Graphing transformations of exponential functions

Sometimes we're given the graph of one exponential function, and asked to sketch the graph of a similar function which is just a few "transformations" away from the given function. Each **transformation** is of one of the following operations:

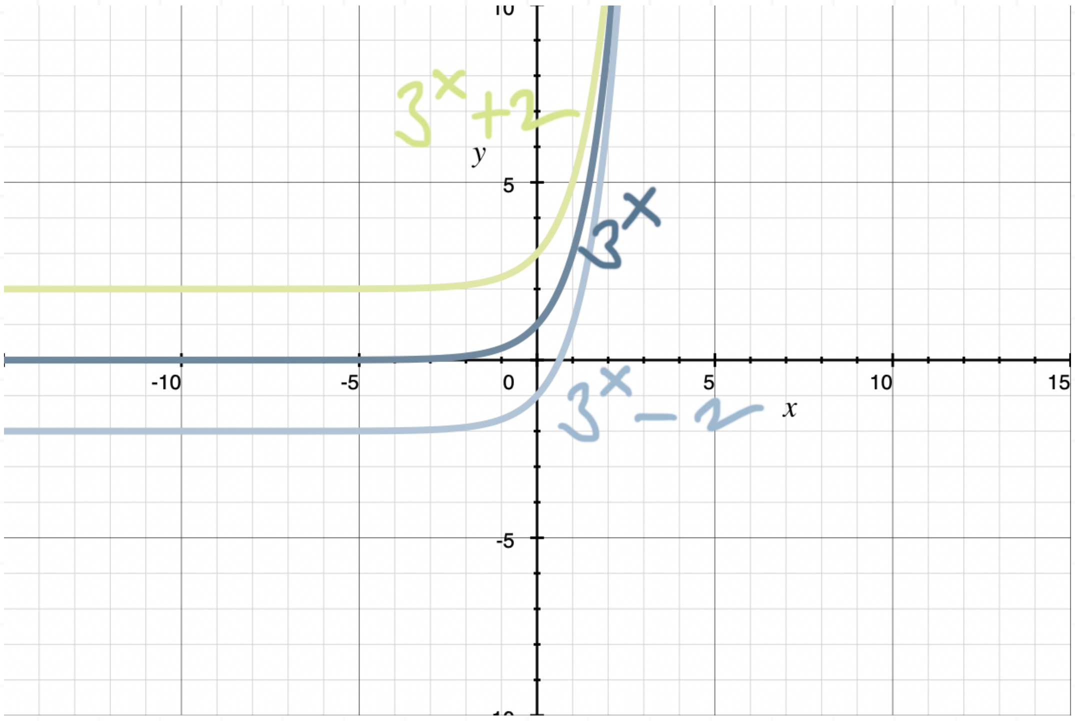
- addition of a constant to the exponent
- addition of a constant to the exponential function
- multiplication of the exponent by a nonzero constant
- multiplication of the exponential function by a nonzero constant

These produce a few different transformations, including vertical and horizontal shifts, a stretch or compression, or reflection. Let's learn how we can graph each transformation.

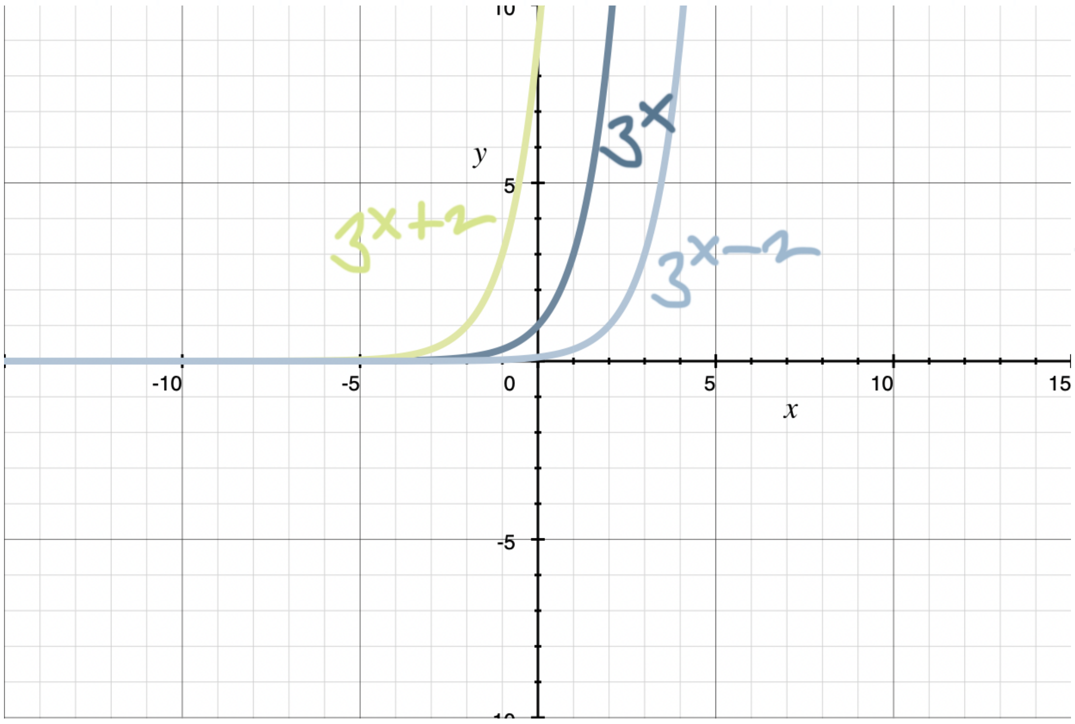
Vertical and horizontal shifts

Let's consider the parent function $f(x)=b^x$. Adding a constant d to the parent function gives us a vertical shift d units in the same direction as the sign of d.

For example, a sketch of the parent function $f(x)=3^x$ and this same function shifted vertically up 2 units and down 2 units gives



To shift a curve horizontally, we can add a constant c to the input of the parent function $f(x) = b^x$, but the direction of the shift is opposite the sign of c. So a sketch of the parent function $f(x) = 3^x$ and this same function shifted horizontally left and right 2 units gives



Let's do an example where we have to sketch a function with both a vertical and a horizontal shift.

Example

Graph $f(x) = 3^{x-1} + 2$.

We have an equation in the form $f(x) = b^{x+c} + d$, with $b = 3$, $c = -1$, and $d = 2$. So the horizontal asymptote is $y = 0$.

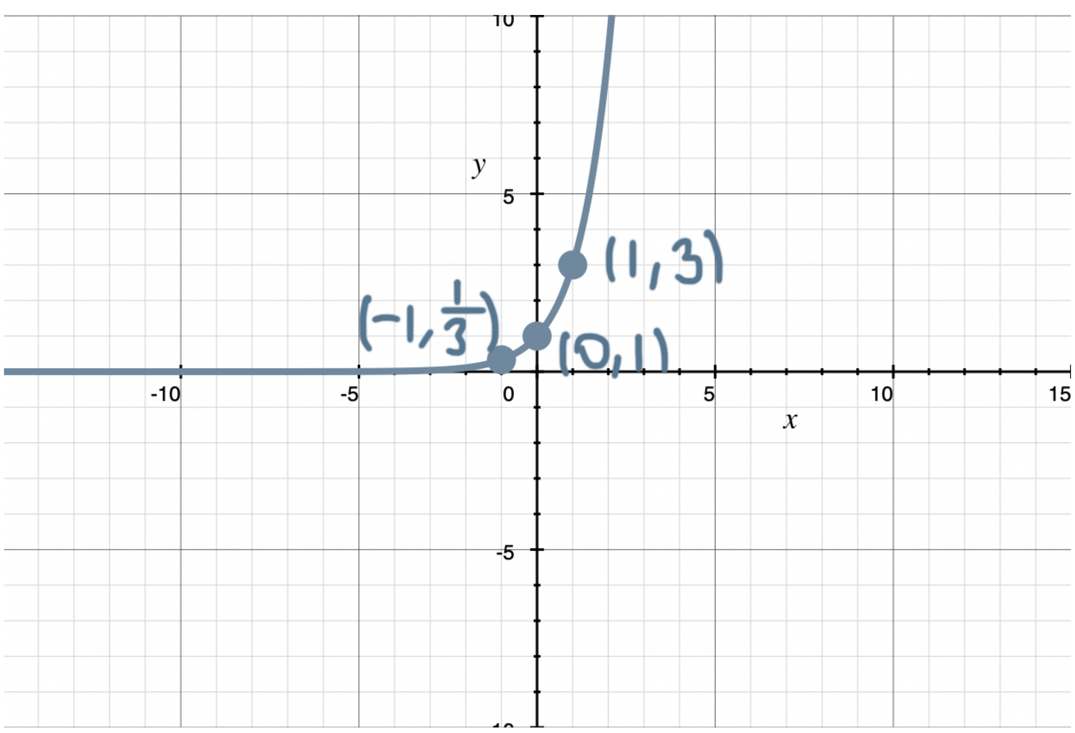
To sketch the graph of $y = 3^x$, we'll plug in a few values of x for which the value of $f(x)$ will be easy to calculate.

For $x = 0$: $f(0) = 3^0 = 1$

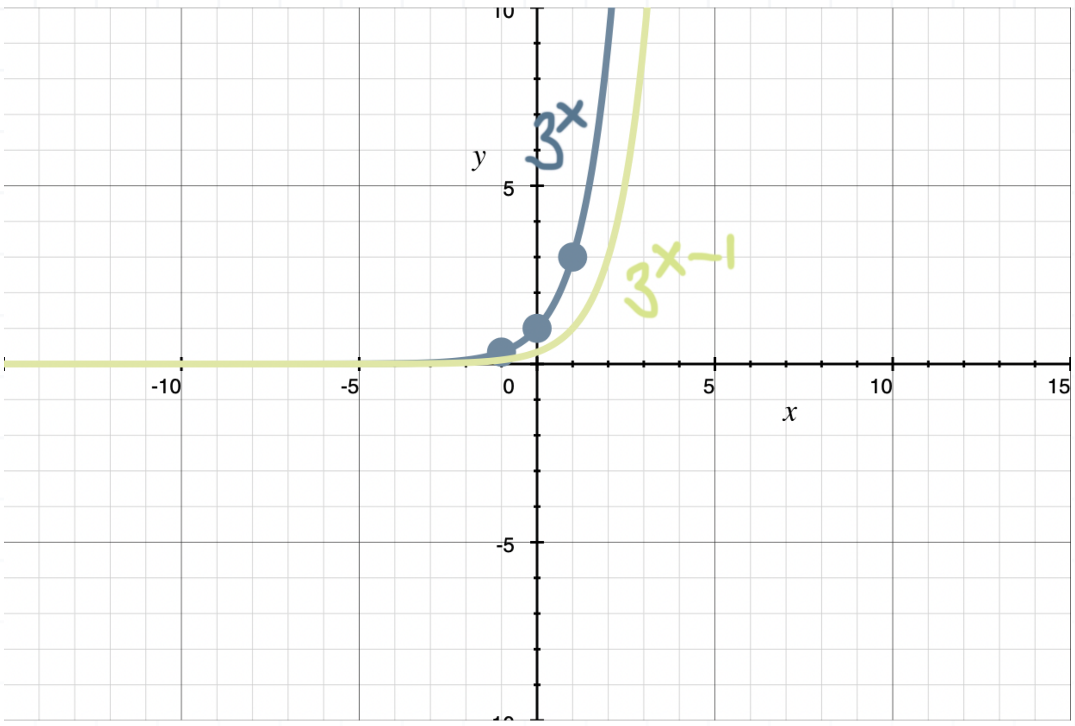
For $x = -1$: $f(-1) = 3^{-1} = 1/3$

For $x = 1$: $f(1) = 3^1 = 3$

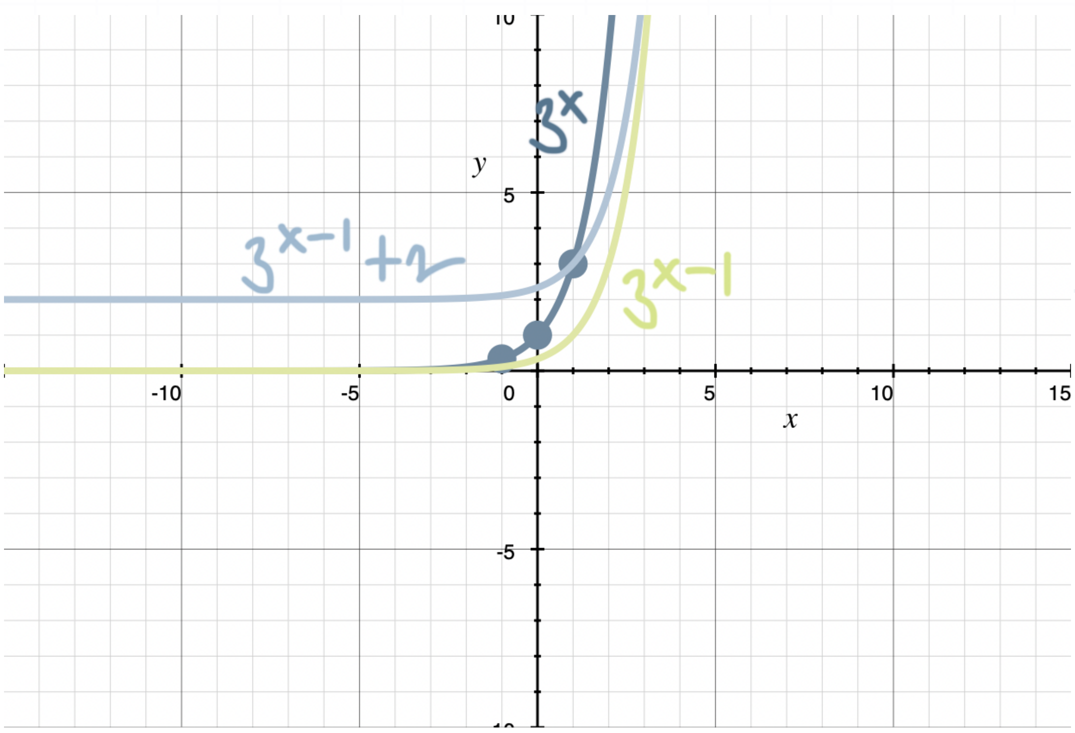
Now we have three points on the graph of f: $(0,1)$, $(-1,1/3)$, and $(1,3)$. If we plot these three points and connect the points with a smooth curve, we get



Then to go from $y = 3^x$ to $y = 3^{x-1}$, we move the graph 1 unit to the right,



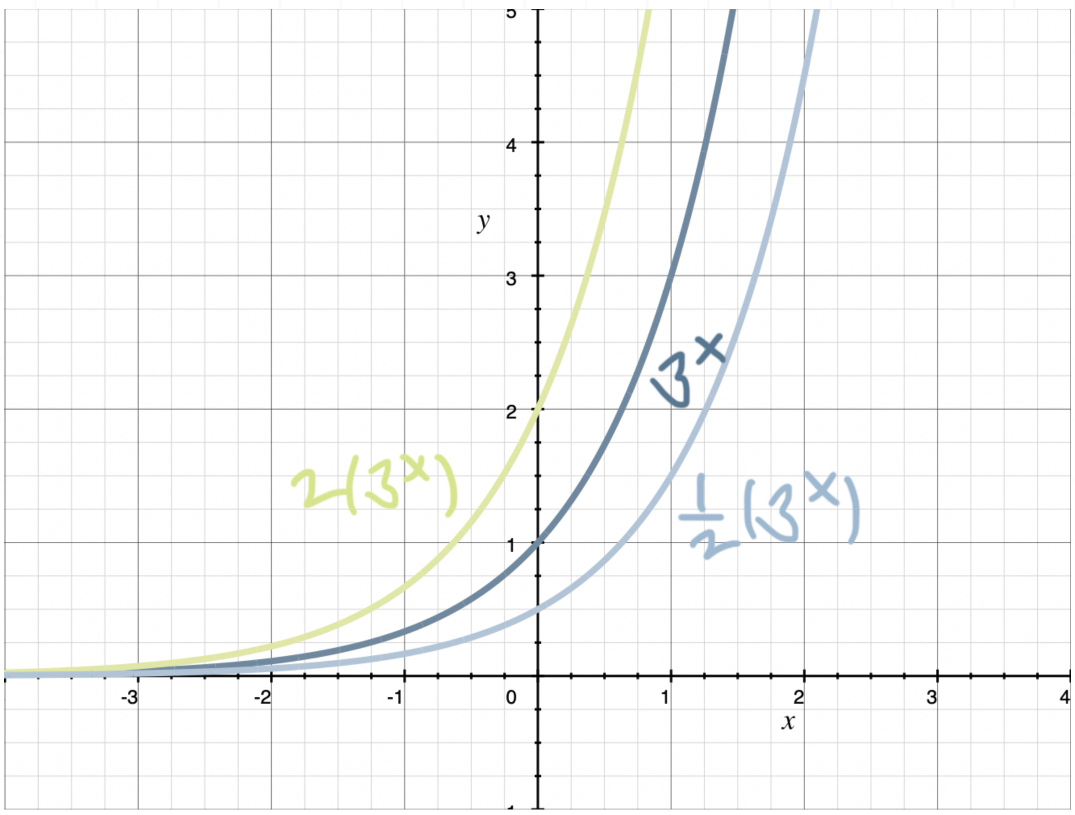
and to go from $y = 3^{x-1}$ to $y = 3^{x-1} + 2$, we move the graph 2 units up vertically.



Vertical and horizontal stretches and compressions

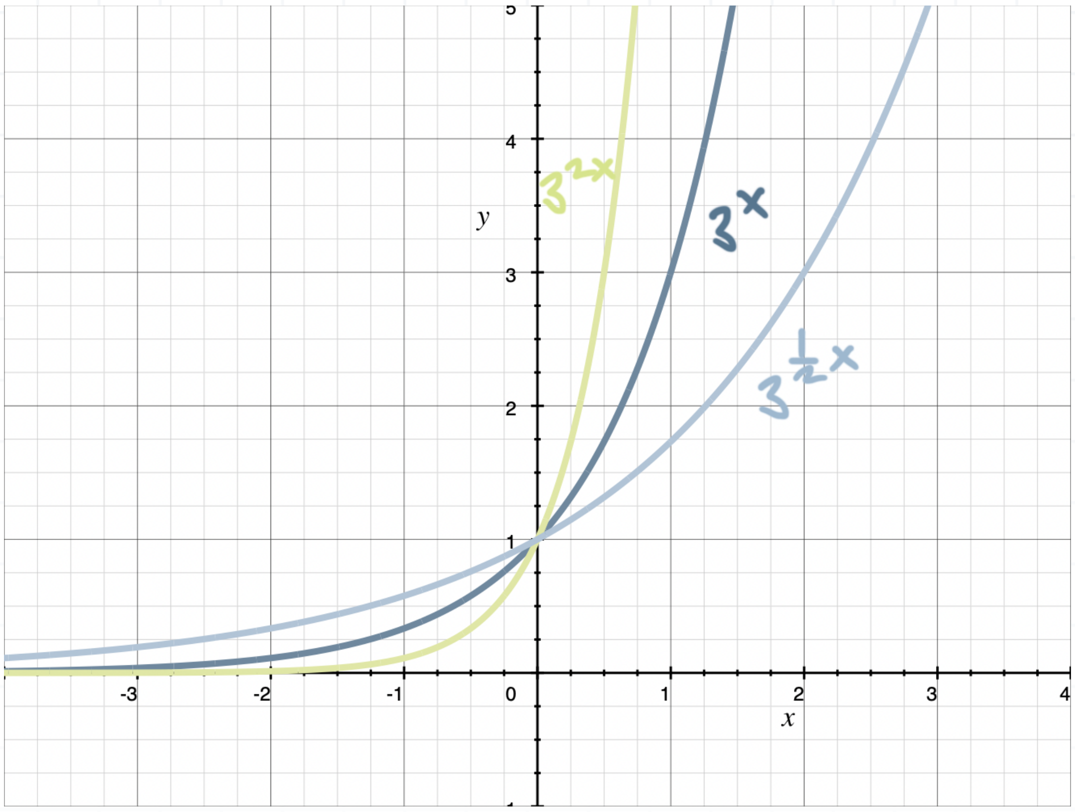
Let's consider the parent function $f(x) = b^x$. If we multiply the parent by some a where $1 < |a|$ to get $f(x) = ab^x$, then $f(x) = b^x$ is being stretched vertically by a factor of a. But when $0 < |a| < 1$, then $f(x) = b^x$ is being vertically compressed.

For example, a sketch of the parent function $f(x) = 3^x$ and this same function stretched and compressed vertically by a factor of 2 gives



If instead we multiply the input x by some k where $0 < |k| < 1$ to get $f(x) = b^{kx}$, then $f(x) = b^x$ is being stretched horizontally by a factor of $\frac{1}{k}$. But when $1 < |k|$, then $f(x) = b^x$ is being compressed.

For example, a sketch of the parent function $f(x) = 3^x$ and this same function stretched and compressed horizontally by a factor of 2 gives

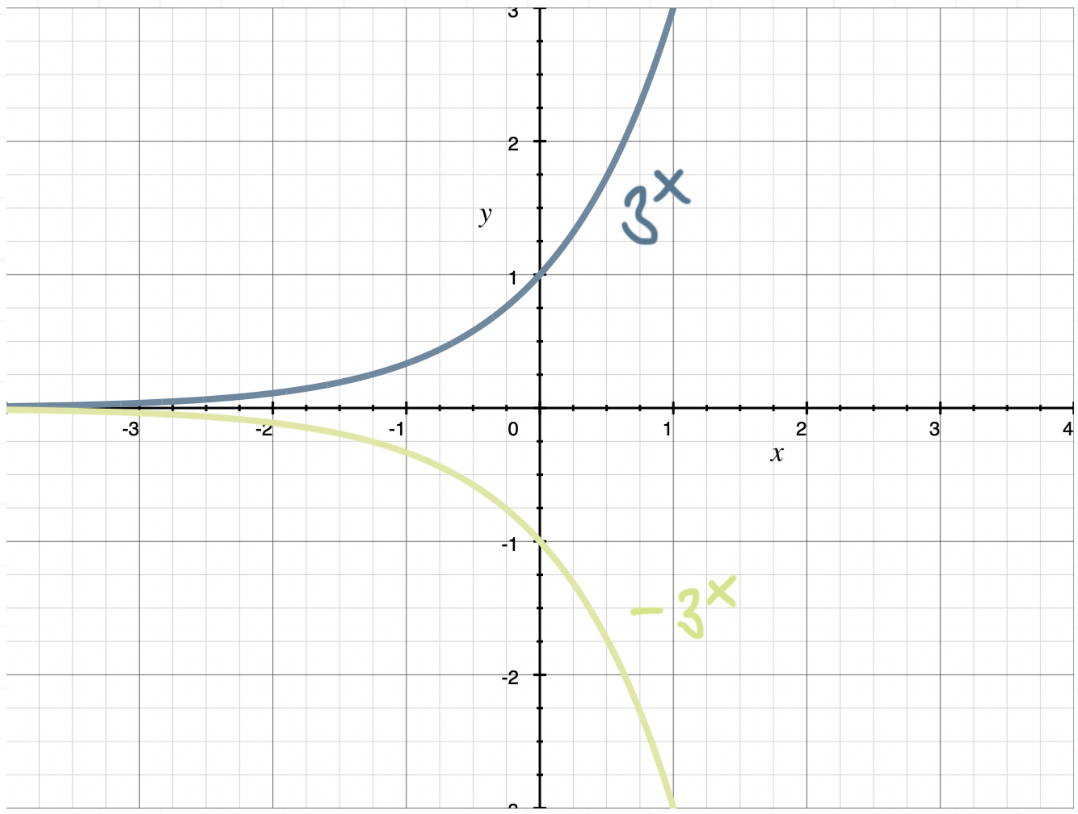


Let's do an example where we have to sketch a function with both a vertical and a horizontal stretch or compression.

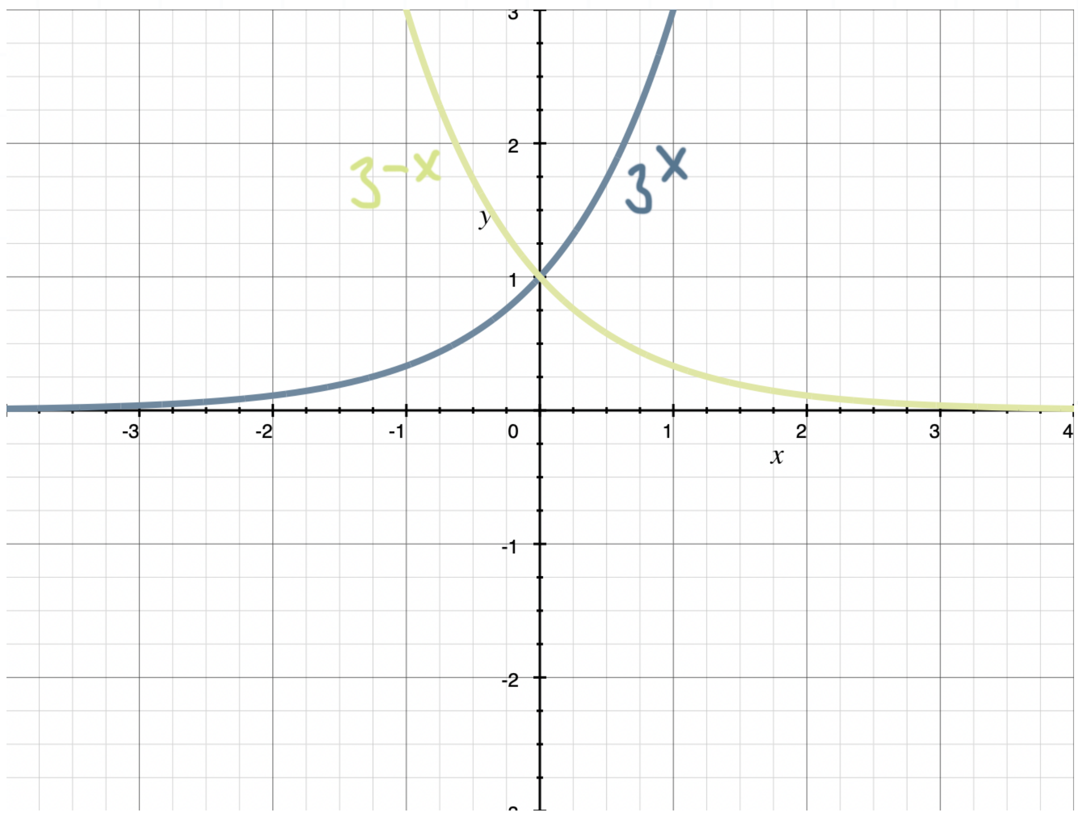
Vertical and horizontal reflections

It's also possible to reflect the graph across the x-axis and/or the y-axis. When we multiply the parent function $f(x) = b^x$ by -1, the graph gets reflected across the x-axis. But when we multiply the input of the parent by -1, the graph gets reflected across the y-axis.

For example, let's choose $f(x) = 3^x$ again as the parent function. Its reflection across the x-axis is $g(x) = -3^x$,



and its reflection across the y-axis is $h(x) = 3^{-x}$.



Combining transformations

Now that we've seen this collection of translations, let's summarize the order in which we should apply them, given multiple translations in the same equation.

1. Horizontal stretch or compression

2. Horizontal shift

3. Horizontal reflection

4. Vertical stretch or compression

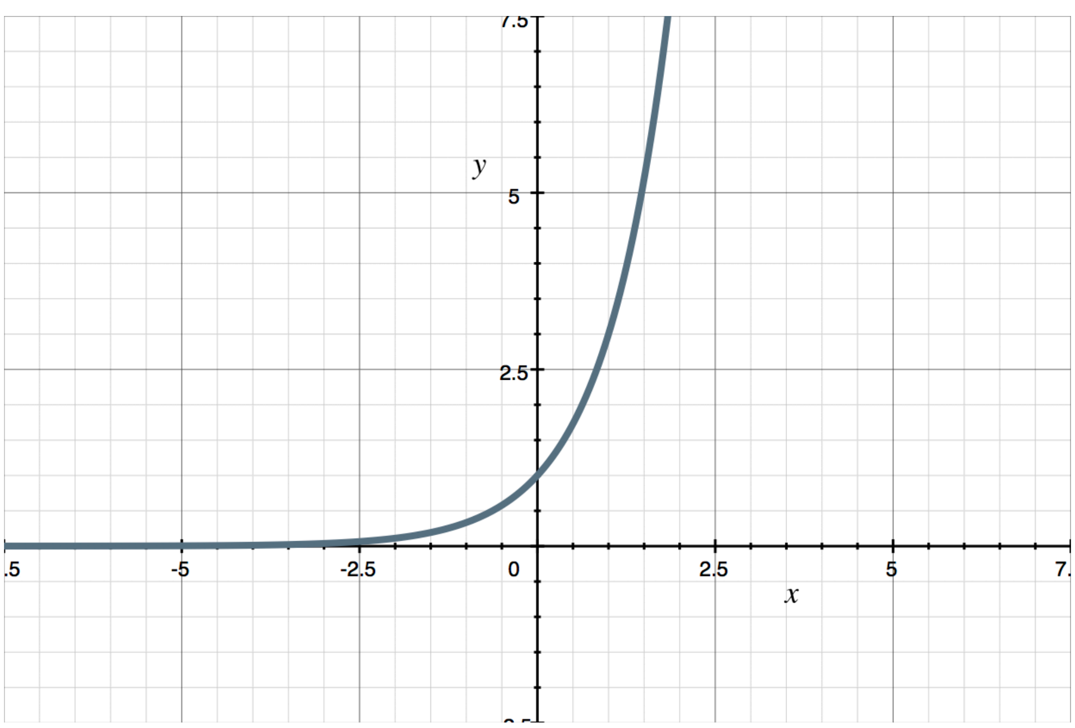
5. Vertical reflection

6. Vertical shift

Let's do an example where we apply these shifts in order.

Example

Use the graph of $f(x)=3^x$ to sketch the graph of $f(x)=6\cdot 3^{-x}+1$.



The function $6 \cdot 3^{-x} + 1$ is the result of applying several transformations to the function 3^x. We'll tackle each transformation as a separate step, and we'll give different names to the functions we obtain in each step.

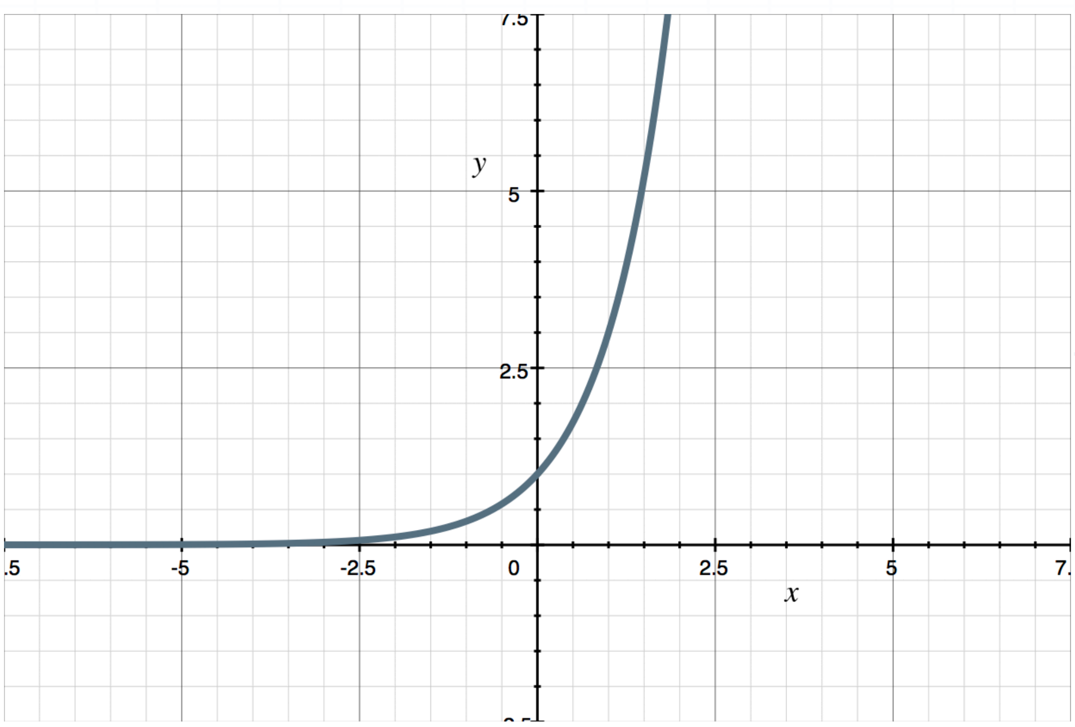
[1] $f(x) = 3^x$

[2] $g(x) = 3^{-x}$

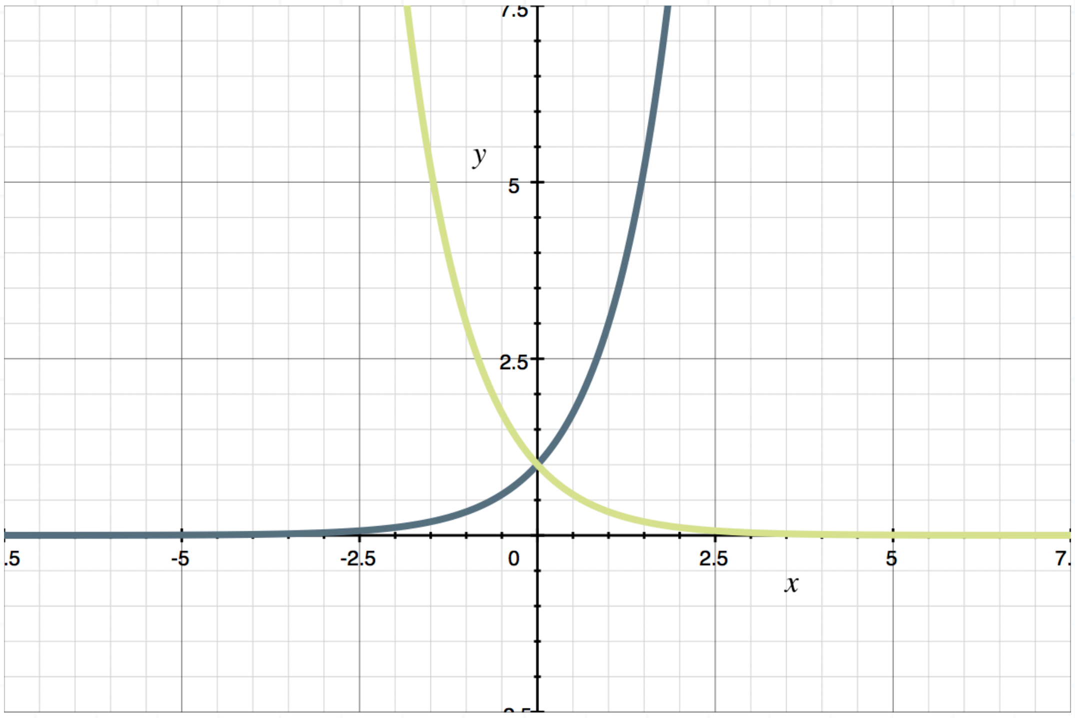
[3] $h(x) = 6 \cdot 3^{-x}$

[4] $k(x) = 6 \cdot 3^{-x} + 1$

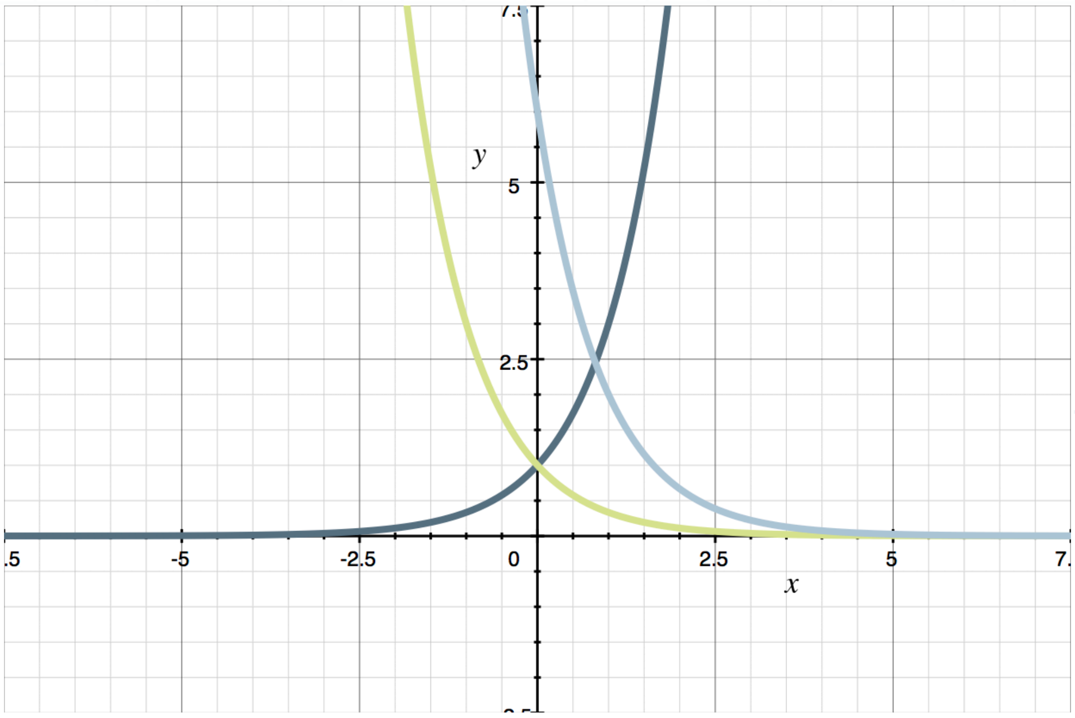
We were given the graph of $f(x)$,



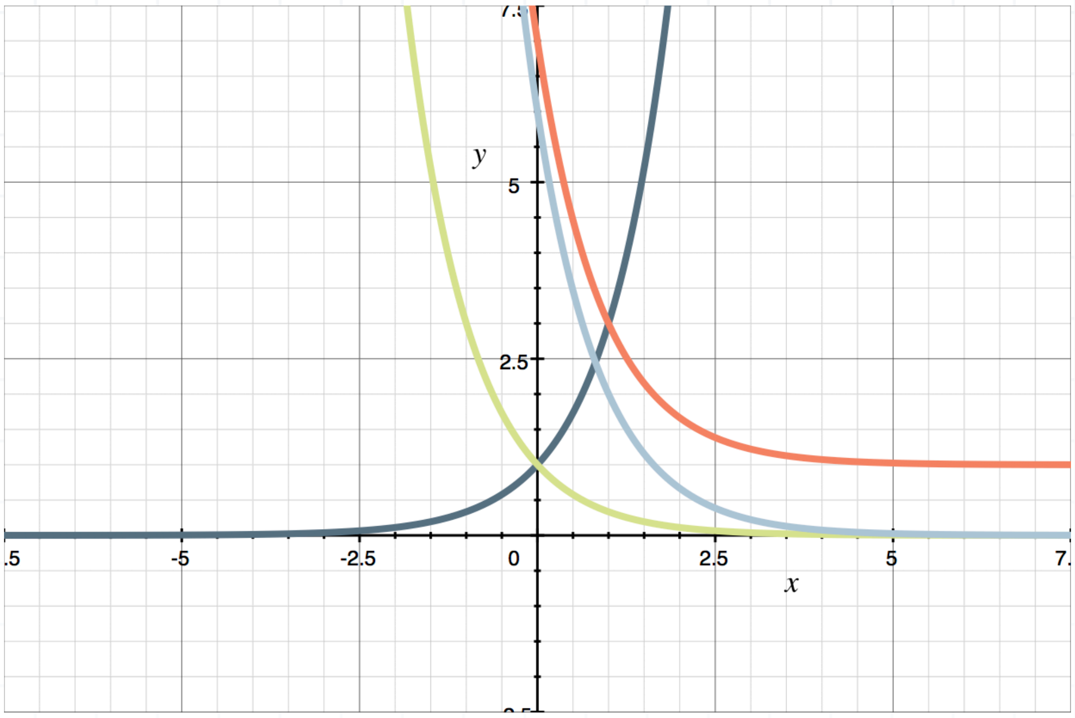
and we can think of $g(x) = 3^{-x}$ as the function we get when we replace x with $-x$, which reflects the graph over the vertical axis. If we graph $g(x) = 3^{-x}$ on the same set of axes as $f(x) = 3^x$, we get



To get $h(x) = 6 \cdot 3^{-x}$, we multiply $g(x)$ by 6. Since the graph of $g(x)$ crosses the y-axis at 1, the graph of $h(x)$ will cross the y-axis at $1 \cdot 6 = 6$.



To get $k(x)$, we add 1 to the value of $h(x)$, so we take the graph of $h(x)$ and shift it up by 1 unit. Since the graph of $h(x)$ crosses the y-axis at 6, the graph of $k(x)$ will cross the y-axis at $6 + 1 = 7$.



To summarize, we started with $f(x) = 3^x$, then applied a horizontal reflection to get $g(x) = 3^{-x}$, a vertical stretch to get $h(x) = 6 \cdot 3^{-x}$, and a vertical shift to get $k(x) = 6 \cdot 3^{-x} + 1$.

[1] $f(x) = 3^x$

[2] $g(x) = f(-x) = 3^{-x}$

[3] $h(x) = 6 \cdot g(x) = 6 \cdot 3^{-x}$

[4] $k(x) = h(x) + 1 = 6 \cdot 3^{-x} + 1$

An alternative procedure

In general, we can also use the following procedure for graphing exponential functions:

1. Plug in $x = 100$ and $x = -100$, and use the values $f(100)$ and $f(-100)$ to determine the “end behavior” of the function, that is, what happens to the value of the function as $x \to \infty$ and as $x \to -\infty$.

2. One of these will result in an infinite value, the other will give a real-number value. The real-number value is the horizontal asymptote of the exponential function.

3. Plug in a few easy-to-calculate values of x, like $x = -1, 0, 1$, in order to get a couple of points that we can plot.

4. Connect the points with an exponential curve, and draw the horizontal asymptote.

Let’s do an example where we walk through each of these steps.

Example

Graph the exponential function.

$$f(x) = -3^{x-1} - 2$$

We’ll start by plugging in $x = 100$ and $x = -100$.

For $x = 100$:

$$f(100) = -3^{100-1} - 2$$

$$f(100) = -3^{99} - 2$$

$$f(100) = -(\text{very large positive number}) - 2$$

$$f(100) = \text{very large negative number} - 2$$

$$f(100) = \text{very large negative number}$$

$$f(100) = -\infty$$

For $x = -100$:

$$f(-100) = -3^{-100-1} - 2$$

$$f(-100) = -3^{-101} - 2$$

$$f(-100) = -\frac{1}{3^{101}} - 2$$

$$f(-100) = -\frac{1}{\text{very large positive number}} - 2$$

$$f(-100) = -(0) - 2$$

$$f(-100) = 0 - 2$$

$$f(-100) = -2$$

We'll plug in a few values of x for which the value of $f(x)$ will be easy to calculate.

For $x = 0$:

$$f(0) = -3^{0-1} - 2$$

$$f(0) = -3^{-1} - 2$$

$$f(0) = -\frac{1}{3^1} - 2$$

$$f(0) = -\frac{1}{3} - \frac{6}{3}$$

$$f(0) = -\frac{7}{3}$$

For $x = -1$:

$$f(-1) = -3^{-1-1} - 2$$

$$f(-1) = -3^{-2} - 2$$

$$f(-1) = -\frac{1}{3^2} - 2$$

$$f(-1) = -\frac{1}{9} - \frac{18}{9}$$

$$f(-1) = -\frac{19}{9}$$

For $x = 1$:

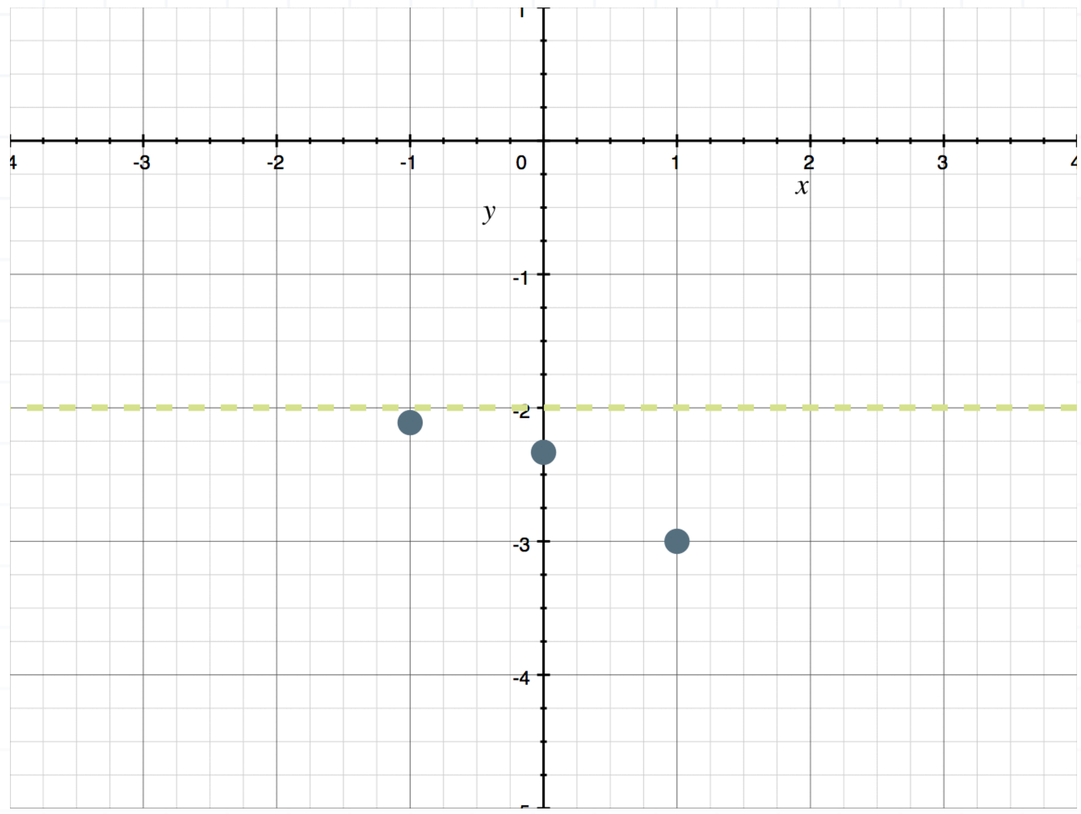
$$f(1) = -3^{1-1} - 2$$

$$f(1) = -3^0 - 2$$

$$f(1) = -1 - 2$$

$$f(1) = -3$$

Now we have three points of the graph of f: $(0, -7/3)$, $(-1, -19/9)$, and $(1, -3)$. If we plot these three points and draw the horizontal asymptote $y = -2$, we get



Based on the asymptote and the points we've found, we can already see what the graph is going to do. We'll simply connect the points to sketch the graph.