EXERCISE 12.1 PAGE NO: 271

1. A point is on the x-axis. What are its y coordinate and z-coordinates?
Solution:
If a point is on the x-axis, then the coordinates of y and z are 0.
So the point is (x, 0, 0).

2. A point is in the XZ-plane. What can you say about its *y*-coordinate?
Solution:
If a point is in XZ plane, then its y-co-ordinate is 0.

3. Name the octants in which the following points lie:
(1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (– 4, 2, –5), (– 4, 2, 5), (–3, –1, 6) (2, – 4, –7).
Solution:
Here is the table which represents the octants:

Octants	I	II	III	IV	V	VI	VII	VIII
x	+	-	-	+	+	-	-	+
y	+	+	-	-	+	+	-	-
z	+	+	+	+	-	-	-	-

(i) (1, 2, 3)
Here x is positive, y is positive and z is positive.
So it lies in I octant.

(ii) (4, -2, 3)
Here x is positive, y is negative and z is positive.
So it lies in IV octant.

(iii) (4, -2, -5)
Here x is positive, y is negative and z is negative.
So it lies in VIII octant.

(iv) (4, 2, -5)
Here x is positive, y is positive and z is negative.
So it lies in V octant.

(v) (-4, 2, -5)

Here x is negative, y is positive and z is negative.
So it lies in VI octant.

(vi) (-4, 2, 5)
Here x is negative, y is positive and z is positive.
So it lies in II octant.

(vii) (-3, -1, 6)
Here x is negative, y is negative and z is positive.
So it lies in III octant.

(viii) (2, -4, -7)
Here x is positive, y is negative and z is negative.
So it lies in VIII octant.

4. Fill in the blanks:
(i) The x-axis and y-axis taken together determine a plane known as _______.
(ii) The coordinates of points in the XY-plane are of the form _______.
(iii) Coordinate planes divide the space into ______ octants.
Solution:
(i) The x-axis and y-axis taken together determine a plane known as XY Plane.

(ii) The coordinates of points in the XY-plane are of the form (x, y, 0).

(iii) Coordinate planes divide the space into eight octants.

EXERCISE 12.2 PAGE NO: 273

1. Find the distance between the following pairs of points:
(i) (2, 3, 5) and (4, 3, 1)
(ii) (–3, 7, 2) and (2, 4, –1)
(iii) (–1, 3, – 4) and (1, –3, 4)
(iv) (2, –1, 3) and (–2, 1, 3)
Solution:
(i) (2, 3, 5) and (4, 3, 1)
Let P be (2, 3, 5) and Q be (4, 3, 1)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 2, y_1 = 3, z_1 = 5$
$x_2 = 4, y_2 = 3, z_2 = 1$
Distance PQ = $\sqrt{[(4 - 2)^2 + (3 - 3)^2 + (1 - 5)^2]}$
$= \sqrt{[(2)^2 + 0^2 + (-4)^2]}$
$= \sqrt{[4 + 0 + 16]}$
$= \sqrt{20}$
$= 2\sqrt{5}$
∴ The required distance is $2\sqrt{5}$ units.

(ii) (–3, 7, 2) and (2, 4, –1)
Let P be (– 3, 7, 2) and Q be (2, 4, – 1)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -3, y_1 = 7, z_1 = 2$
$x_2 = 2, y_2 = 4, z_2 = -1$
Distance PQ = $\sqrt{[(2 - (-3))^2 + (4 - 7)^2 + (-1 - 2)^2]}$
$= \sqrt{[(5)^2 + (-3)^2 + (-3)^2]}$
$= \sqrt{[25 + 9 + 9]}$
$= \sqrt{43}$
∴ The required distance is $\sqrt{43}$ units.

(iii) (–1, 3, – 4) and (1, –3, 4)
Let P be (– 1, 3, – 4) and Q be (1, – 3, 4)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,
$x_1 = -1, y_1 = 3, z_1 = -4$
$x_2 = 1, y_2 = -3, z_2 = 4$
Distance PQ = $\sqrt{[(1 - (-1))^2 + (-3 - 3)^2 + (4 - (-4))^2]}$
$= \sqrt{[(2)^2 + (-6)^2 + (8)^2]}$
$= \sqrt{[4 + 36 + 64]}$
$= \sqrt{104}$
$= 2\sqrt{26}$
∴ The required distance is $2\sqrt{26}$ units.

(iv) (2, –1, 3) and (–2, 1, 3)
Let P be (2, – 1, 3) and Q be (– 2, 1, 3)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 2, y_1 = -1, z_1 = 3$
$x_2 = -2, y_2 = 1, z_2 = 3$
Distance PQ = $\sqrt{[(-2 - 2)^2 + (1 - (-1))^2 + (3 - 3)^2]}$
$= \sqrt{[(-4)^2 + (2)^2 + (0)^2]}$
$= \sqrt{[16 + 4 + 0]}$
$= \sqrt{20}$
$= 2\sqrt{5}$
∴ The required distance is $2\sqrt{5}$ units.

2. Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.
Solution:
If three points are collinear, then they lie on a line.
Firstly let us calculate distance between the 3 points
i.e. PQ, QR and PR

<u>Calculating PQ</u>
P ≡ (– 2, 3, 5) and Q ≡ (1, 2, 3)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -2, y_1 = 3, z_1 = 5$
$x_2 = 1, y_2 = 2, z_2 = 3$
Distance PQ = $\sqrt{[(1 - (-2))^2 + (2 - 3)^2 + (3 - 5)^2]}$
$= \sqrt{[(3)^2 + (-1)^2 + (-2)^2]}$

$= \sqrt{[9 + 1 + 4]}$
$= \sqrt{14}$

<u>Calculating QR</u>
Q ≡ (1, 2, 3) and R ≡ (7, 0, – 1)
By using the formula,
Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 1, y_1 = 2, z_1 = 3$
$x_2 = 7, y_2 = 0, z_2 = -1$
Distance QR = $\sqrt{[(7 - 1)^2 + (0 - 2)^2 + (-1 - 3)^2]}$
$= \sqrt{[(6)^2 + (-2)^2 + (-4)^2]}$
$= \sqrt{[36 + 4 + 16]}$
$= \sqrt{56}$
$= 2\sqrt{14}$

<u>Calculating PR</u>
P ≡ (– 2, 3, 5) and R ≡ (7, 0, – 1)
By using the formula,
Distance PR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -2, y_1 = 3, z_1 = 5$
$x_2 = 7, y_2 = 0, z_2 = -1$
Distance PR = $\sqrt{[(7 - (-2))^2 + (0 - 3)^2 + (-1 - 5)^2]}$
$= \sqrt{[(9)^2 + (-3)^2 + (-6)^2]}$
$= \sqrt{[81 + 9 + 36]}$
$= \sqrt{126}$
$= 3\sqrt{14}$

Thus, PQ = $\sqrt{14}$, QR = $2\sqrt{14}$ and PR = $3\sqrt{14}$
So, PQ + QR = $\sqrt{14} + 2\sqrt{14}$
$= 3\sqrt{14}$
= PR
∴ The points P, Q and R are collinear.

3. Verify the following:
(i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
(ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.

Solution:

(i) (0, 7, –10), (1, 6, – 6) and (4, 9, – 6) are the vertices of an isosceles triangle.
Let us consider the points be
P(0, 7, –10), Q(1, 6, – 6) and R(4, 9, – 6)
If any 2 sides are equal, hence it will be an isosceles triangle
So firstly let us calculate the distance of PQ, QR

Calculating PQ
P ≡ (0, 7, – 10) and Q ≡ (1, 6, – 6)
By using the formula,
Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 0, y_1 = 7, z_1 = -10$
$x_2 = 1, y_2 = 6, z_2 = -6$
Distance PQ = $\sqrt{[(1 - 0)^2 + (6 - 7)^2 + (-6 - (-10))^2]}$
$= \sqrt{[(1)^2 + (-1)^2 + (4)^2]}$
$= \sqrt{[1 + 1 + 16]}$
$= \sqrt{18}$

Calculating QR
Q ≡ (1, 6, – 6) and R ≡ (4, 9, – 6)
By using the formula,
Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 1, y_1 = 6, z_1 = -6$
$x_2 = 4, y_2 = 9, z_2 = -6$
Distance QR = $\sqrt{[(4 - 1)^2 + (9 - 6)^2 + (-6 - (-6))^2]}$
$= \sqrt{[(3)^2 + (3)^2 + (-6+6)^2]}$
$= \sqrt{[9 + 9 + 0]}$
$= \sqrt{18}$
Hence, PQ = QR
18 = 18
2 sides are equal
∴ PQR is an isosceles triangle.

(ii) (0, 7, 10), (–1, 6, 6) and (– 4, 9, 6) are the vertices of a right angled triangle.
Let the points be
P(0, 7, 10), Q(– 1, 6, 6) & R(– 4, 9, 6)
Firstly let us calculate the distance of PQ, OR and PR

Calculating PQ

P ≡ (0, 7, 10) and Q ≡ (– 1, 6, 6)

By using the formula,

Distance PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 0, y_1 = 7, z_1 = 10$

$x_2 = -1, y_2 = 6, z_2 = 6$

Distance PQ = $\sqrt{[(-1 - 0)^2 + (6 - 7)^2 + (6 - 10)^2]}$

$= \sqrt{[(-1)^2 + (-1)^2 + (-4)^2]}$

$= \sqrt{[1 + 1 + 16]}$

$= \sqrt{18}$

Calculating QR

Q ≡ (1, 6, – 6) and R ≡ (4, 9, – 6)

By using the formula,

Distance QR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 1, y_1 = 6, z_1 = -6$

$x_2 = 4, y_2 = 9, z_2 = -6$

Distance QR = $\sqrt{[(4 - 1)^2 + (9 - 6)^2 + (-6 - (-6))^2]}$

$= \sqrt{[(3)^2 + (3)^2 + (-6+6)^2]}$

$= \sqrt{[9 + 9 + 0]}$

$= \sqrt{18}$

Calculating PR

P ≡ (0, 7, 10) and R ≡ (– 4, 9, 6)

By using the formula,

Distance PR = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So here,

$x_1 = 0, y_1 = 7, z_1 = 10$

$x_2 = -4, y_2 = 9, z_2 = 6$

Distance PR = $\sqrt{[(-4 - 0)^2 + (9 - 7)^2 + (6 - 10)^2]}$

$= \sqrt{[(-4)^2 + (2)^2 + (-4)^2]}$

$= \sqrt{[16 + 4 + 16]}$

$= \sqrt{36}$

Now,

$PQ^2 + QR^2 = 18 + 18$

$= 36$

$= PR^2$

By using converse of Pythagoras theorem,
∴ The given vertices P, Q & R are the vertices of a right – angled triangle at Q.

(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.
Let the points be: A(–1, 2, 1), B(1, –2, 5), C(4, –7, 8) & D(2, –3, 4)
ABCD can be vertices of parallelogram only if opposite sides are equal.
i.e. AB = CD and BC = AD
Firstly let us calculate the distance

Calculating AB
A ≡ (– 1, 2, 1) and B ≡ (1, – 2, 5)
By using the formula,
Distance AB = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = -1, y_1 = 2, z_1 = 1$
$x_2 = 1, y_2 = -2, z_2 = 5$
Distance AB = $\sqrt{[(1 - (-1))^2 + (-2 - 2)^2 + (5 - 1)^2]}$
$= \sqrt{[(2)^2 + (-4)^2 + (4)^2]}$
$= \sqrt{[4 + 16 + 16]}$
$= \sqrt{36}$
$= 6$

Calculating BC
B ≡ (1, – 2, 5) and C ≡ (4, – 7, 8)
By using the formula,
Distance BC = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 1, y_1 = -2, z_1 = 5$
$x_2 = 4, y_2 = -7, z_2 = 8$
Distance BC = $\sqrt{[(4 - 1)^2 + (-7 - (-2))^2 + (8 - 5)^2]}$
$= \sqrt{[(3)^2 + (-5)^2 + (3)^2]}$
$= \sqrt{[9 + 25 + 9]}$
$= \sqrt{43}$

Calculating CD
C ≡ (4, – 7, 8) and D ≡ (2, – 3, 4)
By using the formula,
Distance CD = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,

$x_1 = 4$, $y_1 = -7$, $z_1 = 8$
$x_2 = 2$, $y_2 = -3$, $z_2 = 4$
Distance CD $= \sqrt{[(2-4)^2 + (-3-(-7))^2 + (4-8)^2]}$
$= \sqrt{[(-2)^2 + (4)^2 + (-4)^2]}$
$= \sqrt{[4 + 16 + 16]}$
$= \sqrt{36}$
$= 6$

Calculating DA
D ≡ (2, – 3, 4) and A ≡ (– 1, 2, 1)
By using the formula,
Distance DA $= \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = 2$, $y_1 = -3$, $z_1 = 4$
$x_2 = -1$, $y_2 = 2$, $z_2 = 1$
Distance DA $= \sqrt{[(-1-2)^2 + (2-(-3))^2 + (1-4)^2]}$
$= \sqrt{[(-3)^2 + (5)^2 + (-3)^2]}$
$= \sqrt{[9 + 25 + 9]}$
$= \sqrt{43}$

Since AB = CD and BC = DA (given)
So, In ABCD both pairs of opposite sides are equal.
∴ ABCD is a parallelogram.

4. Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).
Solution:
Let A (1, 2, 3) & B (3, 2, – 1)
Let point P be (x, y, z)
Since it is given that point P(x, y, z) is equal distance from point A(1, 2, 3) & B(3, 2, – 1)
i.e. PA = PB
Firstly let us calculate

Calculating PA
P ≡ (x, y, z) and A ≡ (1, 2, 3)
By using the formula,
Distance PA $= \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = x$, $y_1 = y$, $z_1 = z$

$x_2 = 1, y_2 = 2, z_2 = 3$
Distance PA = $\sqrt{[(1 – x)^2 + (2 – y)^2 + (3 – z)^2]}$

Calculating PB
P ≡ (x, y, z) and B ≡ (3, 2, – 1)
By using the formula,
Distance PB = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$
So here,
$x_1 = x, y_1 = y, z_1 = z$
$x_2 = 3, y_2 = 2, z_2 = – 1$
Distance PB = $\sqrt{[(3 – x)^2 + (2 – y)^2 + (-1 – z)^2]}$
Since PA = PB
Square on both the sides, we get
$PA^2 = PB^2$
$(1 – x)^2 + (2 – y)^2 + (3 – z)^2 = (3 – x)^2 + (2 – y)^2 + (– 1 – z)^2$
$(1 + x^2 – 2x) + (4 + y^2 – 4y) + (9 + z^2 – 6z)$
$(9 + x^2 – 6x) + (4 + y^2 - 4y) + (1 + z^2 + 2z)$
$– 2x – 4y – 6z + 14 = – 6x - 4y + 2z + 14$
$4x – 8z = 0$
$x – 2z = 0$
∴ The required equation is $x – 2z = 0$

5. Find the equation of the set of points P, the sum of whose distances from A (4, 0, 0) and B (– 4, 0, 0) is equal to 10.
Solution:
Let A (4, 0, 0) & B (– 4, 0, 0)
Let the coordinates of point P be (x, y, z)

Calculating PA
P ≡ (x, y, z) and A ≡ (4, 0, 0)
By using the formula,
Distance PA = $\sqrt{[(x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2]}$
So here,
$x_1 = x, y_1 = y, z_1 = z$
$x_2 = 4, y_2 = 0, z_2 = 0$
Distance PA = $\sqrt{[(4– x)^2 + (0 – y)^2 + (0 – z)^2]}$

Calculating PB
P ≡ (x, y, z) and B ≡ (– 4, 0, 0)

By using the formula,
Distance PB = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$
So here,
$x_1 = x, y_1 = y, z_1 = z$
$x_2 = -4, y_2 = 0, z_2 = 0$
Distance PB = $\sqrt{[(-4 - x)^2 + (0 - y)^2 + (0 - z)^2]}$
Now it is given that:
PA + PB = 10
PA = 10 – PB
Square on both the sides, we get
$PA^2 = (10 - PB)^2$
$PA^2 = 100 + PB^2 - 20\ PB$
$(4 - x)^2 + (0 - y)^2 + (0 - z)^2$
$100 + (-4 - x)^2 + (0 - y)^2 + (0 - z)^2 - 20\ PB$
$(16 + x^2 - 8x) + (y^2) + (z^2)$
$100 + (16 + x^2 + 8x) + (y^2) + (z^2) - 20\ PB$
$20\ PB = 16x + 100$
$5\ PB = (4x + 25)$

Square on both the sides again, we get
$25\ PB^2 = 16x^2 + 200x + 625$
$25\ [(-4 - x)^2 + (0 - y)^2 + (0 - z)^2] = 16x^2 + 200x + 625$
$25\ [x^2 + y^2 + z^2 + 8x + 16] = 16x^2 + 200x + 625$
$25x^2 + 25y^2 + 25z^2 + 200x + 400 = 16x^2 + 200x + 625$
$9x^2 + 25y^2 + 25z^2 - 225 = 0$
∴ The required equation is $9x^2 + 25y^2 + 25z^2 - 225 = 0$

EXERCISE 12.3 PAGE NO: 277

1. Find the coordinates of the point which divides the line segment joining the points (– 2, 3, 5) and (1, – 4, 6) in the ratio (i) 2: 3 internally, (ii) 2: 3 externally.

Solution:

Let the line segment joining the points P (-2, 3, 5) and Q (1, -4, 6) be PQ.

(i) 2: 3 internally

By using section formula,

We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Upon comparing we have

$x_1 = -2$, $y_1 = 3$, $z_1 = 5$;

$x_2 = 1$, $y_2 = -4$, $z_2 = 6$ and

m = 2, n = 3

So, the coordinates of the point which divides the line segment joining the points P (– 2, 3, 5) and Q (1, – 4, 6) in the ratio 2 : 3 internally is given by:

$$\left(\frac{2\times 1 + 3\times(-2)}{2+3}, \frac{2\times(-4) + 3\times 3}{2+3}, \frac{2\times 6 + 3\times 5}{2+3}\right)$$

$$= \left(\frac{2-6}{5}, \frac{-8+9}{5}, \frac{12+15}{5}\right)$$

$$= \left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$$

Hence, the coordinates of the point which divides the line segment joining the points (-2, 3, 5) and (1, -4, 6) is (-4/5, 1/5, 27/5)

(ii) 2: 3 externally

By using section formula,

We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) externally in the ratio m: n is given by:

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

Upon comparing we have

$x_1 = -2$, $y_1 = 3$, $z_1 = 5$;

$x_2 = 1$, $y_2 = -4$, $z_2 = 6$ and

m = 2, n = 3

So, the coordinates of the point which divides the line segment joining the points P (– 2, 3, 5) and Q (1, – 4, 6) in the ratio 2: 3 externally is given by:

$$\left(\frac{2\times1-3\times(-2)}{2-3},\frac{2\times(-4)-3\times3}{2-3},\frac{2\times6-3\times5}{2-3}\right)$$

$$=\left(\frac{2-(-6)}{-1},\frac{-8-9}{-1},\frac{12-15}{-1}\right)$$

$$=\left(\frac{8}{-1},\frac{-17}{-1},\frac{-3}{-1}\right)$$

= (-8, 17, 3)

∴ The co-ordinates of the point which divides the line segment joining the points (-2, 3, 5) and (1, -4, 6) is (-8, 17, 3).

2. Given that P (3, 2, – 4), Q (5, 4, – 6) and R (9, 8, –10) are collinear. Find the ratio in which Q divides PR.

Solution:

Let us consider Q divides PR in the ratio k: 1.

By using section formula,

We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m : n is given by:

$$\left(\frac{mx_2+nx_1}{m+n},\frac{my_2+ny_1}{m+n},\frac{mz_2+nz_1}{m+n}\right)$$

Upon comparing we have,

$x_1 = 3$, $y_1 = 2$, $z_1 = -4$;

$x_2 = 9$, $y_2 = 8$, $z_2 = -10$ and

m = k, n = 1

So, we have

$$\left(\frac{9k+3}{k+1},\frac{8k+2}{k+1},\frac{-10k-4}{k+1}\right)=(5,4,-6)$$

$$\frac{9k+3}{k+1}=5,\frac{8k+2}{k+1}=4,\frac{-10k-4}{k+1}=-6$$

9k + 3 = 5 (k+1)

9k + 3 = 5k + 5

9k – 5k = 5 - 3

4k = 2

k = 2/4

= ½

Hence, the ratio in which Q divides PR is 1: 2.

3. Find the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8).

Solution:

Let the line segment formed by joining the points P (-2, 4, 7) and Q (3, -5, 8) be PQ.
We know that any point on the YZ-plane is of the form (0, y, z).
So now, let R (0, y, z) divides the line segment PQ in the ratio k: 1.
Then,
Upon comparing we have,
$x_1 = -2, y_1 = 4, z_1 = 7$;
$x_2 = 3, y_2 = -5, z_2 = 8$ and
m = k, n = 1
By using the section formula,
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

So we have,

$$\left(\frac{3k-2}{k+1}, \frac{-5k+4}{k+1}, \frac{8k+7}{k+1}\right) = (0, y, z)$$

$$\frac{3k-2}{k+1} = 0$$

3k – 2 = 0
3k = 2
k = 2/3
Hence, the ratio in which the YZ-plane divides the line segment formed by joining the points (–2, 4, 7) and (3, –5, 8) is 2:3.

4. Using section formula, show that the points A (2, –3, 4), B (–1, 2, 1) and C (0, 1/3, 2) are collinear.

Solution:

Let the point P divides AB in the ratio k: 1.
Upon comparing we have,
$x_1 = 2, y_1 = -3, z_1 = 4$;
$x_2 = -1, y_2 = 2, z_2 = 1$ and
m = k, n = 1
By using section formula,

We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m : n is given by:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

So we have,

The coordinates of P = $\left(\frac{-k+2}{k+1}, \frac{2k-3}{k+1}, \frac{k+4}{k+1}\right)$

Now, we check if for some value of k, the point coincides with the point C.

Put (-k+2)/(k+1) = 0

-k + 2 = 0

k = 2

When k = 2, then (2k-3)/(k+1) = (2(2)-3)/(2+1)

= (4-3)/3

= 1/3

And, (k+4)/(k+1) = (2+4)/(2+1)

= 6/3

= 2

∴ C (0, 1/3, 2) is a point which divides AB in the ratio 2: 1 and is same as P.

Hence, A, B, C are collinear.

5. Find the coordinates of the points which trisect the line segment joining the points P (4, 2, – 6) and Q (10, –16, 6).

Solution:

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) trisect the line segment joining the points P (4, 2, -6) and Q (10, -16, 6).

A divides the line segment PQ in the ratio 1: 2.

Upon comparing we have,

$x_1 = 4$, $y_1 = 2$, $z_1 = -6$;

$x_2 = 10$, $y_2 = -16$, $z_2 = 6$ and

m = 1, n = 2

By using the section formula,

We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m : n is given by:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

So we have,

The coordinates of A = $\left(\frac{1\times10+2\times4}{1+2}, \frac{1\times(-16)+2\times2}{1+2}, \frac{1\times6+2\times(-6)}{1+2}\right)$

= (18/3, -12/3, -6/3)

= (6, -4, -2)

Similarly, we know that B divides the line segment PQ in the ratio 2: 1.
Upon comparing we have,
$x_1 = 4$, $y_1 = 2$, $z_1 = -6$;
$x_2 = 10$, $y_2 = -16$, $z_2 = 6$ and
m = 2, n = 1
By using the section formula,
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

So we have,

The coordinates of B = $\left(\frac{2\times10+1\times4}{2+1}, \frac{2\times(-16)+1\times2}{2+1}, \frac{2\times6+1\times(-6)}{2+1}\right)$
= (24/3, -30/3, 6/3)
= (8, -10, 2)

∴ The coordinates of the points which trisect the line segment joining the points P (4, 2, –6) and Q (10, –16, 6) are (6, -4, -2) and (8, -10, 2).

MISCELLANEOUS EXERCISE PAGE NO: 278

1. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex.

Solution:

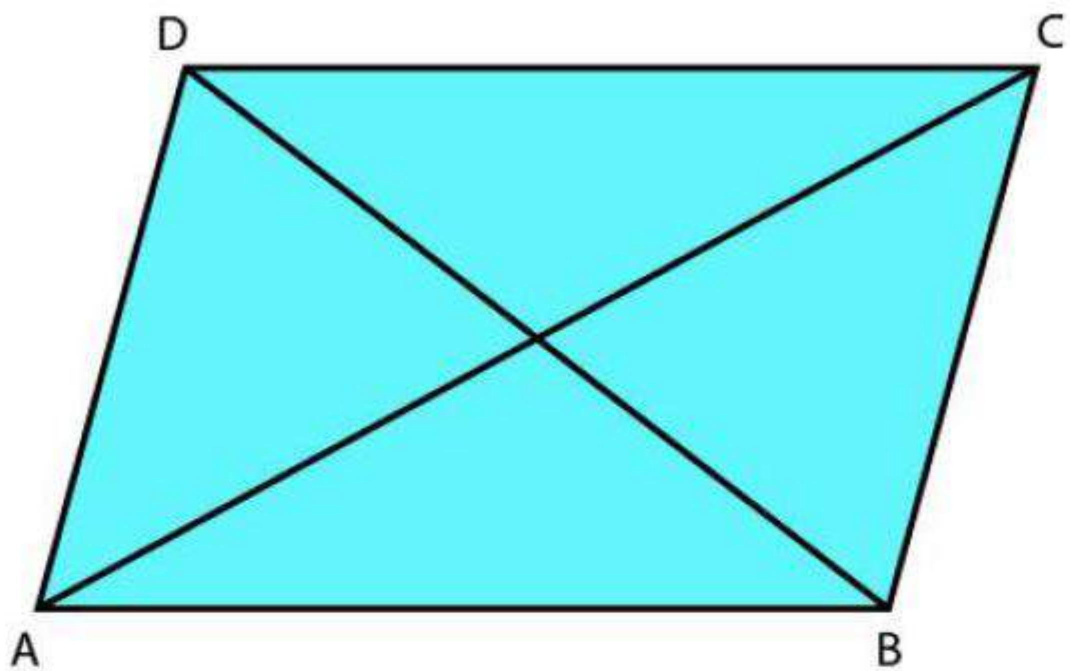
Given:

ABCD is a parallelogram, with vertices A (3, -1, 2), B (1, 2, -4), C (-1, 1, 2).

Where, $x_1 = 3$, $y_1 = -1$, $z_1 = 2$;

$x_2 = 1$, $y_2 = 2$, $z_2 = -4$;

$x_3 = -1$, $y_3 = 1$, $z_3 = 2$



Let the coordinates of the fourth vertex be D (x, y, z).

We also know that the diagonals of a parallelogram bisect each other, so the mid points of AC and BD are equal, i.e. Midpoint of AC = Midpoint of BD ……….(1)

Now, by Midpoint Formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $[(x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2]$

So we have,

Co-ordinates of the midpoint of AC:

$$= \left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right)$$

= (2/2, 0/2, 4/2)

= (1, 0, 2)

Co-ordinates of the midpoint of BD:

$$= \left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right)$$

So, using (1), we have

$$\left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right) = (1,0,2)$$

$$\frac{1+x}{2} = 1, \frac{2+y}{2} = 0, \frac{-4+z}{2} = 2$$

1 + x = 2, 2 + y = 0, -4 + z = 4

x = 1, y = -2, z = 8

Hence, the coordinates of the fourth vertex is D (1, -2, 8).

2. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Solution:

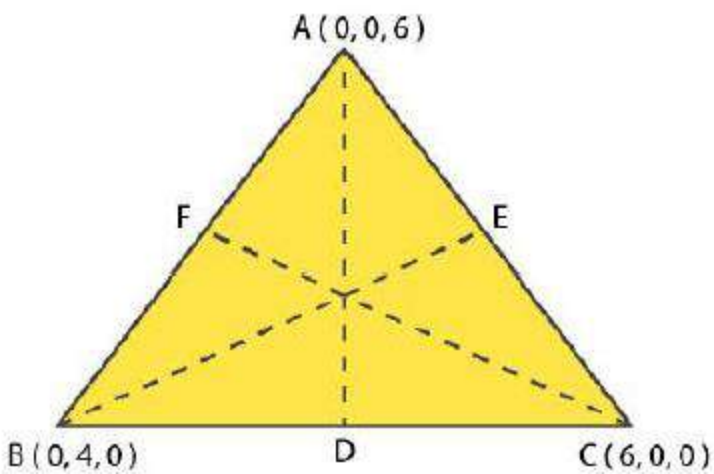
Given:

The vertices of the triangle are A (0, 0, 6), B (0, 4, 0) and C (6, 0, 0).

$x_1 = 0, y_1 = 0, z_1 = 6;$

$x_2 = 0, y_2 = 4, z_2 = 0;$

$x_3 = 6, y_3 = 0, z_3 = 0$



So, let the medians of this triangle be AD, BE and CF corresponding to the vertices A, B and C respectively.

D, E and F are the midpoints of the sides BC, AC and AB respectively.

By Midpoint Formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are [$(x_1+x_2)/2$, $(y_1+y_2)/2$, $(z_1+z_2)/2$]

So we have,

The coordinates of D:

$$= \left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right) = \left(\frac{6}{2}, \frac{4}{2}, \frac{0}{2}\right)$$

$$= (3, 2, 0)$$

The coordinates of E:

$$= \left(\frac{0+6}{2}, \frac{0+0}{2}, \frac{6+0}{2}\right) = \left(\frac{6}{2}, \frac{0}{2}, \frac{6}{2}\right)$$

$$= (3, 0, 3)$$

And the coordinates of F:

$$= \left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = \left(\frac{0}{2}, \frac{4}{2}, \frac{6}{2}\right)$$

$$= (0, 2, 3)$$

By Distance Formula, we know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

So the lengths of the medians are:

$$AD = \sqrt{(3-0)^2 + (2-0)^2 + (0-6)^2} = \sqrt{3^2 + 2^2 + (-6)^2} = \sqrt{9+4+36}$$
$$= \sqrt{49} = 7$$

$$BE = \sqrt{(3-0)^2 + (0-4)^2 + (3-0)^2} = \sqrt{3^2 + (-4)^2 + 3^2} = \sqrt{9+16+9}$$
$$= \sqrt{34}$$

$$CF = \sqrt{(0-6)^2 + (2-0)^2 + (3-0)^2} = \sqrt{(-6)^2 + 2^2 + 3^2} = \sqrt{36+4+9}$$
$$= \sqrt{49} = 7$$

∴ The lengths of the medians of the given triangle are 7, √34 and 7.

3. If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (– 4, 3b, –10) and R(8, 14, 2c), then find the values of a, b and c.

Solution:

Given:

The vertices of the triangle are P (2a, 2, 6), Q (-4, 3b, -10) and R (8, 14, 2c).

Where,

$x_1 = 2a$, $y_1 = 2$, $z_1 = 6$;

$x_2 = -4$, $y_2 = 3b$, $z_2 = -10$;

$x_3 = 8$, $y_3 = 14$, $z_3 = 2c$

We know that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are $[(x_1+x_2+x_3)/3, (y_1+y_2+y_3)/3, (z_1+z_2+z_3)/3]$

So, the coordinates of the centroid of the triangle PQR are

$$\left(\frac{2a-4+8}{3},\frac{2+3b+14}{3},\frac{6-10+2c}{3}\right)=\left(\frac{2a+4}{3},\frac{3b+16}{3},\frac{2c-4}{3}\right)$$

Now, it is given that the origin (0, 0, 0) is the centroid.

So, we have $\left(\frac{2a+4}{3},\frac{3b+16}{3},\frac{2c-4}{3}\right)=(0,0,0)$

$$\frac{2a+4}{3}=0,\frac{3b+16}{3}=0,\frac{2c-4}{3}=0$$

2a + 4 = 0, 3b + 16 = 0, 2c – 4 = 0

a = -2, b = -16/3, c = 2

∴ The values of a, b and c are a = -2, b = -16/3, c = 2

4. Find the coordinates of a point on y-axis which are at a distance of 5√2 from the point P (3, –2, 5).

Solution:

Let the point on y-axis be A (0, y, 0).

Then, it is given that the distance between the points A (0, y, 0) and P (3, -2, 5) is 5√2.

Now, by using distance formula,

We know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

Distance of PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, the distance between the points A (0, y, 0) and P (3, -2, 5) is given by

Distance of AP = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

$= \sqrt{[(3-0)^2 + (-2-y)^2 + (5-0)^2]}$

$= \sqrt{[3^2 + (-2-y)^2 + 5^2]}$

$= \sqrt{[(-2-y)^2 + 9 + 25]}$

$5\sqrt{2} = \sqrt{[(-2-y)^2 + 34]}$

Squaring on both the sides, we get

$(-2 -y)^2 + 34 = 25 \times 2$

$(-2 -y)^2 = 50 – 34$

$4 + y^2 + (2 \times -2 \times -y) = 16$

$y^2 + 4y + 4 -16 = 0$

$y^2 + 4y – 12 = 0$

$y^2 + 6y – 2y – 12 = 0$

y (y + 6) – 2 (y + 6) = 0

(y + 6) (y - 2) = 0

y = -6, y = 2

∴ The points (0, 2, 0) and (0, -6, 0) are the required points on the y-axis.

5. A point R with x-coordinate 4 lies on the line segment joining the points P (2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.
[Hint Suppose R divides PQ in the ratio k: 1. The coordinates of the point R are given by
$$\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)]$$

Solution:
Given:
The coordinates of the points P (2, -3, 4) and Q (8, 0, 10).
$x_1 = 2$, $y_1 = -3$, $z_1 = 4$;
$x_2 = 8$, $y_2 = 0$, $z_2 = 10$
Let the coordinates of the required point be (4, y, z).
So now, let the point R (4, y, z) divides the line segment joining the points P (2, -3, 4) and Q (8, 0, 10) in the ratio k: 1.
By using Section Formula,
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:
$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$
So, the coordinates of the point R are given by $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$

So, we have $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right) = (4, y, z)$
$$\Rightarrow \frac{8k+2}{k+1} = 4$$
8k + 2 = 4 (k + 1)
8k + 2 = 4k + 4
8k – 4k = 4 – 2
4k = 2
k = 2/4
= 1/2
Now let us substitute the values, we get
$$\Rightarrow y = \frac{-3}{\frac{1}{2}+1} = \frac{-3}{\frac{3}{2}} = \frac{-3 \times 2}{3} = -2,$$
$$z = \frac{10\left(\frac{1}{2}\right) + 4}{\frac{1}{2}+1} = \frac{5+4}{\frac{3}{2}} = \frac{9 \times 2}{3} = 3 \times 2$$

= 6

∴ The coordinates of the required point are (4, -2, 6).

6. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Solution:

Given:

The points A (3, 4, 5) and B (-1, 3, -7)

$x_1 = 3, y_1 = 4, z_1 = 5$;

$x_2 = -1, y_2 = 3, z_2 = -7$;

$PA^2 + PB^2 = k^2$ ……….(1)

Let the point be P (x, y, z).

Now by using distance formula,

We know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

So,

$$PA = \sqrt{(3 - x)^2 + (4 - y)^2 + (5 - z)^2}$$

And

$$PB = \sqrt{(-1 - x)^2 + (3 - y)^2 + (-7 - z)^2}$$

Now, substituting these values in (1), we have

$[(3 - x)^2 + (4 - y)^2 + (5 - z)^2] + [(-1 - x)^2 + (3 - y)^2 + (-7 - z)^2] = k^2$

$[(9 + x^2 – 6x) + (16 + y^2 – 8y) + (25 + z^2 – 10z)] + [(1 + x^2 + 2x) + (9 + y^2 – 6y) + (49 + z^2 + 14z)] = k^2$

$9 + x^2 – 6x + 16 + y^2 – 8y + 25 + z^2 – 10z + 1 + x^2 + 2x + 9 + y^2 – 6y + 49 + z^2 + 14z = k^2$

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 = k^2$

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = k^2 – 109$

$2 (x^2 + y^2 + z^2 – 2x – 7y + 2z) = k^2 – 109$

$(x^2 + y^2 + z^2 – 2x – 7y + 2z) = (k^2 – 109)/2$

Hence, the required equation is $(x^2 + y^2 + z^2 – 2x – 7y + 2z) = (k^2 – 109)/2$