Quadratic formula

Now that we know how to complete the square, we can introduce the quadratic formula, which is certainly one of the most famous formulas from Algebra.

The quadratic formula

The quadratic formula gives us any solutions to a quadratic equation $ax^2+bx+c=0$ as

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Just as we saw when we learned to complete the square, the number of solutions to the quadratic can be determined by the value under the square root, **the discriminant**, b^2-4ac.

- When $b^2-4ac=0$, the solution is one real number
- When $b^2-4ac>0$, the solutions are two real numbers
- When $b^2-4ac<0$, the solutions are two real complex numbers

Building the quadratic formula

If we complete the square for a quadratic in standard form, $ax^2+bx+c=0$, we would start by dividing through the equation by a, since we said before that we can only complete the square when the coefficient on x^2 is 1.

$$ax^2+bx+c=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

We'll divide b/a by 2 to get $b/2a$, then we'll square that result to get $(b/2a)^2=b^2/4a^2$. Adding this value to both sides and then subtracting c/a gives

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}=0+\frac{b^2}{4a^2}$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

The left side will always factor as a perfect square,

$$\left(x+\frac{b}{2a}\right)\left(x+\frac{b}{2a}\right)=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

and we can find a common denominator to combine the fractions on the right side.

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}\left(\frac{4a}{4a}\right)$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Take the square root of both sides.

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Subtract $b/2a$ from both sides to solve for x, then combine the fractions.

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=-\frac{b\pm\sqrt{b^2-4ac}}{2a}$$

Just like the method of completing the square, we can always use the quadratic formula to find the roots of a quadratic. Let’s do an example with a positive discriminant.

Example

Solve for x using the quadratic formula.

$$x^2 + 2x - 8 = 0$$

For this particular quadratic equation, $a = 1$, $b = 2$, and $c = -8$. Plugging these values into the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

$$x = \frac{-2 \pm \sqrt{36}}{2}$$

$$x = \frac{-2 \pm 6}{2}$$

We can factor a 2 out of the numerator,

$$x = \frac{2(-1 \pm 3)}{2}$$

and then cancel the common factor of 2 from the numerator and denominator.

$$x = -1 \pm 3$$

$x = -1 - 3$ or $x = -1 + 3$

$x = -4$ or $x = 2$

There are two real roots. That makes sense, since the value of the discriminant is positive, $b^2 - 4ac = 2^2 - 4(1)(-8) = 4 + 32 = 36 > 0$.

Let's try another example, this time where $a \neq 1$.

Example

Solve for the roots of the quadratic using the quadratic formula.

$$3x^2 + 6x + 2 = 0$$

We can identify $a = 3$, $b = 6$, and $c = 2$. Plugging these values into the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$x = \frac{-6 \pm \sqrt{12}}{6}$$

$$x = \frac{-6 \pm \sqrt{4}\sqrt{3}}{6}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{6}$$

We can factor a 2 out of the numerator and denominator,

$$x = \frac{2(-3 \pm \sqrt{3})}{2(3)}$$

and then cancel the common factor of 2 from the numerator and denominator.

$$x = \frac{-3 \pm \sqrt{3}}{3}$$

There are two real roots. That makes sense, since the value of the discriminant is positive, $b^2 - 4ac = 6^2 - 4(3)(2) = 36 - 24 = 12 > 0$.