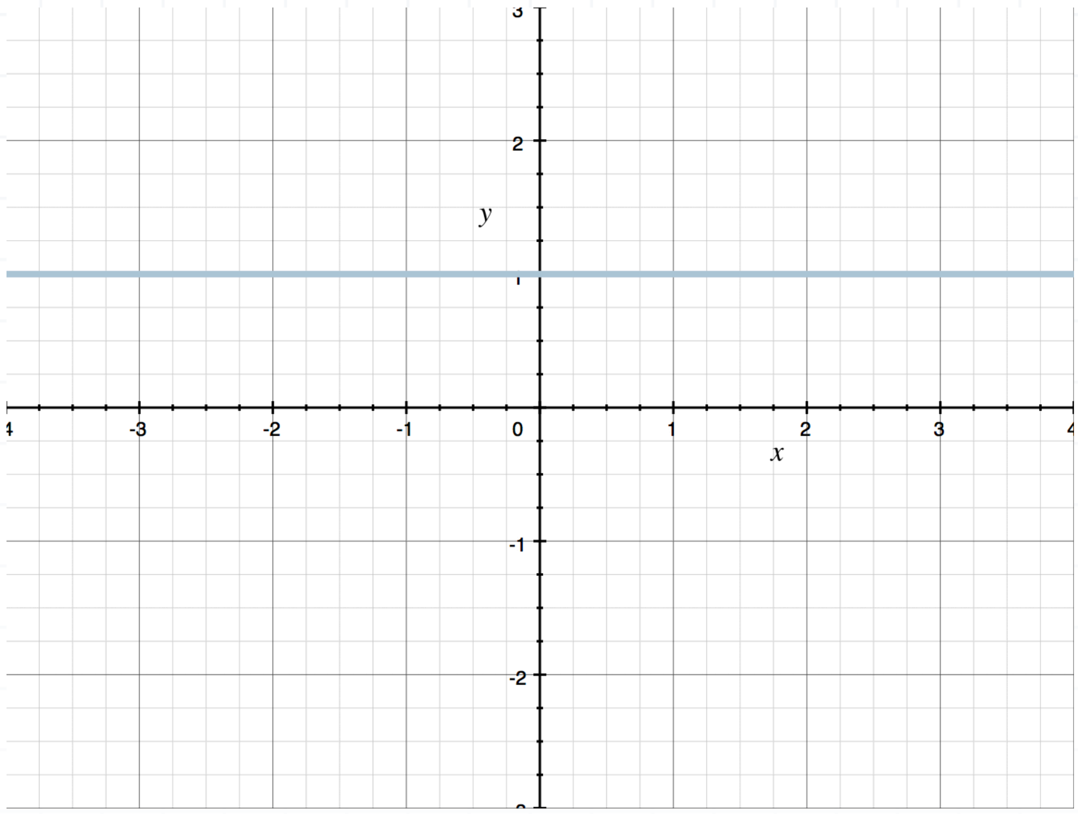

Let's try another example.

**Example**

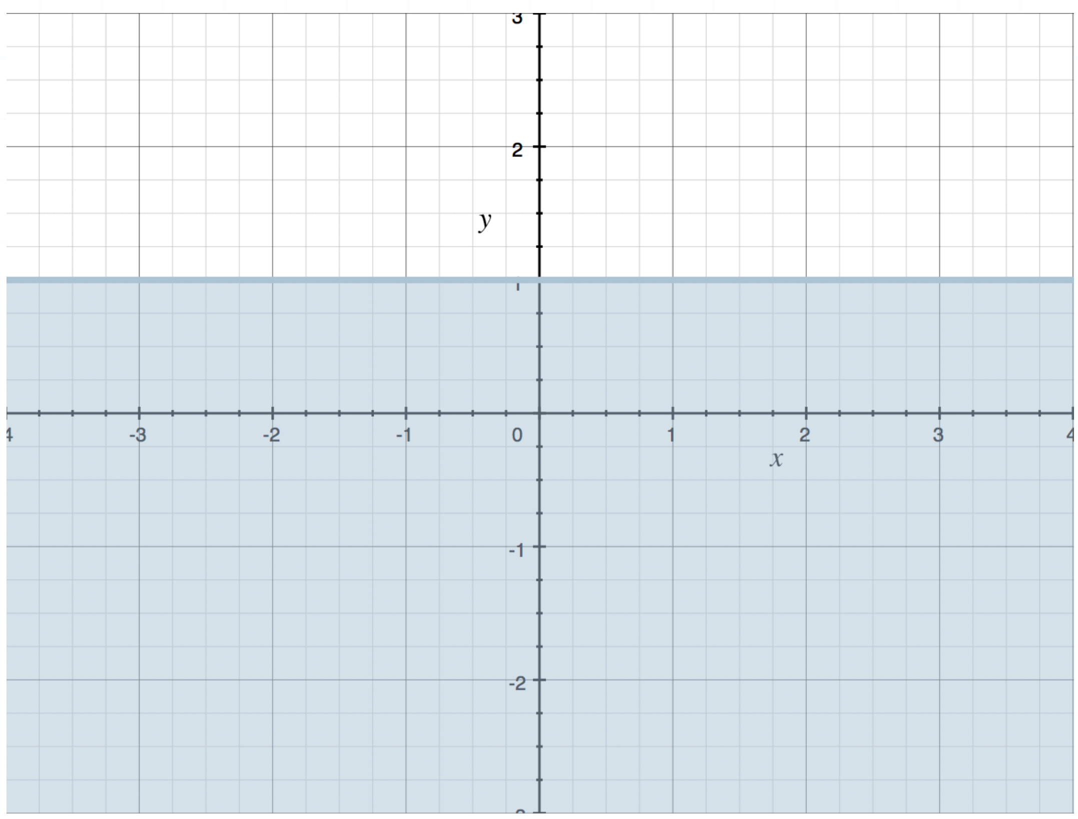
Graph the inequality.

$$y \leq 1$$

This time we start with a solid line at $y = 1$ because we have the $\leq$ sign.




And because we have the $\leq$ sign, we'll shade below the boundary line.

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The boundary line in these last two examples has been perfectly horizontal. Now let's look at a linear inequality where the boundary line isn't horizontal.

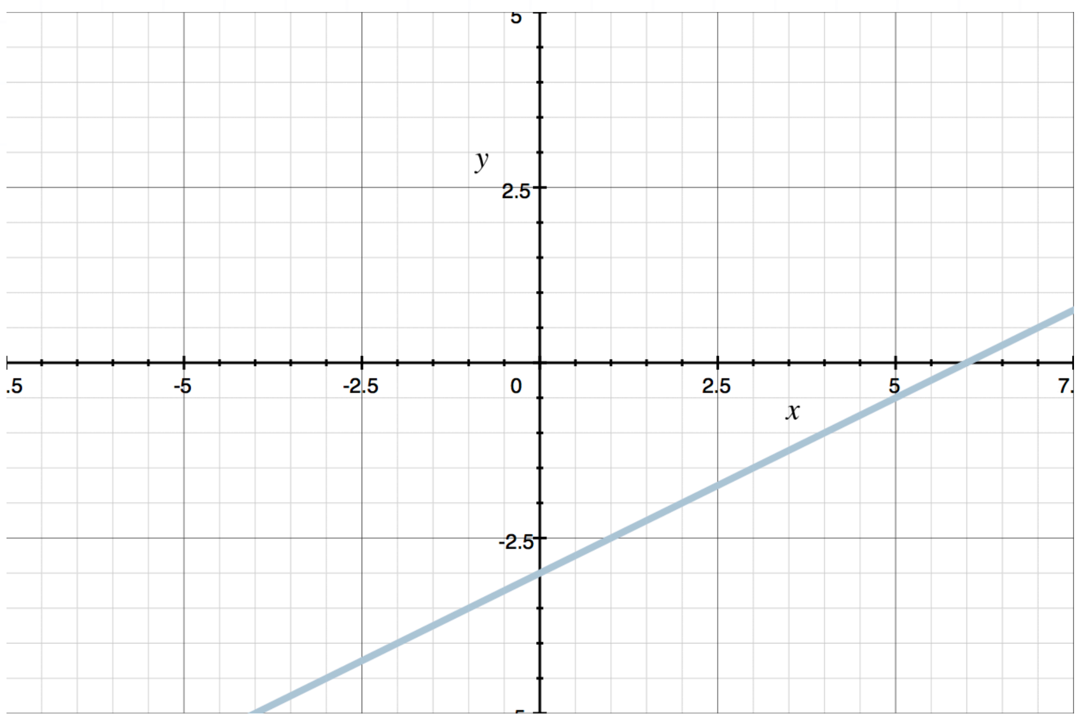
**Example**

Graph the inequality.

$$y \geq \frac{1}{2}x - 3$$

Begin by drawing the solid boundary line, since we have the $\geq$ symbol. To do this, we'll start with the $y$-intercept, which has coordinates $(0, -3)$. Then we'll use the slope to count up $1$ and over $2$ to the right to place a second point at $(2, -2)$.

Then we'll draw a solid line that passes through $(0, -3)$ and $(2, -2)$.



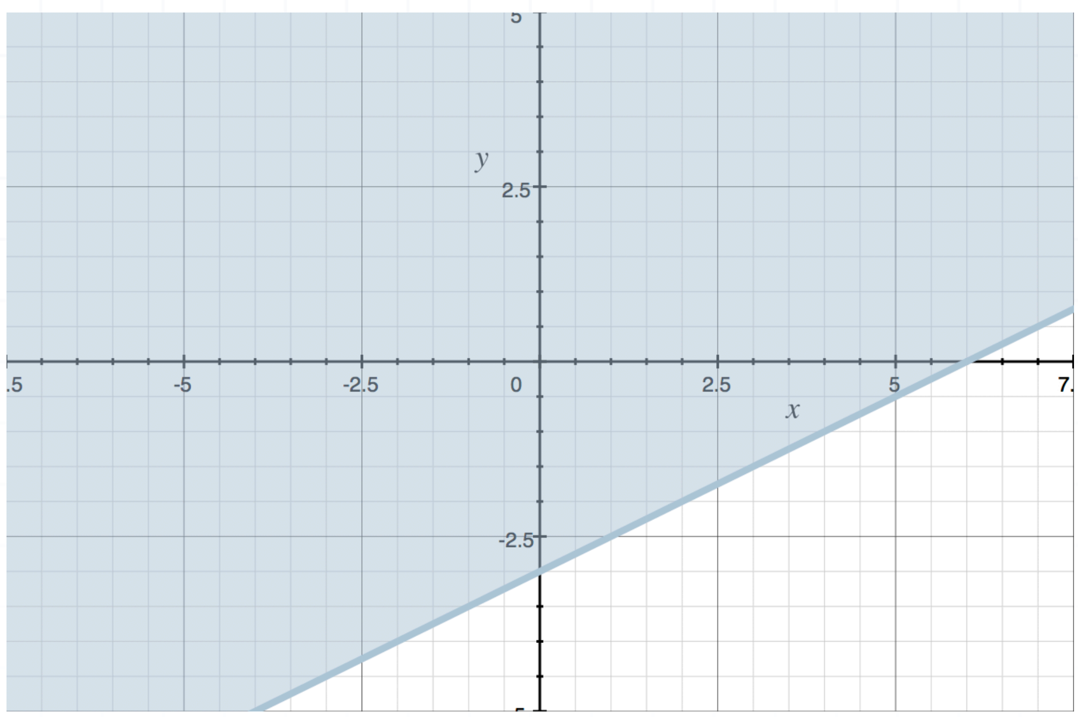

We can shade above the line because we have a $\geq$ symbol. Alternatively, let's pick the point $(0,0)$ and substitute it into the inequality.

$$y \geq \frac{1}{2}x - 3$$

$$0 \geq \frac{1}{2} \cdot 0 - 3$$

$$0 \geq -3$$

Since this inequality is true, we know $(0,0)$ satisfies the inequality and therefore that we need to shade on the side of the line that contains $(0,0)$.



# Absolute value equations

The **absolute value** of any real number $x$, defined as $|x|$, is the distance of $x$ from $0$ (or from the origin). For instance, $|2| = 2$ and $|-2| = 2$.

In other words, opposite values of $x$ have the same absolute value because they're both equally distant from $0$.

To describe absolute value in simple terms, taking the absolute value of something that's positive won't change the value of that positive thing. But taking the absolute value of something that's negative will change that negative thing into a positive thing. It just removes the negative.

## Equations with one absolute value

Given an equation $|f(x)| = a$, with $a > 0$, then $f(x) = a$ or $f(x) = -a$. It's important to say that, even though $|f(x)|$ is a distance, which by definition can't be negative, an absolute value equation can have negative solution, which is why it's possible to find $f(x) = -a$.

- If $a > 0$, then $|f(x)| = a$ has two solutions.
- If $a = 0$, then $|f(x)| = a$ has one solution.
- If $a < 0$, then $|f(x)| = a$ has no solution.

To solve an absolute value equation, we'll follow three steps.

1. Isolate the absolute value expression on one side of the equation.

2. Check the value of $a$. If $a > 0$, then set up and solve two equations, $f(x) = a$ and $f(x) = -a$. If $a = 0$, set up the equation $f(x) = 0$. And if $a < 0$, we know the equation has no solutions.

3. For any values we find in Step 2, verify that they satisfy the original absolute value equation.

Let's look at an example so that we can see how to work through these steps to solve the absolute value equation.

---

**Example**

Solve $|2x - 4| = 4$.

Set up the two related equations,

$2x - 4 = 4$ $\qquad$ $2x - 4 = -4$

then solve both equations for $x$.

$2x - 4 = 4$ $\qquad$ $2x - 4 = -4$

$2x = 8$ $\qquad$ $2x = 0$

$x = 4$ $\qquad$ $x = 0$

Now that we have these values, we'll plug them into the original absolute value equation to see whether they satisfy it.

$|2x - 4| = 4$ $\qquad$ $|2x - 4| = 4$

$$|2(4)-4|=4 \qquad |2(0)-4|=4$$

$$|8-4|=4 \qquad |0-4|=4$$

$$|4|=4 \qquad |-4|=4$$

Because both equations are true, we can conclude that both $x=4$ and $x=0$ are solutions to the absolute value equation.

---

Let's try one more example with one absolute value in the equation.

**Example**

Solve $2|2x+1|-3=7$.

First, isolate the absolute value expression on the left side by adding $3$ to both sides of the equation,

$$2|2x+1|-3+3=7+3$$

$$2|2x+1|=10$$

and dividing both sides by $2$.

$$\frac{2|2x+1|}{2}=\frac{10}{2}$$

$$|2x+1|=5$$

Now set up the two related equations,

$$2x + 1 = 5 \qquad\qquad 2x + 1 = -5$$

then solve both equations for $x$.

$$2x + 1 = 5 \qquad\qquad 2x + 1 = -5$$

$$2x = 4 \qquad\qquad 2x = -6$$

$$x = 2 \qquad\qquad x = -3$$

Now that we have these values, we'll plug them into the original absolute value equation to see whether they satisfy it.

$$2|2x + 1| - 3 = 7 \qquad\qquad 2|2(-3) + 1| - 3 = 7$$

$$2|2(2) + 1| - 3 = 7 \qquad\qquad 2|2(2) + 1| - 3 = 7$$

$$|5| = 5 \qquad\qquad |-5| = 5$$

Because both equations are true, we can conclude that both $x = 2$ and $x = -3$ are solutions to the absolute value equation.

---

## Equations with two absolute values

Now we want to consider equations that contain two absolute value expressions, like $|m| = |n|$.

In an equation like this one, $m$ and $n$ are either equal to each other or opposites (negatives) of each other. So if $|m| = |n|$, then

$m = n$ or $m = -n$

Let's try an example where we work through solving an equation that contains two absolute value expressions.

**Example**

Solve $|-2x+1| = |4x-5|$.

Split the inequality into two separate equations, one where the expressions are equal to each other, and one where the expressions are opposites of each other.

$-2x + 1 = 4x - 5$

$-2x + 1 = -(4x - 5)$

Now we'll solve equation equation for $x$.

$-2x + 1 + 5 = 4x - 5 + 5$

$-2x + 6 = 4x$

$-2x + 2x + 6 = 4x + 2x$

$6 = 6x$

$\frac{6}{6} = \frac{6}{6}x$

$x = 1$

$-2x + 1 = -4x + 5$

$-2x + 1 - 1 = -4x + 5 - 1$

$-2x = -4x + 4$

$-2x + 4x = -4x + 4x + 4$

$2x = 4$

$\frac{2}{2}x = \frac{4}{2}$

$x = 2$

Now that we have these values, we'll plug them into the original absolute value equation to see whether they satisfy it.

$$|-2x+1| = |4x-5| \qquad |-2x+1| = |4x-5|$$

$$|-2(1)+1| = |4(1)-5| \qquad |-2(2)+1| = |4(2)-5|$$

$$|-1| = |-1| \qquad |-3| = |3|$$

Because both equations are true, we can conclude that both $x = 1$ and $x = 2$ are solutions to the absolute value equation.

# Absolute value inequalities

Solving absolute value inequalities is really similar to solving absolute value equations.

## Solving absolute value inequalities

We'll always start by isolating the absolute value expression on one side of the inequality.

Once we have the absolute value isolated on one side, we need to consider the sign of the opposite side of the inequality.

If the value on the other side is negative, the inequality either has no solutions, or the solution is the set of all real numbers.

| Inequality | Solution |
|---|---|
| $\vert$absolute value$\vert$ < negative | No solution |
| $\vert$absolute value$\vert$ > negative | Solution is all real numbers |

If the value on the other side is positive, we need to rewrite the inequality as a compound inequality.

| Inequality | Solution |
|---|---|
| $\vert$absolute value$\vert$ < positive $a$ | Conjunction |
| | $-a <$ absolute value $< a$ |

| | |
|---|---|
| $\vert \text{absolute value} \vert >$ positive $a$ | Disjunction |
| | absolute value $< -a$ |
| | or absolute value $> a$ |

Once we have the correct inequality statement(s) set up, all we have left to do is solve them. Let's work through an example.

### Example

Solve the absolute value inequality.

$$|4x+5|+3<1$$

First, we'll isolate the absolute value on one side of the inequality. In this case, we can do that by subtracting $3$ from both sides.

$$|4x+5|<-2$$

This is the "$\vert$absolute value$\vert$ $<$ negative" case. We know the left side will be positive, and the right side is negative, and it can't be true that we'll have some positive value less than a negative value.

So there are no values that satisfy the inequality, and we can say that it has no solution.

---

Let's try another example.

**Example**

Solve the absolute value inequality.

$$|2x-5|+2\geq 5$$

Isolate the absolute value on one side of the inequality.

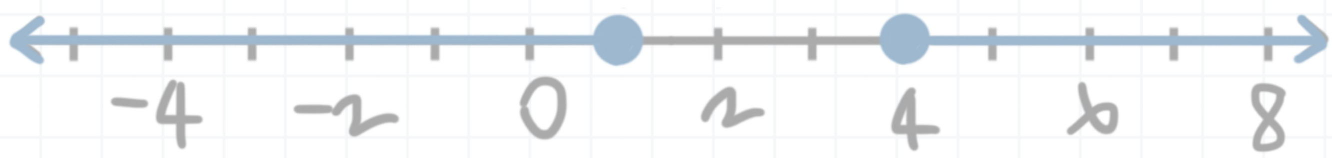
$$|2x-5|+2-2\geq 5-2$$

$$|2x-5|\geq 3$$

This is the "|absolute value| > positive $a$" case, which means the solution is a disjunction, and we can write the set of solutions as

| | | |
|---|---|---|
| absolute value $< -a$ | or | absolute value $> a$ |
| $2x-5\leq -3$ | or | $2x-5\geq 3$ |
| $2x-5+5\leq -3+5$ | or | $2x-5+5\geq 3+5$ |
| $2x\leq 2$ | or | $2x\geq 8$ |
| $\frac{2}{2}x\leq\frac{2}{2}$ | or | $\frac{2}{2}x\geq\frac{8}{2}$ |
| $x\leq 1$ | or | $x\geq 4$ |

We can sketch the solution set as a disjunction on the number line.




Let’s do one more example, this time where the solution is a conjunction.

**Example**

Solve the absolute value inequality.

$$|3x - 1| + 3 < 6$$

Isolate the absolute value on one side of the inequality.

$$|3x - 1| + 3 - 3 < 6 - 3$$

$$|3x - 1| < 3$$

This is the “$|$absolute value$|$ < positive $a$” case, which means the solution is a conjunction, and we can write the set of solutions as

$$-3 < 3x - 1 < 3$$

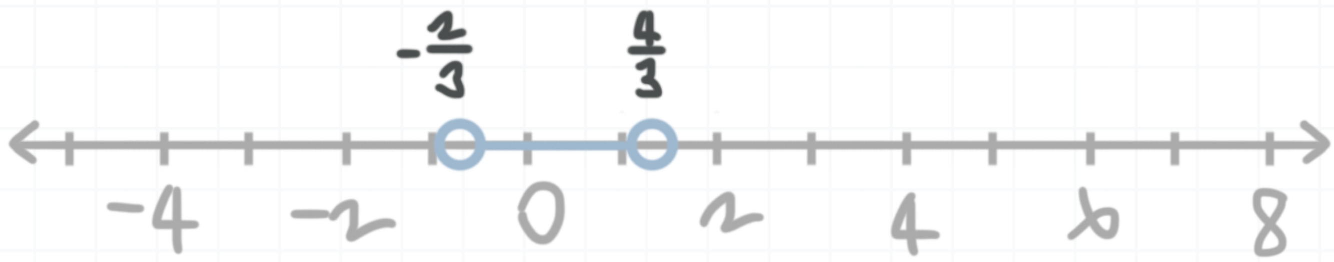
$$-3 + 1 < 3x - 1 + 1 < 3 + 1$$

$$-2 < 3x < 4$$

$$-\frac{2}{3} < \frac{3}{3}x < \frac{4}{3}$$

$$-\frac{2}{3} < x < \frac{4}{3}$$

We can sketch the solution set as a conjunction on the number line.



# Two-step problems

Two-step problems are problems in which we need to solve an equation for the value of a variable, but then also use the solution to evaluate some other expression that depends on that variable.

In other words, we'll start by solving one equation like we normally would, using order of operations, keeping the equation balanced, etc. This will be our first of the two steps.

Then our second step will be to take the solution from that equation and plug it into a different expression, in order to find the value of that expression at the solution of the first equation.

Let's do an example so that we can see these two steps together.

---

**Example**

If $6x - 4 = 8$, what is $x + 3$?

First, solve the equation $6x - 4 = 8$ using order of operations.

$$6x - 4 + 4 = 8 + 4$$

$$6x = 12$$

Divide both sides by $6$.

$$\frac{6x}{6} = \frac{12}{6}$$

$$x = 2$$

This is where we would normally stop, because we found the solution to the first equation. But we're not done! We've been asked to find the value of $x + 3$. We now know $x = 2$, so we need to substitute $x = 2$ into the second expression.

$$x + 3$$

$$2 + 3$$

$$5$$

So we know that the value of $x + 3$ must be $5$, because we were able to find $x = 2$ from the first equation.

---

Let's try one more example of a two-step problem.

---

**Example**

If $-2(3x + 5) = -34$, what is $6x - 7$?

Solve the first equation using order of operations.

$$-2(3x + 5) = -34$$

$$\frac{-2(3x + 5)}{-2} = \frac{-34}{-2}$$

$$3x + 5 = 17$$

Subtract $5$ from both sides.

$$3x + 5 - 5 = 17 - 5$$

$$3x = 12$$

Divide both sides by $3$.

$$\frac{3x}{3} = \frac{12}{3}$$

$$x = 4$$

We could have also found this value by distributing the $-2$ across the parentheses first, instead of starting by dividing by $-2$.

$$-2(3x + 5) = -34$$

$$-6x - 10 = -34$$

$$-6x = -24$$

$$x = 4$$

Either way, we've finished the first step, now our second step is to plug this solution into the expression $6x - 7$.

$$6x - 7$$

$$6(4) - 7$$

$$24 - 7$$

$$17$$

Since we were able to determine that $x = 4$, we know the value of $6x - 7$ must be $17$.

---

# Solving systems with substitution

Now that we understand how to solve two-step problems, we at least have the idea in our heads that two equations or expressions can both be related to the same variable(s).

## Simultaneous equations

With that in mind, we want to start looking at how to solve systems of equations, like

$$y = x + 3$$

$$2x - 3y = 10$$

Notice that we can't solve either of these equations by themselves. If we look at the first equation, there's no way to solve for the value of $y$ without knowing the value of $x$, but there's also no way to solve for the value of $x$ without knowing the value of $y$. So we're stuck. And the same is true for the second equation.

However, if we can consider the equations as a pair, then we can use them to solve for both variables simultaneously. For that reason, **systems of equations** are sometimes also called **simultaneous equations**.

As a rule of thumb, it's only possible to solve systems when we have at least as many equations as we do variables. In other words, up to now we've been working with one equation in terms of one variable, like

$3x + 4 = 10$, so we've been able to solve them. If we move to two variables, one equation won't give us enough information to solve for both values; we'll need two equations. And if we want to solve for the value of three variables, we'll need three equations to do that.

Keep in mind, even if we have $n$ number of equations to solve for $n$ variables, that doesn't *guarantee* that we'll still be able to find a solution for every variable.

Having $n$ number of equations is just the *minimum* requirement for solving for $n$ variables. If we have fewer than $n$ equations, we know immediately that we won't be able to solve for the values of $n$ variables. But depending on the specific equations we have in our system, it's possible that we'll need more than that minimum number of $n$ equations.

For example, if we want to solve for $x$ and $y$, we know we need two equations because we have two variables. But we couldn't just use the system

$$y = x + 3$$

$$2y = 2x + 6$$

It looks like we have two equations here, but if we divide the second equation by $2$, we $y = x + 3$, which is identical to the first equation. Because the equations are actually identical, the second one doesn't give us any new information, which means we won't have enough to solve for $x$ and $y$.

## Solving with substitution

But if we do have $n$ number of equations for $n$ variables, and those equations are sufficiently different, we can use them to solve for the values of $n$ number of variables in the system.

There are a few methods we can use to solve the system, but the first one we'll look at is substitution. This method is similar to the two-step problems we looked at previously, because we'll be solving one equation for an expression, and then plugging that expression to the other equation.

Here are the steps we'll follow when we use the **substitution** method:

1. Get a variable by itself in one of the equations.

2. Substitute the expression from step 1 into the other equation.

3. Solve the equation in step 2 for the remaining variable.

4. Substitute the result from step 3 into the equation from step 1.

Let's look at an example so that we can see these steps in action.

---

**Example**

Find the solution to the system of equations.

$$y = x + 3$$

$$2x - 3y = 10$$

Since $y$ is already solved for in the first equation, step 1 is completed, and we'll go on to step 2 by substituting $x+3$ for $y$ into the second equation.

$$2x-3y=10$$

$$2x-3(x+3)=10$$

Using substitution, we now have one equation in terms of one variable, so we can move on to step 3 and solve this equation like we normally would. Start by distributing the $-3$.

$$2x-3x-9=10$$

$$-x-9=10$$

Add $9$ to both sides.

$$-x-9+9=10+9$$

$$-x=19$$

Multiply both sides by $-1$.

$$-x(-1)=19(-1)$$

$$x=-19$$

Now that we have the value of one of the variables, we can move on to step 4, where we substitute $x=-19$ back into the first equation to find $y$.

$$y=x+3$$

$$y=-19+3$$

$$y = -16$$

The solution to this system is $(-19, -16)$. If we plug these values back into the original pair of equations, they should both be satisfied.

$$y = x + 3$$

$$-16 = -19 + 3$$

$$-16 = -16$$

and

$$2x - 3y = 10$$

$$2(-19) - 3(-16) = 10$$

$$-38 + 48 = 10$$

$$10 = 10$$

Let's try another example where we use substitution to solve the system.

**Example**

Find the solution to the system of equations.

$$x - 2y = 6$$

$$4x + 5y = 32$$

First, we need to get a variable by itself. It's easiest to get $x$ by itself in the first equation by adding $2y$ to both sides.

$$x - 2y + 2y = 6 + 2y$$

$$x = 6 + 2y$$

Now we'll substitute $x = 6 + 2y$ into the second equation.

$$4x + 5y = 32$$

$$4(6 + 2y) + 5y = 32$$

Now that we have one equation in one variable, we can solve it by itself.

$$24 + 8y + 5y = 32$$

$$24 + 13y = 32$$

$$24 - 24 + 13y = 32 - 24$$

$$13y = 8$$

$$y = \frac{8}{13}$$

To find $x$, plug $y = 8/13$ into the first equation and solve it for $x$.

$$x = 6 + 2y$$

$$x = 6 + 2\left(\frac{8}{13}\right)$$

$$x = \frac{78}{13} + \frac{16}{13}$$

$$x = \frac{94}{13}$$

The solution to this system is $(94/13,8/13)$. If we plug these values back into the original pair of equations, they should both be satisfied.

$$x - 2y = 6$$

$$\frac{94}{13} - 2\left(\frac{8}{13}\right) = 6$$

$$\frac{94}{13} - \frac{16}{13} = 6$$

$$\frac{78}{13} = 6$$

$$6 = 6$$

and

$$4x + 5y = 32$$

$$4\left(\frac{94}{13}\right) + 5\left(\frac{8}{13}\right) = 32$$

$$\frac{376}{13} + \frac{40}{13} = 32$$

$$\frac{416}{13} = 32$$

$$32 = 32$$

## Number of solutions

In both of these last examples, we were able to find exactly one unique solution to the system. But this won't always be the case. In fact, there are three possible solutions to a system of equations.

- one solution (called the unique solution), or
- no solutions, or
- infinitely many solutions.

In the one solution case, there *is* a solution to the system, and there is only one solution to the system. If this is the case, we'll be able to find the solution using the substitution method, like we did in these last couple of examples.

In the no solutions case, as we're solving the system, somewhere along the way we'll end up with an equation like

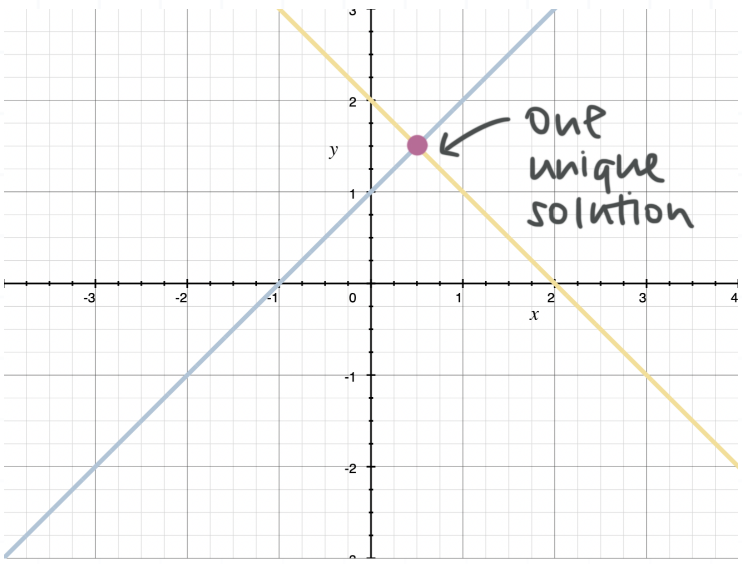
$$a = c$$

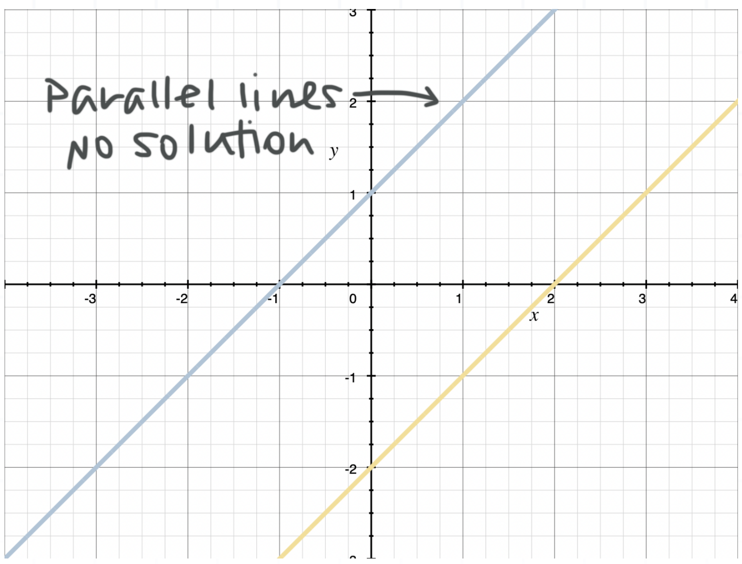
where $a$ and $c$ are constants with different values. For instance, maybe we get to a point in the solution where we find $7 = 11$. This is clearly impossible, so we know the system has no solutions. Graphically, this means we have to parallel lines, so they never cross each other, and there is therefore no intersection of the lines and no solution to the system.

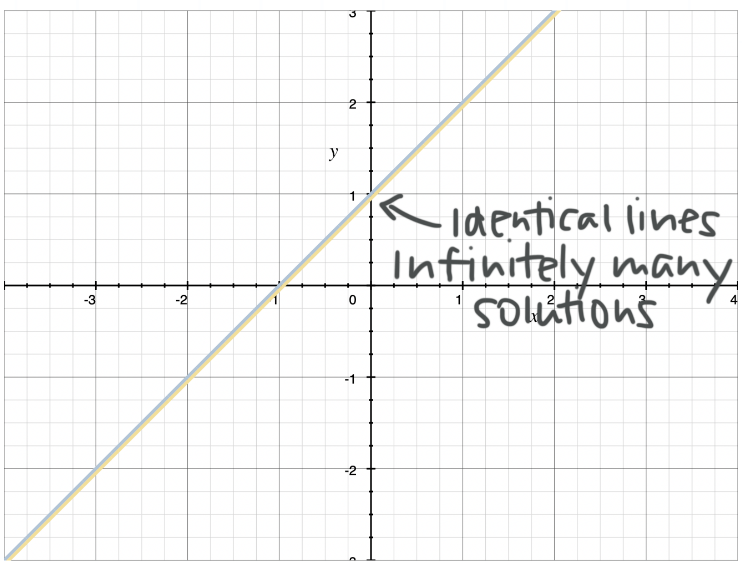
In the infinitely many solutions case, as we're solving the system, somewhere along the way we'll end up with an equation like

$$a = a$$

For instance, maybe we get to a point where we find $7 = 7$, or maybe $0 = 0$. If that's our ending point, then we know the system has infinitely many solutions. Graphically, this means we have two identical lines, so they sit on top of each other and every point along both lines is also a point on the other line, and there are therefore infinitely many solutions.



# Solving systems with elimination

We know how to solve systems using substitution, but now we want to look at a second method: elimination.

## Solving with elimination

As the name suggests, the goal of the elimination method is to eliminate one of the variables from the system by adding or subtracting the equations. With one variable eliminated, we'll be left with just one equation in terms of one variable.

We can solve that equation for the value of that variable, and then use the value we found to go back and find the value of the other variable.

Here are the steps we'll follow when we use the **elimination method** to solve a system of two linear equations:

1. If necessary, rearrange both equations so that the $x$-terms are first, followed by the $y$-terms, the equals sign, and the constant term (in that order). If an equation appears to have no constant term, that means that the constant term is $0$.

2. Multiply one (or both) equations by a constant that will allow either the $x$-terms or the $y$-terms to cancel when the equations are added or subtracted.

3. Add or subtract the equations to eliminate one of the variables.

4. Solve for the remaining variable.

5. Plug the result of step 4 into one of the original equations, then solve for the other variable.

Let's look at an example so that we can see how these steps work.

---

**Example**

Find the solution to the system of equations.

$$3x + 4y = 12$$

$$-3x + 2y = 18$$

Steps 1 and 2 are done, since the individual parts of each equation are in the correct places, and the $x$-terms ($3x$ in the first equation and $-3x$ in the second equation) will cancel when we add the equations. So we'll skip to step $3$ and add the equations.

$$3x + 4y = 12 \qquad + \qquad -3x + 2y = 18$$

$$3x + 4y + (-3x + 2y) = 12 + (18)$$

$$3x + 4y - 3x + 2y = 12 + 18$$

$$3x - 3x + 4y + 2y = 30$$

$$0 + 6y = 30$$

$$6y = 30$$

$$y = 5$$

Now, we'll substitute $y = 5$ into the first equation and solve for $x$.

$$3x + 4y = 12$$

$$3x + 4(5) = 12$$

$$3x + 20 = 12$$

Subtract $20$ from both sides.

$$3x + 20 - 20 = 12 - 20$$

$$3x = -8$$

$$x = -\frac{8}{3}$$

The solution to the system is $(-8/3,5)$.

---

Let's try another example of solving with elimination.

### Example

Find the unique solution to the system of equations.

$$y = 3x - 4$$

$$-x + 2y = 12$$

First, we'll rearrange the first equation so that its individual parts are in the correct places for elimination. Subtract $3x$ from both sides.

$$y = 3x - 4$$

$$-3x + y = 3x - 3x - 4$$

$$-3x + y = -4$$

Now our system is

$$-3x + y = -4$$

$$-x + 2y = 12$$

Next, multiply this new first equation by $2$,

$$2(-3x + y) = 2(-4)$$

$$-6x + 2y = -8$$

such that the new system becomes

$$-6x + 2y = -8$$

$$-x + 2y = 12$$

We multiplied the first equation by $2$ so that the $y$-terms will cancel when we subtract the equations.

$$-6x + 2y = -8 \qquad - \qquad -x + 2y = 12$$

$$-6x + 2y - (-x + 2y) = -8 - (12)$$

$$-6x + 2y + x - 2y = -8 - 12$$

$$-6x + x + 2y - 2y = -20$$

$$-5x + 0 = -20$$

$$-5x = -20$$

$$x = 4$$

To solve for $y$, we'll substitute $x = 4$ into the original first equation.

$$y = 3x - 4$$

$$y = 3(4) - 4$$

$$y = 12 - 4$$

$$y = 8$$

The solution to the system is $(4,8)$.

# Solving systems three ways

There are three ways to solve systems of linear equations: substitution, elimination, and graphing. Let's review the steps for each method.

**Substitution**

1. Get a variable by itself in one of the equations.

2. Take the expression you got for the variable in step 1, and plug it (substitute it using parentheses) into the other equation.

3. Solve the equation in step 2 for the remaining variable.

4. Use the result from step 3 and plug it into the equation from step 1.

**Elimination**

1. If necessary, rearrange both equations so that the $x$-terms are first, followed by the $y$-terms, the equals sign, and the constant term (in that order). If an equation appears to have not constant term, that means that the constant term is $0$.

2. Multiply one (or both) equations by a constant that will allow either the $x$-terms or the $y$-terms to cancel when the equations are added or subtracted (when their left sides and their right sides are added separately, or when their left sides and their right sides are subtracted separately).

3. Add or subtract the equations.

4. Solve for the remaining variable.

5. Plug the result of step 4 into one of the original equations and solve for the other variable.

**Graphing**

1. Solve for $y$ in each equation.

2. Graph both equations on the same Cartesian coordinate system.

3. Find the point of intersection of the lines (the point where the lines cross).

Now let's look at a few examples in which we need to decide which of these three methods to use.

---

**Example**

Which method should we use to solve the system?

$$x = y + 2$$

$$3y - 2x = 15$$

The easiest way to solve this system would be to use substitution since $x$ is already isolated in the first equation. Whenever one equation is already solved for a variable, substitution will be the quickest and easiest method.

If we wanted to solve the system, we'd substitute $x = y + 2$ into the second equation.

$$3y - 2x = 15$$

$$3y - 2(y + 2) = 15$$

$$3y - 2y - 4 = 15$$

$$y - 4 = 15$$

$$y - 4 + 4 = 15 + 4$$

$$y = 19$$

Then we'll substitute $y = 19$ into the first equation.

$$x = y + 2$$

$$x = 19 + 2$$

$$x = 21$$

The unique solution to the system is $(21,19)$.

---

Let's try another example.

**Example**

To solve the system by elimination, what would be a useful first step?

$$x + 3y = 12$$

$$2x - y = 5$$

When we use elimination to solve a system, it means we're going to get rid of (eliminate) one of the variables. So we need to be able to add the equations, or subtract one from the other, and in doing so cancel either the $x$-terms or the $y$-terms.

Any of the following options would be a useful first step:

- Multiply the first equation by $2$, which would give us $2x$ in both equations and cause the $x$-terms to cancel when we subtract one equation from the other.
- Multiply the second equation by $3$, which would give us $-3y$ in the second equation and cause the $y$-terms to cancel when we add the equations.
- Divide the second equation by $2$, which would give us $x$ in both equations and cause the $x$-terms to cancel when we subtract one equation from the other.
- Divide the first equation by $3$, which would give us $y$ in the first equation and cause the $y$-terms to cancel when we add the equations.

---

Let's re-do the last example, but instead of the elimination method, graph both equations to find the solution to the system.

**Example**

Graph both equations to find the solution to the system.

$$x + 3y = 12$$

$$2x - y = 5$$

In order to graph these equations, let's put both of them into slope-intercept form. We get

$$x + 3y = 12$$

$$3y = -x + 12$$

$$y = -\frac{1}{3}x + 4$$

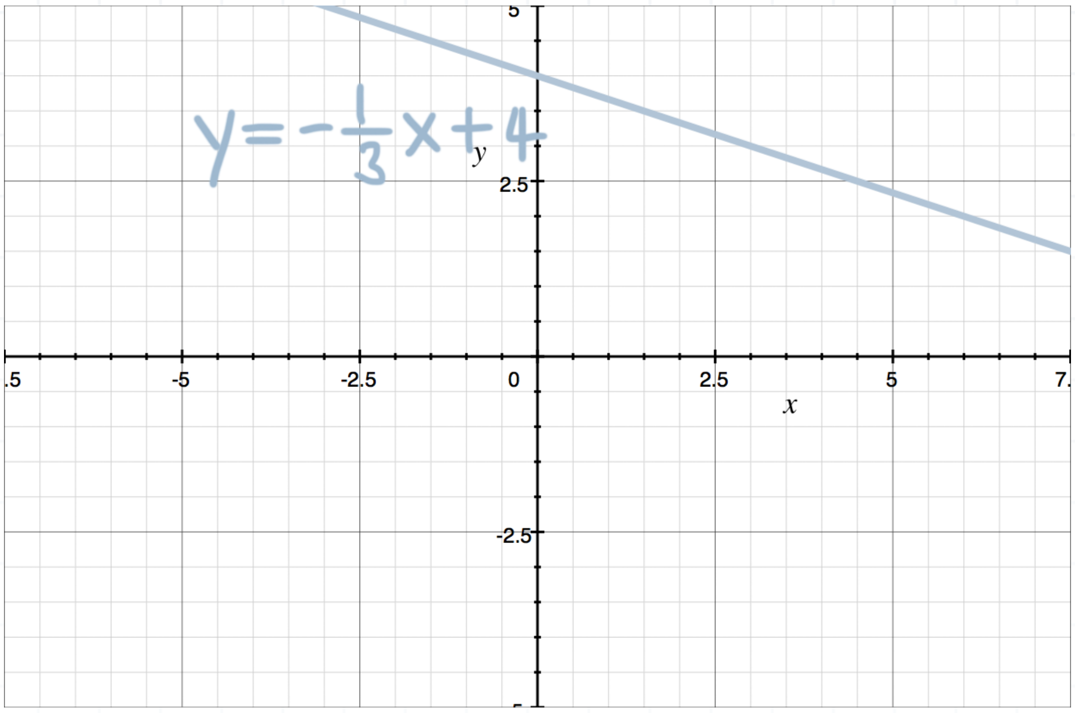
and

$$2x - y = 5$$

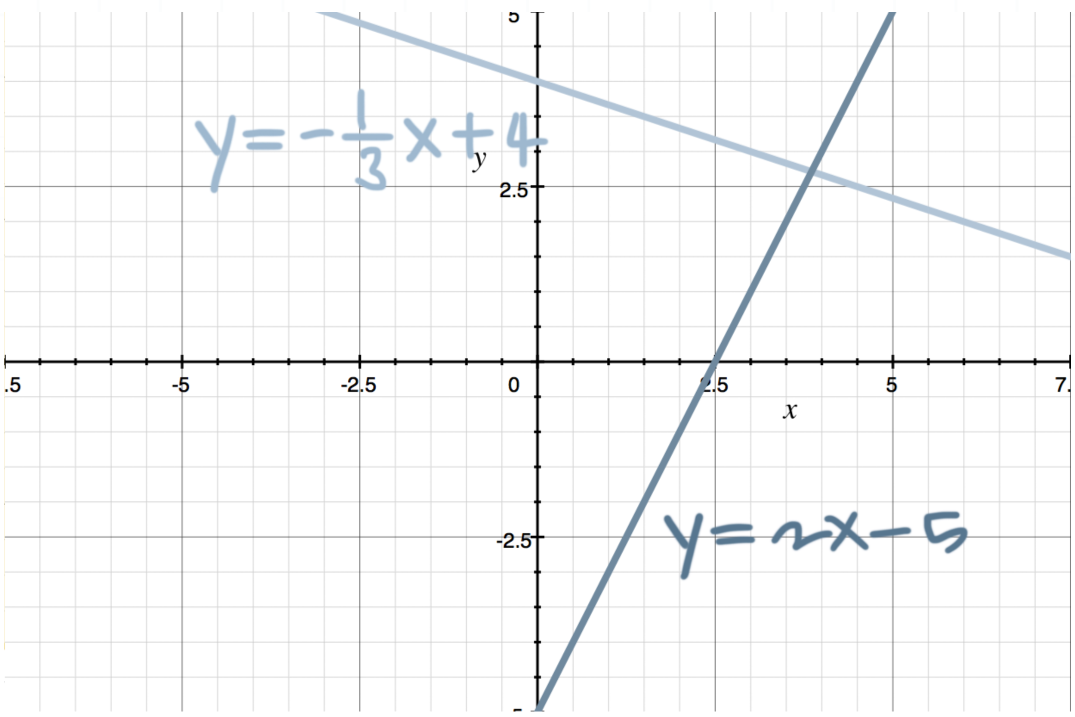
$$-y = -2x + 5$$

$$y = 2x - 5$$

The line $y = -(1/3)x + 4$ intersects the $y$-axis at $4$, and then has a slope of $-1/3$, so its graph is




The line $y = 2x - 5$ intersects the $y$-axis at $-5$, and then has a slope of $2$, so if you add its graph to the graph of $y = -(1/3)x + 4$, you get

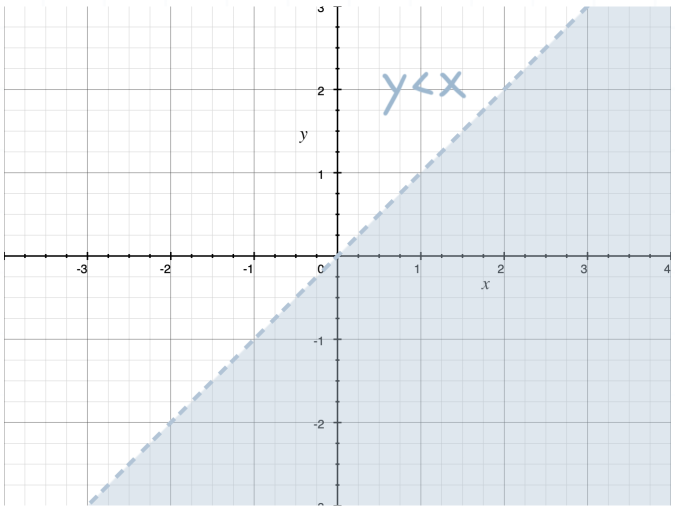


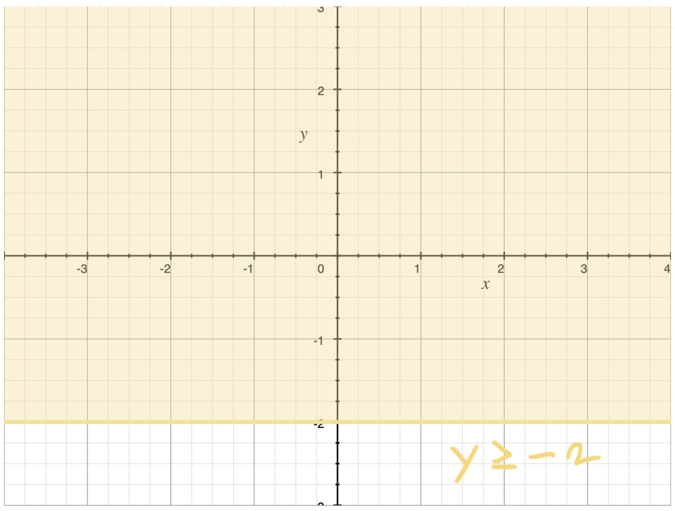

Looking at the intersection point, it appears as though the solution is approximately $(3.75,2.75)$. If we solve the systems with substitution or elimination, the solution is $(27/7,19/7) \approx (3.86,2.71)$, so our visual estimate of $(3.75,2.75)$ isn't that far off.

# Systems of linear inequalities

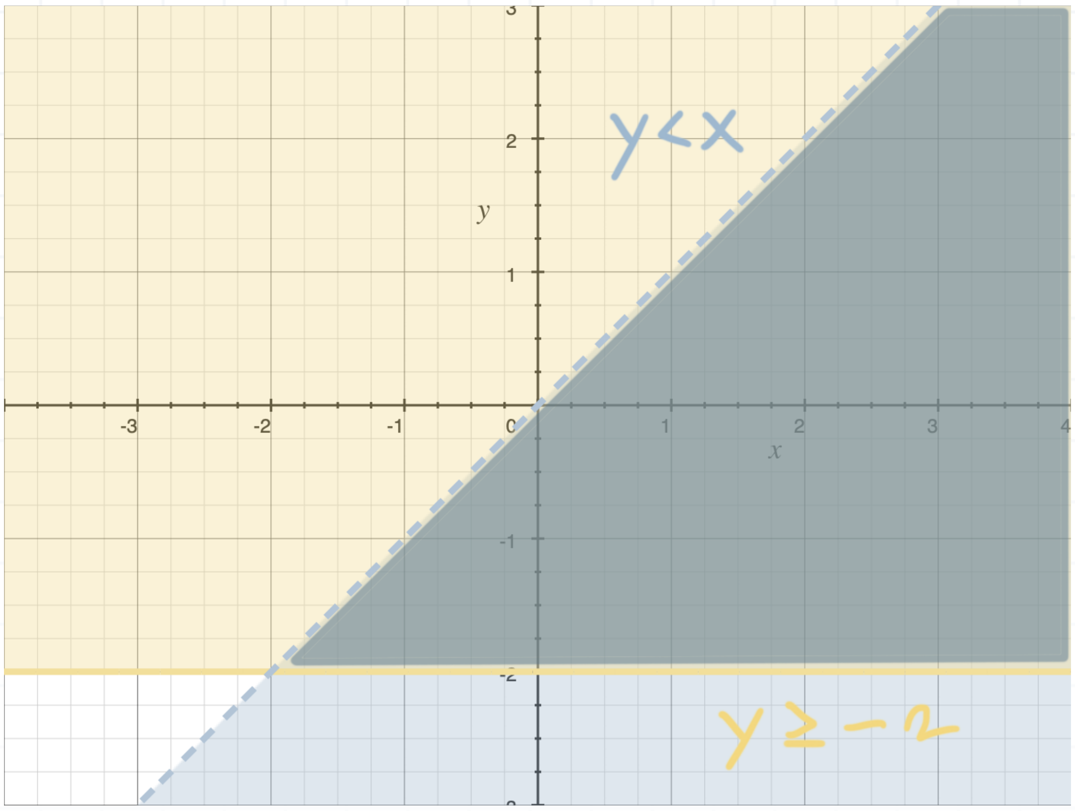
Now that we know how to solve systems of linear equations using substitution, elimination, and graphing, we want to learn to solve systems of inequalities.

We need to realize that the solution to a system of linear inequalities (if the system has a solution) will be a region in the plane. For instance, given two lines $y < x$ and $y \geq -2$, we can sketch each inequality separately in the plane,




and then identify the region of overlap as the solution to the system of inequalities.




## Graphing the solution to the system

To find and graph the region of the solution to the system of inequalities, we'll

1. Graph the boundary lines

2. Determine whether the boundary lines are dashed ($<$ or $>$) or solid ($\leq$ or $\geq$)

3. Determine which side of each boundary line to shade

4. Identify the overlapping shaded region as the solution, keeping only the overlap shaded

Let's work through an example so that we can see these steps in practice.

### Example

Graph the solution to the system of linear inequalities.

$$y < -2x + 3$$

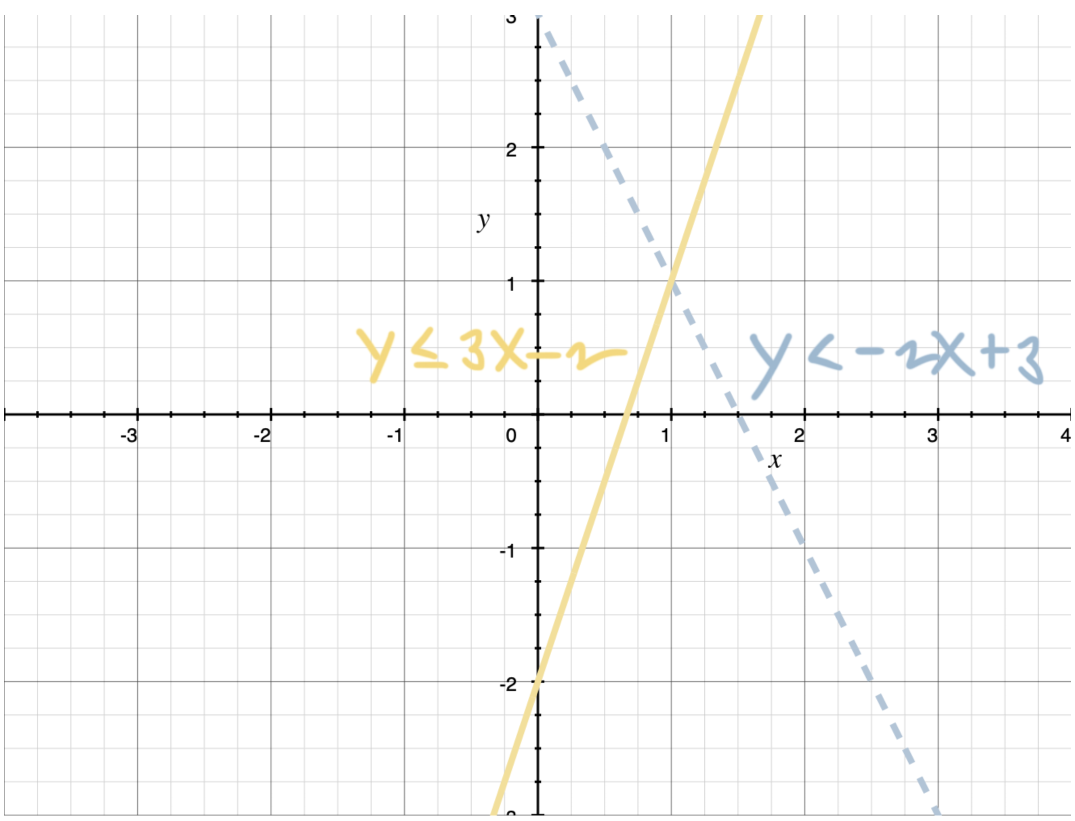
$$y \leq 3x - 2$$

First, we need to sketch the boundary lines of each inequality by graphing the corresponding equations,

$$y = -2x + 3$$

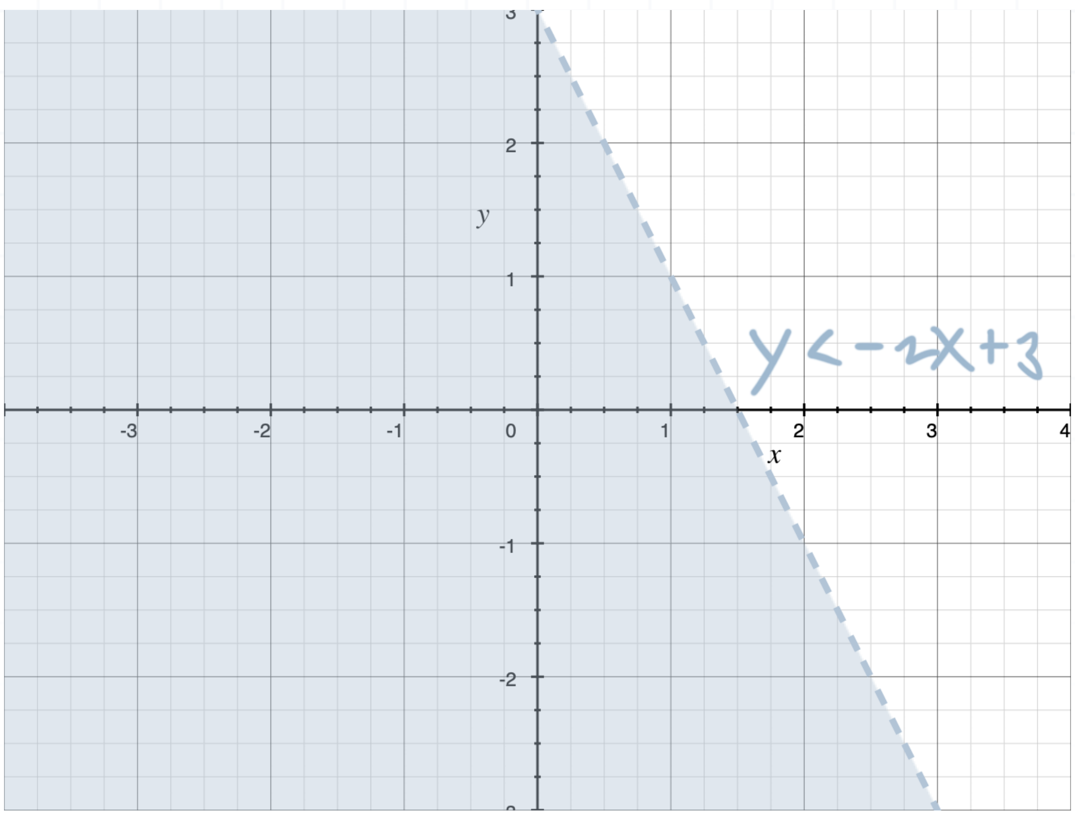
$$y = 3x - 2$$

Both lines are in slope-intercept form, so a sketch of the two of them together is

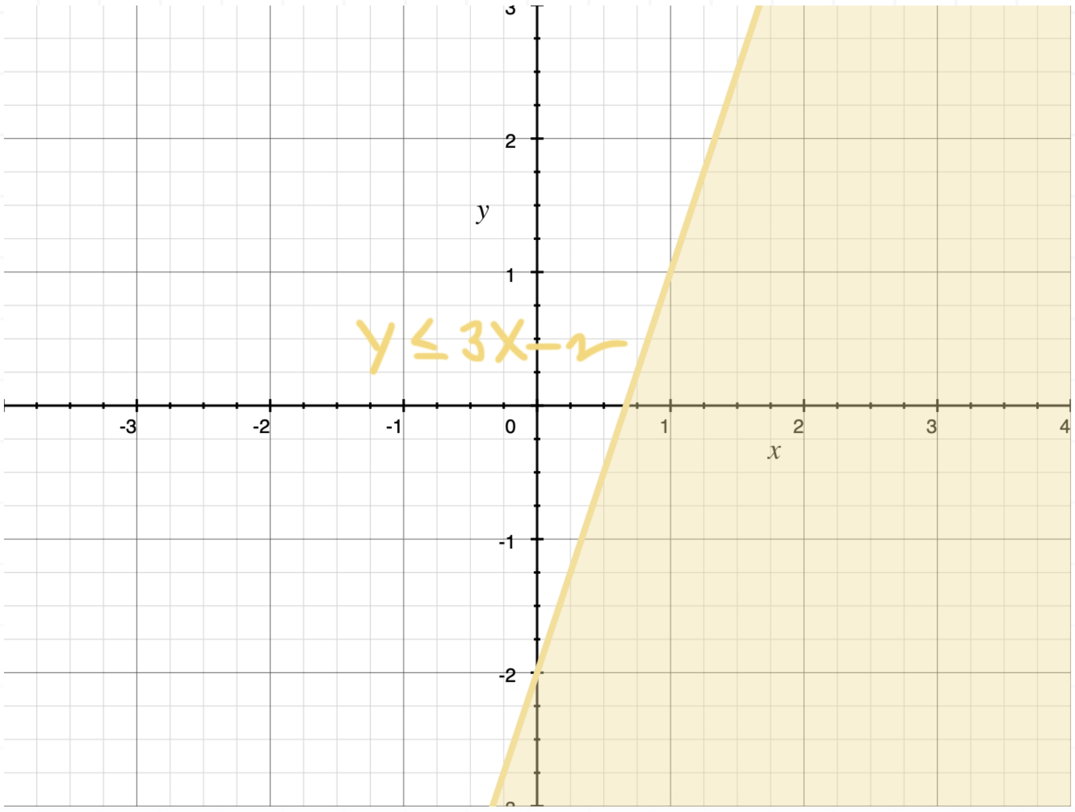


The line associated with $y < -2x + 3$ is dashed, given the “less than” sign, whereas the line associated with $y \leq 3x - 2$ is solid, given the “less than or equal to” sign.

Because $y < -2x + 3$ is a “less than” inequality, we shade below the dashed $y = -2x + 3$ line,




and because $y \leq 3x - 2$ is a “less than or equal to” inequality, we shade below the solid $y = 3x - 2$ line.




Putting these two regions together, we can identify the overlapping region as the solution to the system of inequalities.