Inverse functions

In this lesson we'll look at the definition of an inverse function and how to find a function's inverse.

If we remember from the last lesson, a function is invertible (has an inverse) if and only if it's one-to-one. Now let's look a little more into how to find an inverse and what an inverse does.

When we have a function with points $(x, f(x))$, the inverse function will have points $(f(x), x)$. We could think of the inverse of a function f as the function that "undoes" f. If we first evaluate $f(x)$ at some x in the domain of f, and then evaluate the inverse of f at that value of $f(x)$, what we get is just x (the input we started with). The inverse of a function $f(x)$ is written as $f^{-1}(x)$. Because f^{-1} "undoes" f, we could think of the function $g(x) = x$ as the composite of $f^{-1}(x)$ and $f(x)$, because

$$g(x) = x = f^{-1}(f(x))$$

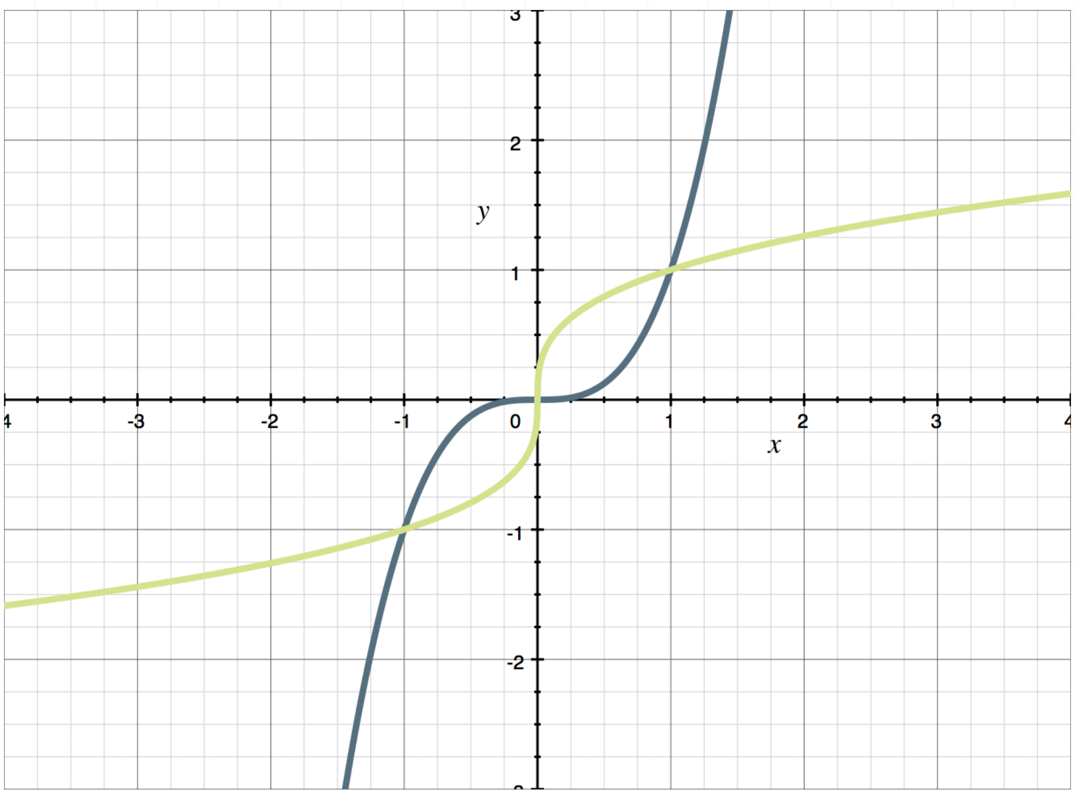
For example, if $g(x)$ and $g^{-1}(x)$ are inverses of one another, then the tables below would give sets of points from each.

x	g(x)
1	4
4	8
10	12
16	2

x	$g^{-1}(x)$
4	1
8	4
12	10
2	16

Now let's look at the graphs of a function and its inverse. Look at the graph of the function $f(x) = x^3$ (in blue) and the graph of its inverse (in green). Notice that in order to "get back to x" from $f(x)$ (to get back to x from x^3), we have to take the cube root of $f(x)$, because

$$x = (x^3)^{\frac{1}{3}} = \sqrt[3]{x^3}$$



Notice that the x- and y-coordinates of the points on the blue curve are the y- and x-coordinates, respectively, of the points on the green curve, that is, the coordinates of the points of the graph of $f^{-1}(x)$ have switched places with the coordinates of the points of the graph of $f(x)$. Now let's look at how to calculate an inverse algebraically.

The inverse is not an exponent

It's worth mentioning the most common mistake students make when studying inverse functions, which is confusing the "-1" notation for an exponent. That notation indicates the inverse function, and it's not an exponent, so

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

How to find the inverse

Here are the steps we'll use to find the inverse function, assuming we're given the function $f(x)$ and want to find its inverse, $f^{-1}(x)$.

1. Replace $f(x)$ with y to make the process easier.

2. Replace every x with a y and every y with an x.

3. Solve the equation from Step 2 for y.

4. Replace y with $f^{-1}(x)$ to show that we've found the inverse function.

5. Double-check to verify that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true.

Domain and range

Lastly, keep in mind that the range of $f(x)$ is the domain of $f^{-1}(x)$, and that the domain of $f(x)$ is the range of $f^{-1}(x)$.

If we have a function and we need to find the domain and range of its inverse, remember that,

1. if the function is one-to-one, we write the range of the original function as the domain of the inverse, and the domain of the original function as the range of the inverse.

2. if we need to restrict the domain of the original function to make it one-to-one, this restricted domain then becomes the range of the inverse function.

Let's work through an example where we find the inverse function.

Example

What is the inverse of the function?

$$f(x) = \frac{2}{3}x - 4$$

First, notice that this function is invertible, because its graph is a line that's neither vertical nor horizontal (so its graph passes both the Vertical Line Test and the Horizontal Line Test, which means that the function is one-to-one).

To find the inverse of this function, first replace $f(x)$ with the variable y.

$$y = \frac{2}{3}x - 4$$

Next, switch x with y.

$$x = \frac{2}{3}y - 4$$

Now solve for y.

$$x + 4 = \frac{2}{3}y$$

$$\frac{3}{2}(x + 4) = \frac{3}{2}\left(\frac{2}{3}y\right)$$

$$\frac{3}{2} \cdot x + \frac{3}{2} \cdot 4 = \frac{3}{2} \cdot \frac{2}{3}y$$

$$\frac{3}{2}x + 6 = y$$

Now we can write the inverse function by replacing y with $f^{-1}(x)$ (and then turning the equation around so that $f^{-1}(x)$ is on the left side).

$$f^{-1}(x) = \frac{3}{2}x + 6$$

Let's do one more example.

Example

Find the inverse of the function.

$$g(x) = \frac{x}{x - 3}$$

First replace $g(x)$ with y.

$$y = \frac{x}{x - 3}$$

At this point in finding the inverse of the function in the other example, we first switched x with y, and then solved for y. When we use algebra to get the inverse of a function, we could just as well first solve for x, and then switch x with y, so we'll do it that way here.

$$y(x - 3) = x$$

$$xy - 3y = x$$

$$xy - x = 3y$$

$$x(y - 1) = 3y$$

$$x = \frac{3y}{y - 1}$$

Now switch x with y.

$$y = \frac{3x}{x - 1}$$

Finally, write the inverse function by replacing y with $g^{-1}(x)$.

$$g^{-1}(x) = \frac{3x}{x - 1}$$
