Faculty of Information Technology, Monash University

FIT2004: Algorithms and Data Structures

Week 11: Network Flow

These slides are prepared by M. A. Cheema and are based on the material developed by Arun Konagurthu and Lloyd Allison.

Announcements

- SETU Feedback and nominations for Teaching Excellence Awards
  - Links on Moodle (on the right)
  - Closes 21st June 2020

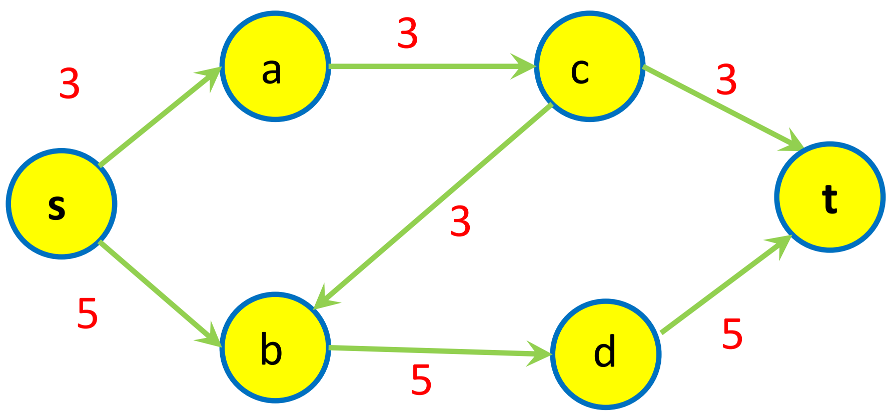
- Real-time anonymous feedback: https://docs.google.com/forms/d/e/1FAIpQLSdAlUwUNPJtccGebDF-fg41oUCWrt-of5A2mg5I7VtEK6vN3A/viewform

Outline

1. Maximum Flow Problem
2. Ford-Fulkerson Algorithm
3. Min-cut Max-flow Theorem

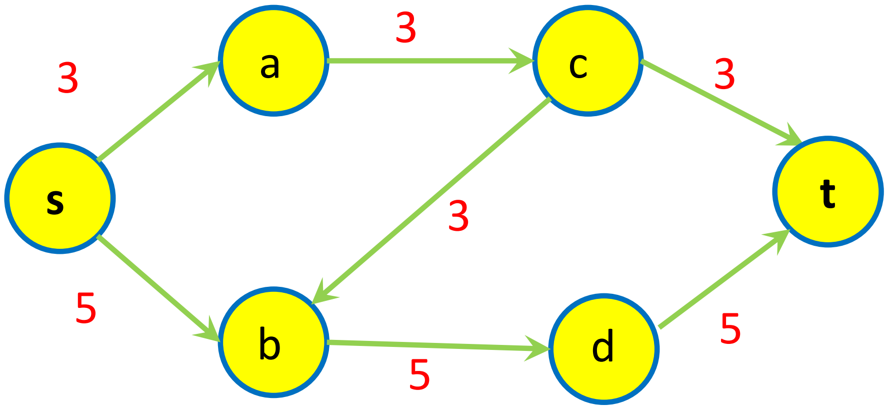
Flow Networks

- A flow network is a connected directed graph where
  - there is a single source vertex and a single sink/destination vertex;
  - each edge has a given (non-negative) capacity (usually integers)
    - giving the maximum amount/rate of flow that the edge can carry;
- Flow networks model many real-world problems
  - Water flowing through an assembly of pipes.
  - Electric current flowing through electrical circuits.
  - Information flowing through communication networks
  - Can be applied to many scenarios (unrelated to physical flows).



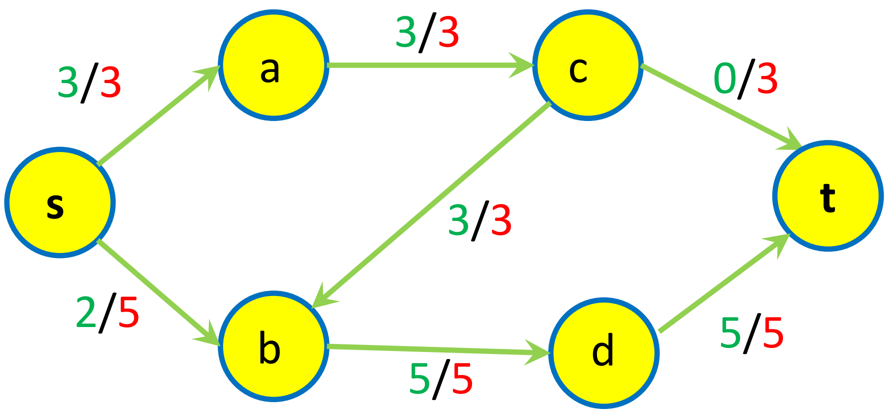
Some basic notations

- Set of all incoming edges to a vertex v: denoted as $E_{in}(v)$
  - $E_{in}(b) = s \rightarrow b, c \rightarrow b$
  - $E_{in}(a) = ?$
- Set of all outgoing edges from a vertex v: denoted as $E_{out}(v)$
  - $E_{out}(b) = b \rightarrow d$
  - $E_{out}(a) = ?$
- Source Vertex: denoted as s (has no incoming edges)
- Sink/target vertex: denoted as t (has no outgoing edges)



Flow

- Flow is an **assignment** of how much material is flowing through each edge in the flow network given its stated edge capacity.



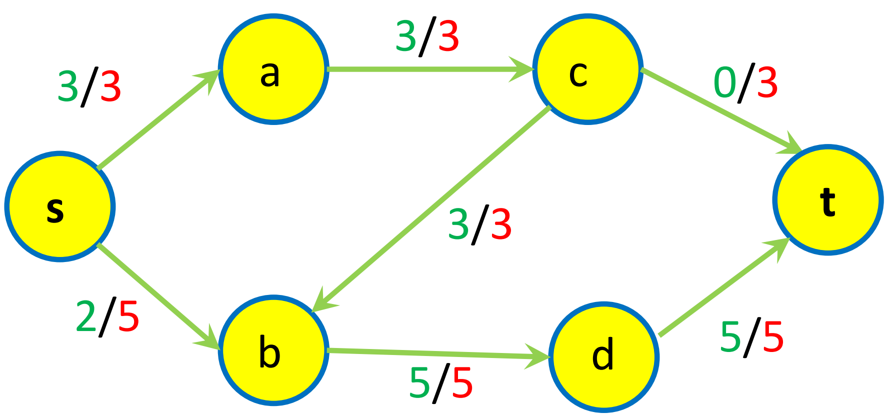
Green numbers indicate flow and red indicate capacity. Flow is not shown if 0

Flow

- All vertices (except source and sink) conserve their flow. That is
  - The total amount flowing into any vertex (through incoming edges)

    IS EQUAL TO

    the total amount flowing out of that vertex (through outgoing edges).
  - This key property is called flow conservation.



Properties of a Flow Network

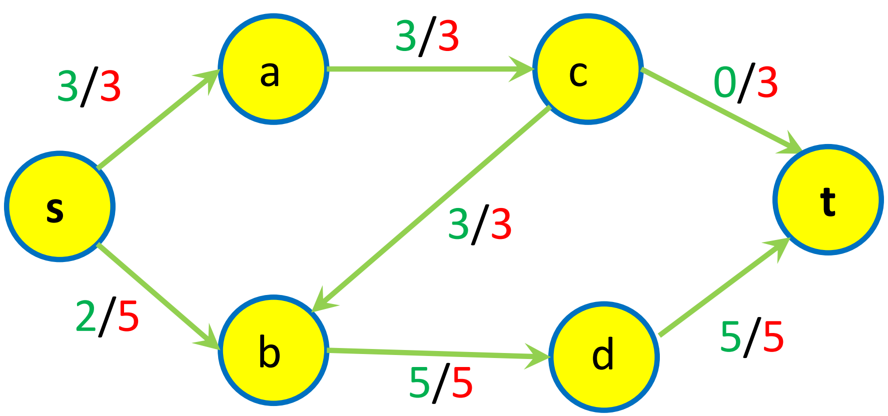
A flow network must satisfy the following two properties.

Property 1: Capacity Constraint

- For each edge e, its flow, denoted as f(e), is bounded by the capacity of its edge, i.e., $0 \leq f(e) \leq c(e)$ where c(e) is the capacity of the edge

Property 2: Flow Conservation

- For any vertex v (except source and sink), the total flow coming into the vertex must be equal to the total flow going out from this vertex – formally

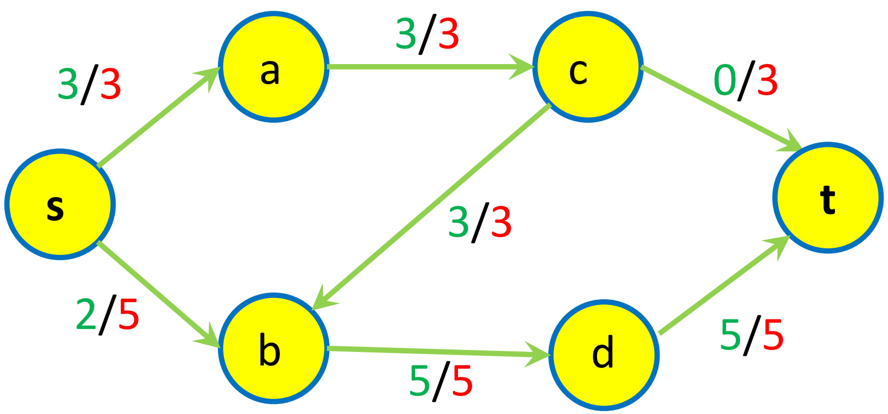


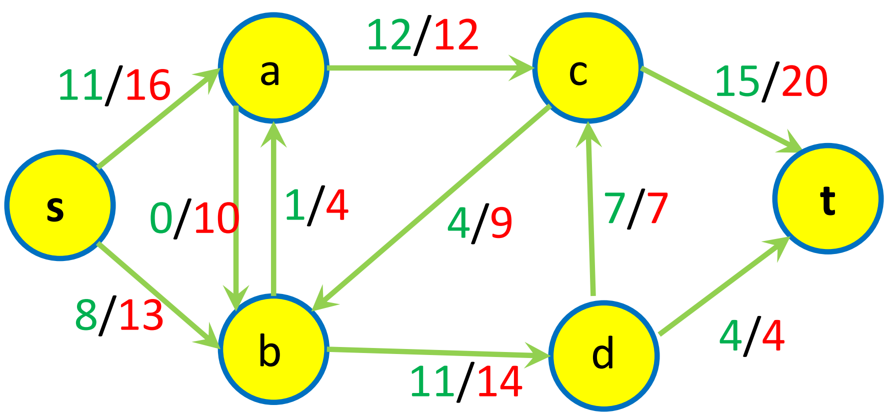
$$\sum_{\forall e_{in} \in E_{in}(v)} f(e_{in}) = \sum_{\forall e_{out} \in E_{out}(v)} f(e_{out}).$$

Maximum-flow Problem

Value of a flow in a network:

- Given that flow network satisfies the capacity constraint and flow conservation properties, flow of a network is the total flow out of the source vertex. Equivalently, this is the same as the total flow into sink vertex.
  - What is the flow value in the flow network at bottom left?
  - What is the flow value in the flow network at bottom right?





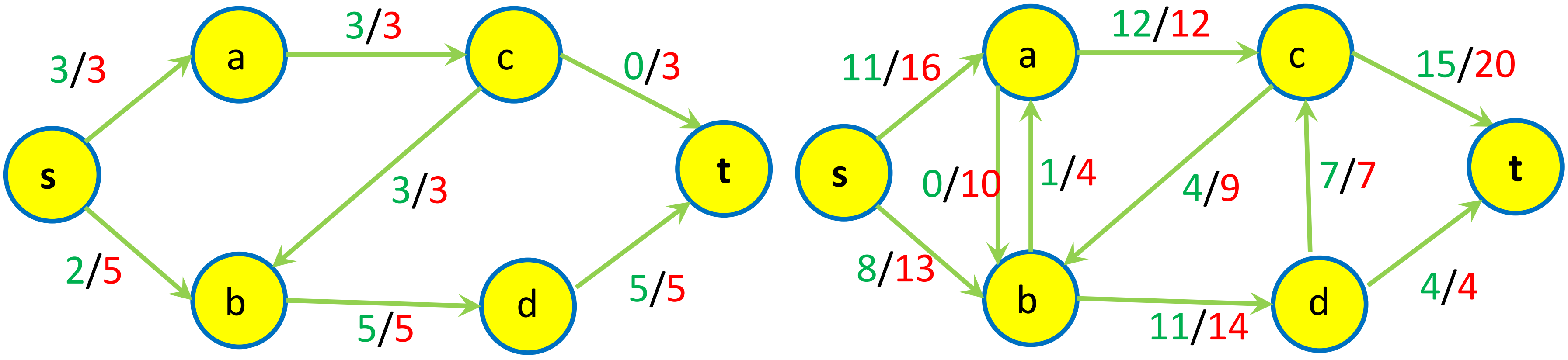
Maximum-flow Problem

Value of a flow in a network:

- Given that flow network satisfies the capacity constraint and flow conservation properties, flow of a network is the total flow out of the source vertex.

Maximum-flow problem

- Given a flow network, determine the maximum value of the flow that can be sent from source s to sink t without violating the flow network properties.



Outline

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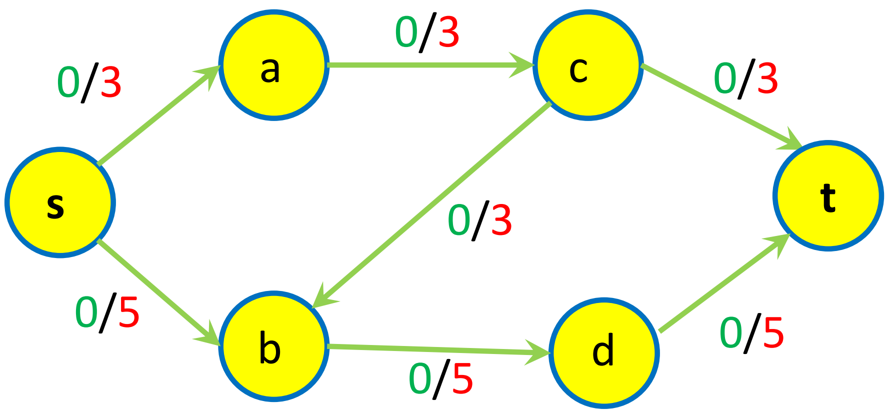
2. Ford-Fulkerson Algorithm
3. Min-cut Max-flow Theorem

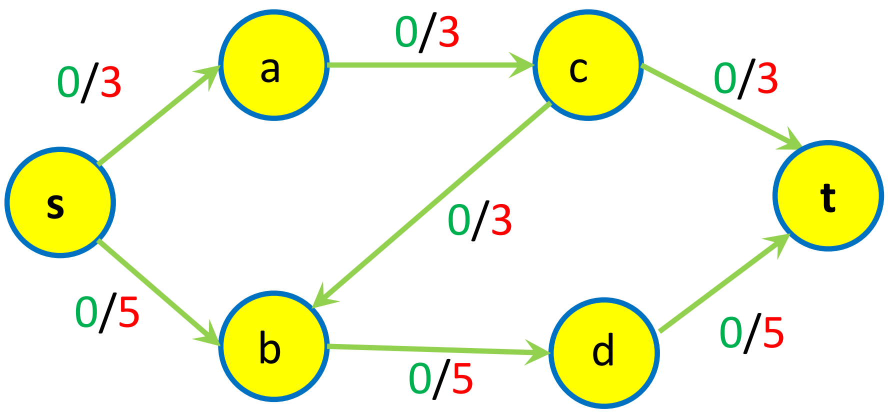
Ford Fulkerson Intuition



How can we increase the flow in the graph above?

Quiz time!
https://flux.qa - RFIBMB

Ford Fulkerson Intuition

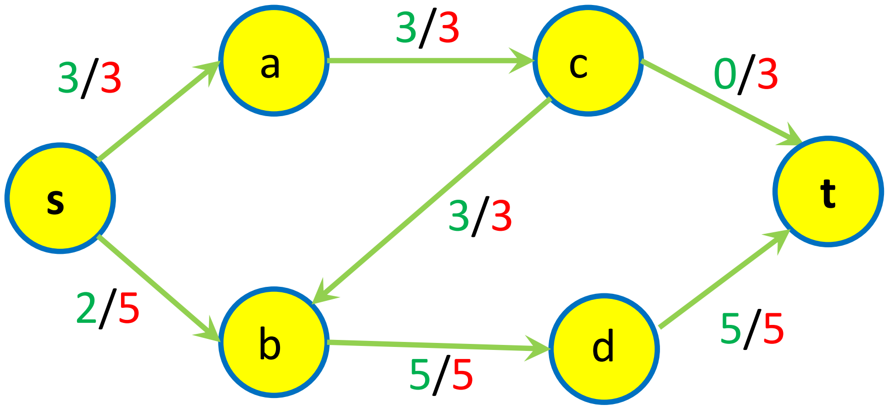


How can we increase the flow in the graph above?

1. Choose a path from source to sink
2. Increase flow along it

Seems easy enough!

Ford Fulkerson Intuition



Can we increase the flow in the above network?

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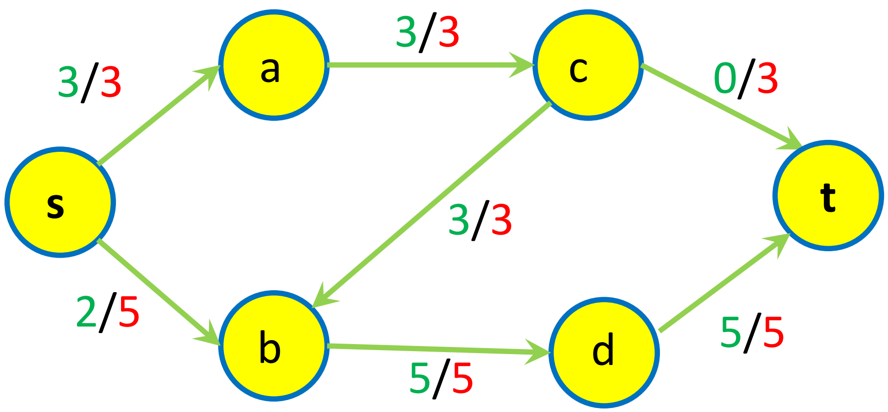
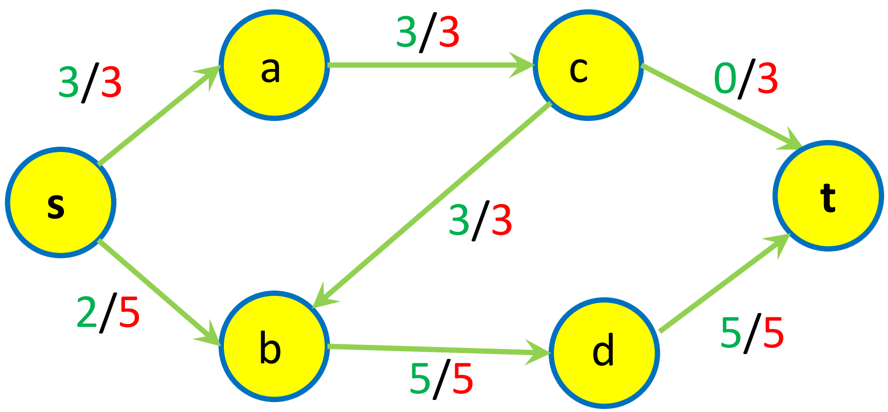
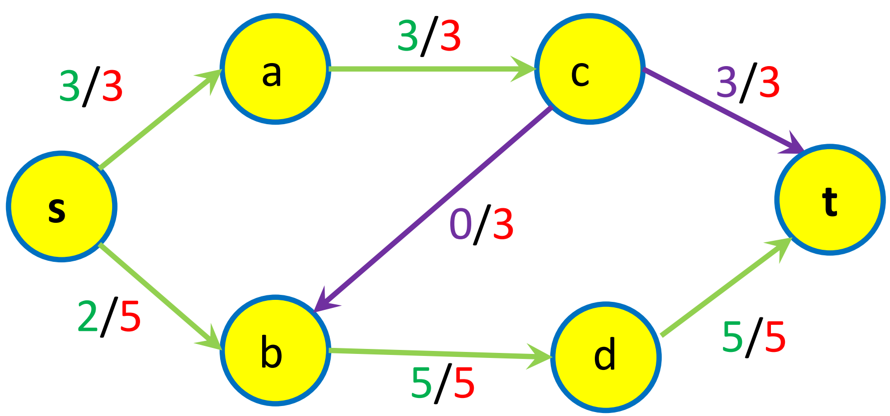


Can we increase the flow in the above network?

We can!
But there is no path from source to sink with spare capacity…

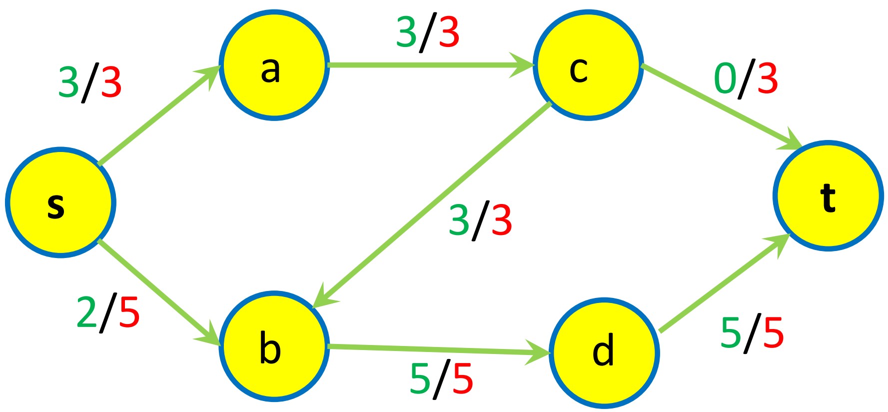
Ford Fulkerson Intuition

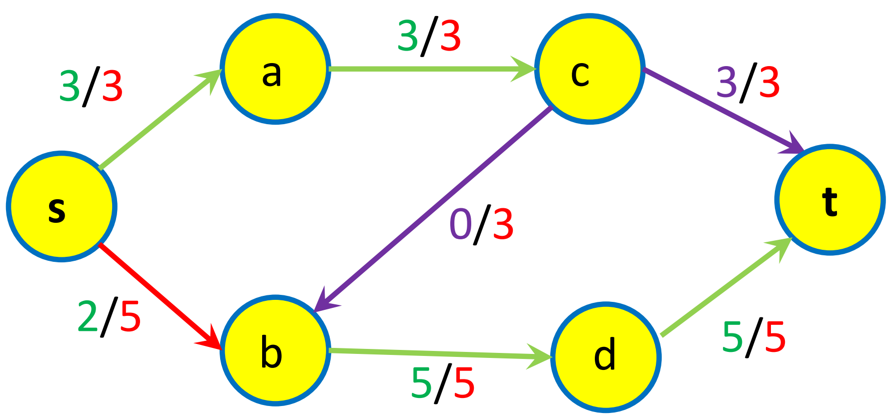




- There is not path from s to t with spare capacity
- Instead, we “redirect” the 3 units on the edge c->b
- They are now going to t

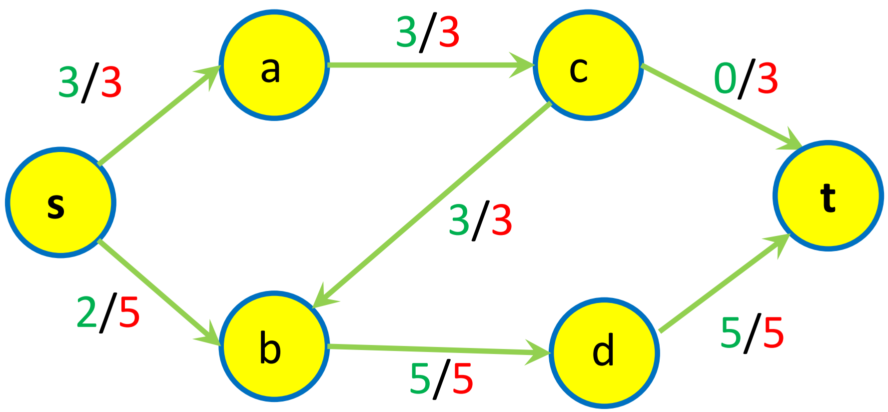
Ford Fulkerson Intuition

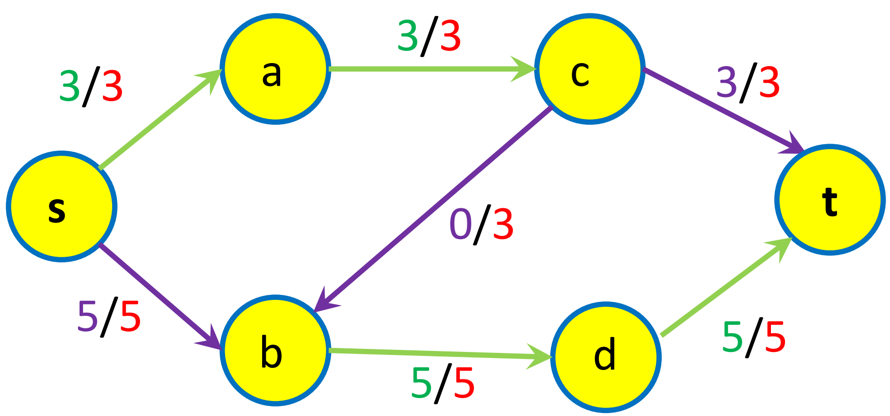




- There is not path from s to t with spare capacity
- Instead, we "redirect" the 3 units on the edge c->b
- They are now going to t
- In the second diagram, the flow through b is not conserved
- We can send 3 more units along s->b to both increase total flow and conserve flow through b

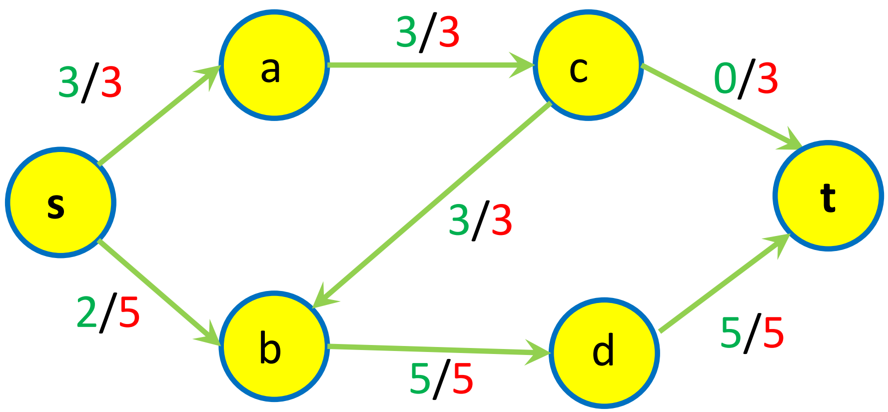
Ford Fulkerson Intuition

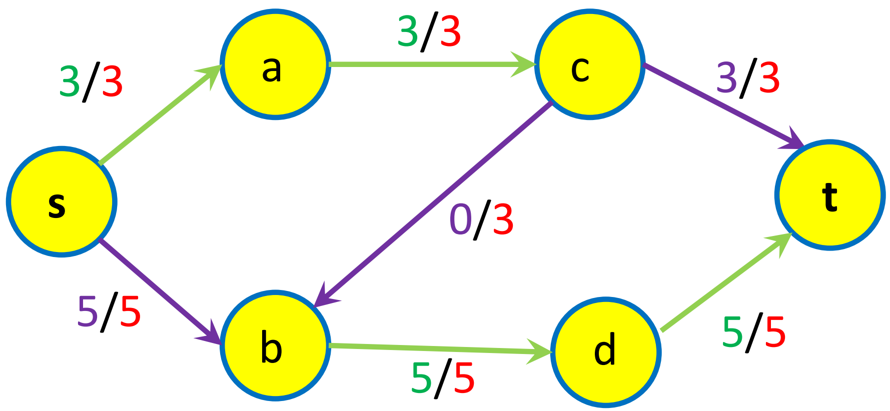




- There is not path from s to t with spare capacity
- Instead, we “redirect” the 3 units on the edge c->b
- They are now going to t
- This means that we need to send 3 more units into b (because of flow conservation)
- In this case, the extra flow comes from s

Ford Fulkerson Intuition

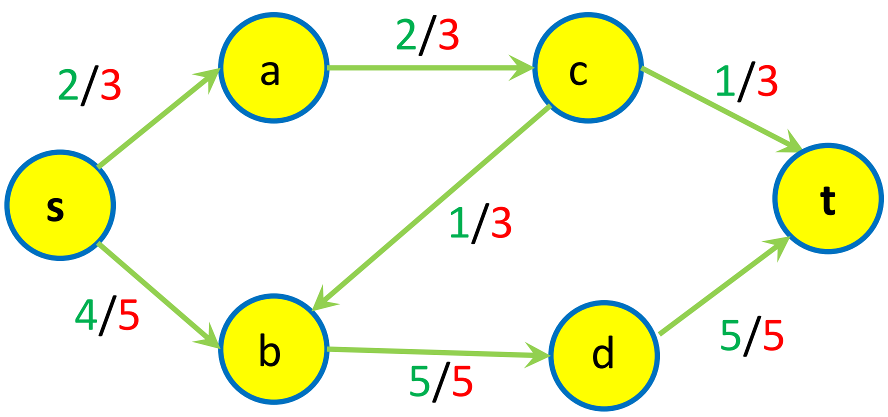




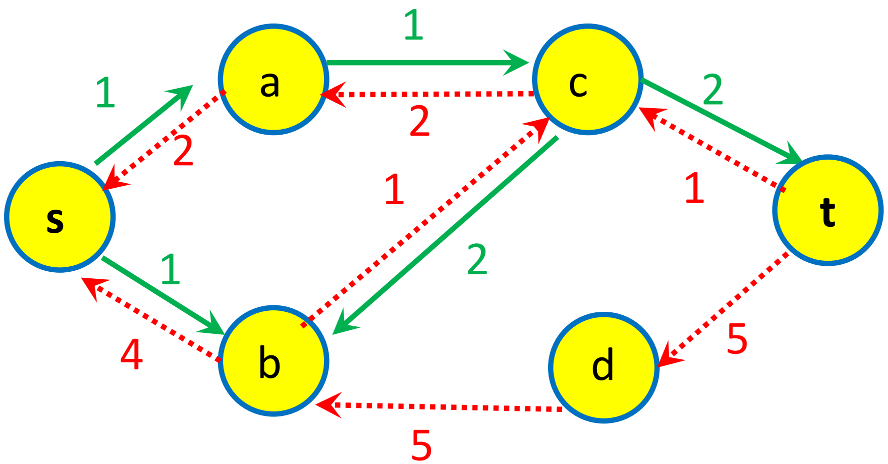
- What actually happened here?
- We increased the total flow by 3:
  - We sent 3 units along s->b, c->t
  - We "removed" 3 units of flow from c->b
- Our path was s->b->c->t, but we had a backwards edge…

Residual Network

- Residual network has the same vertices as the original network.
- For every directed edge u→v in flow network, we add two edges in the residual network:
  - Forward edge/Residual edge: An edge in the same direction as u→v with the residual/remaining capacity
  - Backward edge/Reversible flow edge: An edge in the direction opposite to u→v (i.e., v→u) with weight equal to the current flow of u→v in the flow network



Flow Network: Green numbers indicate flow and red indicate capacity. Flow is not shown if 0

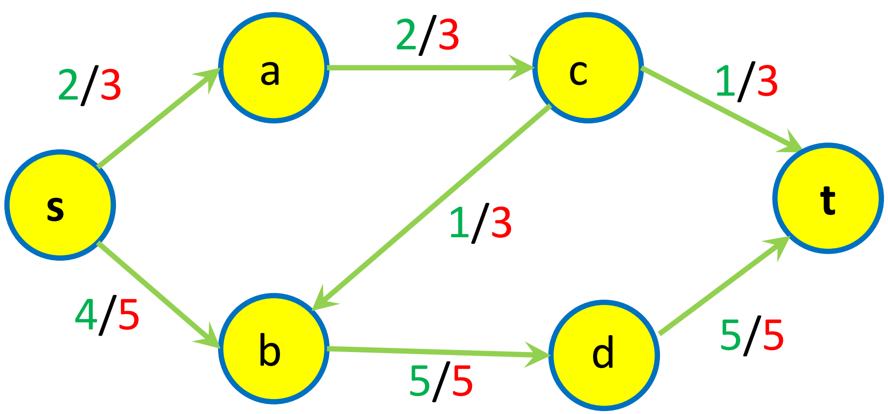


Residual Network: Where possible red edges indicate reversible flow and green indicate residual capacity

Augmenting Path in Residual Network

Augmenting path is any simple path (a path without repeating vertices) from source s to target t.

- E.g., s → b → c → t (shown in purple edges)
- **Residual capacity** of a path is the minimum edge weight on this path (e.g., 4 in the example)
- For each edge along this path, we can push additional flow equal to the "residual capacity of the path" in the flow network, e.g., 1 along each edge on s → b → c → t



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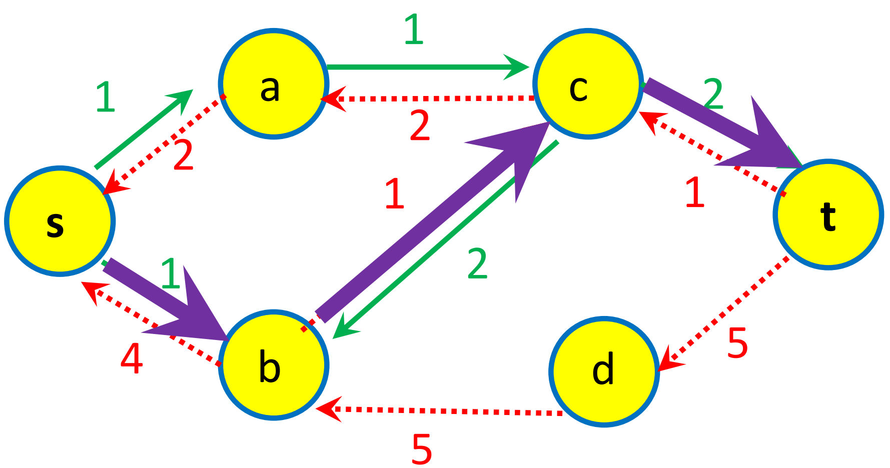


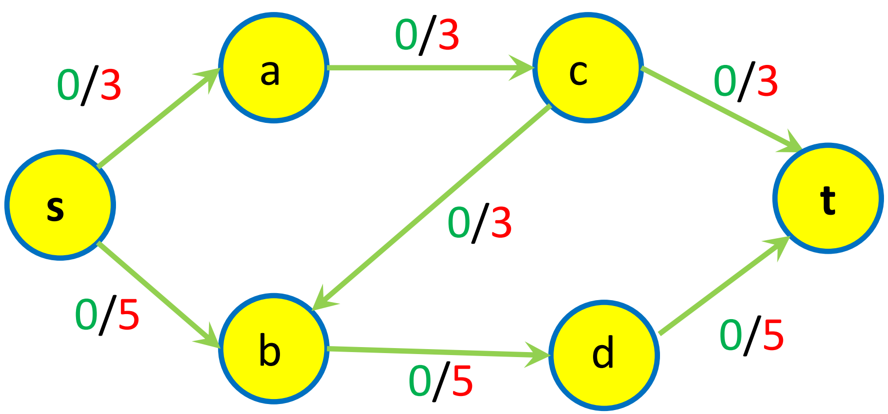
Residual Network: Where possible red edges indicate reversible flow and green indicate residual capacity

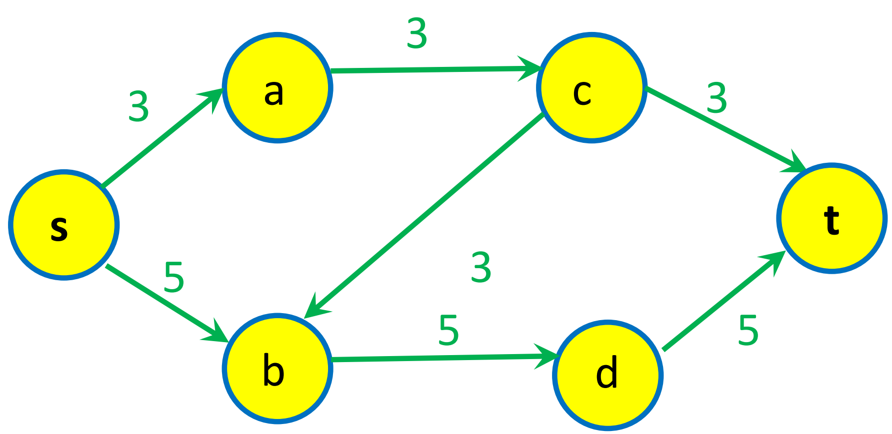
Ford-Fulkerson Method – Worked example (by hand in lecture)

Algorithm 75 The Ford-Fulkerson method

function MAX_FLOW($G = (V, E), s, t$)
  Set initial flow f to 0 on all edges
  while there exists an augmenting path p in the residual network G_f **do**
    Augment the flow f along the augmenting path p
  return f

Total flow: 0





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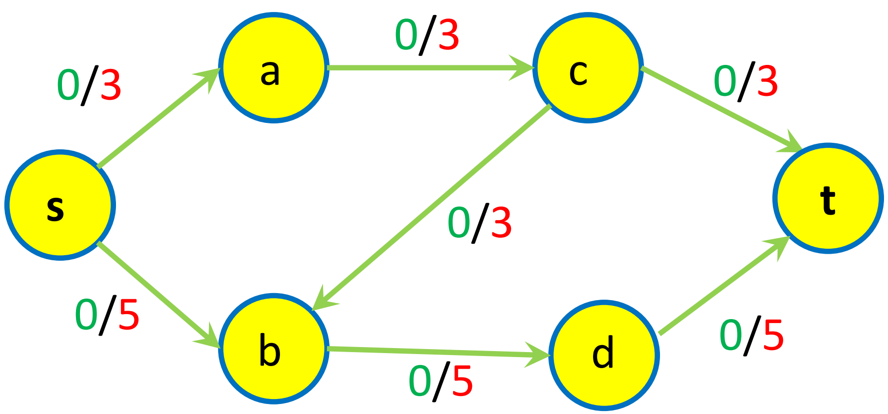

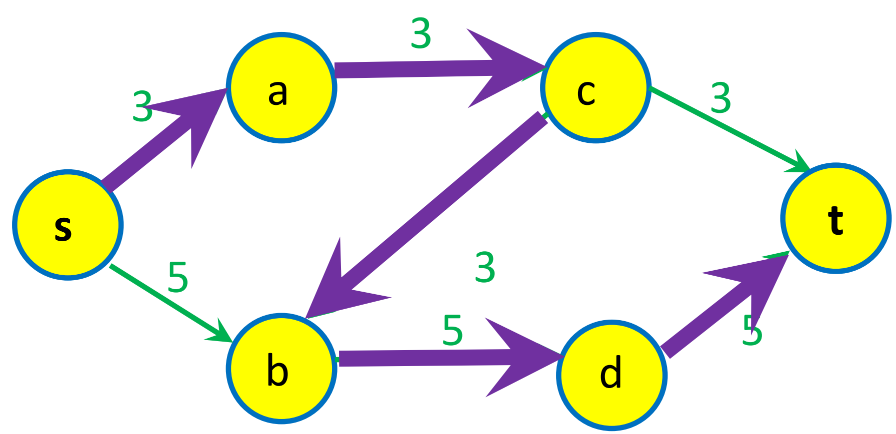

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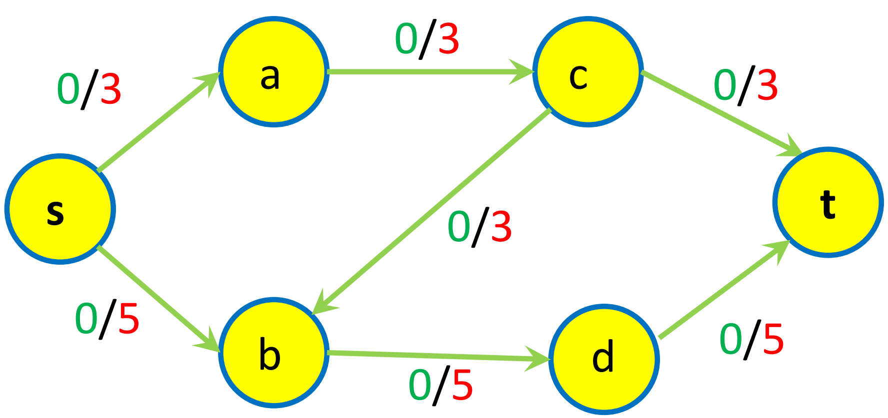

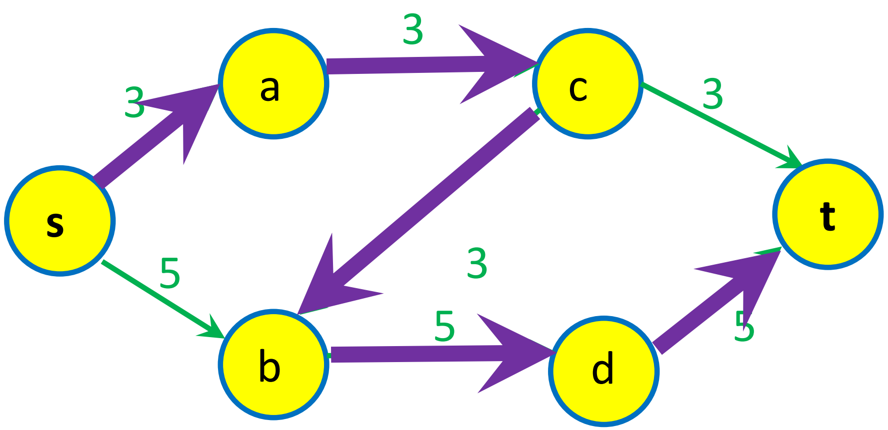

Path found: capacity = 3

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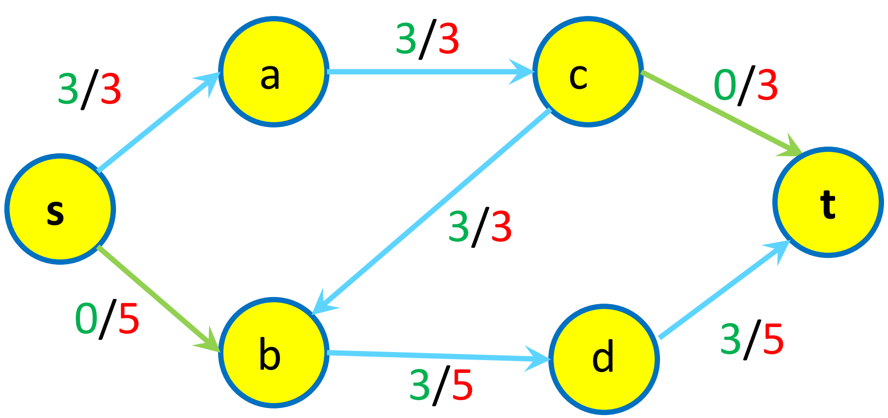

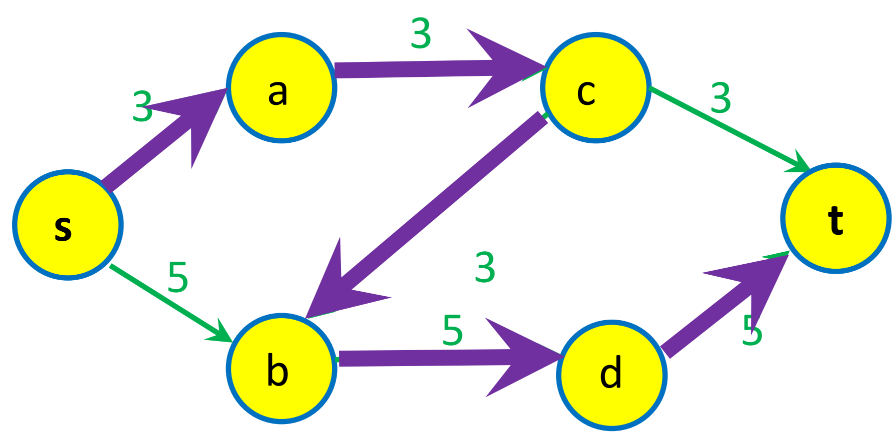

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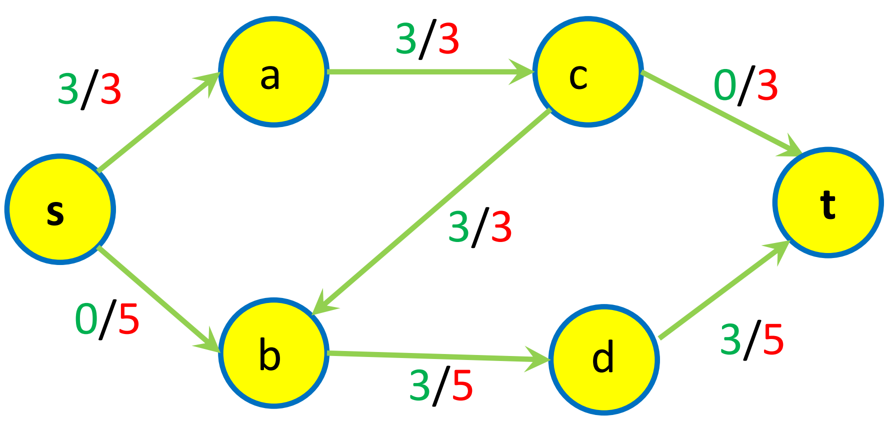

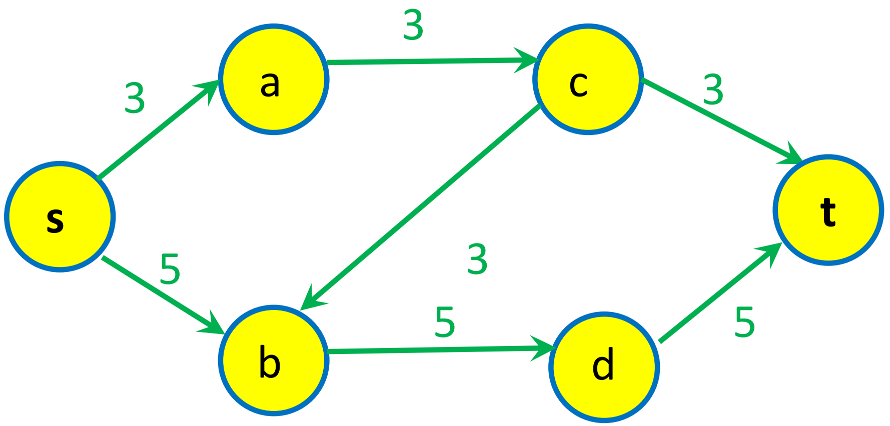

Update residual to match new flows

Ford-Fulkerson Method – Worked example (by hand in lecture)

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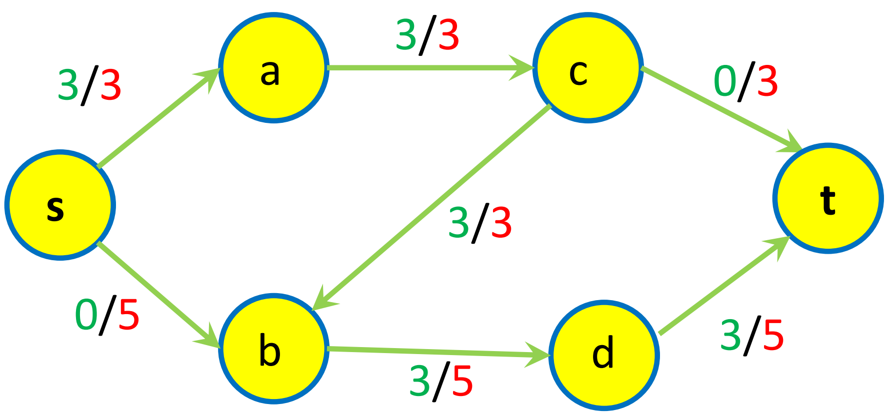

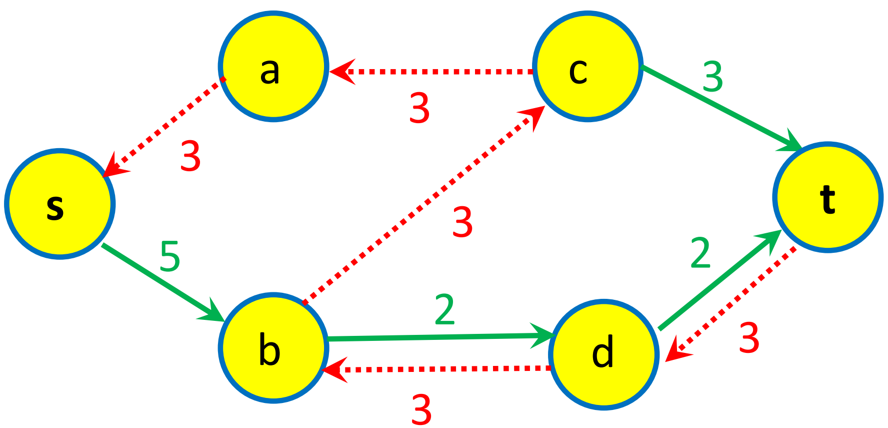

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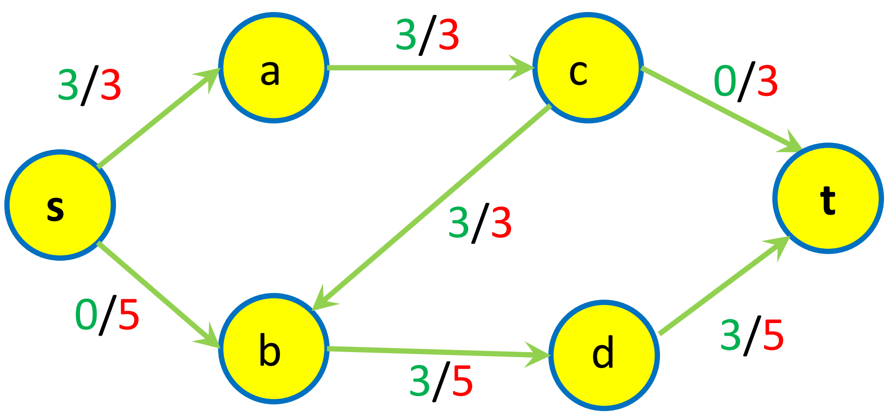

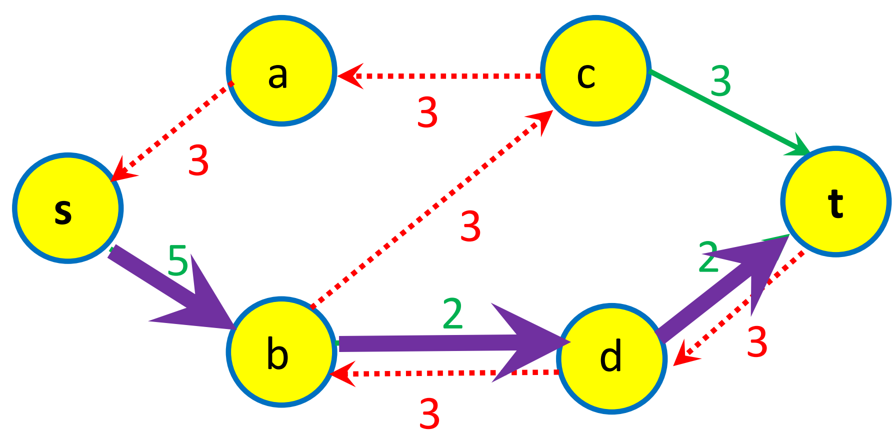

Path found: capacity = 2

Ford-Fulkerson Method – Worked example (by hand in lecture)

Algorithm 75 The Ford-Fulkerson method

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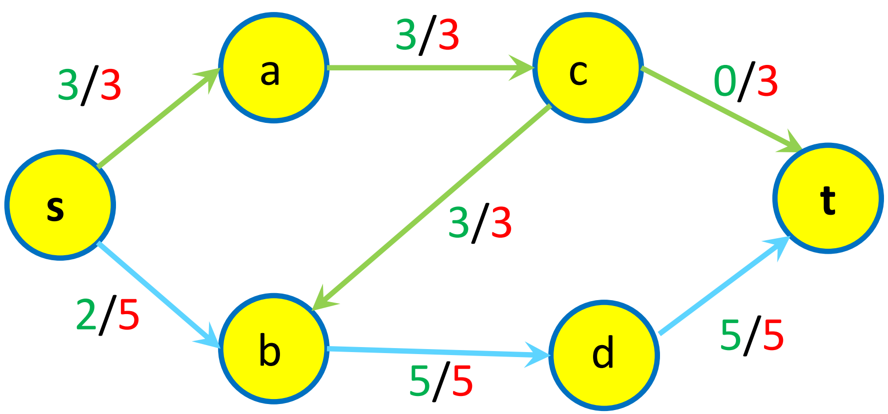
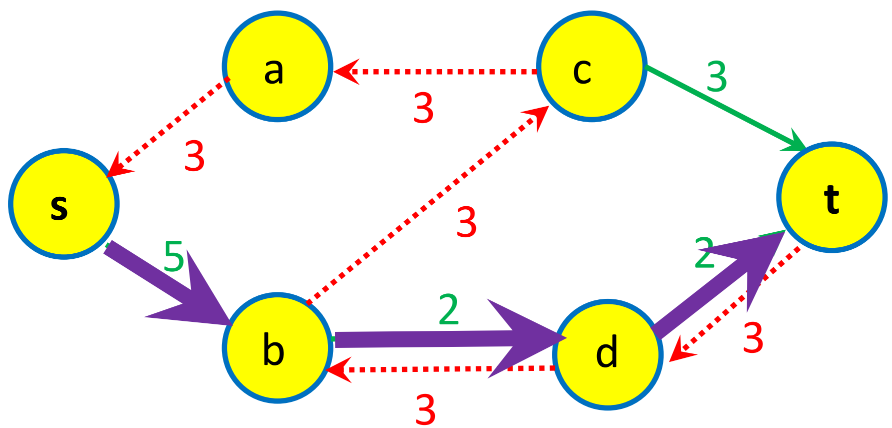
Total flow: 5





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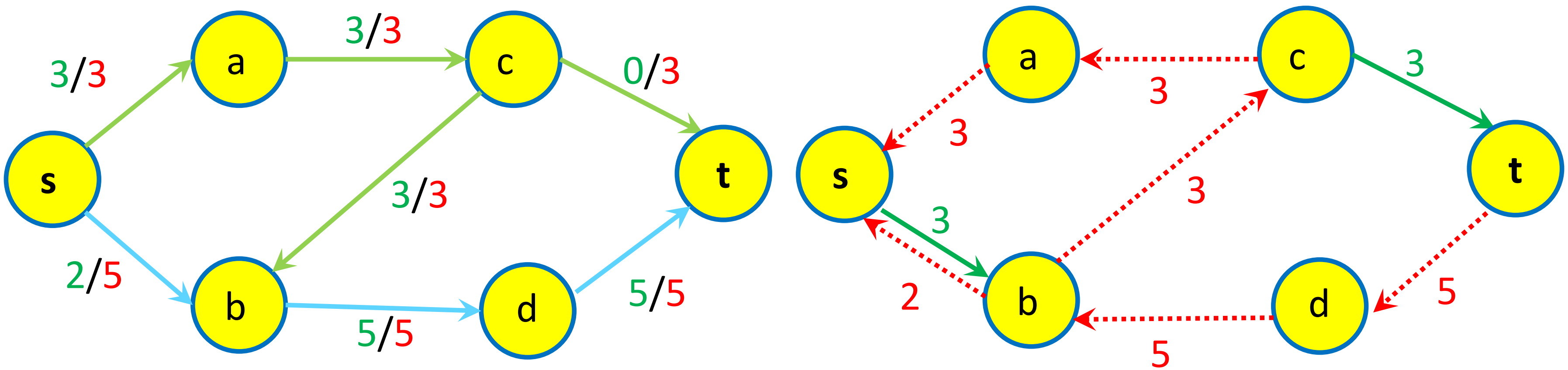

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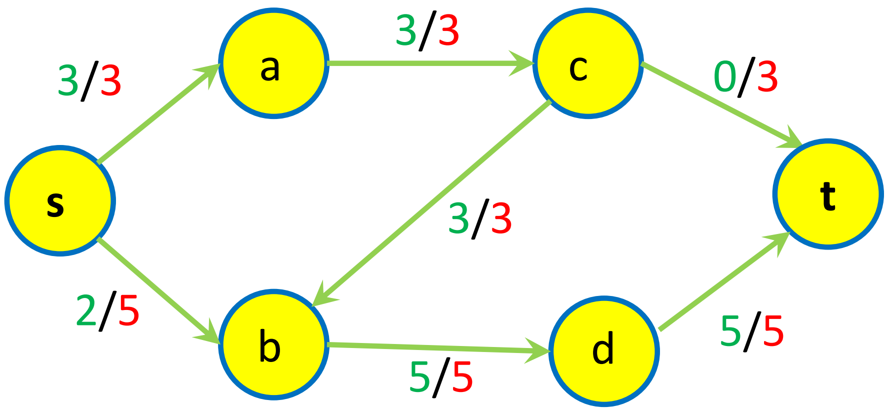

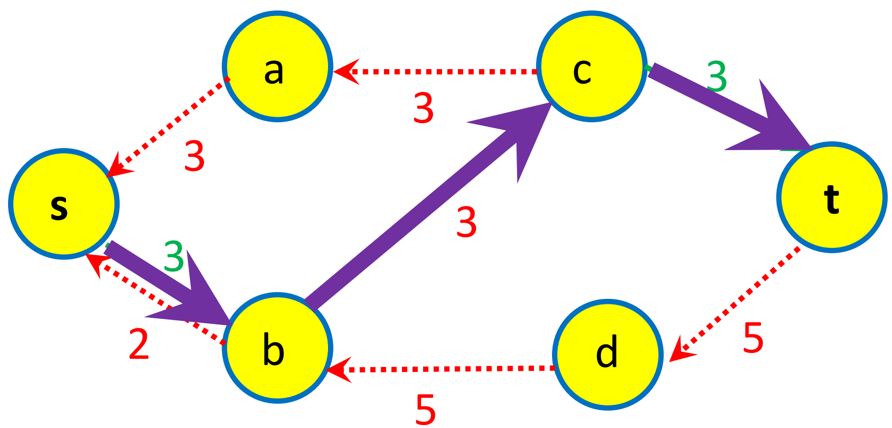

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Ford-Fulkerson Method – Worked example (by hand in lecture)

Algorithm 75 The Ford-Fulkerson method

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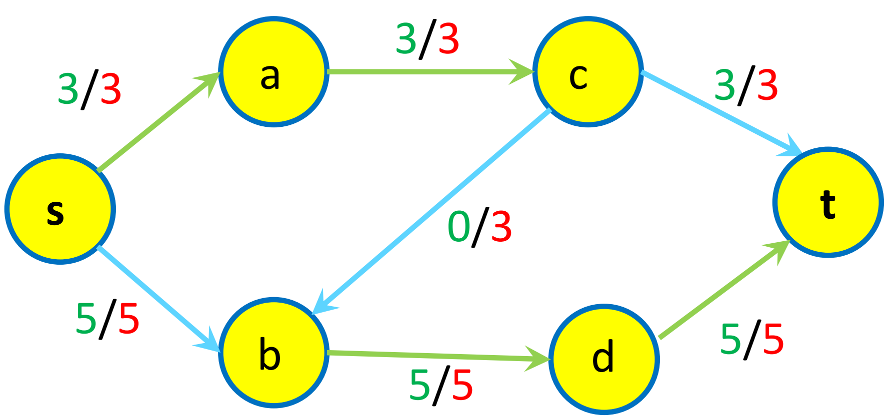
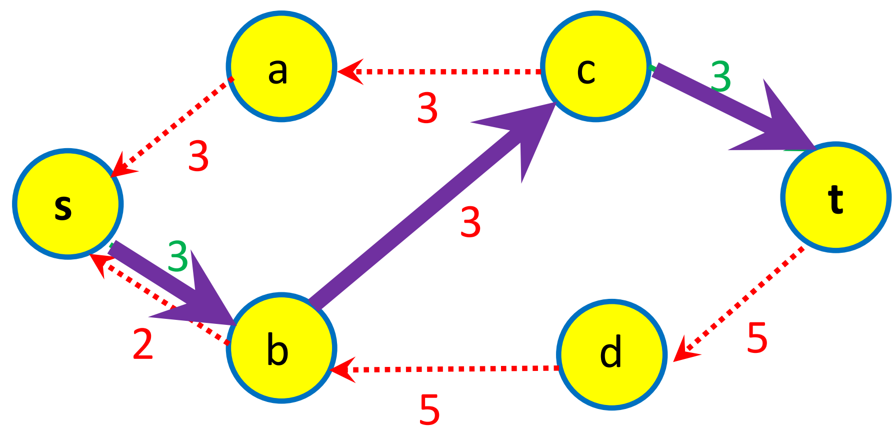
Total flow: 8





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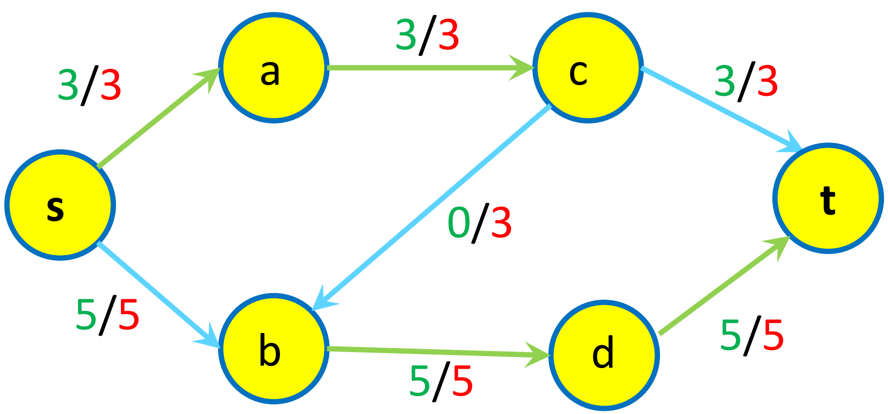
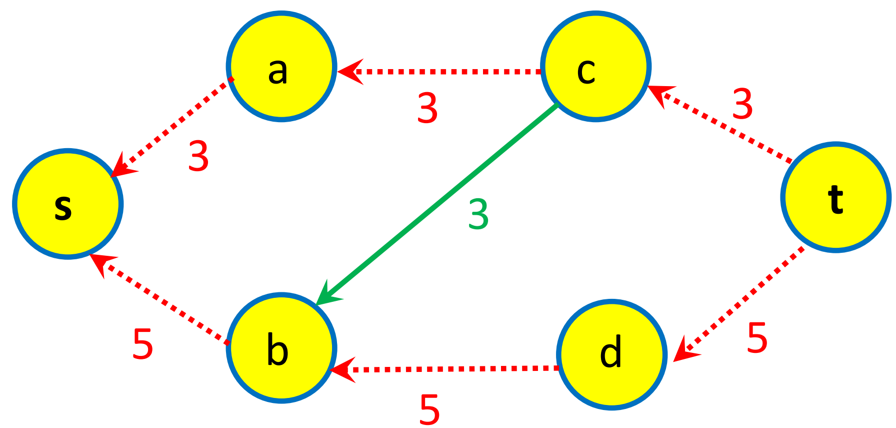




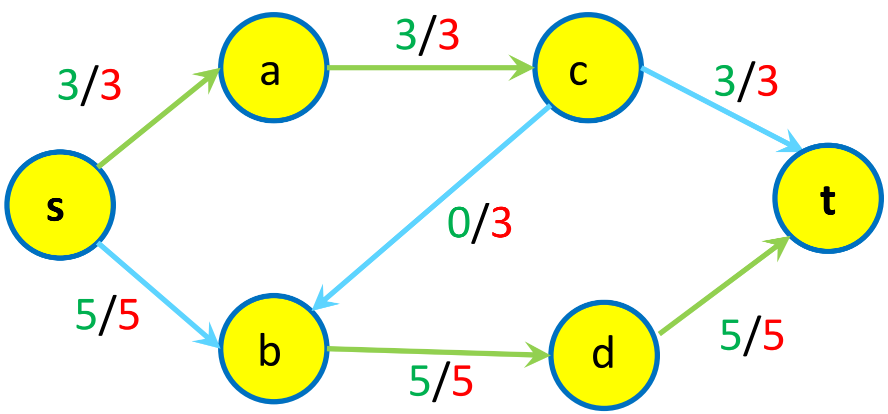
Update residual to match new flows

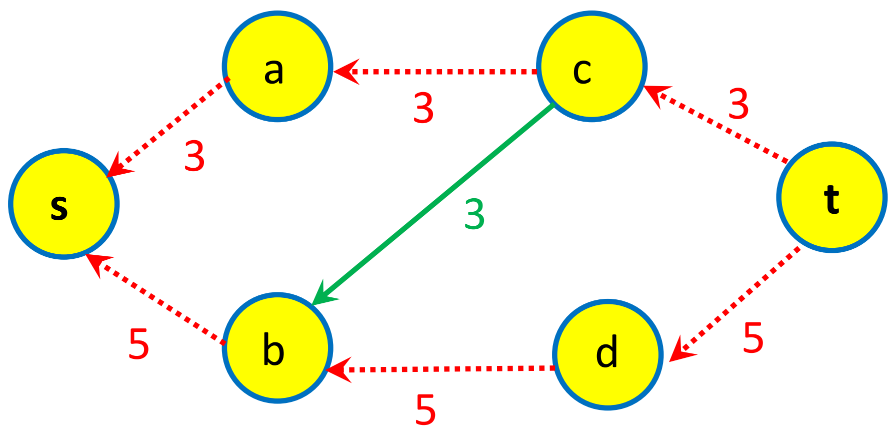
Ford-Fulkerson Method – Worked example (by hand in lecture)

Algorithm 75 The Ford-Fulkerson method

function MAX_FLOW($G = (V, E), s, t$)
    Set initial flow f to 0 on all edges
    while there exists an augmenting path p in the residual network G_f **do**
        Augment the flow f along the augmenting path p
    return f

Total flow: 8





No more augmenting paths!

Complexity Analysis

Algorithm 75 The Ford-Fulkerson method

```
function MAX_FLOW(G = (V, E), s, t)
    Set initial flow f to 0 on all edges
    while there exists an augmenting path p in the residual network G_f do
        Augment the flow f along the augmenting path p
    return f
```

Cost of finding an augmenting path:

- Using BFS, cost is O(V+E) (or just O(E) since the graph is connected)

Augmenting flow along a path:

- length of path ≤ V, so cost is O(V)

Updating the residual:

- Same amount of work as updating the flows along the augmenting path, O(V)

Total work in one iteration of the loop:

- **O(V+E) = O(E)**

Complexity Analysis

Algorithm 75 The Ford-Fulkerson method

```
function MAX_FLOW(G = (V, E), s, t)
    Set initial flow f to 0 on all edges
    while there exists an augmenting path p in the residual network G_f do
        Augment the flow f along the augmenting path p
    return f
```

Total work in one iteration of the loop:

- **O(V+E) = O(E)**

How many iterations?

- Assuming integer flows and capacities…
- Each iteration, flow grows by at least 1
- If maximum flow in graph is **F**
- Maximum number of iterations is also F

Total work:

- **O(EF)**

Complexity Analysis

Algorithm 75 The Ford-Fulkerson method

1: **function** MAX_FLOW($G = (V, E), s, t$)
2: Set initial flow f to 0 on all edges
3: **while** there exists an augmenting path p in the residual network G_f **do**
4: Augment the flow f along the augmenting path p
5: **return** f

Total work:

- **O(EF)**

Not examinable:

- This looks polynomial
- It isn't because F is a number, so its **value** is exponential in the space required to store it
- It can be proven that the complexity is **$O(VE^2)$** when using BFS to find augmenting paths, which is polynomial

Proof of Correctness

Algorithm 75 The Ford-Fulkerson method

function MAX_FLOW($G = (V, E), s, t$)
  Set initial flow f to 0 on all edges
  while there exists an augmenting path p in the residual network G_f **do**
    Augment the flow f along the augmenting path p
  return f

- **Does the algorithm terminate?**
  - Yes (assuming all capacities are integers), because
  - the flow always increases by at least 1 and the algorithm terminates when flow is equal to the maximum flow

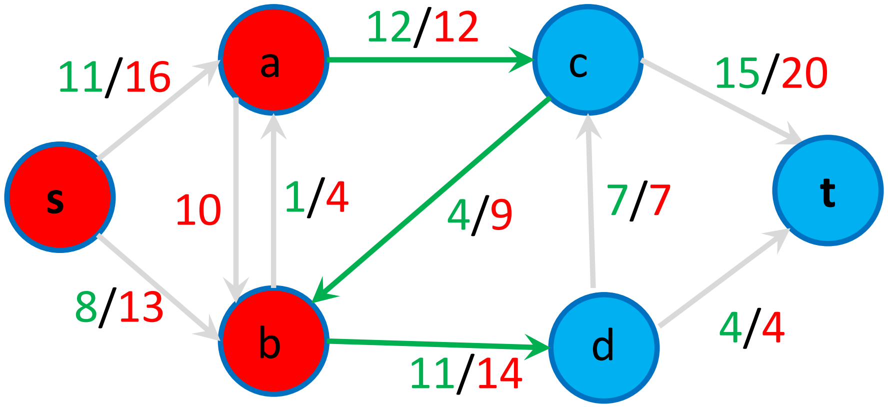
- When the algorithm terminates (i.e., there is no augmenting path in residual network), the flow of the network is the maximum flow.
- We will need to understand "min-cut and max-flow" theorem to prove this fact

Outline

1. Maximum Flow Problem
2. Ford-Fulkerson Algorithm
3. Min-cut Max-flow Theorem

Flow and capacity of a cut

- A cut (S,T) of a flow network partitions the vertices of the network into two disjoint partitions S and T such that source s is in S and target t is in T.
  - E.g., S = {s,a,b} and T = {t, c, d}
- Cut-set of a cut (S,T) is the set of edges that “cross” the cut, i.e., each edge connects one vertex in S with another in T.
  - E.g., the cut-set for the example is a→c, b→d, c→b (green edges)
  - The edges that have direction from a vertex in S to a vertex in T are called outgoing edges of the cut.
    - E.g., a→c and b→d are the outgoing edges of the cut
  - The edges that have direction from a vertex in T to a vertex in S are called incoming edges of the cut.
    - E.g., c→b is an incoming edge of the cut.



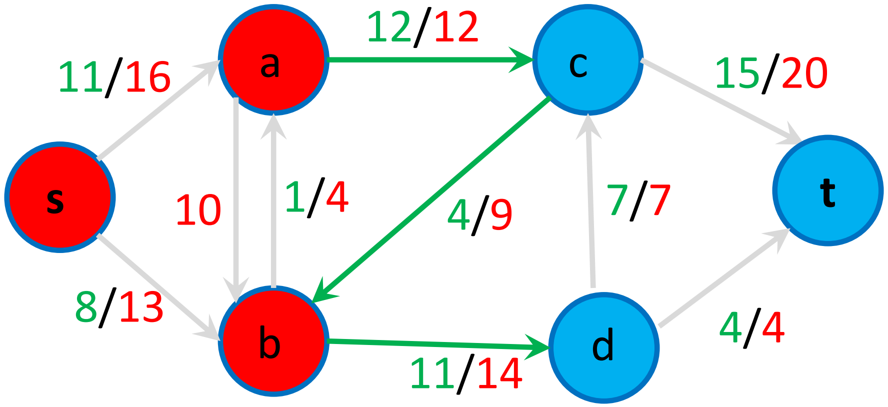
Flow and capacity of a cut

- Capacity of a cut (S,T) is **just** the total capacity of its outgoing edges
  - E.g., capacity of the cut in the example is 12 + 14 = 26
- Flow of a cut (S,T) is Total flow of outgoing edges – total flow of incoming edges
  - E.g., flow in the example is 12 + 11 – 4 = 19

Is it true that flow of a cut is always less than or equal to the capacity of the cut?

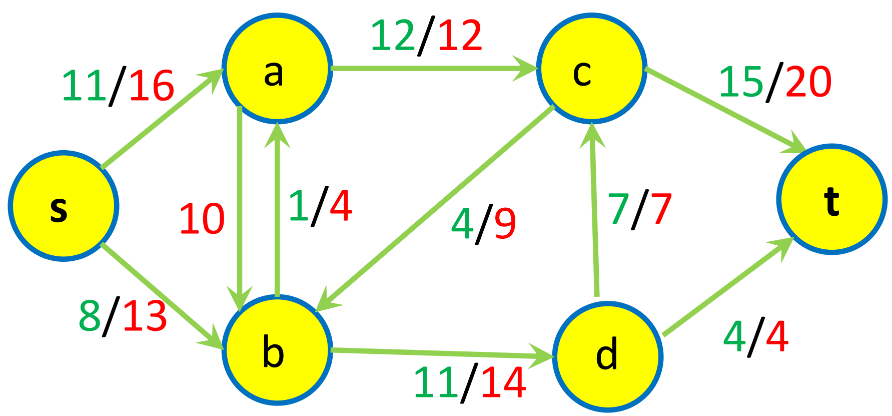
Yes, because

- Flow of an edge ≤ capacity of an edge
- Capacity of a cut does not subtract capacities for incoming edges



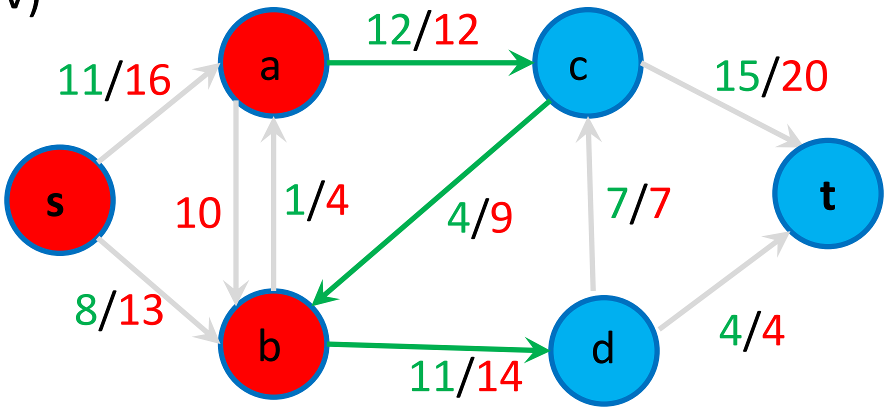
Flow and capacity of a cut

- Capacity of a cut (S,T): the total capacity of its outgoing edges
- Flow of a cut (S,T): Total flow of its outgoing edges – total flow of its incoming edges
- Assume S = {s, a,b,c,d} and T = {t}.
  - What is the capacity of this cut?
  - What is the flow of this cut?
- Assume S = {s, a, b,d} and T = {c,t}.
  - What is the capacity of this cut?
  - What is the flow of this cut?
- Assume S = {s, a} and T = {b,c,d,t}.
  - What is the capacity of this cut?
  - What is the flow of this cut?
- What is the flow value of this network?
- Note: flow of all of the above cuts is 19
  - which is the same as flow of the network.
- I.e., flow of **every** cut = flow of the network
- Let's prove this formally



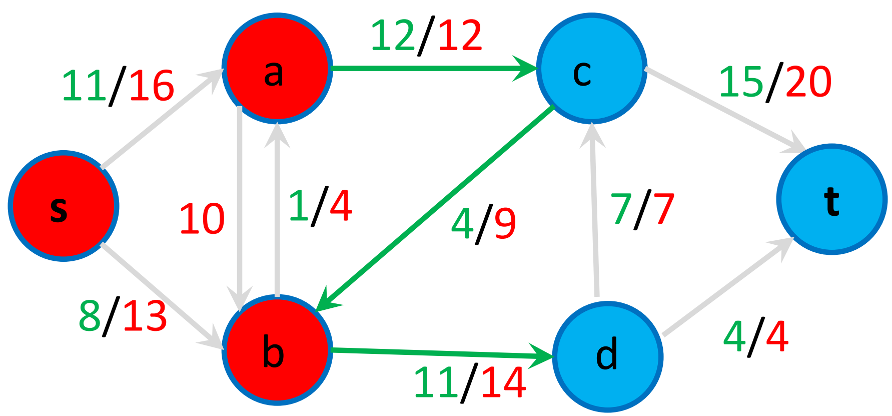
Flow of a cut = Flow of the network

- Let $F^{out}(v)$ be the total flow going out of a vertex and $F^{in}(v)$ be the total flow coming in the vertex
- Recall that flow of a network is the total flow going out from the source s.
  - Flow of the network = $F^{out}(s)$
- Flow conservation property: $F^{out}(v) - F^{in}(v) = 0$ for every vertex except s and t
- Flow of the network = $F^{out}(s)$
- Flow of the network = $F^{out}(s) + \sum_{v \in S \setminus s} F^{out}(v) - Fin(v)$
  - recall S is the cut containing s and excluding t
- Since $F^{in}(s) = 0$, we can rewrite the flow as.
- Flow of the network = $\sum_{v \in S} F^{out}(v) - Fin(v)$



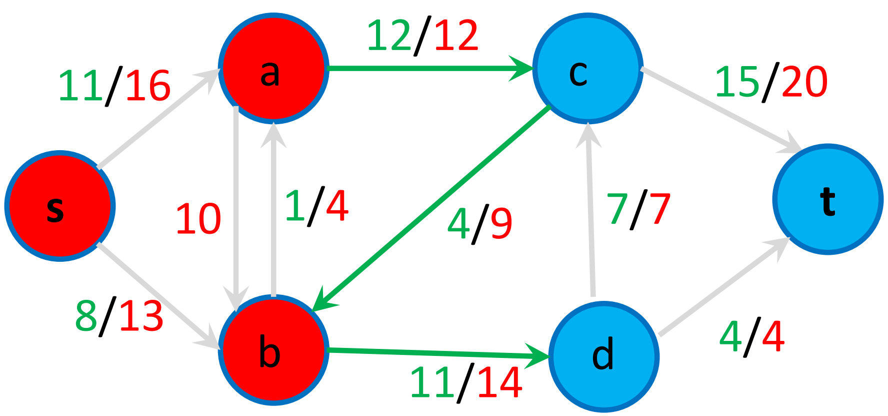
Flow of a cut = Flow of the network

- Flow of the network = $\sum_{v \in S}$ F^{out} (v) −Fin (v)
- Each vertex v in S (red vertices) can have two types of edges
  - Grey edges (the edges that connect the vertex to another vertex in S)
  - Green edges (the edges that connect the vertex to a vertex in T)
  - Let $F^{out-grey}$ (v) be the total flow out from v via grey edges. Similarly, $F^{in-grey}$ (v) be the total flow coming to v via grey edges.
  - Let $F^{out-green}$ (v) be the total flow out from v via green edges. Similarly, $F^{in-green}$ (v) be the total flow coming to v via green edges.
  - We have $F^{out-green}$ (v) + $F^{out-grey}$ (v) = F^{out} (v) and $F^{in-green}$ (v) + $F^{in-grey}$ (v) = F^{in} (v)



Flow of a cut = Flow of the network

- We have $F^{out-green}$ (v) + $F^{out-grey}$ (v) = F^{out} (v) and $F^{in-green}$ (v) + $F^{in-grey}$ (v) = F^{in} (v)
- From before, Flow of the network = $\sum_{v \in S}$ F^{out} (v) −Fin (v)
- Flow of the network = $\sum_{v \in S}$ $F^{out-green}$ (v) + $F^{out-grey}$ (v) − ($F^{in-green}$ (v) + $F^{in-grey}$ (v))
- Flow of the network = $\sum_{v \in S}$ $F^{out-green}$ (v) − Fin^{-green} (v) + $F^{out-grey}$ (v) − $F^{in-grey}$ (v)
- Note that $\sum_{v \in S}$ $F^{out-grey}$ (v) − $F^{in-grey}$ (v) = 0 because each grey edge appears once as an incoming edge for one vertex and once as an outgoing edge for another vertex.
- Flow of the network = $\sum_{v \in S}$ $F^{out-green}$ (v) − Fin^{-green} (v)
- Flow of the network = $\mathit{Flow\ of\ the\ cut}$



Min-cut Max-Flow Theorem

- Min-cut of a flow network is the cut with the minimum capacity

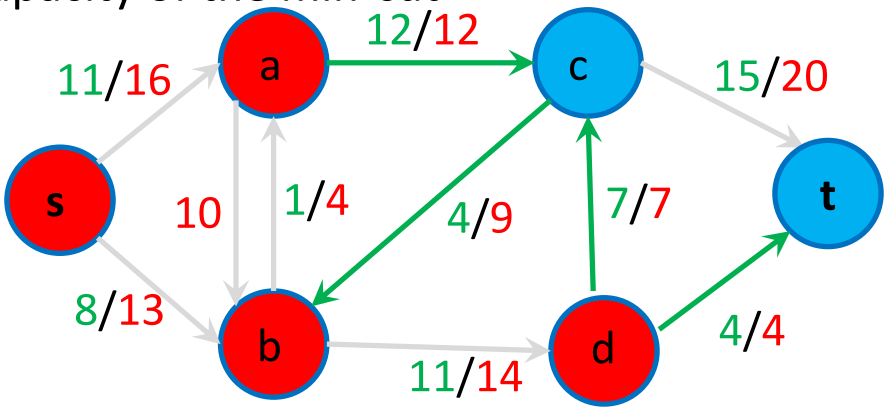
We know that

1. Flow of a cut ≤ capacity of the cut
2. Flow of **<u>every</u>** cut = Flow of the network

- Therefore, Maximum possible flow of the network ≤ capacity of every cut
- Or, Maximum possible flow of the network ≤ capacity of min-cut
- What if we can find a cut such that the flow of the network = capacity of the cut
  - This would mean flow of the network is the maximum possible (we have found maximum possible flow)
  - The cut is the min-cut of the flow network

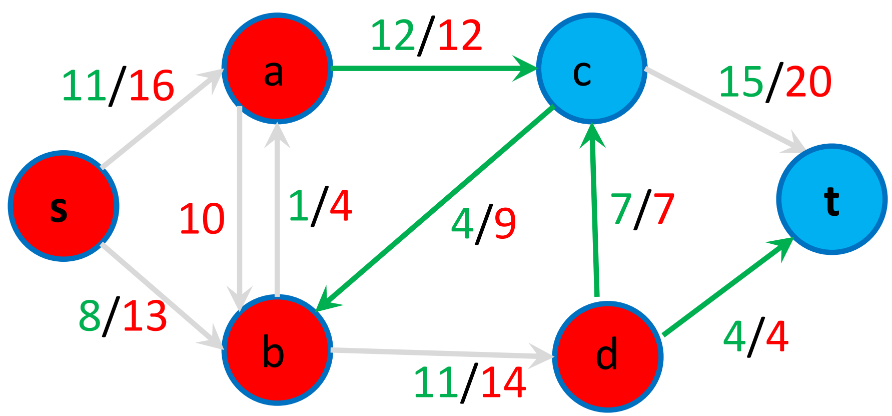
Min-cut Max-Flow Theorem

- Maximum possible flow of a network = capacity of the min-cut



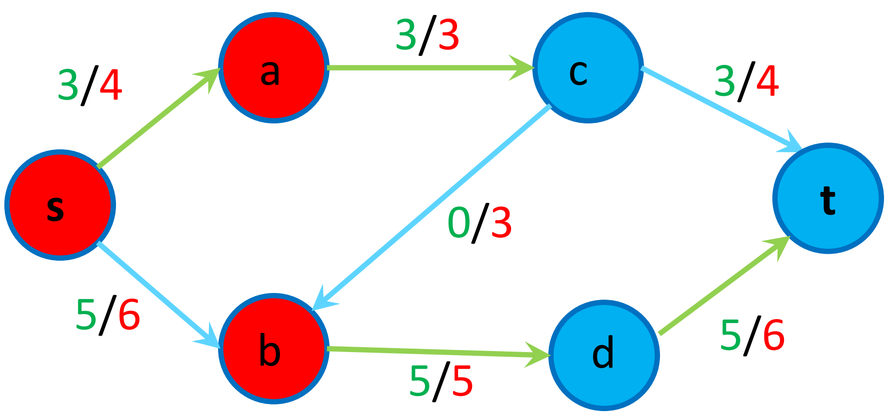
Min-cut Max-Flow Theorem

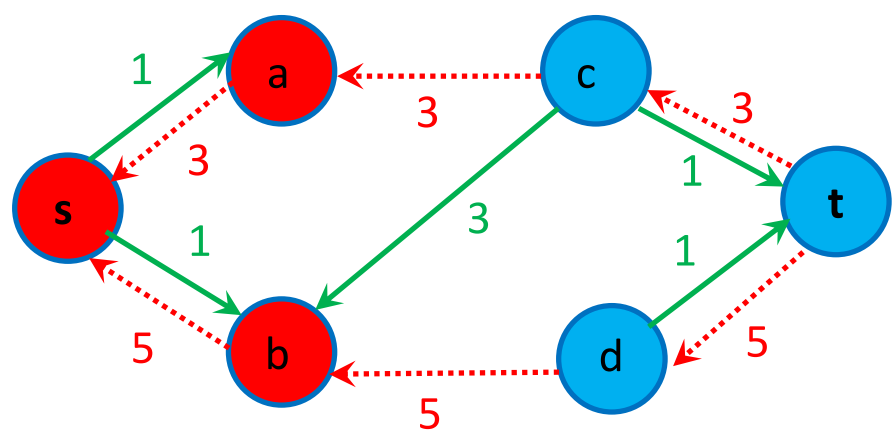
- Capacity of a cut (S,T) is the total capacity of its **outgoing** edges
- Flow of a cut (S,T) = Total flow of its **outgoing** edges – total flow of its **incoming** edges
- Flow of a cut = capacity of the cut when:
  1. Flow on **each** outgoing edge of the cut is equal to the capacity of the edge; AND
  2. Flow on **each** incoming edge of the cut is zero
- We show that when the algorithm terminates, there exists a cut for which both of the above two conditions hold which imply that the flow of the cut is equal to its capacity.
  - **This guarantees that the flow is maximum**



Proof of correctness

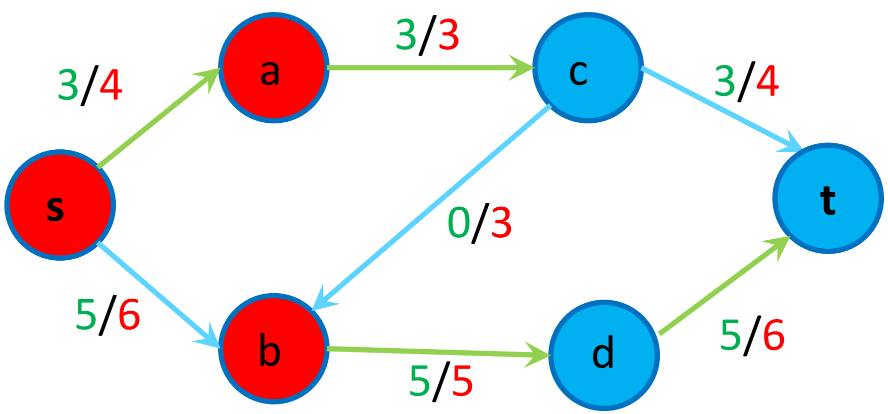
- Suppose the algorithm has terminated (there does not exist any augmenting path in the residual network).
- We define a cut (**S**,T) such that
  - **S** contains every vertex v that is reachable from s in **the residual network**.
  - **T** contains every other vertex. Note t cannot be in S because it is not reachable from S (no augmenting path)

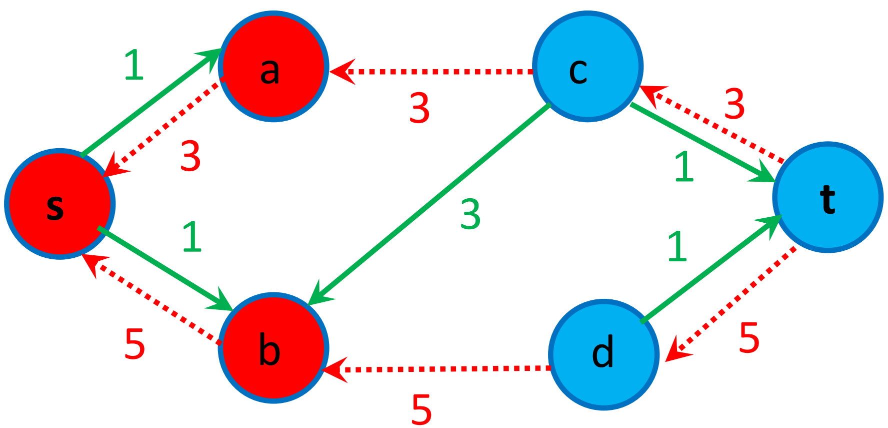




Proof of correctness

- Flow of this cut = Capacity of this cut because
  - For each outgoing edge a→c, its flow is equal to the capacity of the edge
    - Otherwise, we would have an edge a→c in the residual network which would mean c is reachable from s but we know this is not the case as c is not in S.
  - For each incoming edge c → b, its flow is zero.
    - Otherwise, there would be an edge b→c in the residual network implying c is reachable from s but we know this is not the case as c is not in S.
- Therefore, the flow is maximum when the algorithm terminates





Summary

Take home message

- Maximum flow of a network is equal to its min-cut and can be found using Ford-Fulkerson

Things to do (this list is not exhaustive)

- Make sure you understand
  - the two algorithms
  - understand why Ford-Fulkerson is correct
- Start preparing for the final exam

Coming Up Next

- Topological sorting and final exam
- Revision – bring any questions you would like to discuss to next week's lecture!