Laws of logarithms

Up until now, we've always enclosed the argument of the log function in parentheses, just as we enclose the argument of any function in parentheses. However, it's common to eliminate the parentheses in the case of a log function, so we'll sometimes do that as well, especially when the meaning is clear without the parentheses.

For instance, here are four simple laws of logarithms we can use to simplify logarithmic expressions, each written without the parentheses around the argument.

$$\log_b 1 = 0 \qquad \log_b b^x = x$$

$$\log_b b = 1 \qquad b^{\log_b x} = x$$

Let's look at an example where we use these log rules, before diving in depth into the product, quotient, and power rules for logarithms.

Example

Simplify the expression.

$$\log \frac{1}{1{,}000}$$

If we rewrite $1{,}000$ as 10^3, and recognize that the log has base 10, then the expression becomes

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$$\log_{10} \frac{1}{10^3}$$

$$\log_{10} 10^{-3}$$

Comparing this to the law of logs $\log_b b^x = x$ we saw earlier, we can see that the expression simplifies to just -3. So

$$\log \frac{1}{1{,}000} = -3$$

The product rule

When the argument of a log function is a product of two quantities, the value of that log can be written as the sum of the values of the logs of those two quantities.

$$\log_a(xy) = \log_a x + \log_a y$$

Keep in mind that this rule can be used in both directions. Given the expression $\log_a(xy)$, we can expand it to $\log_a x + \log_a y$. And given the expression $\log_a x + \log_a y$, we can condense it to $\log_a(xy)$.

The bases of two log functions must be equal in order to use the product rule. In other words, we can use the product rule to condense $\log_a x + \log_a y$, but we can't use it to condense $\log_a x + \log_b y$.

Example

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Write the expression as a rational number if possible, or if not, as a single logarithm.

$$\log_4 64 + \log_4 16$$

First, we can use the rule

$$\log_a x + \log_a y = \log_a(xy)$$

because the two log functions have the same base.

$$\log_4 64 + \log_4 16$$

$$\log_4(64 \cdot 16)$$

$$\log_4(1{,}024)$$

To simplify further, use the relationship between exponents and logarithms,

If $\log_a(y) = x$ then $a^x = y$

So if we let $x = \log_4 1{,}024$, then

$$4^x = 1{,}024$$

Now we'll factor $1{,}024$ using 4 as each factor:

$$1{,}024 = 4 \cdot 256$$

$$1{,}024 = 4 \cdot 4 \cdot 64$$

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$$1{,}024 = 4 \cdot 4 \cdot 4 \cdot 16$$

$$1{,}024 = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

Therefore, $1{,}024 = 4^5$, so

$$4^x = 4^5$$

$$x = 5$$

The quotient rule

When the argument of a log function is a quotient of two quantities, the value that log can be written as the difference of the values of the logs of those two quantities.

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

As before, this rule can be used in both directions. Given the expression $\log_a(x/y)$, we can expand it to $\log_a x - \log_a y$. And given the expression $\log_a x - \log_a y$, we can condense it to $\log_a(x/y)$.

Just as with the product rule, the bases of two log functions must be equal in order to use the quotient rule. In other words, we can use the quotient rule to condense $\log_a x - \log_a y$, but we can't use it to condense $\log_a x - \log_b y$.

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The power rule

When the argument of a log function is a power function, the exponent can be pulled out in front of the log function.

$$\log_a(x^n) = n\log_a x$$

As before, this rule can be used in both directions. Given the expression $\log_a(x^n)$, we can rewrite it as $n\log_a x$. And given the expression $n\log_a x$, we can rewrite it as $\log_a(x^n)$.

Alternatively, we can write the power rule as

$$\log_{a^n}(x) = \frac{1}{n}\log_a x$$

As before, this rule can be used in both directions. Given the expression $\log_{a^n}(x)$, we can rewrite it as $(1/n)\log_a x$. And given the expression $(1/n)\log_a x$, we can rewrite it as $\log_{a^n}(x)$.

Combining log rules

The product, quotient, and power rules for logarithms, as well as the general rule for logs (and the change of base formula we'll cover in the next lesson), can all be used together, in any combination, in order to solve log problems.

Let's look at an example that requires us to use two of these log rules.

Example

Write the expression as a single logarithm.

$$\log_3 14 - 2\log_3 5$$

First we can use the power rule for logs

$$\log_a(x^n) = n\log_a x$$

on the second term to get

$$\log_3 14 - \log_3 5^2$$

$$\log_3 14 - \log_3 25$$

Because the bases of these logs are the same, we can use the quotient rule for logs

$$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$$

to get

$$\log_3 \frac{14}{25}$$

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