Absolute value equations

The **absolute value** of any real number x, defined as $|x|$, is the distance of x from 0 (or from the origin). For instance, $|2| = 2$ and $|-2| = 2$.

In other words, opposite values of x have the same absolute value because they're both equally distant from 0.

To describe absolute value in simple terms, taking the absolute value of something that's positive won't change the value of that positive thing. But taking the absolute value of something that's negative will change that negative thing into a positive thing. It just removes the negative.

Equations with one absolute value

Given an equation $|f(x)| = a$, with $a > 0$, then $f(x) = a$ or $f(x) = -a$. It's important to say that, even though $|f(x)|$ is a distance, which by definition can't be negative, an absolute value equation can have negative solution, which is why it's possible to find $f(x) = -a$.

- If $a > 0$, then $|f(x)| = a$ has two solutions.
- If $a = 0$, then $|f(x)| = a$ has one solution.
- If $a < 0$, then $|f(x)| = a$ has no solution.

To solve an absolute value equation, we'll follow three steps.

1. Isolate the absolute value expression on one side of the equation.

2. Check the value of a. If $a > 0$, then set up and solve two equations, $f(x) = a$ and $f(x) = -a$. If $a = 0$, set up the equation $f(x) = 0$. And if $a < 0$, we know the equation has no solutions.

3. For any values we find in Step 2, verify that they satisfy the original absolute value equation.

Let's look at an example so that we can see how to work through these steps to solve the absolute value equation.

Example

Solve $|2x - 4| = 4$.

Set up the two related equations,

$2x - 4 = 4$

$2x - 4 = -4$

then solve both equations for x.

$2x - 4 = 4$

$2x = 8$

$x = 4$

$2x - 4 = -4$

$2x = 0$

$x = 0$

Now that we have these values, we'll plug them into the original absolute value equation to see whether they satisfy it.

$|2x - 4| = 4$

$|2x - 4| = 4$

$$|2(4) - 4| = 4 \qquad |2(0) - 4| = 4$$

$$|8 - 4| = 4 \qquad |0 - 4| = 4$$

$$|4| = 4 \qquad |-4| = 4$$

Because both equations are true, we can conclude that both $x = 4$ and $x = 0$ are solutions to the absolute value equation.

Let's try one more example with one absolute value in the equation.

Example

Solve $2|2x + 1| - 3 = 7$.

First, isolate the absolute value expression on the left side by adding 3 to both sides of the equation,

$$2|2x + 1| - 3 + 3 = 7 + 3$$

$$2|2x + 1| = 10$$

and dividing both sides by 2.

$$\frac{2|2x + 1|}{2} = \frac{10}{2}$$

$$|2x + 1| = 5$$

Now set up the two related equations,

$$2x+1=5 \qquad\qquad 2x+1=-5$$

then solve both equations for x.

$$2x+1=5 \qquad\qquad 2x+1=-5$$

$$2x=4 \qquad\qquad 2x=-6$$

$$x=2 \qquad\qquad x=-3$$

Now that we have these values, we'll plug them into the original absolute value equation to see whether they satisfy it.

$$2|2x+1|-3=7 \qquad\qquad 2|2(-3)+1|-3=7$$

$$2|2(2)+1|-3=7 \qquad\qquad 2|2(2)+1|-3=7$$

$$|5|=5 \qquad\qquad |-5|=5$$

Because both equations are true, we can conclude that both $x=2$ and $x=-3$ are solutions to the absolute value equation.

Equations with two absolute values

Now we want to consider equations that contain two absolute value expressions, like $|m|=|n|$.

In an equation like this one, m and n are either equal to each other or opposites (negatives) of each other. So if $|m|=|n|$, then

$m = n$ or $m = -n$

Let's try an example where we work through solving an equation that contains two absolute value expressions.

Example

Solve $|-2x+1| = |4x-5|$.

Split the inequality into two separate equations, one where the expressions are equal to each other, and one where the expressions are opposites of each other.

$$-2x + 1 = 4x - 5$$

$$-2x + 1 = -(4x - 5)$$

Now we'll solve equation equation for x.

$$-2x + 1 + 5 = 4x - 5 + 5$$

$$-2x + 6 = 4x$$

$$-2x + 2x + 6 = 4x + 2x$$

$$6 = 6x$$

$$\frac{6}{6} = \frac{6}{6}x$$

$$x = 1$$

$$-2x + 1 = -4x + 5$$

$$-2x + 1 - 1 = -4x + 5 - 1$$

$$-2x = -4x + 4$$

$$-2x + 4x = -4x + 4x + 4$$

$$2x = 4$$

$$\frac{2}{2}x = \frac{4}{2}$$

$$x = 2$$

Now that we have these values, we'll plug them into the original absolute value equation to see whether they satisfy it.

$$|-2x+1| = |4x-5| \qquad\qquad |-2x+1| = |4x-5|$$

$$|-2(1)+1| = |4(1)-5| \qquad\qquad |-2(2)+1| = |4(2)-5|$$

$$|-1| = |-1| \qquad\qquad |-3| = |3|$$

Because both equations are true, we can conclude that both $x = 1$ and $x = 2$ are solutions to the absolute value equation.