What is a logarithm?

When we first start learning about logarithms, it's helpful to think about how they're related to exponents, since exponents are something we already understand.

Exponents vs. logarithms

Remember that an exponent tells us how many times to multiply the base by itself. In other words, in 2^3, the exponent of 3 tells us that the base 2 should be multiplied three times. And that tells us how to find the value of 2^3:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

We already know that this is what exponents do. What we haven't learned yet is what to do when we have something like this instead:

$$2^x = 8$$

Logarithms are what we would use to solve for x in this equation, because logarithms let us solve for the value of a variable that appears in an exponent. **Logarithms** tell us how many times we multiply one number by itself in order to get a different number. So when we already have the base (in this case 2), and we have the result (in this case 8), logarithms (or "logs," for short) tell us what the value of the exponent needs to be in order to make the equation true.

To express the solution to the equation $2^x = 8$ (which is called an **exponential equation** because the exponent is/includes a variable) in log form, we write

$$\log_2(8) = 3$$

In the equation $\log_2(8) = 3$, the 2 is the **base** and the 8 is the **argument** of the log function. The term "argument" isn't used exclusively with the log function. In fact, it can be used with any function, to mean the input. For example, the argument of the function f in $f(-5x)$ is $-5x$, and the argument of the function g in $g(7)$ is 7. We can read $\log_2(8) = 3$ in either of the following two ways:

"log base 2 of 8 is 3," or

"the log of 8 with base 2 is 3," or

"the base-2 log of 8 is 3"

Realize that the "base" of a log is the same as the "base" in an expression of the form a^b. Remembering that can help us when we're converting back and forth between logs and exponents.

Not multiplication

It's also worth pointing out that, in the equation $\log_2(8) = 3$, $\log_2$ is not a number that's supposed to be multiplied by 8. It's tempting to think that the parentheses in the expression $\log_2(8)$ mean multiplication, but they don't. Instead, the big number that comes after the base is the "argument" of the log function.

Think back to function notation, where we talked about functions like $f(x) = 2x + 1$. Remember that the function notation $f(x)$ doesn't mean that f should be multiplied by x. Instead, it means either "the value of the function f at x" or "f is a function of x."

And the same is true of logarithms. In $\log_2(8) = 3$, the notation $\log_2(8)$ means "the value of the log function at 8."

The general log rule

This basic log rule that relates exponents to logs can be written as follows:

> Given the exponential equation $a^x = y$,
>
> the associated logarithmic equation is $\log_a(y) = x$,
>
> and vice versa.

Let's do an example where we convert from one form to the other.

Example

Convert the equation $5^4 = 625$ to its logarithmic form.

If we match the quantities in the equation $5^4 = 625$ to those in the equation $a^x = y$ from the general log rule, we see that

$$a = 5$$

$$x = 4$$

$$y = 625$$

Plugging these values into the equation $\log_a(y) = x$ from the general log rule, we get

$$\log_5(625) = 4$$

Realize that we could have just as easily started with the equation $\log_5(625) = 4$; identified a, x, and y; and converted it to the form $a^x = y$.

In both forms, the equation says "The multiplication of 5 four times gives 625."

Let's do an example where we solve a log equation for a variable.

Example

Solve the equation.

$$\log_2(16) = x$$

This log equation is asking "How many times do we have to multiply 2 in order to get 16?" Using the general log rule, we can rewrite the equation as

$$2^x = 16$$

We know that

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

so the solution to the equation is $x = 4$.

We can solve log problems regardless of which quantity is unknown. In the last example, $\log_2(16) = x$, the value of $\log_2(16)$ was the unknown quantity. If instead the base were the unknown quantity, we would've had

$$\log_x(16) = 4$$

$$x^4 = 16$$

$$x = 2$$

Or if instead the argument of the log function were the unknown quantity, we would've had

$$\log_2(x) = 4$$

$$2^4 = x$$

$$16 = x$$