Laws of natural logs

All of the rules that we've just learned for manipulating logarithms apply to natural logs.

General rule for logs

The general rule for natural logs is:

Given the exponential equation $e^x = y$,

the associated logarithmic equation is $\log_e(y) = x$,

and vice versa.

Of course, $\log_e$ is the same as the natural log, $\ln$. So we can rewrite the general rule for natural logs:

Given the exponential equation $e^x = y$,

the associated logarithmic equation is $\ln(y) = x$,

and vice versa.

Product rule

The product rule for logs is

$$\log_a(xy) = \log_a x + \log_a y$$

In terms of natural logarithms, this rule is

$$\ln(xy) = \ln x + \ln y$$

Let's use the same example of the product rule as in the previous lesson, but this time with natural logs instead of with logs to base 4.

Example

Simplify the expression.

$$\ln 64 + \ln 16$$

First, we can use the product rule.

$$\ln(xy) = \ln x + \ln y$$

$$\ln(64 \cdot 16)$$

$$\ln(1{,}024)$$

From here, we can use a calculator to find the approximate value of $\ln(1{,}024)$.

$$\ln(1{,}024) \approx 6.9315$$

Or we could let $x = \ln 1{,}024$. Then

$$e^x = 1{,}024$$

The way we solve equations in this form, where the variable is in the exponent in an exponential equation, is to take the logarithm (in this case the natural logarithm) of both sides.

$$\ln e^x = \ln 1{,}024$$

Now note that $\ln(e^x) = x$, because any exponential function a^x and the associated log function ($\log_a x$) are inverses of each other. So if we evaluate a^x, and then take the log (to base a) of the result, we get back to our starting point (x). This means that our equation simplifies to

$$x = \ln 1{,}024$$

$$x \approx 6.9315$$

Quotient rule

The quotient rule for logs is

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

In terms of natural logarithms, this rule is

$$\ln \left(\frac{x}{y}\right) = \ln x - \ln y$$

As with all log rules, we can use it in either direction. If we start with something that matches the the form of the expression on the left side of the equation, we can rewrite it in the form on the right side of the equation, and vice versa.

Example

Write the expression as one logarithm.

$$\ln 32 - \ln 8$$

Using the quotient rule for natural logs,

$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

we can rewrite the expression as

$$\ln \frac{32}{8}$$

$$\ln 4$$

We can leave it this way, as an exact value, or we can use a calculator to find an approximate value.

$$\ln 4 \approx 1.3863$$

The power rule

The power rule for logs is

$$\log_a(x^n) = n\log_a x$$

For natural logs, this rule is

$$\ln(x^n) = n\ln x$$

Let's look at an example.

Example

Write the expression as one logarithm.

$$\ln(8^4) - \ln(8^2)$$

There are multiple ways to approach this problem, but let's start by using the power rule.

$$\ln(8^4) - \ln(8^2)$$

$$4\ln 8 - 2\ln 8$$

$$2\ln 8$$

We can leave it in this form, or we can pull the 2 back in as an exponent.

$$\ln(8^2)$$

$$\ln 64$$

We can also express 64 as a power (for instance, we know that $64 = 2^6$), and then use the power rule.

$$\ln(2^6)$$

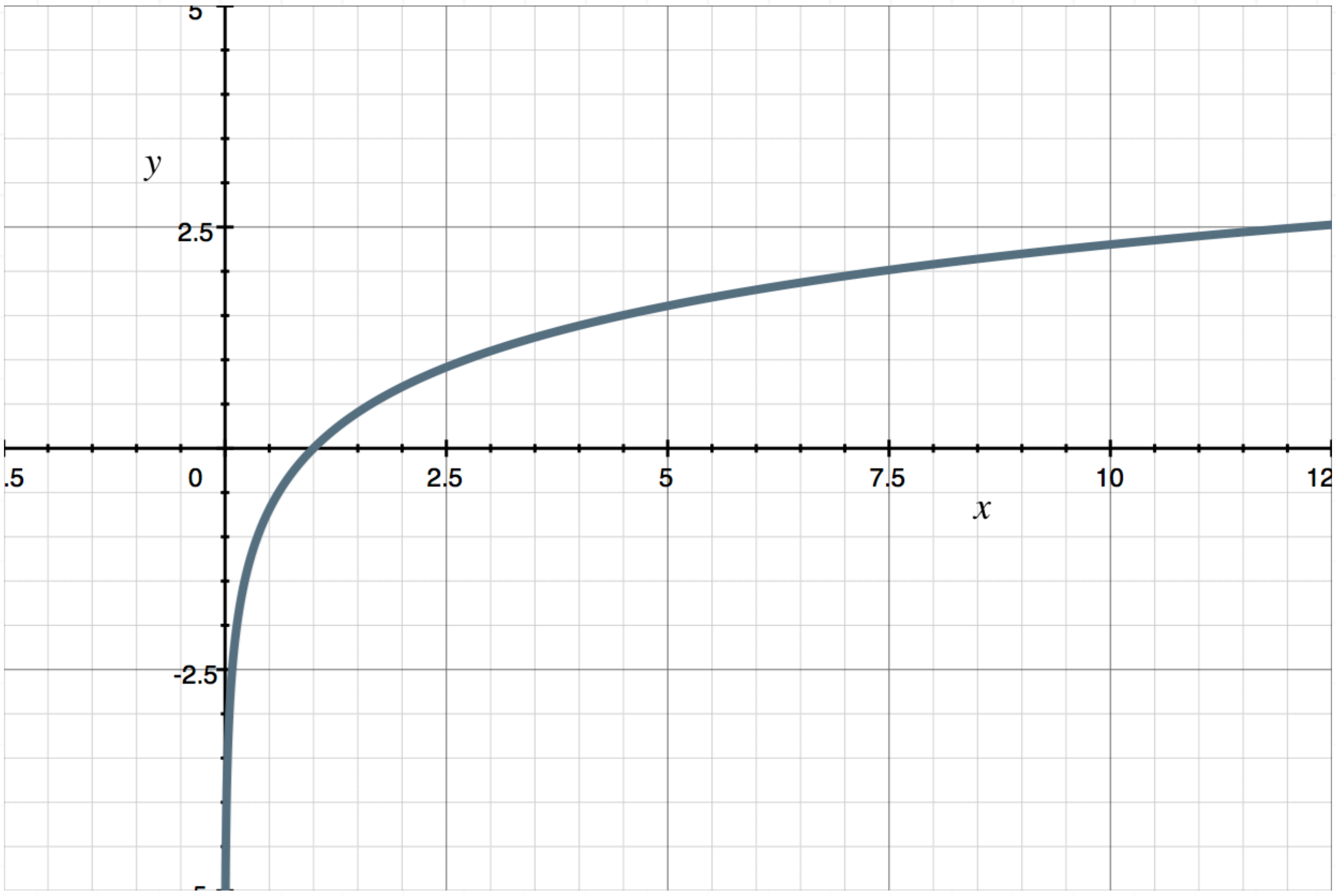
$$6\ln 2$$

Combining natural log rules

The product, quotient, and power rules for logarithms, as well as the general rule for logs, can all be used together, in any combination, in order to solve problems with natural logs.

Special values for natural logs

We'll look at graphs of exponential and logarithmic functions in the next couple of lessons, but for now, let's take a sneak peak at the graph of the natural log function, $\ln x$:



Notice how the graph dips down sharply as it approaches $x = 0$ from the right side (as positive values of x get closer and closer to 0). This shows that the value of the natural log function goes to $-\infty$ as $x \to 0$ from the right side, and that the natural log function is undefined at $x = 0$.

Also, as we've already learned, the natural log function is undefined at all $x < 0$. As $x \to \infty$, the value of the natural log function increases without bound. And the graph of the natural log function crosses the x-axis at $x = 1$, which means that $\ln 1 = 0$. To see this algebraically, we could set $\ln x$ equal to 0 (write the equation $\ln x = 0$) and then solve for x. In exponential form, this equation is $e^0 = x$. Since $e^0 = 1$, we get $1 = x$.

So based on its graph, we can say that the natural log function has a few key features:

$\ln x$ is undefined at all $x \leq 0$

$\ln 1 = 0$

$\ln x \to \infty$ as $x \to \infty$