# Quadratic inequalities

A **quadratic inequality** is simply a quadratic equation where the equal sign has been replaced by an inequality sign.

Which means quadratic inequalities can take four forms, depending on the inequality sign.

$$ax^2 + bx + c > 0 \qquad ax^2 + bx + c < 0$$

$$ax^2 + bx + c \geq 0 \qquad ax^2 + bx + c \leq 0$$

Remember that the graph of any quadratic function $f(x) = ax^2 + bx + c = 0$ is a parabola, so depending on the sign of the inequality, we'll need to determine whether to shade above or below the parabola.

There are two ways to solve quadratic inequalities: algebraically and graphically. Let's review the general steps of each of those methods.

## Solving quadratic inequalities algebraically

To solve a quadratic inequality algebraically,

1. Put the inequality into standard form.

2. Find the critical points, which are the solutions to the related quadratic equation.

3. Using the critical points, divide the number line into intervals.

4. Choose a point from each interval and substitute it into the quadratic expression to find its sign, either positive or negative.

5. Finally, choose the intervals where the inequality is true and write the solution using interval notation.

Let's do an example so we can see these steps in action.

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**Example**

Solve $x^2 - 9x + 14 \leq 0$ algebraically.

The quadratic is already in standard form, so we can proceed with finding the critical points by solving the related quadratic equation.

$$x^2 - 9x + 14 = 0$$

$$(x - 2)(x - 7) = 0$$

$$x - 2 = 0 \qquad\qquad x - 7 = 0$$

$$x = 2 \qquad\qquad x = 7$$

Now we can use $2$ and $7$ to divide the number line into intervals.

Since we have three intervals (left of $x = 2$, between $x = 2$ and $x = 7$, and right of $x = 7$), we'll choose three test points, one from each interval, and substitute them into quadratic equation to find the sign of each interval.

For $x = 0$:

$$0^2 - 9(0) + 14 = 14$$

For $x = 4$:

$$4^2 - 9(4) + 14 = -6$$

For $x = 8$:

$$8^2 - 9(8) + 14 = 6$$

Because we find a positive, then negative, then positive value, we can add these signs to each interval on the number line.


The original inequality $x^2 - 9x + 14 \leq 0$ tells us that we're looking for the interval(s) of the number line that are associated with a negative value (because of the $<$), as well as the points on the number line where the quadratic is equal to $0$ (because of the $\leq$).

Looking at the number line and the signs that we've attached to it, we can see that the solutions are all the values of $x$ between $x = 2$ and $x = 7$, as well as $x = 2$ and $x = 7$ themselves. So the solution to the quadratic inequality is

$$2 \leq x \leq 7$$

## Solving quadratic inequalities graphically

We can also solve quadratic inequalities using the graph of the inequality. In general, we'll follow the steps below.

1. Write the inequality in standard form.

2. Determine the $x$-intercepts by looking at the graph, or by solving $ax^2 + bx + c = 0$.

3. Graph the parabolic function given by $f(x) = ax^2 + bx + c$.

4. Depending on the sign of the inequality, determine whether it asks for the value(s) of $x$ that make the parabola negative (below the horizontal axis) or positive (above the horizontal axis).

Let's do an example.

**Example**

Solve $x^2 + 2x - 8 > 0$ graphically.

The inequality is already in standard form, and the $x$-intercepts are given by

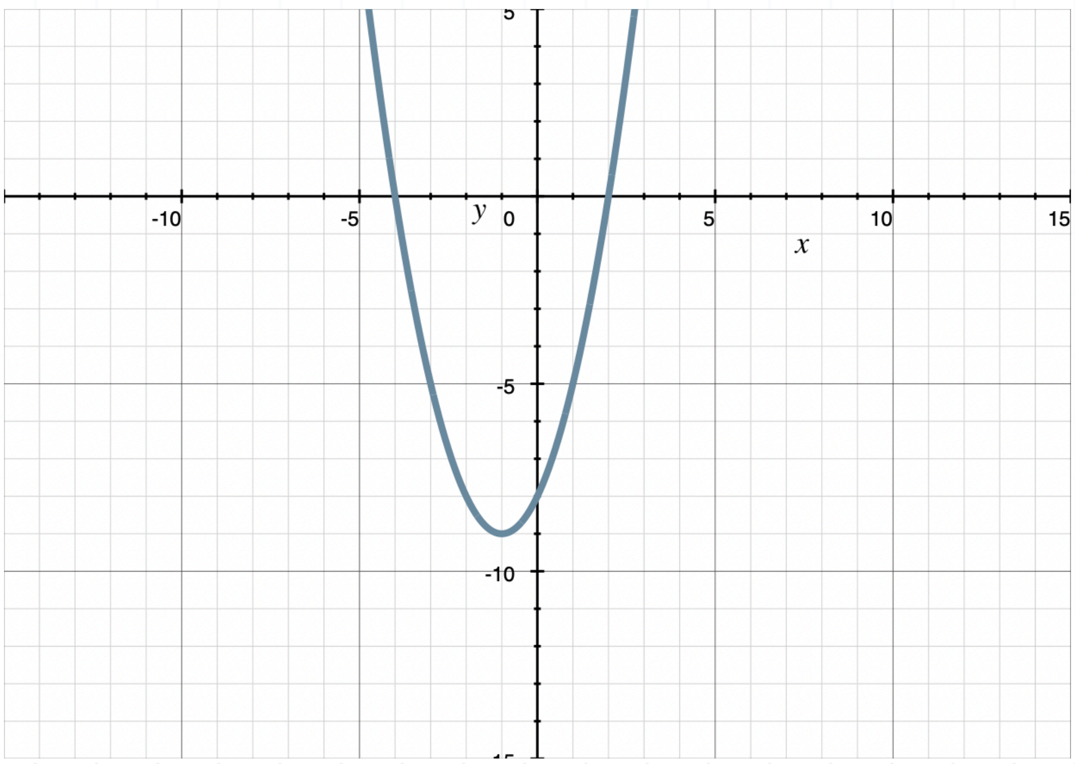
$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$x + 4 = 0 \qquad x - 2 = 0$$

$$x = -4 \qquad x = 2$$

The $x$-intercepts are therefore $(-4,0)$ and $(2,0)$. A sketch of the parabola is




The inequality $x^2 + 2x - 8 > 0$ is asking for values of $x$ that make the quadratic greater than $0$, which means we need to find the values of $x$ where the parabola is above the $x$-axis. Therefore, the solution is all values of $x$ to the left of $x = -4$ (not including $x = -4$ itself), and all values of $x$ to the right of $2$ (not including $x = 2$ itself).

$x < -4$ and $x > 2$

## Using the discriminant

Remember that the value of the discriminant, $b^2 - 4ac$, tells us how many times the parabola will intersect the horizontal axis.

| | |
|---|---|
| $b^2 - 4ac > 0$ | Two zeros |
| $b^2 - 4ac = 0$ | One zero |
| $b^2 - 4ac < 0$ | No zeros |

We can use this fact to help us determine how many solutions there are to the quadratic inequality.

Let's do an example where we use the discriminant to determine the number of solutions.

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**Example**

Solve the quadratic inequality.

$$-x^2 + 3x - 4 > 0$$

Instead of using the set of steps we outline for how to solve quadratic inequalities graphically, let's start by finding the discriminant.

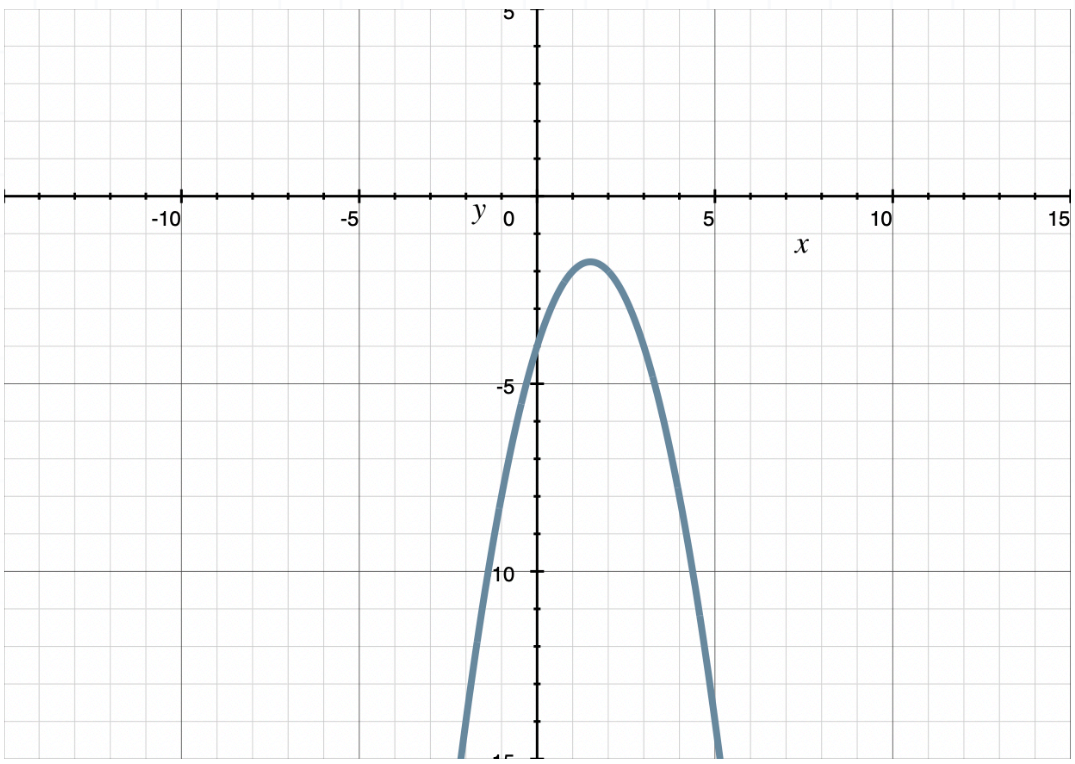
For the given quadratic, $a = -1$, $b = 3$, and $c = -4$, so the discriminant is

$$b^2 - 4ac = 3^2 - 4(-1)(-4)$$

$$b^2 - 4ac = 9 - 16$$

$$b^2 - 4ac = -7$$

Since the discriminant is negative, there are no $x$-intercepts, which means the parabola never crosses the horizontal axis. Because the parabola opens downward (because $a = -1$ is negative), that means the vertex of the parabola must be below the horizontal axis, with the parabola opening down from that point, and therefore never crossing the $x$-axis.




The inequality is asking for the values of $x$ where the parabola is above the horizontal axis, but we know that this never happens. Therefore, there is no solution to the inequality.