Algebra 2 Notes

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Powers of negative bases

There are two cases to think of when we're simplifying powers of negative bases.

Case 1: Actually not a negative base at all

If we have something that looks like $-b^a$, where a and b are both positive real numbers, it's really the same as $-1 \cdot b^a$. Notice how the -1 can be pulled out in front, leaving just b^a. Because b is positive, this base actually isn't negative at all. This is just a case of raising a positive number to a power, and then changing the sign of the result by multiplying by -1. So when we see something like -4^2, it means the same thing as

$$-1(4^2)$$

$$-1(4 \cdot 4)$$

$$-1(16)$$

$$-16$$

This is because PEMDAS and the order of operations tells us that we need to take care of the exponent first, and then multiply by the negative sign.

Case 2: A negative sign included in the parenthesis

If we have something that looks like $(-b)^a$, where a and b are both positive real numbers, then raise the $-b$ inside the parentheses to the power of a. In other words, this is the multiplication in which the negative integer $-b$ appears as a factor a times (and there are no other factors).

This is the case most people think of when they perform operations with exponents. It means, for example, that $(-4)^2$ equals $(-4)(-4)$ or 16.

Be careful not to confuse an expression like $(-b)^a$ with an expression like $-b^a$ (which was our Case 1 example). The $-b^a$ case means that we first raise the positive integer b to the power of a, and then change the sign of the result (by placing a negative sign in front of it).

If a is even, then $(-b)^a = b^a$, and if a is odd, then $(-b)^a = -b^a$.

Example

Simplify the expression.

$$-2^3$$

By PEMDAS and the order of operations, we have to take care of the exponent first, and then apply the negative sign. Remember that applying a negative sign is the same as multiplying by -1.

$$-2^3$$

$$-(2 \cdot 2 \cdot 2)$$

$$-(8)$$

$$-8$$

Let's take a look at an example with a negative number inside parentheses.

Example

Simplify the expression.

$$(-1)^4$$

Remember that -1^4 is different than $(-1)^4$. When we have $(-1)^4$, the negative sign is included in the parentheses. This means we need to raise the -1 inside the parentheses to the power of 4, so it's the same thing as having four factors of -1.

$$(-1)^4$$

$$(-1)(-1)(-1)(-1)$$

Doing the multiplication form left to right (according to the order of operations), we get

$$(1)(-1)(-1)$$

$$(-1)(-1)$$

$$1$$

Or, since 4 is even, then $(-1)^4 = 1^4 = 1$.

Powers of fractions

This lesson will cover how to find the power of a fraction as well as introduce how to work with fractional exponents.

Powers of fractions

Say we have something like

$$\left(\frac{a}{b}\right)^c$$

where a, b, and c are integers. This is like saying that we're doing a multiplication in which a/b appears as a factor c times (and there are no other factors). This turns the power problem into a fraction multiplication problem, where we multiply the numerators and the denominators separately. In this example, a is the numerator in each factor, and b is the denominator in each factor.

Example

Simplify the expression.

$$\left(\frac{3}{4}\right)^2$$

This is an example of a power of a fraction. The way the problem is written, it's like saying that we're multiplying $3/4$ by itself, since the base is $3/4$ and the exponent is 2. So the problem becomes

$$\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)$$

Now we've got a fraction multiplication problem. When we multiply fractions, we multiply the numerators and the denominators separately.

$$\frac{3 \cdot 3}{4 \cdot 4} = \frac{9}{16}$$

Let's look at an example with variables.

Example

Simplify the expression.

$$\left(\frac{x}{y^3}\right)^4$$

This is an example of a power of a fraction. The way the problem is written, it's like saying that we're doing a multiplication in which x/y^3 appears as a factor four times (and there are no other factors), since the base is x/y^3 and the exponent is 4. So the problem becomes

$$\left(\frac{x}{y^3}\right)\left(\frac{x}{y^3}\right)\left(\frac{x}{y^3}\right)\left(\frac{x}{y^3}\right)$$

Now we've got a fraction multiplication problem. Remember, when we multiply fractions, we multiply the numerators and the denominators separately.

$$\frac{x \cdot x \cdot x \cdot x}{y^3 \cdot y^3 \cdot y^3 \cdot y^3}$$

Now we have a like base of x in the numerator and a like base of y in the denominator.

In the numerator we can write x^4 because x appears as a factor four times.

Remember that, when we have like bases, we can add the exponents. We'll need to do this for the denominator. Let's look at the calculation for the denominator:

$$y^3 \cdot y^3 \cdot y^3 \cdot y^3 = y^{3+3+3+3} = y^{12}$$

So the simplified expression is

$$\frac{x^4}{y^{12}}$$

We can also simplify $(a/b)^c$ by rewriting it as a fraction in which the numerator and the denominator are separately raised to the power c.

$$\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$$

Zero as an exponent

This lesson will cover how to find the value of a nonzero number (or a variable) raised to the power 0.

The rule for 0 as an exponent:

> Any nonzero real number raised to the power 0 is equal to 1, which means anything that looks like a^0 is equal to 1 if a is not equal to 0. (It's important for a to be nonzero, because 0^0 is undefined.)

The reason this is true comes from the quotient rule for exponents. We know that x^n/x^n is 1, since the numerator and denominator are equal. But according to the quotient rule for exponents, we also know that

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

Therefore, we know that $x^0 = 1$. As long as $x \neq 0$, the rule will hold, so let's look at an example.

Example

Simplify the expression.

$$9^0$$

Just remember that any nonzero real number raised to the power 0 is equal to 1, so

$$9^0 = 1$$

Let's look at another example.

Example

Simplify the expression.

$$99{,}102^0$$

Look different? Don't worry! Just remember that any nonzero real number raised to the power 0 is equal to 1, so

$$99{,}102^0 = 1$$

Let's try some examples with variables.

Example

Simplify the expression.

$$y^0$$

It's also true that any variable raised to the power 0 is equal to 1 (as long as the value of the variable isn't 0), so

$$y^0 = 1$$

We do need to assume that $y \neq 0$.

Good news! The rule is still true if we have more than one variable, or a combination of variables and numbers.

Example

Simplify the expression.

$$(3xy + a)^0$$

We know that any nonzero real number raised to the power 0 is equal to 1, and that the expression $3xy + a$ really is just a representation of a number. This means that

$$(3xy + a)^0 = 1$$

We do need to make the assumption that the value of the expression $3xy + a$ isn't 0.

Negative exponents

This lesson will cover how to find the value of a positive number raised to a negative power.

Remember that any number can be written as itself divided by 1. For example, 3 is equal to $3/1$. Also, remember that any number can be written as itself multiplied by 1, and that any nonzero number divided by itself is equal to 1.

The rule for negative exponents

If we have two positive real numbers a and b, then

$$a^{-b} = \frac{1}{a^b}$$

First we need to realize that a^{-b} is equal to

$$\frac{a^{-b}}{1}$$

We'll change the exponent in a^{-b} from $-b$ to b, and move the resulting expression from the numerator to the denominator, by performing a series of algebraic operations that we're already familiar with. First, we'll multiply $(a^{-b})/1$ by 1, which doesn't change its value.

$$\frac{a^{-b}}{1} \cdot 1$$

Next, we'll write that new factor of 1 as $(a^b)/(a^b)$.

$$\left(\frac{a^{-b}}{1}\right)\left(\frac{a^b}{a^b}\right)$$

Now we'll multiply the fractions, remembering to multiply their numerators and denominators separately.

$$\frac{(a^{-b})(a^b)}{(1)(a^b)}$$

Using the fact that $(1)(a^b) = a^b$, we get

$$\frac{(a^{-b})(a^b)}{a^b}$$

Remember that when we multiply numbers that have like bases (as in our numerator, where the base of each factor is a), we keep the base and add the exponents.

$$\frac{a^{(-b+b)}}{a^b}$$

By addition of the exponents ($-b$ and b), we get

$$\frac{a^0}{a^b}$$

Since a is nonzero, we have $a^0 = 1$.

$$\frac{1}{a^b}$$

Let's look at a few examples.

Example

Simplify the expression.

$$4^{-2}$$

Remember that 4^{-2} is equal to

$$\frac{4^{-2}}{1}$$

We'll change the exponent in 4^{-2} from -2 to 2 and move the resulting expression from the numerator to the denominator, which gives,

$$\frac{1}{4^2}$$

Now we'll perform the calculation in the denominator.

$$\frac{1}{4^2} = \frac{1}{4 \cdot 4} = \frac{1}{16}$$

Let's look at an example with a negative sign in front of the base.

Example

Simplify the expression.

$$-5^{-3}$$

Remember, we can rewrite -5^{-3} as

$$\frac{-5^{-3}}{1}$$

because they are of equal value.

We'll change the exponent in 5^{-3} from -3 to 3 and move the resulting expression (including the negative sign out in front) from the numerator to the denominator.

$$\frac{1}{-5^3}$$

We have to apply the exponent before we apply the negative sign, so the expression becomes

$$\frac{1}{-125}$$

$$-\frac{1}{125}$$

Let's also take a look at an example with a variable as the base.

Example

Write the expression with with no negative exponents.

$$x^{-3}$$

First, we need to realize that the expression x^{-3} is equal to

$$\frac{x^{-3}}{1}$$

We'll change the exponent in x^{-3} from -3 to 3 and move the resulting expression from the numerator to the denominator.

$$\frac{1}{x^3}$$

Of course, this is defined only if $x \neq 0$.

Negative exponents and product rule

This lesson will cover how to use the product rule for negative exponents to find the value of a positive number raised to a negative power.

Negative exponents in the numerator and denominator

If we have two positive real numbers a and b, then

$$a^{-b} = \frac{1}{a^b}$$

In order to change the exponent in a^{-b} from $-b$ to b, we move the a^{-b} from the numerator to the denominator to get $1/a^b$.

Alternatively, if we have two positive real numbers a and b, then

$$\frac{1}{a^{-b}} = a^b$$

In order to change the exponent in a^{-b} from $-b$ to b, we move the a^{-b} from the denominator to the numerator to get $1 \cdot a^b$, or just a^b.

In other words, if we have a negative exponent in the numerator, we can make it positive by moving that term to the denominator. Or if we have a negative exponent in the denominator, we can make it positive by moving that term to the numerator.

Also, we may have to deal with expressions like $(a/b)^{-n}$, in which case we need to swap the numerator and the denominator, and raise each of them to the nth power.

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n} = \frac{b^n}{a^n}$$

Reciprocals

By the way, a^b and a^{-b} are reciprocals. As we may remember, two numbers whose product is equal to 1 are reciprocals (of each other). Sometimes we'll hear or read about negative exponents and their relationship to reciprocals, and that relationship follows from the product rule for negative exponents. If we multiply a^b by a^{-b}, we can see that the product is 1, which means a^b and a^{-b} are reciprocals of each other.

$$(a^b) \cdot (a^{-b})$$

$$a^{b+(-b)}$$

$$a^{b-b}$$

$$a^0$$

$$1$$

So these two pairs are reciprocals of one another:

a^b and $\dfrac{1}{a^b}$

$$a^{-b} \text{ and } \frac{1}{a^{-b}}$$

Let's look at a few examples.

Example

Rewrite the expression with no negative exponents.

$$2^{-1}$$

In order to get rid of the negative exponent, we change the exponent in 2^{-1} from -1 to 1 and move the resulting expression from the numerator to the denominator, so we get

$$\frac{1}{2^1}$$

Since $2^1 = 2$, we can write this as

$$\frac{1}{2}$$

Let's look at an example with a variable.

Example

Write the expression with no negative exponents.

$$x^{-5}$$

In order to get rid of the negative exponent, we change the exponent in x^{-5} from -5 to 5 and move the resulting expression from the numerator to the denominator. We get

$$\frac{1}{x^5}$$

Let's look at another example.

Example

Write this expression with no negative exponents.

$$\frac{1}{b^{-7}}$$

In order to get rid of the negative exponent, we change the exponent in b^{-7} from -7 to 7 and move the resulting expression from the denominator to the numerator. We get $1 \cdot b^7$, which is equal to b^7.

Let's look at a final example, this time with a number other than 1 in the numerator.

Example

Write the expression with no negative exponents.

$$\frac{3}{x^{-5}}$$

In order to get rid of the negative exponent, we change the exponent in x^{-5} from -5 to 5 and move the resulting expression from the denominator to the numerator. We get

$$3x^5$$

Fractional exponents

In this lesson we'll work with both positive and negative fractional exponents.

When we have a fractional exponent, the numerator (of the exponent) is the power and the denominator is the root. In the expression $x^{\frac{a}{b}}$, where a and b are positive integers, a is the power and b is the root.

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

For instance,

$$x^{\frac{1}{3}} = \sqrt[3]{x}$$

$$x^{\frac{2}{5}} = \sqrt[5]{x^2}$$

$$x^{\frac{4}{3}} = \sqrt[3]{x^4}$$

Let's look at a quick example.

Example

Simplify the expression.

$$4^{\frac{3}{2}}$$

In the fractional exponent $3/2$, 3 is the power and 2 is the root, which means we can rewrite the expression as

$$\sqrt{4^3}$$

$$\sqrt{4 \cdot 4 \cdot 4}$$

$$\sqrt{64}$$

$$8$$

In other words, to make a problem easier to solve, we can break up the exponent by rewriting it as a product of two positive real numbers. For a fractional exponent a/b, two “natural” ways to do this are as follows:

$$\frac{a}{b} = a \cdot \left(\frac{1}{b}\right) \text{ and } \frac{a}{b} = \left(\frac{1}{b}\right) \cdot a$$

There’s a power rule for exponents which tells us that if c and d are positive real numbers, then

$$(x^c)^d = x^{c \cdot d}$$

In the expression $x^{a/b}$, therefore, we can let $c = a$ and $d = 1/b$, so we can rewrite $x^{a/b}$ as

$$[(x)^a]^{\frac{1}{b}}$$

But if we let $c = 1/b$ and $d = a$, we can rewrite $x^{a/b}$ as

$$[(x)^{\frac{1}{b}}]^a$$

Let’s do a few more examples.

Example

Simplify the expression.

$$\left(\frac{1}{9}\right)^{\frac{3}{2}}$$

9 is a perfect square, so we can simplify the problem by finding the square root first. We can rewrite the expression by breaking up the exponent.

$$\left[\left(\frac{1}{9}\right)^{\frac{1}{2}}\right]^3$$

Raising a number to the power $1/2$ is the same as taking the square root of that number, so we get

$$\left[\sqrt{\frac{1}{9}}\right]^3$$

$$\left(\frac{\sqrt{1}}{\sqrt{9}}\right)^3$$

$$\left(\frac{1}{3}\right)^3$$

This is the same as

$$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$$

$$\frac{1}{27}$$

Let's look at another example.

Example

Write the expression with no fractional exponents.

$$\left(\frac{1}{6}\right)^{\frac{3}{2}}$$

We can rewrite the expression by breaking up the exponent.

$$\left[\left(\frac{1}{6}\right)^{3}\right]^{\frac{1}{2}}$$

$$\left(\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\right)^{\frac{1}{2}}$$

Raising a number to the power $1/2$ is the same as taking the square root of that number, so we get

$$\sqrt{\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}}$$

$$\sqrt{\frac{1}{216}}$$

$$\frac{\sqrt{1}}{\sqrt{216}}$$

$$\frac{1}{\sqrt{36 \cdot 6}}$$

$$\frac{1}{\sqrt{36}\sqrt{6}}$$

$$\frac{1}{6\sqrt{6}}$$

What happens if we have a negative fractional exponent?

We should deal with the negative sign first, then use the rules for fractional exponents.

Example

Write the expression with no fractional exponents.

$$4^{-\frac{2}{5}}$$

First, we'll deal with the negative exponent. Remember that when a is a positive real number, both of these equations are true:

$$x^{-a} = \frac{1}{x^a}$$

$$\frac{1}{x^{-a}} = x^a$$

Therefore, we can rewrite $4^{-2/5}$ as

$$\frac{1}{4^{\frac{2}{5}}}$$

In the fractional exponent $2/5$, 2 is the power and 5 is the root, which means we can rewrite the expression as

$$\frac{1}{\sqrt[5]{4^2}}$$

$$\frac{1}{\sqrt[5]{16}}$$

Fractional exponents with like bases

If we start with something like $x^a \cdot x^{c/d}$ (where a, c, and d are integers and x is a variable or a real number), we have like bases because the base of both factors is x. When that's the case, we add the exponents.

$$x^a \cdot x^{\frac{c}{d}} = x^{a+\frac{c}{d}}$$

Now the problem is just about fraction addition.

Let’s look at an example.

Example

Simplify the expression.

$$a^3 \cdot a^{\frac{1}{4}}$$

We have like bases because the base of both factors is a. When that’s the case, we add the exponents.

$$a^{3+\frac{1}{4}}$$

Now the problem is just about fraction addition within the exponent. To find the sum in the exponent, we have to find a common denominator.

$$a^{3\left(\frac{4}{4}\right)+\frac{1}{4}}$$

$$a^{\frac{12}{4}+\frac{1}{4}}$$

$$a^{\frac{13}{4}}$$

Rationalizing the denominator

What is rationalizing the denominator? When we rationalize the denominator, it means we're taking all of the radicals out of the denominator of a fraction and moving them to the numerator. We do this by multiplying the numerator and denominator by the product of all the radicals in the original denominator.

Remember two facts:

Fact 1: We can multiply any number by 1 without changing its value.

$$\frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$$

Fact 2: We can divide any nonzero number by itself, and it'll equal 1.

$$\frac{\sqrt{2}}{\sqrt{2}} = 1$$

Let's look at a simple example so we can see how rationalizing the denominator works.

Example

Rationalize the denominator.

$$\frac{1}{\sqrt{2}}$$

We begin by multiplying the numerator and denominator by the radical in the denominator.

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{1\sqrt{2}}{2}$$

$$\frac{\sqrt{2}}{2}$$

This means

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Notice that we changed only how the number looks; we didn't change its value.

When we have a multistep problem, we might need to perform other operations as well.

Here are some helpful rules to remember about radicals:

Rule 1: $$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Rule 2: $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Rule 3: $a \cdot \sqrt{b} = a\sqrt{b}$

Rule 4: $\sqrt{a} \cdot \sqrt{a} = a$

Example

Simplify the expression, making sure to rationalize the denominator.

$$\sqrt{\frac{8}{49}} + \sqrt{\frac{14}{28}}$$

First simplify the fraction in the second radical to lowest terms.

$$\frac{14}{28}$$

$$\frac{1}{2}$$

Now we have

$$\sqrt{\frac{8}{49}} + \sqrt{\frac{1}{2}}$$

When we take the square root of a fraction, we can take the square roots of the numerator and denominator separately. Therefore, we can rewrite the expression as

$$\frac{\sqrt{8}}{\sqrt{49}}+\frac{\sqrt{1}}{\sqrt{2}}$$

Rewrite this by taking the square roots of any perfect squares.

$$\frac{\sqrt{8}}{7}+\frac{1}{\sqrt{2}}$$

Now we need to find a common denominator. Since we have only two terms, we can do this by multiplying the numerator and denominator of each fraction by the denominator of the other fraction.

$$\frac{\sqrt{8}}{7}\cdot\frac{\sqrt{2}}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cdot\frac{7}{7}$$

$$\frac{\sqrt{8}\sqrt{2}}{7\cdot\sqrt{2}}+\frac{1\cdot 7}{\sqrt{2}\cdot 7}$$

$$\frac{\sqrt{8}\sqrt{2}}{7\cdot\sqrt{2}}+\frac{1\cdot 7}{7\cdot\sqrt{2}}$$

$$\frac{\sqrt{16}}{7\sqrt{2}}+\frac{7}{7\sqrt{2}}$$

Now that we have a common denominator, add the fractions.

$$\frac{\sqrt{16}+7}{7\sqrt{2}}$$

$$\frac{4+7}{7\sqrt{2}}$$

$$\frac{11}{7\sqrt{2}}$$

Now we need to rationalize the denominator.

$$\frac{11}{7\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{11\sqrt{2}}{7\sqrt{2}\sqrt{2}}$$

$$\frac{11\sqrt{2}}{7 \cdot 2}$$

$$\frac{11\sqrt{2}}{14}$$

Let's look at one more example.

Example

Simplify the expression, making sure to rationalize the denominator.

$$4\sqrt{\frac{2}{3}} - 5\sqrt{\frac{3}{2}} + \sqrt{150}$$

We know that when we take the square root of a fraction, we can take the square roots of the numerator and denominator separately.

$$4\frac{\sqrt{2}}{\sqrt{3}}-5\frac{\sqrt{3}}{\sqrt{2}}+\sqrt{150}$$

Now we'll factor the 150 (under the radical sign in the last term) as $25 \cdot 6$, which will ultimately help us to simplify things because 25 is a perfect square.

$$4\frac{\sqrt{2}}{\sqrt{3}}-5\frac{\sqrt{3}}{\sqrt{2}}+\sqrt{25 \cdot 6}$$

$$4\frac{\sqrt{2}}{\sqrt{3}}-5\frac{\sqrt{3}}{\sqrt{2}}+\sqrt{25}\sqrt{6}$$

$$\frac{4\sqrt{2}}{\sqrt{3}}-\frac{5\sqrt{3}}{\sqrt{2}}+5\sqrt{6}$$

We can divide the $5\sqrt{6}$ in the last term by 1, which won't change its value.

$$\frac{4\sqrt{2}}{\sqrt{3}}-\frac{5\sqrt{3}}{\sqrt{2}}+\frac{5\sqrt{6}}{1}$$

Now we need to find a common denominator so that we can combine the fractions. We can use the product of the three denominators ($\sqrt{3}$, $\sqrt{2}$, and 1) as our common denominator:

$$\sqrt{3} \cdot \sqrt{2} \cdot 1$$

$$\sqrt{6}$$

We'll multiply the numerator and denominator of each fraction by whatever gets us $\sqrt{6}$ in the denominator.

$$\frac{4\sqrt{2}}{\sqrt{3}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)-\frac{5\sqrt{3}}{\sqrt{2}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)+\frac{5\sqrt{6}}{1}\left(\frac{\sqrt{6}}{\sqrt{6}}\right)$$

$$\frac{4\sqrt{2}\sqrt{2}}{\sqrt{3}\sqrt{2}}-\frac{5\sqrt{3}\sqrt{3}}{\sqrt{2}\sqrt{3}}+\frac{5\sqrt{6}\sqrt{6}}{1\sqrt{6}}$$

$$\frac{4\cdot 2}{\sqrt{6}}-\frac{5\cdot 3}{\sqrt{6}}+\frac{5\cdot 6}{\sqrt{6}}$$

$$\frac{8}{\sqrt{6}}-\frac{15}{\sqrt{6}}+\frac{30}{\sqrt{6}}$$

$$\frac{23}{\sqrt{6}}$$

Now we need to rationalize the denominator.

$$\frac{23}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}$$

$$\frac{23\sqrt{6}}{6}$$

Rationalizing with conjugate method

Remember that “rationalize the denominator” just means “get the radicals out of the denominator.”

We already know how to rationalize the denominator if the denominator has just one term and it consists of a square root and nothing else.

Example

Rationalize the denominator.

$$\frac{6}{\sqrt{13}}$$

If we multiply the denominator by $\sqrt{13}$, we’ll get rid of the square root there. But in order to keep the value of the fraction the same, we have to multiply both the numerator and the denominator by $\sqrt{13}$.

$$\frac{6}{\sqrt{13}}\left(\frac{\sqrt{13}}{\sqrt{13}}\right)$$

$$\frac{6\sqrt{13}}{13}$$

Now that the square root is out of the denominator, we’ve rationalized the denominator.

But how do we rationalize the denominator when it has two terms, such as the denominator in the expression

$$\frac{3}{5-\sqrt{3}}$$

In a case like this one, where the denominator is the sum or difference of two terms, at least one of which contains a square root (and neither term contains a higher root - a cube root, a fourth root, ...), we can use the *conjugate method* to rationalize the denominator.

The conjugate of a binomial has the same two terms, but with the opposite sign in between. For $5-\sqrt{3}$, we keep the same two terms, 5 and $\sqrt{3}$, but we change the sign in the middle. Since the sign in this case is negative, we'll change it to a positive sign.

Original binomial	$5-\sqrt{3}$
Its conjugate	$5+\sqrt{3}$

To rationalize the denominator of a fraction where the denominator is a binomial, we'll multiply both the numerator and denominator by the conjugate.

As we're doing these problems, let's also remember these facts:

Fact 1: We can multiply any number by 1 without changing its value.

$$\frac{1}{\sqrt{2}-5}\cdot 1 \text{ is the same as } \frac{1}{\sqrt{2}-5}$$

Fact 2: We can write 1 as any nonzero number divided by itself, for example,

$$\frac{\sqrt{2}+5}{\sqrt{2}+5}=1$$

Example

Rationalize the denominator.

$$\frac{3-\sqrt{2}}{\sqrt{2}-5}$$

We want to use the conjugate method to get the radical out of the denominator. Remember that the conjugate of a binomial has the same two terms but with the opposite sign between them. So the conjugate of $\sqrt{2}-5$ is $\sqrt{2}+5$. This is the binomial that both the numerator and denominator have to be multiplied by.

$$\frac{3-\sqrt{2}}{\sqrt{2}-5}$$

$$\frac{3-\sqrt{2}}{\sqrt{2}-5}\cdot\frac{\sqrt{2}+5}{\sqrt{2}+5}$$

Now this becomes a binomial multiplication problem. We need to make sure to multiply our first terms, outer terms, inner terms, and last terms (FOIL).

$$\frac{\left(3-\sqrt{2}\right)\left(\sqrt{2}+5\right)}{\left(\sqrt{2}-5\right)\left(\sqrt{2}+5\right)}$$

$$\frac{3\sqrt{2}+15-2-5\sqrt{2}}{2+5\sqrt{2}-5\sqrt{2}-25}$$

Group together the terms that contain $\sqrt{2}$, and group together the terms that have no radical. Then combine (add) the terms within each group. Do this for the numerator and denominator separately.

$$\frac{\left(3\sqrt{2}-5\sqrt{2}\right)+(15-2)}{\left(5\sqrt{2}-5\sqrt{2}\right)+(2-25)}$$

$$\frac{-2\sqrt{2}+13}{-23}$$

Multiply by $(-1)/(-1)$ to remove the negative sign from the denominator.

$$\frac{-2\sqrt{2}+13}{-23}\cdot\frac{-1}{-1}$$

$$\frac{2\sqrt{2}-13}{23}$$

Let’s do another example.

Example

Rationalize the denominator.

$$\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$

We want to use the conjugate method to get the radical out of the denominator. Remember that the conjugate of a binomial has the same two terms but with the opposite sign between them. So the conjugate of $\sqrt{5}+\sqrt{7}$ is $\sqrt{5}-\sqrt{7}$. This is the binomial that both the numerator and denominator have to be multiplied by.

$$\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$

$$\frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}+\sqrt{7}} \cdot \frac{\sqrt{5}-\sqrt{7}}{\sqrt{5}-\sqrt{7}}$$

Now this becomes a binomial multiplication problem. We need to make sure to multiply our first terms, outer terms, inner terms, and last terms.

$$\frac{\left(\sqrt{5}-\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)}{\left(\sqrt{5}+\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)}$$

$$\frac{5-\sqrt{35}-\sqrt{35}+7}{5-\sqrt{35}+\sqrt{35}-7}$$

$$\frac{5-2\sqrt{35}+7}{5-7}$$

$$\frac{12-2\sqrt{35}}{-2}$$

Factor out a 2 in the numerator.

$$\frac{2\left(6-\sqrt{35}\right)}{-2}$$

Now cancel the 2 in the numerator against the 2 in the denominator.

$$\frac{6-\sqrt{35}}{-1}$$

$$-6+\sqrt{35}$$

$$\sqrt{35}-6$$

Let's try one more with numbers.

Example

Rationalize the denominator.

$$\frac{3}{5-\sqrt{3}}$$

Since the denominator is a binomial in which one of the terms is a square root, we need to multiply the numerator and denominator by the conjugate of the binomial in order to rationalize the denominator.

$$\frac{3}{5-\sqrt{3}}\left(\frac{5+\sqrt{3}}{5+\sqrt{3}}\right)$$

$$\frac{15+3\sqrt{3}}{25+5\sqrt{3}-5\sqrt{3}-3}$$

Using the conjugate method, the two terms in the middle of the denominator will always cancel with each other.

$$\frac{15+3\sqrt{3}}{25-3}$$

$$\frac{15+3\sqrt{3}}{22}$$

Now that the square root is out of the denominator, we've rationalized the denominator.

We can even use the conjugate method with variables, and with square roots in both terms in the denominator.

Example

Rationalize the denominator.

$$\frac{a+\sqrt{3}}{-3\sqrt{a}+\sqrt{3}}$$

First, we'll switch the order of the terms in the denominator so that we lead with a positive term instead of a negative term, just to make things a little simpler.

$$\frac{a+\sqrt{3}}{\sqrt{3}-3\sqrt{a}}$$

Then we'll multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{a+\sqrt{3}}{\sqrt{3}-3\sqrt{a}}\left(\frac{\sqrt{3}+3\sqrt{a}}{\sqrt{3}+3\sqrt{a}}\right)$$

$$\frac{a\sqrt{3}+3a\sqrt{a}+3+3\sqrt{3a}}{3+3\sqrt{3a}-3\sqrt{3a}-9a}$$

Using the conjugate method, the two terms in the middle of the denominator will always cancel with each other.

$$\frac{a\sqrt{3}+3a\sqrt{a}+3+3\sqrt{3a}}{3-9a}$$

Ratios and proportions

In this lesson we'll learn how to set up and solve ratio and proportion word problems.

For word problems, the best thing to do is to look at a few examples, but first let's review a few vocabulary terms.

> **Ratio**: A ratio is a comparison of two numbers (or other mathematical expressions), and it is often written as a fraction. For example, if there are 2 boys and 5 girls in a group, then the ratio can be written as $2:5$ or $2/5$.
>
> **Proportion**: A proportion is an equality between two ratios.

One process that will come in handy in dealing with ratio and proportion is **cross multiplication**. When we cross multiply, we start with a proportion like

$$\frac{a}{b}=\frac{c}{d}$$

Then we multiply the numerator on the left side by the denominator on the right side (we multiply a by d), and we multiply the numerator on the right side by the denominator on the left side (we multiply c by b). And finally, we equate those two products:

$$a\cdot d=c\cdot b$$

To see why this works (and is mathematically legal!), notice that if we start with the equation $a/b = c/d$, we can multiply both sides by both denominators (b and d):

$$\left(\frac{a}{b}\right) \cdot b \cdot d = \left(\frac{c}{d}\right) \cdot b \cdot d$$

On the left side, we can cancel the b's, and on the right side we can cancel the d's, so we get

$$a \cdot d = c \cdot b$$

This is exactly the same equation we get when we start with the proportion $a/b = c/d$ and cross multiply.

Now let's do some problems.

Example

A pet store has 12 dogs, 14 fish, and 32 cats. What is the ratio of cats to total animals?

We know there are 32 cats, and we're looking for the ratio

$$\frac{\text{cats}}{\text{total}}$$

We can find the total number of animals by adding the numbers of animals in the three groups.

$$12 + 14 + 32 = 58$$

So we get the ratio

$$\frac{32}{58}$$

We need to simplify the ratio to lowest terms.

$$\frac{16(2)}{29(2)}$$

$$\frac{16}{29}$$

There are 16 cats for every 29 animals.

Let's look at another type of word problem.

Example

Two numbers have a ratio of 1 to 9, and a sum of 80. What are the two numbers?

Let's call the two unknown numbers x (for the first one) and y (for the second one). Then we'll set up a proportion, by equating the ratio x/y to the ratio $1/9$, and solve for one of the variables in terms of the other.

$$\frac{x}{y} = \frac{1}{9}$$

Cross multiply.

$$9x = 1y$$

$$y = 9x$$

Next, set up an equation for x and y using what we know about their sum.

$$x + y = 80$$

Let's solve this for y in terms of x, since we already know that $y = 9x$.

$$y = 80 - x$$

Now let's use our system of equations and the substitution method. We know:

$$y = 80 - x$$

$$y = 9x$$

Since we have two expressions equal to y, we can set those expressions equal to each other and solve for x.

$$80 - x = 9x$$

$$80 = 9x + x$$

$$80 = 10x$$

$$\frac{80}{10} = \frac{10x}{10}$$

$$8 = x$$

$$x = 8$$

Now we can use $y = 9x$ and the fact that $x = 8$ to solve for y.

$$y = 9(8)$$

$$y = 72$$

Let's check our work.

It's true that

$$\frac{8}{72} = \frac{1}{9}$$

and that

$$8 + 72 = 80$$

Let's do one more like that one.

Example

Two numbers have a ratio of 7 to 13, and a sum of 300. What are the two numbers?

Let's call the two unknown numbers x and y. Set up a proportion, by equation the ratio x/y to the ratio $7/13$, and solve for one of the variables in terms of the other.

$$\frac{x}{y} = \frac{7}{13}$$

Cross multiply.

$$13x = 7y$$

Solve for x.

$$\frac{13x}{13} = \frac{7y}{13}$$

$$x = \frac{7}{13}y$$

Next, set up an equation for x and y using what we know about their sum.

$$x + y = 300$$

Let's solve for x in terms of y, since we already know that $x = (7/13)y$.

$$x = 300 - y$$

Now let's use our system of equations and the substitution method. We know:

$$x = 300 - y$$

$$x = \frac{7}{13}y$$

Since we have two expressions equal to x, we can set those expressions equal to each other and solve for y.

$$\frac{7}{13}y = 300 - y$$

$$\frac{7}{13}y + y = 300$$

$$\frac{7}{13}y + \frac{13}{13}y = 300$$

$$\frac{20}{13}y = 300$$

$$\frac{13}{20} \cdot \frac{20}{13}y = \frac{13}{20} \cdot 300$$

$$y = 13 \cdot 15$$

$$y = 195$$

Now solve for x.

$$x = 300 - 195$$

$$x = 105$$

We can use the original equations to check double-check that

$$\frac{105}{195} = \frac{7}{13}$$

and that

$$105 + 195 = 300$$

Chemical compounds

Chemical compounds are a great real-world illustration of ratio and proportion. A chemical compound like water, H_2O, is made up of two hydrogen atoms and one oxygen atom, and in these kinds of chemical compound problems, we can figure out the proportion of the hydrogen compared to the entire compound, or the proportion of oxygen compared to the entire compound.

In other words, in this lesson we'll learn how to find the molar mass of a chemical compound given the molar masses of the elements (of the periodic table) that make up the compound, and how to find the molar mass of an individual element in a compound given the molar mass of the compound. We'll also learn how to find the weight of an element in a compound given the molar mass and weight of a compound.

The molar mass of a compound is the sum of products of the atomic weight of each element that make up the compound and the numbers of atoms of the corresponding element in one molecule of that compound.

Molar mass is measured in **grams per mole**, which is often abbreviated g/mol.

Let's look at a few examples.

Example

Find the molar mass of table sugar.

Table sugar has the molecular formula $C_{12}H_{22}O_{11}$.

Carbon has a molar mass of 12.01 g/mol.

Hydrogen has a molar mass of 1.01 g/mol.

Oxygen has a molar mass of 16.00 g/mol.

We can see that one molecule of table sugar $C_{12}H_{22}O_{11}$ has 12 carbon atoms, 22 hydrogen atoms, and 11 oxygen atoms, which means the ratio of atoms is $12:22:11$.

Multiply the numbers of atoms by the corresponding molar masses, and add them up.

carbon: $12(12.01) = 144.12$ g in 1 mole of table sugar

hydrogen: $22(1.01) = 22.22$ g in 1 mole of table sugar

oxygen: $11(16.00) = 176.00$ g in 1 mole of table sugar

Find the total.

$$144.12 + 22.22 + 176.00 = 342.34 \text{ g in } 1 \text{ mole of table sugar}$$

Therefore, the molar mass of table sugar is 342.34 g/mol.

Let's look at an example where we know the molar mass of a compound and we want the molar mass of an individual element in that compound.

Example

What is the mass of cadmium in one mole of cadmium carbonate?

Cadmium carbonate is made of cadmium, carbon, and oxygen; its molecular formula is $CdCO_3$

$CdCO_3$ has a molar mass of about 172.42 g/mol.

Carbon has a molar mass of 12.01 g/mol.

Oxygen has a molar mass of 16.00 g/mol.

We can see from the molecular formula that the ratio of atoms is $1:1:3$. We haven't been told the molar mass of cadmium, so we'll use the variable x for that unknown quantity.

Let's summarize the information we have about cadmium carbonate.

cadmium: $1x = x$ g in 1 mole of cadmium carbonate

carbon: $1(12.01) = 12.01$ g in 1 mole of cadmium carbonate

oxygen: $3(16.00) = 48.00$ g in 1 mole of cadmium carbonate

$CdCO_3$: 172.42 g/mol

Now let's set up an equation to solve for the molar mass of cadmium.

$$x + 12.01 + 48.00 = 172.42$$

$$x + 60.01 = 172.42$$

$$x = 112.41 \text{ g/mol}$$

In this case we know that the mass of cadmium in one mole of cadmium carbonate is 112.41 g, because there is only one cadmium atom in a molecule of cadmium carbonate.

Let's do another example.

Example

Imagine we have 250 g of sodium hydrocarbonate, $NaHCO_3$, and we want to know the combined mass of sodium and carbon in this compound.

Sodium hydrocarbonate is made of sodium, hydrogen, carbon, and oxygen; its molecular formula is $NaHCO_3$

$NaHCO_3$ has a molar mass of about 84.007 g/mol.

Sodium has a molar mass of 22.989769 g/mol.

Carbon has a molar mass of 12.01 g/mol.

First, calculate the combined molar mass of the sodium and carbon.

$$22.989769 + 12.01 = 34.999769 \text{ g/mol.}$$

Let's use x as a combined mass of sodium and carbon in sodium hydrocarbonate. We can now set up the proportion which consists of two ratios. The first is the ratio of the combined molar mass of sodium and carbon to the molar mass of the sodium hydrocarbonate, and the second is the ratio of the combined mass of sodium and carbon, x, to the mass of the sodium hydrocarbonate. Therefore, we have

$$\frac{34.999769}{84.007} = \frac{x}{250}$$

$$x = \frac{250(34.999769)}{84.007} \approx 104 \text{ g}$$

The combined mass of sodium and carbon in 250 g of sodium hydrocarbonate is approximately 104 g.

Fractions to decimals to percents

In this lesson we'll learn how to convert between fractions, decimals, and percents. We can always use a proportion to help us.

$$\frac{\text{percent}}{100} = \frac{\text{part}}{\text{whole}}$$

We can also use these rules:

1. A percent means some indicated part out of 100. For instance, $4\,\%$ means 4 out of every 100.

2. To change a percent to a decimal, divide by 100. For instance, to change $49\,\%$ to a decimal, divide it by 100.

$$49\,\% = \frac{49}{100} = 0.49$$

3. To change a decimal to a percent, multiply by 100. For instance, to change 0.05 to a percent, multiply it by 100.

$$0.05 \cdot 100 = 5\,\%$$

4. To change a fraction to a percent, first change the fraction to a decimal, then change the decimal to a percent. For instance, to change $1/4$ to a percent, first change it to 0.25, and then multiply 0.25 by 100 to get the percent.

$$\frac{1}{4} = 0.25$$

$$0.25 \cdot 100 = 25\,\%$$

5. To find a percent of a number in decimal form, change the percent to a decimal and multiply it by that number. For instance, to find 6% of 99, convert 6% to a decimal by dividing by 100.

$$\frac{6}{100}=0.06$$

Then multiply 0.06 by 99.

$$0.06\cdot 99=5.94$$

6% of 99 is 5.94

Let's look at a few other examples of percent problems.

Example

Find a mixed number that represents the given value.

9% of 160

To find 9% of 160, we set it up as

$$\frac{9}{100}\cdot 160$$

$$\frac{9}{5}\cdot 8$$

$$\frac{72}{5}$$

5 goes into 72 fourteen times, with a remainder of 2, so we can change the improper fraction to a mixed number and get

$$14\frac{2}{5}$$

Let's look at one more example of converting fractions to percents.

Example

Convert the fraction to a percent.

$$\frac{120}{180}$$

First, since the fraction isn't already in lowest terms, we'll reduce it to lowest terms.

$$\frac{120 \div 60}{180 \div 60}$$

$$\frac{2}{3}$$

One way we can convert this fraction to a percent is to first convert it to a decimal using long division, and then convert the decimal to a percent by moving the decimal place, or we can set up the proportion

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$$

and use the variable x for the missing piece (the percent).

$$\frac{2}{3} = \frac{x}{100}$$

$$2 \cdot 100 = 3x$$

$$200 = 3x$$

$$x = \frac{200}{3}$$

$$x \approx 66.66...\%$$

We could round a repeating decimal to an indicated decimal place. For example, if we round $66.66...$ to the hundredths place (round it to two decimal places), we'll get 66.67%.

Percent markup

In this lesson we will learn how to calculate a percent markup and how it changes the price of an item.

What is a percent markup? Well in retail, a store will buy an item for a certain price from a manufacturer. In order to make money, the store has to add to the price they paid for it and sell it to the customer for more.

The store usually does this as a percentage of the original amount, and if that's the case we call this a percent markup.

Here are three different formulas we can use to describe this:

$$\text{New Price} = \text{Original Price} + \text{Markup Amount}$$

$$\text{New Price} = \text{Original Price}\left(1 + \frac{\text{Percent Markup}}{100}\right)$$

$$\text{Markup Amount} = \text{Original Price}\left(\frac{\text{Percent Markup}}{100}\right)$$

In this formula, the "new price" is the price the end customer pays the store for the item. This is often called the "selling price."

The "original price" is the price the store paid the manufacturer for the item. This is sometimes called the "manufacturer's price" or the "original purchase price."

The "percent markup" is the percent of the original price by which the store marked up the item in order to get to the new price.

Let's look at a couple of examples.

Example

The manager of a user-car dealership purchases a used car for $\$13{,}500.00$. The percent markup on the car is $45\,\%$. What is the selling price of the car?

First, we'll find the amount of money the company added to the original price of the car. To do this, we'll find $45\,\%$ of the original price of the car.

$45\,\%$ of $\$13{,}500$

$$\frac{45}{100}\cdot \$13{,}500$$

$$\$6{,}075$$

This means the company marked up the original price by $\$6{,}075$, so now we need to add this to the original price to find the selling price.

$$\$13{,}500+\$6{,}075$$

$$\$19{,}575$$

Let's look at an example of how to find the manufacturer's price.

Example

A bakery purchases a set of pre-made cakes from the manufacturer and sells them to their customers for $\$545.82$. The markup was 65%. What price did the bakery pay the manufacturer?

If the bakery paid x for the cakes and marked them up by 65%, then the price they're selling it to their customers for is 1.65 times the price they paid for it. We use 1.65 because the price the bakery will charge their customers for the cakes is 100% of what they paid the manufacturer, plus 65% of that amount for their own markup. The 100% plus the 65% means they're charging 165%, or 1.65 times what they paid.

$$1.65x = \$545.82$$

$$\frac{1.65x}{1.65} = \frac{\$545.82}{1.65}$$

$$x = \$330.80$$

The bakery paid $\$330.80$ for the cakes, marked them up by 65%, which was a markup of $\$215.02$ ($\$330.80 + \$215.02 = \$545.82$), and then sold them to their customers for $\$545.82$.

Percent markdown

In this lesson we'll learn how to calculate the sale price of an item, that is, the price of an item that's sold at a discount.

What is a percent markdown? In retail, a store will discount an item in order to sell it more quickly. The store usually does this as a percentage off the original amount they had planned to charge, and if that's the case we call this a percent markdown.

Here are three different formulas we can use to describe this:

$$\text{Discount Price} = \text{Original Price} - \text{Discount Amount}$$

$$\text{Discount Price} = \text{Original Price}\left(1 - \frac{\text{Percent Markdown}}{100}\right)$$

$$\text{Discount Amount} = \text{Original Price}\left(\frac{\text{Percent Markdown}}{100}\right)$$

A useful proportion:

$$\frac{\text{Discount Amount}}{\text{Original Price}} = \frac{\text{Percent Markdown}}{100}$$

The percent markdown is the percentage off the original price of the item; sometimes we say the item is on sale for ___% off.

The original price (regular price) is the price the person or store was selling the item for before they discounted it.

The discount amount is the amount in dollars that is taken off the original price.

The sale price is the price of the item after the discount is applied.

Let's do a couple of examples.

Example

A shirt has an original price of $\$25.00$, and it's now on sale for $\$22.50$. What is the percent markdown?

Here's what we know:

Original Price: $\$25.00$

Sale Price: $\$22.50$

We can find the discount amount:

$$\$25.00 - \$22.50 = \$2.50$$

Now use the proportion:

$$\frac{\text{Discount Amount}}{\text{Original Price}} = \frac{\text{Percent Markdown}}{100}$$

$$\frac{2.5}{25} = \frac{x}{100}$$

$$\frac{2.5 \cdot 4}{25 \cdot 4} = \frac{x}{100}$$

$$\frac{10}{100} = \frac{x}{100}$$

$$x = 10$$

The percent markdown is 10%.

Let's look at another example.

Example

A doll has an original price of $22.00, but it's on sale for $16.50. What is the percent markdown?

A doll was originally $22.00, but the price is now reduced by $5.50 ($22.00 – $16.50 = $5.50). Use the proportion:

$$\frac{\text{Discount Amount}}{\text{Original Price}} = \frac{\text{Percent Markdown}}{100}$$

$$\frac{5.5}{22} = \frac{x}{100}$$

$$100 \cdot \frac{5.5}{22} = x$$

$$\frac{550}{22} = x$$

$$x = 25$$

The percent markdown is 25%.

Let's look at an example on calculating the sale price.

Example

The regular price of an item is $\$150.00$, and the item is on clearance for 75% off the regular price. What is the sale price of the item?

The item was originally $\$150.00$, but the price is reduced by 75%, which means we need to figure out what 75% of $\$150.00$ is and then subtract that from the regular price.

$$75\% \text{ of } \$150.00$$

$$\frac{75}{100} \cdot \$150.00$$

$$\$112.50$$

That means that the sale price of the item is

$$\$150.00 - \$112.50$$

$$\$37.50$$

Alternatively, since the item is discounted by 75%, we know that it will only cost 25% of its original price (because $100\% - 75\% = 25\%$). So we could also calculate the cost by multiplying the original price by 25%.

25% of $\$150.00$

$$\frac{25}{100} \cdot \$150.00$$

$\$37.50$

Let's look at a final example on calculating the original price.

Example

The sale price of an item is $\$35.00$, and the item is on sale for 30% off the original price. What is the original price of the item?

The item was originally $\$x$, the price was reduced by 30%, and the item now costs $\$35.00$. This means that the current price of $\$35.00$ is 70% of the original price because $100\% - 30\% = 70\%$. Therefore, we can set up an equation to find the original price x.

$$\frac{70}{100} \cdot x = \$35.00$$

$$0.7 \cdot x = \$35.00$$

$$x = \$50.00$$

That means that the regular price of the item was $\$50.00$.

Calculating commission

In this lesson we'll learn how to calculate the commission earned.

What is commission? Commission is the amount of money a salesperson earns on the sale of an item.

The formula for commission:

$$\text{Commission} = \text{Selling Price} \cdot \text{Percent Commission}$$

Let's look at a couple of examples.

Example

A wax warmer business advertises that we can make 25% commission on sales of their product. If we make $\$2,500.00$ in sales, how much money will we earn?

To find the commission, we multiply the selling price by the percent commission.

$$25\% \text{ of } \$2,500.00$$

$$\$2,500.00 \cdot 0.25$$

$$\$625$$

We'll earn $\$625$ for selling $\$2,500.00$ worth of product.

People who sell clothes often work on commission plus an hourly salary.

Example

An employee at a clothing store earned $\$450.25$ in hourly pay for the month. He also sold $\$3,500.00$ worth of merchandise and expects to earn a commission of 8% on those sales. What is the employee's total expected pay before tax deductions?

First find the amount he earned in commissions.

$$8\% \text{ of } \$3,500.00$$

$$\$3,500.00 \cdot 0.08$$

$$\$280.00$$

Now calculate the employee's total expected pay.

$$\$450.25 + \$280.00$$

$$\$730.25$$

The employee expects to be paid a total of $\$730.25$.

People who sell cars often earn a yearly salary and commissions on sales.

Example

Stephanie earns $\$42,000.00$ per year plus a commission of 15% on all the cars she sells. If she wants to earn a total of $\$67,200.00$, how much money in car sales does she need to make?

We know that the amount she wants to earn in commissions is

$$\$67,200.00 - \$42,000.00$$

$$\$25,200.00$$

Then

$$\$25,200.00 = 15\% \text{ of } x$$

where x is the amount of money she needs to make in car sales.

$$\$25,200.00 = 0.15x$$

$$\frac{25,200.00}{0.15} = x$$

$$\$168,000 = x$$

This means that to earn a salary of $\$67,200.00$, Stephanie needs to sell $\$168,000.00$ worth of cars.

Real estate agents often make all their money on only commission. For example, when they help a client sell a house, they earn a commission (as a percentage of the sale price) on the sale.

Example

A real estate agent is working with multiple clients. In March, he helps his first client sell a house for $\$315{,}000$ and earns 3% commission, he helps a second client sell a house for $\$225{,}000$ and earns 4% commission, and he helps a third client sell a house for $\$410{,}000$ and earns 6% commission. How much commission did the agent earn in total in March?

First, calculate the commission on each sale individually.

First client: $\$315{,}000 \cdot 3\% = 315{,}000 \cdot 0.03 = \$9{,}450$

Second client: $\$225{,}000 \cdot 4\% = 225{,}000 \cdot 0.04 = \$9{,}000$

Third client: $\$410{,}000 \cdot 6\% = 410{,}000 \cdot 0.06 = \$24{,}600$

Then add the commission amounts to find the agent's total commission in March.

$$\$9{,}450 + \$9{,}000 + \$24{,}600$$

$$\$18{,}450 + \$24{,}600$$

$$\$43{,}050$$

Calculating simple interest

In this lesson we'll look at simple interest and how it's calculated. We'll also be able to use it to figure out the total amount of money we have in an investment.

What is simple interest? Simple interest is the amount we earn on an investment each year. It's called simple interest because we earn the same amount on the account every year (it doesn't compound).

The formula for simple interest is

$$I = Prt$$

where

I is the amount of interest earned in the account over a certain time period.

P is the principal (the original amount of money in the account).

r is the annual interest rate, expressed as a decimal.

t is the period of time.

Let's look at an example of how simple interest is calculated.

Example

If we deposit $\$200$ into a savings account and it earns 8% simple interest per year, how much interest will we earn on the account in 4 years?

We know

$$P = \$200$$

$$r = \frac{8}{100} = 0.08$$

$t = 4$ years

If we plug these values into the formula for simple interest, we get

$$I = Prt$$

$$I = (200)(0.08)(4)$$

$$I = \$64$$

This means that in four years we'll earn $\$64$ in interest.

Let's look at a different way to do the same problem in case we forget the formula.

First, we need to find the interest earned in one year, by multiplying the initial amount by the interest rate.

$8\,\%$ of $\$200$

$$0.08 \cdot \$200$$

$$\$16$$

Because the account earns simple interest, that means the interest doesn't compound, and the same interest is earned each year. We're looking for the interest earned in four years so we multiply the interest earned in one year by 4.

$$\$16 \cdot 4$$

$$\$64$$

Let's look at another formula. How is the amount in the account calculated? The total amount in the account is calculated by the formula

$$A = P(1 + rt)$$

where

A is the total amount in the account after a given time period

P is the initial amount in the account (the principal)

r is the annual interest rate, expressed as a decimal

t is the time period

Let's do an example with this formula.

Example

If we deposit $\$260$ into a savings account that earns 5% simple interest per year, how much is in the account after 8 years?

We don't know A, but we know

$$P = \$260$$

$$r = \frac{5}{100} = 0.05$$

$$t = 8 \text{ years}$$

If we plug these values into the formula, we get

$$A = P(1 + rt)$$

$$A = \$260(1 + 0.05 \cdot 8)$$

$$A = \$364$$

If we happen to forget this formula, that's okay too; we can think this through as well.

First, we need to find the interest earned in one year, by multiplying the initial amount by the interest rate.

$$5\% \text{ of } \$260$$

$$0.05 \cdot \$260$$

$$\$13$$

Because the account earns simple interest, that means the interest doesn't compound, and the same interest is earned each year. Since we're looking for the account balance after eight years, we add to the principal amount eight times the interest earned in one year.

$$\$260 + 8 \cdot \$13$$

$$\$260 + \$104$$

$$\$364$$

Which means that after 8 years we'll have a total of $\$364$ in the account.

Of course we could also use the formula $I = Prt$ and then just add it to the original amount. That would work too.

Remember that sometimes we'll need to round the answer and that's okay; just round to the nearest cent!

Complex fractions

A complex fraction is an algebraic expression with fraction(s) in either the numerator, the denominator, or both. The main steps of solving complex fractions are

1. Simplify both the numerator and denominator so we can have a simple fraction.

2. Simplify the numerator and the denominator as much as possible.

What do we need to remember?

A reciprocal of a fraction is just that fraction "flipped upside down."

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$

The reciprocal of $\frac{x}{1}$ is $\frac{1}{x}$

A fraction bar can be thought of as a division sign.

$$\frac{x}{y} = x \div y$$

To divide by a fraction, we can multiply by its reciprocal.

$$\frac{x}{\left(\frac{a}{b}\right)} = x \div \frac{a}{b} = x \cdot \frac{b}{a}$$

Any number or variable can be written as itself divided by 1.

$$x = \frac{x}{1}$$

Let's look at a few examples.

Example

Simplify the expression.

$$\frac{\left(\frac{2}{3}\right)}{\left(\frac{3}{4}\right)}$$

Here, we're dividing the fraction in the numerator ($2/3$) by the fraction in the denominator ($3/4$).

$$\frac{2}{3} \div \frac{3}{4}$$

Now that we have a fraction divided by another fraction, instead of dividing by the fraction that was originally in the denominator, we can multiply by its reciprocal.

$$\frac{2}{3} \cdot \frac{4}{3}$$

For fraction multiplication, multiply the numerators and denominators separately.

$$\frac{2 \cdot 4}{3 \cdot 3}$$

$$\frac{8}{9}$$

We can do the same thing with variables.

Example

Simplify the expression.

$$\frac{x}{\left(\frac{a}{b}\right)}$$

We have to rewrite the given fraction.

$$\frac{\left(\frac{x}{1}\right)}{\left(\frac{a}{b}\right)}$$

$$\frac{x}{1} \div \frac{a}{b}$$

Now that we have a fraction divided by another fraction, instead of dividing by the fraction that was originally in the denominator, we can multiply by its reciprocal.

$$\frac{x}{1} \cdot \frac{b}{a}$$

For fraction multiplication, multiply the numerators and denominators separately.

$$\frac{xb}{1a}$$

$$\frac{xb}{a}$$

Let’s now solve a more complex example.

Example

Simplify the expression.

$$\frac{\frac{1}{a} - 1}{\frac{1}{b} - \frac{1}{a}}$$

Simplify just the numerator by finding a common denominator.

$$\frac{1}{a} - 1$$

$$\frac{1}{a} - \frac{a}{a}$$

$$\frac{1-a}{a}$$

Simplify just the denominator by finding a common denominator.

$$\frac{1}{b}-\frac{1}{a}$$

$$\frac{1}{b}\cdot\frac{a}{a}-\frac{1}{a}\cdot\frac{b}{b}$$

$$\frac{a}{ab}-\frac{b}{ab}$$

$$\frac{a-b}{ab}$$

Rewrite the given fraction with the simplified numerator and denominator.

$$\frac{\frac{1-a}{a}}{\frac{a-b}{ab}}$$

Multiply the numerator by the reciprocal of the denominator.

$$\frac{1-a}{a}\div\frac{a-b}{ab}$$

$$\frac{1-a}{a}\cdot\frac{ab}{a-b}$$

$$\frac{1-a}{1}\cdot\frac{b}{a-b}$$

$$\frac{b-ab}{a-b}$$

Ratios and proportions with complex fractions

When we have complex fractions in a proportion that includes a variable, we can solve for the variable in one of two ways. We can either use cross multiplication, or we can multiply by the reciprocals of the fractions in the denominators.

Remember that cross multiplication tells us that $a/b = c/d$ can be rewritten as $ad = bc$.

Let's look at a few examples.

Example

Solve for the variable.

$$\frac{\left(\frac{1}{3}\right)}{x} = \frac{\left(\frac{1}{6}\right)}{\left(\frac{1}{7}\right)}$$

We'll cross multiply.

$$\frac{1}{3} \cdot \frac{1}{7} = x \cdot \frac{1}{6}$$

Now we can simplify by multiplying the fractions.

$$\frac{1 \cdot 1}{3 \cdot 7} = \frac{x}{6}$$

$$\frac{1}{21} = \frac{x}{6}$$

Multiply both sides of this equation by 6 to solve for x.

$$\frac{6}{21} = x$$

$$x = \frac{2}{7}$$

Let's look at one more example.

Example

Solve for the variable.

$$\frac{\left(\frac{x}{4}\right)}{\left(\frac{8}{3}\right)} = \frac{\left(\frac{4}{3}\right)}{\left(\frac{5}{4}\right)}$$

Instead of dividing by the fractions in the denominators, we can multiply by their reciprocals.

$$\frac{x}{4} \cdot \frac{3}{8} = \frac{4}{3} \cdot \frac{4}{5}$$

After multiplying we get

$$\frac{3x}{32} = \frac{16}{15}$$

Multiply both sides by 32.

$$3x = 32 \cdot \frac{16}{15}$$

Divide both sides by 3 to solve for x. Then multiply fractions to simplify.

$$x = \frac{32}{3} \cdot \frac{16}{15}$$

$$x = \frac{512}{45}$$

Imaginary and complex numbers

In this lesson we'll look at the imaginary number i, what it means, and how to use it in expressions.

The **imaginary number** i is defined as the square root of -1, and we can use it in algebraic expressions. An imaginary number (in general) is defined as a number that can be written as a product of a real number and i. For instance, $4i$ and $-15i$ are imaginary numbers.

Properties of imaginary numbers

These are the things we need to know about imaginary numbers.

1. The formulas for i and i^2 are

   $$i = \sqrt{-1} \text{ and } i^2 = -1$$

   We can use these formulas to express i^3 as the imaginary number $-i$, and i^4 as the real number 1.

   $$i^3 = i^{2+1} = (i^2)(i^1) = (-1)(i) = -i$$

   $$i^4 = i^{2+2} = (i^2)(i^2) = (-1)(-1) = 1$$

   In fact, if n is any positive integer, then we can express i^n as either an imaginary number (if n is odd) or a real number (if n is even).

2. When we add or subtract expressions with i raised to the same power, we treat them as like terms.

For example, in the list

$i, 3i^2, 4, 2i, 8, 5i^2$

the like terms are

i and $2i$

$3i^2$ and $5i^2$

4 and 8

We could in turn use $i^2 = -1$ to express $3i^2$ and $5i^2$ as -3 and -5, respectively, so the like terms in the list $i, 3i^2, 4, 2i, 8, 5i^2$ would end up being

i and $2i$

$-3, 4, 8,$ and -5

3. If we have the sum of a real number and an imaginary number, we should write the real number first and the imaginary number second.

So $-6i + 8$ should be written as $8 - 6i$, with the real number first and the imaginary number second. A number that can be written as the sum of a real number and an imaginary number (a number that can be written in the form $a + bi$ where a and b are real numbers) is called a **complex number**.

Arithmetic with imaginary numbers

Now let's look at how we can perform basic operations with complex numbers, like addition and subtraction.

When we want to add complex numbers, we combine the real parts and imaginary parts separately. In general, adding complex numbers looks like

$$z_1 + z_2 = a + ib + c + id$$

$$z_1 + z_2 = (a + c) + (ib + id)$$

$$z_1 + z_2 = (a + c) + i(b + d)$$

So we get the general formula

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

We can subtract complex numbers using the same principle, by subtracting the real parts and imaginary parts separately. Then the subtraction of complex numbers looks like

$$z_1 - z_2 = (a + ib) - (c + id)$$

$$z_1 - z_2 = a + ib - c - id$$

$$z_1 - z_2 = (a - c) + (ib - id)$$

$$z_1 - z_2 = (a - c) + i(b - d)$$

So we get the general formula

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Let's wrap it up by outlining the general steps for adding and subtracting complex numbers.

1. Separate the real and imaginary parts of the complex numbers.
2. Add/subtract the real parts of the complex numbers.
3. Add/subtract the imaginary parts of the complex numbers.
4. Write the final answer in $a+bi$ format.

Let's begin with a simple example.

Example

Simplify the expression.

$$-1-8i-4-i$$

Begin by grouping the like terms.

$$-1-4-8i-i$$

Remember that there's an unwritten 1 in front of the i.

$$-1-4-8i-1i$$

Follow the usual addition and subtraction rules.

$$-5-9i$$

Let's look at another one.

Example

Simplify the expression.

$$\sqrt{-9}+\sqrt{9}+5+3i-\sqrt{-4}$$

Remember that

$$\sqrt{-1}=i$$

Let's start with the square roots.

$$\sqrt{9\cdot -1}+\sqrt{9}+5+3i-\sqrt{4\cdot -1}$$

$$\sqrt{9}\sqrt{-1}+\sqrt{9}+5+3i-\sqrt{4}\sqrt{-1}$$

$$3i+3+5+3i-2i$$

Now group like terms.

$$3+5+3i+3i-2i$$

$$8+4i$$

Let's do one final example.

Example

Simplify the expression.

$$-\sqrt{-25}+8i^3+2i-\sqrt{-4}\sqrt{4}+3\sqrt{-9}$$

Let's start with the square roots.

$$-\sqrt{25\cdot -1}+8i^3+2i-\sqrt{4\cdot -1}\sqrt{4}+3\sqrt{9\cdot -1}$$

$$-\sqrt{25}\sqrt{-1}+8i^3+2i-\sqrt{4}\sqrt{-1}\sqrt{4}+3\sqrt{9}\sqrt{-1}$$

$$-5i+8i^3+2i-2i\cdot 2+3\cdot 3i$$

Let's simplify $8i^3$ to $8i^2i$, which is $8(-1)i$, or $-8i$.

$$-5i-8i+2i-2i\cdot 2+3\cdot 3i$$

Now let's do the rest of the multiplication.

$$-5i-8i+2i-4i+9i$$

Finally, let's combine like terms, by doing the addition and subtraction from left to right:

$$-13i+2i-4i+9i$$

$$-11i-4i+9i$$

$$-15i+9i$$

$$-6i$$

Rationalizing complex denominators

Remember that a **complex number** is a number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary number $\sqrt{-1}$. The number a is the **real part** of the complex number, and bi is the **imaginary part**.

An **imaginary number** (also called a pure imaginary number) is a complex number whose real part is 0. For example, $-6i$ and $4i$ are imaginary numbers. So every complex number can be written as the sum of a real number and an imaginary number.

Multiplying complex numbers

To multiply two or more complex numbers, we use the Distributive Property. For example, if we have two complex numbers $x = a + bi$ and $y = c + di$, then their product is

$$xy = (a + bi)(c + di)$$

Now we apply the Distributive Property of Multiplication to expand the product.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Sometimes we need to multiply complex numbers with only real or imaginary parts. Imagine we multiply two complex numbers, one of which has just a real part while the other has both real and imaginary parts. The product would be given by

$$a(c + di) = ac + adi$$

For example, if we multiply 4 by $2 + 5i$, then we get

$$4(2 + 5i) = 8 + 20i$$

Now let's consider the other scenario. If we need to multiply a pure imaginary number bi by a complex number $c + di$, then the product is

$$(bi)(c + di) = bci - bd$$

For example, if we multiply a complex number $1 + 4i$ by $-2i$, then we get

$$(-2i)(1 + 4i) = -2i - 8i^2$$

$$(-2i)(1 + 4i) = -2i + 8$$

Multiplying by the conjugate

There are a few more things we need to understand about multiplying complex numbers.

1. The **conjugate** of a complex number (usually called the **complex conjugate** of that number) is formed by changing the sign of the imaginary part and leaving the real part unchanged.

   For example, the complex conjugate of $5 + 3i$ is $5 - 3i$.

2. We can multiply any number times 1 without changing its value, and 1 can be written as any nonzero number or expression divided by itself.

For example,

$$\frac{3}{5+3i}=\frac{3}{5+3i}\cdot 1=\frac{3}{5+3i}\cdot\frac{5-3i}{5-3i}$$

because

$$\frac{5-3i}{5-3i}=1$$

To rationalize a fraction that has a complex number in the denominator, we multiply it by the fraction in which both the numerator and the denominator are the complex conjugate of that complex number (just like in this last example). This is called the **conjugate method**.

3. Use FOIL or the Distributive Property to multiply complex numbers, then simplify.

An example of FOIL (multiplying the first terms, the outer terms, the inner terms, and last terms, in that order):

$$(5+3i)(4+2i)$$

$$5\cdot 4+5\cdot 2i+3i\cdot 4+3i\cdot 2i$$

$$20+10i+12i+6i^2$$

Combining like terms and replacing i^2 with -1, we get

$$20 + 22i + 6(-1)$$

$$20 - 6 + 22i$$

$$14 + 22i$$

An example of the Distributive Property.

$$5(3 - 4i)$$

$$5(3) + 5(-4i)$$

$$15 + (-20i)$$

$$15 - 20i$$

Now let's use the conjugate method to simplify a fraction that has a complex number in the denominator.

Example

Use the conjugate method to simplify the expression.

$$\frac{3 - 4i}{-2 + i}$$

We can use the conjugate method to get the imaginary number i, out of the denominator. The complex conjugate of $-2 + i$ is $-2 - i$.

$$\frac{3 - 4i}{-2 + i} \cdot \frac{-2 - i}{-2 - i}$$

$$\frac{(3-4i)(-2-i)}{(-2+i)(-2-i)}$$

Now that we have a binomial multiplication problem, we need to make sure that (in the numerator and denominator separately) we multiply the first terms, outer terms, inner terms, and last terms.

$$\frac{-6-3i+8i+4i^2}{4+2i-2i-i^2}$$

$$\frac{4i^2+5i-6}{-i^2+4}$$

Replacing i^2 with -1, and then combining like terms, we get

$$\frac{4(-1)+5i-6}{-(-1)+4}$$

$$\frac{-4+5i-6}{1+4}$$

$$\frac{5i-10}{5}$$

$$\frac{5i}{5}-\frac{10}{5}$$

$$i-2$$

$$-2+i$$

Let's look at another example of rationalizing a complex denominator.

Example

Simplify.

$$\frac{10i^2 - 5i}{-6 + 6i}$$

First, we'll rewrite the expression as

$$\frac{10(-1) - 5i}{-6 + 6i}$$

$$\frac{-10 - 5i}{-6 + 6i}$$

Now we can use the conjugate method to get the imaginary number i out of the denominator. The complex conjugate of $-6 + 6i$ is $-6 - 6i$.

$$\frac{-10 - 5i}{-6 + 6i} \cdot \frac{-6 - 6i}{-6 - 6i}$$

$$\frac{(-10 - 5i)(-6 - 6i)}{(-6 + 6i)(-6 - 6i)}$$

Now that we have a binomial multiplication problem, we need to make sure that (in the numerator and denominator separately) we multiply the first terms, outer terms, inner terms, and last terms.

$$\frac{60 + 60i + 30i + 30i^2}{36 + 36i - 36i - 36i^2}$$

$$\frac{60 + 90i + 30i^2}{36 - 36i^2}$$

Replacing i^2 with -1, we get

$$\frac{60 + 90i + 30(-1)}{36 - 36(-1)}$$

$$\frac{60 + 90i - 30}{36 + 36}$$

$$\frac{30 + 90i}{72}$$

Divide out 6, which goes evenly into 30, $90i$, and 72.

$$\frac{5 + 15i}{12}$$

We can also write this as the sum of two fractions.

$$\frac{5}{12} + \frac{15i}{12}$$

Then we can reduce the second fraction to lowest terms, which gives

$$\frac{5}{12} + \frac{5i}{4}$$

$$\frac{5}{12} + \frac{5}{4}i$$

Coefficients in quadratics

In this lesson we'll look at methods for factoring quadratic polynomials in which the coefficient of the x^2 term is neither 1 nor -1.

Factoring means we're taking an expression and rewriting it as parts that are being multiplied together (the factors).

Factoring a quadratic polynomial with coefficients means taking a quadratic polynomial ax^2+bx+c where a, b, and c are real numbers with $a\neq 1$ and $a\neq -1$, and writing it in the form $(px+r)(qx+s)$ where p, q, r, s are all real numbers.

Let's do a few examples.

Example

Factor the quadratic.

$$3x^2+5x-2$$

Let's begin by looking at the factors of 3 (the coefficient of the x^2 term) and 2 (the absolute value of the constant term). The only factors of 3 are 3 and 1, so we know we'll have

$$(3x\qquad)(x\qquad)$$

The only factors of 2 are 2 and 1, which means we'll have one of the following:

$$(3x \quad 2)(x \quad 1)$$

$$(3x \quad 1)(x \quad 2)$$

We need to determine the signs of the constant terms in the individual factors such that when each constant term is multiplied by the x term in the opposite factor, and then those two products are added, we get the “middle term” (the x term) in the original quadratic polynomial.

Let’s see what happens if we do the factoring the first way.

$$(3x \quad 2)(x \quad 1) = 3x^2 \quad 3x \quad 2x \quad 2$$

We need to combine $3x$ and $2x$ in such a way that we get the middle term, $5x$. But remember that in $3x^2 + 5x - 2$, the constant term (-2) is negative, which means that the sign of the constant term in exactly one of the factors has to be negative, so there are only two possibilities:

$$(3x + 2)(x - 1) = 3x^2 - 3x + 2x - 2 = 3x^2 - x - 2$$

$$(3x - 2)(x + 1) = 3x^2 + 3x - 2x - 2 = 3x^2 + x - 2$$

But neither of these is correct, because we don’t get $5x$ for the middle term. Let’s try doing the factoring the second way.

$$(3x \quad 1)(x \quad 2) = 3x^2 \quad 6x \quad x \quad 2$$

Can we get $5x$ by combining $6x$ and x? Yes, we can.

$$6x - x = 5x$$

Therefore, we have to use 2 as the constant term in the second factor (because $6x = 3x \cdot 2$), and -1 as the constant term in the first factor (because $-x = -1 \cdot x$), so we get

$$(3x - 1)(x + 2)$$

Let's try one more.

Example

Factor the quadratic.

$$15x^2 + 66x - 45$$

First, we'll factor out a 3, because 3 is the factor that's common to all three terms.

$$3(5x^2 + 22x - 15)$$

Now, let's factor $5x^2 + 22x - 15$.

The only factors of 5 are 5 and 1, so we know we'll have

$$(5x \quad)(x \quad)$$

The only pairs of factors of 15 are $(3,5)$ and $(15,1)$. From the pair $(3,5)$, we get two possibilities:

$$(5x \quad 3)(x \quad 5)$$

$$(5x \quad 5)(x \quad 3)$$

From the pair $(15,1)$, we get two possibilities:

$$(5x \quad 15)(x \quad 1)$$

$$(5x \quad 1)(x \quad 15)$$

For each possibility, let's look at the x terms we'll get when we multiply the constant term in each factor by the x term in the opposite factor, to see which possibility has a combination of x terms that will give us the middle term, $22x$.

Since there are so many possibilities, let's use a table to help keep them organized.

Possibility	Polynomial	x terms	Combine to 22x?
(5x 3)(x 5)	$5x^2$ 25x 3x 15	25x and 3x	Yes: 25x-3x=22x
(5x 5)(x 3)	$5x^2$ 15x 5x 15	15x and 5x	No
(5x 15)(x 1)	$5x^2$ 5x 15x 15	5x and 15x	No
(5x 1)(x 15)	$5x^2$ 75x x 15	75x and x	No

So we need to use $(5x \quad 3)(x \quad 5)$ and set it up to get $25x$ and $-3x$. Therefore, we have to use 5 as the constant term in the second factor (because $25x = 5x \cdot 5$), and -3 as the constant term in the first factor (because $-3x = -3 \cdot x$).

$$15x^2 + 66x - 45$$

$$3(5x^2 + 22x - 15)$$

$$3(5x - 3)(x + 5)$$

Grouping

In this lesson we'll look at factoring a polynomial using a method called grouping.

When we have a polynomial, sometimes we can use grouping to help us find the factors. To do this, we need to look for a way to split the terms of the polynomial into two groups in such a way that each group can be factored separately and there's a factor that's common to the two groups.

Let's look at an example.

Example

Factor by grouping.

$$11z + 11qr + pyz + pyqr$$

Since we've been asked to use grouping to factor the polynomial, we need to look for a way to split the terms of the polynomial into two groups that can be factored separately. Since the first two terms have a factor of 11 in common, and the last two terms have a factor of py in common, we'll group the first two terms separately from the last two terms.

$$11z + 11qr + pyz + pyqr$$

$$(11z + 11qr) + (pyz + pyqr)$$

With our terms grouped, we need to look for the greatest common factor in each group. In this case, those are the factors we identified earlier (11 in the first group, and py in the second group). Factoring these out of the respective groups separately, we get

$$11(z+qr)+py(z+qr)$$

Notice that the two groups of terms do indeed have a factor in common (specifically, $z+qr$), so we can now factor that out of each group.

$$(11+py)(z+qr)$$

This is the correct solution, but it can also be written as $(z+qr)(11+py)$ or even $(qr+z)(11+py)$.

There are usually different ways to group our terms before we factor. We could have used grouping to factor our polynomial this way:

$$11z+11qr+pyz+pyqr$$

$$11qr+pyqr+11z+pyz$$

$$(11qr+pyqr)+(11z+pyz)$$

$$qr(11+py)+z(11+py)$$

$$(qr+z)(11+py)$$

Let's do another example.

Example

Factor by grouping.

$$pqx^2 - psx + qrx - rs$$

Since we've been asked to use grouping to factor the polynomial, we need to look for a way to split the terms of the polynomial into two groups that can be factored separately. Since the first two terms have a factor of px in common, and the last two terms have a factor of r in common, we'll group the first two terms separately from the last two terms.

$$pqx^2 - psx + qrx - rs$$

$$(pqx^2 - psx) + (qrx - rs)$$

With our terms grouped, we need to look for the greatest common factor in each group. In this case, those are the factors we identified earlier (px in the first group, and r in the second group). Factoring these out of the respective groups separately, we get

$$px(qx - s) + r(qx - s)$$

Notice that the two groups of terms do indeed have a factor in common (specifically, $qx - s$), so we can now factor that out of each group.

$$(px + r)(qx - s)$$

As we mentioned before, there are multiple ways of writing this, such as $(-s + qx)(r + px)$, and it all depends on how we choose to group the factors.

We can also use grouping to factor quadratics. We already know how to factor quadratics of the form

$$ax^2 + bx + c$$

by looking at the factors of a and c and trying to figure out which combination of factors can be used to get the coefficient of the middle term, b. But that's not the only way to factor quadratics. We can also use grouping.

To use grouping to factor a quadratic, the first step is to find $a \cdot c$, then look for factors of $a \cdot c$ that sum to b. Then we can rewrite the quadratic, and factor by grouping.

Let's look at an example with numbers to see how this works.

Example

Factor the quadratic.

$$11x^2 + 13x + 2$$

In this case $a = 11$, $c = 2$, and $b = 13$, so we need to find the pair of factors of $a \cdot c = 11 \cdot 2 = 22$ whose sum is 13. The pairs of factors of 22 are $(1,22)$ and $(2,11)$. Which of these pairs of factors have a sum of 13?

$$1 + 22 = 23$$

and

$$2 + 11 = 13$$

So we need the pair $(2,11)$. Let's rewrite $11x^2 + 13x + 2$ as $11x^2 + 2x + 11x + 2$.

Now, since we want to use grouping to factor the polynomial, we need to look for a way to split the terms of the polynomial into two groups that can be factored separately. Since the first two terms have an x in common, we'll group the first two terms together separately from the last two terms.

$$(11x^2 + 2x) + (11x + 2)$$

Let's factor out the x from the terms in the first group.

$$x(11x + 2) + (11x + 2)$$

Remember we can write this as

$$x(11x + 2) + 1(11x + 2)$$

Notice that the two groups of terms do indeed have a factor in common (specifically, $11x + 2$), so we can now factor that out of each group.

$$(11x + 2)(x + 1)$$

If we're struggling with factoring a quadratic, factoring by grouping can give us a nice procedure to follow.

Difference of cubes

In this lesson we'll look at how to recognize a difference of two cubes and then use a formula to factor it.

We know we're dealing with the difference of cubes, because we have two perfect cubes separated by a minus sign to indicate that the second perfect cube is to be subtracted from the first perfect cube. When that's the case, we can take the cube root of each term.

The formula for factoring a difference of cubes is

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example

Factor the expression.

$$c^3 - 8d^{12}$$

To check whether the terms to the left and right of the minus sign are perfect cubes, we'll take the cube root of each of them (we'll raise each of them to the $1/3$ power).

$$\sqrt[3]{c^3} = (c^3)^{\frac{1}{3}} = c$$

$$\sqrt[3]{8d^{12}} = (8d^{12})^{\frac{1}{3}} = (8)^{\frac{1}{3}}(d^{12})^{\frac{1}{3}} = 2d^4$$

We can see that both terms are perfect cubes. The difference of cubes formula says $a^3 - b^3$ is always factored as

$$(a-b)(a^2+ab+b^2)$$

a is the cube root of the first term (in this case $a = c$), and b is the cube root of the second term (in this case $b = 2d^4$), so we get

$$(c-2d^4)\left[(c)^2+(c)(2d^4)+(2d^4)^2\right]$$

$$(c-2d^4)(c^2+2cd^4+4d^8)$$

Let's do one more.

Example

Factor the expression.

$$27x^3y^9 - 216z^{15}$$

To check whether the terms to the left and right of the minus sign are perfect cubes, we'll take the cube root of each of them (we'll raise each of them to the $1/3$ power).

$$\sqrt[3]{27x^3y^9} = (27x^3y^9)^{\frac{1}{3}} = (27)^{\frac{1}{3}}(x^3)^{\frac{1}{3}}(y^9)^{\frac{1}{3}} = 3xy^3$$

$$\sqrt[3]{216z^{15}} = (216z^{15})^{\frac{1}{3}} = (216)^{\frac{1}{3}}(z^{15})^{\frac{1}{3}} = 6z^5$$

We can see that both terms are perfect cubes. The difference of cubes formula says $a^3 - b^3$ is always factored as

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

a is the cube root of the first term, and b is the cube root of the second term, so

$$a = 3xy^3$$

$$b = 6z^5$$

Now we'll apply the formula.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$a^3 - b^3 = (3xy^3 - 6z^5)[(3xy^3)^2 + (3xy^3)(6z^5) + (6z^5)^2]$$

$$a^3 - b^3 = (3xy^3 - 6z^5)(9x^2y^6 + 18xy^3z^5 + 36z^{10})$$

We can check our work by distributing each term in the binomial factor over all the terms in the trinomial factor.

$$(3xy^3)(9x^2y^6) + (3xy^3)(18xy^3z^5) + (3xy^3)(36z^{10})$$

$$+(-6z^5)(9x^2y^6) + (-6z^5)(18xy^3z^5) + (-6z^5)(36z^{10})$$

$$27x^3y^9 + 54x^2y^6z^5 + 108xy^3z^{10} - 54x^2y^6z^5 - 108xy^3z^{10} - 216z^{15}$$

$$27x^3y^9 + 54x^2y^6z^5 - 54x^2y^6z^5 + 108xy^3z^{10} - 108xy^3z^{10} - 216z^{15}$$

$$27x^3y^9 - 216z^{15}$$

Sum of cubes

In this lesson we'll look at how to recognize a sum of two cubes and then use a formula to factor it.

We'll know when we have a sum of cubes because we'll have two perfect cubes separated by a plus sign to indicate that the second perfect cube is to be added to the first perfect cube. When that's the case, we can take the cube root of each term and use a formula to do the factoring.

The formula for the sum of two cubes is

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Let's do an example.

Example

Factor the expression.

$$125x^3 + 512y^3z^9$$

First check to see if each term is a perfect cube.

$$\sqrt[3]{125x^3} = (125x^3)^{\frac{1}{3}} = (125)^{\frac{1}{3}}(x^3)^{\frac{1}{3}} = 5x$$

$$\sqrt[3]{512y^3z^9} = (512y^3z^9)^{\frac{1}{3}} = (512)^{\frac{1}{3}}(y^3)^{\frac{1}{3}}(z^9)^{\frac{1}{3}} = 8yz^3$$

Both terms are perfect cubes, so we can use the formula to do the factoring

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

In this case $a = 5x$ and $b = 8yz^3$, so we can plug these into our formula and get

$$(5x + 8yz^3)\left[(5x)^2 - (5x)(8yz^3) + (8yz^3)^2\right]$$

$$(5x + 8yz^3)(25x^2 - 40xyz^3 + 64y^2z^6)$$

We can check our work by distributing each term in the binomial factor over all the terms in the trinomial factor.

$$5x(25x^2 - 40xyz^3 + 64y^2z^6) + 8yz^3(25x^2 - 40xyz^3 + 64y^2z^6)$$

$$5x(25x^2) - 5x(40xyz^3) + 5x(64y^2z^6) + 8yz^3(25x^2) - 8yz^3(40xyz^3) + 8yz^3(64y^2z^6)$$

$$125x^3 - 200x^2yz^3 + 320xy^2z^6 + 200x^2yz^3 - 320xy^2z^6 + 512y^3z^9$$

$$125x^3 - 200x^2yz^3 + 200x^2yz^3 + 320xy^2z^6 - 320xy^2z^6 + 512y^3z^9$$

$$125x^3 + 512y^3z^9$$

Let's do another example.

Example

Factor the expression.

$$729h^{30}j^9 + 27m^{15}n^3$$

First check to see if each term is a perfect cube.

$$\sqrt[3]{729h^{30}j^9} = (729h^{30}j^9)^{\frac{1}{3}} = (729)^{\frac{1}{3}}(h^{30})^{\frac{1}{3}}(j^9)^{\frac{1}{3}} = 9h^{10}j^3$$

$$\sqrt[3]{27m^{15}n^3} = (27m^{15}n^3)^{\frac{1}{3}} = (27)^{\frac{1}{3}}(m^{15})^{\frac{1}{3}}(n^3)^{\frac{1}{3}} = 3m^5n$$

Both terms are perfect cubes, so we can use the formula for factoring the sum of perfect cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

In this case,

$$a = 9h^{10}j^3$$

$$b = 3m^5n$$

We use the sum of cubes formula to get

$$(9h^{10}j^3 + 3m^5n)\left[(9h^{10}j^3)^2 - (9h^{10}j^3)(3m^5n) + (3m^5n)^2\right]$$

$$(9h^{10}j^3 + 3m^5n)(81h^{20}j^6 - 27h^{10}j^3m^5n + 9m^{10}n^2)$$

Simplifying rational functions

In this lesson we'll look at how to simplify rational functions by cancelling factors. Remember that to cancel any factor, it must be a common factor in both the numerator and denominator.

A common factor in a rational function is a factor that's shared by all the terms of the numerator and all the terms of the denominator. For example, $3x$, $12x^2$, and $9x^3$ have a common factor of $3x$, because, $1 \cdot 3x = 3x$, $4x \cdot 3x = 12x^2$, and $3x^2 \cdot 3x = 9x^3$.

Let's look at a few examples.

Example

Simplify the rational function to lowest terms.

$$\frac{9x^3 - 12x^2 + 3x}{15x^2}$$

Let's look for a common factor. We can factor a $3x$ out of the numerator and the denominator, and then cancel it.

$$\frac{3x(3x^2 - 4x + 1)}{3x(5x)}$$

$$\frac{3x^2 - 4x + 1}{5x}$$

It would also work to take out each factor individually: either first the 3 and then the x, or first the x and then the 3.

Let's look at another example.

Example

Simplify each rational function in the difference.

$$\frac{12ab + 8a^2b^2}{10ab} - \frac{36a^3b^3 - 6a^2b^2}{6ab}$$

We want to factor everything that we can out of the numerator of each rational function in the difference. That means that for each numerator, we need to take out every constant, every factor of a, and every factor of b that's common to all the terms of that numerator; and that for each denominator, we need to take out every constant, every factor of a, and every factor of b that's common to all the terms of that denominator.

$$\frac{12ab + 8a^2b^2}{10ab} - \frac{36a^3b^3 - 6a^2b^2}{6ab}$$

$$\frac{2ab(6 + 4ab)}{2ab(5)} - \frac{6a^2b^2(6ab - 1)}{6ab}$$

$$\frac{2ab(6 + 4ab)}{2ab(5)} - \frac{6ab(ab)(6ab - 1)}{6ab}$$

Then for each rational function, we want to cancel what's common to its numerator and denominator.

$$\frac{6+4ab}{5} - \frac{(ab)(6ab-1)}{1}$$

$$\frac{6+4ab}{5} - ab(6ab-1)$$

Let’s look at another example.

Example

Simplify the rational function to lowest terms.

$$\frac{x^2-x-20}{x^2-7x+10}$$

The quadratic expression in the numerator is factored as $(x+4)(x-5)$ and the quadratic expression in the denominator is factored as $(x-5)(x-2)$. Then we get

$$\frac{(x+4)(x-5)}{(x-2)(x-5)}$$

In this case, we can cancel the common factor $(x-5)$.

$$\frac{x+4}{x-2}$$

Adding and subtracting rational functions

In this lesson we will look at how to add and subtract rational functions. In other words, how to add and subtract fractions that have variables in them as well as numbers.

Remember that when we add and subtract fractions, we need a common denominator. The lowest common denominator is the least common multiple of the denominators in the individual fractions.

Let's look at how to find the least common multiple of a group of terms. For instance, if we have the factors $5x$, xy^2, and $5y^4$, we can set up a table to help us organize the factors as individual terms. Put the terms down the far-left column, and a heading for each piece across the top row. Then fill in the table with the factors of each term.

	Coefficients	x's	y's
5x	5	x	
xy^2		x	y^2
$5y^4$	5		y^4

In order to generate the least common multiple, we have to take the least common multiple of the entries in each column, and then form the product of the results.

- The least common multiple of the entries in the coefficients column is 5. (The coefficient in xy^2 is 1, so we're taking the least common multiple of 1 and 5, which is 5.)

- The least common multiple of the entries in the base-x column is x. ($5x$ and xy^2 can be written as $5x^1$ and x^1y^2, respectively, and $5y^4$ can be written as $5x^0y^4$, so we're taking the least common multiple of x^0 and x^1, which is x^1, or x.

- The least common multiple of the entries in the base-y column is y^4. ($5x$ can be written as $5xy^0$, so we're taking the least common multiple of y^0, y^2, and y^4, which is y^4.)

Therefore, the least common multiple of our terms $5x$, xy^2, and $5y^4$ is

$$5xy^4$$

Now let's look at how to apply this idea to the addition and subtraction of rational functions.

Example

Simplify the expression by combining the three rational functions into a single rational function.

$$\frac{y}{5x}+\frac{a}{xy^2}-\frac{c}{5y^4}$$

We need to combine the three fractions in the expression into one fraction, which we'll do by finding a common denominator.

The lowest common denominator will be the least common multiple of the three denominators in

$$\frac{y}{5x} + \frac{a}{xy^2} - \frac{c}{5y^4}$$

The denominators are $5x$, xy^2, and $5y^4$. We found that their least common multiple is $5xy^4$.

Now we need to multiply the numerator and denominator of each fraction by whatever expression is needed to make its denominator equal to $5xy^4$.

$$\frac{y^4}{y^4} \cdot \frac{y}{5x} + \frac{5y^2}{5y^2} \cdot \frac{a}{xy^2} - \frac{x}{x} \cdot \frac{c}{5y^4}$$

$$\frac{y^5}{5xy^4} + \frac{5ay^2}{5xy^4} - \frac{cx}{5xy^4}$$

$$\frac{y^5 + 5ay^2 - cx}{5xy^4}$$

Let's do another example.

Example

Simplify the expression by combining the three rational functions into a single rational function.

$$\frac{a}{2bc^2} + \frac{m}{3c} + \frac{y}{4bc}$$

We need to combine the three fractions in the expression into one fraction, which we'll do by finding a common denominator.

The lowest common denominator will be the least common multiple of the three denominators in

$$\frac{a}{2bc^2} + \frac{m}{3c} + \frac{y}{4bc}$$

	Coefficients	b's	c's
2bc²	2	b	c²
3c	3		c
4bc	2*2	b	c

In order to generate the least common multiple, we have to take the least common multiple of the entries in each column, and then form the product of the results.

- The least common multiple of the entries in the coefficients column is 12. (We're taking the least common multiple of 2, 3, and $2 \cdot 2$, which is $2 \cdot 2 \cdot 3$, or 12.)

- The least common multiple of the entries in the base-b column is b. ($2bc^2$ and $4bc$ can be written as $2b^1c^2$ and $4b^1c$, respectively, and $3c$ can be written as $3b^0c$, so we're taking the least common multiple of b^0 and b^1, which is b^1, or b.)

- The least common multiple of the entries in the base-c column is c^2. ($3c$ and $4bc$ can be written as $3c^1$ and $4bc^1$, respectively, so we're taking the least common multiple of c^1 and c^2, which is c^2.)

Therefore, the least common multiple of $2bc^2$, $3c$, and $4bc$ is

$$12bc^2$$

So we need to multiply the numerator and denominator of each fraction by whatever expression is needed to make its denominator equal to

$$12bc^2$$

We get

$$\frac{6}{6} \cdot \frac{a}{2bc^2} + \frac{4bc}{4bc} \cdot \frac{m}{3c} + \frac{3c}{3c} \cdot \frac{y}{4bc}$$

$$\frac{6a}{12bc^2} + \frac{4bcm}{12bc^2} + \frac{3cy}{12bc^2}$$

$$\frac{6a + 4bcm + 3cy}{12bc^2}$$

Factoring to find a common denominator

A fraction in which the numerator and denominator are polynomials is known as a **rational expression**. In this lesson, we'll learn how to add rational expressions.

To do this, we need to find a common denominator, just like when we add fractions in which the numerator and denominator are just numbers. The difference is that finding the common denominator of rational expressions can be more complicated because their denominators can include variables.

Often we'll need to factor the denominators of rational expressions in order to find a common denominator.

Let's look at an example.

Example

Simplify the expression by combining the two fractions.

$$\frac{3}{x-3}+\frac{9}{x^2+2x-15}$$

In order to add these fractions, we'll need a common denominator. Start by factoring the denominator of the second fraction.

$$\frac{3}{x-3}+\frac{9}{(x+5)(x-3)}$$

Now we can see that the common denominator is $(x+5)(x-3)$ and we need to multiply the first rational expression by

$$\frac{x+5}{x+5}$$

This is really just multiplying by a well-chosen expression for 1, and therefore doesn't break any rules of math.

$$\frac{3}{x-3}\cdot\frac{x+5}{x+5}+\frac{9}{(x+5)(x-3)}$$

$$\frac{3(x+5)}{(x+5)(x-3)}+\frac{9}{(x+5)(x-3)}$$

Distribute the 3 in the numerator of the first fraction.

$$\frac{3x+15}{(x+5)(x-3)}+\frac{9}{(x+5)(x-3)}$$

Add the numerators, remembering that the denominator will stay the same.

$$\frac{3x+15+9}{(x+5)(x-3)}$$

$$\frac{3x+24}{(x+5)(x-3)}$$

In this case we could simplify the top a little by factoring out a 3.

$$\frac{3(x+8)}{(x+5)(x-3)}$$

Let's try another example of factoring to find a common denominator.

Example

Simplify the expression by combining the two fractions.

$$\frac{x-5}{2x^2+x-10}+\frac{4}{2x+5}$$

In order to these fractions, we'll need a common denominator. Start by factoring the denominator of the first fraction.

$$\frac{x-5}{(2x+5)(x-2)}+\frac{4}{2x+5}$$

Now we can see that the common denominator is $(2x+5)(x-2)$ and we need to multiply the second rational expression by

$$\frac{x-2}{x-2}$$

Remember, this is just like multiplying by 1.

$$\frac{x-5}{(2x+5)(x-2)}+\frac{4}{2x+5}\cdot\frac{x-2}{x-2}$$

$$\frac{x-5}{(2x+5)(x-2)}+\frac{4(x-2)}{(2x+5)(x-2)}$$

Distribute the 4 in the numerator of the second fraction.

$$\frac{x-5}{(2x+5)(x-2)} + \frac{4x-8}{(2x+5)(x-2)}$$

Add the numerators, remembering that the denominator will stay the same.

$$\frac{x-5+4x-8}{(2x+5)(x-2)}$$

$$\frac{5x-13}{(2x+5)(x-2)}$$

Multiplying rational functions

In this lesson we'll look at how to multiply rational functions, including factoring and cancellation to make the multiplication easier.

As we're canceling factors, we need to remember that numbers that are excluded from the domain of at least one of the rational functions in the product (because they result in the denominator of at least one of the functions being 0) won't be in the domain of the product.

After we cancel factors from the denominator of a product of rational functions, however, it may not be obvious (from the simplified form of the product) that the values which make those factors equal to 0 aren't in the domain of the product. Which is why we should explicitly state the "hidden" values of the variable that are excluded from the domain.

For example, in the product

$$\frac{x+2}{x-1} \cdot \frac{x-1}{x+4}$$

the factor $x-1$ will be canceled, so the simplified version should be written as

$$\frac{x+2}{x+4}, x \neq 1$$

Here we canceled a factor of $x-1$. Since there is no factor $x-1$ in the denominator of the simplified form of the product, it isn't obvious (from the simplified form) that the value of x which makes $x-1$ equal to 0 ($x=1$)

isn't in the domain of the product. This is why we should explicitly state that $x \neq 1$ when we give the simplified form of it.

It'll also help us with these kinds of problems to remember other factoring formulas, like the formulas for the difference and sum of squares:

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

Let's do a couple of examples.

Example

Simplify the expression by combining the two rational functions into a single rational function.

$$\frac{16a^2 - 1}{a^2 - 16} \cdot \frac{a - 4}{4a - 1}$$

To simplify the product

$$\frac{16a^2 - 1}{a^2 - 16} \cdot \frac{a - 4}{4a - 1}$$

we have to factor the top and bottom of the first fraction.

$$\frac{(4a - 1)(4a + 1)}{(a - 4)(a + 4)} \cdot \frac{a - 4}{4a - 1}$$

Cancel the factors that appear in both the numerator and the denominator. $4a-1$ will cancel from the numerator of the first fraction and the denominator of the second fraction. $a-4$ will cancel from the denominator of the first fraction and the numerator of the second fraction. So we're left with only

$$\frac{4a+1}{a+4}$$

We canceled a factor $4a-1$ and a factor $a-4$, and there's no factor of either of those two types in the denominator of the simplified form of the product. Therefore, it isn't obvious (from the simplified form) that the values of a which makes $4a-1$ or $a-4$ equal to 0 aren't in the domain of the product, so we should explicitly state the "hidden" values of a that are excluded from the domain.

To determine those values of a, we solve each of the equations $4a-1=0$ and $a-4=0$.

$$4a-1=0$$

$$4a=1$$

$$a=\frac{1}{4}$$

and

$$a-4=0$$

$$a=4$$

So we should write the product as

$$\frac{4a+1}{a+4},\ a \neq \frac{1}{4},\ 4$$

Let's do another example.

Example

Simplify the expression by combining the two rational functions into a single rational function.

$$\frac{a^2+3a-10}{a^2-7a+12} \cdot \frac{a^2-a-6}{a^2+8a+15}$$

To simplify the product, we have to factor the top and bottom of each fraction.

$$\frac{(a+5)(a-2)}{(a-3)(a-4)} \cdot \frac{(a-3)(a+2)}{(a+5)(a+3)}$$

Cancel the factors that appear in both the numerator and the denominator. We can cancel $a+5$ from the numerator of the first fraction and the denominator of the second fraction. We can cancel $a-3$ from the denominator of the first fraction and the numerator of the second fraction. This leaves us with only

$$\frac{(a-2)}{(a-4)} \cdot \frac{(a+2)}{(a+3)}$$

We canceled a factor $a+5$ and a factor $a-3$, and there's no factor of either of those two types in the denominator of the simplified form of the product. Therefore, it isn't obvious (from the simplified form) that the values of a which makes $a+5$ or $a-3$ equal to 0 aren't in the domain of the product, so we should explicitly state the "hidden" values of a that are excluded from the domain.

To determine those values of a, we solve each of the equations $a+5=0$ and $a-3=0$.

$$a+5=0$$

$$a=-5$$

and

$$a-3=0$$

$$a=3$$

So we should write the product as

$$\frac{(a-2)}{(a-4)}\cdot\frac{(a+2)}{(a+3)},\ a\neq-5,\ 3$$

We can further simplify the product by doing the multiplication in the numerator and the denominator separately.

$$\frac{(a-2)(a+2)}{(a-4)(a+3)},\ a\neq-5,\ 3$$

$$\frac{a^2+2a-2a-4}{a^2+3a-4a-12},\ a\neq-5,\ 3$$

$$\frac{a^2-4}{a^2-a-12}, \ a \neq -5, 3$$

Dividing rational functions

In this lesson we'll look at how to simplify and divide rational functions by using factoring and cancellation.

Factoring and cancellation

If we can find equivalent values to factor out of both the numerator and denominator of a fraction, they can both be pulled out into a separate fraction and simplified (or canceled) to 1. Here's a simple example with constants.

$$\frac{4+2}{8}=\frac{2(2)+2(1)}{2(4)}=\frac{2(2+1)}{2(4)}=\frac{2}{2}\cdot\frac{(2+1)}{(4)}=1\cdot\frac{(2+1)}{(4)}=\frac{3}{4}$$

We were able to factor a 2 out of every term in both the numerator and denominator, then pull them out into a separate fraction, simplify that fraction to 1, and we're left with a simpler fraction.

Here's an example with variables:

$$\frac{4x^2+2x}{8x^3}=\frac{2x(2x)+2x(1)}{2x(4x^2)}=\frac{2x(2x+1)}{2x(4x^2)}$$

$$=\frac{2x}{2x}\cdot\frac{2x+1}{4x^2}=1\cdot\frac{2x+1}{4x^2}=\frac{2x+1}{4x^2}$$

Dividing rational functions

When we divide one rational function by another, the first fraction is the **dividend**, the second fraction is the **divisor**, and the result of doing the division gives us the **quotient**.

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient}$$

Just like when we divide one fraction by another, we turn the division problem into a multiplication problem. To do this, we flip the divisor upside down, and change the division into multiplication. So the problem instead becomes

$$\text{dividend} \cdot \frac{1}{\text{divisor}} = \text{quotient}$$

Once we've changed the division into multiplication, we'll multiply the numerators together to get the new numerator, and multiply the denominators together to get the new denominator.

Then, if there are any common factors between the numerator and denominator, we'll cancel those common factors in order to simplify the quotient.

Let's look at an example so that we can see the transition from division to multiplication in action.

Example

Find the quotient of the rational functions.

$$\frac{x+2}{x-1} \div \frac{x+4}{x-1}$$

To find the quotient, we'll flip the divisor (the second fraction) upside down, and simultaneously change the division to multiplication. So we can rewrite the problem as

$$\frac{x+2}{x-1} \cdot \frac{x-1}{x+4}$$

Now that we're doing multiplication instead of division, we can combine the fractions by multiplying across the numerators and across the denominators.

$$\frac{(x+2)(x-1)}{(x-1)(x+4)}$$

We've got a common factor of $x-1$ in the numerator and denominator that can be canceled, leaving us with just

$$\frac{x+2}{x+4}$$

Restrictions on the domain

We've just learned how to divide rational functions, but we left out one really important aspect of these division problems. Whenever we divide rational functions, we have to consider the values for which the quotient will be undefined.

There are three sets of values that will make a quotient undefined:

1. Any values(s) where the **dividend's denominator** is 0

2. Any value(s) where the **divisor's denominator** is 0

3. Any values(s) where the **divisor's numerator** is 0

Remember, any fraction will be undefined whenever its denominator is 0, so these first two value sets make sense. Of course, the dividend will be undefined when its denominator is 0, and the divisor will be undefined when its denominator is 0. We're certainly not going to be able to find the quotient if either the dividend or divisor is undefined, so we need to make sure to state that the quotient won't be defined for any values where either denominator is 0.

The third value set is slightly trickier. If the numerator of a fraction is 0, then the value of the whole fraction will be 0, since 0 divided by anything, is 0. So, when the numerator of the divisor is 0, that makes the entire divisor equal to 0. But if the divisor is 0, then when we try to find the quotient, we'll be taking

$$\frac{\text{dividend}}{0}$$

which we can't do. That's why a zero value in the divisor's numerator will also make the quotient undefined.

So, in actuality, when we divide rational functions, we need to start by considering these restrictions first, and then proceed with the rest of the division problem.

Let's revisit the example we were working with, and this time consider these restrictions on the domain.

Example

Find the quotient of the rational functions.

$$\frac{x+2}{x-1} \div \frac{x+4}{x-1}$$

This time around, we'll look at the restrictions, considering all three value sets that might cause us trouble:

The dividend's denominator will be 0 at

$$x-1=0$$

$$x=1$$

The divisor's denominator will be 0 at

$$x-1=0$$

$$x=1$$

The divisor's numerator will be 0 at

$$x+4=0$$

$$x=-4$$

If we put these results together, we can say that the restrictions on the domain are $x \neq -4,1$.

Once we've established this set of values, we can actually do the division problem, following the same steps as the last example.

$$\frac{x+2}{x-1} \div \frac{x+4}{x-1}$$

$$\frac{x+2}{x-1} \cdot \frac{x-1}{x+4}$$

$$\frac{(x+2)(x-1)}{(x-1)(x+4)}$$

$$\frac{x+2}{x+4}$$

The answer to the problem is the combination of the quotient and the restrictions.

$$\frac{x+2}{x+4} \text{ with } x \neq -4,1$$

However, there's just one more thing we always have to check. The quotient itself, given that the denominator is $x+4$, makes it obvious that $x \neq -4$, since $x=-4$ would make the denominator of the quotient equal to 0.

So there's no need to list that value as a restriction on the quotient's value, because we can see it just by looking at the quotient. The $x \neq 1$ restriction can't be identified from the quotient (it was lost when we canceled common factors), so we have to keep that restriction. Therefore, the final answer is

$$\frac{x+2}{x+4} \text{ with } x \neq 1$$

Let's do a couple more examples where we follow this same process to find the quotient.

Example

Find the quotient of the rational functions.

$$\frac{6a+27}{18a^2+36a} \div \frac{16a+72}{2a+4}$$

Factor the numerator and denominator of both fractions as completely as possible.

$$\frac{3(2a+9)}{18a(a+2)} \div \frac{8(2a+9)}{2(a+2)}$$

Now we'll consider the restrictions. The dividend's denominator tells us that $a \neq -2,0$, the divisor's denominator tells us that $a \neq -2$, and the divisor's numerator tells us that $a \neq -9/2$. So the full set of restrictions is $a \neq -9/2, -2,0$.

Next we'll find the quotient.

$$\frac{3(2a+9)}{18a(a+2)} \cdot \frac{2(a+2)}{8(2a+9)}$$

$$\frac{6(2a+9)(a+2)}{144a(a+2)(2a+9)}$$

$$\frac{6}{144a}$$

$$\frac{1}{24a}$$

Putting this quotient together with the restrictions gives

$$\frac{1}{24a} \text{ with } a \neq -\frac{9}{2}, -2, 0$$

But the quotient still shows us that $a \neq 0$, so we don't need to include it in the list of restrictions.

$$\frac{1}{24a} \text{ with } a \neq -\frac{9}{2}, -2$$

Let's try another example, this time where none of the restrictions on the domain are identifiable from the quotient.

Example

Find the quotient of the rational functions.

$$\frac{3x^2-25x-18}{27x+18} \div \frac{5x-3}{5x^2-33x+18}$$

Factor the numerator and denominator of both fractions as completely as possible.

$$\frac{(3x+2)(x-9)}{9(3x+2)} \div \frac{5x-3}{(5x-3)(x-6)}$$

Now we'll consider the restrictions. The dividend's denominator tells us that $x \neq -2/3$, the divisor's denominator tells us that $x \neq 3/5,6$, and the divisor's numerator tells us that $x \neq 3/5$. So the full set of restrictions is $x \neq -2/3,3/5,6$.

Next we'll find the quotient.

$$\frac{(3x+2)(x-9)}{9(3x+2)} \cdot \frac{(5x-3)(x-6)}{5x-3}$$

$$\frac{(3x+2)(x-9)(5x-3)(x-6)}{9(3x+2)(5x-3)}$$

$$\frac{(x-9)(x-6)}{9}$$

Putting this quotient together with the restrictions gives

$$\frac{(x-9)(x-6)}{9} \text{ with } x \neq -\frac{2}{3},\frac{3}{5},6$$

None of these restrictions are evident from the quotient, so we keep all of them.

We’ll do one more example. This one has a lot of restrictions!

Example

Find the quotient of the rational functions.

$$\frac{x^2-4}{x^2-9x+18} \div \frac{x^2-25}{x^2-8x-20}$$

Factor the numerator and denominator of both fractions as completely as possible.

$$\frac{(x-2)(x+2)}{(x-6)(x-3)} \div \frac{(x-5)(x+5)}{(x-10)(x+2)}$$

Now we’ll consider the restrictions. The dividend’s denominator tells us that $x \neq 3,6$, the divisor’s denominator tells us that $x \neq -2,10$, and the divisor’s numerator tells us that $x \neq -5,5$. So the full set of restrictions is $x \neq -5,-2,3,5,6,10$.

Next we’ll find the quotient.

$$\frac{(x-2)(x+2)}{(x-6)(x-3)} \cdot \frac{(x-10)(x+2)}{(x-5)(x+5)}$$

$$\frac{(x-2)(x+2)^2(x-10)}{(x-6)(x-3)(x-5)(x+5)}$$

Putting this quotient together with the restrictions gives

$$\frac{(x-2)(x+2)^2(x-10)}{(x-6)(x-3)(x-5)(x+5)} \text{ with } x \neq -5,-2,3,5,6,10$$

But the quotient still shows us that $x \neq -5,3,5,6$, so we don't need to include those in the list of restrictions.

$$\frac{(x-2)(x+2)^2(x-10)}{(x-6)(x-3)(x-5)(x+5)} \text{ with } x \neq -2,10$$

Direct variation

In this lesson we'll look at solving equations that express direct variation relationships, which are relationships in the form $kx = y$, where k is a constant. In a direct variation equation, we have two variables, usually x and y, and a constant, usually k.

The main idea in direct variation is that, as one variable increases, the other variable will also increase. That means if x increases y increases, and if y increases x increases. The number k is a constant, so it's the same for all pairs of numbers (x, y) that satisfy the equation.

In a direct variation problem, x and y are said to vary directly, and k is called the constant of proportionality.

This lesson will help us find the value of a variable in a direct variation equation, given other information (such as the corresponding value of the other variable and the constant k, or the corresponding value of the other variable and one pair (x, y) that satisfies the equation).

Let's look at an example.

Example

Two variables x and y vary directly. If the constant of proportionality, k, is 20, what is the value of y when x equals 15?

Remember that the general form of a direction variation equation is $y = kx$. In this example, we know that $k = 20$ and $x = 15$, and that we're looking for y. So

$$y = 20(15)$$

$$y = 300$$

Let's try another one.

Example

In a certain direct variation equation, the constant of proportionality, k, satisfies $5k = 50$. If $y = 85$, what is the value of x?

Remember that the general form of a direct variation equation is $y = kx$. In this example,

$$5k = 50$$

$$\frac{5k}{5} = \frac{50}{5}$$

$$k = 10$$

and

$$y = 85$$

So $85 = 10x$. Now let's solve for x.

$$\frac{85}{10} = \frac{10x}{10}$$

$$8.5 = x$$

Sometimes we'll be given a pair of equations that involve the same direct variation equation (they have the same constant of proportionality) and we'll need to solve for one of the variables. We call this type of problem a two-step problem.

Example

If $2k = 14$ and $kx = 56$, what is the value of x?

We'll solve the first equation for k.

$$2k = 14$$

$$\frac{2k}{2} = \frac{14}{2}$$

$$k = 7$$

Now we'll take the value we found for k and plug it into the second equation to solve for x.

$$kx = 56$$

$$7x = 56$$

$$\frac{7x}{7} = \frac{56}{7}$$

$$x = 8$$

Inverse variation

In this lesson we'll look at solving equations that express inverse variation relationships, which are relationships of the form

$$y = \frac{k}{x}$$

In an inverse variation relationship we have two variables, usually x and y, and a constant, usually k.

The main idea in inverse variation is that as one variable increases the other variable decreases. That means that if x is increasing y is decreasing, and if x is decreasing y is increasing. The number k is a constant, so it's the same for all pairs of numbers (x, y) that satisfy the equation.

Inverse variation can also be called "inverse proportion," because the variables x and y can be said to be inversely proportional to each other. The constant of proportionality is the number k. (Similarly, direct variation can be called "direct proportion." In that case, the variables can be said to be directly proportional to each other.)

An inverse variation can be written in any of the following three forms: $xy = k$, $y = k/x$, and $x = k/y$.

Let's look at an example.

Example

If we know that y varies inversely with x, and $x = 10$ when $y = 4$, what is the constant of proportionality? And if we express y as a function of x, what is that function?

The formula for an inverse variation is $xy = k$.

We know that $x = 10$ and $y = 4$, so

$$10 \cdot 4 = k$$

$$40 = k$$

The constant of proportionality is $k = 40$. If we express y as a function of x, we get

$$y = \frac{k}{x}$$

$$y = \frac{40}{x}$$

Let's look at another example.

Example

The product of two numbers is always 100. Describe the relationship between the two numbers as a function.

The equation $xy = k$ expresses an inverse relationship. It means that for all pairs (x, y) that satisfy this equation, the product of x and y is always the same (namely, k).

In this example $xy = 100$, so $k = 100$. If we express y as a function of x, we get

$$y = \frac{100}{x}$$

Sometimes we'll be given a pair of equations that involve the same inverse variation equation (they have the same constant of proportionality) and we'll need to solve for one of the variables. Just as with direction variation, we call this type of problem a **two-step problem**.

Example

Solve the two-step problem, given that x and y vary inversely.

If $k/5 = 3$ and $k/x = y$, find the value of x when $y = 30$.

We'll solve the first equation for k.

$$\frac{k}{5} = 3$$

$$\frac{k}{5} \cdot 5 = 3 \cdot 5$$

$$k = 15$$

Now we'll take the value we found for k, and we'll plug that and the value of y (30) into the second equation to solve for x.

$$\frac{k}{x} = y$$

$$\frac{15}{x} = 30$$

$$\frac{15}{x}(x) = 30(x)$$

$$15 = 30x$$

$$\frac{15}{30} = \frac{30x}{30}$$

$$\frac{1}{2} = x$$

Decimal equations

In this lesson we'll learn to solve equations with decimals by multiplying by powers of 10.

Remember from Pre-Algebra that we can identify the **place value** of any digit.

0.1 has a 1 in the tenths place

0.02 has a 2 in the hundredths place

0.003 has a 3 in the thousandths place

0.0004 has a 4 in the ten-thousandths place

When we multiply by...

10 move the decimal point one place to the right

100 move the decimal point two places to the right

$1,000$ move the decimal point three places to the right

$10,000$ move the decimal point four places to the right

Notice that we move the decimal point to the right one place for each 0 in the multiple of 10.

One method of solving an equation with decimals is to multiply both sides of the equation by a power of 10. In order to solve an equation involving decimals, the first thing we'll do is get rid of the decimals by changing them to whole numbers.

Let's do a couple of examples.

Example

Solve the decimal equation.

$$0.3x + 5 = 11$$

In order to solve this equation, the first thing we'll do is get rid of the decimal by changing it to a whole number. Since the decimal in our equation ends in the tenths place, we'll need to multiply both sides of the equation by 10 in order to change the decimal to a whole number.

$$0.3x + 5 = 11$$

$$(0.3x + 5)(10) = 11(10)$$

$$0.3x(10) + 5(10) = 11(10)$$

$$3x + 50 = 110$$

$$3x + 50 - 50 = 110 - 50$$

$$3x = 60$$

$$\frac{3x}{3} = \frac{60}{3}$$

$$x = 20$$

Let's do another one.

Example

Solve for the variable.

$$6a + 5a = -1.1$$

In order to solve this equation, the first thing we'll do is get rid of the decimal by changing it to an integer. Since the decimal in our equation ends in the tenths place, we'll need to multiply both sides of the equation by 10 in order to change the decimal to a whole number.

$$6a + 5a = -1.1$$

$$(6a + 5a)(10) = -1.1(10)$$

$$6a(10) + 5a(10) = -1.1(10)$$

$$60a + 50a = -11$$

$$110a = -11$$

$$\frac{110a}{110} = \frac{-11}{110}$$

$$a = \frac{-1}{10}$$

$$a = -0.1$$

Let's look at another example.

Example

Solve for the variable.

$$0.4n + 3.9 = 5.78$$

In order to solve this equation, the first thing we'll do is get rid of the decimals by changing them to whole numbers. Since the longest decimal in our equation ends in the hundredths place, we'll need to multiply both sides of the equation by 100 in order to change all the decimals to whole numbers. This means we'll move each decimal point two places to the right.

$$0.4n + 3.9 = 5.78$$

$$(0.4n + 3.9)(100) = 5.78(100)$$

$$0.4n(100) + 3.9(100) = 5.78(100)$$

$$40n + 390 = 578$$

$$40n + 390 - 390 = 578 - 390$$

$$40n = 188$$

$$\frac{40n}{40} = \frac{188}{40}$$

$$n = \frac{47}{10}$$

$$n = 4.7$$

Fractional equations

In this lesson we'll look at how to solve equations that include fractions as coefficients and standalone terms. There are a couple of things we need to remember about multiplying fractions.

1. Multiplying a fraction by its reciprocal will always give us a value of 1.

   For example, $4/5$ has a reciprocal of $5/4$ because

   $$\frac{4}{5} \cdot \frac{5}{4} = 1$$

2. To clear a fraction from an equation, multiply both sides of the equation by the fraction's denominator.

   For example, to clear the 2 from the fraction in $5x + 1/2 = 12$, multiply both sides of the equation by 2.

   $$2\left(5x + \frac{1}{2}\right) = 2(12)$$

   $$2(5x) + 2\left(\frac{1}{2}\right) = 2(12)$$

   $$10x + 1 = 24$$

In general, we'll use the same set of steps to solve fractional equations.

1. Find the lowest common denominator (LCD).

2. Multiply both sides of the equation by the LCD to remove the fractions.

3. Solve the equation and check the solution.

Let's do a few examples where we solve an equation with a fraction. First let's look at an equation that has a fractional coefficient.

Example

Solve for the variable.

$$\frac{4}{5}n = 20$$

To get rid of a fractional coefficient, we'll multiply both sides of the equation by the fraction's reciprocal, because that'll change the coefficient to 1.

$$\frac{4}{5}n = 20$$

$$\frac{5}{4} \cdot \frac{4}{5}n = \frac{5}{4} \cdot 20$$

$$\frac{20}{20}n = \frac{100}{4}$$

$$1n = 25$$

$$n = 25$$

If we have a fractional coefficient and another term on the same side of the equation, we can isolate the term with the variable and then multiply both sides by the reciprocal of the fractional coefficient.

Example

Solve for the variable.

$$\frac{4}{7}x + 14 = 22$$

First isolate the fractional term.

$$\frac{4}{7}x + 14 - 14 = 22 - 14$$

$$\frac{4}{7}x = 8$$

Now get rid of the fractional coefficient by multiplying both sides of the equation by the reciprocal of $4/7$.

$$\frac{7}{4} \cdot \frac{4}{7}x = \frac{7}{4} \cdot 8$$

$$\frac{28}{28}x = \frac{56}{4}$$

$$1x = 14$$

$$x = 14$$

We could also solve this problem by first clearing the fraction. In order to get rid of the fraction, we have to multiply both sides of the equation by the fraction's denominator.

$$7\left(\frac{4}{7}x + 14\right) = 7(22)$$

$$7 \cdot \frac{4}{7}x + 7 \cdot 14 = 7 \cdot 22$$

$$4x + 98 = 154$$

Now we can solve for the variable using inverse operations.

$$4x + 98 - 98 = 154 - 98$$

$$4x = 56$$

$$x = 14$$

Let's do one more example.

Example

Solve for the variable.

$$\frac{2x}{5} + \frac{x}{3} = \frac{3}{5}$$

The lowest common denominator (LCD) of the three fractions is 15, because 15 is the least common multiple (LCM) of the denominators in the equation, 5 and 3. So in order to remove the fractions but keep the equation balanced, we'll multiply both sides of the equation by 15.

$$15\left(\frac{2x}{5}+\frac{x}{3}\right)=15\left(\frac{3}{5}\right)$$

$$15\left(\frac{2x}{5}\right)+15\left(\frac{x}{3}\right)=15\left(\frac{3}{5}\right)$$

$$6x+5x=9$$

$$11x=9$$

$$x=\frac{9}{11}$$

Rational equations

We just learned how to solve equations with fractions, but the fractions we looked at never included any variables themselves.

Now we want to look at fractional equations where we have variables in the numerator and/or denominator of the fractions, and we'll call these **rational equations**.

Let's review the general steps for how to solve rational equations.

1. Find any value of the variable that would make any denominator equal to zero.

2. Find the least common denominator (LCD).

3. Multiply both sides of the equation to clear the fractions.

4. Solve the resulting equation and check the solutions. Don't forget to discard any values from step 1 if they are algebraic solutions.

We multiply the fractions by the least common denominator because doing so clears all the fractions from the equation, which makes it much easier to solve.

We have to also make sure that we find the values that would make any denominators zero because this would help us know whether there are any algebraic solutions that must be discarded. This type of solution is called an **extraneous solution**, and it can cause parts of the equation to be undefined.

Let's solve some rational equations.

Example

Solve the rational equation.

$$\frac{1}{x}+\frac{1}{5}=\frac{3}{10}$$

If $x=0$, then $1/x$ is undefined, so this equation is true only if $x \neq 0$.

Now we need to find the least common denominator of all denominators in the equation. The LCD is $10x$, so we'll multiply both sides of the equation by $10x$.

$$\left(\frac{1}{x}+\frac{1}{5}=\frac{3}{10}\right)10x$$

$$\frac{1}{x}(10x)+\frac{1}{5}(10x)=\frac{3}{10}(10x)$$

$$10+2x=3x$$

Let's move $2x$ to the right side.

$$10=3x-2x$$

$$10=x$$

$$x=10$$

We can check the solution by substituting $x=10$ into the equation.

$$\frac{1}{x}+\frac{1}{5}=\frac{3}{10}$$

$$\frac{1}{10}+\frac{1}{5}=\frac{3}{10}$$

$$\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$$

$$\frac{3}{10}=\frac{3}{10}$$

Because the solution satisfies the equation, we know it's a real solution, and we can therefore say that the solution is $x=10$.

Let's try another one.

Example

Solve the equation.

$$\frac{x-8}{x^2+2x-8}-\frac{2}{x-2}=\frac{5}{x+4}$$

We can rewrite the equation as

$$\frac{x-8}{(x-2)(x+4)}-\frac{2}{x-2}=\frac{5}{x+4}$$

If $x - 2 = 0$ or $x + 4 = 0$, then the equation is undefined, so this equation is true only if $x \neq -4$ and $x \neq 2$.

Now we need to find the least common denominator of all denominators in the equation. The LCD is $(x - 2)(x + 4)$, so we'll multiply both sides of the equation by $(x - 2)(x + 4)$.

$$\left(\frac{x-8}{(x-2)(x+4)} - \frac{2}{x-2}\right)(x-2)(x+4) = \left(\frac{5}{x+4}\right)(x-2)(x+4)$$

$$\frac{x-8}{(x-2)(x+4)} \cdot (x-2)(x+4) - \frac{2}{x-2} \cdot (x-2)(x+4) = \frac{5}{x+4} \cdot (x-2)(x+4)$$

$$x - 8 - 2(x + 4) = 5(x - 2)$$

$$x - 8 - 2x - 8 = 5x - 10$$

$$-x - 16 = 5x - 10$$

$$-6x - 16 = -10$$

$$-6x = 6$$

$$x = -1$$

We can check the solution by substituting $x = -1$ into the equation to find out whether the equation is true.

$$\frac{x-8}{x^2+2x-8} - \frac{2}{x-2} = \frac{5}{x+4}$$

$$\frac{-1-8}{(-1)^2+2(-1)-8} - \frac{2}{-1-2} = \frac{5}{-1+4}$$

$$\frac{-9}{-9} - \frac{2}{-3} = \frac{5}{3}$$

$$1 + \frac{2}{3} = \frac{5}{3}$$

$$\frac{3}{3} + \frac{2}{3} = \frac{5}{3}$$

$$\frac{5}{3} = \frac{5}{3}$$

Because $x = -1$ satisfies the equation, it’s a solution to the equation.

Radical equations

In this lesson we'll look at how to solve for the variable in a radical equation by isolating the radical, squaring both sides, and then using inverse operations.

The thing to remember about solving a radical equation is that if we can get the radical by itself, then we just need to square both sides and solve for the variable. However, because of the squaring, we could introduce "solutions" that aren't actually solutions of the original equation. (They're called **extraneous solutions**.)

So after we think we've found the solutions, we need to check them by plugging them into the original equation. If the equation we get for a potential solution is true, then that solution is indeed a solution; otherwise, it isn't a solution.

These are general steps for solving a radical equation:

1. Isolate the radical expression on one side of the equation.

2. Square both sides of the equation.

3. Rearrange and solve the equation.

The general strategy of solving a radical equation is raising a radical with index n to the nth power, which will help us eliminate the radical. In other words, for $a \geq 0$,

$$(\sqrt[n]{a})^n = a$$

Let's look at a few examples.

Example

Solve for the variable.

$$\sqrt{x} - 3 = 2$$

We have to keep the equation balanced, so when we add 3 to the left side, we'll also add it to the right side.

$$\sqrt{x} - 3 = 2$$

$$\sqrt{x} - 3 + 3 = 2 + 3$$

$$\sqrt{x} = 5$$

Squaring both sides, we get

$$(\sqrt{x})^2 = 5^2$$

$$x = 25$$

To determine whether this is actually a solution, we'll substitute 25 for x in the original equation:

$$\sqrt{x} - 3 = 2$$

$$\sqrt{25} - 3 = 2$$

$$5 - 3 = 2$$

$$2 = 2$$

This equation is true, so $x = 25$ is indeed a solution.

Let's do another one.

Example

Solve for the variable.

$$\sqrt{x-2} + 5 = 9$$

We have to keep the equation balanced, so when we subtract 5 from the left side, we'll also subtract it from the right side.

$$\sqrt{x-2} + 5 = 9$$

$$\sqrt{x-2} + 5 - 5 = 9 - 5$$

$$\sqrt{x-2} = 4$$

Squaring both sides, we get

$$(\sqrt{x-2})^2 = 4^2$$

$$x - 2 = 16$$

Now add 2 to both sides.

$$x - 2 + 2 = 16 + 2$$

$$x = 18$$

To determine whether this is actually a solution, we'll substitute 18 for x in the original equation.

$$\sqrt{x-2} + 5 = 9$$

$$\sqrt{18-2} + 5 = 9$$

$$\sqrt{16} + 5 = 9$$

$$4 + 5 = 9$$

$$9 = 9$$

This equation is true, so $x = 18$ is indeed a solution.

Let's look at an example where we'll have an x^2 term once we do the squaring.

Example

Solve for the variable.

$$2x + \sqrt{x+1} = 8$$

We'll first get the radical by itself.

$$2x + \sqrt{x+1} = 8$$

$$2x - 2x + \sqrt{x+1} = 8 - 2x$$

$$\sqrt{x+1} = 8 - 2x$$

Squaring both sides, we get

$$(\sqrt{x+1})^2 = (8 - 2x)^2$$

$$x + 1 = 64 - 32x + 4x^2$$

Now let's get all of the terms to one side of the equation. Once we do that, we'll have a quadratic polynomial on one side of the equation. As usual, we'd like the coefficient of the x^2 term in that quadratic polynomial to be positive, so we want to keep the $4x^2$ on the right side of the equation. Therefore, we'll move the $x + 1$ to the right side.

$$x + 1 - (x + 1) = 64 - 32x + 4x^2 - (x + 1)$$

$$0 = 64 - 32x + 4x^2 - x - 1$$

Now we'll combine like terms on the right side.

$$0 = 4x^2 + (-32x - x) + (64 - 1)$$

$$0 = 4x^2 - 33x + 63$$

Factor the quadratic polynomial $4x^2 - 33x + 63$, and then solve the resulting equation for x.

$$0 = (4x - 21)(x - 3)$$

Now we'll set each factor equal to 0 and solve for x.

$$4x - 21 = 0$$

$$4x = 21$$

$$x = \frac{21}{4}$$

and

$$x - 3 = 0$$

$$x = 3$$

Now we'll check to determine whether $x = 21/4$ and $x = 3$ are actually solutions, by plugging each of them into the original equation.

For $x = 21/4$:

$$2x + \sqrt{x+1} = 8$$

$$2\left(\frac{21}{4}\right) + \sqrt{\frac{21}{4} + 1} = 8$$

$$\frac{21}{2} + \sqrt{\frac{21}{4} + \frac{4}{4}} = 8$$

$$\frac{21}{2} + \sqrt{\frac{25}{4}} = 8$$

$$\frac{21}{2} + \frac{5}{2} = 8$$

$$\frac{26}{2} = 8$$

$$13 = 8$$

This equation is false, so $x = 21/4$ is not a solution.

For $x = 3$:

$$2x + \sqrt{x+1} = 8$$

$$2(3) + \sqrt{3+1} = 8$$

$$6 + \sqrt{4} = 8$$

$$6 + 2 = 8$$

$$8 = 8$$

This equation is true, so $x = 3$ is indeed a solution.

More than one root

We understand now how to solve equations that include only one square root, but what if there are two or more square roots? In that case, solving the equation will just take a little longer, because we just need to repeat the same process for each of the square roots.

Let's look at an example.

Example

Solve for the variable.

$$\sqrt{2x-1}-\sqrt{x-9}=3$$

To solve the radical equation, first isolate one of the radicals.

$$\sqrt{2x-1}=3+\sqrt{x-9}$$

Square both sides of the equation. Remember to FOIL the right side of the equation.

$$(\sqrt{2x-1})^2=(3+\sqrt{x-9})^2$$

$$2x-1=9+6\sqrt{x-9}+x-9$$

$$2x-1=6\sqrt{x-9}+x$$

Now isolate the other radical.

$$x-1=6\sqrt{x-9}$$

$$\frac{x-1}{6}=\sqrt{x-9}$$

Square both sides of the equation.

$$\left(\frac{x-1}{6}\right)^2=(\sqrt{x-9})^2$$

$$\frac{x^2-2x+1}{36}=x-9$$

$$x^2 - 2x + 1 = 36x - 324$$

$$x^2 - 38x + 325 = 0$$

$$(x - 13)(x - 25) = 0$$

Then

$$x - 13 = 0$$

$$x = 13$$

or

$$x - 25 = 0$$

$$x = 25$$

Plug both solutions back into the original equation to make sure they satisfy it. Plugging in $x = 13$ satisfies the equation,

$$\sqrt{2(13) - 1} - \sqrt{13 - 9} = 3$$

$$\sqrt{26 - 1} - \sqrt{4} = 3$$

$$\sqrt{25} - 2 = 3$$

$$5 - 2 = 3$$

$$3 = 3$$

and plugging in $x = 25$ also satisfies the equation.

$$\sqrt{2(25) - 1} - \sqrt{25 - 9} = 3$$

$$\sqrt{50-1}-\sqrt{16}=3$$

$$\sqrt{49}-4=3$$

$$7-4=3$$

$$3=3$$

So $x=13$ and $x=25$ are both solutions.

Multivariable equations

In this lesson we'll look at how to solve a multivariable equation for a certain variable in terms of the others.

When we solve an equation for a variable, we need to use inverse operations to isolate the variable we're solving for.

For example, if we have $y = z + w + x$ and we want to solve the equation for w in terms of x, y, and z, we need to move the terms other than w so that the w is by itself on one side of the equation.

$$y = z + w + x$$

Move the z and the x by subtracting them from both sides.

$$y - z - x = z - z + w + x - x$$

$$y - z - x = w$$

Now we've isolated the variable we were solving for (w). If we want to end up with that variable on the left-hand side, we can switch the two sides of this equation.

$$w = y - z - x$$

Let's do a few more examples so we can get the hang of it.

Example

Solve for t in terms of v, w, x, and z.

$$wx = -tvz$$

We want to get t by itself on one side of the equation.

We can do this by dividing both sides by -1, then dividing both sides by vz.

$$wx = -tvz$$

$$\frac{wx}{-1} = \frac{-tvz}{-1}$$

$$-wx = tvz$$

$$\frac{-wx}{vz} = \frac{tvz}{vz}$$

$$\frac{-wx}{vz} = t$$

If we'd like, we can move the t to the left-hand side. Another thing we can do is write the negative sign in front of the fraction.

$$t = -\frac{wx}{vz}$$

Let's do one with terms that have coefficients other than 1.

Example

Isolate the variable s.

$$2s - 3t + 2u = 7$$

We need to get s by itself on one side of the equation. Let's move the $2u$ first.

$$2s - 3t + 2u = 7$$

$$2s - 3t + 2u - 2u = 7 - 2u$$

$$2s - 3t = 7 - 2u$$

Next, let's move the $-3t$.

$$2s - 3t + 3t = 7 - 2u + 3t$$

$$2s = 7 - 2u + 3t$$

Finally, we'll divide both sides by 2.

$$\frac{2s}{2} = \frac{7 - 2u + 3t}{2}$$

$$s = \frac{7 - 2u + 3t}{2}$$

Let's do one more, this time with terms that have coefficients other than 1 on both sides.

Example

Solve for b in terms of a and c.

$$3a - 4c - b = 5a - 3c$$

We need to get b by itself on one side of the equation. Let's move the $3a$ first.

$$3a - 4c - b = 5a - 3c$$

$$3a - 3a - 4c - b = 5a - 3a - 3c$$

$$-4c - b = 2a - 3c$$

Next, let's move the $-4c$.

$$-4c + 4c - b = 2a - 3c + 4c$$

$$-b = 2a + c$$

Finally, we'll multiply by -1 to isolate b.

$$-1(-b) = -1(2a + c)$$

$$b = -2a - c$$

Multivariable rational equations

At times we'd like to take an equation that has at least one fraction with a variable in its denominator and write the equation in a different way. We'll call an equation like this an **abstract fractional equation**. This lesson will look at how to do that.

There are a few things we want to remember about rational functions in general.

1. Multiplying a fraction by its reciprocal will always give us a value of 1.

   For example x/y has a reciprocal of y/x because

   $$\frac{x}{y} \cdot \frac{y}{x} = 1$$

   Remember that we can't divide by 0, so we should explicitly state that neither x nor y can be 0.

   $$\frac{x}{y} \cdot \frac{y}{x} = 1, \ x, y \neq 0$$

2. To clear a fraction from an equation, multiply both sides of the equation by the fraction's denominator.

   For example, to clear the b from the fraction in

   $$ax + \frac{m}{b} = c$$

   multiply both sides of the equation by b.

$$ax + \frac{m}{b} = c$$

$$b\left(ax + \frac{m}{b}\right) = b(c)$$

$$b(ax) + b\left(\frac{m}{b}\right) = b(c)$$

$$abx + m = bc$$

Remember that we can't divide by 0, so this new equation is true only if $b \neq 0$.

Let's solve some abstract fractional equations for specific variables.

Example

Solve the equation for n, if $n \neq 0$.

$$\frac{m}{n} + x + ab = c$$

In order to get rid of the fraction, we have to multiply both sides of the equation by the denominator of m/n.

$$\frac{m}{n} + x + ab = c$$

$$n\left(\frac{m}{n} + x + ab\right) = n(c)$$

$$n \cdot \frac{m}{n} + n(x) + n(ab) = n(c)$$

$$m + nx + nab = nc$$

To solve for n, we'll need to collect all terms containing n on one side of the equation, and then factor out the n.

Let's move m to the right side and nc to the left side.

$$nx + nab - nc = -m$$

Now factor out n.

$$n(x + ab - c) = -m$$

Divide both sides by $(x + ab - c)$.

$$n = \frac{-m}{x + ab - c}, \ n \neq 0$$

We could also write this with the negative sign in front.

$$n = -\frac{m}{x + ab - c}, \ n \neq 0$$

Let's try another one.

Example

Solve for x if $x \neq 0$ and $y \neq 0$.

$$\frac{1}{x} - \frac{m}{y} = p$$

In order to get rid of the fractions, we have to multiply both sides of the equation by the denominators of both fractions, x and y.

$$\frac{1}{x} - \frac{m}{y} = p$$

$$xy\left(\frac{1}{x} - \frac{m}{y}\right) = xy(p)$$

$$xy\left(\frac{1}{x}\right) - xy\left(\frac{m}{y}\right) = xy(p)$$

$$y - mx = xyp$$

To solve for x we'll need to collect all terms containing x on one side of the equation, and then factor out the x.

Let's move mx to the right side.

$$y = mx + xyp$$

It's nice to end up with the variable that we're solving for (in this case x) on the left side, so we'll switch the two sides of this equation.

$$mx + xyp = y$$

Now factor out the x.

$$x(m + yp) = y$$

Divide both sides by $m+yp$.

$$\frac{x(m+yp)}{m+yp}=\frac{y}{m+yp}$$

$$x=\frac{y}{m+yp},\ x,\ y\neq 0$$

Systems with subscripts

In math and science, we might encounter a variable with a subscript, but we shouldn't let the subscripts scare us. They're just a way to keep track of variables that could be related to each other in some way.

What does a variable with a subscript look like?

As an example, t_1, t_2, t_3 are all variables with subscripts. They could represent three different measurements of time for the same experiment. We read them as "time 1," "time 2," and "time 3," but it's shorter to write them with the subscripts instead of writing them out.

Even though the variables can be related in some way, that doesn't mean they have the same value. This means that if we're solving systems of equations that have variables with subscripts, we'll need to solve for each variable.

Let's do a few examples so we can get comfortable with the idea.

Example

Use any method to find the unique solution to the system of equations.

$$R_1T_1 = 500$$

$$R_2 = 10R_1$$

$$R_2T_2 = 800$$

$$T_2 = 4 - T_1$$

Let’s come up with a plan. We know how to solve a pair of equations in two unknowns, so let’s see if we can rewrite the third equation ($R_2T_2 = 800$) as an equation in terms of the variables R_1 and T_1, so we can solve the system that consists of that new equation and the first equation ($R_1T_1 = 500$).

Since the second equation ($R_2 = 10R_1$) is already solved for R_2, and the fourth equation ($T_2 = 4 - T_1$) is already solved for T_2, we can substitute the expressions $10R_1$ and $4 - T_1$ for R_2 and T_2, respectively, in the third equation.

$$R_2T_2 = 800$$

$$(10R_1)(4 - T_1) = 800$$

Use the distributive property.

$$10R_1(4) - 10R_1(T_1) = 800$$

$$40R_1 - 10R_1T_1 = 800$$

We can divide everything by 10 to make it a little easier.

$$4R_1 - R_1T_1 = 80$$

The first equation ($R_1T_1 = 500$) gives the value of R_1T_1 (500). If we substitute 500 for R_1T_1 in the equation we just found ($4R_1 - R_1T_1 = 80$), we have

$$4R_1 - 500 = 80$$

$$4R_1 - 500 + 500 = 80 + 500$$

$$4R_1 = 580$$

$$\frac{4R_1}{4} = \frac{580}{4}$$

$$R_1 = 145$$

We know that $R_2 = 10R_1$, so we get

$$R_2 = 10(145)$$

$$R_2 = 1{,}450$$

We can also use R_1 to find T_1, with the equation $R_1T_1 = 500$.

$$145(T_1) = 500$$

$$\frac{145(T_1)}{145} = \frac{500}{145}$$

$$T_1 = \frac{100}{29}$$

We can use R_2 to find T_2, with the equation $R_2T_2 = 800$.

$$1{,}450(T_2) = 800$$

$$\frac{1{,}450(T_2)}{1{,}450} = \frac{800}{1{,}450}$$

$$T_2 = \frac{16}{29}$$

Collecting all of our results, we get

$$(R_1, T_1) = \left(145, \frac{100}{29}\right)$$

$$(R_2, T_2) = \left(1{,}450, \frac{16}{29}\right)$$

Let's look at a system of two equations with subscripts.

Example

Solve the system of equations for h_t and x_t.

$$h_t = 2x_t - 4$$

$$h_t = \frac{1}{3}x_t + 3$$

Here the expression on the left-hand side of both equations is h_t, so we can equate the expressions on the right-hand side.

$$2x_t - 4 = \frac{1}{3}x_t + 3$$

Let's move the constant terms to the right.

$$2x_t - 4 + 4 = \frac{1}{3}x_t + 3 + 4$$

$$2x_t = \frac{1}{3}x_t + 7$$

Let's move the x_t terms to the left.

$$\frac{6}{3}x_t - \frac{1}{3}x_t = \frac{1}{3}x_t - \frac{1}{3}x_t + 7$$

$$\frac{5}{3}x_t = 7$$

Multiply both sides by $3/5$.

$$\frac{3}{5} \cdot \frac{5}{3}x_t = \frac{3}{5} \cdot 7$$

$$x_t = \frac{21}{5}$$

Now use either equation to solve for h_t. We'll use $h_t = 2x_t - 4$.

$$h_t = 2x_t - 4$$

$$h_t = 2\left(\frac{21}{5}\right) - 4$$

$$h_t = \frac{42}{5} - 4$$

$$h_t = \frac{42}{5} - \frac{20}{5}$$

$$h_t = \frac{22}{5}$$

So we get

$$(x_t, h_t) = \left(\frac{21}{5}, \frac{22}{5}\right)$$

As we can see, if we have subscripts in a system of equations, we can simply use the approach we normally would to solve it.

Uniform motion problems

In this lesson we'll look at how to compare and solve for values of the variables in the distance equation,

Distance = Rate · Time

$D = RT$

when we have a case of uniform motion and a pair of related scenarios, including

- a pair of scenarios with the same distance but different speeds and times,
- a pair of scenarios with the same speed but different distances and times, or
- a pair of scenarios with the same time but different distances and speeds.

Let's talk about the units of each of these values.

Distance has units of inches, feet, miles, etc., or of centimeters, meters, kilometers, etc.

Time has units of seconds, minutes, hours, etc.

Rate has units of distance/time, for example inches/second, miles/hour, or kilometers/hour.

Before we can use the formula $D = RT$, we need to make sure that the units for distance and time are the same as the units in of the rate. If they aren't, we'll need to change them so we're working with the same units.

Let's do an example of a standard distance, rate, and time problem.

Example

Heather ran 56 km in 5 hours. What was Heather's rate in km/hr?

We'll use the formula for distance.

$$\text{Distance} = \text{Rate} \cdot \text{Time}$$

$$D = RT$$

Let's write down what we know.

$$D = 56 \text{ km}$$

$$T = 5 \text{ hr}$$

If we plug these into the distance formula, we get

$$56 \text{ km} = R \cdot 5 \text{ hr}$$

Now solve for the rate.

$$\frac{56 \text{ km}}{5 \text{ hr}} = \frac{R \cdot 5 \text{ hr}}{5 \text{ hr}}$$

$$R = 11.2\ \frac{\text{km}}{\text{hr}}$$

Let's try one with two people.

Example

Susan and Benjamin were 60 miles apart on a straight trail. Susan started walking toward Benjamin at a rate of 5 mph at 7:30 a.m. Benjamin left three hours later, and they met on the trail at 3:30 p.m. How fast did Benjamin walk?

We've been given information about distance, rate, and time, so we'll use the formula

Distance = Rate · Time

$$D = RT$$

We can use subscripts to create unique equations for Susan and Benjamin; we'll use S for Susan, and B for Benjamin.

Susan $\quad D_S = R_S T_S$

Benjamin $\quad D_B = R_B T_B$

We know that in order to meet each other, they must have covered a distance of 60 miles between them. Therefore,

$$D_S + D_B = 60$$

Since we know that $D_S = R_S T_S$ and $D_B = R_B T_B$, we can rewrite this equation as

$$R_S T_S + R_B T_B = 60$$

and then substitute the known quantities (Susan's speed and time, and Benjamin's time) into the equation. The problem tells us that Susan walked at a speed of 5 mph, and that she walked for 8 hours, since she walked from 7:30 a.m. until 3:30 p.m. So

$$(5)(8) + R_B T_B = 60$$

$$40 + R_B T_B = 60$$

$$R_B T_B = 20$$

Benjamin left three hours after Susan, which means he started walking at 10:30 a.m., and he kept walking until they met at 3:30 p.m., which means he walked for 5 hours. So

$$R_B(5) = 20$$

$$R_B = 4$$

Which means that Benjamin walks at a speed of 4 mph.

Let's look at another uniform motion problem.

Example

One train leaves Station A at a constant speed and arrives at Station B in 8 hours. A second train leaves Station A at a constant rate of 40 mph and arrives at Station B in 10 hours. What was the speed of the first train?

Since each train is traveling at a uniform speed, we recognize this as a uniform motion problem. We can use subscripts to create a unique equation for each train. We'll call them Train 1 and Train 2.

Train 1: $D_1 = R_1T_1$

Train 2: $D_2 = R_2T_2$

Let's organize the information we know about each train.

Train 1:

$D_1 = ?$

$R_1 = ?$

$T_1 = 8$ hours

Train 2:

$D_2 = ?$

$R_2 = 40$ mph

$T_2 = 10$ hours

Now let's plug this information into the equations for Train 1 and Train 2.

$$D_1 = R_1T_1$$

$$D_1 = R_1(8 \text{ hrs})$$

and

$$D_2 = R_2T_2$$

$$D_2 = (40 \text{ mph})(10 \text{ hrs})$$

$$D_2 = 400 \text{ miles}$$

The two trains traveled the same distance ($D_1 = D_2$), so we can equate the value we just found for D_2 to the expression we found for D_1 (and then solve for R_1).

$$D_2 = D_1$$

$$400 \text{ miles} = r_1(8 \text{ hrs})$$

$$50 \text{ mph} = r_1$$

The first train traveled at a constant speed of 50 mph from Station A to Station B.

Let's do one more example.

Example

Cassie is driving at a constant rate of 30 mph on the highway. Four hours later, her friend Susan starts from the same point and drives at a constant rate of 60 mph and passes Cassie. How many hours had each woman been traveling at the time that Susan passed Cassie? And how far had each woman traveled at that time?

Since each woman is traveling at a uniform rate, we recognize this as a uniform motion problem, so we can use the equation $D = RT$, where D is the distance each of them traveled, R is the rate at which they traveled, and T is the time it took them to get to the place where Susan passed Cassie. We can use subscripts to set up a unique equation for each woman's travel.

Cassie: $D_c = R_c T_c$

Susan: $D_s = R_s T_s$

The problem tells us that Cassie traveled at a rate of 30 mph, that Susan traveled at a rate of 60 mph, and that it took Susan 4 hours less than Cassie to travel the same distance (because she left 4 hours later).

Let's set up what we know.

Cassie:

$$D_c = ?$$

$$R_c = 30 \text{ mph}$$

$$T_c = ?$$

$$D_c = (30 \text{ mph})T_c$$

Susan:

$$D_s = ?$$

$$R_s = 60 \text{ mph}$$

$$T_s = T_c - 4$$

$$D_s = (60 \text{ mph})(T_c - 4)$$

Cassie and Susan traveled the same distance ($D_c = D_s$), so can equate the expression we found for D_c to the expression we found for D_s (and then solve for T_c).

$$(30 \text{ mph})T_c = (60 \text{ mph})(T_c - 4)$$

$$\frac{(30 \text{ mph})T_c}{60 \text{ mph}} = \frac{(60 \text{ mph})(T_c - 4)}{60 \text{ mph}}$$

$$\frac{1}{2}T_c = T_c - 4$$

$$\frac{1}{2}T_c - T_c = T_c - T_c - 4$$

$$-\frac{1}{2}T_c = -4$$

$$-2\left(-\frac{1}{2}T_c\right) = -2(-4)$$

$$T_c = 8 \text{ hours}$$

We now know that it took Cassie 8 hours to get to the point at which Susan passed her, so we can substitute 8 for T_c in the equation we found for T_s (and then compute the value of T_s).

$$T_s = T_c - 4$$

$$T_s = 8 - 4$$

$$T_s = 4 \text{ hours}$$

Now that we have a rate and a time for both Cassie and Susan, we can find the distance that each of them traveled (and verify that it was the same for both of them).

Cassie:

$$D_c = R_c T_c$$

$$D_c = 30(8)$$

$$D_c = 240 \text{ miles}$$

Susan:

$$D_s = R_s T_s$$

$$D_s = 60(4)$$

$$D_s = 240 \text{ miles}$$

Number word problems

The purpose of number word problems is to give us practice in taking the words (about numbers) in problem statements, translating them to mathematical notation (variables, expressions, equations), using mathematics to solve the problems, and answering the specific questions that were asked in the problem statements.

For some people these are fun number games, but they do appear on tests and in math classes from time to time, so it's good to be comfortable with them.

In a number word problem, we're given information about a pair or group of numbers, and we usually need to translate the information into equations to solve for the numbers.

Some helpful vocabulary:

> Consecutive integers are integers that are in ascending order and have no integer between them, such as 4 and 5.
>
> Consecutive even numbers are even numbers that are in ascending order and have no even number between them, such as 4 and 6.
>
> Consecutive odd numbers are odd numbers that are ascending order and have no odd number between them, such as 5 and 7.

And of course "sum" means the result of addition, "difference" means the result of subtraction, and "product" means the result of a multiplication. Place value can also be used in these problems.

Let's do an example.

Example

The sum of two consecutive integers is 25. Find the numbers.

In solving any word problem, we should start by defining the variable(s). In this case, let's let X be the smaller integer. Then the larger integer is $X+1$.

Let's write an equation for what we know about their sum.

$$X+(X+1)=25$$

Now let's solve for X.

$$X+X+1=25$$

$$2X+1=25$$

$$2X+1-1=25-1$$

$$2X=24$$

$$\frac{2X}{2}=\frac{24}{2}$$

$$X=12$$

The smaller integer is 12, so the larger one is $12+1=13$. Therefore, the integers are 12 and 13. We can double check our answer and see that $12+13=25$.

Let's look at another style of problem.

Example

The sum of the digits of a certain two-digit number is 17. Reversing the digits gives a number which is 9 less than the original number. What is the original number?

Let T be the tens digit of the original number, and let U be its unit digit.

The value of the original number is

$$10T + U$$

Reversing the digits gives us a number whose value is

$$10U + T$$

The second number is 9 less than the original number, so we can write

Original number $- 9 =$ Second number

In mathematical notation, that translates to the following equation:

$$(10T + U) - 9 = (10U + T)$$

$$10T + U - 9 = 10U + T$$

Move the $10U$ and the T to the left-hand side.

$$10T - T + U - 10U - 9 = 0$$

$$9T - 9U - 9 = 0$$

Dividing through by 9 gives

$$T - U - 1 = 0$$

$$T = U + 1$$

We know that the sum of the digits is 17, so we'll substitute the expression we just found for T into the equation $T + U = 17$, and then solve for U.

$$T + U = 17$$

$$(U + 1) + U = 17$$

$$2U + 1 = 17$$

$$2U = 16$$

$$U = 8$$

Substitute 8 for U in the equation $T + U = 17$, and then solve for T.

$$T + U = 17$$

$$T + 8 = 17$$

$$T = 9$$

The original number is 98. Also, the second number is 89, which is indeed 9 less than the original number: $89 = 98 - 9$.

Age word problems

Age word problems are like number word problems. We'll still need to translate problem statements in English to mathematical notation (variables, expressions, equations), use mathematics to solve the problems, and answer the specific questions about people's ages that were asked in the problem statements. In this lesson we'll look at how to do that.

One helpful way to organize these types of problems is by making a table.

Let's do a few examples.

Example

In 18 years, Sasha will be four times as old as she is now. How old is she now?

In 18 years, Sasha's age will be four times her current age, so we can relate her age now, which we'll denote by s, to her age in 18 years, which is $4s$.

So we can write the equation $s+18=4s$.

We can also organize the information by making a table.

	Sasha	Equation
Now	s	
Future	s+18, her age in 18 years 4s, four times her current age	s+18=4s

Now solve for s in the equation.

$$s+18=4s$$

$$s-s+18=4s-s$$

$$18=3s$$

$$6=s$$

Sasha is 6 now, and in 18 years she'll be 24, which is four times her current age: $24=4\cdot 6$.

Let's do an example with more than one person.

Example

April is 12 years older than Eric. In 5 years, April will be twice as old as Eric. How old are Eric and April now?

Because April is 12 years older than Eric, we can write

$$A=E+12$$

In 5 years, April's age will be $A+5$ and Eric's age will be $E+5$. At that time, April will be twice as old as Eric, so, we need to double Eric's age in 5 years to get April's age in 5 years. The next equation is

$$2(E+5) = A+5$$

This is a table that summarizes their ages and our equations:

	Eric	April	More info	Equation
Age now	E	A	April is 12 years older than Eric	A=E+12
Age in 5 years	E+5	A+5	April is twice as old as Eric	2(E+5)=A +5

Now we can substitute $E+12$ for A in the equation $2(E+5)=A+5$ and then solve for E.

$$2(E+5) = A+5$$

$$2(E+5) = (E+12)+5$$

$$2E+10 = E+17$$

$$E = 7$$

Substitute the value we found for E into the equation $A=E+12$, and then compute the value of A.

$$A = E+12$$

$$A = 7+12$$

$$A = 19$$

Right now, Eric is 7 and April is 19. Note that in 5 years, Eric will be 12 and April will be 24, so at that time April will indeed by twice as old as Eric: $24 = 2 \cdot 12$.

Systems with non-linear equations

In this lesson we'll look at the algebraic way to solve a pair of equations, where one is a linear equation and the other is a non-linear equation in which at least one of the variables is squared.

Remember that an equation of a circle or an ellipse has both an x^2 term and a y^2 term. It might look like $x^2 + 4y^2 = 100$. On the other hand, an equation of a line has an x term and a y term. An example of a linear equation would be $y = -(3/2)x - 5$.

If we take the two equations and put them together,

$$\begin{cases} x^2 + 4y^2 = 100 \\ y = -\frac{3}{2}x - 5 \end{cases}$$

then we have a system of equations.

The solutions to a system of equations are the points (x, y) where the graphs of the equations in the system intersect.

Let's look at how to solve the system that was given above.

Example

Solve the system for x and y.

$$\begin{cases} x^2 + 4y^2 = 100 \\ y = -\frac{3}{2}x - 5 \end{cases}$$

In this case the second equation is already solved for y, so we can begin by substituting that expression for y into the first equation.

$$x^2 + 4y^2 = 100$$

$$x^2 + 4\left(-\frac{3}{2}x - 5\right)^2 = 100$$

Expand the square.

$$x^2 + 4\left(-\frac{3}{2}x - 5\right)\left(-\frac{3}{2}x - 5\right) = 100$$

$$x^2 + 4\left(\frac{9}{4}x^2 + 15x + 25\right) = 100$$

Distribute the 4 over everything inside the parentheses.

$$x^2 + 9x^2 + 60x + 100 = 100$$

$$x^2 + 9x^2 + 60x + 100 - 100 = 100 - 100$$

$$10x^2 + 60x = 0$$

Factor out a $10x$ to help solve for x.

$$10x(x + 6) = 0$$

To solve this equation, we set the factors, $10x$ and $x + 6$, to 0 separately, and then solve the resulting equations.

$10x = 0$ gives $x = 0$

$$x + 6 = 0 \text{ gives } x = -6$$

Plug these x-values into the equation $y = -(3/2)x - 5$, to find the y-values that go with them.

For $x = 0$:

$$y = -\frac{3}{2}x - 5$$

$$y = -\frac{3}{2}(0) - 5$$

$$y = 0 - 5$$

$$y = -5$$

So we have the solution $(0, -5)$.

For $x = -6$:

$$y = -\frac{3}{2}x - 5$$

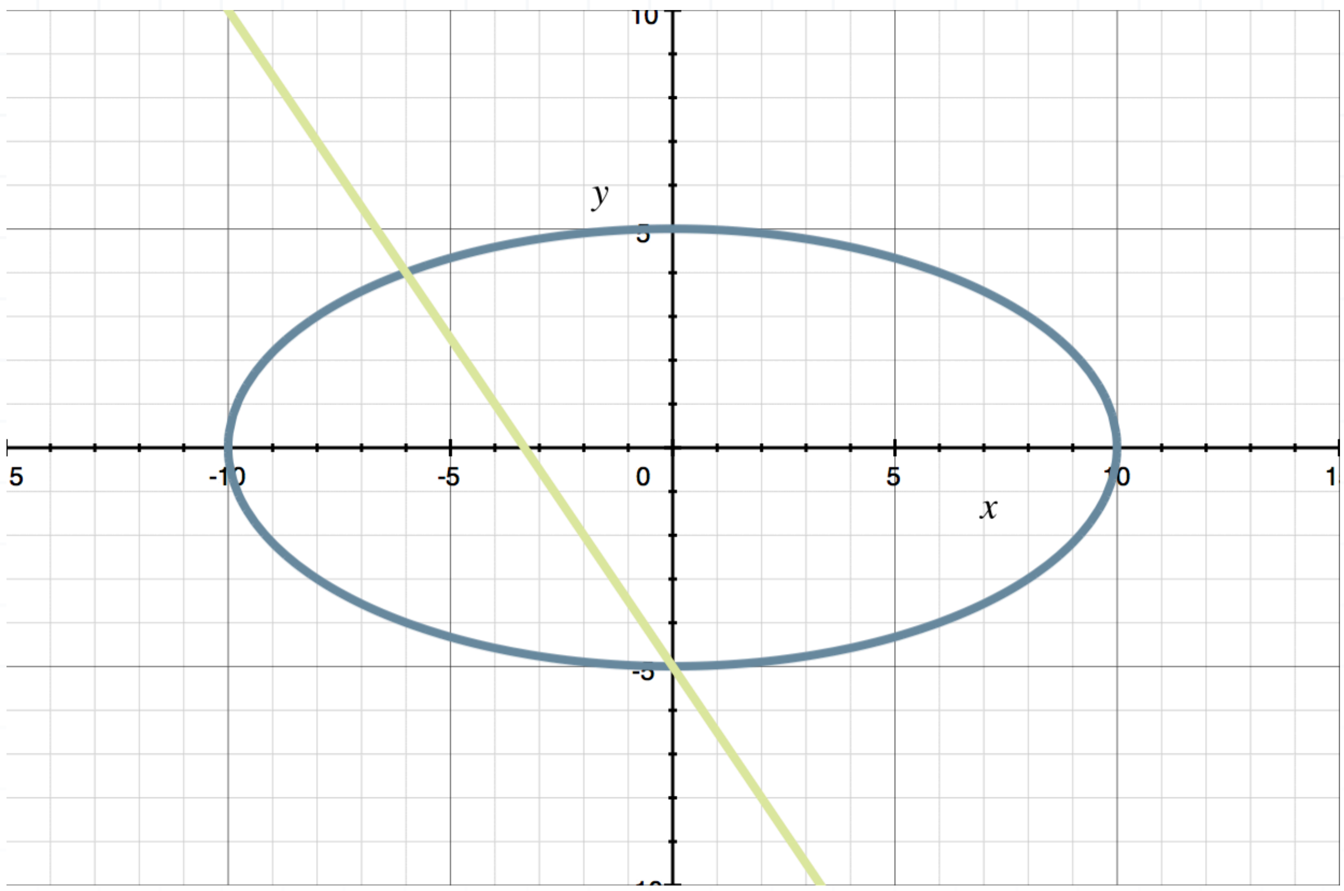
$$y = -\frac{3}{2}(-6) - 5$$

$$y = 9 - 5$$

$$y = 4$$

So we have the solution $(-6,4)$.

The non-linear equation in this system is the equation of an ellipse, and (as always) the linear equation is the equation of a line. We can look at this picture of the system to see that the solutions are the points (x, y) where the ellipse and line intersect.



Let's do an example that involves a few more steps.

Example

Solve the system for x and y.

$$3x^2 + 2y^2 - 54y = 143$$

$$x - 3y = 3$$

Let's solve this system by solving the second equation for x, and then substituting the resulting expression for x into the first equation.

$$x - 3y = 3$$

$$x = 3y + 3$$

Plug this expression for x into the first equation, and then solve for y.

$$3x^2 + 2y^2 - 54y = 143$$

$$3(3y + 3)^2 + 2y^2 - 54y = 143$$

Expand the square.

$$3(9y^2 + 18y + 9) + 2y^2 - 54y = 143$$

$$27y^2 + 54y + 27 + 2y^2 - 54y = 143$$

$$29y^2 = 116$$

$$y^2 = 4$$

$$y = \pm 2$$

Plug these values of y into the expression we found for x to get the corresponding x-values.

For $y = -2$:

$$x = 3y + 3$$

$$x = 3(-2) + 3$$

$$x = -3$$

So one solution is $(-3, -2)$.

For $y = 2$:

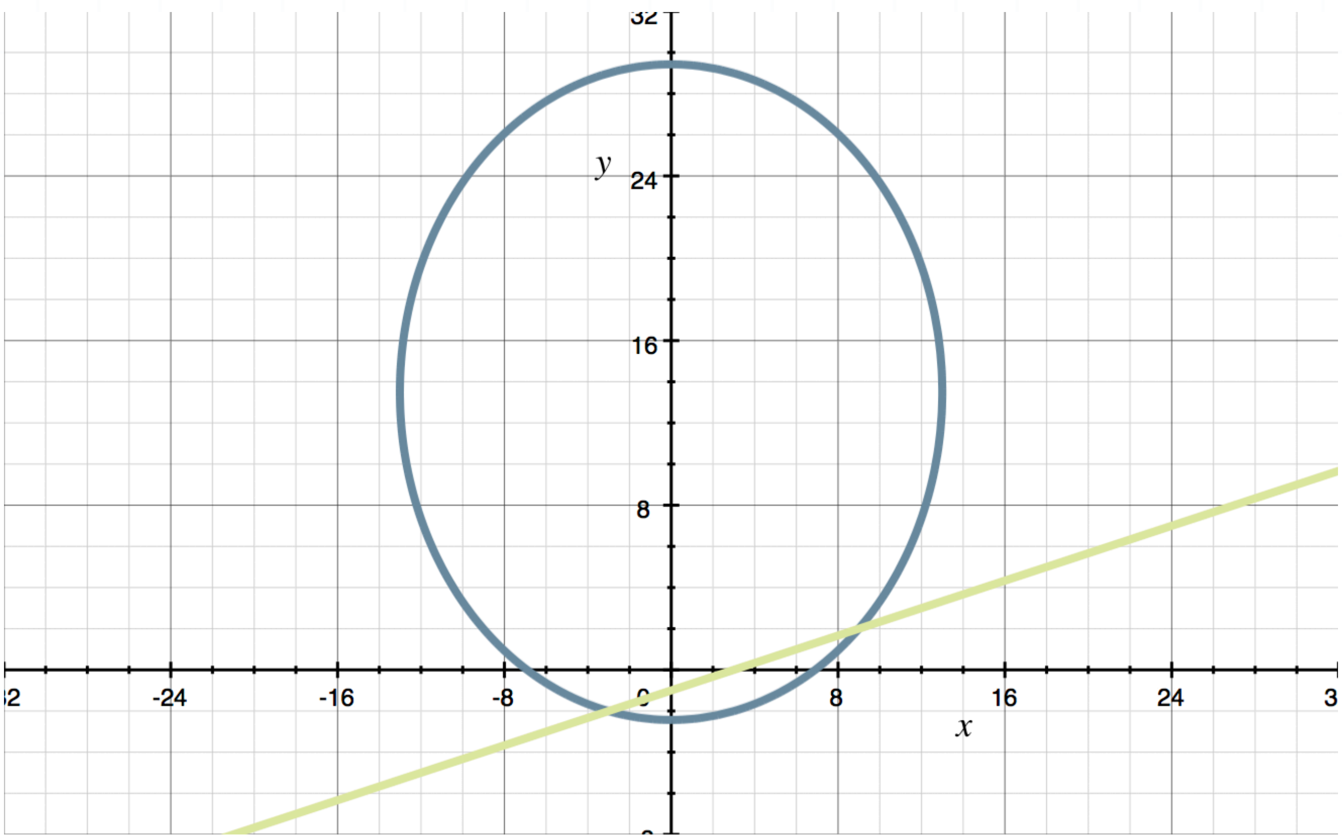
$$x = 3y + 3$$

$$x = 3(2) + 3$$

$$x = 9$$

So the other solution is $(9,2)$.

Sometimes it's nice to have a visual of what we did algebraically. Here are the graphs of the non-linear equation in this system (which is the equation of an ellipse) and the linear equation (which is the equation of a line). Notice that they intersect at the solution points $(-3,-2)$ and $(9,2)$.



Systems of three equations

In this lesson we'll look at how to solve systems of three linear equations in three variables.

If a system of three linear equations has solutions, each solution will consist of one value for each variable.

If the three equations in such a linear system are "independent of one another," the system will have either one solution or no solutions. All the systems of three linear equations that we'll encounter in this lesson have at most one solution.

Let's look at an example.

Example

Solve the system of equations.

$$\begin{cases} -x - 5y + z = 17 \\ -5x - 5y + 5z = 5 \\ 2x + 5y - 3z = -10 \end{cases}$$

So that we can stay organized, let's number the equations.

[1] $-x - 5y + z = 17$

[2] $-5x - 5y + 5z = 5$

[3] $2x + 5y - 3z = -10$

Notice that the coefficients of y in equations [1] and [3] are -5 and 5, respectively. If we add these two equations, the y terms will cancel (we'll eliminate the variable y) and we'll get an equation in only the variables x and z.

$$(-x - 5y + z) + (2x + 5y - 3z) = 17 + (-10)$$

Remove parentheses and combine like terms.

$$-x - 5y + z + 2x + 5y - 3z = 17 - 10$$

$$-x + 2x - 5y + 5y + z - 3z = 17 - 10$$

$$x - 2z = 7$$

We might have also noticed that the coefficients of y in equations [2] and [3] are -5 and 5, respectively, so we can add these two equations to get another equation in only the variables x and z.

$$(-5x - 5y + 5z) + (2x + 5y - 3z) = (5) + (-10)$$

Remove parentheses and combine like terms.

$$-5x - 5y + 5z + 2x + 5y - 3z = 5 - 10$$

$$-5x + 2x - 5y + 5y + 5z - 3z = 5 - 10$$

$$-3x + 2z = -5$$

The coefficients of z in our two new equations are -2 and 2, respectively. If we add these two equations, we can eliminate the variable z, and then solve for x.

$$x - 2z = 7$$

$$-3x + 2z = -5$$

$$(x - 2z) + (-3x + 2z) = 7 + (-5)$$

Remove parentheses and combine like terms.

$$x - 2z - 3x + 2z = 7 - 5$$

$$x - 3x - 2z + 2z = 7 - 5$$

$$-2x = 2$$

$$x = -1$$

Choose one of the new equations, and plug in -1 for x, and then solve for z. We'll choose $x - 2z = 7$.

$$-1 - 2z = 7$$

$$-2z = 8$$

$$z = -4$$

Now choose one of the three original equations, and plug in -1 for x and -4 for z, and then solve for y. We'll choose equation **[1]**.

[1] $-x - 5y + z = 17$

$$-(-1) - 5y + (-4) = 17$$

Simplify and solve for y.

$$1 - 5y - 4 = 17$$

$$-5y + 1 - 4 = 17$$

$$-5y - 3 = 17$$

$$-5y = 20$$

$$y = -4$$

The solution is $(-1, -4, -4)$ or $x = -1$, $y = -4$, and $z = -4$.

Let's do one more.

Example

Use any method to solve the system of equations.

$$\begin{cases} 3a - 3b + 4c = -23 \\ a + 2b - 3c = 25 \\ 4a - b + c = 25 \end{cases}$$

So that we can stay organized, let's number the equations.

[1] $3a - 3b + 4c = -23$

[2] $a+2b-3c=25$

[3] $4a-b+c=25$

None of the terms with the same variable have the same coefficient (or coefficients that are equal in absolute value but opposite in sign). So we'll need to multiply one of the equations by some number such that by combining the resulting equation with one of the other two equations, we'll be able to eliminate a variable. Let's multiply equation **[2]** by 3, so we can eliminate the variable a by subtracting the resulting equation from equation **[1]**.

$$3(a+2b-3c)=3(25)$$

[4] $3a+6b-9c=75$

Now let's subtract equation **[4]** from equation **[1]**, which will give us an equation in only the variables b and c.

[1] $3a-3b+4c=-23$

[4] $3a+6b-9c=75$

$$(3a-3b+4c)-(3a+6b-9c)=(-23)-(75)$$

Eliminate the parentheses, and then combine like terms.

$$3a-3b+4c-3a-6b+9c=-23-75$$

$$3a-3a-3b-6b+4c+9c=-23-75$$

[5] $-9b+13c=-98$

We need to get another equation in only the variables b and c. Let's use equations **[2]** and **[3]**.

This time we need to multiply equation **[2]** by 4, so we can subtract it from equation **[3]** and eliminate the variable a.

[2] $a + 2b - 3c = 25$

$4(a + 2b - 3c) = 4(25)$

[6] $4a + 8b - 12c = 100$

Now we'll subtract equation **[6]** from equation **[3]**.

[3] $4a - b + c = 25$

$(4a - b + c) - (4a + 8b - 12c) = (25) - (100)$

Eliminate the parentheses, and then combine like terms.

$4a - b + c - 4a - 8b + 12c = 25 - 100$

$4a - 4a - b - 8b + c + 12c = 25 - 100$

[7] $-9b + 13c = -75$

With **[5]** and **[7]**, we now have a system of two equations in the variables b and c.

[5] $-9b + 13c = -98$

[7] $-9b + 13c = -75$

If we subtract **[7]** from **[5]**, we get

$$(-9b + 13c) - (-9b + 13c) = -98 - (-75)$$

Eliminate the parentheses, and combine like terms.

$$-9b + 13c + 9b - 13c = -98 + 75$$

$$0 = -23$$

Isn't that impossible?

Actually, it isn't impossible, but if something like that happens, it means that the original system of three equations has no solution.

Parallel and perpendicular lines

In this lesson we'll learn about the characteristics of parallel and perpendicular lines and how to identify them on a graph or in an equation.

The equation of a line in slope-intercept form is $y = mx + b$, where m is the slope of the line and b is the y-coordinate of the y-intercept. Remember, the y-intercept is the point at which the line crosses the y-axis. The x-coordinate of every point on the y-axis is 0, so the x-coordinate of the y-intercept is always 0. Therefore, we sometimes call b the y-intercept, even though (technically speaking) b is just the y-coordinate of the y-intercept.

Parallel lines

For two lines to be parallel, their slopes must be equal but their y-intercepts must be different (otherwise, they are the same line).

Algebraically, in slope-intercept form the equations of a pair of parallel lines would look like

$$\begin{cases} y = mx + b_1 \\ y = mx + b_2 \end{cases}$$

where $b_1 \neq b_2$.

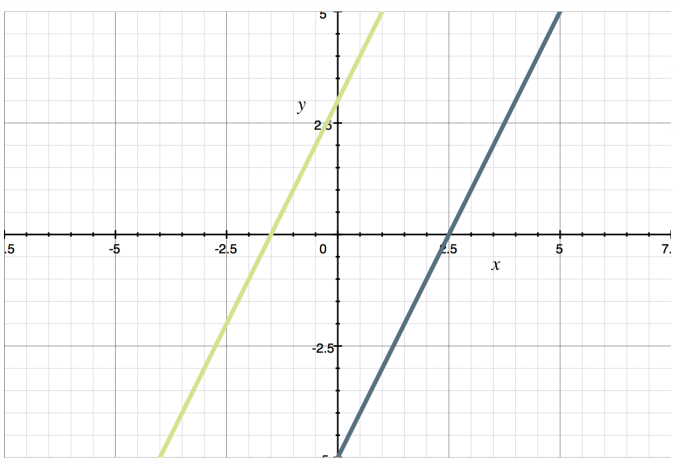
Here are two examples of equations of a pair of parallel lines in slope-intercept form:

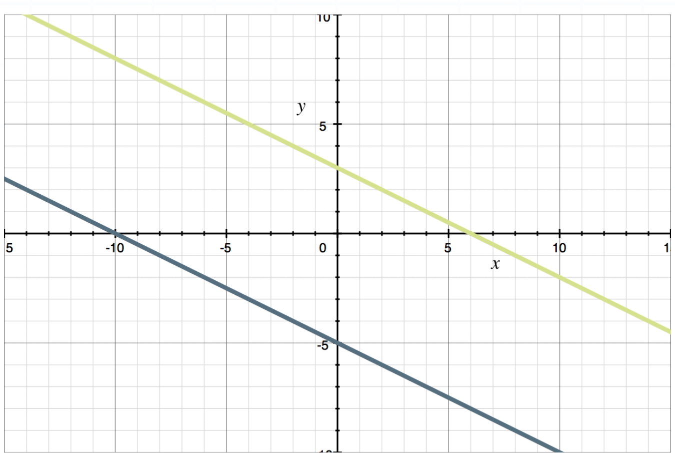
$$\begin{cases} y = 2x - 5 \\ y = 2x + 3 \end{cases}$$

$$\begin{cases} y = -\frac{1}{2}x - 5 \\ y = -\frac{1}{2}x + 3 \end{cases}$$

Parallel lines go on forever in the same direction and never cross each other. Here are the graphs of those two examples of equations of a pair of parallel lines.

$$\begin{cases} y = 2x - 5 & \text{blue} \\ y = 2x + 3 & \text{green} \end{cases} \qquad \begin{cases} y = -\frac{1}{2}x - 5 & \text{blue} \\ y = -\frac{1}{2}x + 3 & \text{green} \end{cases}$$





Perpendicular lines

Perpendicular lines have slopes that are negative reciprocals of each other. Remember, two numbers c, d are reciprocals of each other if $d = 1/c$. Therefore, if the slope of one of the lines in a pair of perpendicular lines is m, the slope of the other line is $-1/m$. Since the slopes are different, the y-intercepts could be the same.

Algebraically, in slope-intercept form the equations of a pair of perpendicular lines would look like

$$\begin{cases} y = mx + b_1 \\ y = -\frac{1}{m}x + b_2 \end{cases}$$

where b_1 and b_2 could be the same.

Here are two examples of equations of a pair of perpendicular lines in slope-intercept form:

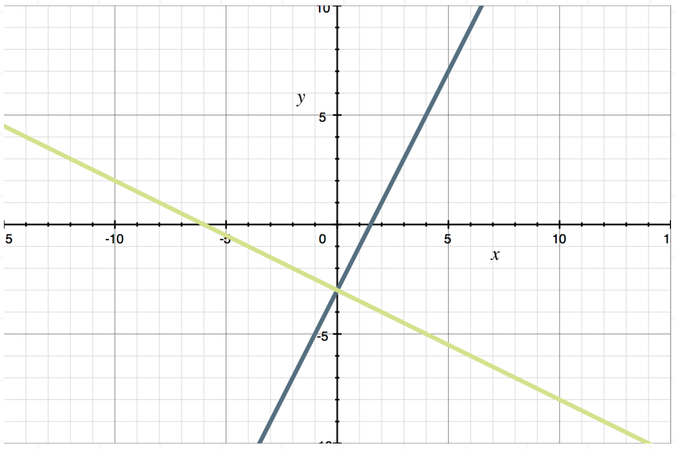
$$\begin{cases} y = 2x - 3 \\ y = -\frac{1}{2}x - 3 \end{cases}$$

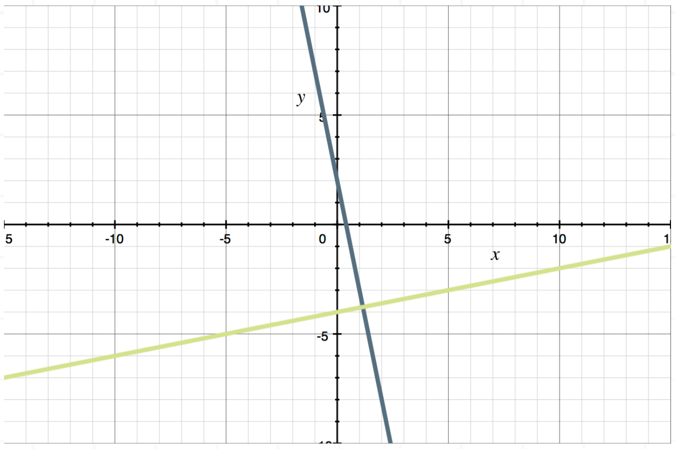
$$\begin{cases} y = -5x + 2 \\ y = \frac{1}{5}x - 4 \end{cases}$$

Perpendicular lines cross each other at a single point and form four right angles (four 90-degree angles) at their point of intersection. Be aware that checking this on a calculator might not always help, because the graphing window settings can disguise right angles.

Here are the graphs of those two examples of equations of a pair of perpendicular lines.

$$\begin{cases} y = 2x - 3 & \text{blue} \\ y = -\frac{1}{2}x - 3 & \text{green} \end{cases} \qquad \begin{cases} y = -5x + 2 & \text{blue} \\ y = \frac{1}{5}x - 4 & \text{green} \end{cases}$$





Let's go ahead and look at a few of the types of problems involving parallel or perpendicular lines that we'll need to know how to solve.

Example

Write the equation of the line that's parallel to $5x + 2y = 10$ and has a y-intercept of 4.

For two lines to be parallel, their slopes must be equal.

Remember that the equation of a line in slope-intercept form is given by

$$y = mx + b$$

where m is the slope and b is the y-intercept. To get the slope of the original line (which will also be the slope of the new line), we'll first convert its equation to slope-intercept form.

$$5x + 2y = 10$$

$$5x - 5x + 2y = -5x + 10$$

$$2y = -5x + 10$$

$$\frac{1}{2} \cdot 2y = \frac{1}{2} \cdot -5x + \frac{1}{2} \cdot 10$$

$$y = -\frac{5}{2}x + 5$$

Therefore, the slope is $-5/2$.

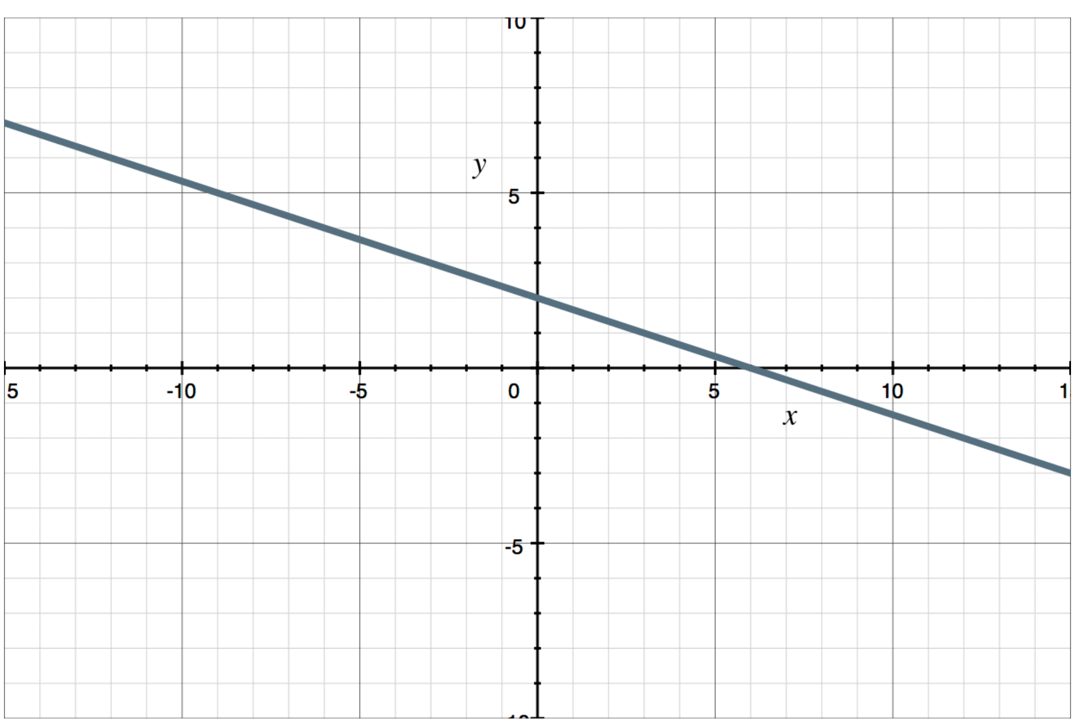
We want to write the equation of the line that has a slope of $-5/2$ and a y-intercept of 4. So $m = -5/2$ and $b = 4$. Therefore, the equation is

$$y = -\frac{5}{2}x + 4$$

Let's look at another example.

Example

Graph the line which is parallel to the line shown in the graph and has a y-intercept of -2.

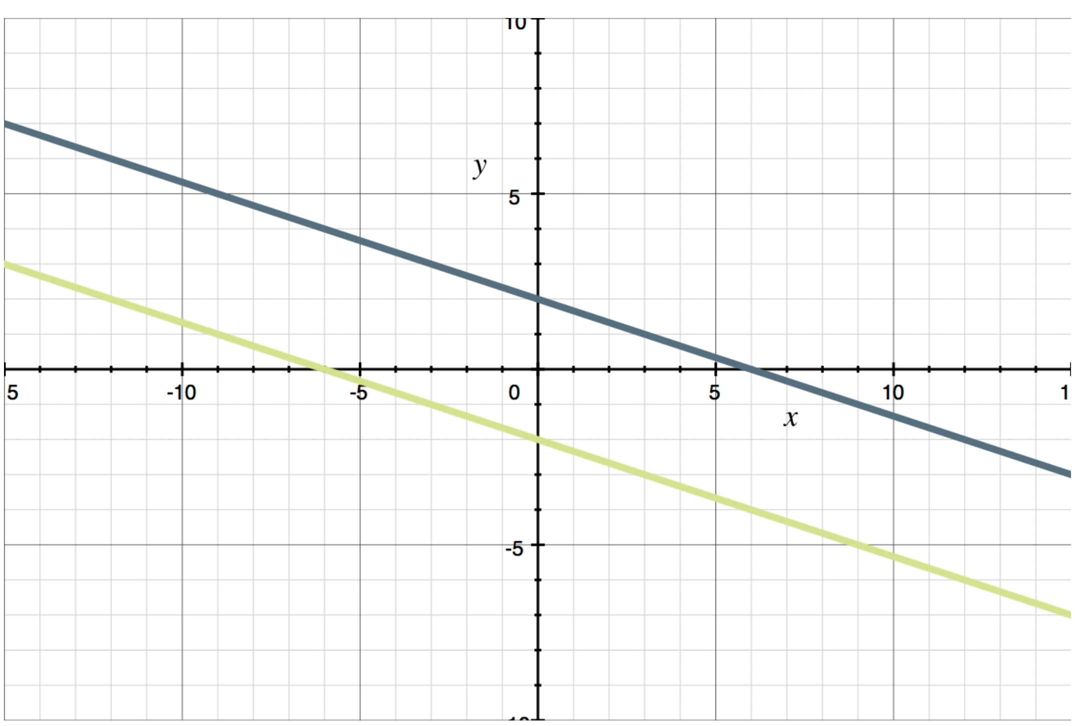


Parallel lines have the same slope. We want to draw the line that has the same slope as this one but goes through the point $(0, -2)$.

Start by graphing the point $(0, -2)$. To get the slope of the original line, we'll start from its y-intercept (the point $(0,2)$) and use the fact that the slope of a line is given by the ratio of the "rise" to the "run." Note that the point $(3,1)$ is on the original line, and that we get there from the point $(0,2)$ by going 1 unit down (from $y = 2$ to $y = 1$, which is a rise of -1) and 3 units to the right (from $x = 0$ to $x = 3$, which is a run of 3 units). Therefore, the slope of the given line (and also of the new line) is $-1/3$.

Then we'll start from the y-intercept of the new line (the point $(0, -2)$) and find the point that we get to by using that same rise and run (1 unit down and 3 units to the right). The coordinates of that point are $(0 + 3, -2 - 1) = (3, -3)$.

Connect the points $(0, -2)$ and $(3, -3)$, and draw the new line.



The equation of the new line is

$$y = -\frac{1}{3}x - 2$$

Let's look at an example of perpendicular lines.

Example

Write the equation of the line that passes through the point $(-2,5)$ and is perpendicular to the line

$$y = -\frac{4}{7}x - 2$$

Remember, perpendicular lines have slopes that are negative reciprocals of each other. In other words, the slope of the new line needs to be the negative reciprocal of $-4/7$, which means the slope of the new line is

$$\frac{7}{4}$$

The equation of a line in slope-intercept form is $y = mx + b$. For our new line, we know the slope, $7/4$, and one point on it, $(-2,5)$.

We can plug the slope and the coordinates of that point into the equation $y = mx + b$ and solve for b.

$$y = mx + b$$

$$5 = \frac{7}{4}(-2) + b$$

$$5 = -\frac{7}{2} + b$$

$$5 + \frac{7}{2} = -\frac{7}{2} + \frac{7}{2} + b$$

$$\frac{10}{2} + \frac{7}{2} = -\frac{7}{2} + \frac{7}{2} + b$$

$$\frac{17}{2} = b$$

The equation of the new line is

$$y = \frac{7}{4}x + \frac{17}{2}$$

Graphing parabolas

In this lesson we'll learn how to identify the characteristics of a parabola and go back and forth between the equation of a parabola and its graph.

A parabola is the solution to an equation $y = f(x)$ where $f(x)$ is a quadratic polynomial. Therefore, a parabola is the solution to a non-linear equation. There are two forms of the equation of a parabola that are especially helpful when we want to know something about a parabola.

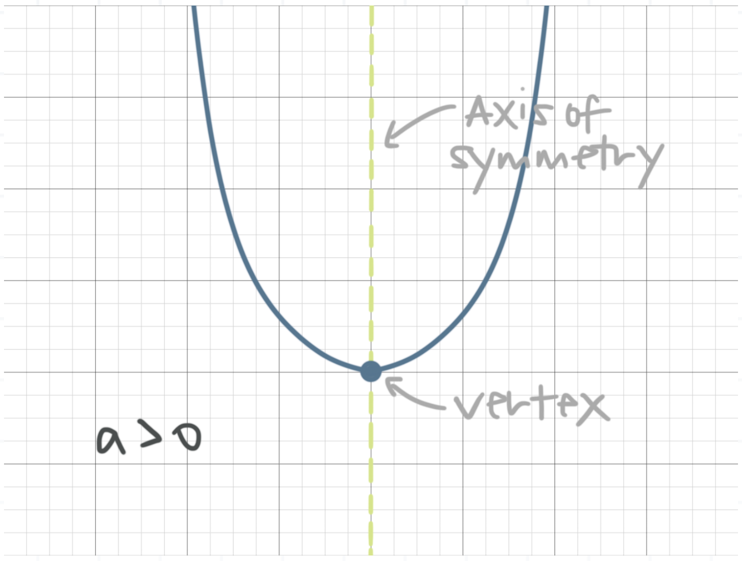
	Standard form	**Vertex form**
Equation	$y = ax^2 + bx + c$	$y = a(x - h)^2 + k$
Axis of symmetry	$x = -b/2a$	$x = h$
Vertex	$(-b/2a, f(-b/2a))$	(h, k)

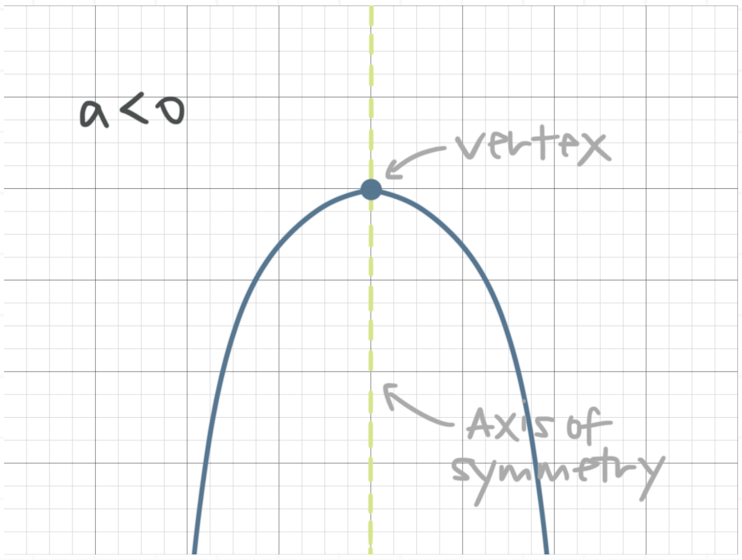
To convert from standard form to vertex form we complete the square, but to convert from vertex form to standard form we expand the square and then distribute and simplify.

Let's talk about the different parts of a parabola.

In both forms (standard and vertex), if $a > 0$ the parabola opens upwards and the vertex is the point at the bottom of the parabola (the point with the minimum value of y).

In both forms (standard and vertex), if $a < 0$ the parabola opens downwards and the vertex is the point at the top of the parabola (the point with the maximum value of y).





Let’s do a few problems.

Example

Write the equation in vertex form.

$$y = 2x^2 + 36x + 170$$

To convert the standard equation of the parabola to vertex form from standard form, we’ll need to complete the square.

Before we complete the square, we’ll factor the coefficient of the x^2 term, which is 2, out of the first two terms on the right-hand side of the given equation.

$$y = 2x^2 + 36x + 170$$

$$y = 2(x^2 + 18x) + 170$$

To complete the square, we need to find the number d that satisfies the equation

$$x^2 + 18x + d^2 = (x + d)^2$$

That is, we need to find the number d for which

$$x^2 + 18x + d^2 = x^2 + 2dx + d^2$$

This means that the coefficient of the x term of the expression inside the parentheses must be equal to $2d$. That coefficient is 18, so we'll set $2d$ to 18 and solve for d.

$$2d = 18 \quad \to \quad d = 9$$

To keep our equation balanced, we need to add and subtract d^2 (81) inside the parentheses, and then distribute, regroup, and simplify.

$$y = 2(x^2 + 18x) + 170$$

$$y = 2(x^2 + 18x + 81 - 81) + 170$$

$$y = 2(x^2 + 18x + 81) + 2(-81) + 170$$

$$y = 2(x^2 + 18x + 81) - 162 + 170$$

$$y = 2(x^2 + 18x + 81) + 8$$

Finally, we'll factor the expression that's now inside the parentheses $(x^2 + 18x + 81)$. By construction ("completing the square"), that expression factors as $(x + d)^2$.

$$x^2 + 18x + 81 = (x + d)^2$$

$$x^2 + 18x + 81 = (x + 9)^2$$

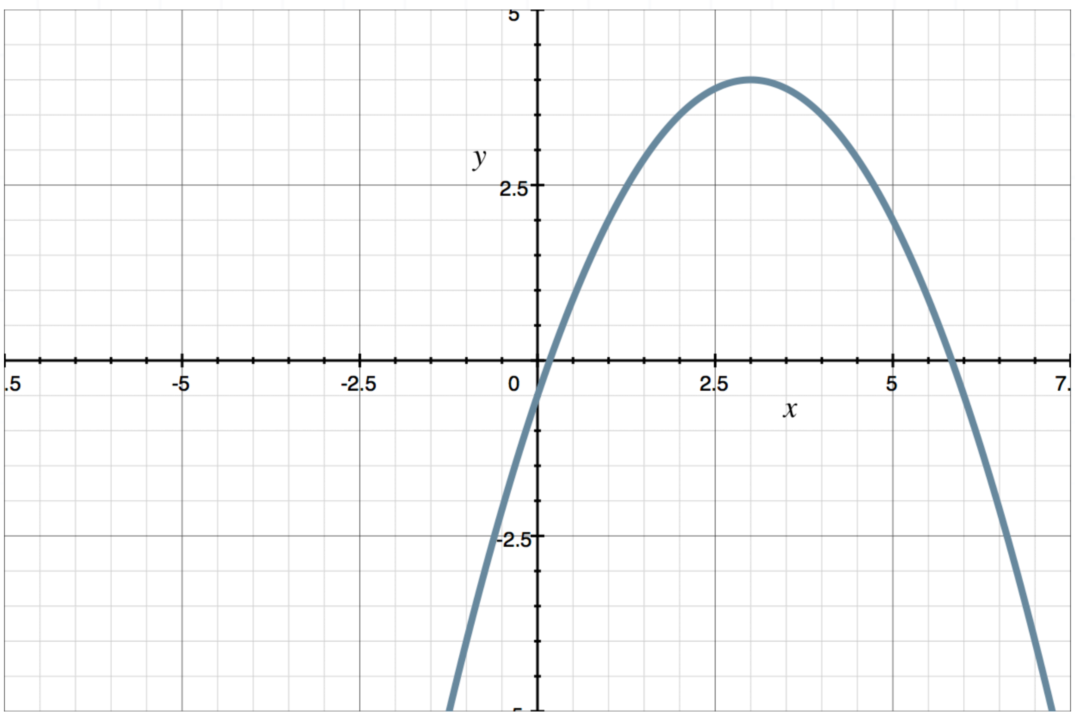
The vertex form of the equation is

$$y = 2(x+9)^2 + 8$$

Let's try one where we need to interpret a graph.

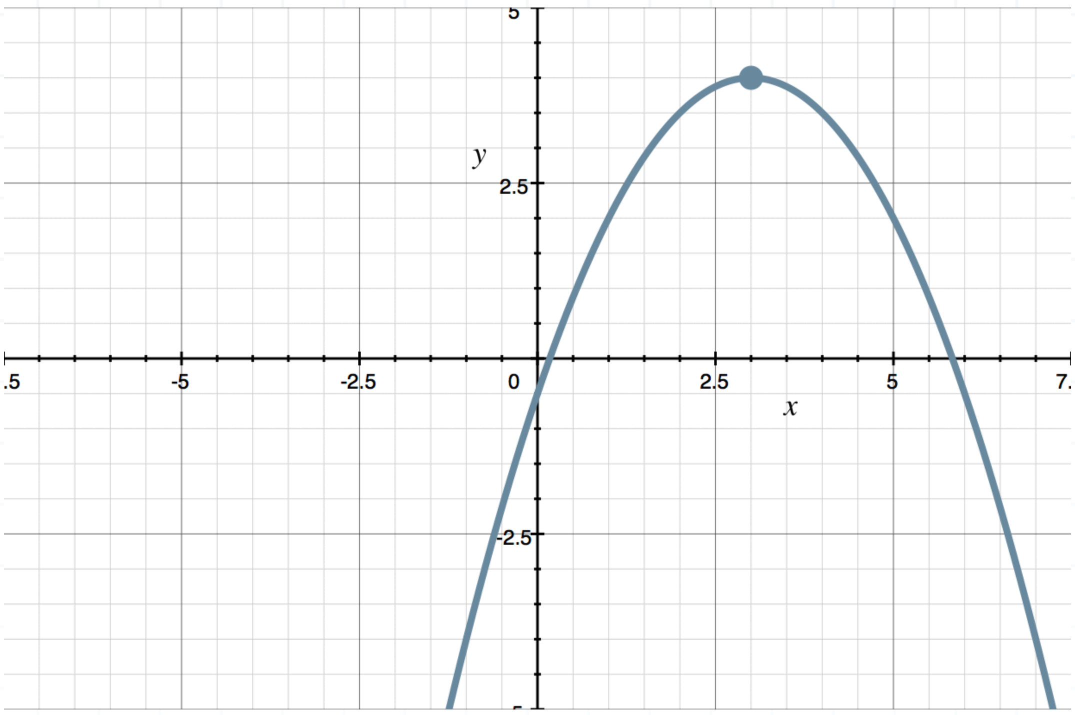
Example

The coefficient of the x^2 term in the equation of the parabola shown in the graph is $-1/2$. What is the equation of the parabola in standard form?



Remember, the vertex form of the equation of a parabola is $y = a(x-h)^2 + k$, where (h,k) are the coordinates of the vertex.

We know that $a = -1/2$, and we can read the coordinates of the vertex from the graph: $(3,4)$.



So we know that $h=3$ and $k=4$. Let's put what we know into the vertex form of the equation of a parabola.

$$y=a(x-h)^2+k$$

$$y=-\frac{1}{2}(x-3)^2+4$$

Now we want to go from vertex form to standard form, so we'll expand the square:

$$y=-\frac{1}{2}(x-3)(x-3)+4$$

$$y=-\frac{1}{2}(x^2-6x+9)+4$$

Distribute the $-1/2$ over all the terms inside the parentheses.

$$y=-\frac{1}{2}(x^2)-\frac{1}{2}(-6x)-\frac{1}{2}(9)+4$$

$$y = -\frac{1}{2}x^2 + 3x - \frac{9}{2} + 4$$

$$y = -\frac{1}{2}x^2 + 3x - \frac{9}{2} + \frac{8}{2}$$

$$y = -\frac{1}{2}x^2 + 3x - \frac{1}{2}$$

Sketching parabolas

Let’s outline the general steps for sketching parabolas.

1. Find the coordinates of the vertex.

2. Find the value of the y-intercept.

3. Find the x-coordinates of the x-intercepts by solving the equation $f(x) = 0$.

4. Make sure we’ve found at least one point to either side of the vertex. This helps make a better sketch. If we already have two x-intercepts from the previous step, we can use those. Otherwise, if we have zero or just one x-intercept, we can then use another x value or use the axis of symmetry and the y-intercept to get the second point.

5. Sketch the graph.

Let's do an example.

Example

Graph the parabola.

$$y = -x^2 + 2x + 8$$

Comparing this equation to the standard form of a parabola, we can identify $a = -1$, $b = 2$, and $c = 8$. Since $a < 0$, the parabola opens downwards and the vertex is the point at the top of the parabola.

The vertex lies on the axis of symmetry, so its x-coordinate is $-b/2a$.

$$x = -\frac{2}{2(-1)} = 1$$

$$y = -1^2 + 2(1) + 8 = -1 + 2 + 8 = 9$$

So the vertex is $(1,9)$. The equation of the axis of symmetry is $x = 1$. Now we need to find x-intercepts, which we can do by substituting $y = 0$ into the equation and solving for x.

$$y = -x^2 + 2x + 8$$

$$0 = -x^2 + 2x + 8$$

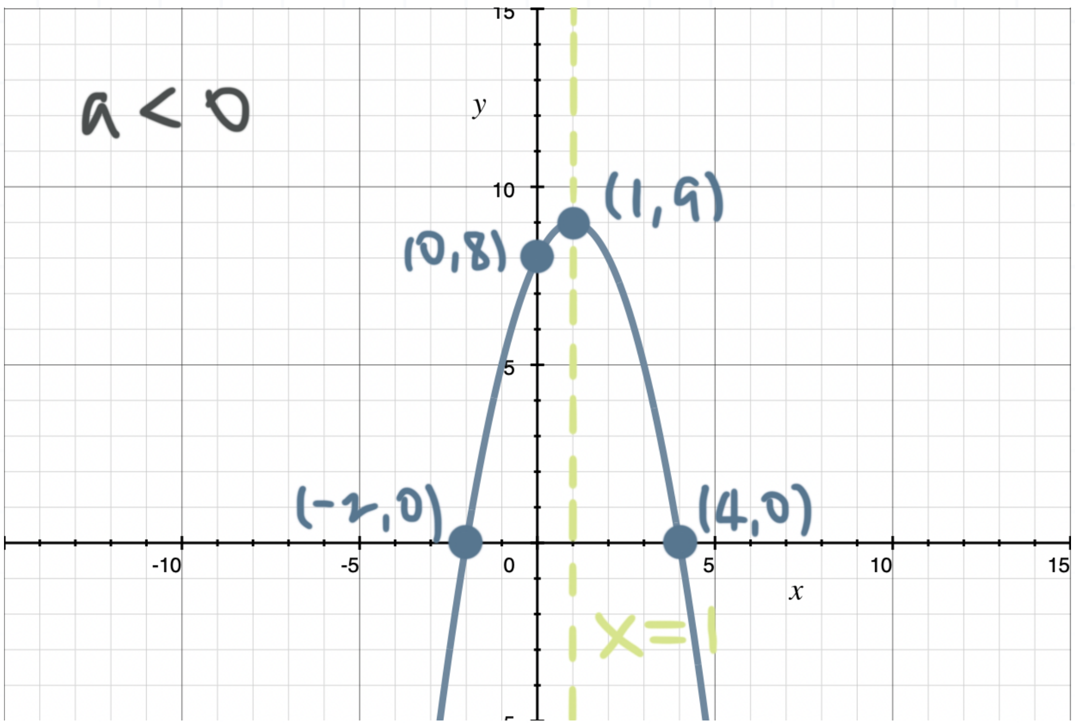
$$0 = (x + 2)(x - 4)$$

$x = -2$ and $x = 4$

So the x-intercepts are $(-2,0)$ and $(4,0)$. To find the y-intercept, we need to find $y(0)$.

$$y(0) = -0^2 + 2(0) + 8 = 8$$

So the y-intercept is $(0,8)$. We have all the information we need to sketch the graph, so a sketch of the parabola is



Center and radius of a circle

In this lesson we'll look at how to write the equation of a circle in standard form in order to find the center and radius of the circle.

The standard form for the equation of a circle is

$$(x-h)^2+(y-k)^2=r^2$$

where r is the radius and (h,k) are the coordinates of the center.

Sometimes in order to write the equation of a circle in standard form, we'll need to complete the square - on x only, on y only, or on both x and y (separately).

Example

Find the center and radius of the circle.

$$x^2+y^2+24x+10y+160=0$$

In order to find the center and radius, we need to convert the equation of the circle to standard form, $(x-h)^2+(y-k)^2=r^2$, where h and k are the coordinates of the center and r is the radius.

In order to get the given equation into standard form, we have to complete the square on both x and y.

Grouping the terms in x separately from the terms in y, and moving the constant term to the right side, we get

$$(x^2 + 24x) + (y^2 + 10y) = -160$$

To complete the square on x, we need to find the number a that satisfies the equation

$$x^2 + 24x + a^2 = (x + a)^2$$

That is, we need to find the number a for which

$$x^2 + 24x + a^2 = x^2 + 2ax + a^2$$

This means that the coefficient of the x term of the expression inside the first set of parentheses must be equal to $2a$. That coefficient is 24, so we'll set $2a$ equal to 24 and solve for a.

$$2a = 24 \quad \to \quad a = 12$$

To keep our equation balanced, we need to add and subtract a^2 (144) inside that set of parentheses and then regroup.

$$(x^2 + 24x) + (y^2 + 10y) = -160$$

$$(x^2 + 24x + 144 - 144) + (y^2 + 10y) = -160$$

$$(x^2 + 24x + 144) - 144 + (y^2 + 10y) = -160$$

To complete the square on y, we need to find the number b that satisfies the equation

$$y^2 + 10y + b^2 = (y + b)^2$$

That is, we need to find the number b for which

$$y^2 + 10y + b^2 = y^2 + 2by + b^2$$

This means that the coefficient of the y term of the expression inside the second set of parentheses must be equal to $2b$. That coefficient is 10, so we'll set $2b$ equal to 10 and solve for b.

$$2b = 10 \quad \to \quad b = 5$$

To keep our equation balanced, we need to add and subtract b^2 (25) inside that set of parentheses and then regroup.

$$(x^2 + 24x + 144) - 144 + (y^2 + 10y) = -160$$

$$(x^2 + 24x + 144) - 144 + (y^2 + 10y + 25 - 25) = -160$$

$$(x^2 + 24x + 144) - 144 + (y^2 + 10y + 25) - 25 = -160$$

Moving the -144 and -25 to the right side, we have

$$(x^2 + 24x + 144) + (y^2 + 10y + 25) = -160 + 144 + 25$$

Factoring the expressions in parentheses and simplifying the right side, we obtain.

$$(x + 12)^2 + (y + 5)^2 = 9$$

If we think of $x + 12$, $y + 5$, and 9 as $x - (-12)$, $y - (-5)$, and 3^2, respectively, we'll see that the center of the circle is at $(h, k) = (-12, -5)$ and its radius is $r = 3$. Remember that r must be positive, because it's a length, so we can rule out the possibility that $r = -\sqrt{9} = -3$.

Let's do another.

Example

Find the center and radius of the circle.

$$6x^2 + 6y^2 + 12x - 13 = 0$$

In order to find the center and radius, we need to convert the equation of the circle to standard form, $(x-h)^2 + (y-k)^2 = r^2$, where h and k are the coordinates of the center and r is the radius.

Let's begin by grouping the terms in x and moving the -13 to the right side.

$$6x^2 + 12x + 6y^2 = 13$$

In standard form, the coefficients of the x^2 term and the y^2 term must be equal to 1. Since the coefficient of each of those terms is now 6, we'll first factor out a 6 on the left side of the equation and then divide both sides by 6.

$$6(x^2 + 2x + y^2) = 13$$

$$x^2 + 2x + y^2 = \frac{13}{6}$$

Now we'll complete the square on x. There's no need to complete the square on y, because y^2 is already a perfect square.

$$(x^2 + 2x) + y^2 = \frac{13}{6}$$

To complete the square on x, we need to find the number a that satisfies the equation

$$x^2 + 2x + a^2 = (x + a)^2$$

That is, we need to find the number a for which

$$x^2 + 2x + a^2 = x^2 + 2ax + a^2$$

This means that the coefficient of the x term of the expression inside the parentheses must be equal to $2a$. that coefficient is 2, so we'll set $2a$ equal to 2 and solve for a.

$$2a = 2 \quad \to \quad a = 1$$

To keep our equation balanced, we need to add and subtract a^2 (1) inside the parentheses and then regroup.

$$(x^2 + 2x) + y^2 = \frac{13}{6}$$

$$(x^2 + 2x + 1 - 1) + y^2 = \frac{13}{6}$$

$$(x^2 + 2x + 1) - 1 + y^2 = \frac{13}{6}$$

We'll therefore add 1 to both sides, and get

$$(x^2 + 2x + 1) + y^2 = \frac{13}{6} + 1$$

Factoring the expression in parentheses and simplifying the right hand side, we get

$$(x+1)^2 + y^2 = \frac{19}{6}$$

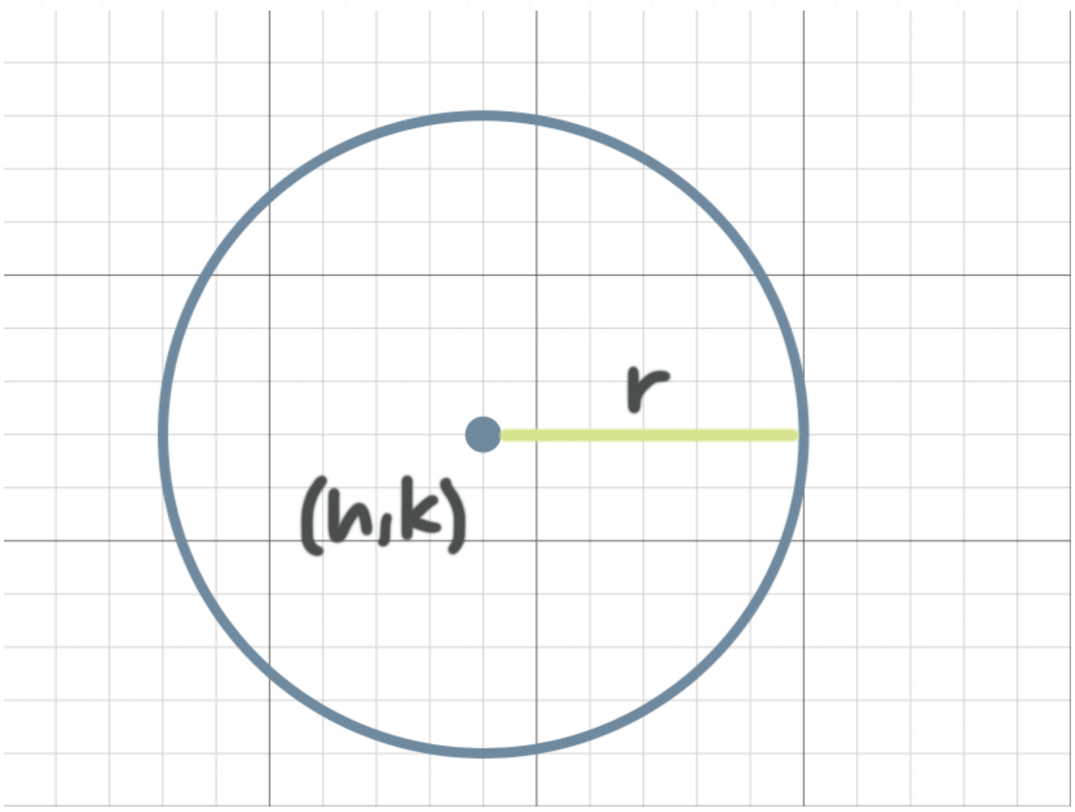
If we think of $x+1$, y, and $19/6$ as $x-(-1)$, $y-0$, and $(\sqrt{19/6})^2$, respectively, we'll see that the center of the circle is at $(h,k) = (-1,0)$ and the radius is $r = \sqrt{19/6}$.

Graphing circles

In this lesson we'll look at how the equation of a circle in standard form relates to its graph.

Remember that the equation of a circle in standard form is $(x-h)^2+(y-k)^2=r^2$, where (h,k) are the coordinates of the center of the circle and r is the radius.

As we can see in the figure, the center of a circle is a point, and the radius of a circle is the distance from the center of the circle to any point on its circumference.

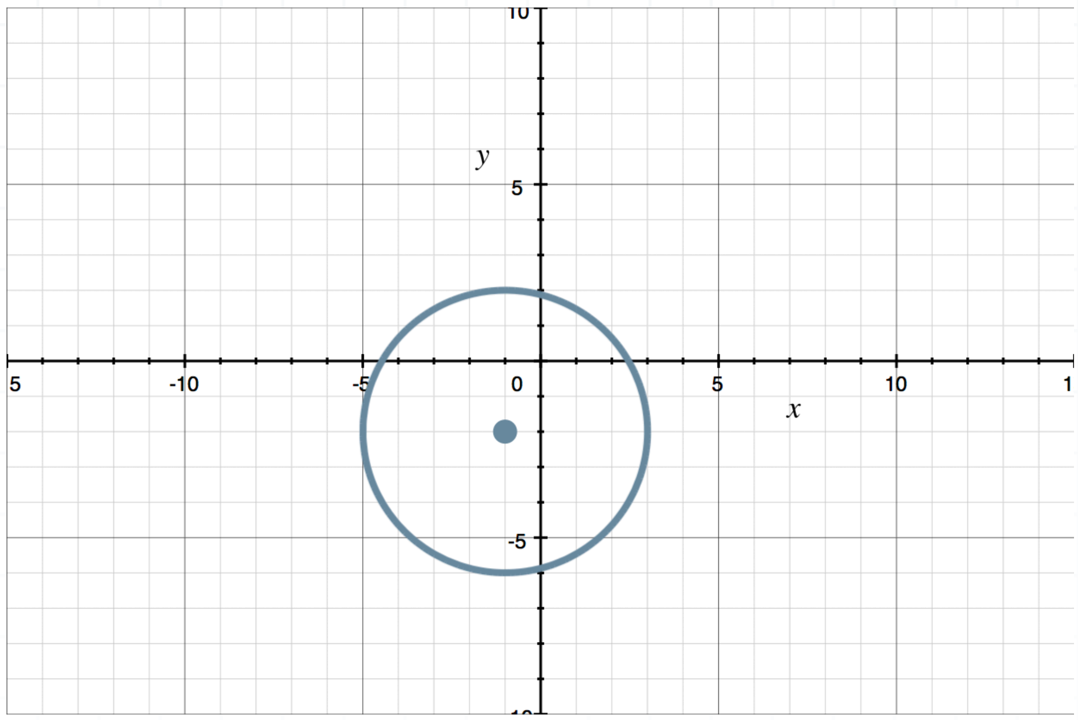


This means that if we have a graph of a circle, we can write its equation in standard form.

Let's do a few examples.

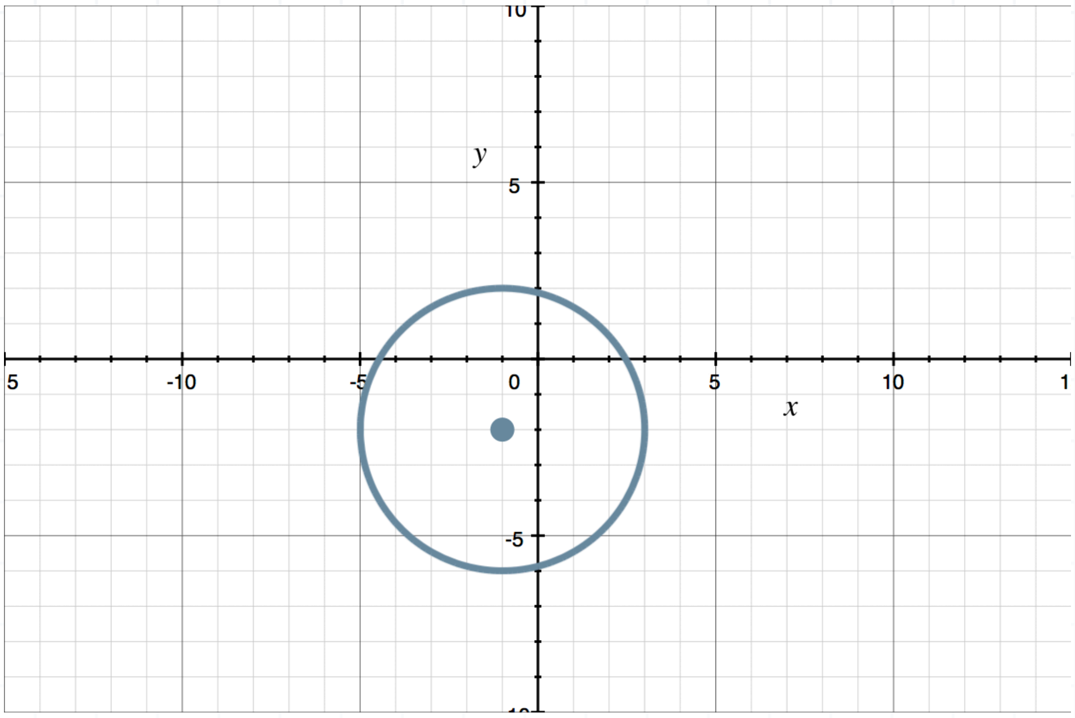
Example

What is the equation of the circle shown in the graph?



We need to find the equation of this circle in the form $(x-h)^2+(y-k)^2=r^2$, which means we need to find the coordinates of the center of the circle and its radius.

Let's find the coordinates of the center first.



The center is at the point $(-1,-2)$, so $h=-1$ and $k=-2$. Now let's find the radius, by determining the distance from the center of the circle to some point on the circumference. One point on the circumference of this circle is $(3,-2)$. The y-coordinate of this point is equal to the y-coordinate of the center of the circle. Therefore, the distance of the point $(3,-2)$ from the center is the difference in their x-coordinates, which is $3-(-1)=4$, so $r=4$.

Now let's plug everything into the standard form of the equation of a circle.

$$(x-h)^2+(y-k)^2=r^2$$

$$(x-(-1))^2+(y-(-2))^2=4^2$$

$$(x+1)^2+(y+2)^2=16$$

Let's try another example.

Example

Graph the circle.

$$(x-2)^2+(y+3)^2=9$$

In order to graph a circle, we need to know the coordinates of its center and its radius. In standard form, the equation of a circle is

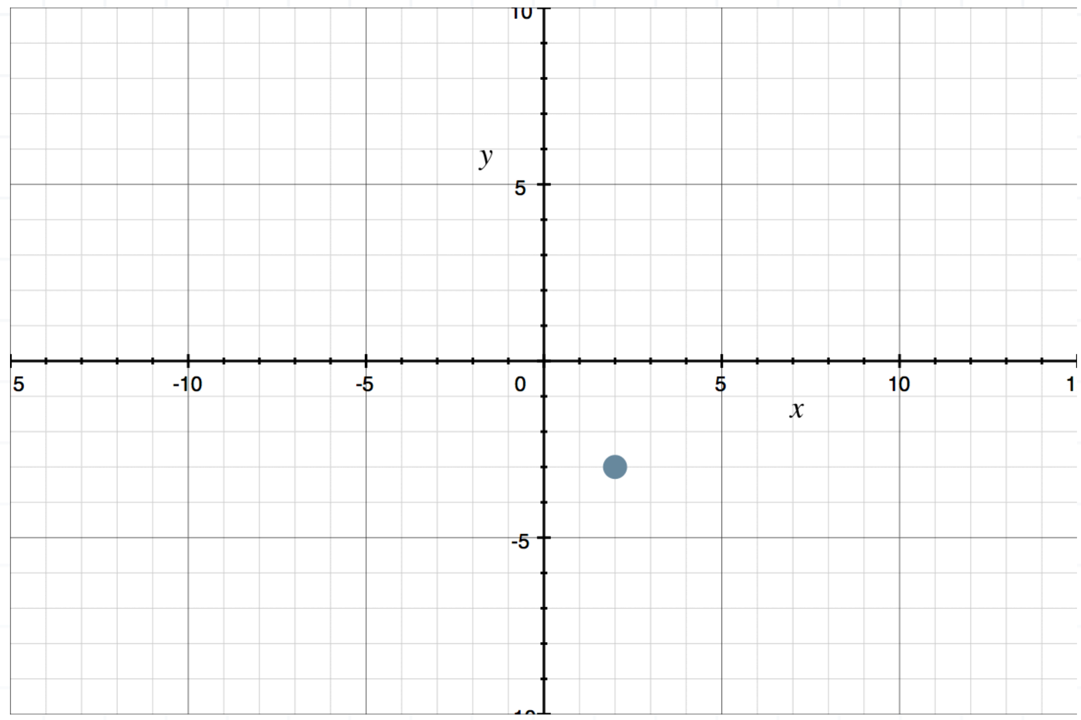
$$(x-h)^2+(y-k)^2=r^2$$

where (h,k) are the coordinates of the center and r is the radius. Let's write the equation of the circle in this form, by writing the constant term on the right-hand side, which is 9, as 3^2.

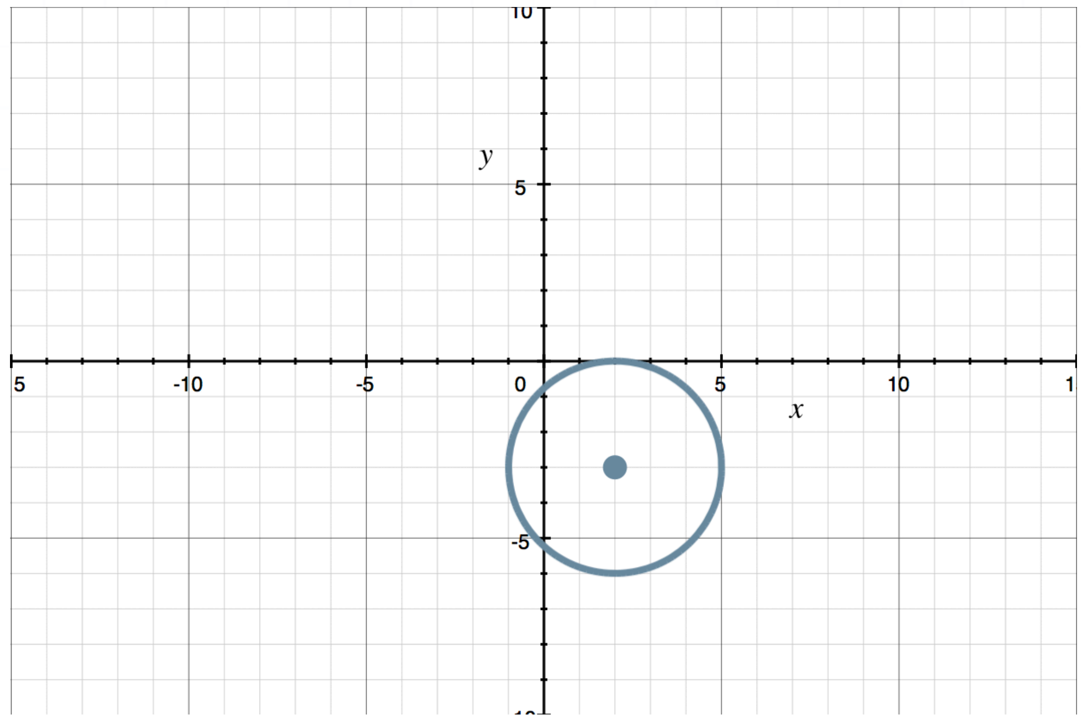
$$(x-2)^2+(y+3)^2=9$$

$$(x-2)^2+(y+3)^2=3^2$$

Now we can see that the coordinates of the center are $(h,k)=(2,-3)$ and the radius is $r=3$. Let's graph the circle, starting with the center.

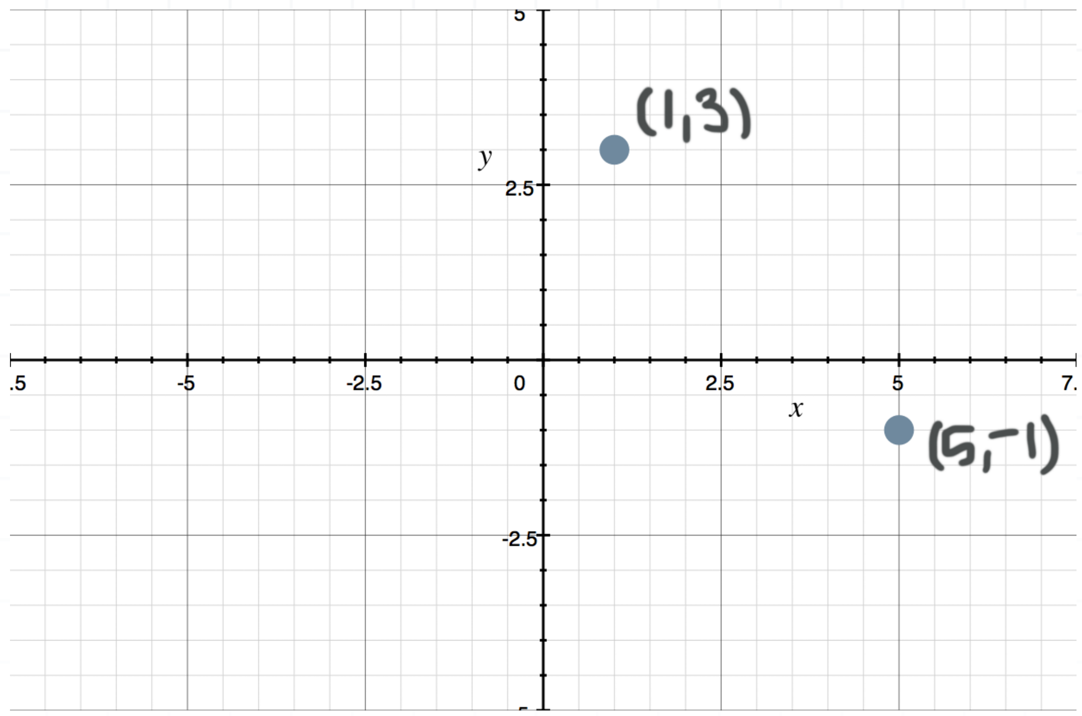


Since the radius is $r = 3$, we can plot several points that are 3 units from the center and use them to construct a rough sketch of the circle, or we can use a compass to draw a more perfect circle.

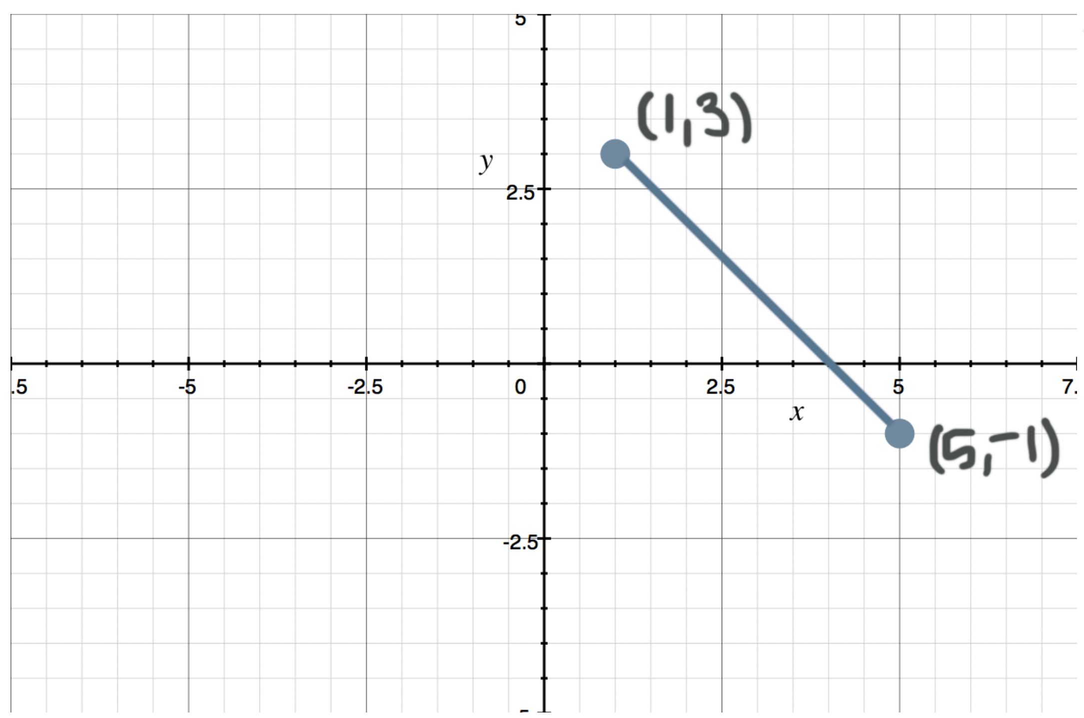


Distance between two points

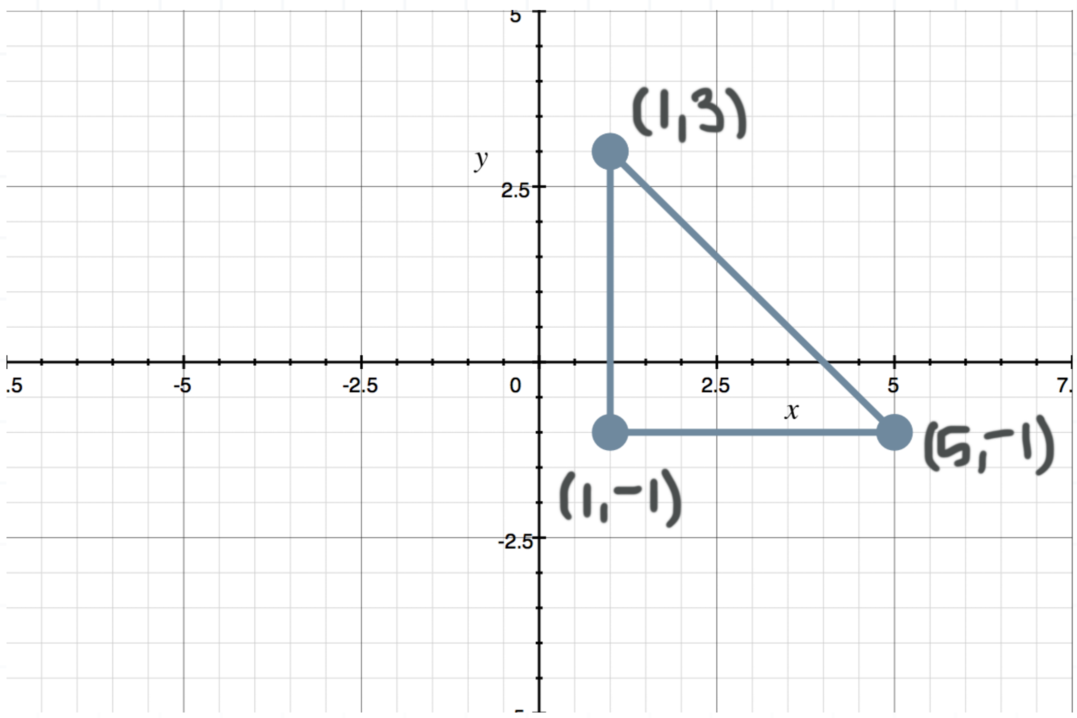
In this lesson we'll learn how to use the distance formula to calculate the distance between two points.



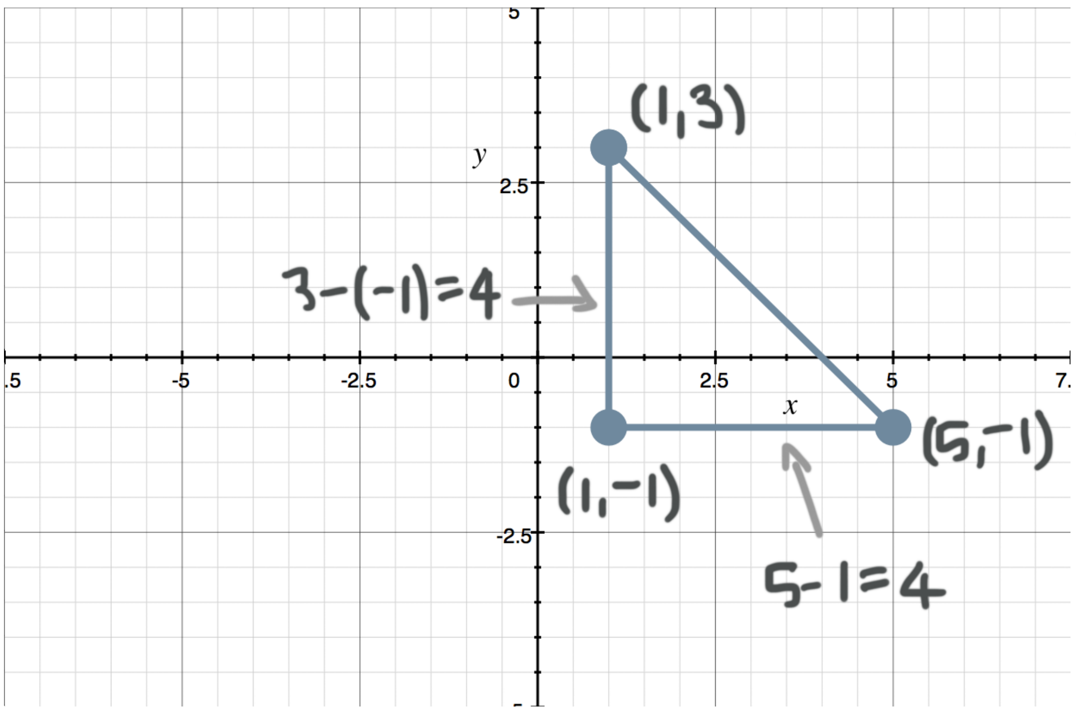
Let's begin by looking at two points on a graph. The distance formula will calculate the length of the straight line between the two points.



We can draw a right triangle that has the line between these two points as its hypotenuse.



We can find the lengths of the legs of the right triangle, a and b (where a is the absolute value of the difference in the x-coordinates of the original two points and b is the absolute value of the difference in their y-coordinates),



and use the Pythagorean Theorem, $a^2 + b^2 = c^2$, to find the length c of the hypotenuse (the straight line between the original two points).

In this case, the original two points are $(1,3)$ and $(5,-1)$. The absolute value of the difference in their x-coordinates is

$$a = |5-1| = |4| = 4$$

and the absolute value of the difference in their y-coordinates is

$$b = |-1-3| = |-4| = 4$$

Substitute these values into the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

$$4^2 + 4^2 = c^2$$

$$32 = c^2$$

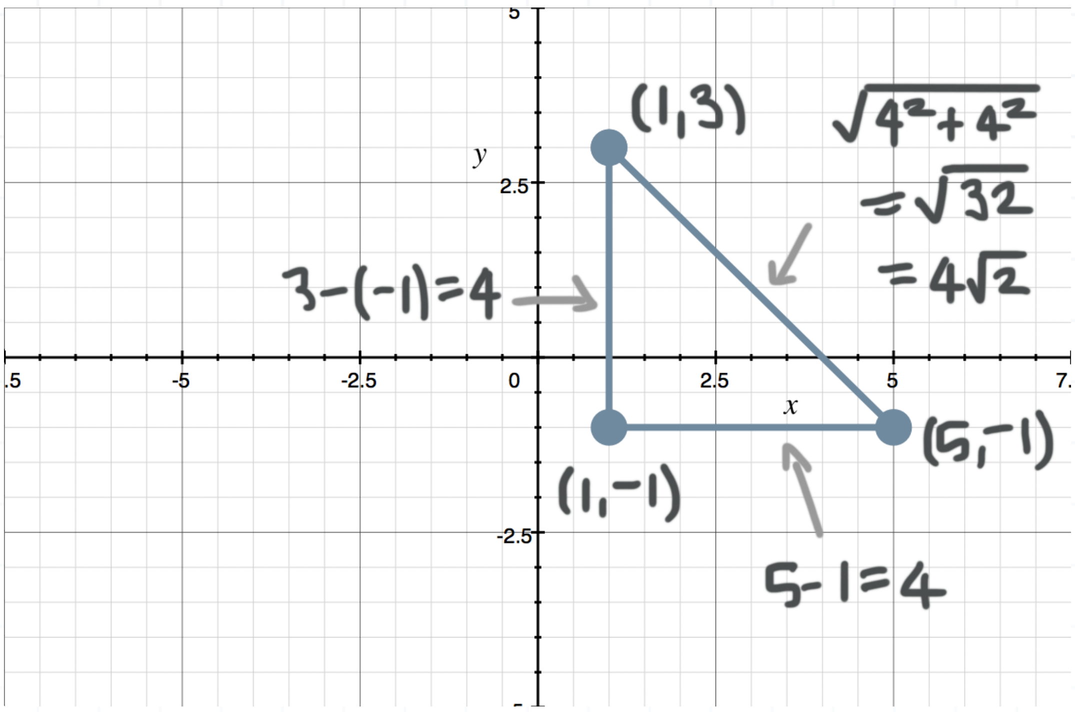
$$c = \sqrt{32}$$

$$c = \sqrt{16 \cdot 2}$$

$$c = \sqrt{16} \cdot \sqrt{2}$$

$$c = 4\sqrt{2}$$

Now we have all three side lengths of the triangle.



So what did we do to get the length between the two points? When we used the Pythagorean Theorem, we got

$$c = \sqrt{(5-1)^2 + (-1-3)^2}$$

or in other words,

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

And that's the distance formula! However, the distance formula uses d instead of c. It says that the distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let's do a few examples where we use the distance formula directly.

Example

What is the distance between the two points?

$$(5,7)$$

$$(-3,5)$$

We'll plug the given points $(x_1, y_1) = (5,7)$ and $(x_2, y_2) = (-3,5)$ into the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(-3 - 5)^2 + (5 - 7)^2}$$

$$d = \sqrt{(-8)^2 + (-2)^2}$$

$$d = \sqrt{64 + 4}$$

$$d = \sqrt{68}$$

$$d = \sqrt{4 \cdot 17}$$

$$d = \sqrt{4} \cdot \sqrt{17}$$

$$d = 2\sqrt{17}$$

The distance formula works with irrational numbers as well.

Example

Find the distance between the points.

$$(3,\sqrt{2})$$

$$(2,-\sqrt{2})$$

We'll plug the given points $(x_1, y_1) = (3,\sqrt{2})$ and $(x_2, y_2) = (2,-\sqrt{2})$ into the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(2-3)^2 + (-\sqrt{2} - \sqrt{2})^2}$$

$$d = \sqrt{(-1)^2 + (-2\sqrt{2})^2}$$

$$d = \sqrt{1 + [(-2)^2(\sqrt{2})^2]}$$

$$d = \sqrt{1 + (4 \cdot 2)}$$

$$d = \sqrt{1+8}$$

$$d = \sqrt{9}$$

$$d = 3$$

Midpoint between two points

We can also find the midpoint between two points, and doing so is just an extension of what we've already done to find the distance between two points.

The midpoint between points will always fall exactly half way between the x-values of their coordinate points, and half way between the y-values of their coordinate points, which means the midpoint M is given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

For instance, let's continue on with the last example, and find the midpoint between the points

$(3,\sqrt{2})$ and $(2,-\sqrt{2})$

Plugging both points into the midpoint formula will give the midpoint between them.

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M = \left(\frac{3 + 2}{2}, \frac{\sqrt{2} - \sqrt{2}}{2}\right)$$

$$M = \left(\frac{5}{2}, \frac{0}{2}\right)$$

$$M = \left(\frac{5}{2}, 0\right)$$

Equation modeling

In this lesson we'll look at how to solve a word problem in which we're told to write an equation that gives one of the quantities in the problem in terms of another one.

Let's start by working through an example.

Example

An RV and a motorcycle were driven for a month. The motorcycle traveled $1{,}000$ miles more than the RV. The fuel mileage for the RV was 15 miles per gallon (mpg), and the fuel mileage for the motorcycle was 43 mpg.

Write an equation which gives the total amount of fuel, g (in gallons), that was used by the two vehicles during that month in terms of the distance m (in miles) traveled by the motorcycle. We can use a table to show the fuel mileages and distances traveled.

	RV	Motorcycle
Mileage	15 mpg	43 mpg
Distance	r miles	m miles

We'll start by writing an equation that gives the distance r (in miles) traveled by the RV in terms of m. We know that $m = r + 1{,}000$ because the motorcycle traveled $1{,}000$ miles more than the RV. So $r = m - 1{,}000$, and we'll replace "r miles" in our table with "$m - 1{,}000$ miles."

	RV	Motorcycle
Mileage	15 mpg	43 mpg
Distance	m-1,000 miles	m miles

We can calculate the amount of fuel used by each vehicle, by dividing its distance by its fuel mileage. We'll add a row to our table, to show the amounts of fuel used by the vehicles.

	RV	Motorcycle
Mileage	15 mpg	43 mpg
Distance	m-1,000 miles	m miles
Fuel used	(m-1,000)/15 gallons	m/43 gallons

Now we know that

$$g = \frac{m - 1{,}000}{15} + \frac{m}{43}$$

$$g = \frac{43}{43} \cdot \frac{m - 1{,}000}{15} + \frac{15}{15} \cdot \frac{m}{43}$$

$$g = \frac{43m - 43{,}000}{645} + \frac{15m}{645}$$

$$g = \frac{43m - 43{,}000 + 15m}{645}$$

$$g = \frac{58m - 43{,}000}{645}$$

Let's look at a few more.

Example

A rock is thrown at a speed of 16 ft/s straight downward from a high platform. The distance through which it has fallen is $D = 16t^2 + 16t$, where t is the amount of time (in seconds) that it's been falling. The average speed of any falling object between time 0 (the time at which it starts falling) and time t is given by the ratio of the distance through which it has fallen to the elapsed time (t). Therefore, the average speed of this rock between time 0 and time t is $V = D/t$. Write an equation that gives t in terms of V.

Start with $V = D/t$, and substitute $16t^2 + 16t$ for D.

$$V = \frac{16t^2 + 16t}{t}$$

$$V = 16t + 16$$

Solve for t in terms of V.

$$16t = V - 16$$

$$t = \frac{V - 16}{16}$$

One last example.

Example

Each employee at a certain level of employment with a company is paid a salary of $\$42{,}000$ per year. In addition, the owner of the company wants to divide a bonus of $\$120{,}000$ evenly among these employees over the course of a year.

If each employee receives his/her salary and bonus in equal monthly installments throughout the year, write an equation which gives the total amount a that each employee is paid per month in terms of e, the number of employees.

Each employee is paid a salary of $\$42{,}000 \div 12 = \$3{,}500$ per month.

The total amount of the bonus is $\$120{,}000.00 \div 12 = \$10{,}000.00$ per month, but that has to be divided by the number of employees. So the monthly amount of the bonus for each employee is

$$\frac{\$10{,}000}{e}$$

The total amount that each employee is paid per month is

$$a = \$3{,}500 + \frac{\$10{,}000}{e}$$

Modeling a piecewise-defined function

In this lesson we'll look at piecewise-defined functions and how to write the equation of the definition of such a function given its graph.

A piecewise-defined function (also called a piecewise function) is a function that's made up of different "pieces," each of which has its own "sub-function" (its own algebraic expression) and its own "sub-domain" (its own part of the domain of the entire piecewise function).

We'll call the "sub-function" for each piece the *function* for that piece. A piecewise function is defined by giving the algebraic expression for the function for each piece and its domain. The domain of a piece of a piecewise function can be either an interval or just a single point.

The definition of a piecewise function is written in this form:

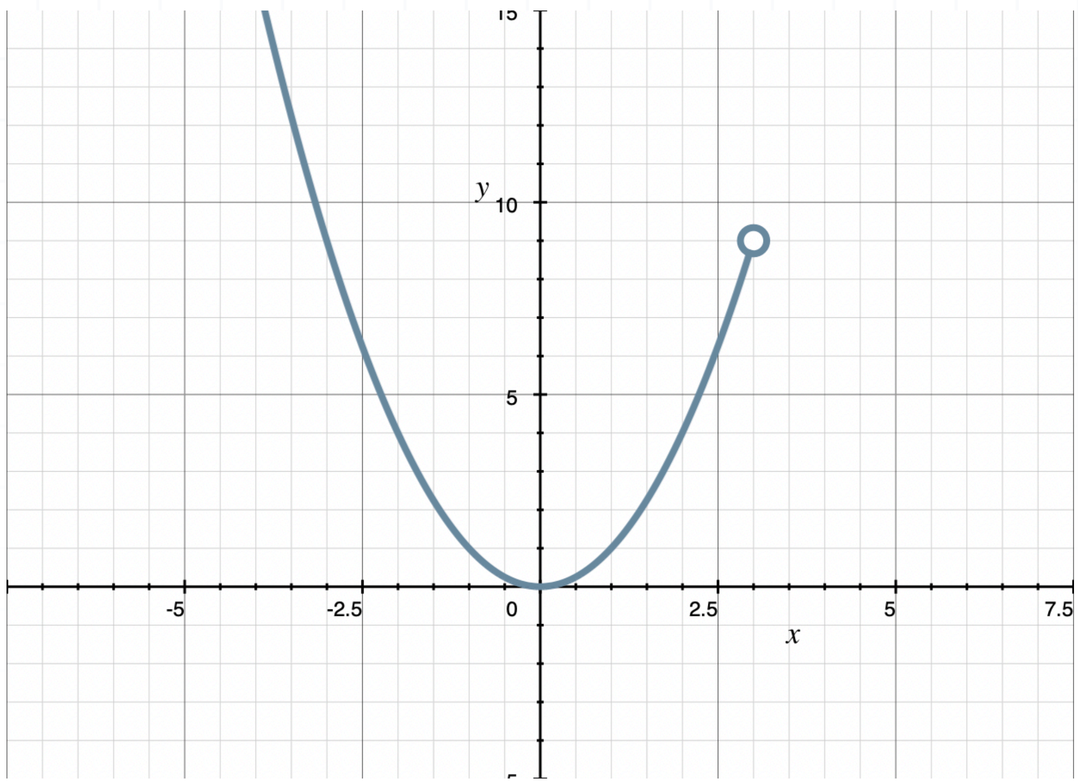
$$f(x) = \begin{cases} \text{Function}_1 & \text{if} \quad \text{Domain}_1 \\ \text{Function}_2 & \text{if} \quad \text{Domain}_2 \\ \text{Function}_3 & \text{if} \quad \text{Domain}_3 \\ \text{Function}_4 & \text{if} \quad \text{Domain}_4 \end{cases}$$

Of course a piecewise function doesn't need to have four pieces. It can have anywhere from two pieces to an infinite number of pieces. Usually, there are only two or three.

Let's look at an example of a definition of a piecewise function and how to graph the function.

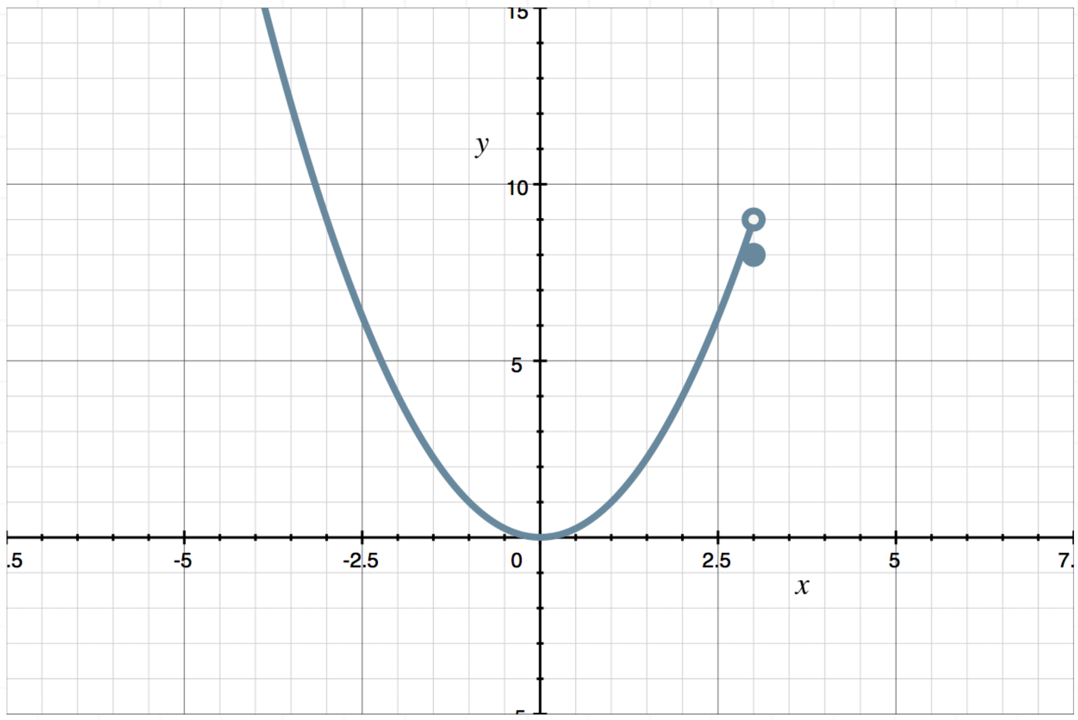
$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 3 \\ 8 & \text{if} \quad x = 3 \\ 2x + 4 & \text{if} \quad x > 3 \end{cases}$$

To graph a piecewise function, we graph each piece on its domain. Let's start by graphing the piece with function x^2 and domain $x < 3$, which is (part of) a parabola that opens upwards and has its vertex at the origin.

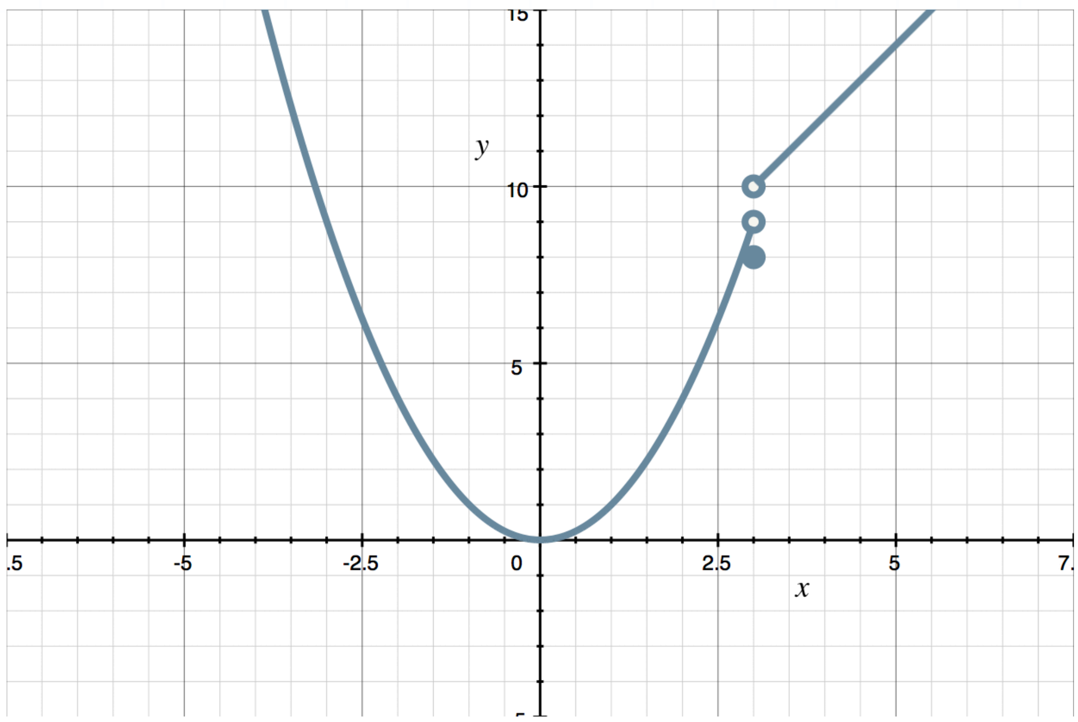


For our graph, we'll need to draw an open circle at the "right end" of the parabola, that is, at the point with $x = 3$, which has coordinates $(3,3^2)$, or $(3,9)$, to show that the point $(3,9)$ isn't a point of the graph of this piecewise function, because the domain is $x < 3$.

The domain of the next piece is just $x = 3$, and we're given that $f(3) = 8$, so we'll plot the point $(3,8)$.



Next we need to graph the piece with function $2x + 4$ and domain $x > 3$, which is (part of) a line. When we graph this piece of the function, we'll need to draw an open circle at the point of this line with $x = 3$, which has coordinates $(3,2(3) + 4)$, or $(3,10)$, because it isn't a point of the graph of this piecewise function.

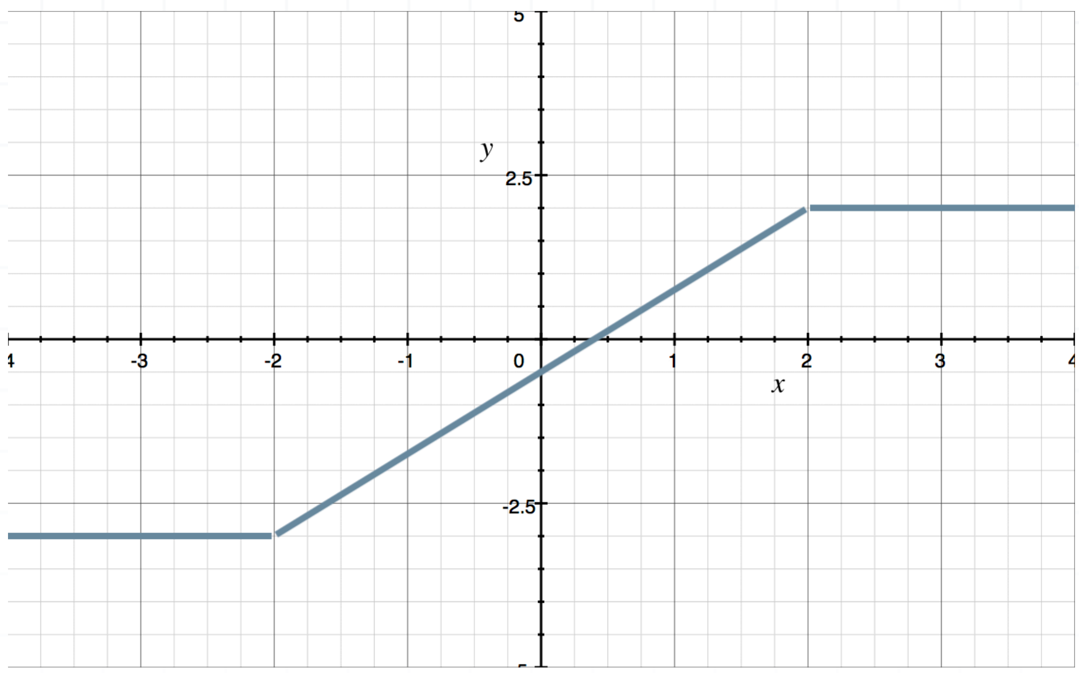


We can also write the definition of a piecewise function when we're given its graph.

Let's look at a few examples.

Example

What is the definition of the piecewise function shown in the graph?



We'll work on the graph from left to right. The horizontal line on the left has a y-value of -3 and includes all values of x in the interval $x < -2$ (all real numbers x that are less than -2). For this piece, we write -3 for the function (the constant function whose value is -3) and $x < -2$ for its domain.

$$f(x) = \begin{cases} -3 & x < -2 \\ & -2 \leq x \leq 2 \\ & x > 2 \end{cases}$$

The slanted line has a slope of $5/4$ and a y-intercept of $-1/2$. To see how to get the slope, notice that the points $(-2,-3)$ and $(2,2)$ are on this line, so

$$y = \frac{5}{4}x - \frac{1}{2}$$

For this piece, we write

$$\frac{5}{4}x - \frac{1}{2}$$

for the function and $-2 \leq x \leq 2$ for its domain.

$$f(x) = \begin{cases} -3 & x < -2 \\ \frac{5}{4}x - \frac{1}{2} & -2 \leq x \leq 2 \\ & x > 2 \end{cases}$$

The horizontal line on the right has a y-value of 2 and includes all values of x in the interval $x > 2$. For this piece, we write 2 for the function and $x > 2$ for its domain.

Putting the three pieces together, we define this piecewise function as follows:

$$f(x) = \begin{cases} -3 & x < -2 \\ \frac{5}{4}x - \frac{1}{2} & -2 \leq x \leq 2 \\ 2 & x > 2 \end{cases}$$

We might wonder how we decide which piece of this function gets the $\leq$ or $\geq$ sign and which piece gets the $<$ or $>$ sign. The truth is that it doesn't matter, as long as each x in the domain of the entire piecewise function is included in the domain of exactly one of its pieces - and, of course, that the function for that piece gives the correct value of $f(x)$. We could write it this way, too:

$$f(x) = \begin{cases} -3 & x \leq -2 \\ \frac{5}{4}x - \frac{1}{2} & -2 < x < 2 \\ 2 & x \geq 2 \end{cases}$$

But it could not be written as

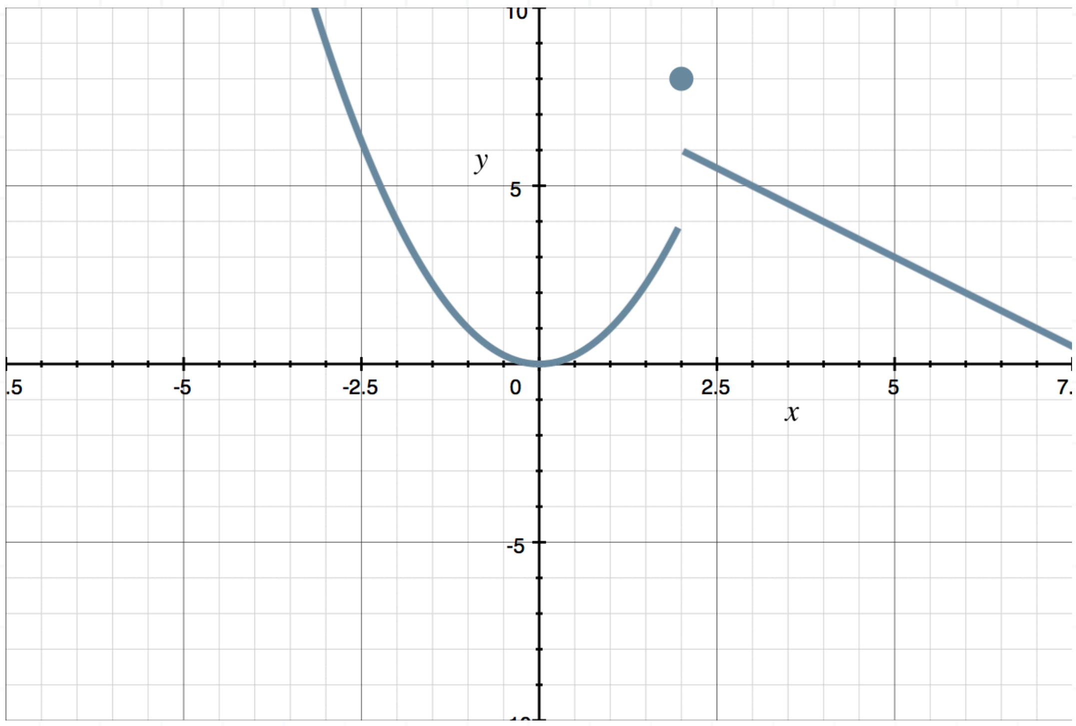
$$f(x) = \begin{cases} -3 & x \leq -2 \\ \frac{5}{4}x - \frac{1}{2} & -2 \leq x \leq 2 \\ 2 & x \geq 2 \end{cases}$$

because here -2 is included in the domains of two different pieces of the function, and so is 2.

Let's do one more example.

Example

What is the definition of the piecewise function shown in the graph.



Going from left to right, the first part of the graph is (part of) the parabola $y = x^2$, which has its vertex at the origin. (To see that $y = x^2$ is the equation of this parabola, note that it passes through the point $(-2,4)$ and that $4 = (-2)^2$.) So the function for this piece is x^2, and its domain is $x < 2$.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ & \text{if} \quad x = 2 \\ & \text{if} \quad x > 2 \end{cases}$$

The second part of the graph is the point $(2,8)$, so the function for this piece is 8 and its domain is $x = 2$.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ 8 & \text{if} \quad x = 2 \\ & \text{if} \quad x > 2 \end{cases}$$

The last part of the graph is part of the line $y = -x + 8$. To see this, we'll first compute the slope from the points $(3,5)$ and $(2,6)$, both of which are on this line. Then we'll use the slope and the point $(3,5)$ to get the point-slope form of the equation of the line (and then use that to get the slope-intercept form). The slope is

$$m = \frac{6-5}{2-3} = \frac{1}{-1} = -1$$

The point-slope form of the equation of a line is

$$y - y_1 = m(x - x_1)$$

Using $(x_1, y_1) = (3,5)$ and the fact that $m = -1$, we get

$$y - 5 = -1(x - 3)$$

$$y - 5 = -x + 3$$

$$y = -x + 8$$

Therefore, the function for the last piece is $-x + 8$ and its domain is $x > 2$.

Putting the three pieces together, we define this piecewise function as follows:

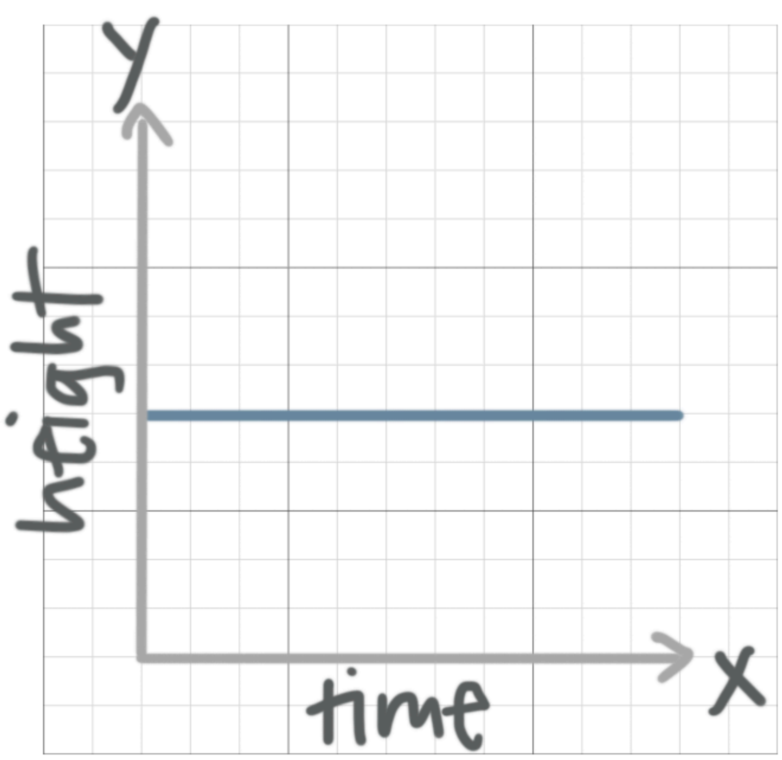
$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ 8 & \text{if} \quad x = 2 \\ -x + 8 & \text{if} \quad x > 2 \end{cases}$$

Sketching graphs from story problems

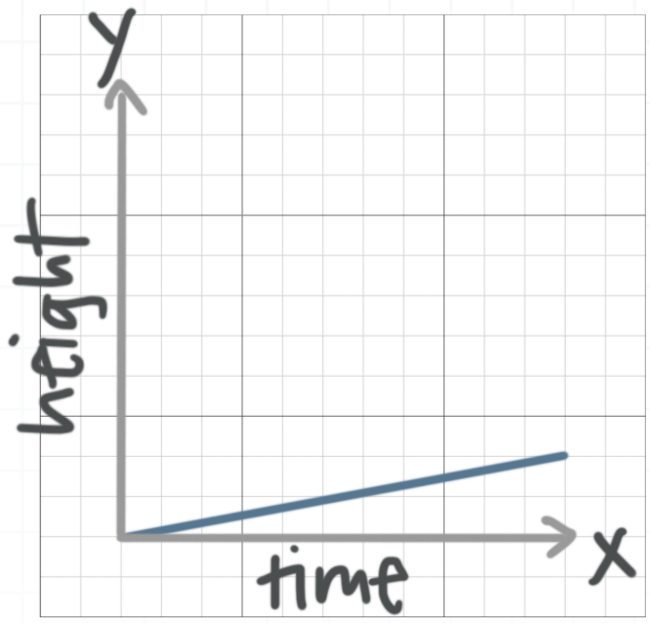
In this lesson we'll look at how we can use simple concepts to identify the graph of a piecewise function that represents a situation described in a word problem, to compose a word problem that's represented by the graph of a piecewise function, and to identify a verbal description of a situation that's represented by the graph of a piecewise function.

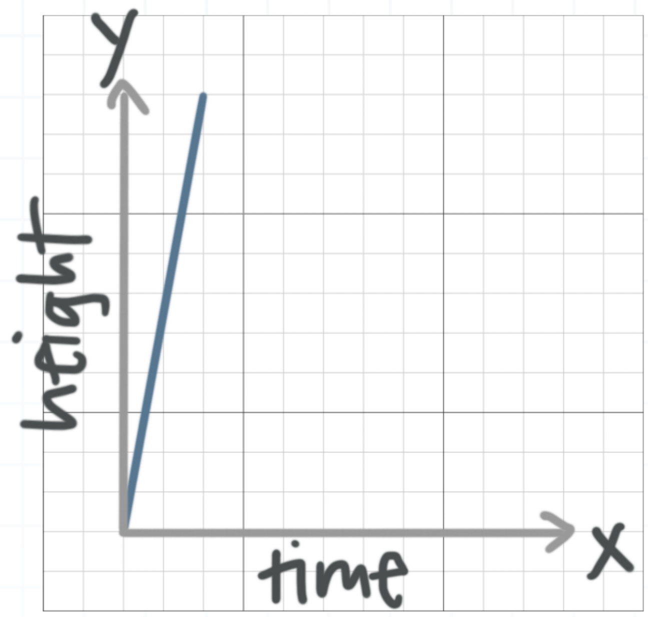
In a graph, a horizontal line represents variable whose value doesn't change (it stays the same). A positive slope shows a variable whose value is increasing, and a negative slope shows a variable whose value is decreasing. The steepness of the slope shows how fast the value of a variable is changing.

Let's look at some example graphs to get an idea of how this works. First, we'll consider graphs that show the height of an object above ground level as a function of time. The graph below tells us that, as time goes on, the height of the object doesn't change.



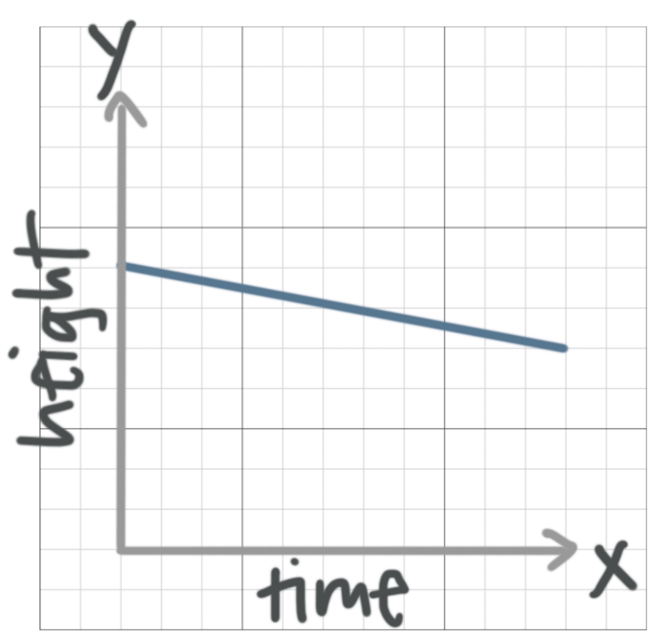
In both of the following graphs the height of the object is increasing, because the line has a positive slope.

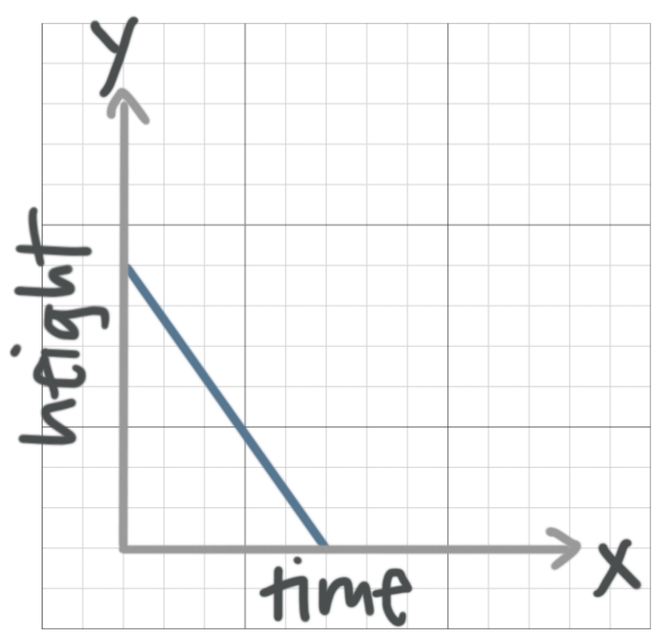




A shallower line, like the first graph, means that height is increasing at a slower rate. A steeper line, like the second graph, means that the height is increasing at a faster rate.

In both of the following graphs the height of the object is decreasing, because the line has a negative slope.



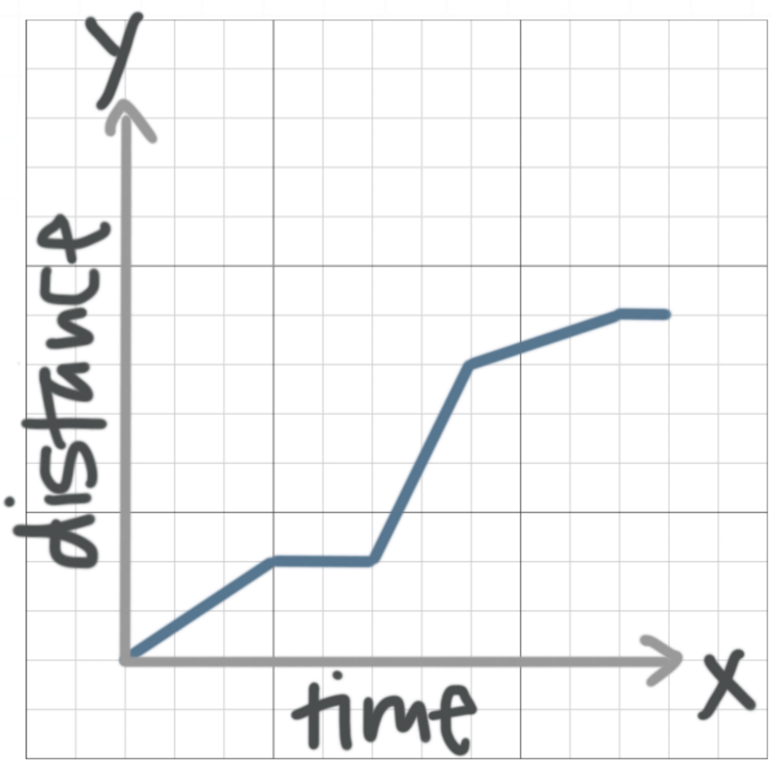


A shallower line, like the first graph, means the height is decreasing at a slower rate. A steep line, like the second graph, means that the height is decreasing at a faster rate.

Now let's look at word problems that involve either composing a story that goes with a given graph of a piecewise function or identifying the graph of a piecewise function that goes with a given story.

Example

Emily left on a trip to go to her grandmother's house in her car. The graph below shows her distance from her house as a function of time. Write a possible story to go along with the graph.



Looking at the graph, we see that the piecewise function it represents has five pieces. On three of the pieces (the ones that are represented by a line with a positive slope), Emily's distance is increasing (in one of these three pieces increasing at a considerably faster rate than in the other two), and on the other two pieces (the ones that are represented by a horizontal line) her distance is constant (she isn't moving at all). The second piece that represents no change in distance (the piece on the far right) probably represents being at her grandmother's house. We can make up a story around the situation that's represented by the piecewise function shown in the graph. Any story that corresponds to this graph would work.

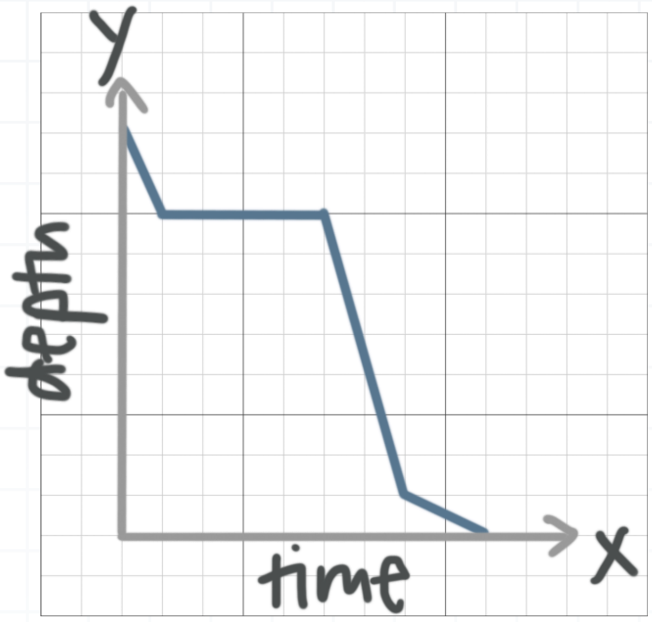
We could say something like this:

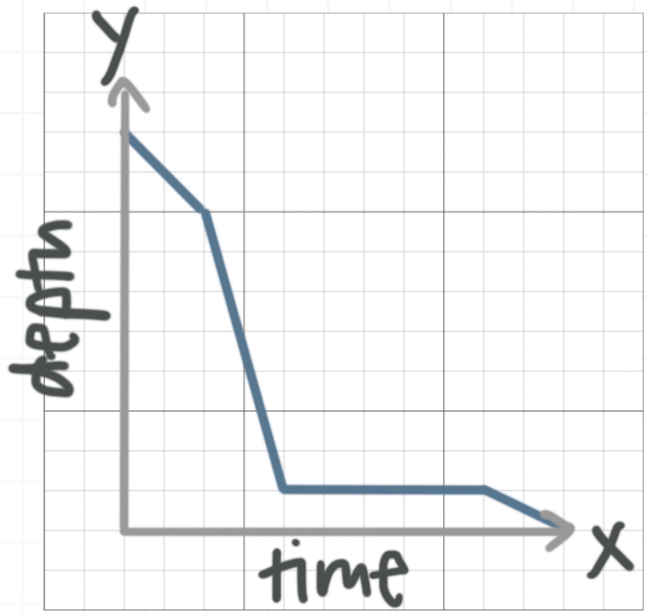
> Emily left her house and drove along a local road, and then stopped for gas. After stopping for gas, she drove on a highway, where she was able to drive faster. When she got off the highway, she drove more slowly through a neighborhood, and then arrived at her grandmother's house, where she stopped.

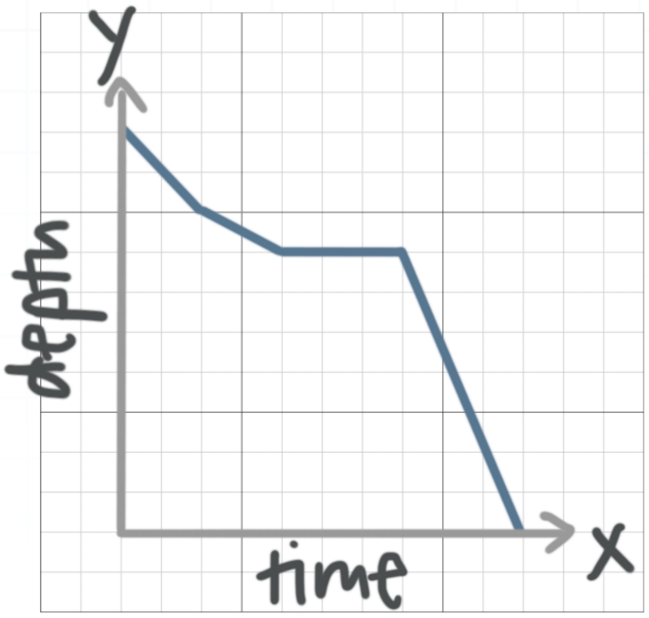
Let's try an example we're given a story and we have to identify the graph that goes with it.

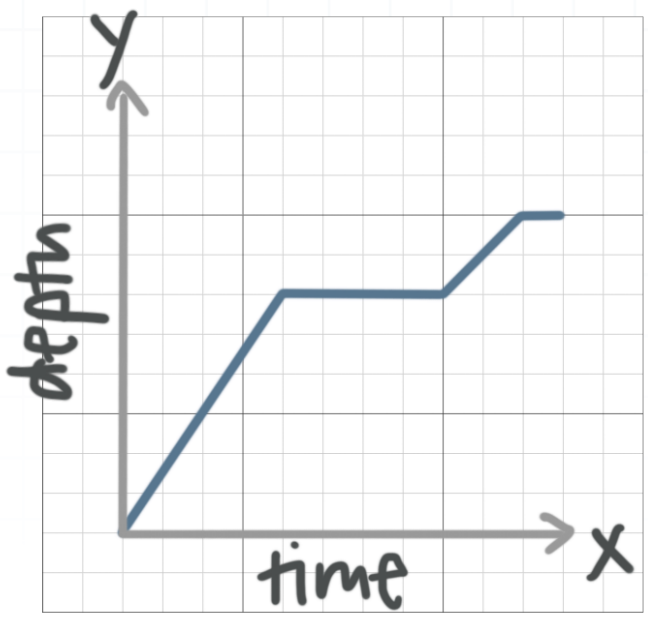
Example

A bulldozer is filling in a hole, and then it slows down and eventually breaks. After it's repaired, it fills in the hole quickly. Which graph best shows how the depth of the hole changes as time goes on?

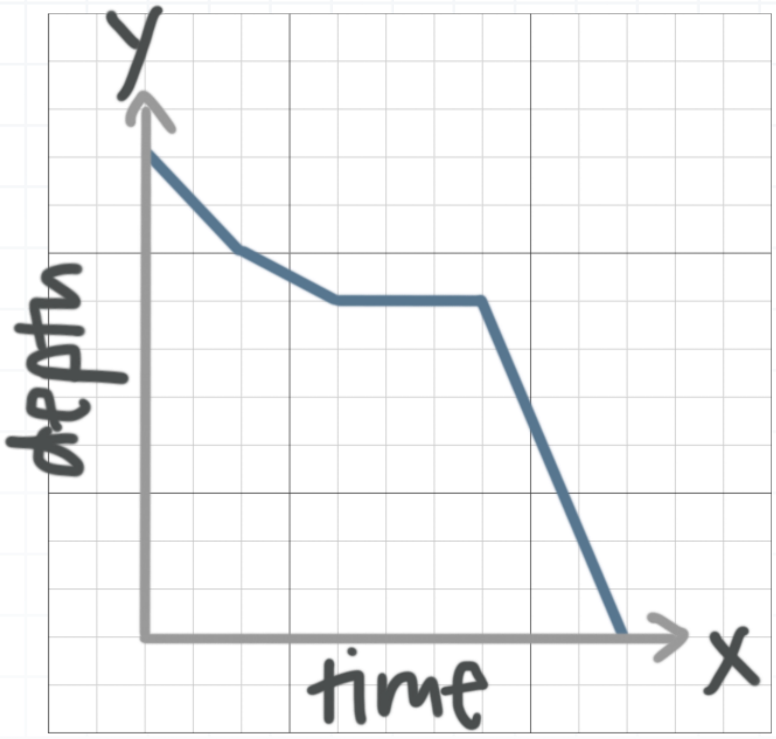








Since the bulldozer is filling in the hole, the depth of the hole will be decreasing as time goes on, so we need a graph that has negative slopes. At first the bulldozer is working, so the slope should be steep and negative. Then the bulldozer slows down, so the slope should be less steep but still negative. While the bulldozer is being fixed, the depth of the hole remains constant. Then after the bulldozer is fixed, it fills in the hole quickly, so the slope will again be steep and negative. The only graph that fits this story is


y
depth
time
x

Quadratic inequalities

A **quadratic inequality** is simply a quadratic equation where the equal sign has been replaced by an inequality sign.

Which means quadratic inequalities can take four forms, depending on the inequality sign.

$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c < 0$$

$$ax^2 + bx + c \geq 0 \qquad ax^2 + bx + c \leq 0$$

Remember that the graph of any quadratic function $f(x) = ax^2 + bx + c = 0$ is a parabola, so depending on the sign of the inequality, we'll need to determine whether to shade above or below the parabola.

There are two ways to solve quadratic inequalities: algebraically and graphically. Let's review the general steps of each of those methods.

Solving quadratic inequalities algebraically

To solve a quadratic inequality algebraically,

1. Put the inequality into standard form.

2. Find the critical points, which are the solutions to the related quadratic equation.

3. Using the critical points, divide the number line into intervals.

4. Choose a point from each interval and substitute it into the quadratic expression to find its sign, either positive or negative.

5. Finally, choose the intervals where the inequality is true and write the solution using interval notation.

Let's do an example so we can see these steps in action.

Example

Solve $x^2 - 9x + 14 \leq 0$ algebraically.

The quadratic is already in standard form, so we can proceed with finding the critical points by solving the related quadratic equation.

$$x^2 - 9x + 14 = 0$$

$$(x - 2)(x - 7) = 0$$

$$x - 2 = 0 \qquad x - 7 = 0$$

$$x = 2 \qquad x = 7$$

Now we can use 2 and 7 to divide the number line into intervals.

Since we have three intervals (left of $x = 2$, between $x = 2$ and $x = 7$, and right of $x = 7$), we'll choose three test points, one from each interval, and substitute them into quadratic equation to find the sign of each interval.

For $x = 0$:

$$0^2 - 9(0) + 14 = 14$$

For $x = 4$:

$$4^2 - 9(4) + 14 = -6$$

For $x = 8$:

$$8^2 - 9(8) + 14 = 6$$

Because we find a positive, then negative, then positive value, we can add these signs to each interval on the number line.

The original inequality $x^2 - 9x + 14 \leq 0$ tells us that we're looking for the interval(s) of the number line that are associated with a negative value (because of the $<$), as well as the points on the number line where the quadratic is equal to 0 (because of the $\leq$).

Looking at the number line and the signs that we've attached to it, we can see that the solutions are all the values of x between $x = 2$ and $x = 7$, as well as $x = 2$ and $x = 7$ themselves. So the solution to the quadratic inequality is

$$2 \le x \le 7$$

Solving quadratic inequalities graphically

We can also solve quadratic inequalities using the graph of the inequality. In general, we'll follow the steps below.

1. Write the inequality in standard form.

2. Determine the x-intercepts by looking at the graph, or by solving $ax^2 + bx + c = 0$.

3. Graph the parabolic function given by $f(x) = ax^2 + bx + c$.

4. Depending on the sign of the inequality, determine whether it asks for the value(s) of x that make the parabola negative (below the horizontal axis) or positive (above the horizontal axis).

Let's do an example.

Example

Solve $x^2 + 2x - 8 > 0$ graphically.

The inequality is already in standard form, and the x-intercepts are given by

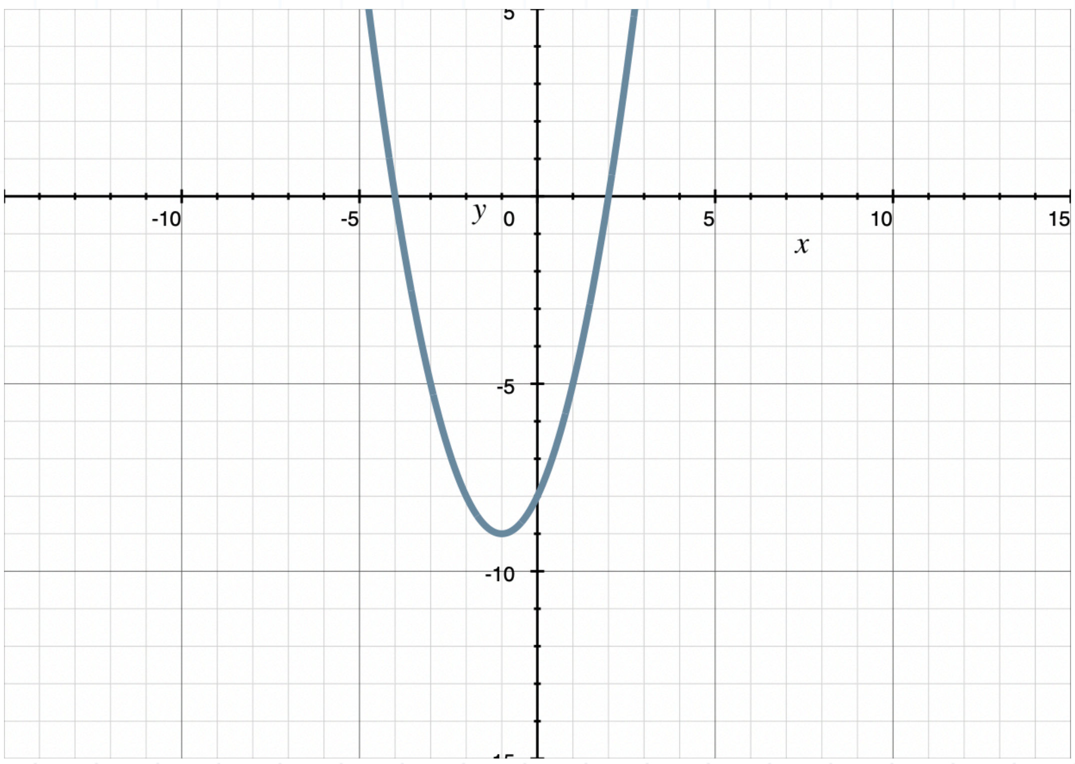
$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$x + 4 = 0 \qquad x - 2 = 0$$

$$x = -4 \qquad x = 2$$

The x-intercepts are therefore $(-4,0)$ and $(2,0)$. A sketch of the parabola is



The inequality $x^2 + 2x - 8 > 0$ is asking for values of x that make the quadratic greater than 0, which means we need to find the values of x where the parabola is above the x-axis. Therefore, the solution is all values of x to the left of $x = -4$ (not including $x = -4$ itself), and all values of x to the right of 2 (not including $x = 2$ itself).

$x < -4$ and $x > 2$

Using the discriminant

Remember that the value of the discriminant, $b^2 - 4ac$, tells us how many times the parabola will intersect the horizontal axis.

$b^2 - 4ac > 0$	Two zeros
$b^2 - 4ac = 0$	One zero
$b^2 - 4ac < 0$	No zeros

We can use this fact to help us determine how many solutions there are to the quadratic inequality.

Let's do an example where we use the discriminant to determine the number of solutions.

Example

Solve the quadratic inequality.

$$-x^2 + 3x - 4 > 0$$

Instead of using the set of steps we outline for how to solve quadratic inequalities graphically, let's start by finding the discriminant.

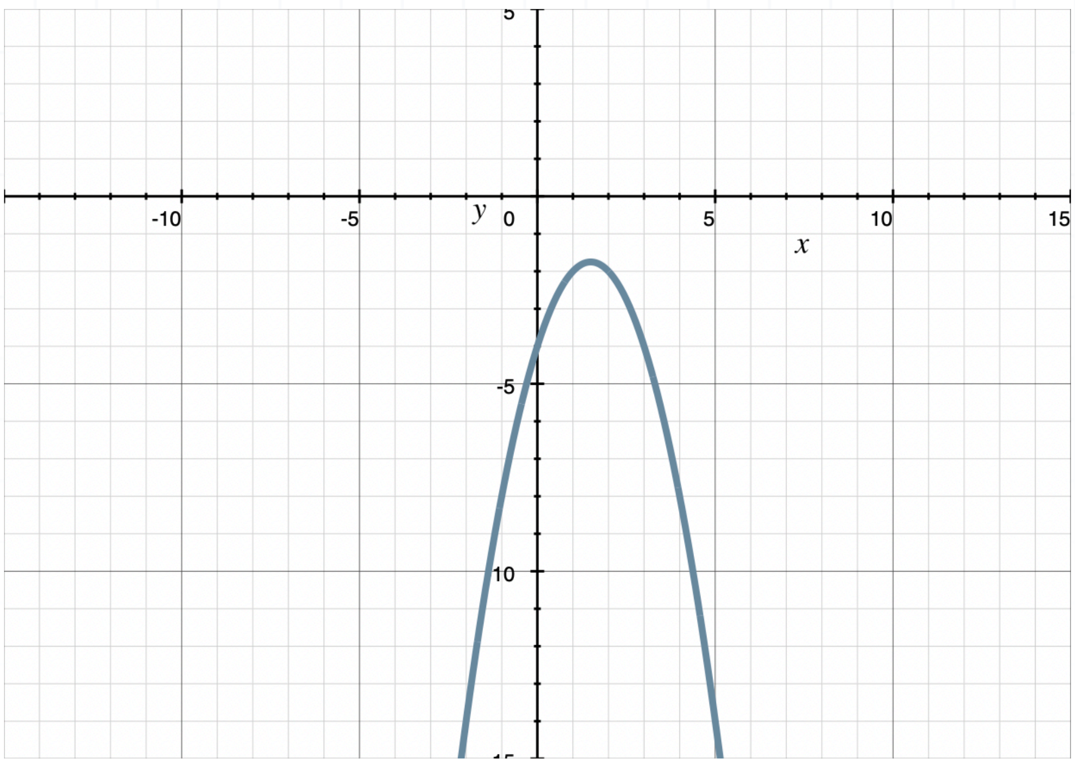
For the given quadratic, $a = -1$, $b = 3$, and $c = -4$, so the discriminant is

$$b^2 - 4ac = 3^2 - 4(-1)(-4)$$

$$b^2 - 4ac = 9 - 16$$

$$b^2 - 4ac = -7$$

Since the discriminant is negative, there are no x-intercepts, which means the parabola never crosses the horizontal axis. Because the parabola opens downward (because $a = -1$ is negative), that means the vertex of the parabola must be below the horizontal axis, with the parabola opening down from that point, and therefore never crossing the x-axis.



The inequality is asking for the values of x where the parabola is above the horizontal axis, but we know that this never happens. Therefore, there is no solution to the inequality.

Systems with quadratic inequalities

We know how to solve systems of equations, including systems with a quadratic equation.

All we want to do now is translate that into systems of inequalities, where at least one of the inequalities in the system is a quadratic inequality.

Solving by graphing

To solve a system of inequalities by graphing, we'll start by sketching the equation associated with each inequality.

Each curve that we sketch will be solid if the inequality is a $\leq$ or $\geq$ inequality, and dashed if the inequality is a $<$ or $>$ inequality.

We'll shade above the curve for $>$ or $\geq$ inequalities, and below the curve for $<$ or $\leq$ inequalities. And we can always use the origin as a test point to determine where to shade.

The solution to the system of inequalities will be any region where the shading from both inequalities overlaps.

Let's look at an example to see how this works.

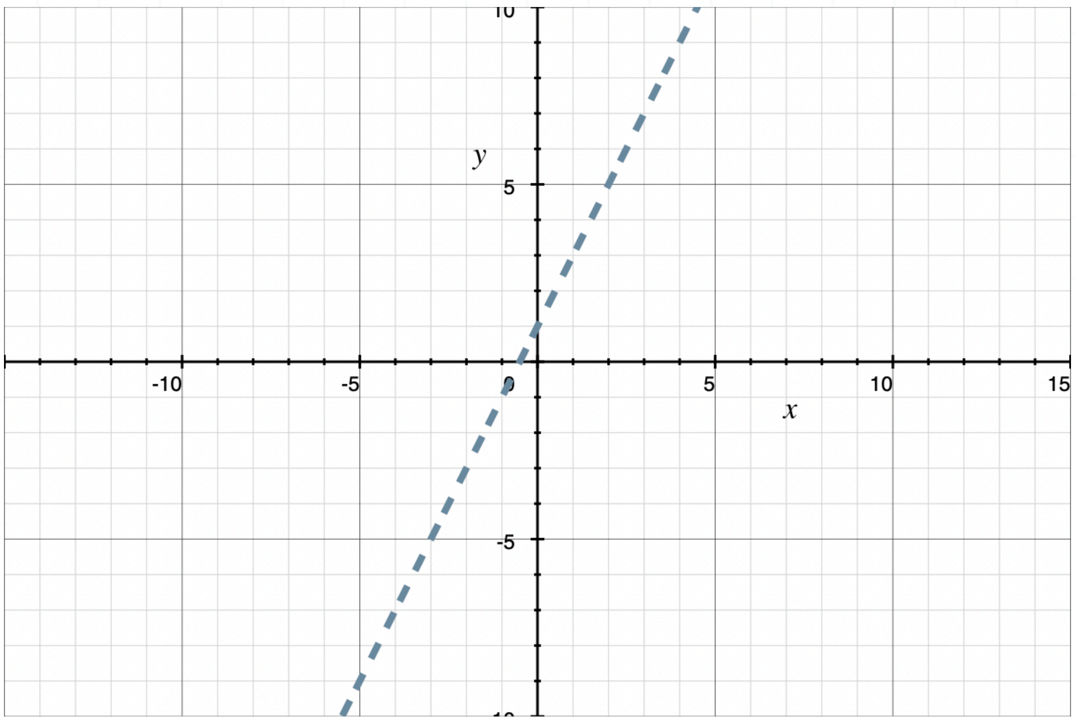
Example

Sketch the solution to the system of inequalities.

$$y > 2x + 1$$

$$y \leq (2x + 1)(x - 3)$$

Begin by graphing the line $y = 2x + 1$ using the y-intercept of 1 and the slope of 2. The line will be dashed because of the $>$ sign.



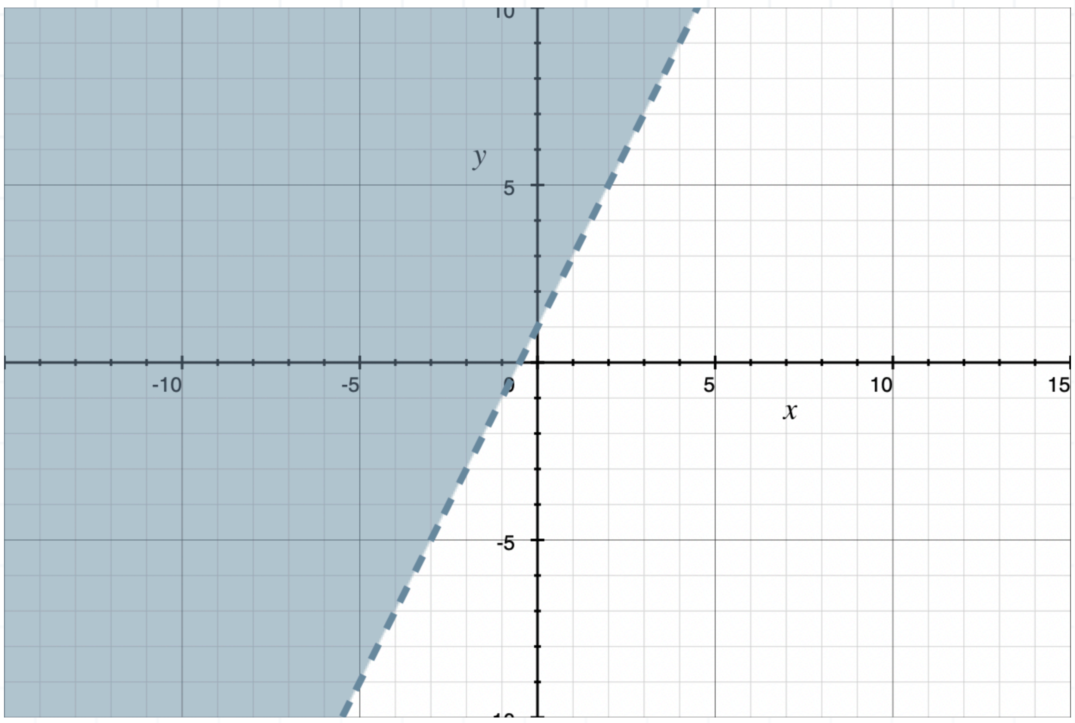
Let's test the origin to help us determine where to shade.

$$y > 2x + 1$$

$$0 > 2(0) + 1$$

$$0 > 1$$

Because this is a false statement, we shade away from the origin.



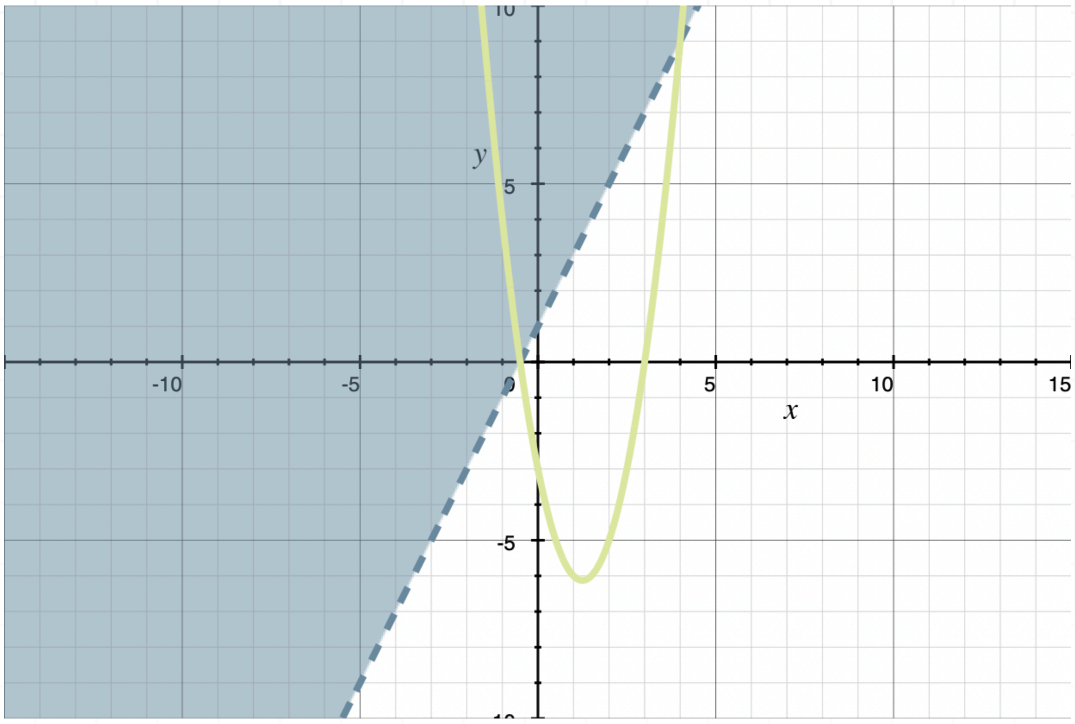
We can find the x-intercepts of the quadratic by solving the corresponding quadratic equation.

$$(2x+1)(x-3)=0$$

$$2x+1=0 \quad \text{and} \quad x-3=0$$

$$x=-\frac{1}{2} \quad \text{and} \quad x=3$$

Now we can graph the parabola. The curve will be solid because of the $\leq$ sign.



Let's test the origin to help us determine where to shade.

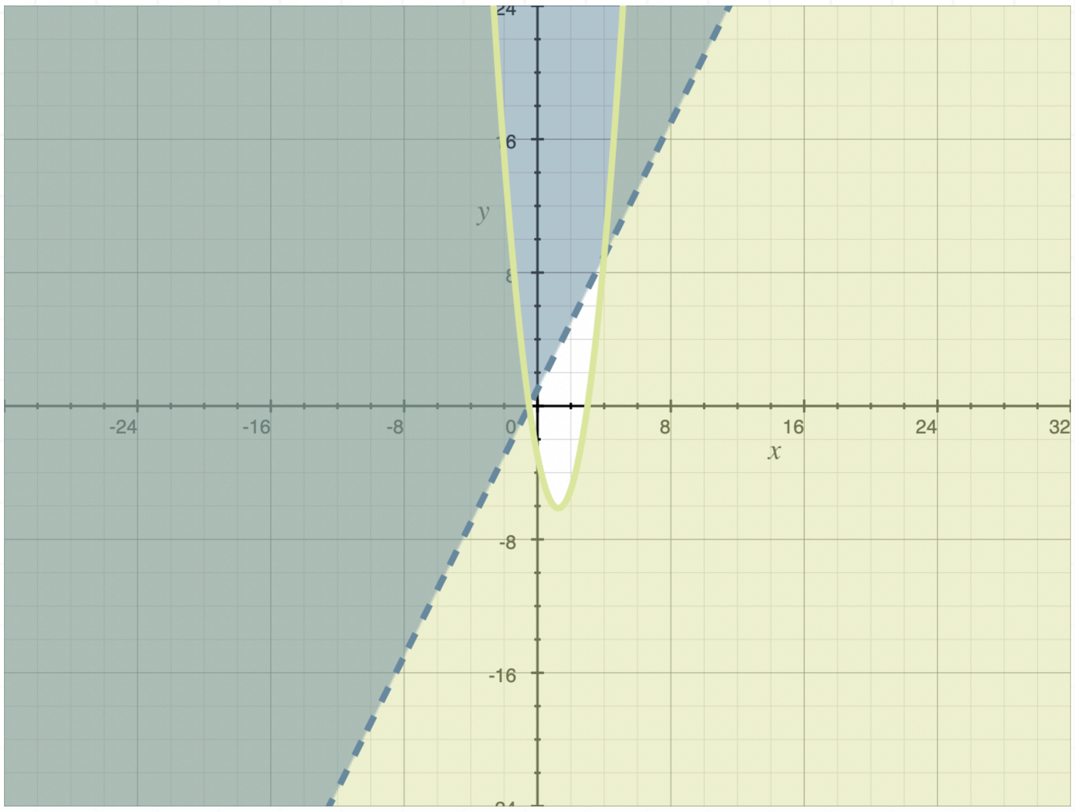
$$y \leq (2x+1)(x-3)$$

$$0 \leq (2(0)+1)(0-3)$$

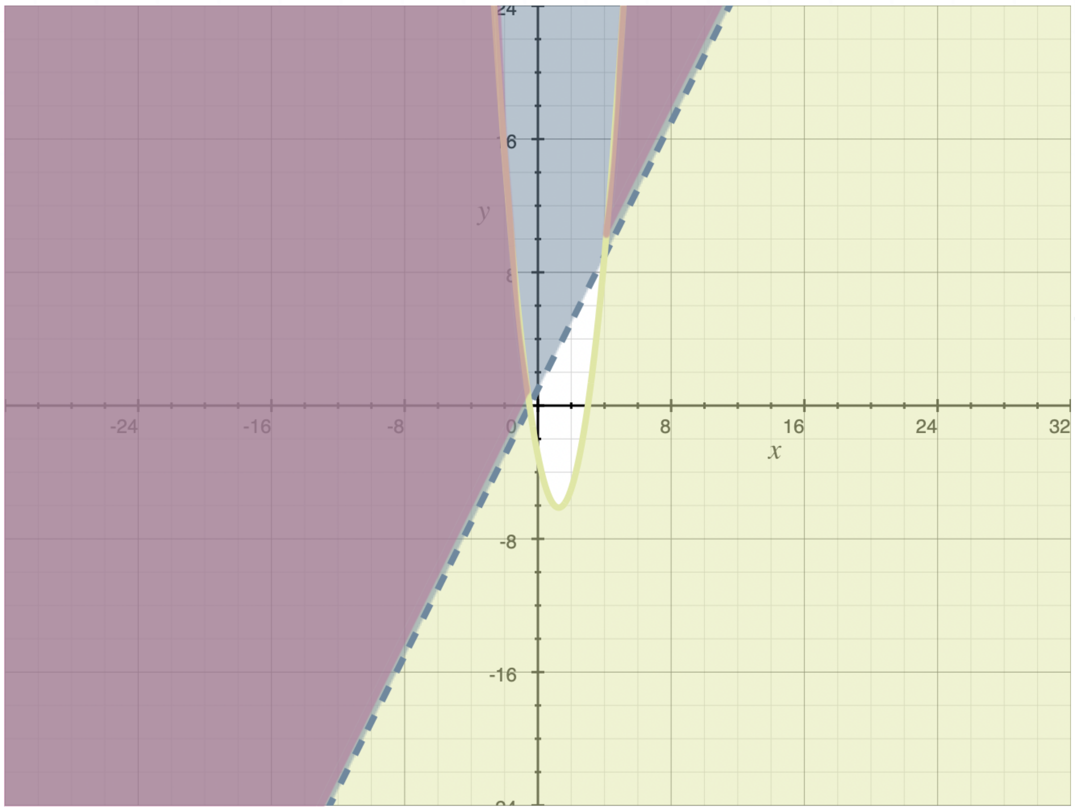
$$0 \leq (1)(-3)$$

$$0 \leq -3$$

Because this is a false statement, we shade away from the origin.



The regions where the shading overlaps is the solution to the system of inequalities.



Let's do another example.

Example

Graph the solution to the system of inequalities.

$$y + 6 > 5x^2$$

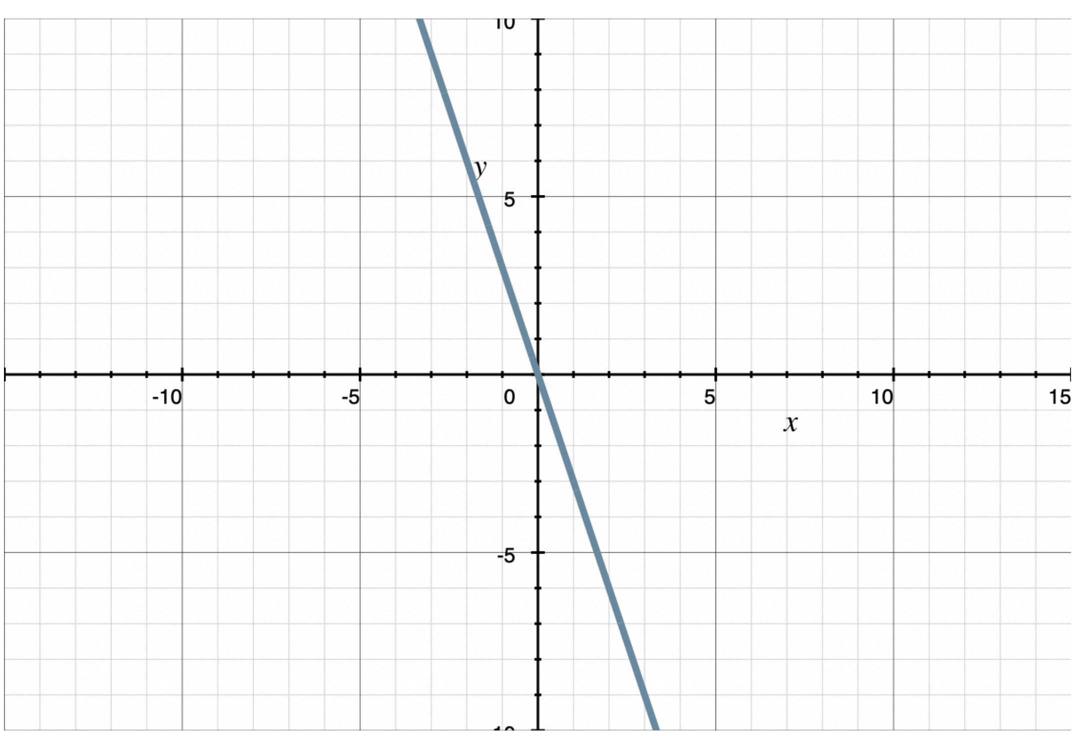
$$-\frac{1}{3}y \geq x$$

First we need to rewrite both inequalities by isolating y.

$$y > 5x^2 - 6$$

$$y \leq -3x$$

Begin by graphing the line $y = -3x$ using the y-intercept of 0 and the slope of -3. The line will be solid because of the $\leq$ sign.



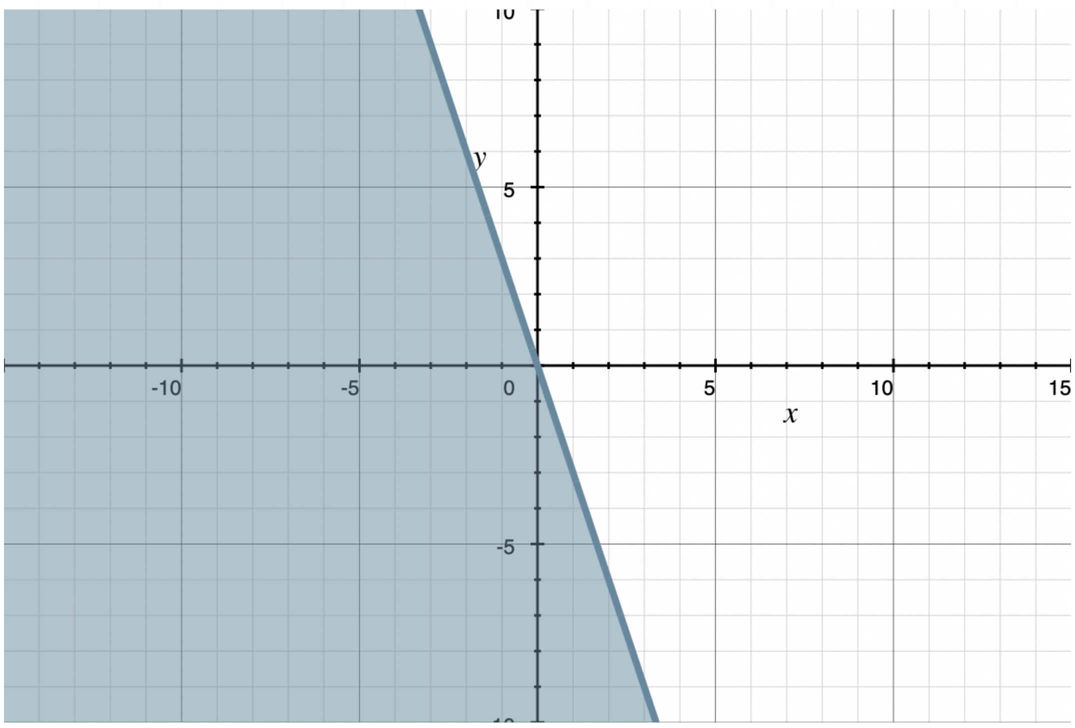
We can't use the origin as a test point because the line intersects the origin, so let's use $(0,1)$ instead.

$$y \le -3x$$

$$1 \le -3(0)$$

$$1 \le 0$$

Because this is a false statement, we shade on the opposite side of the line from the test point $(0,1)$.



We can find the x-intercepts of the quadratic by solving the corresponding quadratic equation.

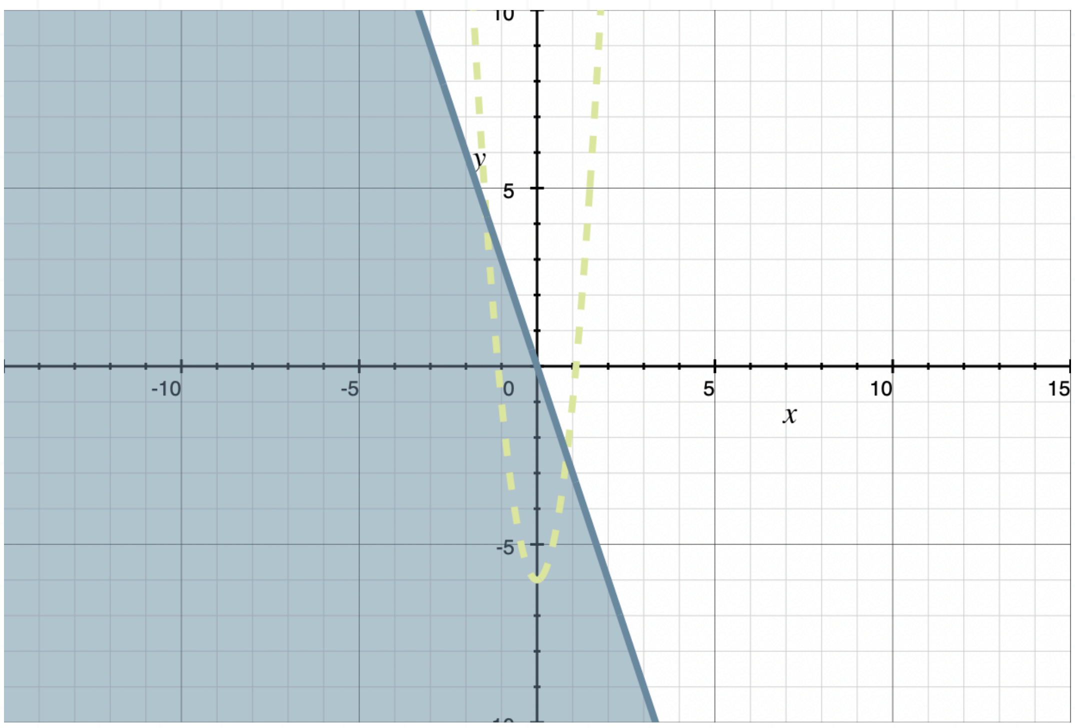
$$5x^2 - 6 = 0$$

$$5x^2 = 6$$

$$x^2 = \frac{6}{5}$$

$$x = \pm\sqrt{\frac{6}{5}}$$

Now we can graph the parabola. The curve will be dashed because of the $>$ sign.



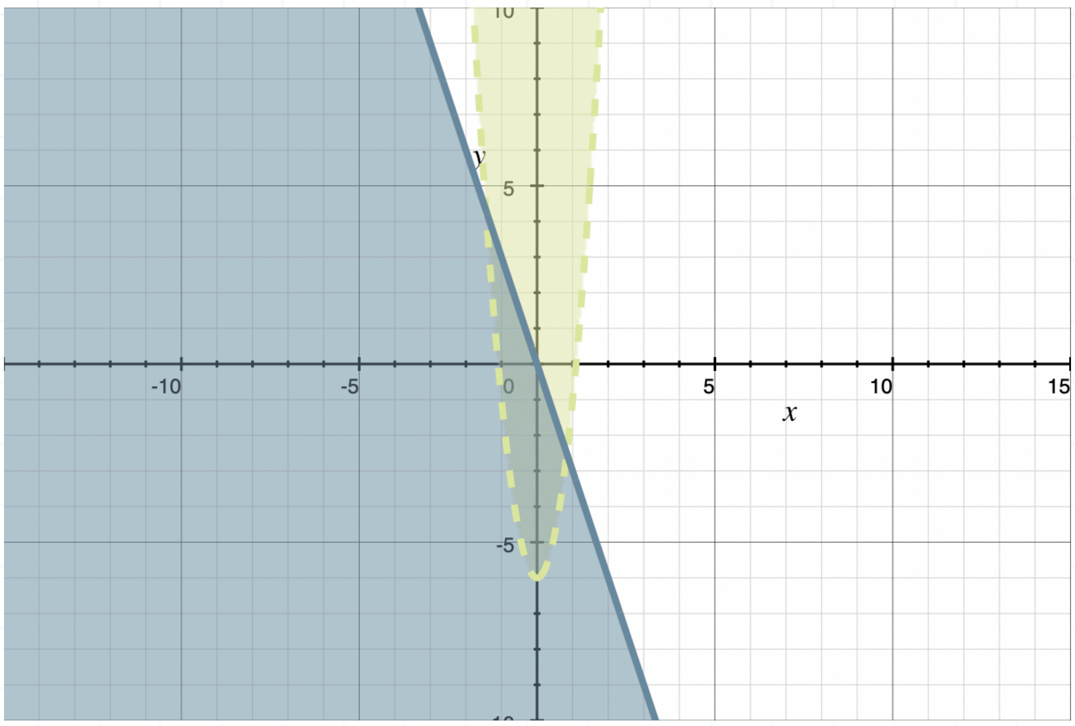
Let's test the origin to help us determine where to shade.

$$y > 5x^2 - 6$$

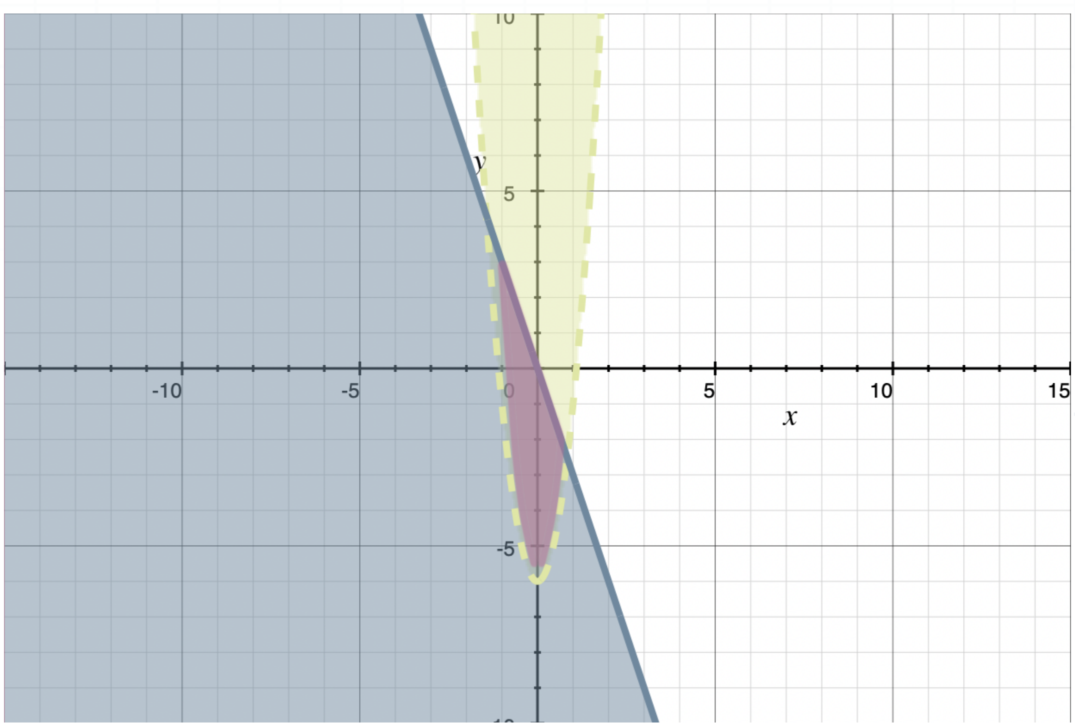
$$0 > 5(0^2) - 6$$

$$0 > -6$$

Because this is a true statement, we shade toward the origin.



The regions where the shading overlaps is the solution to the system of inequalities.



Combinations of functions

In this lesson we'll learn how to combine functions and use combination notation.

Let's say we have two functions, $f(x)$ and $g(x)$. We can find the sum, difference, product, or quotient of f and g, and each of these operations creates a **combination** of the functions. Let's look at each operation and how it's defined.

Sum	$(f+g)(x) = f(x) + g(x)$
Difference	$(f-g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$(f \div g)(x) = \dfrac{f(x)}{g(x)}$

Just as with these same operations on numbers, the order of the functions in addition or multiplication doesn't matter, but the order of the functions in subtraction or division does matter. With subtraction or division of two functions, if the f comes first, then we need to start with $f(x)$; if the g comes first, then we need to start with $g(x)$.

The domain of each of these combinations is given by the intersection of the domain of f and the domain of g. In other words, the combination won't be defined at $x = a$ unless $f(a)$ exists and $g(a)$ exists. And for the quotient in particular to be defined, the denominator can't be 0, which means $g(x)$ can't be 0.

We'll look at composite functions in the next lesson, and the notation for a composite function can look very similar to the notation for a product. Let's look closely here to be sure we see the difference.

Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Composite	$(f \circ g)(x) = f(g(x))$

There's a big difference in meaning between an open circle (for a composite function) and a closed circle (for a product of functions).

Let's do an example of each of the four combination operations.

Example

Find $(f - g)(x)$, if $f(x) = 3x^2 + 2x - 4$ and $g(x) = x^2 - 3x + 2$.

The combination $(f - g)(x)$ means the same thing as $f(x) - g(x)$, so we can find the difference.

$$(f - g)(x) = (3x^2 + 2x - 4) - (x^2 - 3x + 2)$$

$$(f - g)(x) = 3x^2 + 2x - 4 - x^2 + 3x - 2$$

$$(f - g)(x) = 3x^2 - x^2 + 2x + 3x - 4 - 2$$

$$(f - g)(x) = 2x^2 + 5x - 6$$

In the next example, we'll look at the sum of functions.

Example

Find $(f+g)(x)$, if $f(x) = 3x - 4$ and $g(x) = 3x^2 + 5x + 3$.

Remember that $(f+g)(x)$ is the same thing as $f(x) + g(x)$. Therefore,

$$(f+g)(x) = (3x - 4) + (3x^2 + 5x + 3)$$

$$(f+g)(x) = 3x - 4 + 3x^2 + 5x + 3$$

$$(f+g)(x) = 3x^2 + 3x + 5x - 4 + 3$$

$$(f+g)(x) = 3x^2 + 8x - 1$$

It's possible to use names other than f and g for the functions in a combination. Let's use some different function names for the product.

Example

Find $(h \cdot m)(x)$, if $h(x) = 4x - 3$ and $m(x) = -3x^2 - 1$.

The combination $(h \cdot m)(x)$ is the same as $h(x) \cdot m(x)$. Therefore,

$$(h \cdot m)(x) = (4x - 3)(-3x^2 - 1)$$

We can find this product using the FOIL method.

$$(h \cdot m)(x) = -12x^3 - 4x + 9x^2 + 3$$

$$(h \cdot m)(x) = -12x^3 + 9x^2 - 4x + 3$$

Evaluating combinations

Since we already learned how to find the combination of functions, we can now easily evaluate combinations, and there are two ways to do that.

1. We can find the combination of the functions, and then substitute the value at which we want to evaluate the combination.

2. We can evaluate each function individually, and then find the combination of the results.

Let's try a problem where we find the quotient at a particular value.

Example

Find $(b \div w)(3)$, if $w(x) = 2x$ and $b(x) = 3$.

Remember that $(b \div w)(x)$ is the quotient $b(x)/w(x)$, so we could find the combination,

$$(b \div w)(x) = \frac{3}{2x}$$

and then evaluate it at $x = 3$.

$$(b \div w)(3) = \frac{3}{2(3)} = \frac{1}{2}$$

Alternatively, we could have evaluated each function individually at $x = 3$,

$$w(3) = 2(3) = 6$$

$$b(3) = 3$$

and then found the combination of those results.

$$(b \div w)(3) = \frac{3}{6} = \frac{1}{2}$$

Composite functions

In a **composite function**, one function is used as a variable in the other function. For instance, the composite $f(g(x))$ treats $g(x)$ as the variable in $f(x)$, and we can also write $f(g(x))$ as $(f \circ g)(x)$.

We also know that the composites $f(g(x))$ and $g(f(x))$ are different functions. It's possible that $f(g(x)) = g(f(x))$, but that's usually not the case, and we should always treat them as different functions.

Remember, the composite of two functions is not the same as the product of two functions.

Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Composite	$(f \circ g)(x) = f(g(x))$

Let's look at a few examples where we find the composite function.

Example

Find the composite function $(g \circ f)(x)$.

$$g(x) = \frac{2}{x^4}$$

$$f(x) = \sqrt[4]{x-3}$$

To find the composite function $(g \circ f)(x)$, we plug $f(x)$ into $g(x)$, which means that we take the algebraic expression for $f(x)$ and substitute it for x into the algebraic expression for $g(x)$.

$$(g \circ f)(x) = g(f(x)) = \frac{2}{(\sqrt[4]{x-3})^4}$$

$$(g \circ f)(x) = g(f(x)) = \frac{2}{x-3}$$

Here's another example.

Example

Find $h(g(x))$ if $h(x) = 3x^2 - 2$ and $g(x) = x - 4$.

To find the composite function $h(g(x))$, we plug $g(x)$ into $h(x)$, which means that we take the algebraic expression for $g(x)$ and substitute it for x into the algebraic expression for $h(x)$.

$$h(g(x)) = 3(x-4)^2 - 2$$

$$h(g(x)) = 3(x^2 - 8x + 16) - 2$$

$$h(g(x)) = 3x^2 - 24x + 48 - 2$$

$$h(g(x)) = 3x^2 - 24x + 46$$

Evaluating composites

To evaluate the composite function, we start from the inside. For instance, to evaluate $f(g(x))$ at $x = a$, we start on the inside and first find $g(a)$. Then we plug $g(a)$ into f to find $f(g(a))$. In general, we'll use these steps:

1. Evaluate the inside function using the given input value or variable.

2. Use the output obtained in the first step as the input into the outside function.

Let's work through an example.

Example

Find $h(g(1))$ if $h(x) = x^2 - 2x$ and $g(x) = 4x - 3$.

To find the composite function $h(g(x))$, we plug $g(x)$ into $h(x)$, which means that we take the algebraic expression for $g(x)$ and substitute it for x into the algebraic expression for $h(x)$.

$$h(g(x)) = (4x - 3)^2 - 2(4x - 3)$$

$$h(g(x)) = (4x - 3)(4x - 3) - 8x + 6$$

$$h(g(x)) = 16x^2 - 12x - 12x + 9 - 8x + 6$$

$$h(g(x)) = 16x^2 - 32x + 15$$

Now substitute $x = 1$ into composite to find $h(g(1))$.

$$h(g(1)) = 16(1)^2 - 32(1) + 15$$

$$h(g(1)) = 16 - 32 + 15$$

$$h(g(1)) = -1$$

Alternatively, we could have started by finding $g(1)$,

$$g(1) = 4(1) - 3$$

$$g(1) = 1$$

and then plugged this result into $f(x)$ to find $h(g(1))$ as

$$h(g(1)) = 1^2 - 2(1)$$

$$h(g(1)) = 1 - 2$$

$$h(g(1)) = -1$$

Domains of composite functions

In this lesson we'll look at how to find the domain of a composite function.

Remember that the **domain** of a function is the set of x-values where the function is defined. To determine the domain of a composite function, we need to consider the domains of the original functions.

The domain of a composite $f(g(x))$ must exclude all values of x that aren't in the domain of the "inside" function g, and all values of x for which $g(x)$ isn't in the domain of the "outside" function f. In other words, given the composite $f(g(x))$, the domain will exclude all values of x where $g(x)$ is undefined, and all values of x where $f(g(x))$ is undefined.

Therefore, to find the domain of a composite function $f(g(x))$, we'll

1. Find the domain of g.

2. Find the domain of f.

3. Set g equal to any values that are excluded from the domain of f and solve that equation for x.

Any values excluded from the domain of g, as well as any values where g is equivalent to values excluded from the domain of f, will be excluded from the domain of the composite $f(g(x))$.

Let's look at a few examples.

Example

What is the domain of $f \circ g$, if $f(x) = x^2 - 3$ and $g(x) = \sqrt{x + 9}$?

First, find the domain of $g(x)$. The expression $\sqrt{x+9}$ is undefined where $x + 9$ is negative. For example, if $x = -10$, then $x + 9$ is -1. In general, if x is any number less than -9, then $x + 9$ is negative. However, -9 itself is okay, because $\sqrt{-9+9} = 0$. Therefore, the domain of $g(x)$ is all real numbers x such that $x \geq -9$.

The composite function is

$$f(g(x)) = (\sqrt{x+9})^2 - 3$$

$$f(g(x)) = (x + 9) - 3$$

$$f(g(x)) = x + 6$$

For this simple binomial $(x + 6)$, no real numbers are excluded, so its domain is all real numbers. But because the domain of $g(x)$ excludes all $x < -9$, those values of x also have to be excluded from the domain of the composite function $f(g(x))$.

That means the domain of $f(g(x))$ is $x \geq -9$.

Let's try another example.

Example

What is the domain of $f \circ g$?

$$f(x) = \frac{2}{2x + 4}$$

$$g(x) = \frac{3}{x - 5}$$

First, find the domain of $g(x)$. The expression $3/(x - 5)$ is undefined if the denominator is 0. That means $x = 5$ isn't in the domain of $g(x)$. Therefore, the domain of $g(x)$ is all real numbers x such that $x \neq 5$.

The composite function is

$$f(g(x)) = \frac{2}{2\left(\frac{3}{x-5}\right) + 4}$$

$$f(g(x)) = \frac{2}{\left(\frac{6}{x-5}\right) + 4\left(\frac{x-5}{x-5}\right)}$$

$$f(g(x)) = \frac{2}{\left(\frac{6 + 4x - 20}{x-5}\right)}$$

$$f(g(x)) = \frac{2}{\frac{4x - 14}{x - 5}}$$

$$f(g(x)) = 2\left(\frac{x - 5}{4x - 14}\right)$$

$$f(g(x)) = \frac{2(x - 5)}{2(2x - 7)}$$

$$f(g(x)) = \frac{x-5}{2x-7}$$

For this rational function, any numbers that make the denominator 0 are excluded from the domain.

$$2x - 7 = 0 \quad \rightarrow \quad 2x = 7 \quad \rightarrow \quad x = \frac{7}{2}$$

Putting both exclusions together, the domain of the composite is all real numbers except $7/2$ and 5, so

$$f(g(x)) = \frac{x-5}{2x-7}, \; x \neq \frac{7}{2}, \; 5$$

Alternatively, if we didn't need the composite function and only needed its domain, we could first find the domain of $g(x) = 3/(x-5)$ by recognizing that g is undefined when $x - 5 = 0$, which means its domain is $x \neq 5$.

Then we would find the domain of $f(x) = 2/(2x + 4)$ by recognizing that f is undefined when $2x + 4 = 0$, which means its domain is $x \neq -2$.

Given the domain of $f(x)$, we have to exclude from the domain of the composite any values where $g(x) = -2$.

$$\frac{3}{x-5} = -2$$

$$-\frac{3}{2} = x - 5$$

$$-\frac{3}{2} + 5 = x$$

$$x = \frac{7}{2}$$

Merging both constraints, we know that the domain of the composite is all real numbers except $7/2$ and 5.

$$x \neq \frac{7}{2}, 5$$

Decomposing composite functions

Now that we understand how to build a composite function from two other functions, we want to learn to go the other way and decompose a composite into its component functions.

To write a function as a composite of two other functions, we're essentially looking for a "function within a function."

And it's important to remember that there are almost always multiple ways to decompose the composite, and breaking the composite into one pair of functions isn't necessarily better than breaking it into a different pair of functions.

Let's work through an example.

Example

Write $f(x) = \sqrt{x^3 - 3}$ as the composite of two functions.

We're looking for two functions, $g(x)$ and $h(x)$, such that $f(x) = h(g(x))$. To do this, we look for a function inside a function within $f(x)$.

If we notice that the expression $x^3 - 3$ is inside the square root, it's fairly straightforward to decompose the function into

$$g(x) = x^3 - 3 \text{ and } h(x) = \sqrt{x}$$

Alternatively, we could also have decomposed the composite into

$$g(x) = x^3 \text{ and } h(x) = \sqrt{x-3}$$

Let's look at another example.

Example

Write $f(x) = (x-1)^3 + 4(x-1)^2 - 3(x-1) + 5$ as the composite of two functions.

We're looking for two functions, $g(x)$ and $h(x)$, such that $f(x) = h(g(x))$. To do this, we look for a function inside a function within $f(x)$.

If we notice that the expression $x-1$ is raised to power of 3, 2, 1, and 0, it's fairly straightforward to decompose the function into

$$g(x) = x - 1 \text{ and } h(x) = x^3 + 4x^2 - 3x + 5$$

Alternatively, we could have also decomposed the composite into

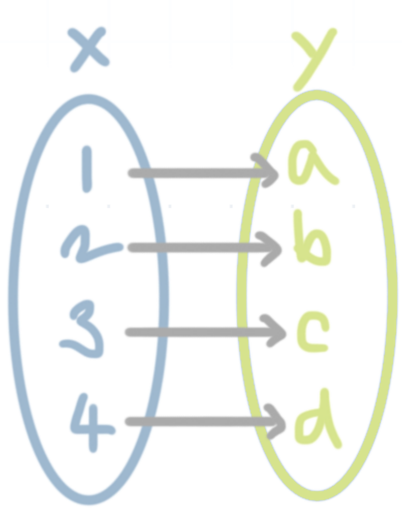
$$g(x) = (x-1)^3 + 4(x-1)^2 - 3(x-1) \text{ and } h(x) = x + 5$$

One-to-one functions and the Horizontal Line Test

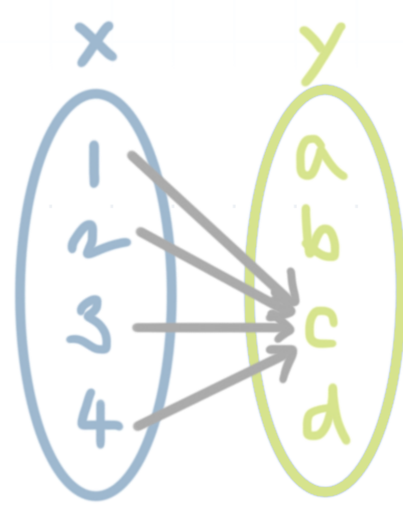
In this lesson we'll talk about how to determine whether a relation represents a one-to-one function. If a relation is a **function**, then it has exactly one y-value in its range for each x-value in its domain. If a function is **one-to-one**, it also has exactly one x-value in its domain for each y-value in its range.

The relation shown on the left below is a one-to-one function because each input corresponds to exactly one output. The relation on the right represents a **many-to-one function** because multiple inputs correspond to one output. This case still represents a function because each input corresponds to precisely one output.

One-to-one function



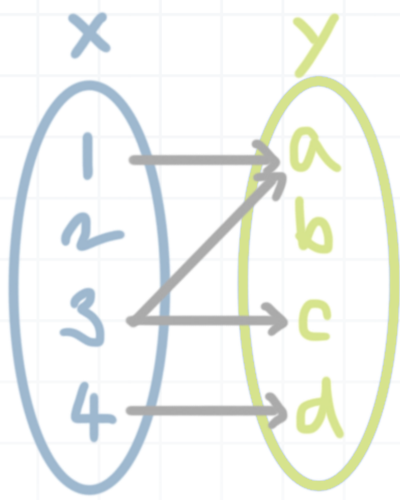
Many-to-one function

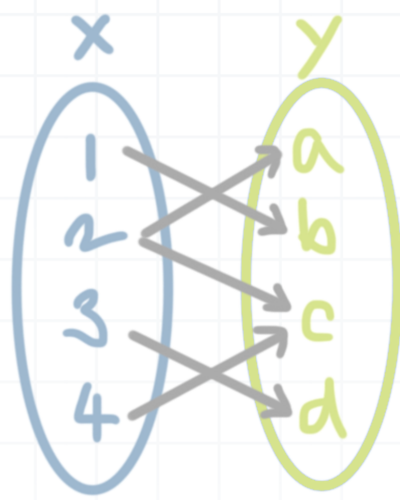


In the diagram below, both cases are relations, but not functions. They represent **one-to-many** and **many-to-many** relations, respectively, which are not functions because some of the inputs correspond to more than one output.

One-to-many relation

Many-to-many relation





We care about one-to-one functions because they're the only type of function that have a defined inverse (a concept we'll talk about in the next lesson). If a function is not one-to-one, then some restrictions on its domain will be needed to make it invertible.

The first method we'll use to check whether or not a function is one-to-one is the Horizontal Line Test.

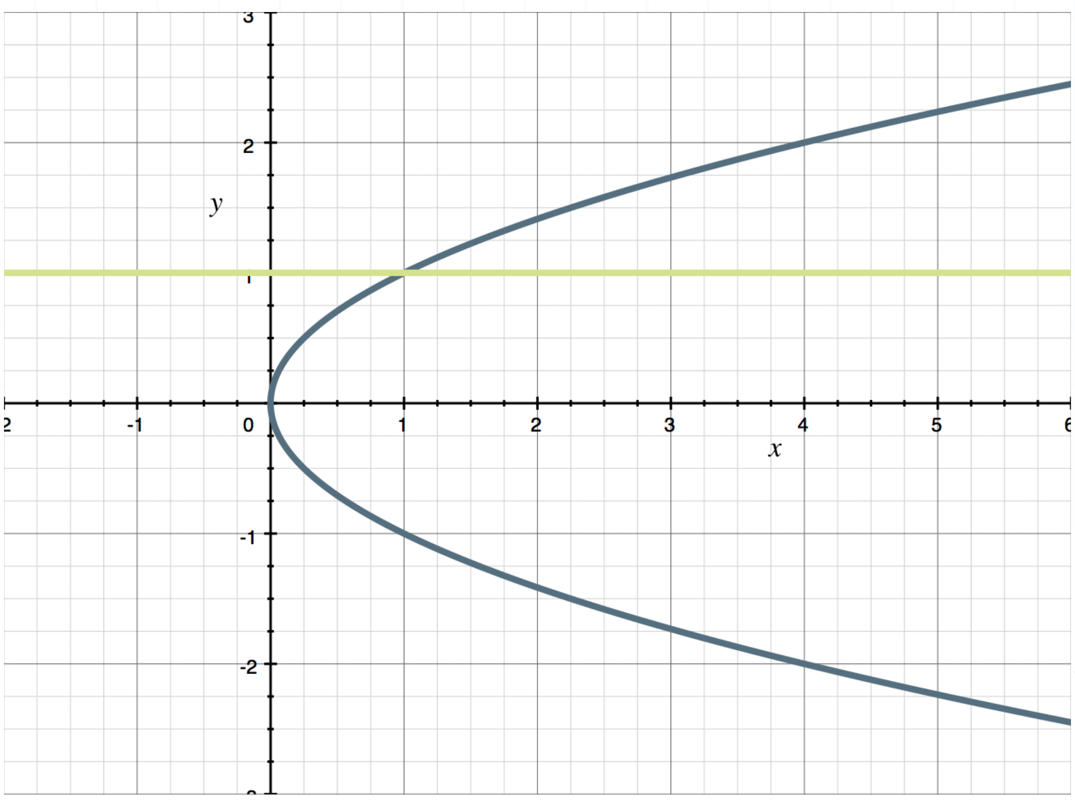
One-to-one functions and the Horizontal Line Test

Remember that we've already talked about the Vertical Line Test, which is a test we use to tell us whether or not a graph represents a function. A graph passes the Vertical Line Test if no two points of the graph have the same x-coordinate (if no vertical line intersects the graph at more than one point).

In the same way that the Vertical Line Test tells us whether or not a graph represents a function, the **Horizontal Line Test** tells us whether or not the function represented by a graph is one-to-one. A graph passes the Horizontal Line Test if no two points of the graph have the same y

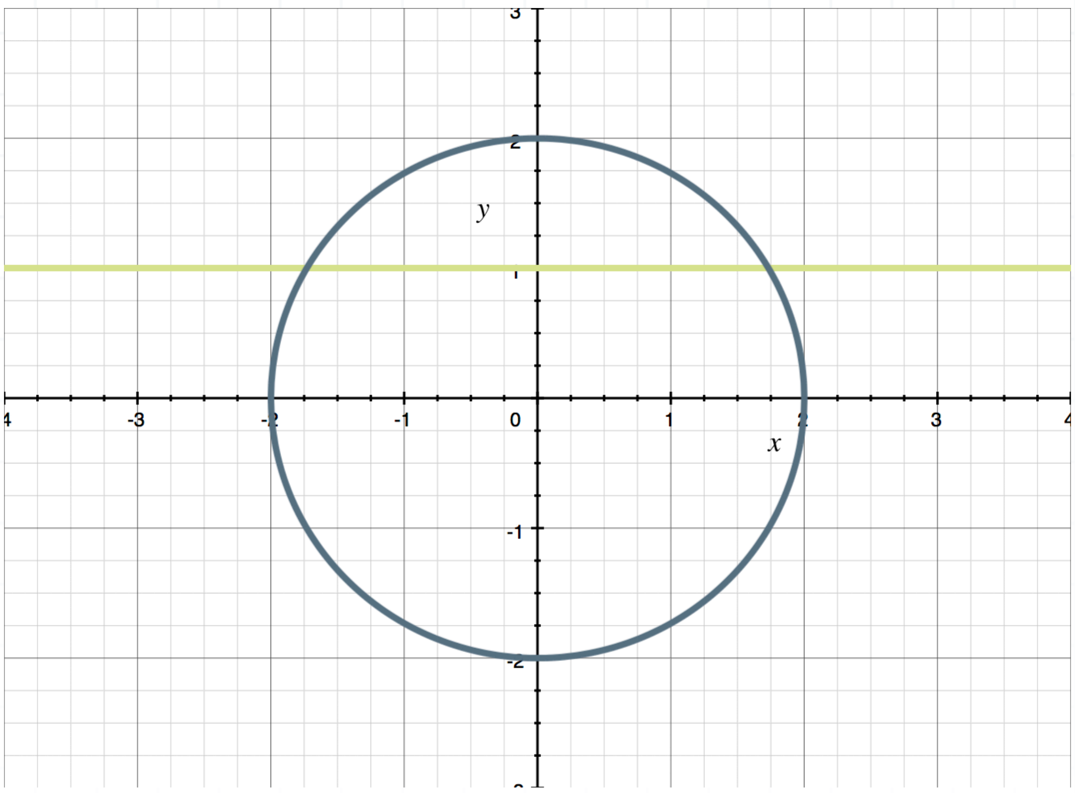
-coordinate (if no horizontal line intersects the graph at more than one point).

The graph below passes the Horizontal Line Test, because no horizontal line intersects it at more than one point. Note, however, that this particular graph doesn't represent *any* function (one-to-one or otherwise), because it fails the Vertical Line Test. This shows that even if a graph passes the Horizontal Line Test, it doesn't necessarily represent a one-to-one function.



A graph represents a **one-to-one function** if and only if it passes the Vertical Line Test *and* the Horizontal Line Test. Passing the Vertical Line Test ensures that the graph represents a *function*, and then also passing the Horizontal Line Test ensures that the function it represents is *one-to-one*.

The next graph doesn't pass the Horizontal Line Test, because any horizontal line between the line $y = -2$ and the line $y = 2$ intersects it at two points.

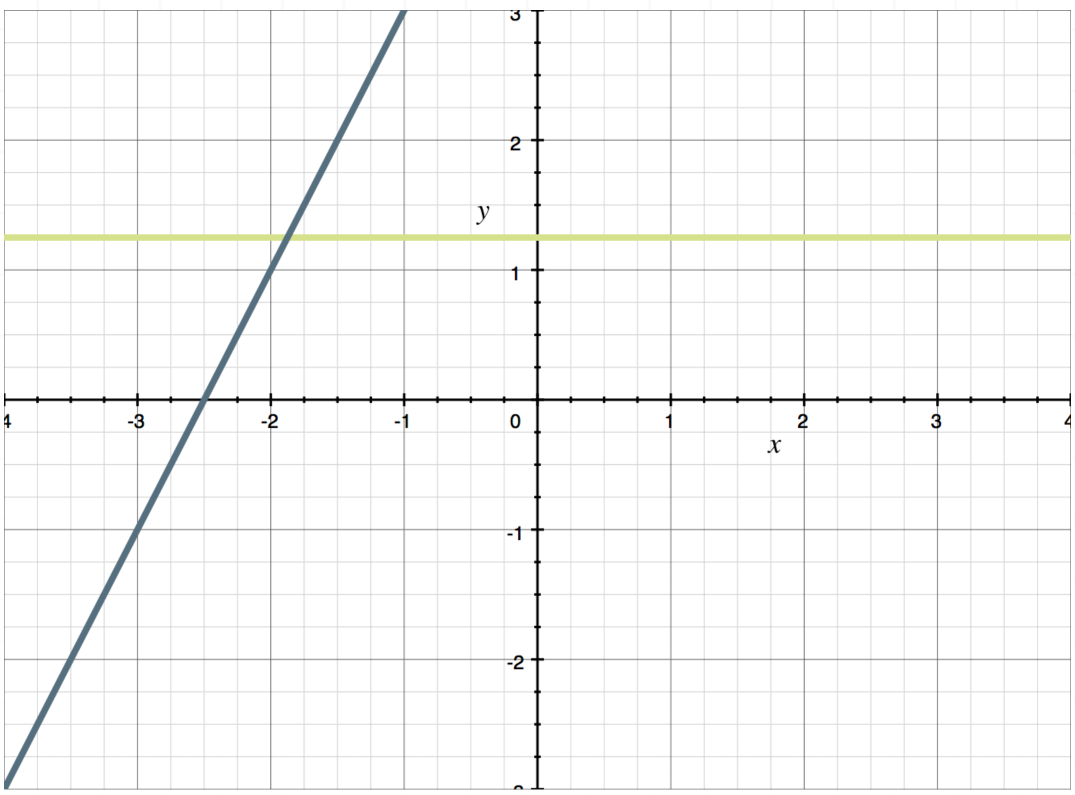


No vertical line is the graph of a function (one-to-one or otherwise), because all the points on a vertical line have the same x-coordinate, which means that a vertical line fails the Vertical Line Test.

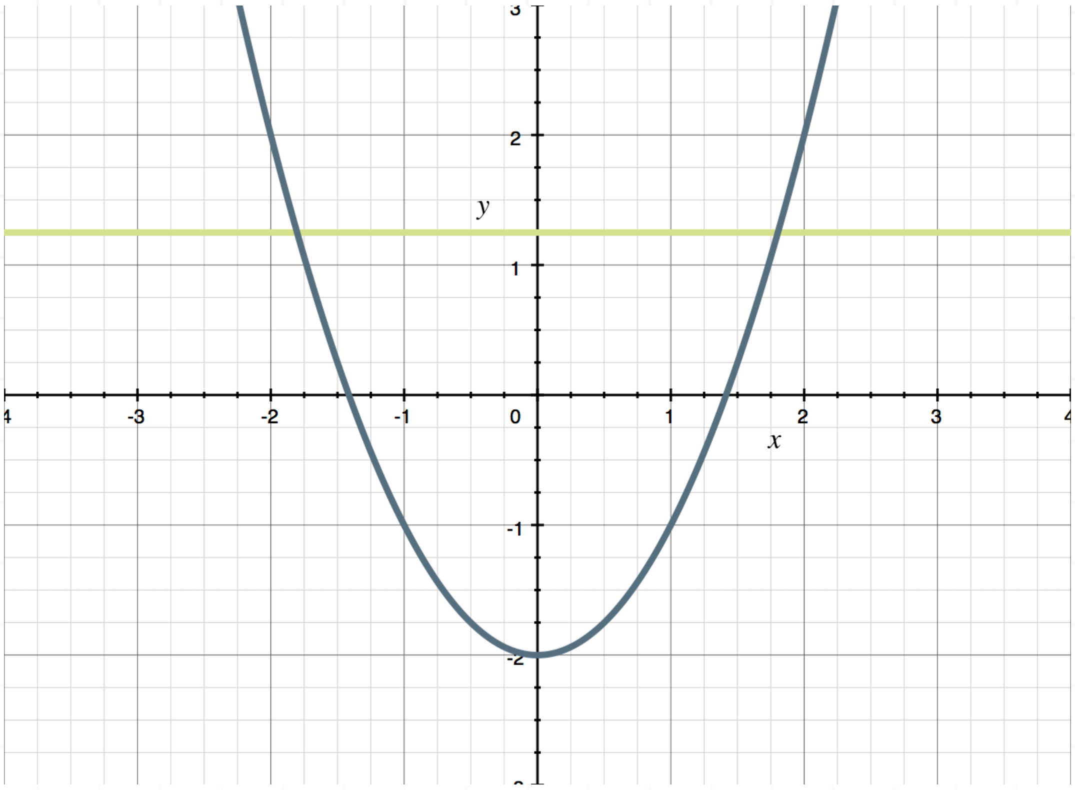
Every horizontal line is the graph of a function, because all the points on a horizontal line have different x-coordinates, which means that it passes the Vertical Line Test. However, no horizontal line is the graph of a one-to-one function, because all the points on a horizontal line have the same y-coordinate, which means that a horizontal line fails the Horizontal Line Test.

Every line that's neither vertical nor horizontal is the graph of a one-to-one function, because no two points on such a line have the same x-coordinate (which means that the line passes the Vertical Line Test) and no two points on such a line have the same y-coordinate (which means that the line passes the Horizontal Line Test as well).

A look at this next graph tells us that there's no horizontal line that intersects the graph at more than one point, so the relation is a function.



On the other hand, no quadratic function is one-to-one. A look at the next graph shows us that it's easy to find a horizontal line that intersects the graph at more than point, thereby proving that the function is not one-to-one.



This is one reason why it's good to have an idea of what the graphs of various "function families" look like. If we're familiar with what the graphs of a certain group of functions look like, then we can think about the graph of a function of that type and decide whether it represents a one-to-one function. For example, the graph of a quadratic function is a parabola so there are horizontal lines that intersect its graph at two points. That's why quadratic functions are never one-to-one.

One-to-one functions algebraically

Another method of checking for a one-to-one function is through the use of algebra to determine whether $f(a)=f(b)$ implies $a=b$, which is true if and only if f is a one-to-one function.

Say we want to know if $f(x)=\sqrt{x-2}$ is one-to-one without drawing or visualizing the graph.

Then we could use algebra to determine whether $f(a) = f(b)$ implies that $a = b$. We know that $f(a) = \sqrt{a-2}$ and $f(b) = \sqrt{b-2}$, so we could say

$$f(a) = f(b)$$

$$\sqrt{a-2} = \sqrt{b-2}$$

$$(\sqrt{a-2})^2 = (\sqrt{b-2})^2$$

$$a - 2 = b - 2$$

$$a = b$$

So $f(x)$ is a one-to-one function.

Let's try another one of those.

Example

Show that the function is one-to-one by showing that $f(a) = f(b)$ leads to $a = b$.

$$f(x) = \frac{x-3}{x+4}$$

We'll start by substituting a for x, and then setting the resulting expression equal to the expression we get when we substitute b for x.

$$\frac{a-3}{a+4} = \frac{b-3}{b+4}$$

$$(a-3)(b+4)=(b-3)(a+4)$$

$$ab+4a-3b-12=ab+4b-3a-12$$

$$4a-3b=4b-3a$$

$$4a+3a=4b+3b$$

$$7a=7b$$

$$a=b$$

This means $f(x)$ is a one-to-one function.

Let's try another example.

Example

Show that $f(x)$ is not one-to-one by showing that $f(a)=f(b)$ does not imply that $a=b$.

$$f(x)=x^2$$

All we need is one case to show that $f(a)=f(b)$ does not imply that $a=b$. That means we can choose one example where $f(a)=f(b)$ but $a\neq b$. Consider the case when $a=2$ and $b=-2$. Then $a\neq b$ but

$$f(a)=f(2)=2^2=4$$

and

$$f(b) = f(-2) = (-2)^2 = 4$$

Therefore, $f(2) = f(-2)$ but $2 \neq -2$. Since we've found one case, the function is not one-to-one.

Inverse functions

In this lesson we'll look at the definition of an inverse function and how to find a function's inverse.

If we remember from the last lesson, a function is invertible (has an inverse) if and only if it's one-to-one. Now let's look a little more into how to find an inverse and what an inverse does.

When we have a function with points $(x, f(x))$, the inverse function will have points $(f(x), x)$. We could think of the inverse of a function f as the function that "undoes" f. If we first evaluate $f(x)$ at some x in the domain of f, and then evaluate the inverse of f at that value of $f(x)$, what we get is just x (the input we started with). The inverse of a function $f(x)$ is written as $f^{-1}(x)$. Because f^{-1} "undoes" f, we could think of the function $g(x) = x$ as the composite of $f^{-1}(x)$ and $f(x)$, because

$$g(x) = x = f^{-1}(f(x))$$

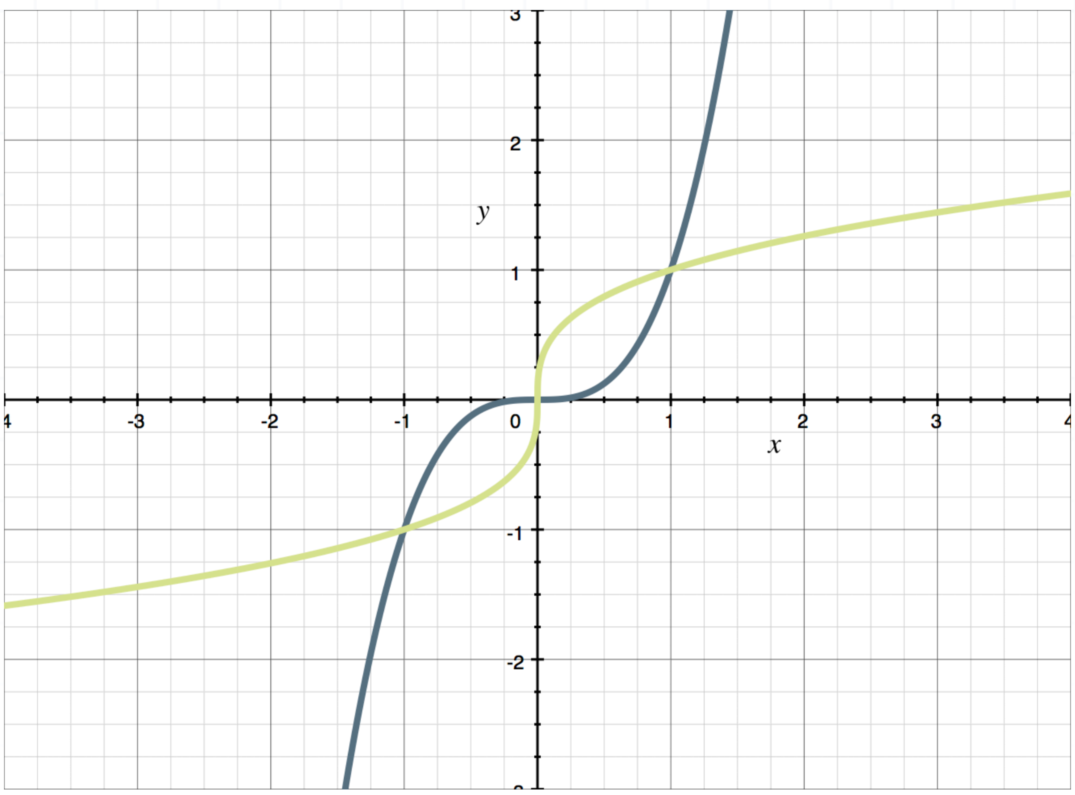
For example, if $g(x)$ and $g^{-1}(x)$ are inverses of one another, then the tables below would give sets of points from each.

x	g(x)
1	4
4	8
10	12
16	2

x	$g^{-1}(x)$
4	1
8	4
12	10
2	16

Now let's look at the graphs of a function and its inverse. Look at the graph of the function $f(x) = x^3$ (in blue) and the graph of its inverse (in green). Notice that in order to "get back to x" from $f(x)$ (to get back to x from x^3), we have to take the cube root of $f(x)$, because

$$x = (x^3)^{\frac{1}{3}} = \sqrt[3]{x^3}$$



Notice that the x- and y-coordinates of the points on the blue curve are the y- and x-coordinates, respectively, of the points on the green curve, that is, the coordinates of the points of the graph of $f^{-1}(x)$ have switched places with the coordinates of the points of the graph of $f(x)$. Now let's look at how to calculate an inverse algebraically.

The inverse is not an exponent

It's worth mentioning the most common mistake students make when studying inverse functions, which is confusing the "-1" notation for an exponent. That notation indicates the inverse function, and it's not an exponent, so

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

How to find the inverse

Here are the steps we'll use to find the inverse function, assuming we're given the function $f(x)$ and want to find its inverse, $f^{-1}(x)$.

1. Replace $f(x)$ with y to make the process easier.
2. Replace every x with a y and every y with an x.
3. Solve the equation from Step 2 for y.
4. Replace y with $f^{-1}(x)$ to show that we've found the inverse function.
5. Double-check to verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true.

Domain and range

Lastly, keep in mind that the range of $f(x)$ is the domain of $f^{-1}(x)$, and that the domain of $f(x)$ is the range of $f^{-1}(x)$.

If we have a function and we need to find the domain and range of its inverse, remember that,

1. if the function is one-to-one, we write the range of the original function as the domain of the inverse, and the domain of the original function as the range of the inverse.

2. if we need to restrict the domain of the original function to make it one-to-one, this restricted domain then becomes the range of the inverse function.

Let's work through an example where we find the inverse function.

Example

What is the inverse of the function?

$$f(x) = \frac{2}{3}x - 4$$

First, notice that this function is invertible, because its graph is a line that's neither vertical nor horizontal (so its graph passes both the Vertical Line Test and the Horizontal Line Test, which means that the function is one-to-one).

To find the inverse of this function, first replace $f(x)$ with the variable y.

$$y = \frac{2}{3}x - 4$$

Next, switch x with y.

$$x = \frac{2}{3}y - 4$$

Now solve for y.

$$x + 4 = \frac{2}{3}y$$

$$\frac{3}{2}(x + 4) = \frac{3}{2}\left(\frac{2}{3}y\right)$$

$$\frac{3}{2} \cdot x + \frac{3}{2} \cdot 4 = \frac{3}{2} \cdot \frac{2}{3}y$$

$$\frac{3}{2}x + 6 = y$$

Now we can write the inverse function by replacing y with $f^{-1}(x)$ (and then turning the equation around so that $f^{-1}(x)$ is on the left side).

$$f^{-1}(x) = \frac{3}{2}x + 6$$

Let's do one more example.

Example

Find the inverse of the function.

$$g(x) = \frac{x}{x - 3}$$

First replace $g(x)$ with y.

$$y = \frac{x}{x-3}$$

At this point in finding the inverse of the function in the other example, we first switched x with y, and then solved for y. When we use algebra to get the inverse of a function, we could just as well first solve for x, and then switch x with y, so we'll do it that way here.

$$y(x-3) = x$$

$$xy - 3y = x$$

$$xy - x = 3y$$

$$x(y-1) = 3y$$

$$x = \frac{3y}{y-1}$$

Now switch x with y.

$$y = \frac{3x}{x-1}$$

Finally, write the inverse function by replacing y with $g^{-1}(x)$.

$$g^{-1}(x) = \frac{3x}{x-1}$$

Finding a function from its inverse

The nice thing about functions and their inverses is that if we know two points, say (a_1, b_1) and (a_2, b_2), of the inverse of a function $f(x)$, then we also know that two of the points of $f(x)$ are (b_1, a_1), and (b_2, a_2). This works out very nicely if we know two points of the inverse of a linear function and we want to find that linear function.

Now we may be wondering if the inverse of a linear function is also a linear function, and the answer to this question is Yes.

To find $f^{-1}(x)$, we can first replace $f(x)$ with y, then switch x with y,

$$y = mx + b$$

$$x = my + b$$

solve for y,

$$x - b = my$$

$$\frac{x - b}{m} = y$$

$$\frac{1}{m} \cdot x - \frac{b}{m} = y$$

and finally replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \frac{1}{m} \cdot x - \frac{b}{m}$$

Let's look at an example.

Example

Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(3) = 4$$

$$f^{-1}(-1) = 5$$

This means that $(3,4)$ and $(-1,5)$ are points of the function $f^{-1}(x)$, which is the inverse of $f(x)$. Therefore, $(4,3)$ and $(5,-1)$ are points of $f(x)$. Now we can use these points on the line that represents $f(x)$ to find the equation of the line. Let's begin by finding the slope m.

$$m = \frac{3-(-1)}{4-5} = \frac{4}{-1} = -4$$

Let's find the y-intercept. We can use the slope we just found ($m = -4$) and the slope-intercept form of the equation of a line ($y = mx + b$), together with the coordinates of one point on the line, to solve for b. Let's use the point $(4,3)$.

$$3 = -4(4) + b$$

$$3 = -16 + b$$

$$3 + 16 = b$$

$$19 = b$$

The equation of the line that represents $f(x)$ is then

$$f(x) = -4x + 19$$

If we like, we can also use the points of the inverse function to find the equation of the line that represents $f^{-1}(x)$ first, and then use that to find $f(x)$.

Example

Use the given information to find $f(x)$ if $f^{-1}(x)$ is a linear function.

$$f^{-1}(-2) = 8$$

$$f^{-1}(-5) = 14$$

Let's begin by finding the equation of the line that represents $f^{-1}(x)$.

Use the points $(-2,8)$ and $(-5,14)$ to find the slope of that line.

$$m = \frac{14 - 8}{-5 - (-2)} = \frac{6}{-3} = -2$$

Let's use the point-slope form of the equation of a line ($y - y_1 = m(x - x_1)$) to solve for the y-intercept this time (although we could still use the slope-intercept form to solve for the y-intercept). To get the point-slope form, we need the slope and the coordinates of one point. We know that $m = -2$, and we can use the point $(-2,8)$.

$$y - y_1 = m(x - x_1)$$

$$y - 8 = -2(x - (-2))$$

$$y - 8 = -2(x + 2)$$

$$y - 8 = -2x - 4$$

$$y = -2x + 4$$

Remember, this is the equation of the line that represents $f^{-1}(x)$. To get $f(x)$, we'll switch x with y, then solve for y, and finally replace y with $f(x)$.

$$x = -2y + 4$$

$$x - 4 = -2y$$

$$-\frac{1}{2}(x - 4) = -\frac{1}{2}(-2y)$$

$$-\frac{1}{2}x + 2 = y$$

$$f(x) = -\frac{1}{2}x + 2$$

As we can see, there's more than one way to solve these types of problems, so we should just use whichever method we're most comfortable with.

What is a logarithm?

When we first start learning about logarithms, it's helpful to think about how they're related to exponents, since exponents are something we already understand.

Exponents vs. logarithms

Remember that an exponent tells us how many times to multiply the base by itself. In other words, in 2^3, the exponent of 3 tells us that the base 2 should be multiplied three times. And that tells us how to find the value of 2^3:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

We already know that this is what exponents do. What we haven't learned yet is what to do when we have something like this instead:

$$2^x = 8$$

Logarithms are what we would use to solve for x in this equation, because logarithms let us solve for the value of a variable that appears in an exponent. **Logarithms** tell us how many times we multiply one number by itself in order to get a different number. So when we already have the base (in this case 2), and we have the result (in this case 8), logarithms (or "logs," for short) tell us what the value of the exponent needs to be in order to make the equation true.

To express the solution to the equation $2^x = 8$ (which is called an **exponential equation** because the exponent is/includes a variable) in log form, we write

$$\log_2(8) = 3$$

In the equation $\log_2(8) = 3$, the 2 is the **base** and the 8 is the **argument** of the log function. The term "argument" isn't used exclusively with the log function. In fact, it can be used with any function, to mean the input. For example, the argument of the function f in $f(-5x)$ is $-5x$, and the argument of the function g in $g(7)$ is 7. We can read $\log_2(8) = 3$ in either of the following two ways:

"log base 2 of 8 is 3," or

"the log of 8 with base 2 is 3," or

"the base-2 log of 8 is 3"

Realize that the "base" of a log is the same as the "base" in an expression of the form a^b. Remembering that can help us when we're converting back and forth between logs and exponents.

Not multiplication

It's also worth pointing out that, in the equation $\log_2(8) = 3$, $\log_2$ is not a number that's supposed to be multiplied by 8. It's tempting to think that the parentheses in the expression $\log_2(8)$ mean multiplication, but they don't. Instead, the big number that comes after the base is the "argument" of the log function.

Think back to function notation, where we talked about functions like $f(x) = 2x + 1$. Remember that the function notation $f(x)$ doesn't mean that f should be multiplied by x. Instead, it means either "the value of the function f at x" or "f is a function of x."

And the same is true of logarithms. In $\log_2(8) = 3$, the notation $\log_2(8)$ means "the value of the log function at 8."

The general log rule

This basic log rule that relates exponents to logs can be written as follows:

Given the exponential equation $a^x = y$,

the associated logarithmic equation is $\log_a(y) = x$,

and vice versa.

Let's do an example where we convert from one form to the other.

Example

Convert the equation $5^4 = 625$ to its logarithmic form.

If we match the quantities in the equation $5^4 = 625$ to those in the equation $a^x = y$ from the general log rule, we see that

$$a = 5$$

$$x = 4$$

$$y = 625$$

Plugging these values into the equation $\log_a(y) = x$ from the general log rule, we get

$$\log_5(625) = 4$$

Realize that we could have just as easily started with the equation $\log_5(625) = 4$; identified a, x, and y; and converted it to the form $a^x = y$.

In both forms, the equation says "The multiplication of 5 four times gives 625."

Let's do an example where we solve a log equation for a variable.

Example

Solve the equation.

$$\log_2(16) = x$$

This log equation is asking "How many times do we have to multiply 2 in order to get 16?" Using the general log rule, we can rewrite the equation as

$$2^x = 16$$

We know that

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

so the solution to the equation is $x = 4$.

We can solve log problems regardless of which quantity is unknown. In the last example, $\log_2(16) = x$, the value of $\log_2(16)$ was the unknown quantity. If instead the base were the unknown quantity, we would've had

$$\log_x(16) = 4$$

$$x^4 = 16$$

$$x = 2$$

Or if instead the argument of the log function were the unknown quantity, we would've had

$$\log_2(x) = 4$$

$$2^4 = x$$

$$16 = x$$

Common bases and restricted values

In the last section, we looked at logarithmic equations written as

$$\log_8(64) = 2$$

Remember that, in this case, the number 8 is called the "base." There are some bases that we use much more often than all others, so we need to give them some special attention.

Base 10

Sometimes we'll see logs written with no base at all, like

$$\log(100) = 2$$

When there's no subscript on the "log" (to indicate the base), it means that we're dealing with the **common logarithm**, which always has a base of 10. Common logs are used so much in the real world that we've decided to save ourselves some time and simplify $\log_{10}$ to just $\log$, and understand that base 10 is implied. This means that we can rewrite the equation $\log(100) = 2$ in either of the following ways:

$$\log_{10}(100) = 2$$

$$10^2 = 100$$

Base e

Perhaps the most basic logarithms are those that have a base called "e." e is known as Euler's number, and it's equal to about 2.72. Here are some more digits of e.

$$e \approx 2.71828182845904523536028747135\ldots$$

Like π, e is an irrational number, so it has an infinite number of digits to the right of the decimal point and they don't repeat. Logarithms to base e are called **natural logarithms**, and we write them with $\ln$ (note the "n" for "natural") instead of with log. In other words,

$$\log_e(x) = \ln(x)$$

Because e is the base, whenever we have a natural log, we're asking "How many times does e need to appear as a factor in order to get a certain result?". For instance,

$$\ln(54.598) = \log_e(54.598) \approx 4, \text{ because } e^4 \approx 2.71828^4 \approx 54.598$$

Example

Solve the equation for x.

$$\log(1{,}000) = x$$

We can use the general rule to rewrite the equation.

$$\log(1{,}000) = x$$

$$10^x = 1{,}000$$

$$x = 3$$

Restricted values

For logarithms to any base, there are two rules we always have to follow.

Let's remember the general rule that relates an exponential equation to the associated logarithmic equation:

Given the exponential equation $a^x = y$,

the associated logarithmic equation is $\log_a(y) = x$,

and vice versa.

In a logarithmic expression $\log_a(y)$,

the base a must be positive (and not equal to 1), and

the argument y must also be positive.

If we don't follow these rules, we can run into trouble and end up with equations that aren't true.

Evaluating logs

We already know how to evaluate simple logs like $\log_2 8$, because we understand that this is asking us the question "To what power do we have to raise 2 in order to get 8?" To answer it, we only need to solve the exponential equation

$$2^x = 8$$

We know that $2^3 = 8$, so it's easy to see that $x = 3$. Also, all the quantities in this problem are whole numbers, but log problems can be more complicated than this, and that's what we want to talk about here.

The base is greater than the argument

In all our examples so far, the argument has been greater than the base. In $\log_2(8)$, the base is 2 and the argument is 8, so argument > base. But what happens when the base is greater than the argument?

$$\log_{27}(3)$$

To evaluate this expression, we'll solve the exponential equation

$$27^x = 3$$

To solve this equation, it would help if we could rewrite it in such a way that we have the same base on both sides. We know that 27 is the same as 3^3, so we'll use that to rewrite the equation.

$$(3^3)^x = 3$$

Now remember the general rule of exponents which tells us that $(a^b)^c = a^{b\cdot c}$. Therefore, our equation becomes

$$3^{3x} = 3$$

$$3^{3x} = 3^1$$

If the bases are equal, then the exponents must also be equal in order for the equation to be true.

$$3x = 1$$

$$x = \frac{1}{3}$$

So we can see that

$$27^{\frac{1}{3}} = 3$$

In terms of logs, this translates to

$$\log_{27}(3) = \frac{1}{3}$$

Let's try another example.

Example

Find the value of the expression.

$$\log_{243}(3)$$

To evaluate this expression, we'll solve the exponential equation

$$243^x = 3$$

To do this, we'll express 243 as a power of 3.

$$243 = 3 \cdot 81 = 3 \cdot 9 \cdot 9 = 3^1 \cdot 3^2 \cdot 3^2 = 3^5$$

$$(3^5)^x = 3$$

$$3^{5x} = 3$$

$$3^{5x} = 3^1$$

Since the bases are equal, the only way to make this equation true is for the exponents to also be equal.

$$5x = 1$$

$$x = \frac{1}{5}$$

So we can see that

$$\log_{243}(3) = \frac{1}{5}$$

The argument is a fraction

Sometimes the argument will be a fraction, like this:

$$\log_2\left(\frac{1}{64}\right)$$

We'll evaluate this expression by solving the exponential equation

$$2^x = \frac{1}{64}$$

To do this, we'll express 64 as a power of 2.

$$2^x = \frac{1}{2^6}$$

$$2^x = 2^{-6}$$

The bases are equal, so the exponents must be equal. Therefore, $x = -6$ and

$$\log_2\left(\frac{1}{64}\right) = -6$$

Let's try another example.

Example

Find the value of the expression.

$$\log_5\left(\frac{1}{625}\right)$$

We'll evaluate this expression by solving the exponential equation

$$5^x = \frac{1}{625}$$

To do this, we'll express 625 as a power of 5.

$$625 = 5 \cdot 125 = 5 \cdot 5 \cdot 25 = 25 \cdot 25 = 5^2 \cdot 5^2 = 5^4$$

$$5^x = \frac{1}{5^4}$$

$$5^x = 5^{-4}$$

The bases are equal, so the exponents must be equal. Therefore, $x = -4$ and

$$\log_5\left(\frac{1}{625}\right) = -4$$

This method for evaluating logs will always work when we can solve an exponential equation that can be converted to a form in which the base is the same on both sides. Let's show a summary of the steps with one last example, in which we evaluate $\log_{32}(16)$.

$$\log_{32}(16)$$

$$32^x = 16$$

$$(2^5)^x = 2^4$$

$$2^{5x} = 2^4$$

$$5x = 4$$

$$x = \frac{4}{5}$$

So we can say

$$\log_{32}(16) = \frac{4}{5}$$

Let's try one more example.

Example

Find the value of the expression.

$$\log_{\frac{1}{625}}(5)$$

We'll evaluate this expression by solving the exponential equation

$$\left(\frac{1}{625}\right)^x = 5$$

$$\frac{1}{625} = \frac{1}{5^4} = 5^{-4}$$

$$(5^{-4})^x = 5$$

$$5^{-4x} = 5^1$$

Since the bases are equal, we can equate the exponents.

$$-4x = 1$$

$$x = -\frac{1}{4}$$

Therefore,

$$\log_{\frac{1}{625}}(5) = -\frac{1}{4}$$

Laws of logarithms

Up until now, we've always enclosed the argument of the log function in parentheses, just as we enclose the argument of any function in parentheses. However, it's common to eliminate the parentheses in the case of a log function, so we'll sometimes do that as well, especially when the meaning is clear without the parentheses.

For instance, here are four simple laws of logarithms we can use to simplify logarithmic expressions, each written without the parentheses around the argument.

$$\log_b 1 = 0 \qquad\qquad \log_b b^x = x$$

$$\log_b b = 1 \qquad\qquad b^{\log_b x} = x$$

Let's look at an example where we use these log rules, before diving in depth into the product, quotient, and power rules for logarithms.

Example

Simplify the expression.

$$\log \frac{1}{1{,}000}$$

If we rewrite $1{,}000$ as 10^3, and recognize that the log has base 10, then the expression becomes

$$\log_{10} \frac{1}{10^3}$$

$$\log_{10} 10^{-3}$$

Comparing this to the law of logs $\log_b b^x = x$ we saw earlier, we can see that the expression simplifies to just -3. So

$$\log \frac{1}{1{,}000} = -3$$

The product rule

When the argument of a log function is a product of two quantities, the value of that log can be written as the sum of the values of the logs of those two quantities.

$$\log_a(xy) = \log_a x + \log_a y$$

Keep in mind that this rule can be used in both directions. Given the expression $\log_a(xy)$, we can expand it to $\log_a x + \log_a y$. And given the expression $\log_a x + \log_a y$, we can condense it to $\log_a(xy)$.

The bases of two log functions must be equal in order to use the product rule. In other words, we can use the product rule to condense $\log_a x + \log_a y$, but we can't use it to condense $\log_a x + \log_b y$.

Example

Write the expression as a rational number if possible, or if not, as a single logarithm.

$$\log_4 64 + \log_4 16$$

First, we can use the rule

$$\log_a x + \log_a y = \log_a(xy)$$

because the two log functions have the same base.

$$\log_4 64 + \log_4 16$$

$$\log_4(64 \cdot 16)$$

$$\log_4(1{,}024)$$

To simplify further, use the relationship between exponents and logarithms,

If $\log_a(y) = x$ then $a^x = y$

So if we let $x = \log_4 1{,}024$, then

$$4^x = 1{,}024$$

Now we'll factor $1{,}024$ using 4 as each factor:

$$1{,}024 = 4 \cdot 256$$

$$1{,}024 = 4 \cdot 4 \cdot 64$$

$$1{,}024 = 4 \cdot 4 \cdot 4 \cdot 16$$

$$1{,}024 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

Therefore, $1{,}024 = 4^5$, so

$$4^x = 4^5$$

$$x = 5$$

The quotient rule

When the argument of a log function is a quotient of two quantities, the value that log can be written as the difference of the values of the logs of those two quantities.

$$\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

As before, this rule can be used in both directions. Given the expression $\log_a(x/y)$, we can expand it to $\log_a x - \log_a y$. And given the expression $\log_a x - \log_a y$, we can condense it to $\log_a(x/y)$.

Just as with the product rule, the bases of two log functions must be equal in order to use the quotient rule. In other words, we can use the quotient rule to condense $\log_a x - \log_a y$, but we can't use it to condense $\log_a x - \log_b y$.

The power rule

When the argument of a log function is a power function, the exponent can be pulled out in front of the log function.

$$\log_a(x^n) = n\log_a x$$

As before, this rule can be used in both directions. Given the expression $\log_a(x^n)$, we can rewrite it as $n\log_a x$. And given the expression $n\log_a x$, we can rewrite it as $\log_a(x^n)$.

Alternatively, we can write the power rule as

$$\log_{a^n}(x) = \frac{1}{n}\log_a x$$

As before, this rule can be used in both directions. Given the expression $\log_{a^n}(x)$, we can rewrite it as $(1/n)\log_a x$. And given the expression $(1/n)\log_a x$, we can rewrite it as $\log_{a^n}(x)$.

Combining log rules

The product, quotient, and power rules for logarithms, as well as the general rule for logs (and the change of base formula we'll cover in the next lesson), can all be used together, in any combination, in order to solve log problems.

Let's look at an example that requires us to use two of these log rules.

Example

Write the expression as a single logarithm.

$$\log_3 14 - 2\log_3 5$$

First we can use the power rule for logs

$$\log_a(x^n) = n\log_a x$$

on the second term to get

$$\log_3 14 - \log_3 5^2$$

$$\log_3 14 - \log_3 25$$

Because the bases of these logs are the same, we can use the quotient rule for logs

$$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$$

to get

$$\log_3 \frac{14}{25}$$

Laws of natural logs

All of the rules that we've just learned for manipulating logarithms apply to natural logs.

General rule for logs

The general rule for natural logs is:

Given the exponential equation $e^x = y$,

the associated logarithmic equation is $\log_e(y) = x$,

and vice versa.

Of course, $\log_e$ is the same as the natural log, $\ln$. So we can rewrite the general rule for natural logs:

Given the exponential equation $e^x = y$,

the associated logarithmic equation is $\ln(y) = x$,

and vice versa.

Product rule

The product rule for logs is

$$\log_a(xy) = \log_a x + \log_a y$$

In terms of natural logarithms, this rule is

$$\ln(xy) = \ln x + \ln y$$

Let's use the same example of the product rule as in the previous lesson, but this time with natural logs instead of with logs to base 4.

Example

Simplify the expression.

$$\ln 64 + \ln 16$$

First, we can use the product rule.

$$\ln(xy) = \ln x + \ln y$$

$$\ln(64 \cdot 16)$$

$$\ln(1{,}024)$$

From here, we can use a calculator to find the approximate value of $\ln(1{,}024)$.

$$\ln(1{,}024) \approx 6.9315$$

Or we could let $x = \ln 1{,}024$. Then

$$e^x = 1{,}024$$

The way we solve equations in this form, where the variable is in the exponent in an exponential equation, is to take the logarithm (in this case the natural logarithm) of both sides.

$$\ln e^x = \ln 1{,}024$$

Now note that $\ln(e^x) = x$, because any exponential function a^x and the associated log function ($\log_a x$) are inverses of each other. So if we evaluate a^x, and then take the log (to base a) of the result, we get back to our starting point (x). This means that our equation simplifies to

$$x = \ln 1{,}024$$

$$x \approx 6.9315$$

Quotient rule

The quotient rule for logs is

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

In terms of natural logarithms, this rule is

$$\ln \left(\frac{x}{y}\right) = \ln x - \ln y$$

As with all log rules, we can use it in either direction. If we start with something that matches the the form of the expression on the left side of the equation, we can rewrite it in the form on the right side of the equation, and vice versa.

Example

Write the expression as one logarithm.

$$\ln 32 - \ln 8$$

Using the quotient rule for natural logs,

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

we can rewrite the expression as

$$\ln\frac{32}{8}$$

$$\ln 4$$

We can leave it this way, as an exact value, or we can use a calculator to find an approximate value.

$$\ln 4 \approx 1.3863$$

The power rule

The power rule for logs is

$$\log_a(x^n) = n\log_a x$$

For natural logs, this rule is

$$\ln(x^n) = n\ln x$$

Let's look at an example.

Example

Write the expression as one logarithm.

$$\ln(8^4) - \ln(8^2)$$

There are multiple ways to approach this problem, but let's start by using the power rule.

$$\ln(8^4) - \ln(8^2)$$

$$4\ln 8 - 2\ln 8$$

$$2\ln 8$$

We can leave it in this form, or we can pull the 2 back in as an exponent.

$$\ln(8^2)$$

$$\ln 64$$

We can also express 64 as a power (for instance, we know that $64 = 2^6$), and then use the power rule.

$$\ln(2^6)$$

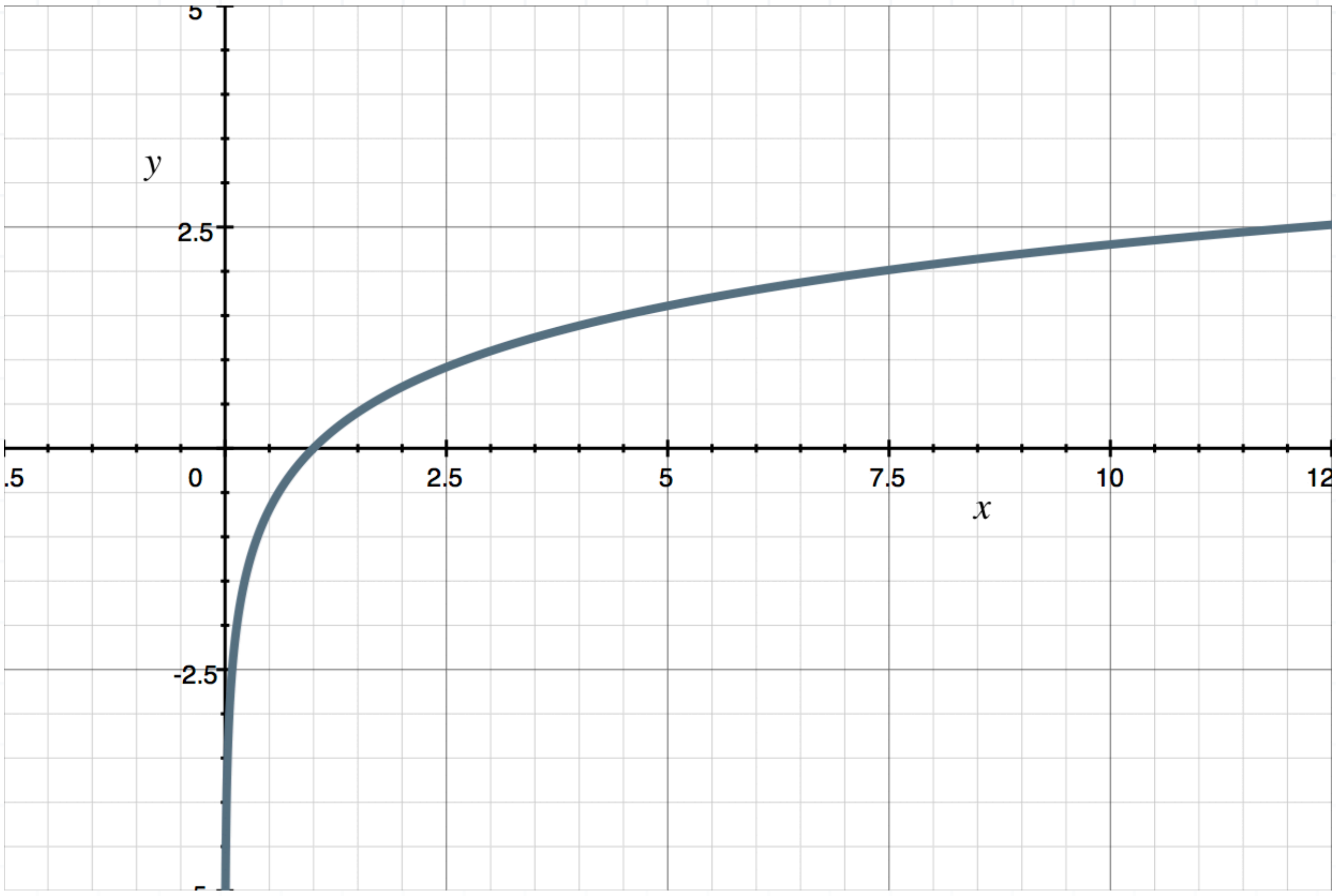
$$6 \ln 2$$

Combining natural log rules

The product, quotient, and power rules for logarithms, as well as the general rule for logs, can all be used together, in any combination, in order to solve problems with natural logs.

Special values for natural logs

We'll look at graphs of exponential and logarithmic functions in the next couple of lessons, but for now, let's take a sneak peak at the graph of the natural log function, $\ln x$:



Notice how the graph dips down sharply as it approaches $x = 0$ from the right side (as positive values of x get closer and closer to 0). This shows that the value of the natural log function goes to $-\infty$ as $x \to 0$ from the right side, and that the natural log function is undefined at $x = 0$.

Also, as we've already learned, the natural log function is undefined at all $x < 0$. As $x \to \infty$, the value of the natural log function increases without bound. And the graph of the natural log function crosses the x-axis at $x = 1$, which means that $\ln 1 = 0$. To see this algebraically, we could set $\ln x$ equal to 0 (write the equation $\ln x = 0$) and then solve for x. In exponential form, this equation is $e^0 = x$. Since $e^0 = 1$, we get $1 = x$.

So based on its graph, we can say that the natural log function has a few key features:

$\ln x$ is undefined at all $x \leq 0$

$\ln 1 = 0$

$\ln x \to \infty$ as $x \to \infty$

Solving logarithmic equations

Logarithmic equations are simply equations that include one or more logarithms, and sometimes we'll want to solve a logarithmic equation for that equation's variable.

The simplest way to solve a logarithmic equation is often to rewrite the equation until we have just one logarithm set equal to one logarithm, and those logarithms have the same bases.

$$\log_b x = \log_b y$$

If we can put the equation in this form, then the only way the equation can be true is if the arguments of the log functions are equivalent, so we set $x = y$, which eliminates the logarithms completely and allows us to solve an equation without any logs.

Once we have an equation without logarithms, we simply collect like terms and solve the equation for the variable. Once we've solved the equation, we should check the answer by plugging the solution back into the original logarithmic equation.

Let's look at an example.

Example

Solve the logarithmic equation.

$$\log(7x) = \log(3x + 4)$$

Because we have two logarithms set equal to each other, and those two logs have the same base, the only way this equation can be true is if the arguments of the log functions are equal.

In other words, given $\log_b x = \log_b y$, we know $x = y$, so we set the arguments equal to each other and then solve the equation for x.

$$7x = 3x + 4$$

$$4x = 4$$

$$x = 1$$

Then we plug $x = 1$ back into the logarithmic equation to make sure it's an actual solution.

$$\log(7(1)) = \log(3(1) + 4)$$

$$\log 7 = \log 7$$

Let's look at another example.

Example

Solve the logarithmic equation.

$$\log_2(x + 3) - \log_2(2x) = \log_2 9$$

Our goal is to manipulate the equation until we have two logarithms with the same base set equal to each other, which means we should try to simplify the left side into one logarithm.

Since we have the difference of two logs with the same base, we can apply the law of logs,

$$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$$

to rewrite the equation as

$$\log_2\left(\frac{x+3}{2x}\right)=\log_2 9$$

Now we have two logarithms with the same base set equal to each other, which means the arguments must be equal.

$$\frac{x+3}{2x}=9$$

$$x+3=18x$$

$$3=17x$$

$$x=\frac{3}{17}$$

Then we plug $x=3/17$ back into the logarithmic equation to make sure it's an actual solution.

$$\log_2\left(\frac{3}{17}+3\right)-\log_2\left(2\left(\frac{3}{17}\right)\right)=\log_2 9$$

$$\log_2\left(\frac{3}{17}+\frac{51}{17}\right)-\log_2\frac{6}{17}=\log_2 9$$

$$\log_2\frac{54}{17}-\log_2\frac{6}{17}=\log_2 9$$

Apply the law of logs

$$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$$

to rewrite the left side of the equation.

$$\log_2\frac{\frac{54}{17}}{\frac{6}{17}}=\log_2 9$$

$$\log_2\left(\frac{54}{17}\left(\frac{17}{6}\right)\right)=\log_2 9$$

$$\log_2\frac{54}{6}=\log_2 9$$

$$\log_2 9=\log_2 9$$

Change of base

It's easier for us to evaluate logs to base 10 or base e, because calculators usually have $\log$ and $\ln$ buttons for these. When the base is anything other than 10 or e, we can use the **change of base** formula.

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Notice that, given a log function with base a and argument b, we can pick any value that we'd like to be the new base, c. Which is really helpful, because we can pick a new base of 10 or e if either of them is convenient for us.

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \frac{\log b}{\log a} \qquad \log_a b = \frac{\log_e b}{\log_e a} = \frac{\ln b}{\ln a}$$

Let's look at an example where we use the change of base formula.

Example

Estimate the log to four decimal places.

$$\log_5 4$$

We can use the change of base formula.

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_5 4 = \frac{\log_{10} 4}{\log_{10} 5}$$

Now we can use a calculator to get the answer.

$$\log_5 4 \approx \frac{0.6021}{0.6990}$$

$$\log_5 4 \approx 0.8614$$

Realize that we can also work backwards, backing our way into the change of base formula.

Example

Simplify the expression to a single real number without using a calculator.

$$\frac{\log 625}{\log 25}$$

If we use the change of base formula, we can rewrite this expression in the form $\log_a b$.

$$\log_{25} 625$$

Let $x = \log_{25} 625$. Then, using the general rule for logarithms, we have

$$25^x = 625$$

Now we want to rewrite both sides of the equation in terms of the same base.

$$(5^2)^x = 5^4$$

$$5^{2x} = 5^4$$

Since the bases are equal, the exponents must also be equal in order for the equation to be true.

$$2x = 4$$

$$x = 2$$

Therefore, the value of the original expression is 2:

$$\frac{\log 625}{\log 25} = 2$$

We can also solve other kinds of exponential equations using logs and the change of base formula.

Example

Use logs to solve the equation.

$$10 \cdot 5^{2x} = 300$$

In problems like this, we have an equation, and we need to solve for the variable, x, which means we need to get x by itself on one side of the equation. In this particular example, we can start by dividing both sides by 10.

$$10 \cdot 5^{2x} = 300$$

$$5^{2x} = 30$$

Now we can use the general rule for logs to change this into a logarithmic equation.

$$\log_5 30 = 2x$$

We'll apply the change of base formula,

$$2x = \frac{\log 30}{\log 5}$$

And then we can solve for the variable.

$$x = \frac{\log 30}{2 \log 5}$$

This is the exact value of the variable, but we can also use a calculator to find the decimal value.

$$x \approx 1.0566$$

Graphing exponential functions

We want to be able to graph **exponential functions**, which are functions in which the variable is in the exponent. These are all exponential functions:

$$f(x) = 3^x$$

$$f(x) = 2\left(\frac{1}{3}\right)^{-x+2} - 4$$

$$f(x) = -3^{x-1} - 2$$

$$f(x) = 6 \cdot 2^{-x} + 1$$

Characteristics of the graph

To learn how to graph exponential functions, let's start by graphing $f(x) = ab^x$, where a is a nonzero number and b is a positive real number not equal to 1.

The coefficient a is the **initial value**, because $f = a$ when $x = 0$, which means the y-intercept of an exponential function $f(x) = ab^x$ is the point $(x, y) = (0,a)$.

The base b is the **growth factor**.

- When $b > 1$, the function grows at rate proportional to its size.
- When $0 < b < 1$, the function decays at a rate proportional to its size.

The base has to be positive in order to ensure that the function will have a real-number output. For instance, if $b = -16$ and $x = 1/2$, then $f(x) = ab^x$ becomes

$$f\left(\frac{1}{2}\right) = (-16)^{\frac{1}{2}} = \sqrt{-16}$$

and we can't take the square root of a negative number. The base also can't be equal to 1 because $f(x) = a(1^x)$ gives $f(x) = a$ for all values of x, and then the function is no longer an exponential, it's a constant function.

The domain of the exponential function is $(-\infty, \infty)$, and the range is all positive real numbers $(0,\infty)$ if $a > 0$ and all negative real numbers $(-\infty,0)$ if $a < 0$. The exponential function always has a horizontal asymptote at $y = 0$.

Let's do an example so that we can see how to graph a simple exponential function.

Example

Graph the exponential function.

$$f(x) = 2^x$$

If we rewrite the function as $f(x) = 1(2^x)$, then we can identify the initial value $a = 1$ and the growth factor $b = 2$.

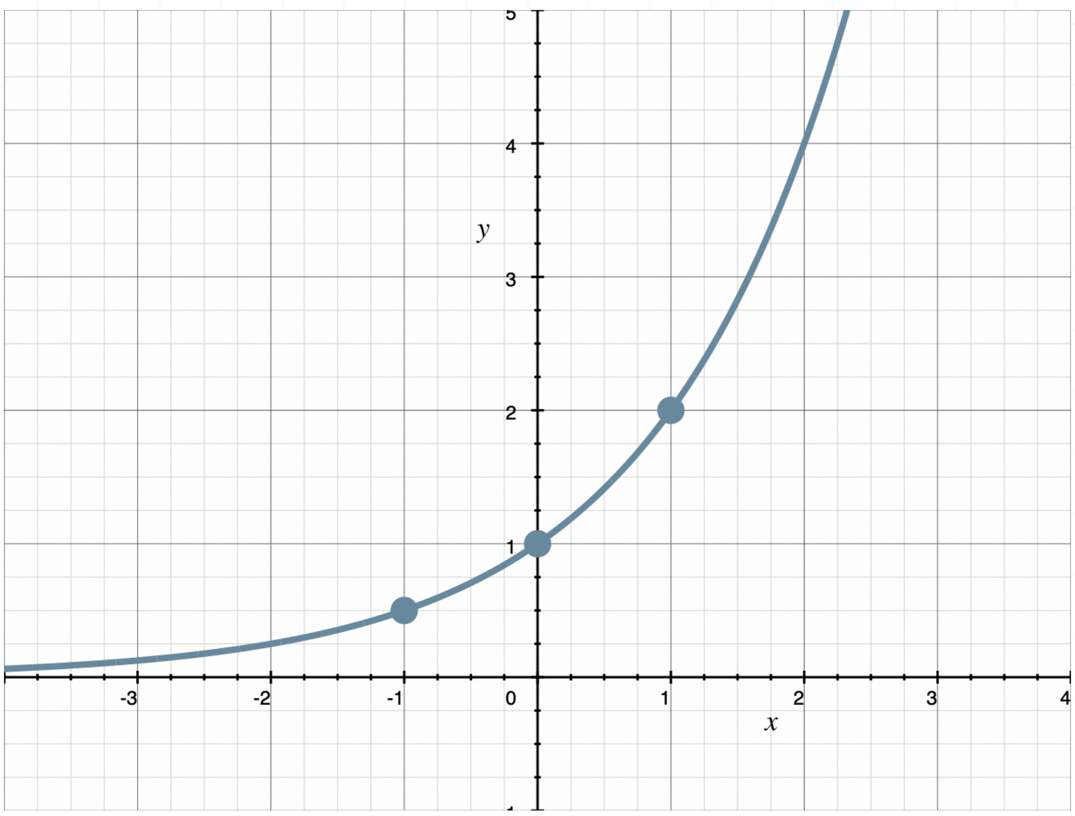
The initial value tells us that the function passes through $(0,1)$. And since $b > 1$, we know the function is increasing above the horizontal asymptote at $y = 0$.

We'll plug in a couple more values of x for which the value of $f(x)$ will be easy to calculate.

For $x = -1$, $f(-1) = 2^{-1} = 1/2$

For $x = 1$, $f(1) = 2^1 = 2$

Now we have three points on the graph of f: the y-intercept $(0,1)$, and $(-1,1/2)$ and $(1,2)$. If we plot these points and connect them with a smooth curve, we get



Graphing transformations of exponential functions

Sometimes we're given the graph of one exponential function, and asked to sketch the graph of a similar function which is just a few "transformations" away from the given function. Each **transformation** is of one of the following operations:

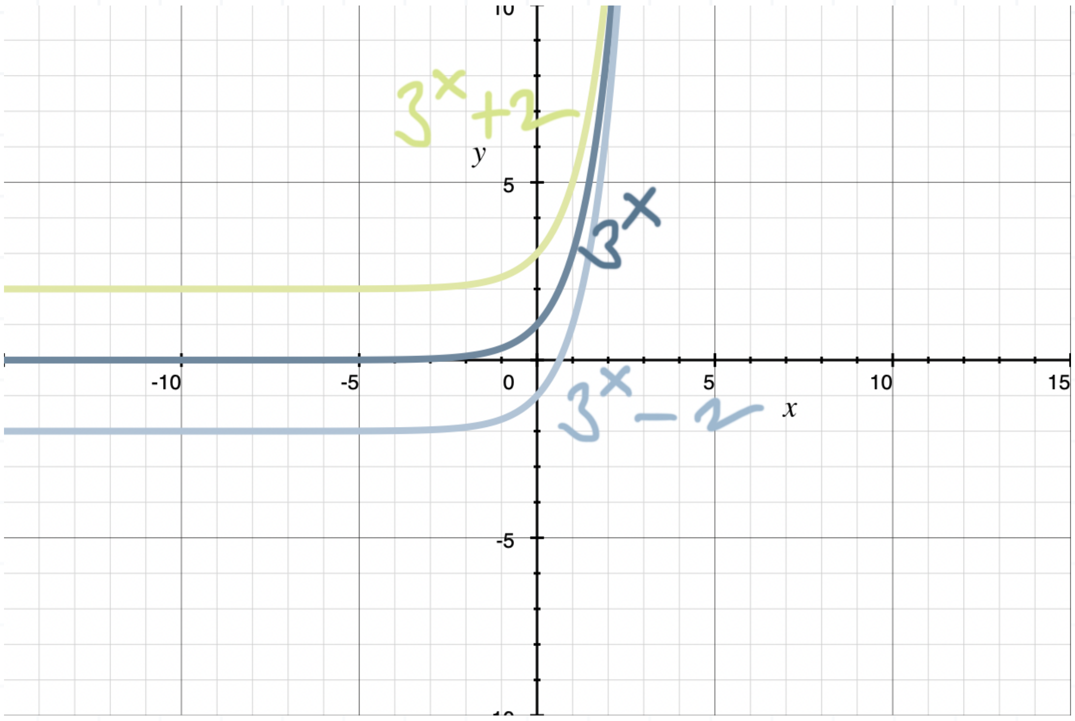
- addition of a constant to the exponent
- addition of a constant to the exponential function
- multiplication of the exponent by a nonzero constant
- multiplication of the exponential function by a nonzero constant

These produce a few different transformations, including vertical and horizontal shifts, a stretch or compression, or reflection. Let's learn how we can graph each transformation.

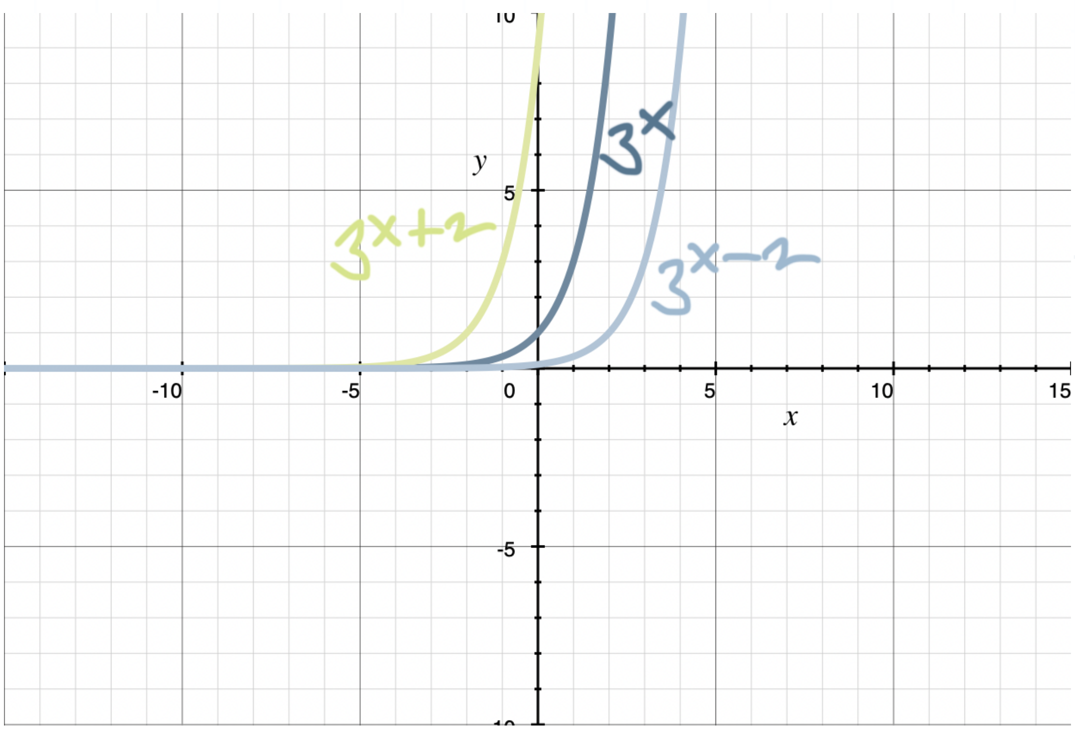
Vertical and horizontal shifts

Let's consider the parent function $f(x) = b^x$. Adding a constant d to the parent function gives us a vertical shift d units in the same direction as the sign of d.

For example, a sketch of the parent function $f(x) = 3^x$ and this same function shifted vertically up 2 units and down 2 units gives



To shift a curve horizontally, we can add a constant c to the input of the parent function $f(x) = b^x$, but the direction of the shift is opposite the sign of c. So a sketch of the parent function $f(x) = 3^x$ and this same function shifted horizontally left and right 2 units gives



Let's do an example where we have to sketch a function with both a vertical and a horizontal shift.

Example

Graph $f(x) = 3^{x-1} + 2$.

We have an equation in the form $f(x) = b^{x+c} + d$, with $b = 3$, $c = -1$, and $d = 2$. So the horizontal asymptote is $y = 0$.

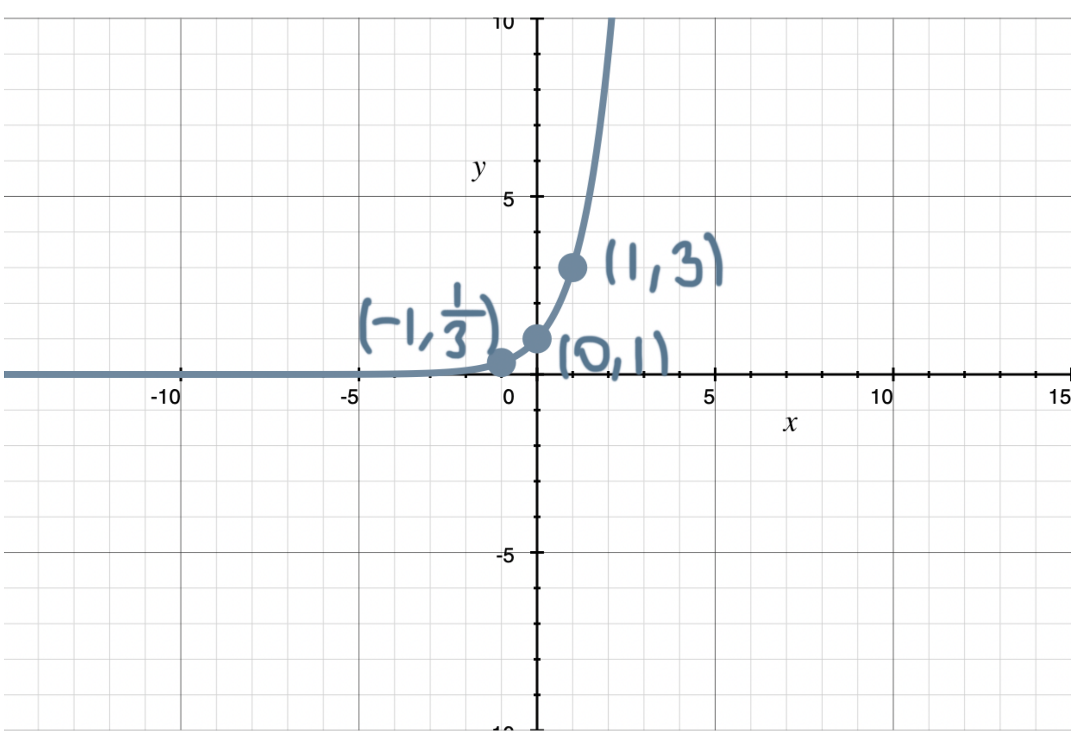
To sketch the graph of $y = 3^x$, we'll plug in a few values of x for which the value of $f(x)$ will be easy to calculate.

For $x = 0$: $f(0) = 3^0 = 1$

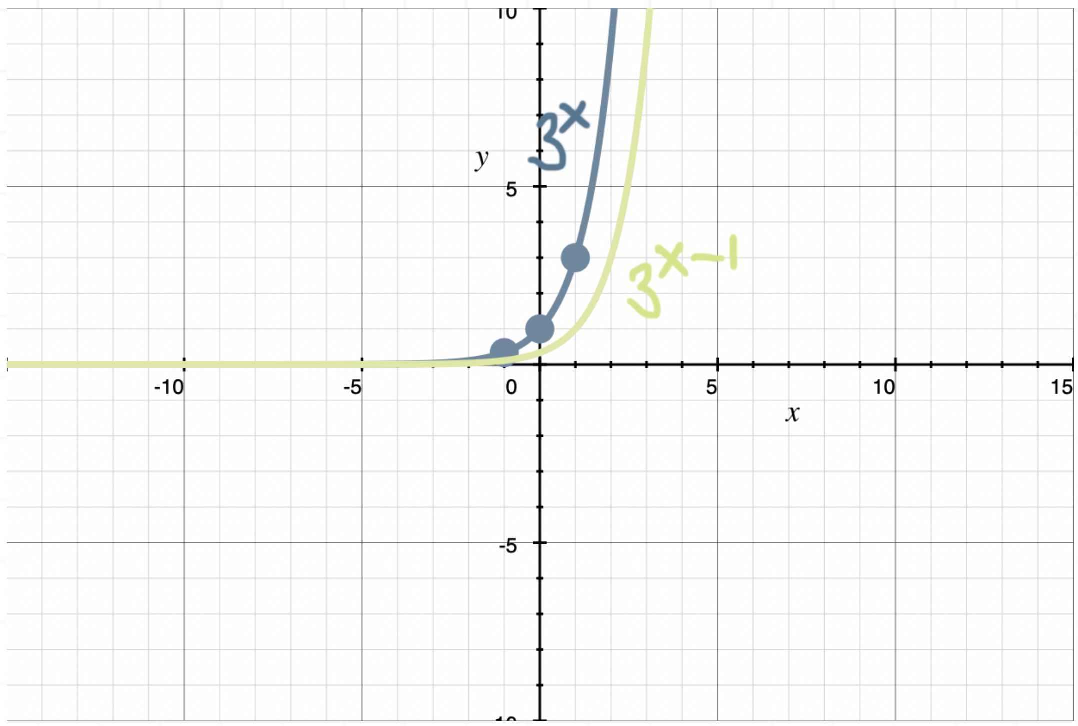
For $x = -1$: $f(-1) = 3^{-1} = 1/3$

For $x = 1$: $f(1) = 3^1 = 3$

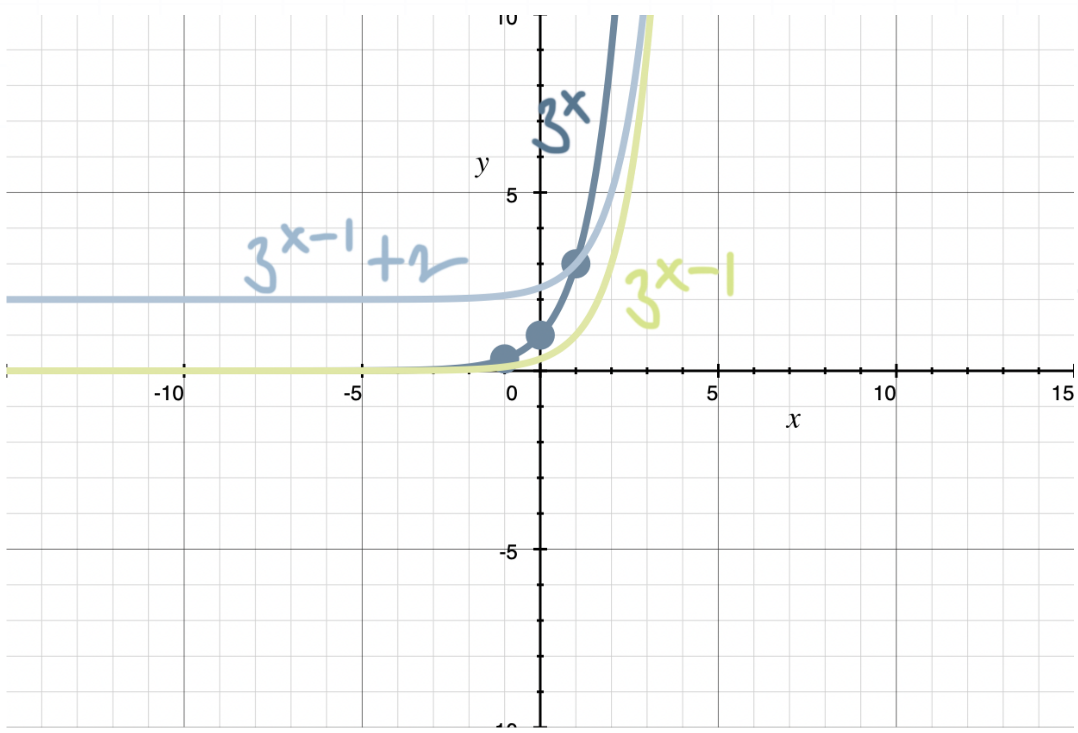
Now we have three points on the graph of f: $(0,1)$, $(-1,1/3)$, and $(1,3)$. If we plot these three points and connect the points with a smooth curve, we get



Then to go from $y = 3^x$ to $y = 3^{x-1}$, we move the graph 1 unit to the right,



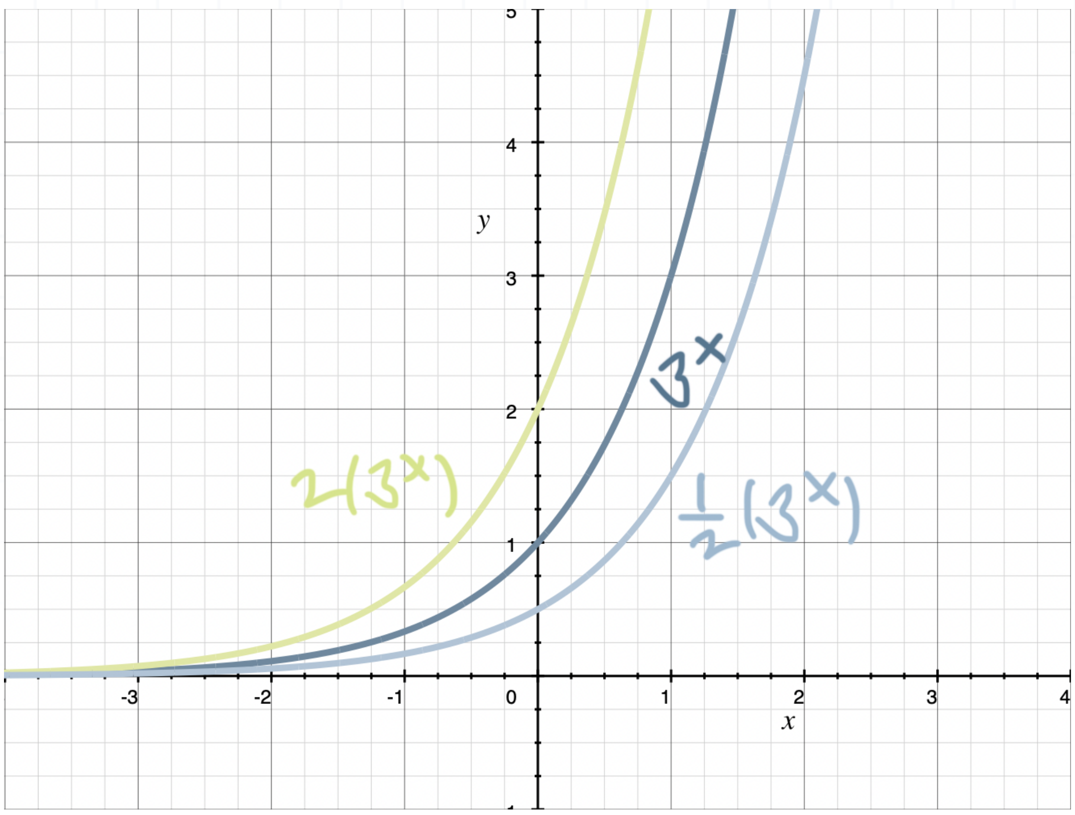
and to go from $y = 3^{x-1}$ to $y = 3^{x-1} + 2$, we move the graph 2 units up vertically.



Vertical and horizontal stretches and compressions

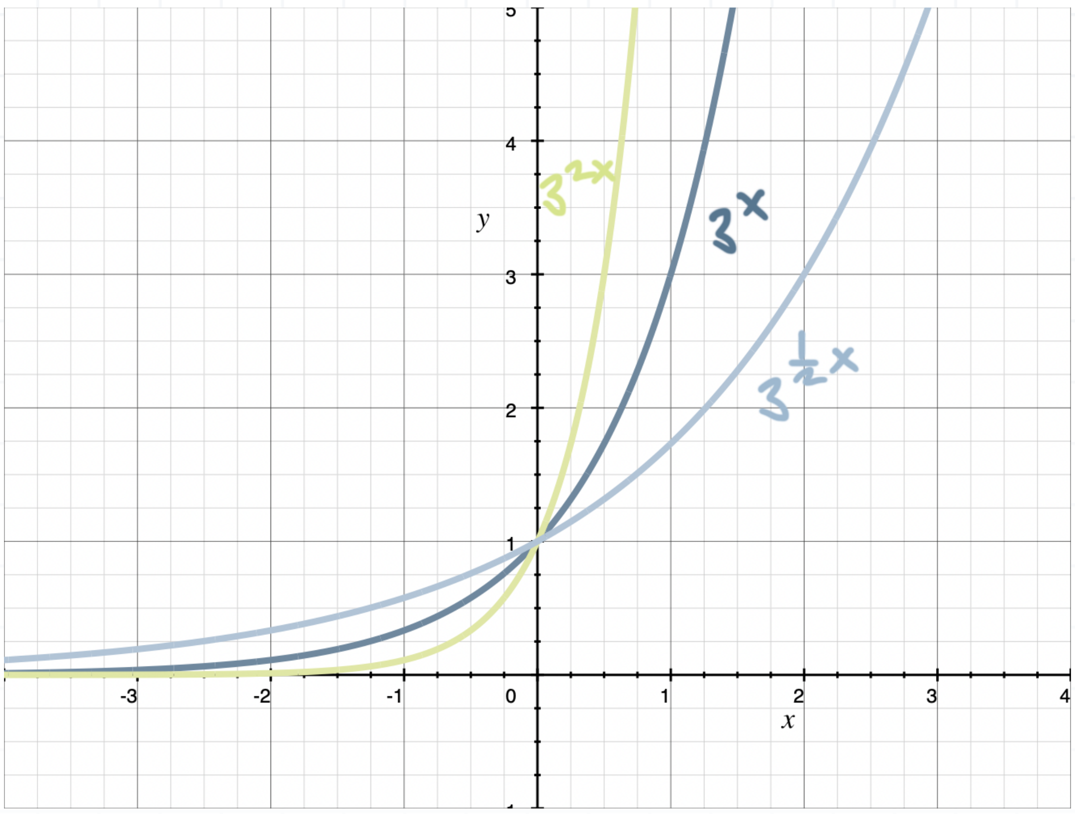
Let's consider the parent function $f(x) = b^x$. If we multiply the parent by some a where $1 < |a|$ to get $f(x) = ab^x$, then $f(x) = b^x$ is being stretched vertically by a factor of a. But when $0 < |a| < 1$, then $f(x) = b^x$ is being vertically compressed.

For example, a sketch of the parent function $f(x) = 3^x$ and this same function stretched and compressed vertically by a factor of 2 gives



If instead we multiply the input x by some k where $0 < |k| < 1$ to get $f(x) = b^{kx}$, then $f(x) = b^x$ is being stretched horizontally by a factor of $\frac{1}{k}$. But when $1 < |k|$, then $f(x) = b^x$ is being compressed.

For example, a sketch of the parent function $f(x) = 3^x$ and this same function stretched and compressed horizontally by a factor of 2 gives

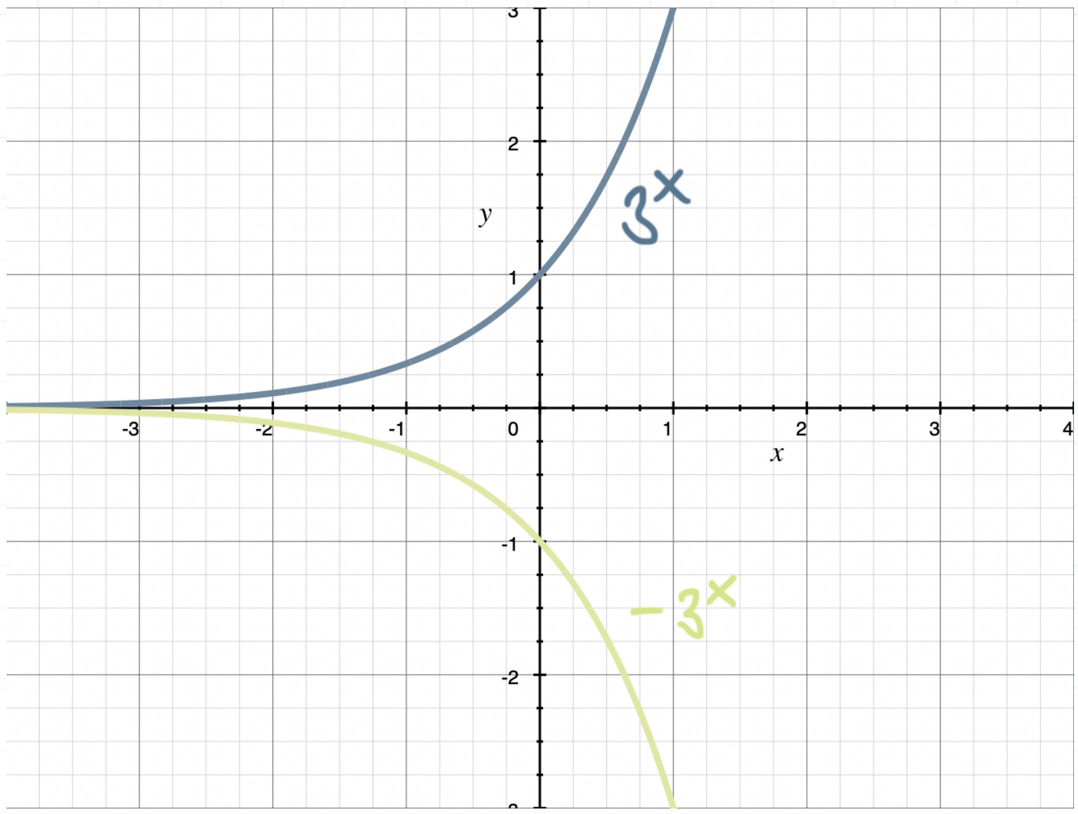


Let's do an example where we have to sketch a function with both a vertical and a horizontal stretch or compression.

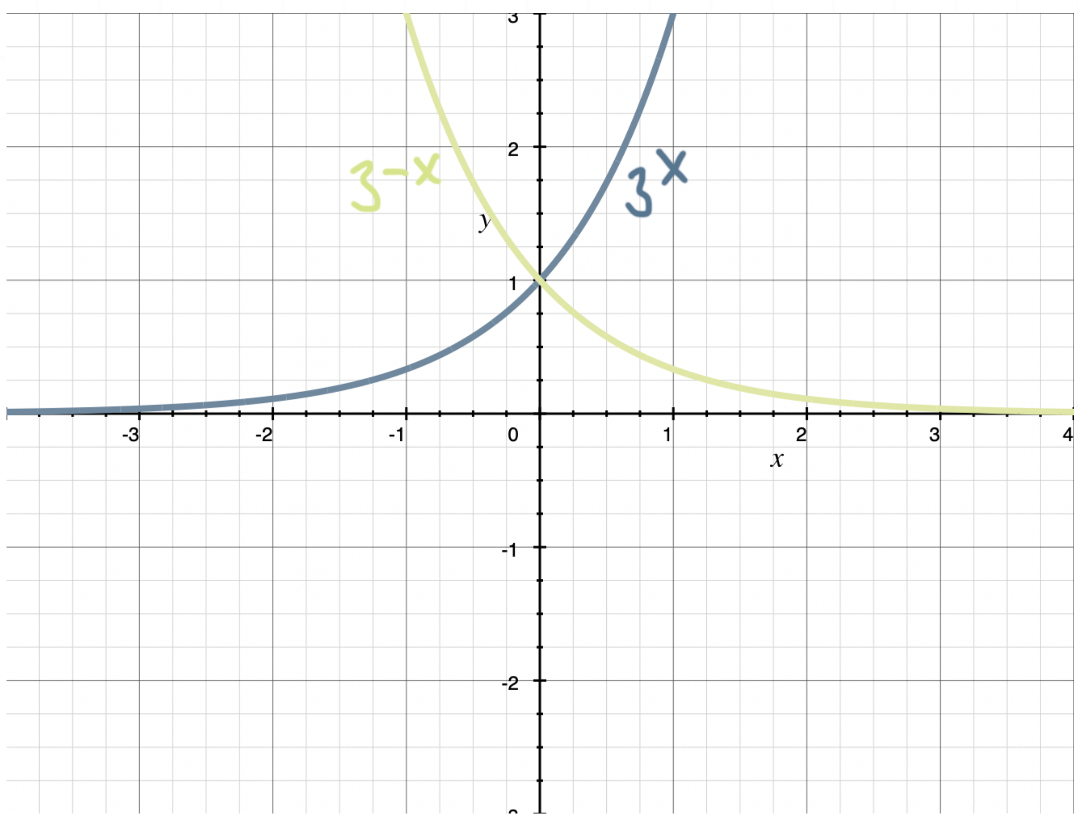
Vertical and horizontal reflections

It's also possible to reflect the graph across the x-axis and/or the y-axis. When we multiply the parent function $f(x) = b^x$ by -1, the graph gets reflected across the x-axis. But when we multiply the input of the parent by -1, the graph gets reflected across the y-axis.

For example, let's choose $f(x) = 3^x$ again as the parent function. Its reflection across the x-axis is $g(x) = -3^x$,



and its reflection across the y-axis is $h(x) = 3^{-x}$.



Combining transformations

Now that we've seen this collection of translations, let's summarize the order in which we should apply them, given multiple translations in the same equation.

1. Horizontal stretch or compression

2. Horizontal shift

3. Horizontal reflection

4. Vertical stretch or compression

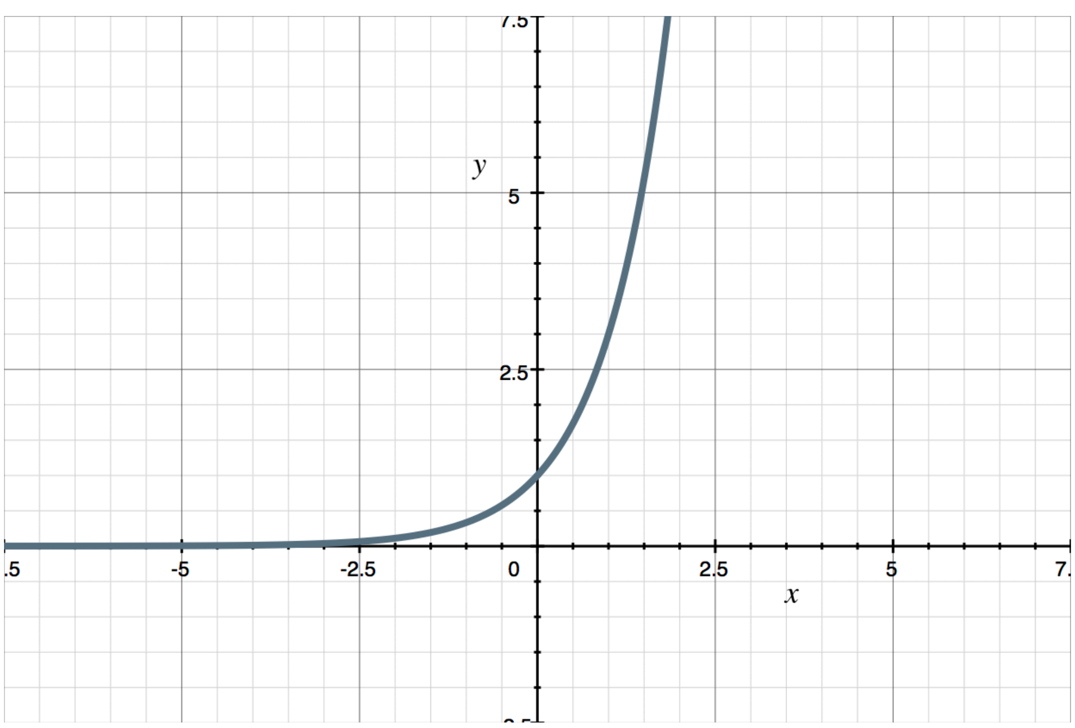
5. Vertical reflection

6. Vertical shift

Let's do an example where we apply these shifts in order.

Example

Use the graph of $f(x) = 3^x$ to sketch the graph of $f(x) = 6 \cdot 3^{-x} + 1$.



The function $6 \cdot 3^{-x} + 1$ is the result of applying several transformations to the function 3^x. We'll tackle each transformation as a separate step, and we'll give different names to the functions we obtain in each step.

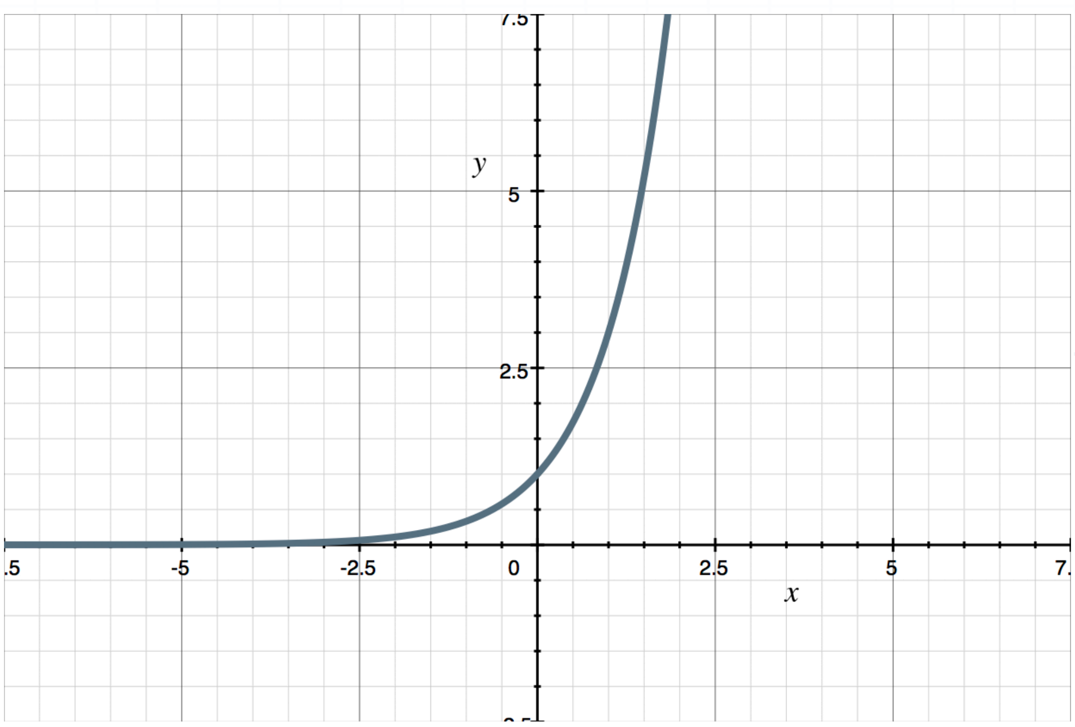
[1] $f(x) = 3^x$

[2] $g(x) = 3^{-x}$

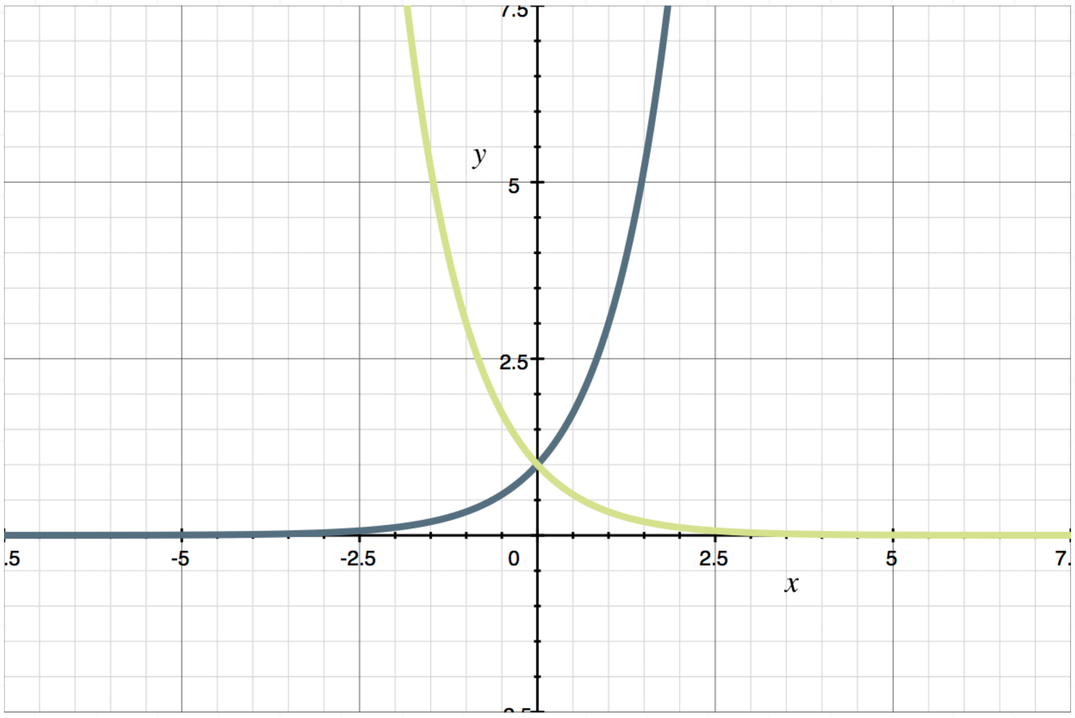
[3] $h(x) = 6 \cdot 3^{-x}$

[4] $k(x) = 6 \cdot 3^{-x} + 1$

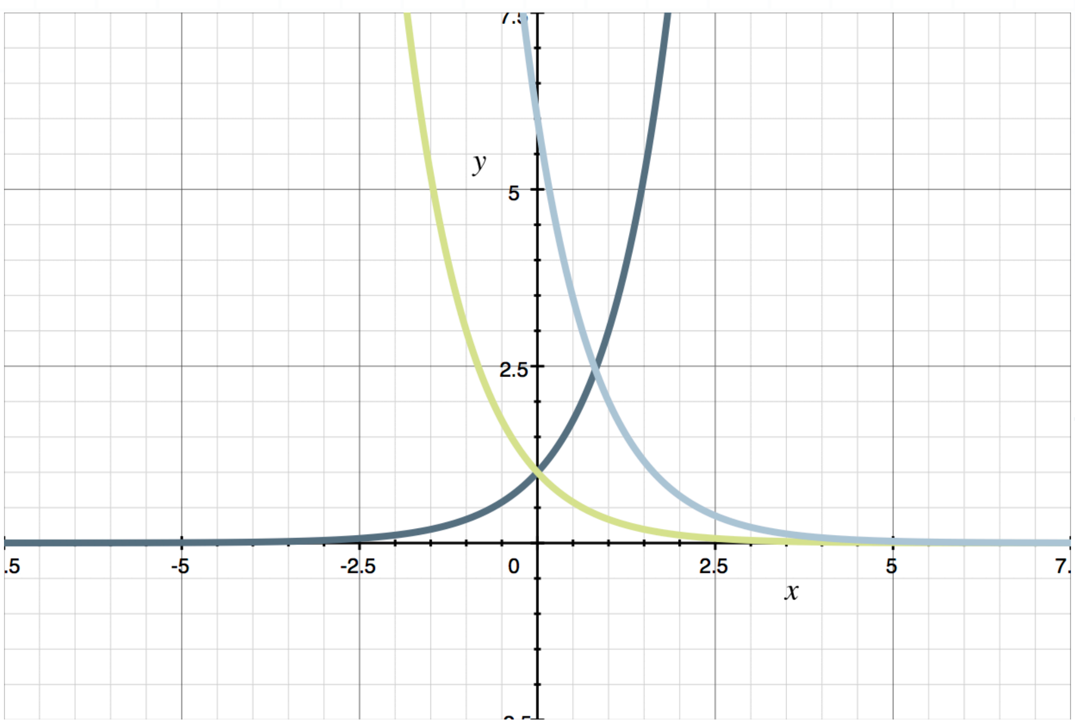
We were given the graph of $f(x)$,



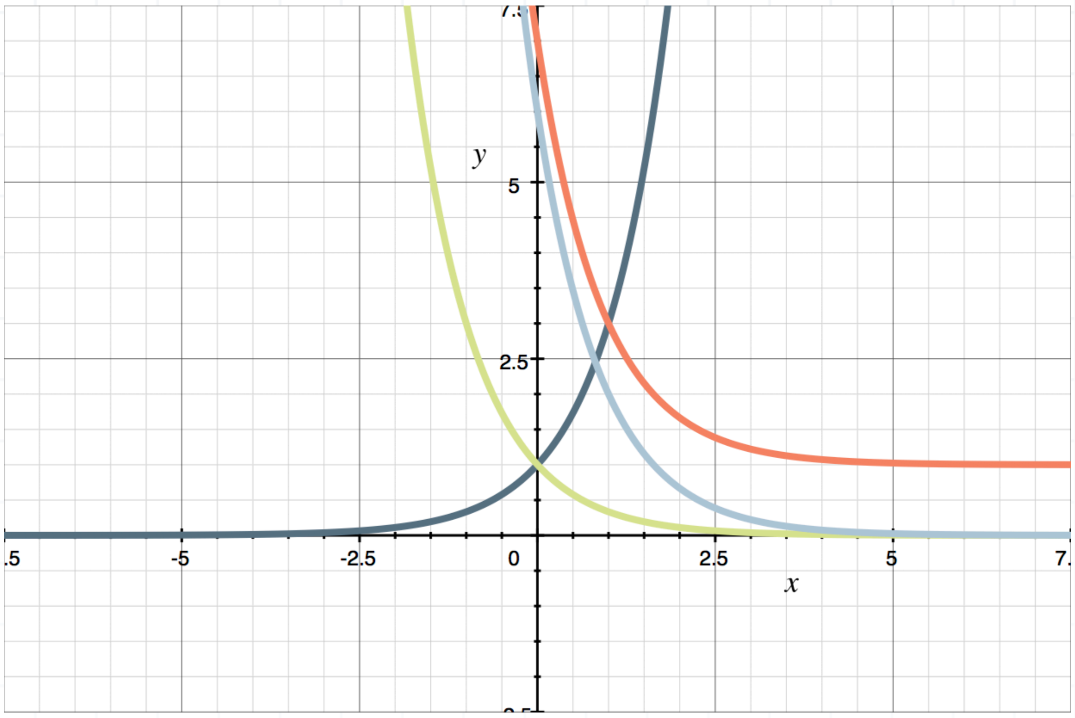
and we can think of $g(x) = 3^{-x}$ as the function we get when we replace x with $-x$, which reflects the graph over the vertical axis. If we graph $g(x) = 3^{-x}$ on the same set of axes as $f(x) = 3^x$, we get



To get $h(x) = 6 \cdot 3^{-x}$, we multiply $g(x)$ by 6. Since the graph of $g(x)$ crosses the y-axis at 1, the graph of $h(x)$ will cross the y-axis at $1 \cdot 6 = 6$.



To get $k(x)$, we add 1 to the value of $h(x)$, so we take the graph of $h(x)$ and shift it up by 1 unit. Since the graph of $h(x)$ crosses the y-axis at 6, the graph of $k(x)$ will cross the y-axis at $6 + 1 = 7$.



To summarize, we started with $f(x) = 3^x$, then applied a horizontal reflection to get $g(x) = 3^{-x}$, a vertical stretch to get $h(x) = 6 \cdot 3^{-x}$, and a vertical shift to get $k(x) = 6 \cdot 3^{-x} + 1$.

[1] $f(x) = 3^x$

[2] $g(x) = f(-x) = 3^{-x}$

[3] $h(x) = 6 \cdot g(x) = 6 \cdot 3^{-x}$

[4] $k(x) = h(x) + 1 = 6 \cdot 3^{-x} + 1$

An alternative procedure

In general, we can also use the following procedure for graphing exponential functions:

1. Plug in $x = 100$ and $x = -100$, and use the values $f(100)$ and $f(-100)$ to determine the "end behavior" of the function, that is, what happens to the value of the function as $x \to \infty$ and as $x \to -\infty$.

2. One of these will result in an infinite value, the other will give a real-number value. The real-number value is the horizontal asymptote of the exponential function.

3. Plug in a few easy-to-calculate values of x, like $x = -1, 0, 1$, in order to get a couple of points that we can plot.

4. Connect the points with an exponential curve, and draw the horizontal asymptote.

Let's do an example where we walk through each of these steps.

Example

Graph the exponential function.

$$f(x) = -3^{x-1} - 2$$

We'll start by plugging in $x = 100$ and $x = -100$.

For $x = 100$:

$$f(100) = -3^{100-1} - 2$$

$$f(100) = -3^{99} - 2$$

$$f(100) = -(\text{very large positive number}) - 2$$

$$f(100) = \text{very large negative number} - 2$$

$$f(100) = \text{very large negative number}$$

$$f(100) = -\infty$$

For $x = -100$:

$$f(-100) = -3^{-100-1} - 2$$

$$f(-100) = -3^{-101} - 2$$

$$f(-100) = -\frac{1}{3^{101}} - 2$$

$$f(-100) = -\frac{1}{\text{very large positive number}} - 2$$

$$f(-100) = -(0) - 2$$

$$f(-100) = 0 - 2$$

$$f(-100) = -2$$

We'll plug in a few values of x for which the value of $f(x)$ will be easy to calculate.

For $x = 0$:

$$f(0) = -3^{0-1} - 2$$

$$f(0) = -3^{-1} - 2$$

$$f(0) = -\frac{1}{3^1} - 2$$

$$f(0) = -\frac{1}{3} - \frac{6}{3}$$

$$f(0) = -\frac{7}{3}$$

For $x = -1$:

$$f(-1) = -3^{-1-1} - 2$$

$$f(-1) = -3^{-2} - 2$$

$$f(-1) = -\frac{1}{3^2} - 2$$

$$f(-1) = -\frac{1}{9} - \frac{18}{9}$$

$$f(-1) = -\frac{19}{9}$$

For $x = 1$:

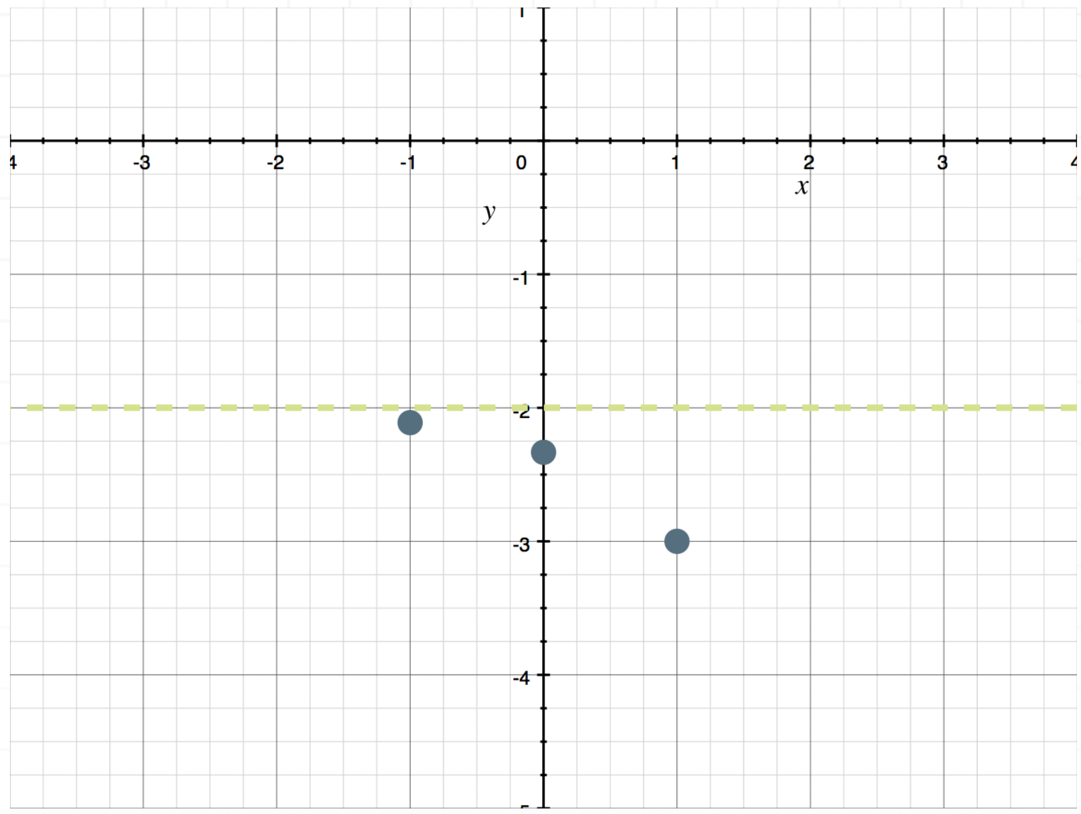
$$f(1) = -3^{1-1} - 2$$

$$f(1) = -3^0 - 2$$

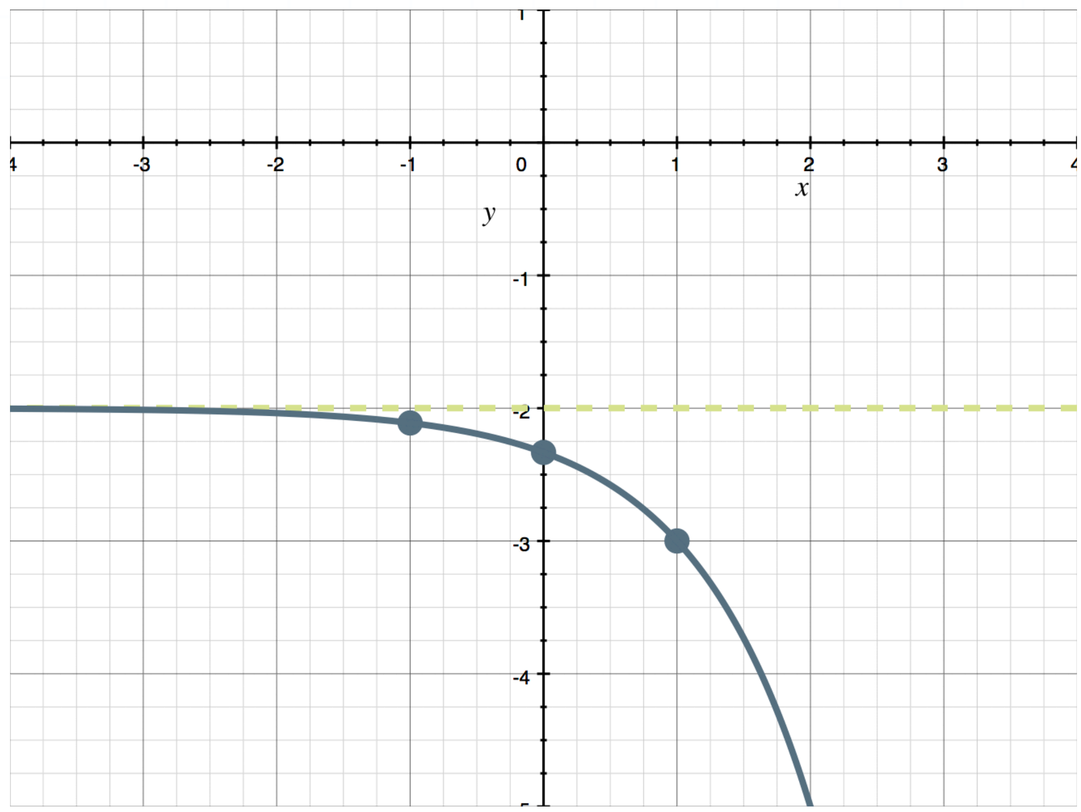
$$f(1) = -1 - 2$$

$$f(1) = -3$$

Now we have three points of the graph of f: $(0, -7/3)$, $(-1, -19/9)$, and $(1, -3)$. If we plot these three points and draw the horizontal asymptote $y = -2$, we get



Based on the asymptote and the points we've found, we can already see what the graph is going to do. We'll simply connect the points to sketch the graph.



The general log rule

The general log rule that we introduced earlier was

> Given the equation $a^x = y$, the associated log is $\log_a(y) = x$, and vice versa.

What this tells us is that

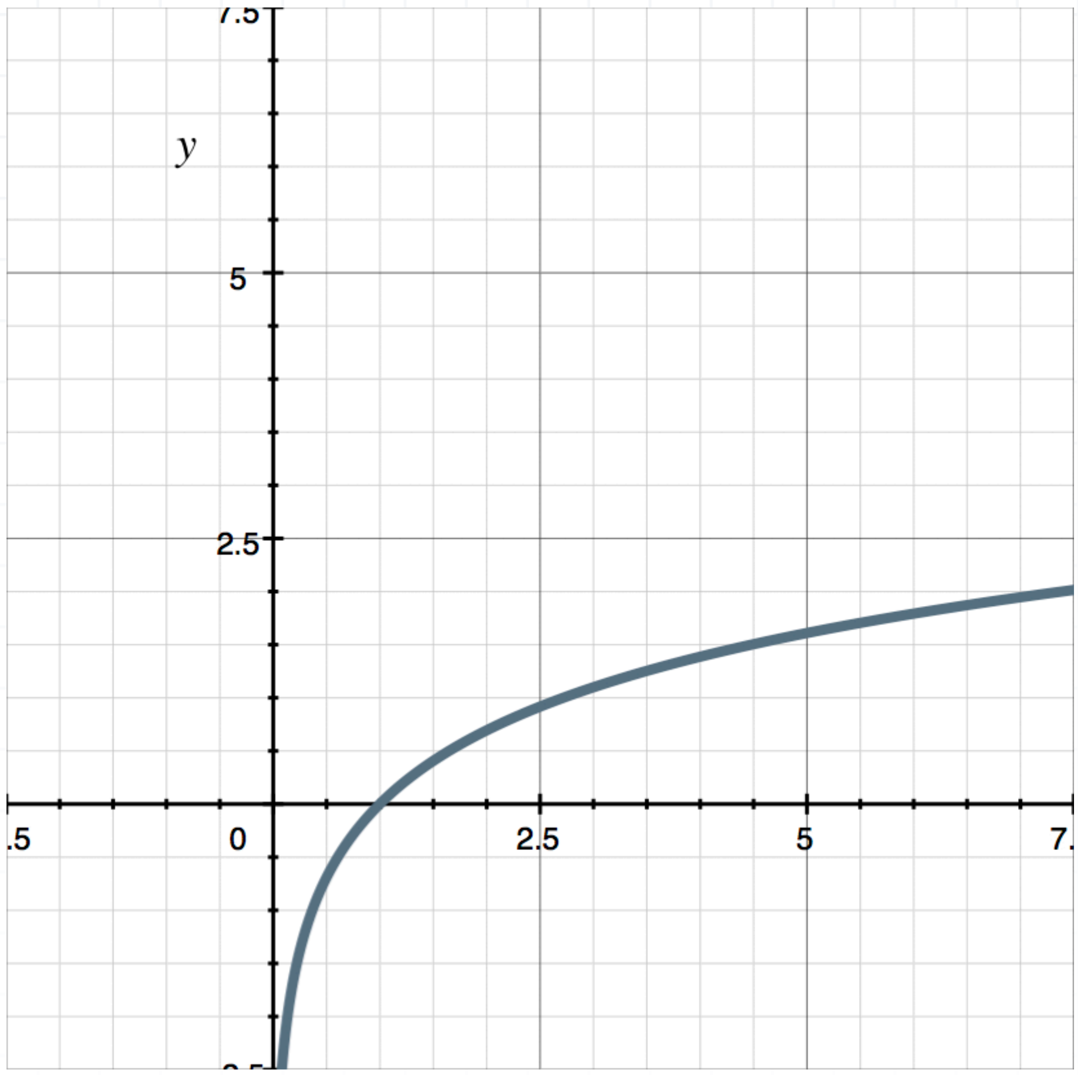
> $\log_a(y) = x$ and $a^x = y$ are equivalent
>
> $\log_a(x) = y$ and $a^y = x$ are equivalent

Remember that inverse functions have their x- and y-values swapped. This means that when we graph inverse functions on the same set of axes, the graphs are mirror images of one another, just reflected over the line $y = x$.

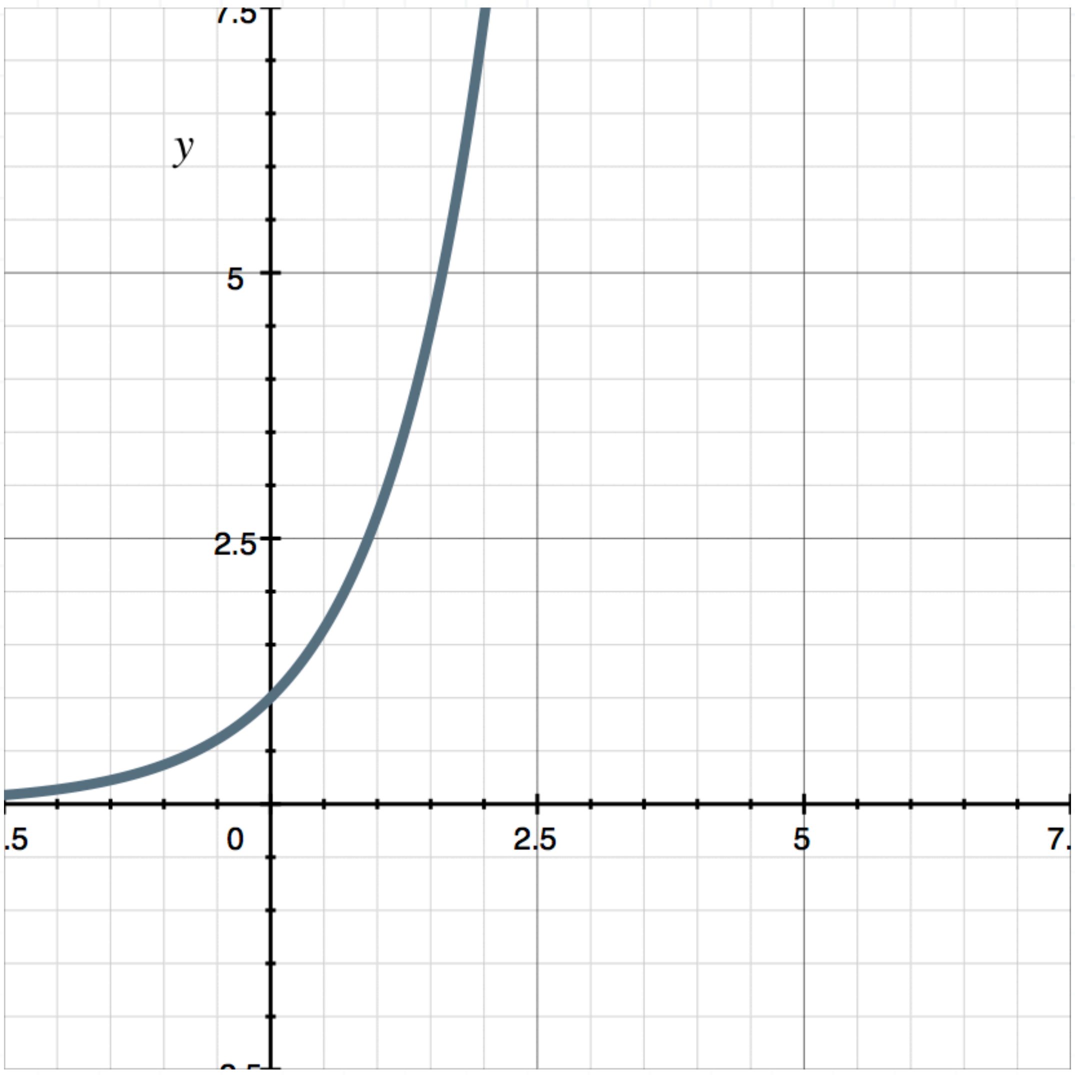
We can see that $\log_a(y) = x$ and $\log_a(x) = y$ have their x- and y-values swapped, and that $a^x = y$ and $a^y = x$ have their x- and y-values swapped. Which means that

> Both $\log_a(x) = y$ and $a^y = x$ are inverses of $\log_a(y) = x$
>
> Both $\log_a(x) = y$ and $a^y = x$ are inverses of $a^x = y$
>
> Both $\log_a(y) = x$ and $a^x = y$ are inverses of $\log_a(x) = y$
>
> Both $\log_a(y) = x$ and $a^x = y$ are inverses of $a^y = x$

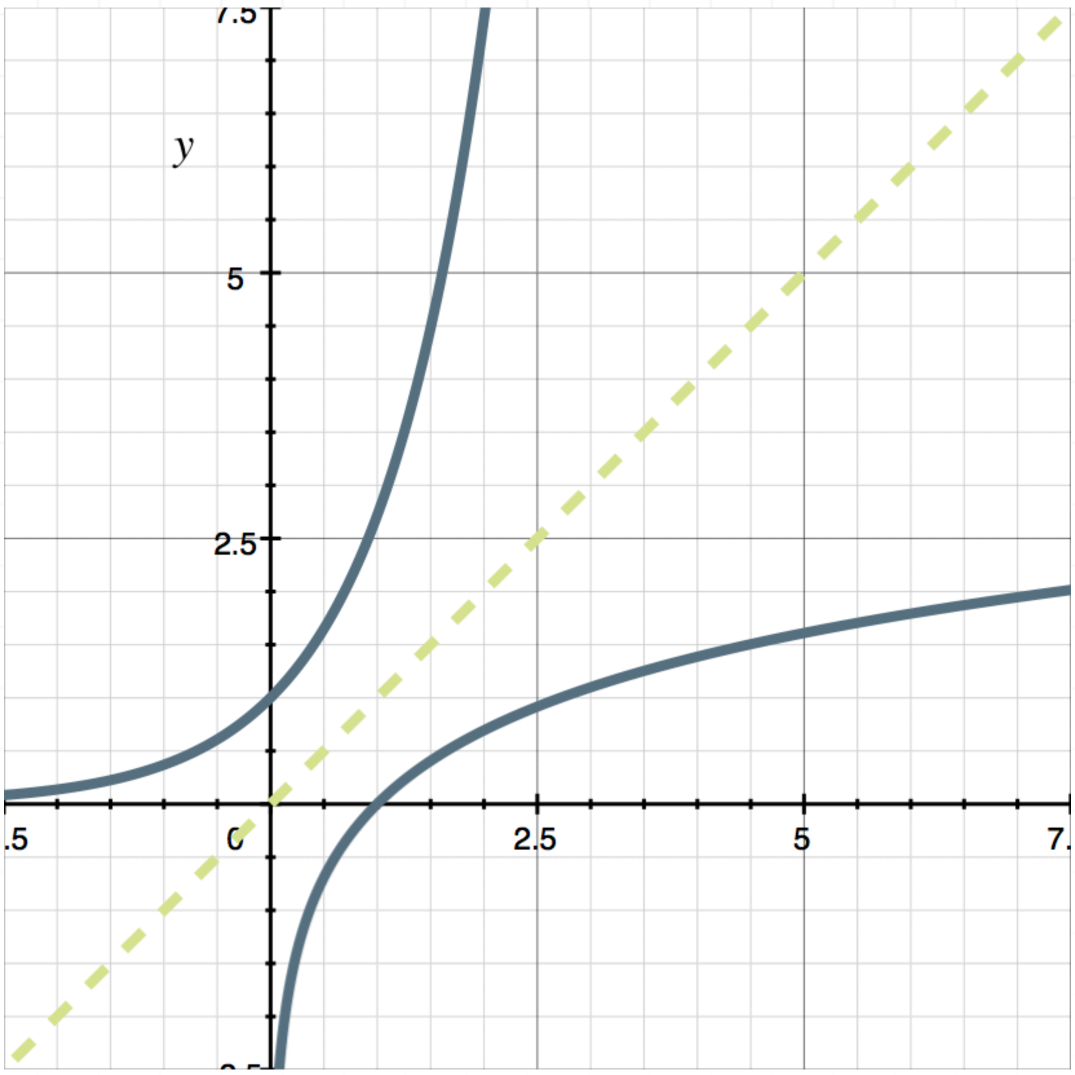
For example, the graph of $\log_a(x) = y$ (or equivalently $a^y = x$) is



And the graph of $\log_a(y) = x$ (or equivalently $a^x = y$) is



And we can see that these are inverses of one another, because they are a reflection of each other over the line $y = x$.



When functions are inverses of one another, we can also express their points in tables. For instance, given the equations $a^x = y$ and $\log_a(x) = y$, we can express points that satisfy each of these equations in tables.

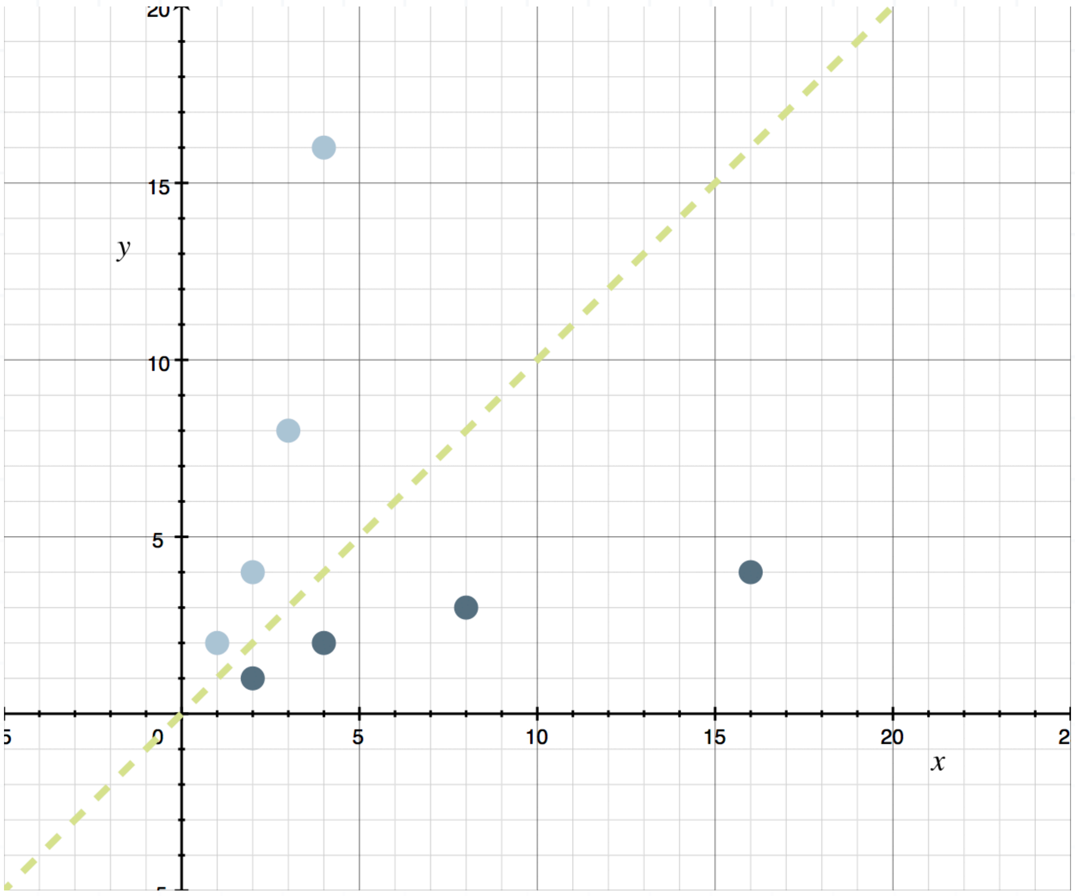
If a point set that satisfies $a^x = y$ is

x	1	2	3	4
$y=2^x$	2	4	8	16

then the point set satisfying its inverse $\log_a(x) = y$ is

x	2	4	8	16
$y=\log_2 x$	1	2	3	4

And if we sketch these points on a graph, we can see again how they are mirror images of one another over the line $y = x$.



Graphing log functions

Previously, we talked about the fact that exponential and logarithmic functions are inverses of each other. This is implied by the general log rule,

$$a^x = y \quad \iff \quad \log_a(y) = x$$

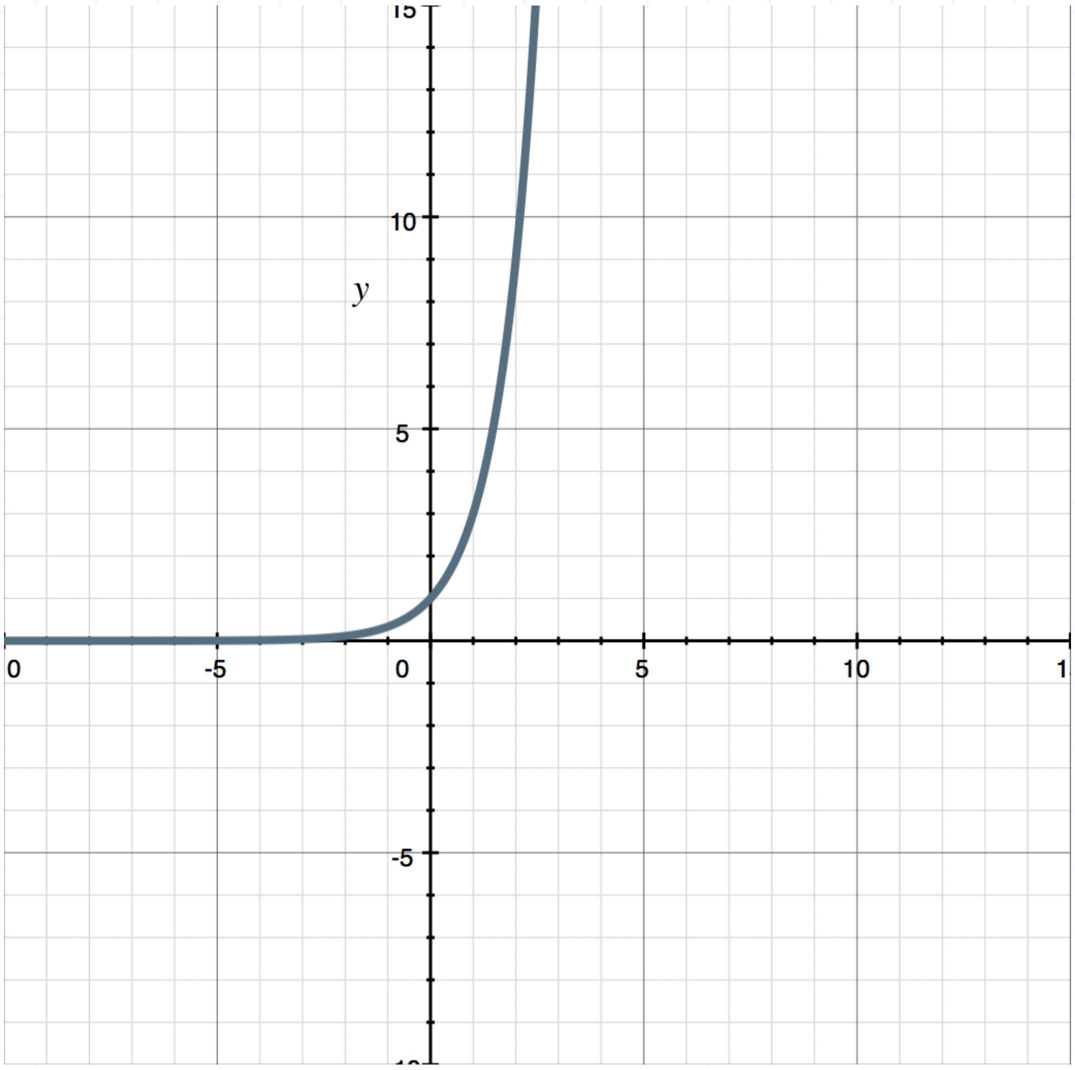
which allows us to convert back and forth between an exponential equation and the associated logarithmic equation. Remember that this is also true for natural logs, as

$$e^x = y \quad \iff \quad \log_e(y) = x$$

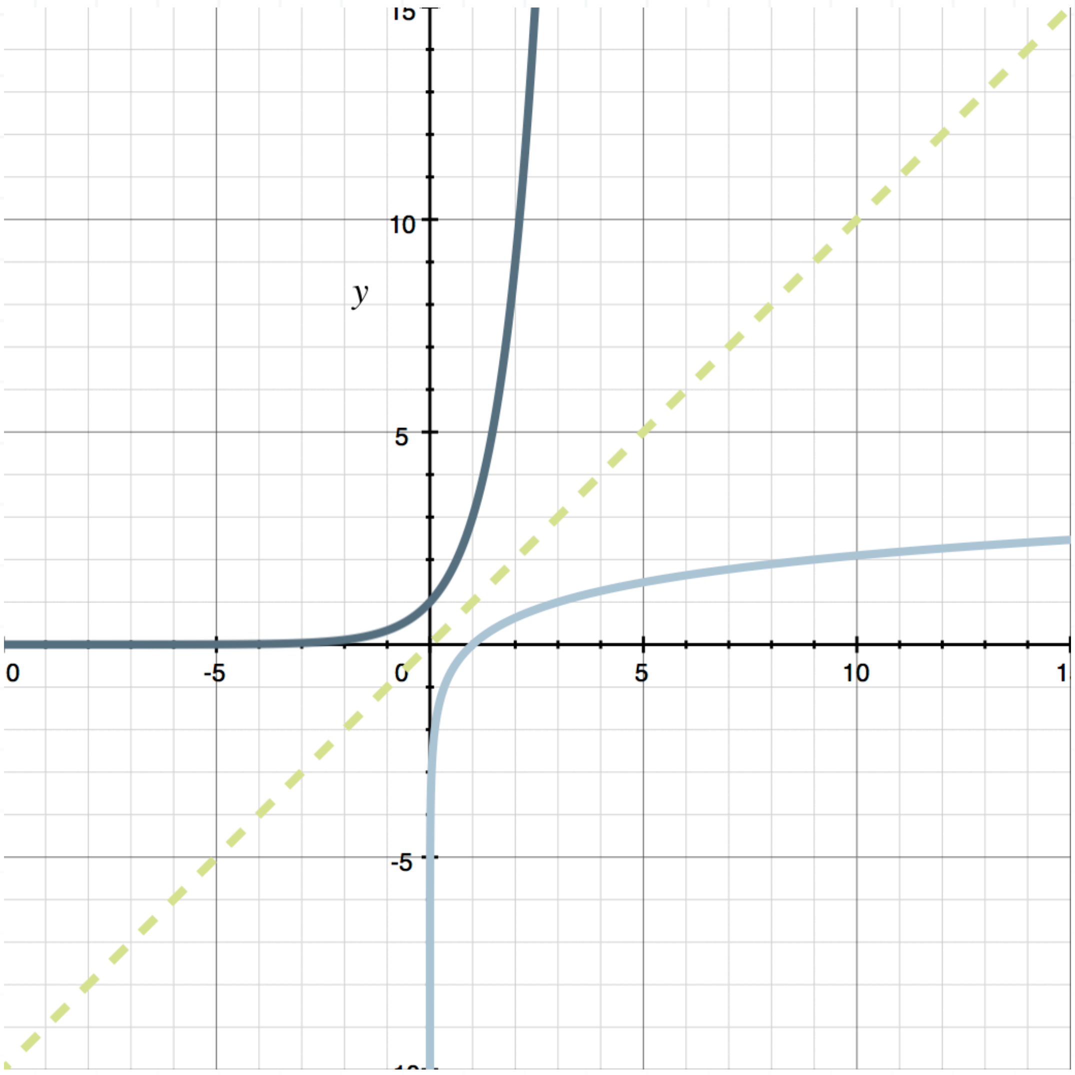
In earlier lessons, we talked about the graphs of exponential functions, including how to graph transformations of exponential functions.

Because exponential and logarithmic functions are inverses of each other, if we have the graph of an exponential function, we can get the graph of the corresponding log function in two ways: by reflecting the graph of the exponential function with respect to the line $y = x$, or by switching the x- and y-coordinates of all the points.

For instance, we already know that the graph of the exponential function $y = 3^x$ is



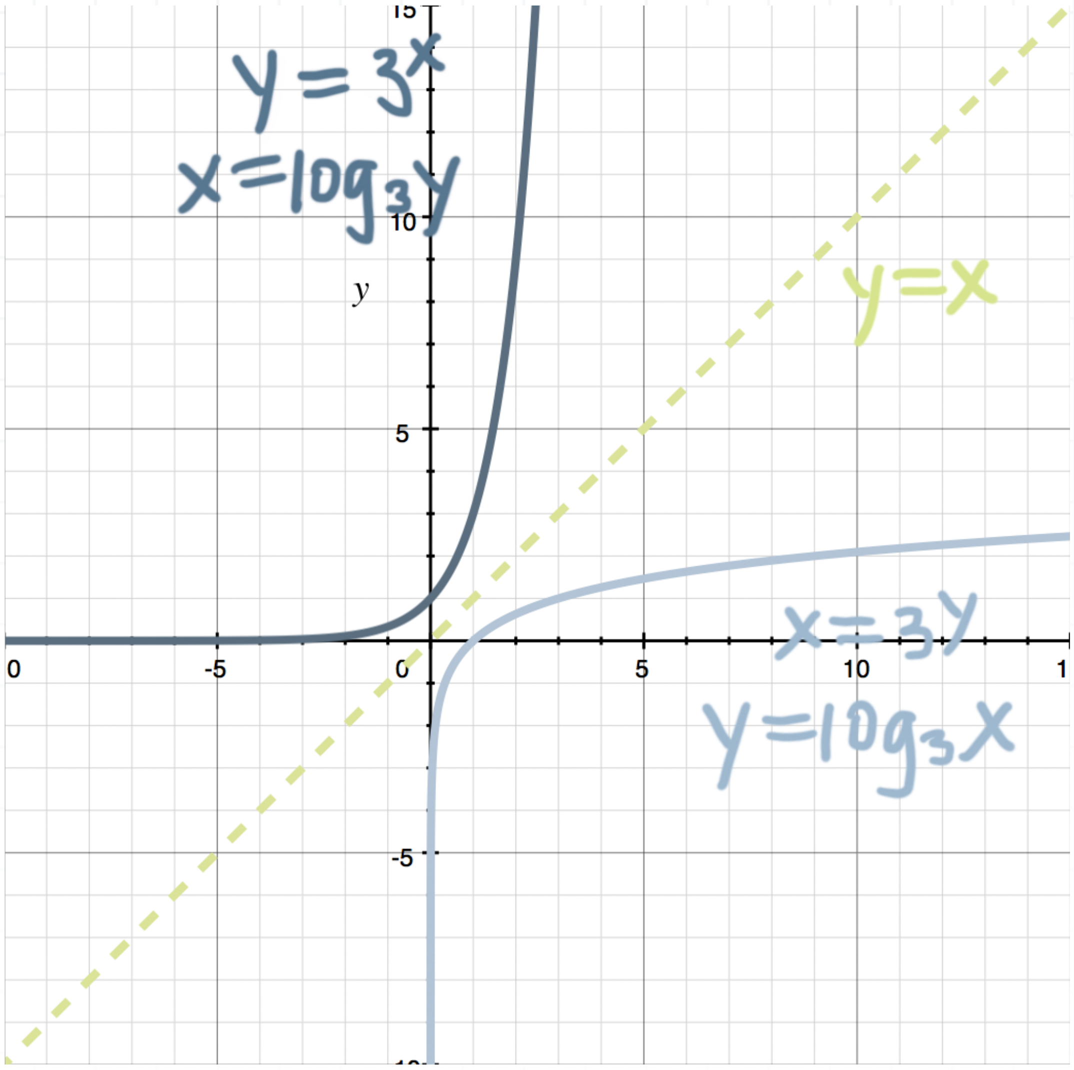
Let's now find the inverse of this exponential function. Functions which are inverses of each other simply have their x- and y-values flipped, which means that the inverse of $y = 3^x$ can simply be given by $x = 3^y$ (the same equation, just with x and y swapped). If we sketch $x = 3^y$, we see that we get the mirror image of $y = 3^x$, reflected over the line $y = x$.



The interesting thing is that, by the general log rule applied to the function $y = \log_3 x$, we get

$$x = 3^y \quad \Longleftrightarrow \quad y = \log_3 x$$

Therefore, we can actually say that $y = 3^x$ and $y = \log_3 x$ are inverses of one another. Similarly, $x = 3^y$ and $x = \log_3 y$ are inverses of one another. Let's show both the exponential and logarithmic expression of both functions, and how they are inverses of each other, reflected over $y = x$.



Let's walk through a couple of examples of graphing logarithmic functions, keeping in mind that we can always use the general log rule to convert logarithmic equations to their exponential form, and then graph the resulting exponential equations using the steps we used in the previous lesson.

Example

Graph the logarithmic function.

$$y = \log_3 x$$

There are several ways to go about this. First, we could use the general rule for logs to convert the logarithmic equation $y = \log_3 x$ to its exponential form, $x = 3^y$. Then we can follow the steps from the previous lesson, but this time by plugging in values of y to get values of x, starting with $y = 100$ and $y = -100$.

For $y = 100$:

$$x = 3^{100}$$

$$x = \text{very large positive number}$$

$$x = \infty$$

For $y = -100$:

$$x = 3^{-100}$$

$$x = \frac{1}{3^{100}}$$

$$x = \frac{1}{\text{very large positive number}}$$

$$x = \text{very small positive number}$$

$$x = 0$$

This basically allowed us to evaluate end behavior, and we've learned that the function has a vertical asymptote at $x = 0$, and heads up toward ∞ as y gets very large.

We'll plug in a few simple-to-calculate values for y.

For $y = -1$:

$$x = 3^{-1}$$

$$x = \frac{1}{3}$$

For $y = 0$:

$$x = 3^0$$

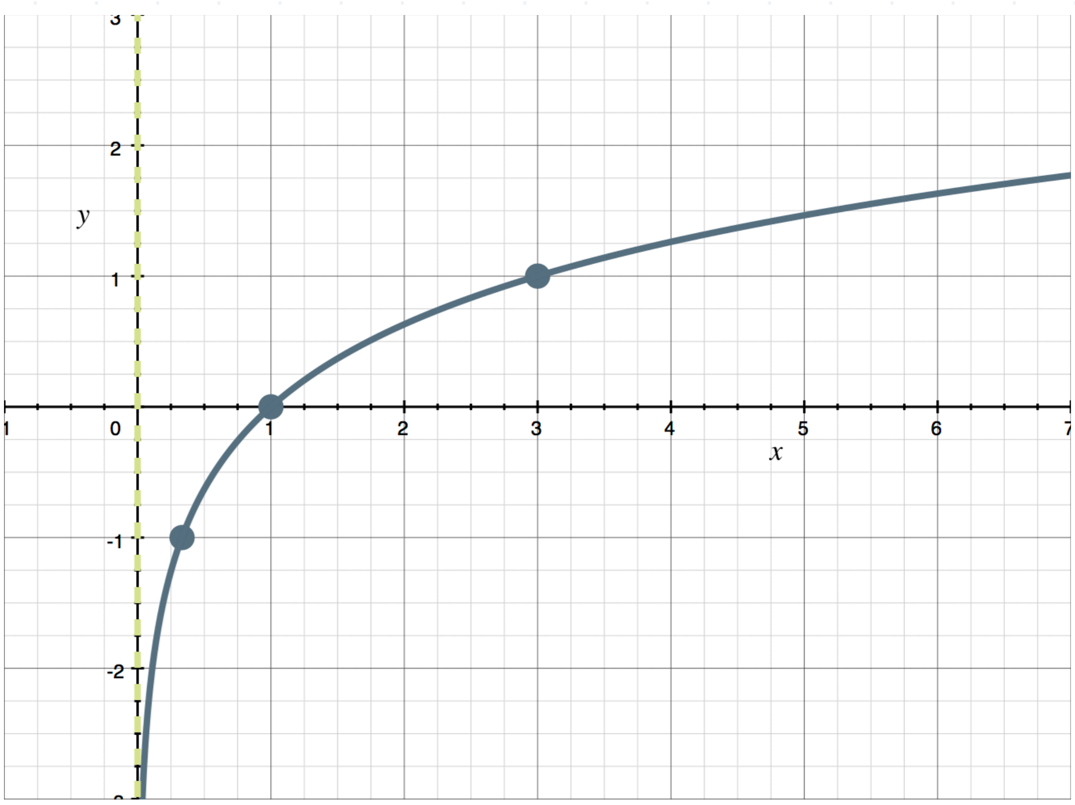
$$x = 1$$

For $y = 1$:

$$x = 3^1$$

$$x = 3$$

If we plot these points, along with the vertical asymptote $x = 0$, and then connect the points, we get the graph of $x = 3^y$.



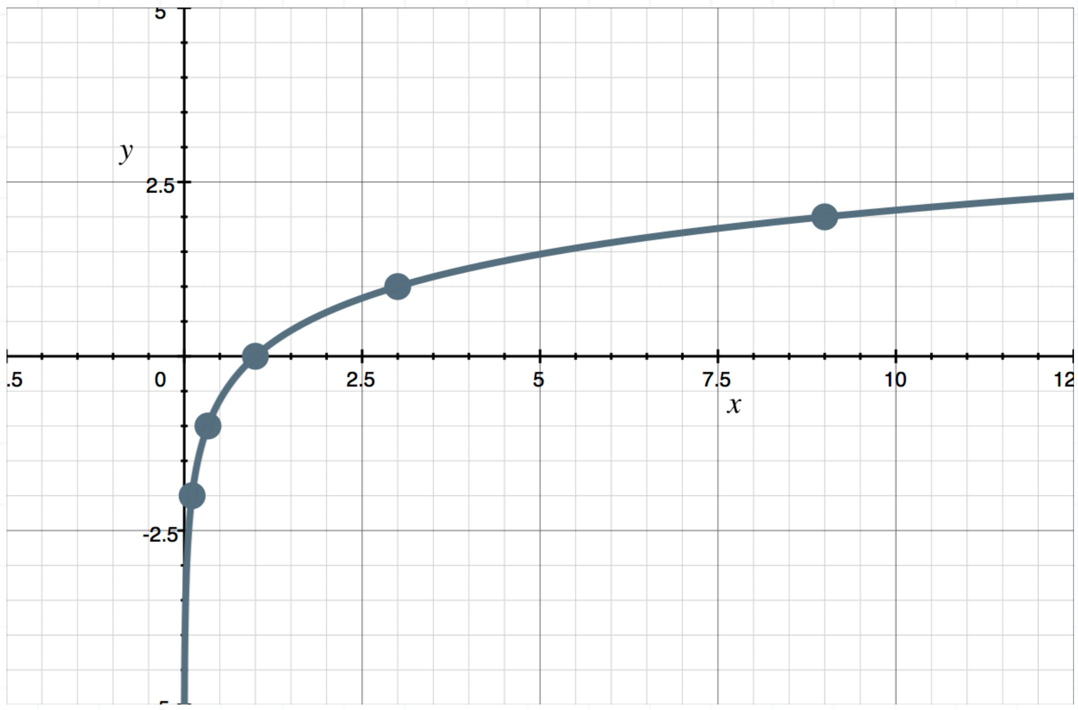
We could also graph the log function using a table of points. Since the logarithmic equation $y = \log_3 x$ corresponds to the exponential equation $x = 3^y$, we can find the coordinates of some points that satisfy this exponential equation. It'll be easier for us to plug in values of y, and see which x-values come out of the equation. For instance, if we plug $y = 0$ into $x = 3^y$, we get $x = 1$.

x			1		
y			0		

Let's fill out the rest of the chart with a few other easy-to-calculate values. It doesn't matter which values we pick, because we're just trying to get a few points that we can use to graph the logarithmic function.

x	1/9	1/3	1	3	9
y	-2	-1	0	1	2

With these five points, we should be able to pretty easily sketch the graph.



Properties of the graphs of log functions

For log functions in the form $y = \log_b(x)$ with $b > 1$,

- The x-intercept is $(1,0)$ because $0 = \log_b(1)$ means that $b^0 = 1$, which is true for any value of b.

- The point $(b,1)$ satisfies the function because $1 = \log_b(b)$ means that $b^1 = b$, which is true for any value of b.

- The point $(1/b, -1)$ satisfies the function because $-1 = \log_b(1/b)$ means that $b^{-1} = 1/b$, which is true for any value of b.

- The y-axis is a vertical asymptote.

- The domain is $(0,\infty)$ and the range is $(-\infty, \infty)$.

Additionally, we know that for log functions in the form $y = \log_b(x)$ with $b > 0$ and $b \neq 1$, the function is one-to-one, and the graph is decreasing for $0 < b < 1$ and increasing for $b > 1$.

So to graph the logarithmic function $f(x) = \log_b(x)$, we'll

1. Set up the vertical asymptote, $x = 0$, and plot the x-intercept $(1,0)$, as well as the points $(b,1)$ and $(1/b, -1)$.

2. Rewrite $f(x) = \log_b(x)$ in exponential form $b^y = x$ to find additional points if needed. Choose a few small values of y to find the corresponding x-values. Plot the points on the graph.

3. Connect the points with a smooth curve.

Just as we've built graphs of exponential functions in the previous lesson, we can use the general rule for logs to convert the logarithmic equation into its exponential form in order to graph the function. Let's do one more example.

Example

Graph the function $y = -\log_3(x-1)$.

First, we could use the general rule for logs to convert the logarithmic equation $y = -\log_3(x-1)$ to its exponential form,

$$x = 1 + \frac{1}{3^y}$$

Then we can follow the steps from the previous lesson, but this time by plugging in values of y to get values of x, starting with $y = 100$ and $y = -100$.

For $y = 100$:

$$x = 1 + \frac{1}{3^{100}}$$

$x = 1 +$ very small positive number

$$x = 1$$

For $y = -100$:

$$x = 1 + \frac{1}{3^{-100}}$$

$$x = 1 + 3^{100}$$

$x = 1 +$ very large positive number

$$x = \infty$$

Doing this allowed us to evaluate end behavior, and we've learned that the function has a vertical asymptote at $x = 1$, and heads up toward ∞ as y gets very small.

We'll plug in a few simple-to-calculate values for y.

For $y = -1$:

$$x = 1 + \frac{1}{3^{-1}}$$

$$x = 1 + 3$$

$$x = 4$$

For $y = 0$:

$$x = 1 + \frac{1}{3^0}$$

$$x = 1 + 1$$

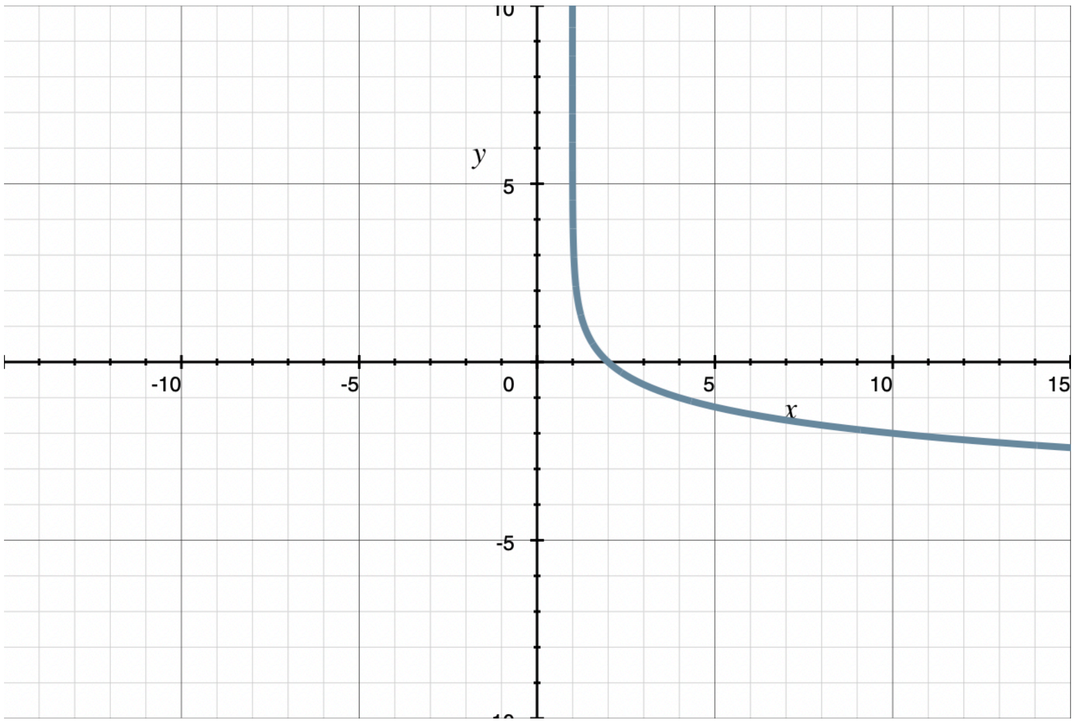
$$x = 2$$

For $y = 1$:

$$x = 1 + \frac{1}{3^1}$$

$$x = \frac{4}{3}$$

If we plot these points, along with the vertical asymptote $x = 1$, and then connect the points, we get the graph of the function.



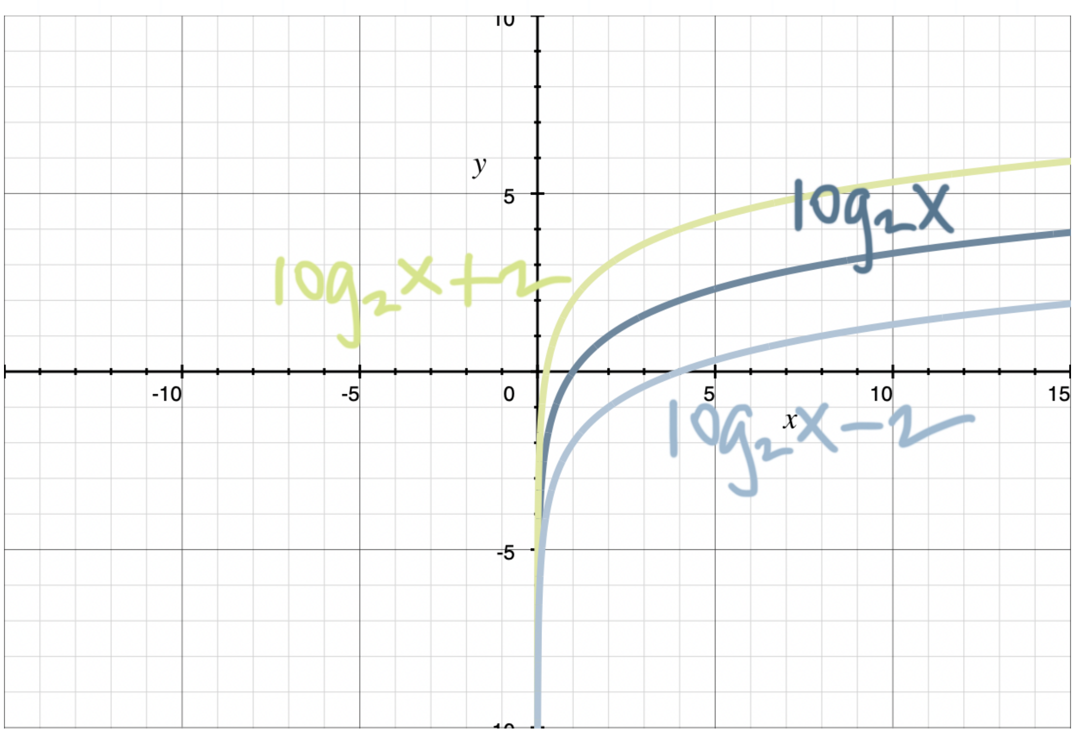
Graphing transformations of log functions

In the same way that we learned to transform exponential functions, we can also transform logarithmic functions. We'll again consider vertical and horizontal shifts, stretch and compressions, and reflections.

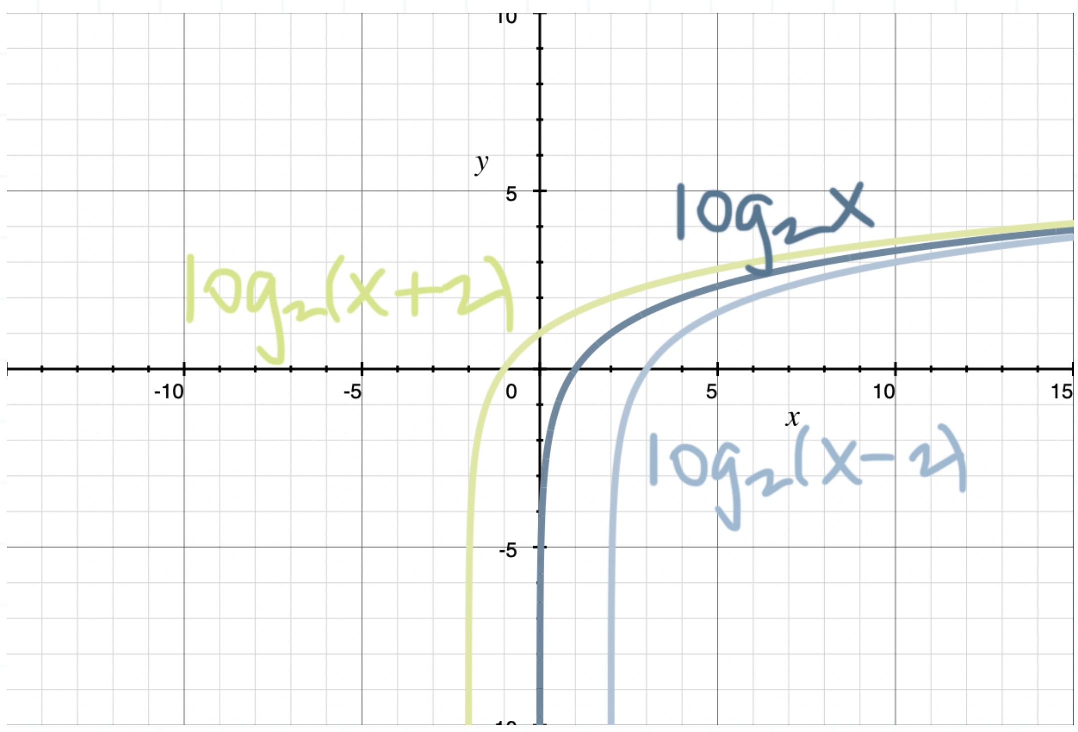
Vertical and horizontal shifts

Let's consider the parent function $f(x) = \log_b(x)$. Adding a constant d to the parent function gives us a vertical shift d units in the same direction as the sign of d.

For example, a sketch of the parent function $f(x) = \log_2(x)$ and this same function shifted vertically up 2 units and down 2 units gives



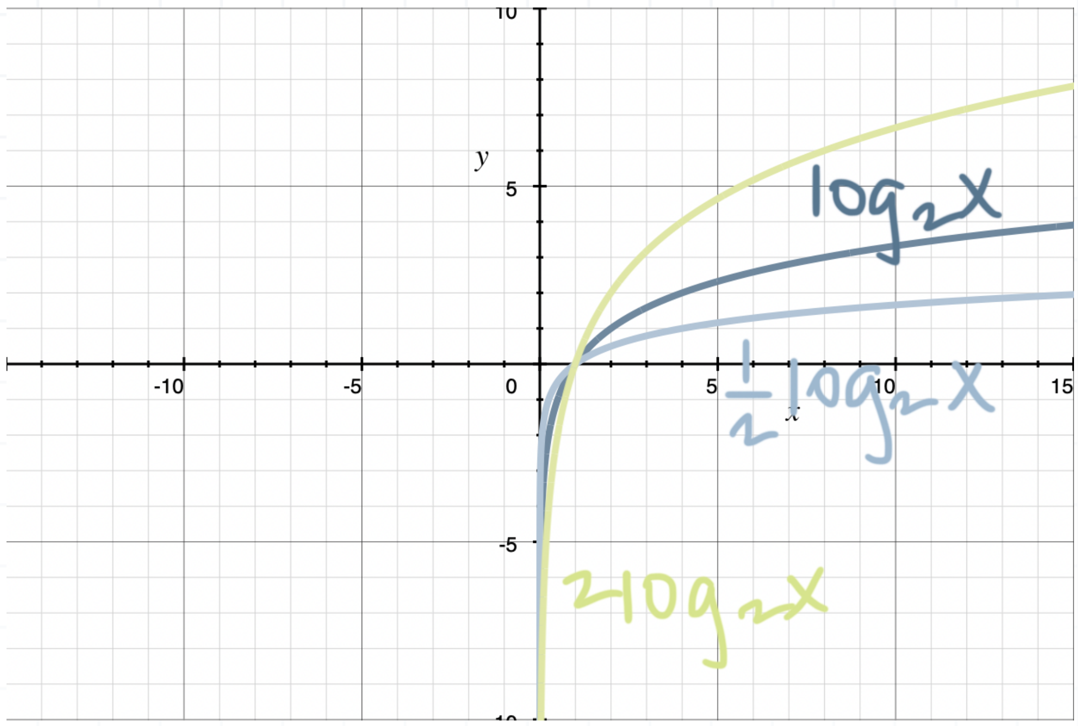
To shift a curve horizontally, we can add a constant c to the input of the parent function $f(x) = \log_b(x)$, but the direction of the shift is opposite the sign of c. So a sketch of the parent function $f(x) = \log_2(x)$ and this same function shifted horizontally left and right 2 units gives



Vertical and horizontal stretches and compressions

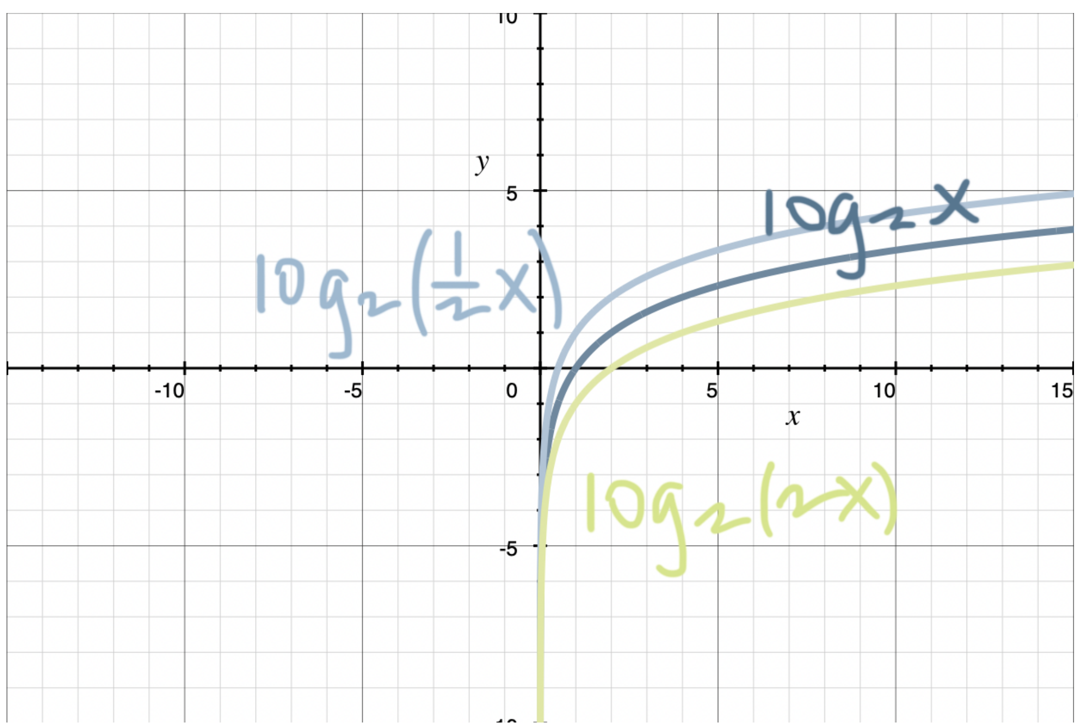
Let's consider the parent function $f(x) = \log_b(x)$. If we multiply the parent by some a where $0 < |a| < 1$ to get $f(x) = a\log_b(x)$, then $f(x) = \log_b(x)$ is being compressed vertically by a factor of a. But when $1 < |a|$, then $f(x) = \log_b(x)$ is being stretched.

For example, a sketch of the parent function $f(x) = \log_2(x)$ and this same function stretched and compressed vertically by a factor of 2 gives



If instead we multiply the input x by some k where $0 < |k| < 1$ to get $f(x) = \log_b(kx)$, then $f(x) = \log_b(x)$ is being stretched horizontally by a factor of k. But when $1 < |k|$, then $f(x) = \log_b(x)$ is being compressed.

For example, a sketch of the parent function $f(x) = \log_2(x)$ and this same function stretched and compressed horizontally by a factor of 2 gives

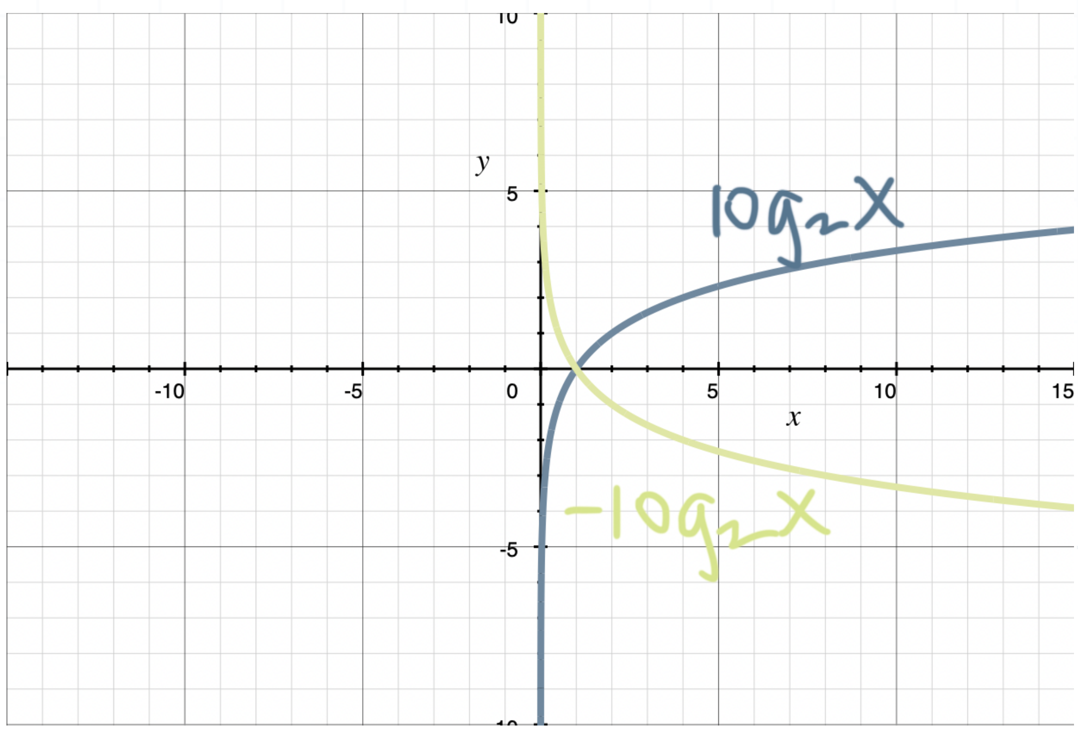


Let's do an example where we have to sketch a function with both a vertical and a horizontal stretch or compression.

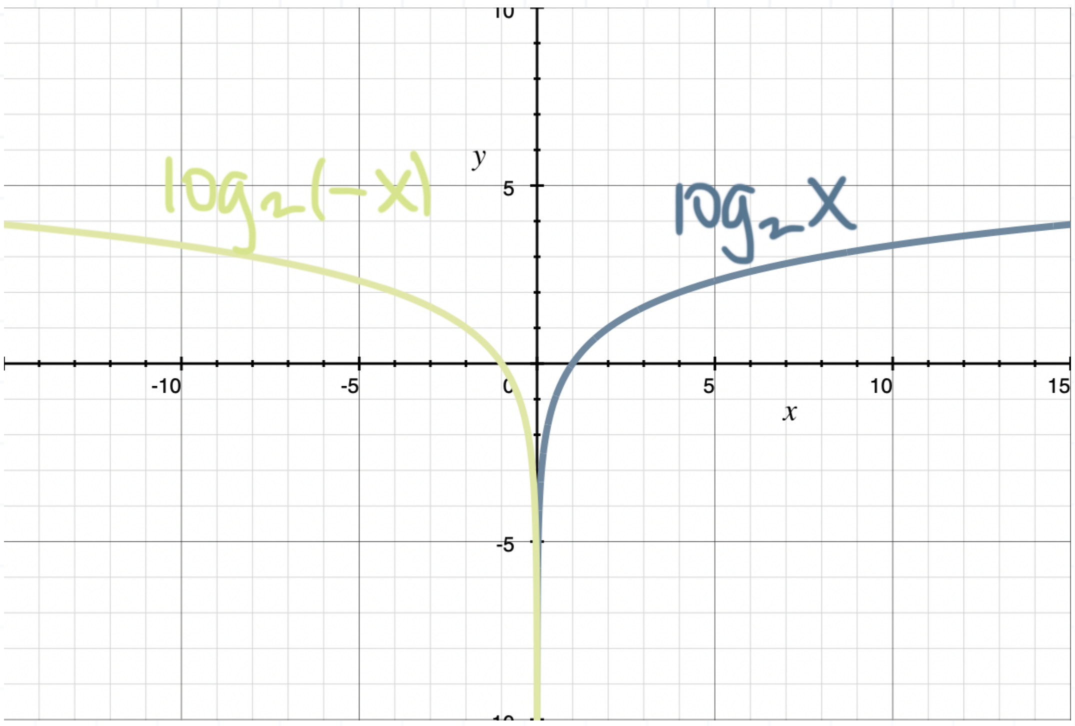
Vertical and horizontal reflections

It's also possible to reflect the graph across the x-axis and/or the y-axis. When we multiply the parent function $f(x) = \log_b(x)$ by -1, the graph gets reflected across the x-axis. But when we multiply the input of the parent by -1, the graph gets reflected across the y-axis.

For example, let's choose $f(x) = \log_2(x)$ again as the parent function. Its reflection across the x-axis is $g(x) = -\log_2(x)$,



and its reflection across the y-axis is $h(x) = \log_2(-x)$.



Combining transformations

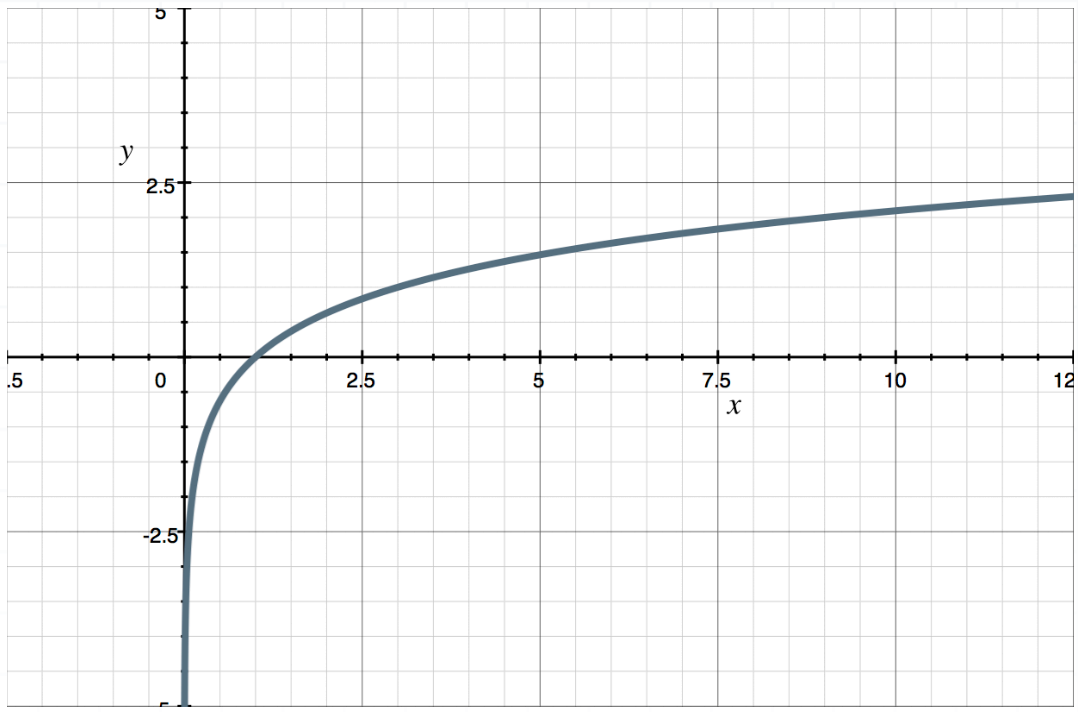
Now that we've seen this collection of transformations, let's summarize the order in which we should apply them, given multiple transformations in the same equation.

1. Horizontal stretch or compression
2. Horizontal shift
3. Horizontal reflection
4. Vertical stretch or compression
5. Vertical reflection
6. Vertical shift

Let's do an example where we apply these transformations in order.

Example

The graph of the logarithmic function $y = \log_3 x$ is given. Use that graph to sketch the graph of the function $y = -\log_3(x-1)$.



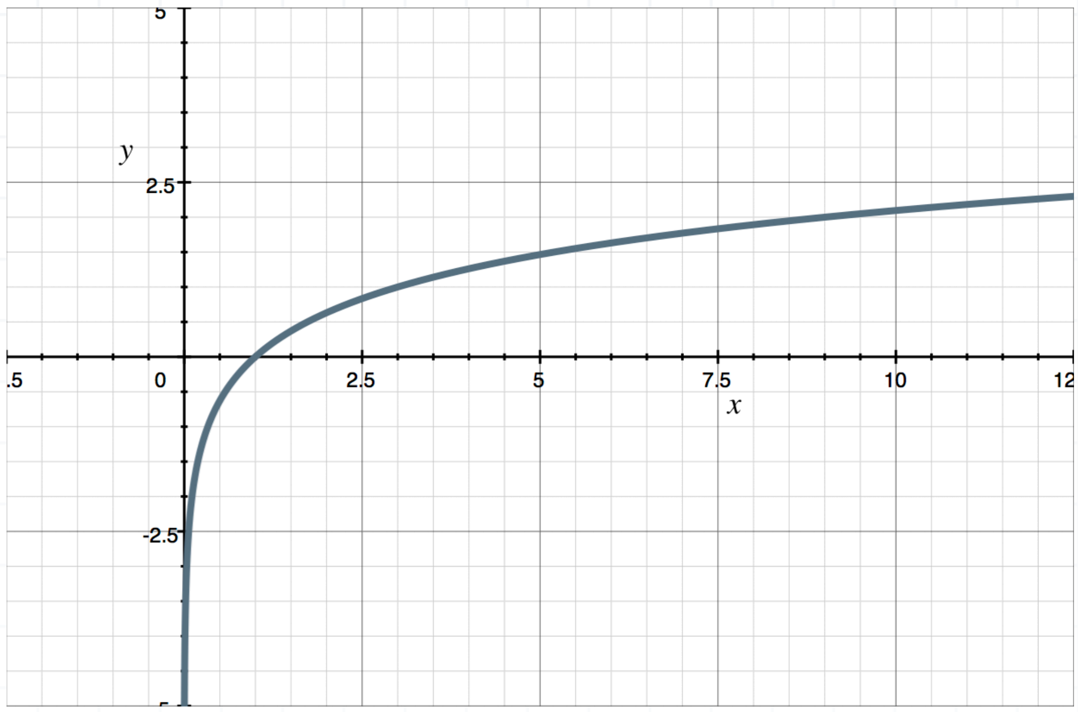
The function $-\log_3(x-1)$ is the result of applying a couple of transformations to the logarithmic function $\log_3 x$ in turn. We'll treat each transformation in a separate step, and we'll give different names to the function we obtain in different steps.

[1] $f(x) = \log_3 x$

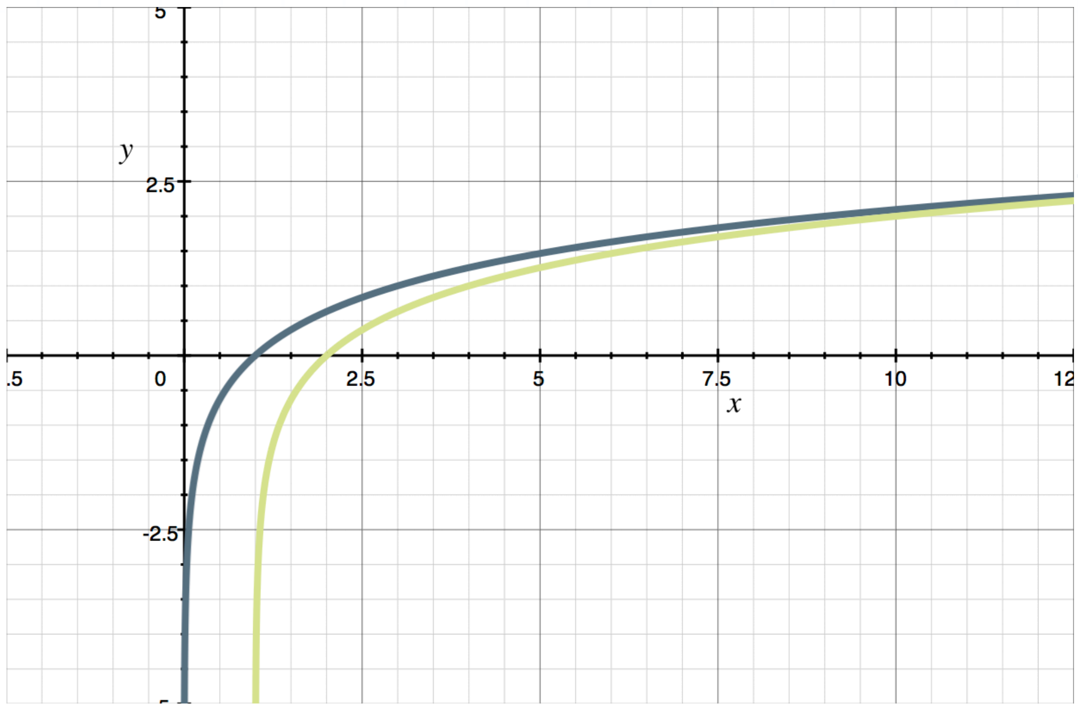
[2] $g(x) = \log_3(x-1)$

[3] $h(x) = -\log_3(x-1)$

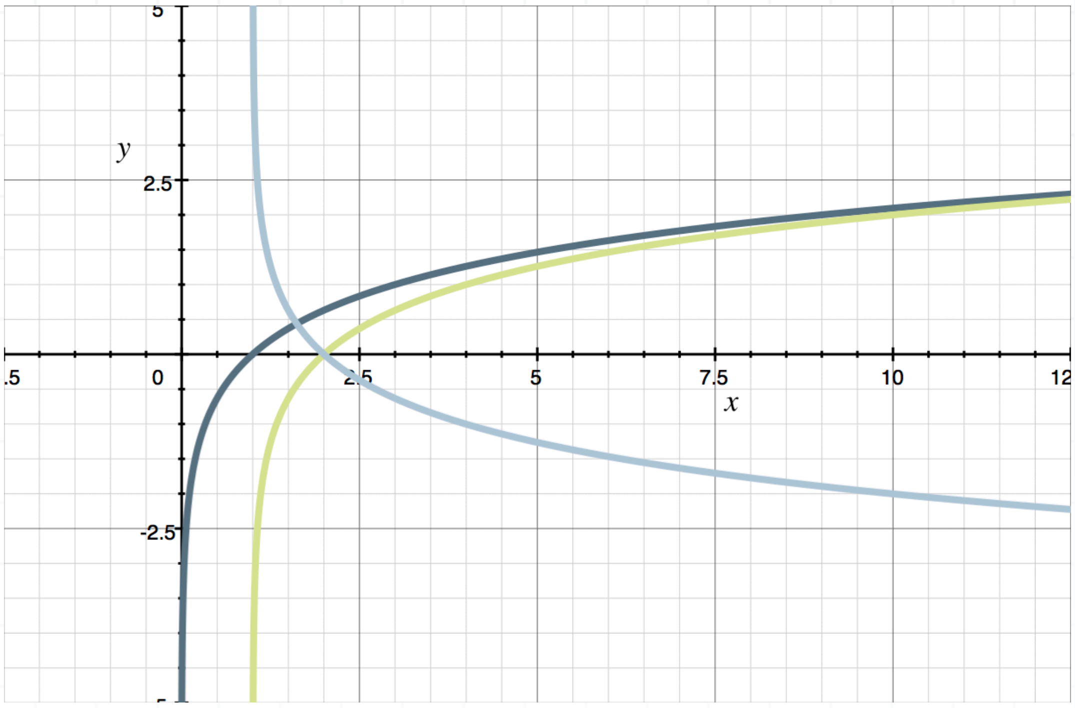
We were given the graph of $f(x)$.



If we substitute $x - 1$ for x in the expression for $f(x)$, we get the expression for $g(x)$: $\log_3(x - 1)$. This means that to get the graph of $g(x)$, we take the graph of $f(x)$ and shift it one unit to the right. If you're not sure about this, try plugging a few values of x into the function $g(x) = \log_3(x - 1)$. If we graph $g(x)$ on the same set of axes as $f(x)$, we get



To get the value of $h(x)$, we multiply the value of $g(x)$ by -1, which means that the graph of $h(x)$ is just the reflection of the graph of $g(x)$ with respect to the x-axis.



To summarize, we started with the function $\log_3 x$ and its graph. To get the graph of $-\log_3(x-1)$, we applied one transformation at a time.

[1] $f(x) = \log_3 x$

[2] $g(x) = f(x-1) = \log_3(x-1)$

[3] $h(x) = -1 \cdot g(x) = -\log_3(x-1)$

Finding the vertical asymptote

We know that the vertical asymptote of the log function $f(x) = \log_b(x)$ is always $x = 0$. But if that log function has undergone any kind of horizontal shift, the vertical asymptote will shift as well.

To find the shifted asymptote, we can set the argument of the shifted log function equal to 0. Let's do an example.

Example

What is the vertical asymptote of $f(x) = -2\ln(2x+6) - 4$?

To find the vertical asymptote of the logarithmic function, we'll set its argument equal to zero, then solve for x.

$$2x + 6 = 0$$

$$2x = -6$$

$$x = -3$$

The vertical asymptote is $x = -3$.