Dividing multivariable polynomials

Dividing multivariable polynomials is very similar to dividing single-variable polynomials. We'll still use long division, but now we'll have more than one variable.

Filling in for missing terms in the dividend

When we were working with only one variable, we could easily fill in missing terms in advance. For example, if the divided contained an x^3 term and an x term, we can fill in the missing x^2 term in advance before we start the division, because we can easily see the progression of x^3, x^2, x, ...

But as soon as the divided includes more than one variable, filling in these missing terms can be difficult. For that reason, we'll usually begin the division, and then fill in any missing terms that we need as we go.

For instance, in this long division problem,

$$\begin{array}{r|l}
 & x^2 \quad + xy \quad - y^2 \\
\hline
x+y & x^3 + 2x^2y + 0xy^2 - y^3 \\
 & -(x^3 + x^2y) \\
\hline
 & \qquad x^2y + 0xy^2 \\
 & \quad -(x^2y + xy^2) \\
\hline
 & \qquad\qquad -xy^2 - y^3 \\
 & \qquad -(-xy^2 - y^3) \\
\hline
 & \qquad\qquad\qquad 0
\end{array}$$

we'll only add the $0xy^2$ term into the dividend once we get the $x^2y + xy^2$ expression on the fourth line. When we arrive at $x^2y + xy^2$, we realize that we have to subtract the xy^2 term, and that we don't have a like term in the dividend. So it's at that point that we add the $0xy^2$ term into the dividend.

Ordering the dividend's terms

Again, when we were working with only one variable, it was simple to order the terms in the dividend by listing them from highest power to lowest power. The x^3 term would come first, then the x^2 term, then the x term, etc.

But when we have two variables, we should put the terms in order of descending power, based on the power of the variable that leads the divisor.

So if the divisor is $x + y$, we should order the terms in the dividend by descending powers of x. Let's do an example.

Example

Find the quotient.

$$\frac{2x^3 + 12x^2y + 15xy^2 - 9y^3}{x + 3y}$$

If we order the divisor as $x+3y$, then x is the leading variable in the divisor, and we therefore want to order the terms in the dividend by decreasing powers of x.

$$
\begin{array}{r|l}
 & 2x^2 + 6xy - 3y^2 \\ \hline
x+3y & 2x^3 + 12x^2y + 15xy^2 - 9y^3 \\
 & \underline{-(2x^3 + 6x^2y)} \\
 & \qquad 6x^2y + 15xy^2 \\
 & \qquad \underline{-(6x^2y + 18xy^2)} \\
 & \qquad\qquad -3xy^2 - 9y^3 \\
 & \qquad\qquad \underline{-(-3xy^2 - 9y^3)} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

But if we order the divisor as $3y+x$ instead, then y is the leading variable in the divisor, and we therefore want to order the terms in the dividend by decreasing powers of y.

$$
\begin{array}{r|l}
 & -3y^2 + 6xy + 2x^2 \\ \hline
3y+x & -9y^3 + 15xy^2 + 12x^2y + 2x^3 \\
 & \underline{-(-9y^3 - 3xy^2)} \\
 & \qquad 18xy^2 + 12x^2y \\
 & \qquad \underline{-(18xy^2 + 6x^2y)} \\
 & \qquad\qquad 6x^2y + 2x^3 \\
 & \qquad\qquad \underline{-(6x^2y + 2x^3)} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

In both cases, the result is the same, we find the same quotient, but the terms in the quotient show up in the opposite order.