Rational equations

We just learned how to solve equations with fractions, but the fractions we looked at never included any variables themselves.

Now we want to look at fractional equations where we have variables in the numerator and/or denominator of the fractions, and we'll call these **rational equations**.

Let's review the general steps for how to solve rational equations.

1. Find any value of the variable that would make any denominator equal to zero.

2. Find the least common denominator (LCD).

3. Multiply both sides of the equation to clear the fractions.

4. Solve the resulting equation and check the solutions. Don't forget to discard any values from step 1 if they are algebraic solutions.

We multiply the fractions by the least common denominator because doing so clears all the fractions from the equation, which makes it much easier to solve.

We have to also make sure that we find the values that would make any denominators zero because this would help us know whether there are any algebraic solutions that must be discarded. This type of solution is called an **extraneous solution**, and it can cause parts of the equation to be undefined.

Let’s solve some rational equations.

Example

Solve the rational equation.

$$\frac{1}{x}+\frac{1}{5}=\frac{3}{10}$$

If $x=0$, then $1/x$ is undefined, so this equation is true only if $x\neq 0$.

Now we need to find the least common denominator of all denominators in the equation. The LCD is $10x$, so we’ll multiply both sides of the equation by $10x$.

$$\left(\frac{1}{x}+\frac{1}{5}=\frac{3}{10}\right)10x$$

$$\frac{1}{x}(10x)+\frac{1}{5}(10x)=\frac{3}{10}(10x)$$

$$10+2x=3x$$

Let’s move $2x$ to the right side.

$$10=3x-2x$$

$$10=x$$

$$x=10$$

We can check the solution by substituting $x=10$ into the equation.

$$\frac{1}{x}+\frac{1}{5}=\frac{3}{10}$$

$$\frac{1}{10}+\frac{1}{5}=\frac{3}{10}$$

$$\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$$

$$\frac{3}{10}=\frac{3}{10}$$

Because the solution satisfies the equation, we know it's a real solution, and we can therefore say that the solution is $x=10$.

Let's try another one.

Example

Solve the equation.

$$\frac{x-8}{x^2+2x-8}-\frac{2}{x-2}=\frac{5}{x+4}$$

We can rewrite the equation as

$$\frac{x-8}{(x-2)(x+4)}-\frac{2}{x-2}=\frac{5}{x+4}$$

If $x-2=0$ or $x+4=0$, then the equation is undefined, so this equation is true only if $x\neq-4$ and $x\neq2$.

Now we need to find the least common denominator of all denominators in the equation. The LCD is $(x-2)(x+4)$, so we'll multiply both sides of the equation by $(x-2)(x+4)$.

$$\left(\frac{x-8}{(x-2)(x+4)}-\frac{2}{x-2}\right)(x-2)(x+4)=\left(\frac{5}{x+4}\right)(x-2)(x+4)$$

$$\frac{x-8}{(x-2)(x+4)}\cdot(x-2)(x+4)-\frac{2}{x-2}\cdot(x-2)(x+4)=\frac{5}{x+4}\cdot(x-2)(x+4)$$

$$x-8-2(x+4)=5(x-2)$$

$$x-8-2x-8=5x-10$$

$$-x-16=5x-10$$

$$-6x-16=-10$$

$$-6x=6$$

$$x=-1$$

We can check the solution by substituting $x=-1$ into the equation to find out whether the equation is true.

$$\frac{x-8}{x^2+2x-8}-\frac{2}{x-2}=\frac{5}{x+4}$$

$$\frac{-1-8}{(-1)^2+2(-1)-8}-\frac{2}{-1-2}=\frac{5}{-1+4}$$

$$\frac{-9}{-9}-\frac{2}{-3}=\frac{5}{3}$$

$$1+\frac{2}{3}=\frac{5}{3}$$

$$\frac{3}{3}+\frac{2}{3}=\frac{5}{3}$$

$$\frac{5}{3}=\frac{5}{3}$$

Because $x=-1$ satisfies the equation, it’s a solution to the equation.