Rationalizing complex denominators

Remember that a **complex number** is a number that can be written in the form $a + bi$, where a and b are real numbers and i is the imaginary number $\sqrt{-1}$. The number a is the **real part** of the complex number, and bi is the **imaginary part**.

An **imaginary number** (also called a pure imaginary number) is a complex number whose real part is 0. For example, $-6i$ and $4i$ are imaginary numbers. So every complex number can be written as the sum of a real number and an imaginary number.

Multiplying complex numbers

To multiply two or more complex numbers, we use the Distributive Property. For example, if we have two complex numbers $x = a + bi$ and $y = c + di$, then their product is

$$xy = (a + bi)(c + di)$$

Now we apply the Distributive Property of Multiplication to expand the product.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Sometimes we need to multiply complex numbers with only real or imaginary parts. Imagine we multiply two complex numbers, one of which has just a real part while the other has both real and imaginary parts. The product would be given by

$$a(c + di) = ac + adi$$

For example, if we multiply 4 by $2 + 5i$, then we get

$$4(2 + 5i) = 8 + 20i$$

Now let's consider the other scenario. If we need to multiply a pure imaginary number bi by a complex number $c + di$, then the product is

$$(bi)(c + di) = bci - bd$$

For example, if we multiply a complex number $1 + 4i$ by $-2i$, then we get

$$(-2i)(1 + 4i) = -2i - 8i^2$$

$$(-2i)(1 + 4i) = -2i + 8$$

Multiplying by the conjugate

There are a few more things we need to understand about multiplying complex numbers.

1. The **conjugate** of a complex number (usually called the **complex conjugate** of that number) is formed by changing the sign of the imaginary part and leaving the real part unchanged.

   For example, the complex conjugate of $5 + 3i$ is $5 - 3i$.

2. We can multiply any number times 1 without changing its value, and 1 can be written as any nonzero number or expression divided by itself.

For example,

$$\frac{3}{5+3i} = \frac{3}{5+3i} \cdot 1 = \frac{3}{5+3i} \cdot \frac{5-3i}{5-3i}$$

because

$$\frac{5-3i}{5-3i} = 1$$

To rationalize a fraction that has a complex number in the denominator, we multiply it by the fraction in which both the numerator and the denominator are the complex conjugate of that complex number (just like in this last example). This is called the **conjugate method**.

3. Use FOIL or the Distributive Property to multiply complex numbers, then simplify.

An example of FOIL (multiplying the first terms, the outer terms, the inner terms, and last terms, in that order):

$$(5+3i)(4+2i)$$

$$5 \cdot 4 + 5 \cdot 2i + 3i \cdot 4 + 3i \cdot 2i$$

$$20 + 10i + 12i + 6i^2$$

Combining like terms and replacing i^2 with -1, we get

$$20 + 22i + 6(-1)$$

$$20 - 6 + 22i$$

$$14 + 22i$$

An example of the Distributive Property.

$$5(3 - 4i)$$

$$5(3) + 5(-4i)$$

$$15 + (-20i)$$

$$15 - 20i$$

Now let's use the conjugate method to simplify a fraction that has a complex number in the denominator.

Example

Use the conjugate method to simplify the expression.

$$\frac{3 - 4i}{-2 + i}$$

We can use the conjugate method to get the imaginary number i, out of the denominator. The complex conjugate of $-2 + i$ is $-2 - i$.

$$\frac{3 - 4i}{-2 + i} \cdot \frac{-2 - i}{-2 - i}$$

$$\frac{(3-4i)(-2-i)}{(-2+i)(-2-i)}$$

Now that we have a binomial multiplication problem, we need to make sure that (in the numerator and denominator separately) we multiply the first terms, outer terms, inner terms, and last terms.

$$\frac{-6-3i+8i+4i^2}{4+2i-2i-i^2}$$

$$\frac{4i^2+5i-6}{-i^2+4}$$

Replacing i^2 with -1, and then combining like terms, we get

$$\frac{4(-1)+5i-6}{-(-1)+4}$$

$$\frac{-4+5i-6}{1+4}$$

$$\frac{5i-10}{5}$$

$$\frac{5i}{5}-\frac{10}{5}$$

$$i-2$$

$$-2+i$$

Let's look at another example of rationalizing a complex denominator.

Example

Simplify.

$$\frac{10i^2 - 5i}{-6 + 6i}$$

First, we'll rewrite the expression as

$$\frac{10(-1) - 5i}{-6 + 6i}$$

$$\frac{-10 - 5i}{-6 + 6i}$$

Now we can use the conjugate method to get the imaginary number i out of the denominator. The complex conjugate of $-6 + 6i$ is $-6 - 6i$.

$$\frac{-10 - 5i}{-6 + 6i} \cdot \frac{-6 - 6i}{-6 - 6i}$$

$$\frac{(-10 - 5i)(-6 - 6i)}{(-6 + 6i)(-6 - 6i)}$$

Now that we have a binomial multiplication problem, we need to make sure that (in the numerator and denominator separately) we multiply the first terms, outer terms, inner terms, and last terms.

$$\frac{60 + 60i + 30i + 30i^2}{36 + 36i - 36i - 36i^2}$$

$$\frac{60 + 90i + 30i^2}{36 - 36i^2}$$

Replacing i^2 with -1, we get

$$\frac{60 + 90i + 30(-1)}{36 - 36(-1)}$$

$$\frac{60 + 90i - 30}{36 + 36}$$

$$\frac{30 + 90i}{72}$$

Divide out 6, which goes evenly into 30, $90i$, and 72.

$$\frac{5 + 15i}{12}$$

We can also write this as the sum of two fractions.

$$\frac{5}{12} + \frac{15i}{12}$$

Then we can reduce the second fraction to lowest terms, which gives

$$\frac{5}{12} + \frac{5i}{4}$$

$$\frac{5}{12} + \frac{5}{4}i$$