Completing the square

We just saw how to use the Zero Theorem to find the roots of a quadratic equation. All we had to do was factor the quadratic, set each factor equal to 0 individually, and then solve each of those equations to find the roots.

But how do we find the roots of a quadratic when it won't easily factor? If the quadratic won't factor, or even if we just can't figure out how to factor it, we can always complete the square in order to find the roots.

How to complete the square

Completing the square is one method we can use when we can't find the roots of a quadratic by factoring.

In standard form, we write a quadratic equation as $ax^2 + bx + c = 0$. But in order to complete the square, we really need the coefficient on x^2 to be $a = 1$. So if the quadratic we're working on has $a \neq 1$, then our first step has to be to divide through both sides of the equation by a, so that the coefficient on x^2 becomes 1.

Once that's done, we could define the quadratic with new coefficients as $x^2 + bx + c = 0$. From here, we'll follow the same set of steps every time in order to complete the square.

1. Calculate $b/2$, then square the result to get $(b/2)^2$.

2. Add $(b/2)^2$ to both sides of the equation to get

$$x^2+bx+\left(\frac{b}{2}\right)^2+c=\left(\frac{b}{2}\right)^2$$

3. Subtract c from both sides.

$$x^2+bx+\left(\frac{b}{2}\right)^2=\left(\frac{b}{2}\right)^2-c$$

4. Factor the left side. It will always factor as a perfect square.

$$\left(x+\frac{b}{2}\right)\left(x+\frac{b}{2}\right)=\left(\frac{b}{2}\right)^2-c$$

$$\left(x+\frac{b}{2}\right)^2=\left(\frac{b}{2}\right)^2-c$$

5. Take the square root of both sides, remembering to include a $\pm$ sign on the right side, then subtract $b/2$ from both sides.

$$x+\frac{b}{2}=\pm\sqrt{\left(\frac{b}{2}\right)^2-c}$$

$$x=-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c}$$

The number of solutions

Using the method of completing the square, we now know that, given a quadratic equation in the form $x^2 + bx + c = 0$, the root(s) of the quadratic will always take the form

$$x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$$

Based on the form of this solution, we can see that the quadratic can have either zero real roots, one real root, or two real roots, depending on the value of the square root in this solution equation.

- If $\left(\frac{b}{2}\right)^2 - c < 0$, then the quadratic has zero real roots (the roots are complex)
- If $\left(\frac{b}{2}\right)^2 - c = 0$, then the quadratic has one root, $x = -\frac{b}{2}$
- If $\left(\frac{b}{2}\right)^2 - c > 0$, then the quadratic has two roots, $x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}$

Let's do an example so that we can see these steps in action.

Example

Solve for x by completing the square.

$$x^2 + 6x + 4 = 0$$

There is no pair of factors of 4 that have a sum of 6, so we'll need to solve by completing the square. In this quadratic, $b = 6$, so $b/2 = 6/2 = 3$, and therefore $(b/2)^2 = 3^2 = 9$.

Add 9 to both sides of the equation, then subtract 4 from both sides.

$$x^2 + 6x + 9 + 4 = 0 + 9$$

$$x^2 + 6x + 9 = 5$$

Factor what remains on the left side as a perfect square.

$$(x + 3)(x + 3) = 5$$

$$(x + 3)^2 = 5$$

Take the square root of both sides, then subtract 3.

$$x + 3 = \pm\sqrt{5}$$

$$x = -3 \pm \sqrt{5}$$

These values of x, $x = -3 - \sqrt{5}$ and $x = -3 + \sqrt{5}$, are the roots of the quadratic. Because the value under the square root was positive, $5 > 0$, we found two roots for the quadratic equation.

Complex roots

We said earlier that the quadratic would have no real roots when $(b/2)^2 - c < 0$. That's because we can't take the square root of negative number. At least, we can't do it with real numbers alone.

But there are actually another set of numbers that we can work with, complex numbers. **Complex numbers** are numbers that include both real and imaginary numbers. An **imaginary number** is any number that includes the imaginary number i, where $i = \sqrt{-1}$.

Let's try an example where the roots of the quadratic are complex.

Example

Solve for x by completing the square.

$$x^2 - 4x + 7 = 0$$

There is no pair of factors of 7 that have a sum of -4, so we'll need to solve by completing the square. In this quadratic, $b = -4$, so $b/2 = -4/2 = -2$, and therefore $(b/2)^2 = (-2)^2 = 4$.

Add 4 to both sides of the equation, then subtract 7 from both sides.

$$x^2 - 4x + 4 + 7 = 0 + 4$$

$$x^2 - 4x + 4 = -3$$

Factor what remains on the left side as a perfect square.

$$(x - 2)(x - 2) = -3$$

$$(x-2)^2 = -3$$

Take the square root of both sides, then subtract 2.

$$x - 2 = \pm\sqrt{-3}$$

$$x = 2 \pm \sqrt{-3}$$

This is where imaginary numbers come in. We can rewrite $\sqrt{-3}$ as $\sqrt{3}\sqrt{-1}$. Then, because $i = \sqrt{-1}$, we can rewrite the solutions as

$$x = 2 \pm \sqrt{3}\sqrt{-1}$$

$$x = 2 \pm \sqrt{3}i$$

So, while the quadratic has no solutions in terms of real numbers only, it does have the two complex roots $x = 2 - \sqrt{3}i$ and $x = 2 + \sqrt{3}i$.