Imaginary and complex numbers

In this lesson we'll look at the imaginary number i, what it means, and how to use it in expressions.

The **imaginary number** i is defined as the square root of -1, and we can use it in algebraic expressions. An imaginary number (in general) is defined as a number that can be written as a product of a real number and i. For instance, $4i$ and $-15i$ are imaginary numbers.

Properties of imaginary numbers

These are the things we need to know about imaginary numbers.

1. The formulas for i and i^2 are

   $i=\sqrt{-1}$ and $i^2=-1$

   We can use these formulas to express i^3 as the imaginary number $-i$, and i^4 as the real number 1.

   $$i^3=i^{2+1}=(i^2)(i^1)=(-1)(i)=-i$$

   $$i^4=i^{2+2}=(i^2)(i^2)=(-1)(-1)=1$$

   In fact, if n is any positive integer, then we can express i^n as either an imaginary number (if n is odd) or a real number (if n is even).

2. When we add or subtract expressions with i raised to the same power, we treat them as like terms.

For example, in the list

$i, 3i^2, 4, 2i, 8, 5i^2$

the like terms are

i and $2i$

$3i^2$ and $5i^2$

4 and 8

We could in turn use $i^2 = -1$ to express $3i^2$ and $5i^2$ as -3 and -5, respectively, so the like terms in the list $i, 3i^2, 4, 2i, 8, 5i^2$ would end up being

i and $2i$

$-3, 4, 8,$ and -5

3. If we have the sum of a real number and an imaginary number, we should write the real number first and the imaginary number second.

So $-6i + 8$ should be written as $8 - 6i$, with the real number first and the imaginary number second. A number that can be written as the sum of a real number and an imaginary number (a number that can be written in the form $a + bi$ where a and b are real numbers) is called a **complex number**.

Arithmetic with imaginary numbers

Now let's look at how we can perform basic operations with complex numbers, like addition and subtraction.

When we want to add complex numbers, we combine the real parts and imaginary parts separately. In general, adding complex numbers looks like

$$z_1 + z_2 = a + ib + c + id$$

$$z_1 + z_2 = (a + c) + (ib + id)$$

$$z_1 + z_2 = (a + c) + i(b + d)$$

So we get the general formula

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

We can subtract complex numbers using the same principle, by subtracting the real parts and imaginary parts separately. Then the subtraction of complex numbers looks like

$$z_1 - z_2 = (a + ib) - (c + id)$$

$$z_1 - z_2 = a + ib - c - id$$

$$z_1 - z_2 = (a - c) + (ib - id)$$

$$z_1 - z_2 = (a - c) + i(b - d)$$

So we get the general formula

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Let's wrap it up by outlining the general steps for adding and subtracting complex numbers.

1. Separate the real and imaginary parts of the complex numbers.
2. Add/subtract the real parts of the complex numbers.
3. Add/subtract the imaginary parts of the complex numbers.
4. Write the final answer in $a+bi$ format.

Let's begin with a simple example.

Example

Simplify the expression.

$$-1-8i-4-i$$

Begin by grouping the like terms.

$$-1-4-8i-i$$

Remember that there's an unwritten 1 in front of the i.

$$-1-4-8i-1i$$

Follow the usual addition and subtraction rules.

$$-5-9i$$

Let's look at another one.

Example

Simplify the expression.

$$\sqrt{-9}+\sqrt{9}+5+3i-\sqrt{-4}$$

Remember that

$$\sqrt{-1}=i$$

Let's start with the square roots.

$$\sqrt{9\cdot -1}+\sqrt{9}+5+3i-\sqrt{4\cdot -1}$$

$$\sqrt{9}\sqrt{-1}+\sqrt{9}+5+3i-\sqrt{4}\sqrt{-1}$$

$$3i+3+5+3i-2i$$

Now group like terms.

$$3+5+3i+3i-2i$$

$$8+4i$$

Let's do one final example.

Example

Simplify the expression.

$$-\sqrt{-25}+8i^3+2i-\sqrt{-4}\sqrt{4}+3\sqrt{-9}$$

Let's start with the square roots.

$$-\sqrt{25\cdot -1}+8i^3+2i-\sqrt{4\cdot -1}\sqrt{4}+3\sqrt{9\cdot -1}$$

$$-\sqrt{25}\sqrt{-1}+8i^3+2i-\sqrt{4}\sqrt{-1}\sqrt{4}+3\sqrt{9}\sqrt{-1}$$

$$-5i+8i^3+2i-2i\cdot 2+3\cdot 3i$$

Let's simplify $8i^3$ to $8i^2i$, which is $8(-1)i$, or $-8i$.

$$-5i-8i+2i-2i\cdot 2+3\cdot 3i$$

Now let's do the rest of the multiplication.

$$-5i-8i+2i-4i+9i$$

Finally, let's combine like terms, by doing the addition and subtraction from left to right:

$$-13i+2i-4i+9i$$

$$-11i-4i+9i$$

$$-15i+9i$$

$$-6i$$