Solving systems with substitution

Now that we understand how to solve two-step problems, we at least have the idea in our heads that two equations or expressions can both be related to the same variable(s).

Simultaneous equations

With that in mind, we want to start looking at how to solve systems of equations, like

$$y = x + 3$$

$$2x - 3y = 10$$

Notice that we can't solve either of these equations by themselves. If we look at the first equation, there's no way to solve for the value of y without knowing the value of x, but there's also no way to solve for the value of x without knowing the value of y. So we're stuck. And the same is true for the second equation.

However, if we can consider the equations as a pair, then we can use them to solve for both variables simultaneously. For that reason, **systems of equations** are sometimes also called **simultaneous equations**.

As a rule of thumb, it's only possible to solve systems when we have at least as many equations as we do variables. In other words, up to now we've been working with one equation in terms of one variable, like

$3x + 4 = 10$, so we've been able to solve them. If we move to two variables, one equation won't give us enough information to solve for both values; we'll need two equations. And if we want to solve for the value of three variables, we'll need three equations to do that.

Keep in mind, even if we have n number of equations to solve for n variables, that doesn't *guarantee* that we'll still be able to find a solution for every variable.

Having n number of equations is just the *minimum* requirement for solving for n variables. If we have fewer than n equations, we know immediately that we won't be able to solve for the values of n variables. But depending on the specific equations we have in our system, it's possible that we'll need more than that minimum number of n equations.

For example, if we want to solve for x and y, we know we need two equations because we have two variables. But we couldn't just use the system

$$y = x + 3$$

$$2y = 2x + 6$$

It looks like we have two equations here, but if we divide the second equation by 2, we $y = x + 3$, which is identical to the first equation. Because the equations are actually identical, the second one doesn't give us any new information, which means we won't have enough to solve for x and y.

Solving with substitution

But if we do have n number of equations for n variables, and those equations are sufficiently different, we can use them to solve for the values of n number of variables in the system.

There are a few methods we can use to solve the system, but the first one we'll look at is substitution. This method is similar to the two-step problems we looked at previously, because we'll be solving one equation for an expression, and then plugging that expression to the other equation.

Here are the steps we'll follow when we use the **substitution** method:

1. Get a variable by itself in one of the equations.

2. Substitute the expression from step 1 into the other equation.

3. Solve the equation in step 2 for the remaining variable.

4. Substitute the result from step 3 into the equation from step 1.

Let's look at an example so that we can see these steps in action.

Example

Find the solution to the system of equations.

$$y = x + 3$$

$$2x - 3y = 10$$

Since y is already solved for in the first equation, step 1 is completed, and we'll go on to step 2 by substituting $x+3$ for y into the second equation.

$$2x-3y=10$$

$$2x-3(x+3)=10$$

Using substitution, we now have one equation in terms of one variable, so we can move on to step 3 and solve this equation like we normally would. Start by distributing the -3.

$$2x-3x-9=10$$

$$-x-9=10$$

Add 9 to both sides.

$$-x-9+9=10+9$$

$$-x=19$$

Multiply both sides by -1.

$$-x(-1)=19(-1)$$

$$x=-19$$

Now that we have the value of one of the variables, we can move on to step 4, where we substitute $x=-19$ back into the first equation to find y.

$$y=x+3$$

$$y=-19+3$$

$$y = -16$$

The solution to this system is $(-19, -16)$. If we plug these values back into the original pair of equations, they should both be satisfied.

$$y = x + 3$$

$$-16 = -19 + 3$$

$$-16 = -16$$

and

$$2x - 3y = 10$$

$$2(-19) - 3(-16) = 10$$

$$-38 + 48 = 10$$

$$10 = 10$$

Let's try another example where we use substitution to solve the system.

Example

Find the solution to the system of equations.

$$x - 2y = 6$$

$$4x + 5y = 32$$

First, we need to get a variable by itself. It's easiest to get x by itself in the first equation by adding $2y$ to both sides.

$$x - 2y + 2y = 6 + 2y$$

$$x = 6 + 2y$$

Now we'll substitute $x = 6 + 2y$ into the second equation.

$$4x + 5y = 32$$

$$4(6 + 2y) + 5y = 32$$

Now that we have one equation in one variable, we can solve it by itself.

$$24 + 8y + 5y = 32$$

$$24 + 13y = 32$$

$$24 - 24 + 13y = 32 - 24$$

$$13y = 8$$

$$y = \frac{8}{13}$$

To find x, plug $y = 8/13$ into the first equation and solve it for x.

$$x = 6 + 2y$$

$$x = 6 + 2\left(\frac{8}{13}\right)$$

$$x = \frac{78}{13} + \frac{16}{13}$$

$$x = \frac{94}{13}$$

The solution to this system is $(94/13,8/13)$. If we plug these values back into the original pair of equations, they should both be satisfied.

$$x - 2y = 6$$

$$\frac{94}{13} - 2\left(\frac{8}{13}\right) = 6$$

$$\frac{94}{13} - \frac{16}{13} = 6$$

$$\frac{78}{13} = 6$$

$$6 = 6$$

and

$$4x + 5y = 32$$

$$4\left(\frac{94}{13}\right) + 5\left(\frac{8}{13}\right) = 32$$

$$\frac{376}{13} + \frac{40}{13} = 32$$

$$\frac{416}{13} = 32$$

$$32 = 32$$

Number of solutions

In both of these last examples, we were able to find exactly one unique solution to the system. But this won't always be the case. In fact, there are three possible solutions to a system of equations.

- one solution (called the unique solution), or
- no solutions, or
- infinitely many solutions.

In the one solution case, there *is* a solution to the system, and there is only one solution to the system. If this is the case, we'll be able to find the solution using the substitution method, like we did in these last couple of examples.

In the no solutions case, as we're solving the system, somewhere along the way we'll end up with an equation like

$$a = c$$

where a and c are constants with different values. For instance, maybe we get to a point in the solution where we find $7 = 11$. This is clearly impossible, so we know the system has no solutions. Graphically, this means we have to parallel lines, so they never cross each other, and there is therefore no intersection of the lines and no solution to the system.

In the infinitely many solutions case, as we're solving the system, somewhere along the way we'll end up with an equation like

$$a = a$$

For instance, maybe we get to a point where we find $7 = 7$, or maybe $0 = 0$. If that's our ending point, then we know the system has infinitely many solutions. Graphically, this means we have two identical lines, so they sit on top of each other and every point along both lines is also a point on the other line, and there are therefore infinitely many solutions.