Solving logarithmic equations

Logarithmic equations are simply equations that include one or more logarithms, and sometimes we'll want to solve a logarithmic equation for that equation's variable.

The simplest way to solve a logarithmic equation is often to rewrite the equation until we have just one logarithm set equal to one logarithm, and those logarithms have the same bases.

$$\log_b x = \log_b y$$

If we can put the equation in this form, then the only way the equation can be true is if the arguments of the log functions are equivalent, so we set $x = y$, which eliminates the logarithms completely and allows us to solve an equation without any logs.

Once we have an equation without logarithms, we simply collect like terms and solve the equation for the variable. Once we've solved the equation, we should check the answer by plugging the solution back into the original logarithmic equation.

Let's look at an example.

Example

Solve the logarithmic equation.

$$\log(7x) = \log(3x + 4)$$

Because we have two logarithms set equal to each other, and those two logs have the same base, the only way this equation can be true is if the arguments of the log functions are equal.

In other words, given $\log_b x = \log_b y$, we know $x = y$, so we set the arguments equal to each other and then solve the equation for x.

$$7x = 3x + 4$$

$$4x = 4$$

$$x = 1$$

Then we plug $x = 1$ back into the logarithmic equation to make sure it's an actual solution.

$$\log(7(1)) = \log(3(1) + 4)$$

$$\log 7 = \log 7$$

Let's look at another example.

Example

Solve the logarithmic equation.

$$\log_2(x + 3) - \log_2(2x) = \log_2 9$$

Our goal is to manipulate the equation until we have two logarithms with the same base set equal to each other, which means we should try to simplify the left side into one logarithm.

Since we have the difference of two logs with the same base, we can apply the law of logs,

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

to rewrite the equation as

$$\log_2\left(\frac{x+3}{2x}\right) = \log_2 9$$

Now we have two logarithms with the same base set equal to each other, which means the arguments must be equal.

$$\frac{x+3}{2x} = 9$$

$$x + 3 = 18x$$

$$3 = 17x$$

$$x = \frac{3}{17}$$

Then we plug $x = 3/17$ back into the logarithmic equation to make sure it's an actual solution.

$$\log_2\left(\frac{3}{17} + 3\right) - \log_2\left(2\left(\frac{3}{17}\right)\right) = \log_2 9$$

$$\log_2\left(\frac{3}{17}+\frac{51}{17}\right)-\log_2\frac{6}{17}=\log_2 9$$

$$\log_2\frac{54}{17}-\log_2\frac{6}{17}=\log_2 9$$

Apply the law of logs

$$\log_a\left(\frac{x}{y}\right)=\log_a x-\log_a y$$

to rewrite the left side of the equation.

$$\log_2\frac{\frac{54}{17}}{\frac{6}{17}}=\log_2 9$$

$$\log_2\left(\frac{54}{17}\left(\frac{17}{6}\right)\right)=\log_2 9$$

$$\log_2\frac{54}{6}=\log_2 9$$

$$\log_2 9=\log_2 9$$