

EXERCISE 4.1**PAGE: 94**

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

1.

$$1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{(3^n - 1)}{2}$$

Solution:

We can write the given statement as

$$P(n): 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{(3^n - 1)}{2}$$

If $n = 1$ we get

$$P(1): 1 = \frac{(3^1 - 1)}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$1 + 3 + 3^2 + \dots + 3^{k-1} = \frac{(3^k - 1)}{2} \quad \dots(i)$$

Now let us prove that $P(k + 1)$ is true.

Here

$$1 + 3 + 3^2 + \dots + 3^{k-1} + 3^{(k+1)-1} = (1 + 3 + 3^2 + \dots + 3^{k-1}) + 3^k$$

By using equation (i)

$$= \frac{(3^k - 1)}{2} + 3^k$$

Taking LCM

$$= \frac{(3^k - 1) + 2 \cdot 3^k}{2}$$

Taking the common terms out

$$= \frac{(1 + 2)3^k - 1}{2}$$

We get

$$= \frac{3 \cdot 3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

2.

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Solution:

We can write the given statement as

$$P(n): 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

If n = 1 we get

$$P(1): 1^3 = 1 = \left(\frac{1(1+1)}{2} \right)^2 = \left(\frac{1 \cdot 2}{2} \right)^2 = 1^2 = 1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2} \right)^2 \quad \dots (i)$$

Now let us prove that P (k + 1) is true.

Here

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = (1^3 + 2^3 + 3^3 + \dots + k^3) + (k+1)^3$$

By using equation (i)

$$= \left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3$$

So we get

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

Taking LCM

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

Taking the common terms out

$$= \frac{(k+1)^2 \{k^2 + 4(k+1)\}}{4}$$

We get

$$= \frac{(k+1)^2 \{k^2 + 4k + 4\}}{4}$$

$$= \frac{(k+1)^2 (k+2)^2}{4}$$

By expanding using formula

$$= \frac{(k+1)^2 (k+1+1)^2}{4}$$

$$= \left(\frac{(k+1)(k+1+1)}{2} \right)^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

3.

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \dots + \frac{1}{(1+2+3+\dots+n)} = \frac{2n}{(n+1)}$$

Solution:

We can write the given statement as

$$P(n): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+n} = \frac{2n}{n+1}$$

If n = 1 we get

$$P(1): 1 = \frac{2 \cdot 1}{1+1} = \frac{2}{2} = 1$$

Which is true.

Consider P (k) be true for some positive integer k

$$1 + \frac{1}{1+2} + \dots + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} = \frac{2k}{k+1} \quad \dots (i)$$

Now let us prove that P (k + 1) is true.

Here

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} + \frac{1}{1+2+3+\dots+k+(k+1)}$$

$$= \left(1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} \right) + \frac{1}{1+2+3+\dots+k+(k+1)}$$

By using equation (i)

$$= \frac{2k}{k+1} + \frac{1}{1+2+3+\dots+k+(k+1)}$$

We know that

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

So we get

$$= \frac{2k}{k+1} + \frac{1}{\left(\frac{(k+1)(k+1+1)}{2} \right)}$$

It can be written as

$$= \frac{2k}{(k+1)} + \frac{2}{(k+1)(k+2)}$$

Taking the common terms out

$$= \frac{2}{(k+1)} \left(k + \frac{1}{k+2} \right)$$

By taking LCM

$$= \frac{2}{k+1} \left(\frac{k(k+2)+1}{k+2} \right)$$

We get

$$= \frac{2}{(k+1)} \left(\frac{k^2+2k+1}{k+2} \right)$$

$$= \frac{2 \cdot (k+1)^2}{(k+1)(k+2)}$$

$$= \frac{2(k+1)}{(k+2)}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

4.

$$1.2.3 + 2.3.4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Solution:

We can write the given statement as

$$P(n): 1.2.3 + 2.3.4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

If n = 1 we get

$$P(1): 1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad \dots (i)$$

Now let us prove that P (k + 1) is true.

Here

$$1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \{1.2.3 + 2.3.4 + \dots + k(k+1)(k+2)\} + (k+1)(k+2)(k+3)$$

By using equation (i)

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

So we get

$$= (k+1)(k+2)(k+3) \left(\frac{k}{4} + 1 \right)$$

It can be written as

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

By further simplification

$$= \frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

$P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

5.

$$1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

Solution:

We can write the given statement as

$$P(n) : 1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

If $n = 1$ we get

$$P(1) : 1.3 = 3 = \frac{(2.1-1)3^{1+1} + 3}{4} = \frac{3^2 + 3}{4} = \frac{12}{4} = 3$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k = \frac{(2k-1)3^{k+1} + 3}{4} \quad \dots (i)$$

Now let us prove that $P(k + 1)$ is true.

Here

$$1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k + (k+1)3^{k+1} = (1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k) + (k+1)3^{k+1}$$

By using equation (i)

$$= \frac{(2k-1)3^{k+1} + 3}{4} + (k+1)3^{k+1}$$

By taking LCM

$$= \frac{(2k-1)3^{k+1} + 3 + 4(k+1)3^{k+1}}{4}$$

Taking the common terms out

$$= \frac{3^{k+1} \{2k-1+4(k+1)\} + 3}{4}$$

By further simplification

$$= \frac{3^{k+1} \{6k+3\} + 3}{4}$$

Taking 3 as common

$$\begin{aligned} &= \frac{3^{k+1} \cdot 3 \{2k+1\} + 3}{4} \\ &= \frac{3^{(k+1)+1} \{2k+1\} + 3}{4} \\ &= \frac{\{2(k+1)-1\} 3^{(k+1)+1} + 3}{4} \end{aligned}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

6.

$$1.2 + 2.3 + 3.4 + \dots + n.(n+1) = \left[\frac{n(n+1)(n+2)}{3} \right]$$

Solution:

We can write the given statement as

$$P(n): 1.2 + 2.3 + 3.4 + \dots + n.(n+1) = \left[\frac{n(n+1)(n+2)}{3} \right]$$

If n = 1 we get

$$P(1): 1.2 = 2 = \frac{1(1+1)(1+2)}{3} = \frac{1.2.3}{3} = 2$$

Which is true.

Consider P (k) be true for some positive integer k

$$1.2 + 2.3 + 3.4 + \dots + k.(k+1) = \left[\frac{k(k+1)(k+2)}{3} \right] \quad \dots (i)$$

Now let us prove that P (k + 1) is true.

Here

$$1.2 + 2.3 + 3.4 + \dots + k.(k+1) + (k+1).(k+2) = [1.2 + 2.3 + 3.4 + \dots + k.(k+1)] + (k+1).(k+2)$$

By using equation (i)

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

We can write it as

$$= (k+1)(k+2)\left(\frac{k}{3}+1\right)$$

We get

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

By further simplification

$$= \frac{(k+1)(k+1+1)(k+1+2)}{3}$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

7.

$$1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}$$

Solution:

We can write the given statement as

$$P(n): 1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}$$

If $n = 1$ we get

$$P(1): 1.3 = 3 = \frac{1(4.1^2 + 6.1 - 1)}{3} = \frac{4 + 6 - 1}{3} = \frac{9}{3} = 3$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1) = \frac{k(4k^2 + 6k - 1)}{3} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$(1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1)) + \{2(k+1)-1\} \{2(k+1)+1\}$$

By using equation (i)

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+2-1)(2k+2+1)$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k + 2 - 1)(2k + 2 + 1)$$

On further calculation

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k + 1)(2k + 3)$$

By multiplying the terms

$$= \frac{k(4k^2 + 6k - 1)}{3} + (4k^2 + 8k + 3)$$

Taking LCM

$$= \frac{k(4k^2 + 6k - 1) + 3(4k^2 + 8k + 3)}{3}$$

By further simplification

$$= \frac{4k^3 + 6k^2 - k + 12k^2 + 24k + 9}{3}$$

So we get

$$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

It can be written as

$$= \frac{4k^3 + 14k^2 + 9k + 4k^2 + 14k + 9}{3}$$
$$= \frac{k(4k^2 + 14k + 9) + 1(4k^2 + 14k + 9)}{3}$$

Separating the terms

$$= \frac{(k + 1)\{4k^2 + 8k + 4 + 6k + 6 - 1\}}{3}$$

Taking the common terms out

$$= \frac{(k + 1)\{4(k^2 + 2k + 1) + 6(k + 1) - 1\}}{3}$$

Using the formula

$$= \frac{(k+1)\{4(k+1)^2 + 6(k+1) - 1\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

8. $1.2 + 2.2^2 + 3.2^2 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$

Solution:

We can write the given statement as

P (n): $1.2 + 2.2^2 + 3.2^2 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$

If n = 1 we get

P (1): $1.2 = 2 = (1 - 1) 2^{1+1} + 2 = 0 + 2 = 2$

Which is true.

Consider P (k) be true for some positive integer k

$1.2 + 2.2^2 + 3.2^2 + \dots + k.2^k = (k - 1) 2^{k+1} + 2 \dots$ (i)

Now let us prove that P (k + 1) is true.

Here

$$\{1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k\} + (k+1) \cdot 2^{k+1}$$

By using equation (i)

$$= (k - 1) 2^{k+1} + 2 + (k + 1) 2^{k+1}$$

Taking the common terms out

$$= 2^{k+1} \{(k - 1) + (k + 1)\} + 2$$

So we get

$$= 2^{k+1} \cdot 2k + 2$$

It can be written as

$$= k \cdot 2^{(k+1)+1} + 2$$

$$= \{(k + 1) - 1\} 2^{(k+1)+1} + 2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

9.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

If $n = 1$ we get

$$P(1): \frac{1}{2} = 1 - \frac{1}{2^1} = \frac{1}{2}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} \right) + \frac{1}{2^{k+1}}$$

By using equation (i)

$$= \left(1 - \frac{1}{2^k} \right) + \frac{1}{2^{k+1}}$$

We can write it as

$$= 1 - \frac{1}{2^k} + \frac{1}{2 \cdot 2^k}$$

Taking the common terms out

$$= 1 - \frac{1}{2^k} \left(1 - \frac{1}{2} \right)$$

So we get

$$= 1 - \frac{1}{2^k} \left(\frac{1}{2} \right)$$

It can be written as

$$= 1 - \frac{1}{2^{k+1}}$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

10.

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

If $n = 1$ we get

$$P(1) = \frac{1}{2.5} = \frac{1}{10} = \frac{1}{6.1+4} = \frac{1}{10}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

By using equation (i)

$$= \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)}$$

By simplification of terms

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

Taking 2 as common

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

Taking the common terms out

$$= \frac{1}{(3k+2)} \left(\frac{k}{2} + \frac{1}{3k+5} \right)$$

Taking LCM

$$= \frac{1}{(3k+2)} \left(\frac{k(3k+5)+2}{2(3k+5)} \right)$$

By multiplication

$$= \frac{1}{(3k+2)} \left(\frac{3k^2+5k+2}{2(3k+5)} \right)$$

Separating the terms

$$= \frac{1}{(3k+2)} \left(\frac{(3k+2)(k+1)}{2(3k+5)} \right)$$

By further calculation

$$= \frac{(k+1)}{6k+10}$$

So we get

$$= \frac{(k+1)}{6(k+1)+4}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

11.

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

If n = 1 we get

$$P(1): \frac{1}{1.2.3} = \frac{1 \cdot (1+3)}{4(1+1)(1+2)} = \frac{1 \cdot 4}{4 \cdot 2 \cdot 3} = \frac{1}{1.2.3}$$

Which is true.

Consider P (k) be true for some positive integer k

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left[\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{k(k+1)(k+2)} \right] + \frac{1}{(k+1)(k+2)(k+3)}$$

By using equation (i)

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

Taking out the common terms

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+3)}{4} + \frac{1}{k+3} \right\}$$

Taking LCM

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+3)^2 + 4}{4(k+3)} \right\}$$

Expanding using formula

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k^2 + 6k + 9) + 4}{4(k+3)} \right\}$$

By further calculation

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k^3 + 6k^2 + 9k + 4}{4(k+3)} \right\}$$

We can write it as

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k^3 + 2k^2 + k + 4k^2 + 8k + 4}{4(k+3)} \right\}$$

Taking the common terms

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k^2 + 2k + 1) + 4(k^2 + 2k + 1)}{4(k+3)} \right\}$$

We get

$$= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+1)^2 + 4(k+1)^2}{4(k+3)} \right\}$$

Here

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

12.

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Solution:

We can write the given statement as

$$P(n): a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

If $n = 1$ we get

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad \dots (i)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\{a + ar + ar^2 + \dots + ar^{k-1}\} + ar^{(k+1)-1}$$

By using equation (i)

$$= \frac{a(r^k - 1)}{r - 1} + ar^k$$

Taking LCM

$$= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$$

Multiplying the terms

$$= \frac{a(r^k - 1) + ar^{k+1} - ar^k}{r - 1}$$

So we get

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r - 1}$$

By further simplification

$$= \frac{ar^{k+1} - a}{r - 1}$$

Taking the common terms out

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

13.

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Solution:

We can write the given statement as

$$P(n) : \left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2n+1)}{n^2}\right) = (n+1)^2$$

If n = 1 we get

$$P(1) : \left(1 + \frac{3}{1}\right) = 4 = (1+1)^2 = 2^2 = 4,$$

Which is true.

Consider P (k) be true for some positive integer k

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2k+1)}{k^2}\right) = (k+1)^2 \quad \dots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$\left[\left(1 + \frac{3}{1}\right) \left(1 + \frac{5}{4}\right) \left(1 + \frac{7}{9}\right) \dots \left(1 + \frac{(2k+1)}{k^2}\right) \right] \left\{ 1 + \frac{\{2(k+1)+1\}}{(k+1)^2} \right\}$$

By using equation (i)

$$= (k+1)^2 \left(1 + \frac{2(k+1)+1}{(k+1)^2} \right)$$

Taking LCM

$$= (k+1)^2 \left[\frac{(k+1)^2 + 2(k+1)+1}{(k+1)^2} \right]$$

So we get

$$= (k+1)^2 + 2(k+1)+1$$

By further simplification

$$= \{(k+1)+1\}^2$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

14.

$$\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = (n+1)$$

Solution:

We can write the given statement as

$$P(n): \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{n}\right) = (n+1)$$

If n = 1 we get

$$P(1): \left(1 + \frac{1}{1}\right) = 2 = (1+1)$$

Which is true.

Consider P (k) be true for some positive integer k

$$P(k): \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right) = (k+1) \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left[\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{k}\right) \right] \left(1 + \frac{1}{k+1}\right)$$

By using equation (i)

$$= (k+1) \left(1 + \frac{1}{k+1}\right)$$

Taking LCM

$$= (k+1) \left(\frac{(k+1)+1}{(k+1)} \right)$$

By further simplification

$$= (k+1) + 1$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

15.

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Solution:

We can write the given statement as

$$P(n) = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

If $n = 1$ we get

$$P(1) = 1^2 = 1 = \frac{1(2 \cdot 1 - 1)(2 \cdot 1 + 1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1,$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k) = 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\{1^2 + 3^2 + 5^2 + \dots + (2k-1)^2\} + \{2(k+1)-1\}^2$$

By using equation (i)

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2$$

So we get

$$= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

Taking LCM

$$= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3}$$

Taking the common terms out

$$= \frac{(2k+1)\{k(2k-1) + 3(2k+1)\}}{3}$$

By further simplification

$$= \frac{(2k+1)\{2k^2 - k + 6k + 3\}}{3}$$

So we get

$$= \frac{(2k+1)\{2k^2 + 5k + 3\}}{3}$$

We can write it as

$$= \frac{(2k+1)\{2k^2 + 2k + 3k + 3\}}{3}$$

Splitting the terms

$$= \frac{(2k+1)\{2k(k+1) + 3(k+1)\}}{3}$$

We get

$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$

$$= \frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

16.

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

If $n = 1$ we get

$$P(1) = \frac{1}{1.4} = \frac{1}{3.1+1} = \frac{1}{4} = \frac{1}{1.4}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left\{ \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right\} + \frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

By using equation (i)

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$

So we get

$$= \frac{1}{(3k+1)} \left\{ k + \frac{1}{(3k+4)} \right\}$$

Taking LCM

$$= \frac{1}{(3k+1)} \left\{ \frac{k(3k+4)+1}{(3k+4)} \right\}$$

Multiplying the terms

$$= \frac{1}{(3k+1)} \left\{ \frac{3k^2 + 4k + 1}{(3k+4)} \right\}$$

It can be written as

$$= \frac{1}{(3k+1)} \left\{ \frac{3k^2 + 3k + k + 1}{(3k+4)} \right\}$$

Separating the terms

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

By further calculation

$$= \frac{(k+1)}{3(k+1)+1}$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

17.

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

Solution:

We can write the given statement as

$$P(n): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

If $n = 1$ we get

$$P(1): \frac{1}{3.5} = \frac{1}{3(2.1+3)} = \frac{1}{3.5}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$P(k): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$\left[\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} \right] + \frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

By using equation (i)

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)}$$

So we get

$$= \frac{1}{(2k+3)} \left[\frac{k}{3} + \frac{1}{(2k+5)} \right]$$

Taking LCM

$$= \frac{1}{(2k+3)} \left[\frac{k(2k+5)+3}{3(2k+5)} \right]$$

Multiplying the terms

$$= \frac{1}{(2k+3)} \left[\frac{2k^2+5k+3}{3(2k+5)} \right]$$

It can be written as

$$= \frac{1}{(2k+3)} \left[\frac{2k^2+2k+3k+3}{3(2k+5)} \right]$$

Separating the terms

$$= \frac{1}{(2k+3)} \left[\frac{2k(k+1)+3(k+1)}{3(2k+5)} \right]$$

By further calculation

$$= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$= \frac{(k+1)}{3\{2(k+1)+3\}}$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

18.

$$1+2+3+\dots+n < \frac{1}{8}(2n+1)^2$$

Solution:

We can write the given statement as

$$P(n): 1 + 2 + 3 + \dots + n < \frac{1}{8}(2n+1)^2$$

If $n = 1$ we get

$$1 < \frac{1}{8}(2 \cdot 1 + 1)^2 = \frac{9}{8}$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$1 + 2 + \dots + k < \frac{1}{8}(2k+1)^2 \quad \dots (1)$$

Now let us prove that $P(k+1)$ is true.

Here

$$(1 + 2 + \dots + k) + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$$

By using equation (i)

$$< \frac{1}{8}\{(2k+1)^2 + 8(k+1)\}$$

Expanding terms using formula

$$< \frac{1}{8}\{4k^2 + 4k + 1 + 8k + 8\}$$

By further calculation

$$< \frac{1}{8}\{4k^2 + 12k + 9\}$$

So we get

$$< \frac{1}{8}(2k+3)^2$$

$$< \frac{1}{8}\{2(k+1)+1\}^2$$

$$(1 + 2 + 3 + \dots + k) + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$$

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

19. $n(n+1)(n+5)$ is a multiple of 3

Solution:

We can write the given statement as

$P(n)$: $n(n+1)(n+5)$, which is a multiple of 3

If $n = 1$ we get

$1(1+1)(1+5) = 12$, which is a multiple of 3

Which is true.

Consider $P(k)$ be true for some positive integer k

$k(k+1)(k+5)$ is a multiple of 3

$k(k+1)(k+5) = 3m$, where $m \in \mathbb{N} \dots\dots (1)$

Now let us prove that $P(k+1)$ is true.

Here

$(k+1)\{(k+1)+1\}\{(k+1)+5\}$

We can write it as

$= (k+1)(k+2)\{(k+5)+1\}$

By multiplying the terms

$= (k+1)(k+2)(k+5) + (k+1)(k+2)$

So we get

$= \{k(k+1)(k+5) + 2(k+1)(k+5)\} + (k+1)(k+2)$

Substituting equation (1)

$= 3m + (k+1)\{2(k+5) + (k+2)\}$

By multiplication

$= 3m + (k+1)\{2k + 10 + k + 2\}$

On further calculation

$= 3m + (k+1)(3k + 12)$

Taking 3 as common

$= 3m + 3(k+1)(k+4)$

We get

$= 3\{m + (k+1)(k+4)\}$

$= 3 \times q$ where $q = \{m + (k+1)(k+4)\}$ is some natural number

$(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

20. $10^{2n-1} + 1$ is divisible by 11

Solution:

We can write the given statement as

$P(n)$: $10^{2n-1} + 1$ is divisible by 11

If $n = 1$ we get

$P(1) = 10^{2 \cdot 1 - 1} + 1 = 11$, which is divisible by 11

Which is true.

Consider $P(k)$ be true for some positive integer k

$10^{2k-1} + 1$ is divisible by 11

$10^{2k-1} + 1 = 11m$, where $m \in \mathbb{N} \dots\dots (1)$

Now let us prove that $P(k+1)$ is true.

Here

$10^{2(k+1)-1} + 1$

We can write it as

$= 10^{2k+2-1} + 1$

$= 10^{2k+1} + 1$

By addition and subtraction of 1

$$= 10^{2k} (10^{2k-1} + 1 - 1) + 1$$

We get

$$= 10^{2k} (10^{2k-1} + 1) - 10^{2k} + 1$$

Using equation 1 we get

$$= 10^{2k} \cdot 11m - 100 + 1$$

$$= 100 \times 11m - 99$$

Taking out the common terms

$$= 11 (100m - 9)$$

= 11 r, where r = (100m - 9) is some natural number

$10^{2(k+1)-1} + 1$ is divisible by 11

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

21. $x^{2n} - y^{2n}$ is divisible by $x + y$

Solution:

We can write the given statement as

P (n): $x^{2n} - y^{2n}$ is divisible by $x + y$

If n = 1 we get

$$P (1) = x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y) (x - y), \text{ which is divisible by } (x + y)$$

Which is true.

Consider P (k) be true for some positive integer k

$x^{2k} - y^{2k}$ is divisible by $x + y$

$$x^{2k} - y^{2k} = m (x + y), \text{ where } m \in \mathbb{N} \dots\dots (1)$$

Now let us prove that P (k + 1) is true.

Here

$$x^{2(k+1)} - y^{2(k+1)}$$

We can write it as

$$= x^{2k} \cdot x^2 - y^{2k} \cdot y^2$$

By adding and subtracting y^{2k} we get

$$= x^{2k} (x^{2k} - y^{2k} + y^{2k}) - y^{2k} \cdot y^2$$

From equation (1) we get

$$= x^{2k} \{m (x + y) + y^{2k}\} - y^{2k} \cdot y^2$$

By multiplying the terms

$$= m (x + y) x^{2k} + y^{2k} \cdot x^{2k} - y^{2k} \cdot y^2$$

Taking out the common terms

$$= m (x + y) x^{2k} + y^{2k} (x^{2k} - y^2)$$

Expanding using formula

$$= m (x + y) x^{2k} + y^{2k} (x + y) (x - y)$$

So we get

$$= (x + y) \{mx^{2k} + y^{2k} (x - y)\}, \text{ which is a factor of } (x + y)$$

P (k + 1) is true whenever P (k) is true.

Therefore, by the principle of mathematical induction, statement P (n) is true for all natural numbers i.e. n.

22. $3^{2n+2} - 8n - 9$ is divisible by 8

Solution:

We can write the given statement as

$P(n)$: $3^{2n+2} - 8n - 9$ is divisible by 8

If $n = 1$ we get

$P(1) = 3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64$, which is divisible by 8

Which is true.

Consider $P(k)$ be true for some positive integer k

$3^{2k+2} - 8k - 9$ is divisible by 8

$3^{2k+2} - 8k - 9 = 8m$, where $m \in \mathbb{N}$ (1)

Now let us prove that $P(k+1)$ is true.

Here

$3^{2(k+1)+2} - 8(k+1) - 9$

We can write it as

$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

By adding and subtracting $8k$ and 9 we get

$= 3^2 (3^{2k+2} - 8k - 9 + 8k + 9) - 8k - 17$

On further simplification

$= 3^2 (3^{2k+2} - 8k - 9) + 3^2 (8k + 9) - 8k - 17$

From equation (1) we get

$= 9 \cdot 8m + 9 (8k + 9) - 8k - 17$

By multiplying the terms

$= 9 \cdot 8m + 72k + 81 - 8k - 17$

So we get

$= 9 \cdot 8m + 64k + 64$

By taking out the common terms

$= 8 (9m + 8k + 8)$

$= 8r$, where $r = (9m + 8k + 8)$ is a natural number

So $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8

$P(k+1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

23. $41^n - 14^n$ is a multiple of 27

Solution:

We can write the given statement as

$P(n)$: $41^n - 14^n$ is a multiple of 27

If $n = 1$ we get

$P(1) = 41^1 - 14^1 = 27$, which is a multiple by 27

Which is true.

Consider $P(k)$ be true for some positive integer k

$41^k - 14^k$ is a multiple of 27

$41^k - 14^k = 27m$, where $m \in \mathbb{N}$ (1)

Now let us prove that $P(k+1)$ is true.

Here

$41^{k+1} - 14^{k+1}$

We can write it as

$= 41^k \cdot 41 - 14^k \cdot 14$

By adding and subtracting 14^k we get

$= 41 (41^k - 14^k + 14^k) - 14^k \cdot 14$

On further simplification

$$= 41 (41^k - 14^k) + 41 \cdot 14^k - 14^k \cdot 14$$

From equation (1) we get

$$= 41 \cdot 27m + 14^k (41 - 14)$$

By multiplying the terms

$$= 41 \cdot 27m + 27 \cdot 14^k$$

By taking out the common terms

$$= 27 (41m - 14^k)$$

$= 27r$, where $r = (41m - 14^k)$ is a natural number

So $41^{k+1} - 14^{k+1}$ is a multiple of 27

$P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .

24. $(2n + 7) < (n + 3)^2$

Solution:

We can write the given statement as

$$P(n): (2n + 7) < (n + 3)^2$$

If $n = 1$ we get

$$2 \cdot 1 + 7 = 9 < (1 + 3)^2 = 16$$

Which is true.

Consider $P(k)$ be true for some positive integer k

$$(2k + 7) < (k + 3)^2 \dots (1)$$

Now let us prove that $P(k + 1)$ is true.

Here

$$\{2(k + 1) + 7\} = (2k + 7) + 2$$

We can write it as

$$= \{2(k + 1) + 7\}$$

From equation (1) we get

$$(2k + 7) + 2 < (k + 3)^2 + 2$$

By expanding the terms

$$2(k + 1) + 7 < k^2 + 6k + 9 + 2$$

On further calculation

$$2(k + 1) + 7 < k^2 + 6k + 11$$

$$\text{Here } k^2 + 6k + 11 < k^2 + 8k + 16$$

We can write it as

$$2(k + 1) + 7 < (k + 4)^2$$

$$2(k + 1) + 7 < \{(k + 1) + 3\}^2$$

$P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e. n .