Expansion

2 x 2

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

3 x 3

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3$$
$$+ a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

n x n

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

where $C_{ij} = (-1)^{i+j}M_{ij}$

$M_{ij} \rightarrow$ Determinant formed after eliminating i^{th} row and j^{th} column

Properties

1. If A be any square matrix, then
   $|A| = |A^T|$
   where A^T denotes the transpose of A
2. If all the elements of a row or a column are zero, then the value of the determinant is zero.
   $$\begin{vmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$
3. If each element of a row or a column is multiplied by 'k', then the value of the determinant changes by k times
   $$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
4. If any two rows or columns are interchanged, then the determinant changes its sign
   $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
5. If corresponding elements of any two rows or columns are equal or proportional, then the value of the determinant is zero.
   $$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 & ka_2 & ka_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_1 & b_3 \\ c_1 & c_1 & c_3 \end{vmatrix} = 0$$
6. If some or all elements of a row or a column are expressed as sum of two or more terms, then the determinant can be expressed as sum of two or more determinants
   $$\begin{vmatrix} a_1 + l & a_2 & a_3 \\ b_1 + m & b_2 & b_3 \\ c_1 + n & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} l & a_2 & a_3 \\ m & b_2 & b_3 \\ n & c_2 & c_3 \end{vmatrix}$$
7. If, to each element of any row/column, the equimultiples of corresponding elements of another row/column are added, then value of determinant remains the same
   $$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \begin{vmatrix} a + kb & b & c \\ p + kq & q & r \\ x + ky & y & z \end{vmatrix} + \begin{vmatrix} a + kp & b + kq & c + kr \\ p & q & r \\ x & y & z \end{vmatrix}$$

Product of two Determinants

$$\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}\begin{vmatrix} a' & b' & c' \\ p' & q' & r' \\ x' & y' & z' \end{vmatrix}$$

$$= \begin{vmatrix} aa' + bp' + cx' & ab' + bq' + cy' & ac' + br' + cz' \\ pa' + qp' + rx' & pb' + qq' + ry' & pc' + qr' + rz' \\ xa' + yp' + zx' & xb' + yq' + zy' & xc' + yr' + zz' \end{vmatrix}$$

Summation

If $\Delta(r) = \begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ d & e & f \end{vmatrix}$, then

$$\sum \Delta(r) = \begin{vmatrix} \sum f(r) & \sum g(r) & \sum h(r) \\ a & b & c \\ a' & b' & c' \end{vmatrix}$$

where a, b, c, a', b', c' are constants, independent of r

Differentiation

If $F(x) = \begin{vmatrix} f_1 & g_1 & \cdots & h_1 \\ f_2 & g_2 & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & g_n & \cdots & h_n \end{vmatrix}$, then F'(x) =

$$\begin{vmatrix} f_1' & g_1' & \cdots & h_1' \\ f_2 & g_2 & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & g_n & \cdots & h_n \end{vmatrix} + \begin{vmatrix} f_1 & g_1 & \cdots & h_1 \\ f_2' & g_2' & \cdots & h_2' \\ \vdots & \vdots & \ddots & \vdots \\ f_n & g_n & \cdots & h_n \end{vmatrix} + \cdots + \begin{vmatrix} f_1 & g_1 & \cdots & h_1 \\ f_2 & g_2 & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n' & g_n' & \cdots & h_n' \end{vmatrix}$$

or $$\begin{vmatrix} f_1' & g_1 & \cdots & h_1 \\ f_2' & g_2 & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n' & g_n & \cdots & h_n \end{vmatrix} + \begin{vmatrix} f_1 & g_1' & \cdots & h_1 \\ f_2 & g_2' & \cdots & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ f_n & g_n' & \cdots & h_n \end{vmatrix} + \cdots + \begin{vmatrix} f_1 & g_1 & \cdots & h_1' \\ f_2 & g_2 & \cdots & h_2' \\ \vdots & \vdots & \ddots & \vdots \\ f_n & g_n & \cdots & h_n' \end{vmatrix}$$

where f_i, g_i ... h_i (i = 1,2,..,n) are functions of x