Equations of a Circle

Center: (0, 0) and Radius: a
$$x^2 + y^2 = a^2 \text{ (Standard)}$$

Center: (x_1, y_1) and Radius: r
$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

Center: (-g, -f) and Radius: $\sqrt{g^2 + f^2 - c}$
$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ (General)}$$

End points of diameter: (x_1, y_1) and (x_2, y_2)
$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Notations

Standard: $T = xx_1 + yy_1 - a^2$, $S_1 = x_1^2 + y_1^2 - a^2$
General: $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$
$$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

Position of a point (x_1, y_1) w.r.t. a circle

Outside: $S_1 > 0$, On: $S_1 = 0$, Inside: $S_1 < 0$

Tangent

Equation of the tangent at the point (x_1, y_1)
$$T = 0 \qquad\qquad (S_1 = 0)$$

Equation of the tangent of given slope m

Standard: $y = mx \pm a\sqrt{1 + m^2}$
General: $y + f = m(x + g) \pm \sqrt{g^2 + f^2 - c}\sqrt{1 + m^2}$

Length of the tangent from the point (x_1, y_1)
$$\sqrt{S_1}$$

Normal

Equation of the normal at the point (x_1, y_1)
Standard: $\dfrac{y}{y_1} = \dfrac{x}{x_1}$

General: $\dfrac{y - y_1}{y_1 + f} = \dfrac{x - x_1}{x_1 + g}$

Equation of the normal of given slope m
Standard: $y = mx$
General: $y + f = m(x + g)$

Chord of contact w.r.t the point (x_1, y_1)
$$T = 0 \qquad\qquad (S_1 > 0)$$

Chord with mid-point (x_1, y_1)
$$T = S_1 \qquad\qquad (S_1 < 0)$$

Family of Circles

Passing through the points of intersection of a circle
S = 0 and a line L = 0
$$S + \lambda L = 0$$

Passing through the points of intersection of two circles S = 0 and S' = 0
$$S + \lambda S' = 0$$

Passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$
$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$$
$$+ \lambda[(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)] = 0$$

Touching the line L = 0 at the point (x_1, y_1)
$$(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$$

Two circles

$S: x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
Center: C_1, Radius: r_1

$S': x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
Center: C_2, Radius: r_2

Angle of Intersection

$$\cos(\pi - \theta) = \frac{r_1^2 + r_2^2 - (C_1C_2)^2}{2r_1r_2}$$

Orthogonal Intersection

$$r_1^2 + r_2^2 = (C_1C_2)^2 \text{ or}$$
$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

Radical axis

$$S - S' = 0 \text{ or}$$
$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

Represents equation of common chord when the circles intersect, and the common tangent (at the point of contact) when the circles touch each other.

Relative Position   (Common Tangents)

Outside one another: $C_1C_2 > r_1 + r_2$ (4)
Touching externally: $C_1C_2 = r_1 + r_2$ (3)
Intersecting at 2 points: $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ (2)
Touching internally: $C_1C_2 = |r_1 - r_2|$ (1)
One inside the other: $C_1C_2 < |r_1 - r_2|$ (0)