Dividing polynomials

We've looked at adding, subtracting, and multiplying polynomials, and now we finally want to jump into how to divide polynomials. Think about dividing polynomials as long division, but with variables.

Review of long division

In lower-level math, when we first learned how to divide real numbers, we learned to use long division. Let's review long division by working through an example where we divide 146 by 13.

Example

Use long division to divide 146 by 13.

The long division will look like,

$$
\begin{array}{r}
11\ \text{R}3 \\
13\,\overline{)\,146} \\
\underline{-13} \\
16 \\
\underline{-13} \\
3
\end{array}
$$

but to work through this step-by-step, we need to start by thinking "How many times does 13 go into 14?" It goes in 1 time, so we write a 1 above the long division sign and line it up with the 4.

Then we multiply 13×1 and get 13, which means we subtract 13 from 14 and get 1. We bring down the 6 so that the remaining 1 becomes 16.

How many times does 13 go into 16? It goes in 1 time, so we write another 1 above the long division sign, this time lined up with the 6.

We multiply $13 \times 1 = 13$, which means we subtract 13 from 16 and get 3. Since 13 doesn't go into 3, and there's nothing left to bring down, we have a remainder of 3.

Our answer to $146 \div 13$ is 11 with a remainder of 3, or

$$\frac{146}{13} = 11 + \frac{3}{13}$$

As a reminder, the value being divided, 146, is the **dividend**. The value we divide by, 13, is the **divisor**. The result of the division, $11 + 3/13$, is the **quotient**, and if the divisor doesn't divide evenly into the dividend, then the quotient will include a **remainder**. In this example, the remainder is 3.

Quotient $\longrightarrow$ 11 R3 $\longleftarrow$ Remainder

$$
\begin{array}{rl}
 & 11 \quad R3 \\
\text{Divisor} \longrightarrow 13 & \overline{)\,146} \longleftarrow \text{Dividend} \\
 & \underline{-13} \downarrow \\
 & 16 \\
 & \underline{-13} \\
 & 3 \longleftarrow \text{Remainder}
\end{array}
$$

Polynomial long division

Now let's look at this same problem again, but this time using polynomial long division. This time we'll divide $x^2 + 4x + 6$ (notice that the coefficients of this polynomial are 1, 4, and 6) by $x + 3$ (notice that x has a coefficient of 1, and we follow that by a constant 3).

$$
\begin{array}{rl}
 & \phantom{x^2+{}} x + 1 \quad R3 \\
x+3 & \overline{)\,x^2 + 4x + 6} \\
 & \underline{-(x^2 + 3x)} \downarrow \\
 & \phantom{x^2+{}} x + 6 \\
 & \underline{-(x + 3)} \\
 & \phantom{x^2+4x+{}} 3
\end{array}
$$

The leading term in the dividend x^2+4x+6 is x^2, and the leading term in the divisor $x+3$ is x. So we start by thinking, "What do we need to multiply by x to get x^2?" The answer is x, so we write x above the long division sign and line it up with the x^2.

Then we multiply $x+3$ by x and get x^2+3x, which means we subtract x^2+3x from x^2+4x and get x. Bring down the $+6$.

What do we need to multiply x by to get x? We need to multiply by 1, so we write $+1$ next to the x above the long division sign.

We multiply $(x+3)\cdot 1 = x+3$, so we subtract $x+3$ from $x+6$ and get 3.

Our answer is the quotient $x+1$ with a remainder of 3. When we do polynomial long division, we should write the remainder as a fraction, with the remainder in the numerator and the divisor in the denominator, so we should write this answer as

$$\frac{x^2+4x+6}{x+3} = x+1+\frac{3}{x+3}$$

Let's do another example where we divide a trinomial by a binomial.

Example

Simplify the expression using polynomial long division.

$$(x^2+3x-5) \div (x-2)$$

Use polynomial long division to simplify.

$$
\begin{array}{rl}
 & x+5 \qquad R5 \\
x-2 & \overline{)\,x^2+3x-5} \\
 & -(x^2-2x) \downarrow \\
 & \qquad 5x-5 \\
 & \quad -(5x-10) \\
 & \qquad\qquad 5
\end{array}
$$

We get a quotient of $x+5$ with a remainder of 5, so

$$\frac{x^2+3x-5}{x-2} = x+5+\frac{5}{x-2}$$

Let's try another example of dividing polynomials.

Example

Use polynomial long division to simplify the expression.

$$(2x^3+x^2+4) \div (x+1)$$

Use polynomial long division to simplify.

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & 2x^2 - x + 1 \qquad R3 \\ \cline{2-2}
x+1 & 2x^3 + x^2 + 0x + 4 \\
 & -(2x^3+2x^2) \\ \cline{2-2}
 & \qquad -x^2+0x \\
 & \qquad -(-x^2-x) \\ \cline{2-2}
 & \qquad\qquad x+4 \\
 & \qquad\qquad -(x+1) \\ \cline{2-2}
 & \qquad\qquad\qquad 3
\end{array}
$$

When we multiplied the second term in the quotient $-x$, by the divisor $x+1$, we found $-x^2-x$. There was no x term in the dividend that we could bring down, so we added a “placeholder” $0x$ into the dividend. That way, we could bring down the $0x$ to pair with the $-x^2$, and then subtract $-x^2-x$ from x^2+0x.

We get a quotient of $2x^2-x+1$ with a remainder of 3, so

$$\frac{2x^3+x^2+4}{x+1} = 2x^2 - x + 1 + \frac{3}{x+1}$$