EXERCISE 13.1 PAGE NO: 301

1. Evaluate the Given limit: $\lim_{x\to 3} x+3$

Solution:

Given

$\lim_{x\to 3} x+3$

Substituting x = 3, we get

= 3 + 3

= 6

2. Evaluate the Given limit: $\lim_{x\to\pi}\left(x-\frac{22}{7}\right)$

Solution:

Given limit: $\lim_{x\to\pi}\left(x-\frac{22}{7}\right)$

Substituting x = π, we get

$\lim_{x\to\pi}\left(x-\frac{22}{7}\right)$ = (π – 22 / 7)

3. Evaluate the Given limit: $\lim_{r\to 1}\pi r^2$

Solution:

Given limit: $\lim_{r\to 1}\pi r^2$

Substituting r = 1, we get

$\lim_{r\to 1}\pi r^2$ = $\pi(1)^2$

= π

4. Evaluate the Given limit: $\lim_{x\to 4}\frac{4x+3}{x-2}$

Solution:

Given limit: $\lim_{x\to 4}\frac{4x+3}{x-2}$

Substituting x = 4, we get

$\lim_{x\to 4}\frac{4x+3}{x-2}$ = [4(4) + 3] / (4 – 2)

= (16 + 3) / 2

= 19 / 2

5. Evaluate the Given limit: $\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Solution:

Given limit: $\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Substituting x = -1, we get

$\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

= $[(-1)^{10} + (-1)^5 + 1] / (-1 – 1)$

= (1 – 1 + 1) / - 2

= - 1 / 2

6. Evaluate the Given limit: $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

Solution:

Given limit: $\lim_{x \to 0} \frac{(x+1)^5 - 1}{x}$

= $[(0 + 1)^5 – 1] / 0$

=0

Since, this limit is undefined

Substitute x + 1 = y, then x = y – 1

$\lim_{y \to 1} \frac{(y)^5 - 1}{y - 1}$

= $\lim_{y \to 1} \frac{(y)^5 - 1^5}{y - 1}$

We know that,

$\lim_{x \to a} \frac{x^n - a^n}{x - a}$ = na^{n-1}

Hence,

$\lim_{y \to 1} \frac{(y)^5 - 1^5}{y - 1}$

$=5(1)^{5-1}$

$= 5(1)^4$

= 5

7. Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Solution:

By evaluating the limit at x = 2, we get

$\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$ = $[3(2)^2 - x - 10] / 4 - 4$

= 0

Now, by factorising numerator, we get

$\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4} = \lim_{x\to 2}\frac{3x^2-6x+5x-10}{x^2-2^2}$

We know that,

$a^2 - b^2 = (a - b)(a + b)$

$= \lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$

$= \lim_{x\to 2}\frac{(3x+5)}{(x+2)}$

By substituting x = 2, we get,

= [3(2) + 5] / (2 + 2)

= 11 / 4

8. Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Solution:

First substitute x = 3 in the given limit, we get

$$= \lim_{x \to 3} \frac{(3)^4 - 81}{2(3)^2 - 5 \times 3 - 3}$$

= (81 – 81) / (18 – 18)

= 0 / 0

Since the limit is of the form 0 / 0, we need to factorise the numerator and denominator

$$\lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{2x^2 - 6x + x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(2x + 1)(x - 3)}$$

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$

Now substituting x = 3, we get

$$= \frac{(3 + 3)(3^2 + 9)}{(2 \times 3 + 1)}$$

= 108 / 7

Hence,

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = 108 / 7$$

9. Evaluate the Given limit: $\lim_{x \to 0} \frac{ax + b}{cx + 1}$

Solution:

$$\lim_{x \to 0} \frac{ax + b}{cx + 1}$$

= [a (0) + b] / c (0) + 1

= b / 1

= b

10. Evaluate the Given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

Solution:

$\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = (1 - 1) / (1 - 1)$

$= 0$

Let the value of $z^{1/6}$ be x

$(z^{1/6})^2 = x^2$

$z^{1/3} = x^2$

Now, substituting $z^{1/3} = x^2$ we get

$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{x^2 - 1^2}{x - 1}$

We know that,

$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

$\lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} = 2\,(1)^{2-1}$

$= 2$

11. Evaluate the Given limit: $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

Solution:

Given limit: $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

Substituting x = 1

$\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$

$= [a\,(1)^2 + b\,(1) + c] / [c\,(1)^2 + b\,(1) + a]$

$= (a + b + c) / (a + b + c)$

Given

$[a + b + c \neq 0]$

$= 1$

12. Evaluate the Given limit: $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

Solution:

By substituting x = - 2, we get

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2} = 0/0$$

Now,

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \frac{\frac{2+x}{2x}}{x+2}$$

= 1 / 2x

= 1 / 2(-2)

= - 1 / 4

13. Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Solution:

Given $\lim_{x\to 0}\frac{\sin ax}{bx}$

Formula used here

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

By applying the limits in the given expression

$$\lim_{x\to 0}\frac{\sin ax}{bx} = \frac{0}{0}$$

By multiplying and dividing by 'a' in the given expression, we get

$$\lim_{x\to 0}\frac{\sin ax}{bx}\times\frac{a}{a}$$

We get,

$$\lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{a}{b}$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$= \frac{a}{b}\lim_{ax\to 0}\frac{\sin ax}{ax} = \frac{a}{b}\times 1$$

= a / b

14. Evaluate the given limit: $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$

Solution:

$\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} = 0 / 0$

By multiplying ax and bx in numerator and denominator, we get

$\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} = \lim\limits_{x \to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}$

Now, we get

$\frac{a}{b} \frac{\lim\limits_{ax \to 0} \frac{\sin ax}{ax}}{\lim\limits_{bx \to 0} \frac{\sin bx}{bx}}$

We know that,

$\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

Hence, a / b × 1

= a / b

15. Evaluate the given limit: $\lim\limits_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

Solution:

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\lim_{\pi-x\to 0}\frac{\sin(\pi-x)}{(\pi-x)}\times\frac{1}{\pi}$$

$$=\frac{1}{\pi}\lim_{\pi-x\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

We know that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\frac{1}{\pi}\lim_{\pi-x\to 0}\frac{\sin(\pi-x)}{(\pi-x)}=\frac{1}{\pi}\times 1$$

$= 1/\pi$

16. Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi-x}$

Solution:

$$\lim_{x\to 0}\frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0}$$

$= 1/\pi$

17. Evaluate the given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Solution:

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1}$$

($\cos 2x = 1 - 2\sin^2 x$)

$$\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\frac{\sin^2 x\times x^2}{x^2}}{\frac{\sin^2\frac{x}{2}\times\frac{x^2}{4}}{(\frac{x}{2})^2}}$$

$$= 4\,\frac{\lim_{x\to 0}\frac{\sin^2 x}{x^2}}{\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}$$

$$= 4\,\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)^2}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{(\frac{x}{2})^2}\right)^2}$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$= 4 \times 1^2/ 1^2$

$= 4$

18. Evaluate the given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Solution:

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to 0}\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\lim\limits_{x\to 0}\frac{\sin x}{x}}\times\lim_{x\to 0}(a+\cos x)$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\frac{1}{b}\times(a+\cos 0)$$

= (a + 1) / b

19. Evaluate the given limit: $\lim\limits_{x\to 0} x\sec x$

Solution:

$$\lim_{x\to 0}x\sec x=\lim_{x\to 0}\frac{x}{\cos x}$$

$$=\lim_{x\to 0}\frac{0}{\cos 0}=\frac{0}{1}$$

= 0

20. Evaluate the given limit: $\lim\limits_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}a,b,a+b\neq 0$

Solution:

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}=\frac{0}{0}$$

Hence,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}=\lim_{x\to 0}\frac{(\sin\frac{ax}{ax})ax+bx}{ax+(\sin\frac{bx}{bx})}$$

$$=\frac{\left(\lim\limits_{ax\to 0}\sin\frac{ax}{ax}\right)\times\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx}{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx\times(\lim\limits_{bx\to 0}\sin\frac{bx}{bx})}$$

We know that,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$=\frac{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx}{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx}$$

We get,

$$=\frac{\lim\limits_{x\to 0}(ax+bx)}{\lim\limits_{x\to 0}(ax+bx)}$$

= 1

21. Evaluate the given limit: $\lim\limits_{x\to 0}(\operatorname{cosec}x-\cot x)$

Solution:

$$\lim_{x \to 0}(\text{cosec } x - \cot x)$$

Applying the formulas for cosec x and cot x, we get

$$\text{cosec } x = \frac{1}{\sin x} \; and \; \cot x = \frac{\cos x}{\sin x}$$

$$\lim_{x \to 0}(\text{cosec } x - \cot x) = \lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$\lim_{x \to 0}(cosec\, x - cot\, x) = \lim_{x \to 0}\frac{1 - \cos x}{\sin x}$$

Now, by applying the formula we get,

$$1 - \cos x = 2\sin^2\frac{x}{2} \; and \; \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\lim_{x \to 0}(\text{cosec } x - \cot x) = \lim_{x \to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$\lim_{x \to 0}(cosec\, x - cot\, x) = \lim_{x \to 0}\tan\frac{x}{2}$$

= 0

22. Evaluate the given limit: $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

Solution:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}} = \frac{0}{0}$$

Let x – (π / 2) = y

Then, x→ (π/2) = y→ 0

Now, we get

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0}\frac{\tan 2(y+\frac{\pi}{2})}{y}$$

$$= \lim_{y\to 0}\frac{\tan(2y+\pi)}{y}$$

$$= \lim_{y\to 0}\frac{\tan(2y)}{y}$$

We know that,

tan x = sin x / cos x

$$= \lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$$

By multiplying and dividing by 2, we get

$$=\lim_{y\to 0}\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}$$

$$=\lim_{2y\to 0}\frac{\sin 2y}{2y}\times\lim_{y\to 0}\frac{2}{\cos 2y}$$

= 1 × 2 / cos 0

= 1 × 2 / 1

= 2

23. Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} 2x+3 & x \le 0 \\ 3(x+1) & x > 0 \end{cases}$

$\lim_{x \to 0} f(x)$:

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0}(2x+3)$

$= 2(0) + 3$

$= 0 + 3$

$= 3$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} 3(x+1)$

$= 3(0+1)$

$= 3(1)$

$= 3$

Hence, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 3$

Now, for $\lim_{x\to 1} f(x)$:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1)$

$= 3\ (1 + 1)$

$= 3\ (2)$

$= 6$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1)$

$= 3\ (1 + 1)$

$= 3\ (2)$

$= 6$

Hence, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$

$\lim_{x\to 0} f(x)=3$ and $\lim_{x\to 1} f(x)=6$

24. Find $\lim_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} x^2 - 1 & x \le 1 \\ -x^2 - 1 & x > 1 \end{cases}$$

Solution:

Given function is:

$$f(x)=\begin{cases} x^2-1 & x\le 1 \\ -x^2-1 & x>1 \end{cases}$$

$\lim\limits_{x\to 1} f(x)$:

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} x^2 - 1$

$= 1^2 - 1$

$= 1 - 1$

$= 0$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1}(-x^2 - 1)$

$= (-1^2 - 1)$

$= -1 - 1$

$= -2$

We find,

$\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$

Hence, $\lim\limits_{x\to 1} f(x)$ does not exist

25. Evaluate $\lim\limits_{x\to 0} f(x)$**, where f(x) =** $\begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:

Given function is $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

We know that,

$\lim\limits_{x \to a} f(x)$ exists only when $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

Now, we need to prove that: $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$

We know,

|x| = x, if x > = -x, if x < 0

Hence,

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \frac{|x|}{x}$

$= \lim\limits_{x \to 0} \frac{-x}{x} = \lim\limits_{x \to 0} (-1)$

= -1

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{|x|}{x}$

$= \lim\limits_{x \to 0} \frac{x}{x} = \lim\limits_{x \to 0} (1)$

= 1

We find here,

$\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ does not exist.

26. Find $\lim_{x\to 0} f(x)$**, where f (x) =** $\begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:

Given function is:

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|}$$

$$= \lim_{x\to 0} \frac{x}{-x} = \lim_{x\to 0} \frac{1}{-1}$$

$$= -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{|x|}$$

$= \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0}(1)$

$= 1$

We find here,

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ does not exist.

27. Find $\lim_{x\to 5} f(x)$, where $f(x) = |x| - 5$

Solution:

Given function is:

$f(x) = |x| - 5$

$\lim_{x\to 5} f(x)$:

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} |x| - 5$

$= \lim_{x\to 5}(x - 5) = 5 - 5$

$= 0$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} |x| - 5$

$= \lim_{x\to 5}(x - 5)$

$= 5 - 5$

$= 0$

Hence, $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 0$

28. Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax & x > 1 \end{cases}$ **and if** $\lim_{x \to 1} f(x) = f(1)$ **what are possible values of a and b**

Solution:

Given function is:

$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$ and

$\lim_{x \to 1} f(x) = f(1)$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} a + bx$

$= a + b\ (1)$

$= a + b$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} b - ax$

$= b - a\ (1)$

$= b - a$

Here,

$f(1) = 4$

Hence, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$

Then, $a + b = 4$ and $b - a = 4$

By solving the above two equations, we get,

$a = 0$ and $b = 4$

Therefore, the possible values of a and b is 0 and 4 respectively

29. Let $a_1, a_2, \ldots\ldots a_n$ be fixed real numbers and define a function $f(x) = (x - a_1)(x - a_2) \ldots\ldots (x - a_n)$.

What is $\lim_{x\to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots\ldots a_n$, compute $\lim_{x\to a} f(x)$

Solution:

Given function is:

$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

$\lim_{x \to a_1} f(x)$:

$\lim_{x \to a_1} f(x) = \lim_{x \to a_1} [(x - a_1)(x - a_2) \ldots (x - a_n)]$

$= [\lim_{x \to a_1}(x - a_1)] \left[\lim_{x \to a_1}(x - a_2)\right] \ldots \left[\lim_{x \to a_1}(x - a_n)\right]$

We get,

$= (a_1 - a_1)(a_1 - a_2) \ldots (a_1 - a_n) = 0$

Hence, $\lim_{x \to a_1} f(x) = 0$

$\lim_{x \to a} f(x)$:

$\lim_{x \to a} f(x) = \lim_{x \to a} [(x - a_1)(x - a_2) \ldots (x - a_n)]$

$= [\lim_{x \to a}(x - a_1)] \left[\lim_{x \to a}(x - a_2)\right] \ldots \left[\lim_{x \to a}(x - a_n)\right]$

We get,

$= (a - a_1)(a - a_2) \ldots\ldots (a - a_n)$

Hence, $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_n)$

Therefore, $\lim_{x \to a_1} f(x) = 0$ and $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_n)$

30. If $f(x)=\begin{cases}|x|+1, x<0\\ 0, \quad x=0\\ |x|-1, x>0\end{cases}$ **For what value (s) of a does** $\lim_{x\to a} f(x)$ **exists?**

Solution:

Given function is:

$$f(x)=\begin{cases}|x|+1, x<0\\ 0, \quad x=0\\ |x|-1, x>0\end{cases}$$

There are three cases.

Case 1:

When $a=0$

$\lim_{x\to 0} f(x)$:

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(|x|+1)$

$= \lim_{x\to 0}(-x+1) = -0+1$

$= 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(|x| - 1)$

$= \lim_{x\to 0}(x - 1) = 0 - 1$

= -1

Here, we find

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ does not exit.

Case 2:

When a < 0

$\lim_{x\to a} f(x)$:

$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x| + 1)$

$= \lim_{x\to a}(-x + 1) = -a + 1$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x| + 1)$

$= \lim_{x\to a}(-x + 1) = -a + 1$

Hence, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = -a + 1$

Therefore, lim (f(x)) exists at x = a and a < 0

Case 3:

When a > 0

$\lim_{x \to a} f(x)$:

$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| - 1)$

$= \lim_{x \to a} (x - 1) = a - 1$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| - 1)$

$= \lim_{x \to a} (x - 1) = a - 1$

Hence, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = a - 1$

Therefore, lim (f (x)) exists at x = a when a > 0

31. If the function f(x) satisfies $\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$

Solution:

Given function that f (x) satisfies $\lim\limits_{x\to 1}\dfrac{f(x)-2}{x^2-1}=\pi$

$$\frac{\lim\limits_{x\to 1} f(x)-2}{\lim\limits_{x\to 1} x^2-1}=\pi$$

$$\lim_{x\to 1}(f(x)-2)=\pi(\lim_{x\to 1}(x^2-1))$$

Substituting x = 1, we get,

$$\lim_{x\to 1}(f(x)-2)=\pi(1^2-1)$$

$$\lim_{x\to 1}(f(x)-2)=\pi(1-1)$$

$$\lim_{x\to 1}(f(x)-2)=0$$

$$\lim_{x\to 1} f(x)-\lim_{x\to 1} 2=0$$

$$\lim_{x\to 1} f(x)-2=0$$

$$=2$$

32. If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m. & x>1 \end{cases}$ **For what integers m and n does both** $\lim\limits_{x\to 0} f(x)$ **and** $\lim\limits_{x\to 1} f(x)$ **exist?**

Solution:

Given function is $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m. & x>1 \end{cases}$

$\lim_{x\to 0} f(x)$:

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(mx^2+n)$

$= m\,(0) + n$

$= 0 + n$

$= n$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(nx+m)$

$= n\,(0) + m$

$= 0 + m$

$= m$

Hence,

$\lim_{x\to 0} f(x)$ exists if n = m.

Now,

$\lim_{x\to 1} f(x)$:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(nx + m)$

$= n\,(1) + m$

$= n + m$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(nx^3 + m)$

$= n\,(1)^3 + m$

$= n\,(1) + m$

$= n + m$

Therefore $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$

Hence, for any integral value of m and n $\lim_{x\to 1} f(x)$ exists.

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1. Find the derivative of $x^2 – 2$ at x = 10

Solution:

Let f (x) = $x^2 – 2$

From first principle

From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(10)}{h}$$

Put x = 10, we get

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$

$$= \lim_{h\to 0}\frac{[(10+h)^2-2]-(10^2-2)}{h}$$

$$= \lim_{h\to 0}\frac{10^2+2\times 10\times h+h^2-2-10^2+2}{h}$$

$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$

$$= \lim_{h\to 0}(20+h)$$

$$= 20 + 0$$

$$= 20$$

2. Find the derivative of x at x = 1.

Solution:

Let f (x) = x

Then,

From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(10)}{h}$$

Let f (x) = x

From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(10)}{h}$$

Put x = 1, we get

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h)-1}{h}$$

$$= \lim_{h\to 0}\frac{1+h-1}{h}$$

$$= \lim_{h\to 0}\frac{h}{h}$$

$$= \lim_{h\to 0} 1$$

$$= 1$$

3. Find the derivative of 99x at x = l00.
Solution:
Let f (x) = 99x,
From first principle

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(10)}{h}$$

Put x = 100, we get

$$f'(100) = \lim_{h\to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h\to 0} \frac{99(100+h) - 99 \times 100}{h}$$

$$= \lim_{h\to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$

$$= \lim_{h\to 0} \frac{99 \times h}{h}$$

$$= \lim_{h\to 0} 99$$

= 99

4. Find the derivative of the following functions from first principle
(i) $x^3 – 27$
(ii) $(x – 1) (x – 2)$
(iii) $1 / x^2$
(iv) $x + 1 / x – 1$
Solution:
(i) Let $f(x) = x^3 - 27$
From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{[(x+h)^3-27]-(x^3-27)}{h}$$

$$= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$

$$= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h}$$

$$= \lim_{h\to 0}(h^2+3x^2+3xh)$$

$$= 0 + 3x^2$$

$$= 3x^2$$

(ii) Let f (x) = (x – 1) (x – 2)

From first principle

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}$$

$$= \lim_{h\to 0}\frac{(x^2+hx-2x+hx+h^2-2h-x-h+2)-(x^2-2x-x+2)}{h}$$

$$= \lim_{h\to 0}\frac{hx+hx+h^2-2h-h}{h}$$

$$= \lim_{h\to 0}(h+2x-3)$$

$= 0 + 2x - 3$

$= 2x - 3$

(iii) Let f (x) = 1 / x^2

From first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h^2 - 2hx}{x^2(x+h)^2}\right]$$

$$= \lim_{h \to 0} \left[\frac{-h - 2x}{x^2(x+h)^2}\right]$$

$= (0 - 2x) / [x^2 (x + 0)^2]$

$= (- 2 / x^3)$

(iv) Let f (x) = x + 1 / x – 1

From first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}}{h}$$

$$=\lim_{h\to 0}\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx+x-x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{-2h}{h(x-1)(x+h-1)}$$

$$=\lim_{h\to 0}\frac{-2}{(x-1)(x+h-1)}$$

$$=-\frac{2}{(x-1)(x-1)}$$

$$=-\frac{2}{(x-1)^2}$$

5. For the function $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots\frac{x^2}{2}+x+1$ **.Prove that f' (1) =100 f' (0).**

Solution:

Given function is:

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots\frac{x^{2}}{2}+x+1$$

By differentiating both sides, we get

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\cdots+\frac{x^{2}}{2}+x+1\right]$$

$$=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\cdots+\frac{d}{dx}\left(\frac{x^{2}}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

We know that,

$$\frac{d}{dx}(x^{n})=nx^{n-1}$$

$$\therefore \frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\cdots+\frac{2x}{2}+1+0$$

$f'(x) = x^{99} + x^{98} + \cdots + x + 1$

At $x = 0$, we get

$f'(0) = 0 + 0 + \ldots + 0 + 1$

$f'(0) = 1$

At $x = 1$, we get

$f'(1) = 1^{99} + 1^{98} + \ldots + 1 + 1 = [1 + 1 \ldots. + 1]$ 100 times $= 1 \times 100 = 100$

Hence, $f'(1) = 100\, f'(0)$

6. Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$ for some fixed real number a.
Solution:

Given function is:

f(x) = $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n\right)$$

$$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \cdots + a^{n-1}\frac{d}{dx}(x) + a^n\frac{d}{dx}(1)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1} + a^n(0)$$

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \ldots + a^{n-1}$$

7. For some constants a and b, find the derivative of
(i) (x − a) (x − b)
(ii) $(ax^2 + b)^2$
(iii) x – a / x – b
Solution:
(i) (x – a) (x – b)

Let f (x) = (x – a) (x – b)

$f(x) = x^2 - (a + b)x + ab$

Now, by differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(x^2 - (a+b)x + ab)$$

$$= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$f'(x) = 2x - (a + b) + 0$

$= 2x - a - b$

(ii) $(ax^2 + b)^2$

Let $f(x) = (ax^2 + b)^2$

$f(x) = a^2x^4 + 2abx^2 + b^2$

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$$

$$f'(x) = \frac{d}{dx}(x^4) + (2ab)\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$f'(x) = a^2 \times 4x^3 + 2ab \times 2x + 0$

$= 4a^2x^3 + 4abx$

$= 4ax(ax^2 + b)$

(iii) $x - a / x - b$

Let $f(x) = \frac{(x-a)}{(x-b)}$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

By further calculation, we get

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

8. Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a.
Solution:

Let $f(x) = \frac{x^n - a^n}{x-a}$

By differentiating both sides and using quotient rule, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x-a}\right)$$

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n - a^n) - (x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

By further calculation, we get

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

9. Find the derivative of
(i) 2x – 3 / 4
(ii) $(5x^3 + 3x - 1)(x - 1)$
(iii) $x^{-3}(5 + 3x)$
(iv) $x^5(3 - 6x^{-9})$
(v) $x^{-4}(3 - 4x^{-5})$
(vi) $(2 / x + 1) - x^2 / 3x - 1$

Solution:

(i)

Let f (x) = 2x – 3 / 4

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(2x - \frac{3}{4}\right)$$

$$= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right)$$

= 2 – 0

= 2

(ii)

Let f (x) = $(5x^3 + 3x – 1) (x – 1)$

By differentiating both sides and using the product rule, we get

$$f'(x) = (5x^3 + 3x - 1)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}(5x^3 + 3x + 1)$$

$$= (5x^3 + 3x - 1) \times 1 + (x-1) \times (15x^2 + 3)$$

$$= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3)$$

$$= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3$$

$$= 20x^3 - 15x^2 + 6x - 4$$

(iii)

Let f (x) = x^{-3} (5 + 3x)

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}(x^{-3})$$

$$= x^{-3}(0+3)+(5+3x)(-3x^{-3-1})$$

By further calculation, we get

$$= x^{-3}(3)+(5+3x)(-3x^{-4})$$

$$= 3x^{-3}-15x^{-4}-9x^{-3}$$

$$= -6x^{-3}-15x^{-4}$$

$$= -3x^{-3}\left(2+\frac{5}{x}\right)$$

$$= \frac{-3x^{-3}}{x}(2x+5)$$

$$= \frac{-3}{x^{4}}(5+2x)$$

(iv)

Let $f(x) = x^5 (3 - 6x^{-9})$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^5 \frac{d}{dx}\left(3 - 6x^{-9}\right) + \left(3 - 6x^{-9}\right)\frac{d}{dx}\left(x^5\right)$$

$$= x^5\left\{0 - 6(-9)x^{-9-1}\right\} + \left(3 - 6x^{-9}\right)\left(5x^4\right)$$

By further calculation, we get

$$= x^5\left(54x^{-10}\right) + 15x^4 - 30x^{-5}$$

$$= 54x^{-5} + 15x^4 - 30x^{-5}$$

$$= 24x^{-5} + 15x^4$$

$$= 15x^4 + \frac{24}{x^5}$$

(v)

Let $f(x) = x^{-4} (3 - 4x^{-5})$

By differentiating both sides and using Leibnitz product rule, we get

$$f'(x) = x^{-4} \frac{d}{dx}\left(3 - 4x^{-5}\right) + \left(3 - 4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$$

$$= x^{-4}\left\{0 - 4(-5)x^{-5-1}\right\} + \left(3 - 4x^{-5}\right)(-4)x^{-4-1}$$

By further calculation, we get

$$= x^{-4}\left(20x^{-6}\right) + \left(3 - 4x^{-5}\right)\left(-4x^{-5}\right)$$

$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$

$$= 36x^{-10} - 12x^{-5}$$

$$= -\frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi)

Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

By differentiating both sides we get,

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1} - \frac{x^2}{3x-1}\right)$$

Using quotient rule we get,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)\times 3}{(3x-1)^2}\right]$$

$$= -\frac{2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= -\frac{2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

10. Find the derivative of cos x from first principle
Solution:

Let f (x) = cos x

Accordingly, f (x + h) = cos (x + h)

By first principle, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

So, we get

$$= \lim_{h\to 0} \frac{1}{h}[\cos(x+h) - \cos(x)]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]$$

By further calculation, we get

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0} -\sin\left(\frac{2x+h}{2}\right) \times \lim_{h\to 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}}$$

$$= -\sin\left(\frac{2x+0}{2}\right) \times 1$$

= - sin (2x / 2)

= - sin (x)

11. Find the derivative of the following functions:

(i) sin x cos x

(ii) sec x

(iii) 5 sec x + 4 cos x

(iv) cosec x

(v) 3 cot x + 5 cosec x

(vi) 5 sin x – 6 cos x + 7

(vii) 2 tan x – 7 sec x

Solution:

(i) sin x cos x

Let f (x) = sin x cos x

Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h}$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\sin(x+h)\cos(x+h) - 2\sin x \cos x\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[\sin 2(x+h) - \sin 2x\right]$$

$$= \lim_{h \to 0} \frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2} \cdot \sin\frac{2x+2h-2x}{2}\right]$$

By further calculation, we get

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cos(2x+h)\sin h\right]$$

$$= \lim_{h \to 0} \cos(2x+h) . \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos(2x+0).1$$

$$= \cos 2x$$

(ii) sec x

Let f (x) = sec x

= 1 / cos x

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

Using quotient rule, we get

$$f'(x) = \frac{\cos x \frac{d}{dx}(1) - 1\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \times 0 - (-\sin x)}{\cos^2 x}$$

We get

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$

= tan x sec x

(iii) 5 sec x + 4 cos x

Let f (x) = 5 sec x + 4 cos x

By differentiating both sides, we get

$$f'(x) = \frac{d}{dx}(5\sec x + 4\cos x)$$

By further calculation, we get

$$= 5\frac{d}{dx}(\sec x) + 4\frac{d}{dx}(\cos x)$$

$$= 5\sec x\tan x + 4 \times (-\sin x)$$

$$= 5\sec x\tan x - 4\sin x$$

(iv) cosec x

Let f (x) = cosec x

Accordingly f (x + h) = cosec (x + h)

By first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\operatorname{cosec}(x+h) - \operatorname{cosec} x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

By further calculation, we get

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=-\frac{1}{\sin x}\times 1\times\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=-\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

= -cosec x cot x

(v) 3 cot x + 5 cosec x

Let f (x) = 3 cot x + 5 cosec x

$f'(x) = 3\,(\cot x)' + 5\,(\text{cosec } x)'$

Let $f_1(x) = \cot x$,

Accordingly $f_1(x + h) = \cot(x + h)$

By using first principle, we get

$$f_1'(x)=\lim_{x\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)$$

By further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right)$$

$$=1/\sin x\ \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

$$=-\frac{1}{\sin x}\times 1\times\frac{1}{\sin(x+0)}$$

$$=-\frac{1}{\sin^2 x}$$

$$=-\operatorname{cosec}^2 x$$

Let $f_2(x) = \operatorname{cosec} x$,

Accordingly $f_2(x+h) = \operatorname{cosec}(x+h)$

By using first principle, we get

$$f_2'(x)=\lim_{h\to 0}\frac{f_2(x+h)-f_2(x)}{h}$$

$$=\lim_{h\to 0}\frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)}\right]$$

By further calculation, we get

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$=\frac{1}{\sin x}\lim_{h\to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right)\sin(x+h)}\right]$$

$$=-\frac{1}{\sin x}\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\lim_{h\to 0}\frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$=-\frac{1}{\sin x}\times 1\times\frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$=-\frac{1}{\sin x}\times\frac{\cos x}{\sin x}$$

$= -\text{cosec } x \cot x$

Now, substitute the value of (cot x)’ and (cosec x)’ in f’(x), we get

$f'(x) = 3\,(\cot x)' + 5\,(\text{cosec } x)'$

$f'(x) = 3 \times (-\text{cosec}^2 x) + 5 \times (-\text{cosec } x \cot x)$

$f'(x) = -3\text{cosec}^2 x - 5\text{cosec } x \cot x$

(vi)5 sin x – 6 cos x + 7

Let f (x) = 5 sin x – 6 cos x + 7

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h)-6\cos(x+h)+7-5\sin x+6\cos x-7\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[5\{\sin(x+h)-\sin x\}-6\{\cos(x+h)-\cos x\}\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right]-6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

By further calculation, we get

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-6\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right]-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

Now, we get

$$=5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)-6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]-6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$

$$=5\cos x.1-6\left[(-\cos x).(0)-\sin x.1\right]$$
$$=5\cos x+6\sin x$$

(vii) 2 tan x – 7 sec x

Let f (x) = 2 tan x – 7 sec x

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\left\{\tan(x+h)-\tan x\right\}-7\left\{\sec(x+h)-\sec x\right\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

By further calculation, we get

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

Now, we get

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$= 2.1.\frac{1}{\cos x \cos x} - 7.1\left(\frac{\sin x}{\cos x \cos x}\right)$$

$$= 2\sec^2 x - 7\sec x \tan x$$

MISCELLANEOUS EXERCISE PAGE NO: 317

1. Find the derivative of the following functions from first principle:

(i) $-x$

(ii) $(-x)^{-1}$

(iii) $\sin(x+1)$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Solution:

(i) - x

Let f (x) = - x

Accordingly, f (x + h) = - (x + h)

Using first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{-(x+h)-(-x)}{h}$$

Now, we get

$$=\lim_{h\to 0}\frac{-x-h+x}{h}$$

$$=\lim_{h\to 0}\frac{-h}{h}$$
$$=\lim_{h\to 0}(-1)=-1$$

(ii) $(-x)^{-1}$

Let $f(x)=(-x)^{-1}=\frac{1}{-x}=\frac{-1}{x}$

Accordingly, $f(x+h)=\frac{-1}{(x+h)}$

Using first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right]$$

By further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right]$$
$$=\lim_{h\to 0}\frac{1}{x(x+h)}$$
$$=\frac{1}{x\cdot x}$$
$$= 1 / x^2$$

(iii) sin (x + 1)

Let f (x) = sin (x + 1)

Accordingly, f (x + h) = sin (x + h + 1)

By using first principle, we get

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

We get,

$$= \lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

We know that,

$$h\to 0 \Rightarrow \frac{h}{2}\to 0$$

$$= \cos\left(\frac{2x+0+2}{2}\right)\cdot 1$$
$$= \cos(x+1)$$

(iv) $\cos\left(x-\frac{\pi}{8}\right)$

Let $f(x) = \cos\left(x-\frac{\pi}{8}\right)$

Accordingly, $f(x+h) = \cos\left(x+h-\frac{\pi}{8}\right)$

By using first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

We get,

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

Further we get,

$$= \lim_{h\to 0} \frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

So,

$$= \lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$= \lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$\left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

Hence, we get

$$= -\sin\left(x-\frac{\pi}{8}\right)$$

Find the derivative of the following functions (it is to be understood that *a, b, c, d, p,* q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

2. (*x* + *a*)

Solution:

Let f (x) = x + a

Accordingly, f (x + h) = x + h + a

Using first principle, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So, now we get

$$= \lim_{h \to 0} \frac{x + h + a - x - a}{h}$$
$$= \lim_{h \to 0} \left(\frac{h}{h}\right)$$
$$= \lim_{h \to 0} (1)$$
$$= 1$$

3. (px + q) (r / x + s)
Solution:

Let $f(x) = (px+q)\left(\frac{r}{x}+s\right)$

Using Leibnitz product rule, we get

$$f'(x) = (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)'$$

We get,

$$= (px+q)\left(rx^{-1}+s\right)' + \left(\frac{r}{x}+s\right)(p)$$

By further calculation, we get

$$= (px+q)\left(-rx^{-2}\right) + \left(\frac{r}{x}+s\right)p$$
$$= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p$$

Now, we get

$$= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps$$
$$= ps - \frac{qr}{x^2}$$

4. (*ax* + *b*) (*cx* + *d*)^2
Solution:

Let $f(x) = (ax+b)(cx+d)^2$

By using Leibnitz product rule, we get

$$f'(x) = (ax+b)\frac{d}{dx}(cx+d)^2 + (cx+d)^2\frac{d}{dx}(ax+b)$$

We get,

$$= (ax+b)\frac{d}{dx}(c^2x^2 + 2cdx + d^2) + (cx+d)^2\frac{d}{dx}(ax+b)$$

By differentiating separately, we get

$$= (ax+b)\left[\frac{d}{dx}(c^2x^2) + \frac{d}{dx}(2cdx) + \frac{d}{dx}d^2\right] + (cx+d)^2\left[\frac{d}{dx}ax + \frac{d}{dx}b\right]$$

So,

$$= (ax+b)(2c^2x + 2cd) + (cx+d^2)a$$
$$= 2c(ax+b)(cx+d) + a(cx+d)^2$$

5. (ax + b) / (cx + d)
Solution:

Let $f(x) = \frac{ax+b}{cx+d}$

Using quotient rule, we get

$$f'(x)=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

Further we get

$$=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

So, now we get

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$

Hence,

$$=\frac{ad-bc}{(cx+d)^2}$$

6. (1 + 1 / x) / (1 – 1 / x)
Solution:

Let $f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}$, where $x\neq 0$

Using quotient rule, we get

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x\neq 0,\ 1$$

Further, we get

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},\ x\neq 0,\ 1$$

So,

$$= \frac{x-1-x-1}{(x-1)^2},\ x \neq 0,\ 1$$

$$= \frac{-2}{(x-1)^2},\ x \neq 0,\ 1$$

7. 1 / (ax^2 + bx + c)

Solution:

$$\text{Let } f(x) = \frac{1}{ax^2+bx+c}$$

Using quotient rule, we get

$$f'(x) = \frac{(ax^2+bx+c)\frac{d}{dx}(1) - \frac{d}{dx}(ax^2+bx+c)}{(ax^2+bx+c)^2}$$

By further calculation, we get

$$= \frac{(ax^2+bx+c)(0)-(2ax+b)}{(ax^2+bx+c)^2}$$

$$= \frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

8. (ax + b) / px^2 + qx + r

Solution:

$$\text{Let } f(x) = \frac{ax+b}{px^2+qx+r}$$

Using quotient rule, we get

$$f'(x) = \frac{(px^2+qx+r)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(px^2+qx+r)}{(px^2+qx+r)^2}$$

Further we get,

$$= \frac{(px^2+qx+r)(a)-(ax+b)(2px+q)}{(px^2+qx+r)^2}$$

Again by further calculation, we get

$$= \frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{(px^2+qx+r)^2}$$

$$= \frac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}$$

9. (px^2 + qx + r) / ax + b

Solution:

Let $f(x) = \frac{px^2+qx+r}{ax+b}$

Using quotient rule, we get

$$f'(x) = \frac{(ax+b)\frac{d}{dx}(px^2+qx+r)-(px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

By further calculation, we get

$$= \frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

So, we get

$$= \frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$= \frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

10. (a / x^4) – (b / x^2) + cox x

Solution:

Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

By differentiating we get,

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

On further calculation, we get

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

We know that,

$$\left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

So,

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x)$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

11. $4\sqrt{x}-2$

Solution:

$$\text{Let } f(x)=4\sqrt{x}-2$$

By differentiating we get,

$$f'(x)=\frac{d}{dx}\left(4\sqrt{x}-2\right)=\frac{d}{dx}\left(4\sqrt{x}\right)-\frac{d}{dx}(2)$$

Further, we get

$$=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0$$

$$= 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= \left(2x^{-\frac{1}{2}}\right)$$

$$= \frac{2}{\sqrt{x}}$$

12. $(ax + b)^n$
Solution:

Let $f(x) = (ax+b)^n$

Accordingly, $f(x+h) = \{a(x+h)+b\}^n = (ax+ah+b)^n$

Using first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

Further we get,

$$= \lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$
$$= (ax+b)^n\lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

By using binomial theorem, we get

$$= (ax+b)^n\lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\underline{|2}}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

Now, we get

$$= (ax+b)^n \lim_{h\to 0} \frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{\underline{|2}(ax+b)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$

So, we get

$$= (ax+b)^n \lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{\underline{|2}(ax+b)^2}+\ldots\right]$$

On further calculation, we get

$$= (ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$= na\frac{(ax+b)^n}{(ax+b)}$$

$$= na(ax+b)^{n-1}$$

13. $(ax + b)^n (cx + d)^m$

Solution:

Let $f(x) = (ax+b)^n (cx+d)^m$

By using Leibnitz product rule, we get

$$f'(x) = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n$$

let $f_1(x) = (cx+d)^m$

Then, $f_1(x+h) = (cx+ch+d)^m$

$$f_1'(x) = \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$= \lim_{h\to 0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}$$

By taking $(cx + d)^m$ as common, we get

$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m - 1\right]$$

On further calculation, we get

$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+\ldots\right)-1\right]$$

Now, we get

$$= (cx+d)^m \lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+\ldots(\text{Terms containing higher degrees of } h)\right]$$

We know that,

$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1}$$

Similarly, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}$

$$= (cx+d)^m \lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+\ldots\right]$$

Now, we get

$$= (cx+d)^m\left[\frac{mc}{cx+d}+0\right]$$

$$= \frac{mc(cx+d)^m}{(cx+d)}$$

$$= mc(cx+d)^{m-1}$$

Hence, we get

$$f'(x) = (ax+b)^n \left\{mc(cx+d)^{m-1}\right\} + (cx+d)^m \left\{na(ax+b)^{n-1}\right\}$$
$$= (ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b) + na(cx+d)\right]$$

14. sin (*x* + *a*)
Solution:

Let $f(x) = \sin(x+a)$

$f(x+h) = \sin(x+h+a)$

By using first principle, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}$$

On further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

So, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

By taking limits, we get

$$= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}$$

Hence, we get

$$= \cos\left(\frac{2x+2a}{2}\right)\times 1$$
$$= \cos(x+a)$$

15. cosec *x* cot *x*

Solution:

Let $f(x) = \operatorname{cosec} x \cot x$

By using Leibnitz product rule, we get

$$f'(x) = \operatorname{cosec} x(\cot x)' + \cot x(\operatorname{cosec} x)' \quad \text{...(1)}$$

Let $f_1(x) = \cot x$.

Accordingly, $f_1(x+h) = \cot(x+h)$

By using first principle, we get

$$f_1'(x) = \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

On further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right)$$

Now, we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

We get

$$=\frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right]$$
$$=\frac{-1}{\sin x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right)$$

So, we get

$$= \frac{-1}{\sin x} . 1 . \left(\frac{1}{\sin(x+0)} \right)$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x$$

Hence, we get

$$(\cot x)' = -\operatorname{cosec}^2 x \qquad ...(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By using first principle, we get

$$f_2'(x) = \lim_{h \to 0} \frac{f_2(x+h) - f_2(x)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h} \left[\operatorname{cosec}(x+h) - \operatorname{cosec} x \right]$$

By calculating further, we get

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$
$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right]$$

So,

$$= \frac{1}{\sin x} . \lim_{h \to 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2} \right) \sin\left(\frac{x-x-h}{2} \right)}{\sin(x+h)} \right]$$
$$= \frac{1}{\sin x} . \lim_{h \to 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2} \right) \sin\left(\frac{-h}{2} \right)}{\sin(x+h)} \right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \right]$$

We get,

$$= \frac{-1}{\sin x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$
$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$
$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$
$$= -\operatorname{cosec} x \cdot \cot x$$

Hence,

$$(\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x \quad \ldots(3)$$

From equations (1) (2) and (3) we get,

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$
$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

16. $\frac{\cos x}{1+\sin x}$

Solution:

Let $f(x) = \frac{\cos x}{1+\sin x}$

By using quotient rule, we get

$$f'(x) = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$

We get,

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

Now, we get

$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{-1}{(1+\sin x)}$$

17. $\frac{\sin x + \cos x}{\sin x - \cos x}$

Solution:

Let $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(\sin x - \cos x)\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

On further calculation, we get

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

By expanding the terms, we get

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

We get

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

18. $\frac{\sec x-1}{\sec x+1}$

Solution:

Let $f(x)=\frac{\sec x-1}{\sec x+1}$

Now, this can be written as

$$f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

On multiplying we get

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

This can be written as

$$= \frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}$$

On taking L.C.M we get

$$= \frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

On further calculation, we get

$$= \frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$= \frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$= \frac{2\sec x\tan x}{(\sec x+1)^2}$$

19. $\sin^n x$

Solution:

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

We know that,

$$\frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx}\sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

So, $\frac{dy}{dx} = \frac{d}{dx}(\sin x\sin x)$

By Leibnitz product rule, we get

$$= (\sin x)'\sin x + \sin x(\sin x)'$$

$$= \cos x\sin x + \sin x\cos x$$

$$= 2\sin x\cos x \qquad ...(1)$$

For $n = 3$, $y = \sin^3 x$.

So, $\frac{dy}{dx} = \frac{d}{dx}(\sin x\sin^2 x)$

By Leibnitz product rule, we get

$= (\sin x)' \sin^2 x + \sin x (\sin^2 x)'$

From equation (1) we get

$= \cos x \sin^2 x + \sin x (2 \sin x \cos x)$

$= \cos x \sin^2 x + 2 \sin^2 x \cos x$

$= 3 \sin^2 x \cos x$

We state that, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

For n = k, let our assertion be true

i.e., $\frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x$ ...(2)

Now, consider

$\frac{d}{dx}(\sin^{k+1} x) = \frac{d}{dx}(\sin x \sin^k x)$

By using Leibnitz product rule, we get

$= (\sin x)' \sin^k x + \sin x (\sin^k x)'$

From equation (2) we get

$= \cos x \sin^k x + \sin x \left(k \sin^{(k-1)} x \cos x\right)$

$= \cos x \sin^k x + k \sin^k x \cos x$

$= (k+1) \sin^k x \cos x$

Hence, our assertion is true for n = k + 1

Therefore, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

20. $\frac{a + b \sin x}{c + d \cos x}$

Solution:

Let $f(x) = \frac{a + b \sin x}{c + d \cos x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(c + d\cos x)\frac{d}{dx}(a + b \sin x) - (a + b \sin x)\frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

On multiplying we get

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

Now, taking bd as common we get

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

21. $\frac{\sin(x+a)}{\cos x}$

Solution:

Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By differentiating and using quotient rule, we get

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}\left[\sin(x+a)\right]-\sin(x+a)(-\sin x)}{\cos^2 x} \quad \text{... (i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$

On further calculation, we get

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

Now, taking limits we get

$$= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

We know that,

$$\left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$= \left(\cos\frac{2x+2a}{2}\right)\times 1$$

$$= \cos(x+a) \qquad \ldots \text{(ii)}$$

From equation (i) and (ii) we get

$$f'(x)=\frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$=\frac{\cos(x+a-x)}{\cos^2 x}$$

$$=\frac{\cos a}{\cos^2 x}$$

22. x^4 (5 sin x – 3 cos x)

Solution:

Let $f(x)=x^4(5\sin x-3\cos x)$

By differentiating and using product rule, we get

$$f'(x)=x^4\frac{d}{dx}(5\sin x-3\cos x)+(5\sin x-3\cos x)\frac{d}{dx}(x^4)$$

On further calculation, we get

$$= x^4\left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

So, we get

$$= x^4\left[5\cos x - 3(-\sin x)\right] + (5\sin x - 3\cos x)(4x^3)$$

By taking x^3 as common, we get

$$= x^3\left[5x\cos x + 3x\sin x + 20\sin x - 12\cos x\right]$$

23. $(x^2 + 1)\cos x$

Solution:

Let $f(x) = (x^2+1)\cos x$

By differentiating and using product rule, we get

$$f'(x) = (x^2+1)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2+1)$$

On further calcualtion, we get

$$= (x^2+1)(-\sin x) + \cos x(2x)$$

By multiplying we get

$$= -x^2\sin x - \sin x + 2x\cos x$$

24. $(ax^2 + \sin x)(p + q\cos x)$

Solution:

Let $f(x) = (ax^2 + \sin x)(p + q\cos x)$

By differentiating and using product rule, we get

$$f'(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$

On further calculation, we get

$$= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x)$$

$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

25. $(x + \cos x)(x - \tan x)$

Solution:

Let $f(x)=(x+\cos x)(x-\tan x)$

By differentiating and using product rule, we get

$$f'(x)=(x+\cos x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\cos x)$$

$$=(x+\cos x)\left[\frac{d}{dx}(x)-\frac{d}{dx}(\tan x)\right]+(x-\tan x)(1-\sin x)$$

Now, we get

$$=(x+\cos x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)(1-\sin x) \qquad \text{... (i)}$$

Let $g(x)=\tan x$. Accordingly, $g(x+h)=\tan(x+h)$

By using first principle, we get

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

On further calculation, we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

Now, we get

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right]$$

So, we get

$$=\frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right)$$

We get

$$= \frac{1}{\cos x} . 1 . \frac{1}{\cos(x+0)}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x \qquad \dots \text{(ii)}$$

Hence, from equation (i) and (ii) we get

$$f'(x) = (x+\cos x)(1-\sec^2 x) + (x-\tan x)(1-\sin x)$$

$$= (x+\cos x)(-\tan^2 x) + (x-\tan x)(1-\sin x)$$

$$= -\tan^2 x(x+\cos x) + (x-\tan x)(1-\sin x)$$

26. $\frac{4x+5\sin x}{3x+7\cos x}$

Solution:

Let $f(x) = \frac{4x+5\sin x}{3x+7\cos x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x) - (4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

On further calculation, we get

$$= \frac{(3x+7\cos x)\left[4\frac{d}{dx}(x) + 5\frac{d}{dx}(\sin x)\right] - (4x+5\sin x)\left[3\frac{d}{dx}x + 7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$= \frac{(3x+7\cos x)(4+5\cos x) - (4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

On multiplying we get

$$= \frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

We get

$$= \frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2}$$

$$= \frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

27. $\dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

Solution:

Let $f(x) = \dfrac{x^2 \cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

By differentiating and using quotient rule, we get

$$f'(x) = \cos\frac{\pi}{4}.\left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

By further calculation, we get

$$= \cos\frac{\pi}{4}.\left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x}\right]$$

By taking x as common, we get

$$= \frac{x \cos\frac{\pi}{4}[2\sin x - x\cos x]}{\sin^2 x}$$

28. $\dfrac{x}{1+\tan x}$

Solution:

Let $f(x) = \dfrac{x}{1+\tan x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x) = \frac{(1+\tan x) - x \cdot \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \quad \text{... (i)}$$

Let $g(x) = 1+\tan x$. Accordingly, $g(x+h) = 1+\tan(x+h)$.

Using first principle, we get

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h \to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

By taking L.C.M we get

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

We get

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h \to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

So, we get

$$= \left(\lim_{h \to 0}\frac{\sin h}{h}\right).\left(\lim_{h \to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(1+\tan x) = \sec^2 x \quad \text{... (ii)}$$

From equation (i) and (ii) we get

$$f'(x) = \frac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$$

29. (x + sec x) (x – tan x)

Solution:

Let $f(x) = (x+\sec x)(x-\tan x)$

By differentiating and using product rule, we get

$$f'(x) = (x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$

So, we get

$$= (x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$

$$= (x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \qquad \text{... (i)}$$

Let $f_1(x)=\tan x,\ f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$
$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

By further calculation, we get

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

Now, by taking L.C.M we get

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$
$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$
$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

Hence we get

$$\frac{d}{dx}\tan x=\sec^2 x \qquad \text{... (ii)}$$

Now, take

$$f_2'(x)=\lim_{h\to 0}\left(\frac{f_2(x+h)-f_2(x)}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{\sec(x+h) - \sec x}{h} \right)$$

This can be written as

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

By taking L.C.M we get

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x} \right]$$

On further calculation, we get

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right]$$

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right]$$

We get

$$= \frac{1}{\cos x} \cdot \lim_{h \to 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\cos(x+h)} \right]$$

By taking limits, we get

$$= \sec x \cdot \frac{\left\{ \lim_{h \to 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\lim_{h \to 0} \cos(x+h)}$$

We get

$$= \sec x.\frac{\sin x.1}{\cos x}$$

$$\frac{d}{dx}\sec x = \sec x \tan x \quad \text{.. (iii)}$$

From equation (i) (ii) and (iii) we get

$$f'(x) = (x+\sec x)(1-\sec^2 x)+(x-\tan x)(1+\sec x\tan x)$$

30. $\frac{x}{\sin^n x}$

Solution:

Let $f(x) = \frac{x}{\sin^n x}$

By differentiating and using quotient rule, we get

$$f'(x) = \frac{\sin^n x\frac{d}{dx}x - x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

Easily, it can be shown that,

$$\frac{d}{dx}\sin^n x = n\sin^{n-1} x\cos x$$

Hence,

$$f'(x) = \frac{\sin^n x\frac{d}{dx}x - x\frac{d}{dx}\sin^n x}{\sin^{2n} x}$$

By further calculation, we get

$$= \frac{\sin^n x.1 - x\left(n\sin^{n-1} x\cos x\right)}{\sin^{2n} x}$$

By taking common terms, we get

$$= \frac{\sin^{n-1} x(\sin x - nx\cos x)}{\sin^{2n} x}$$

Hence, we get

$$= \frac{\sin x - nx\cos x}{\sin^{n+1} x}$$