Change of base

It's easier for us to evaluate logs to base 10 or base e, because calculators usually have $\log$ and $\ln$ buttons for these. When the base is anything other than 10 or e, we can use the **change of base** formula.

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Notice that, given a log function with base a and argument b, we can pick any value that we'd like to be the new base, c. Which is really helpful, because we can pick a new base of 10 or e if either of them is convenient for us.

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \frac{\log b}{\log a} \qquad \log_a b = \frac{\log_e b}{\log_e a} = \frac{\ln b}{\ln a}$$

Let's look at an example where we use the change of base formula.

Example

Estimate the log to four decimal places.

$$\log_5 4$$

We can use the change of base formula.

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_5 4 = \frac{\log_{10} 4}{\log_{10} 5}$$

Now we can use a calculator to get the answer.

$$\log_5 4 \approx \frac{0.6021}{0.6990}$$

$$\log_5 4 \approx 0.8614$$

Realize that we can also work backwards, backing our way into the change of base formula.

Example

Simplify the expression to a single real number without using a calculator.

$$\frac{\log 625}{\log 25}$$

If we use the change of base formula, we can rewrite this expression in the form $\log_a b$.

$$\log_{25} 625$$

Let $x = \log_{25} 625$. Then, using the general rule for logarithms, we have

$$25^x = 625$$

Now we want to rewrite both sides of the equation in terms of the same base.

$$(5^2)^x = 5^4$$

$$5^{2x} = 5^4$$

Since the bases are equal, the exponents must also be equal in order for the equation to be true.

$$2x = 4$$

$$x = 2$$

Therefore, the value of the original expression is 2:

$$\frac{\log 625}{\log 25} = 2$$

We can also solve other kinds of exponential equations using logs and the change of base formula.

Example

Use logs to solve the equation.

$$10 \cdot 5^{2x} = 300$$

In problems like this, we have an equation, and we need to solve for the variable, x, which means we need to get x by itself on one side of the equation. In this particular example, we can start by dividing both sides by 10.

$$10 \cdot 5^{2x} = 300$$

$$5^{2x} = 30$$

Now we can use the general rule for logs to change this into a logarithmic equation.

$$\log_5 30 = 2x$$

We'll apply the change of base formula,

$$2x = \frac{\log 30}{\log 5}$$

And then we can solve for the variable.

$$x = \frac{\log 30}{2 \log 5}$$

This is the exact value of the variable, but we can also use a calculator to find the decimal value.

$$x \approx 1.0566$$
