Faculty of Information Technology,
Monash University

FIT2004: Algorithms and Data Structures

Week 4: Dynamic Programming

These slides are prepared by M. A. Cheema and are based on the material developed by Arun Konagurthu and Lloyd Allison.

Recommended Reading

- Unit Notes (Chapter 5)
- Weiss "Data Structures and Algorithm Analysis" (Pages 462-466.)

Things to remember/note

- Next week is mid-sem break
- If you don't understand some lecture content, come to consultations! Times are available on Moodle
- Assignment 1 is due at the end of week 4
- Assignment 2 will be released at the end of week 4
- Assignment 2 will be due at the end of week 6, so get started early!

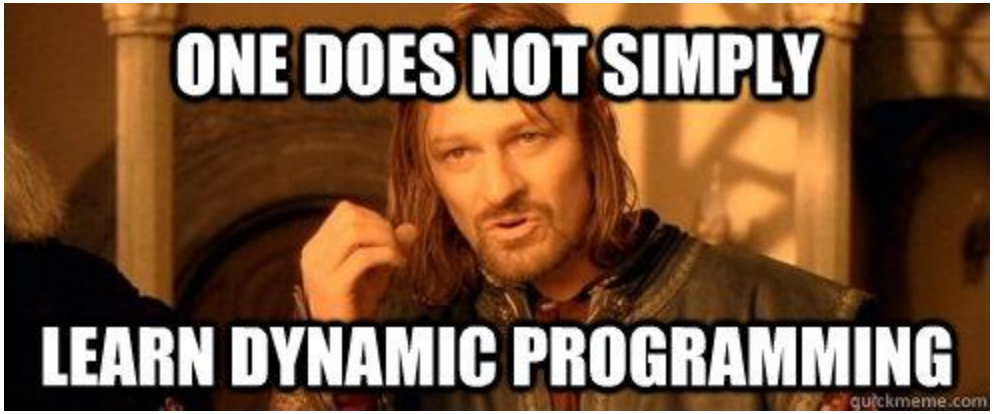
Outline

1. Introduction to Dynamic Programming
2. Coins Change
3. Unbounded Knapsack
4. 0/1 Knapsack
5. Edit Distance
6. Constructing Optimal Solution

Dynamic Programming Paradigm

- A powerful optimization technique in computer science
- Applicable to a wide-variety of problems that exhibit certain properties.

- Practice is the key to be good at dynamic programming



Core Idea

- Divide a complicated problem by breaking it down into simpler subproblems in a recursive manner and solve these.
- Question: But how does this differ from `Divide and Conquer' approach?
- **Overlapping subproblems**: the same subproblem needs to be (potentially) be used multiple times
- We also need **Optimal substructure**: optimal solutions to subproblems help us find optimal solutions to larger problems.

N-th Fibonacci Number

```
fib(N)
    if N == 0 or N == 1
        return N
    else
        return fib(N - 1) + fib(N - 2)
```

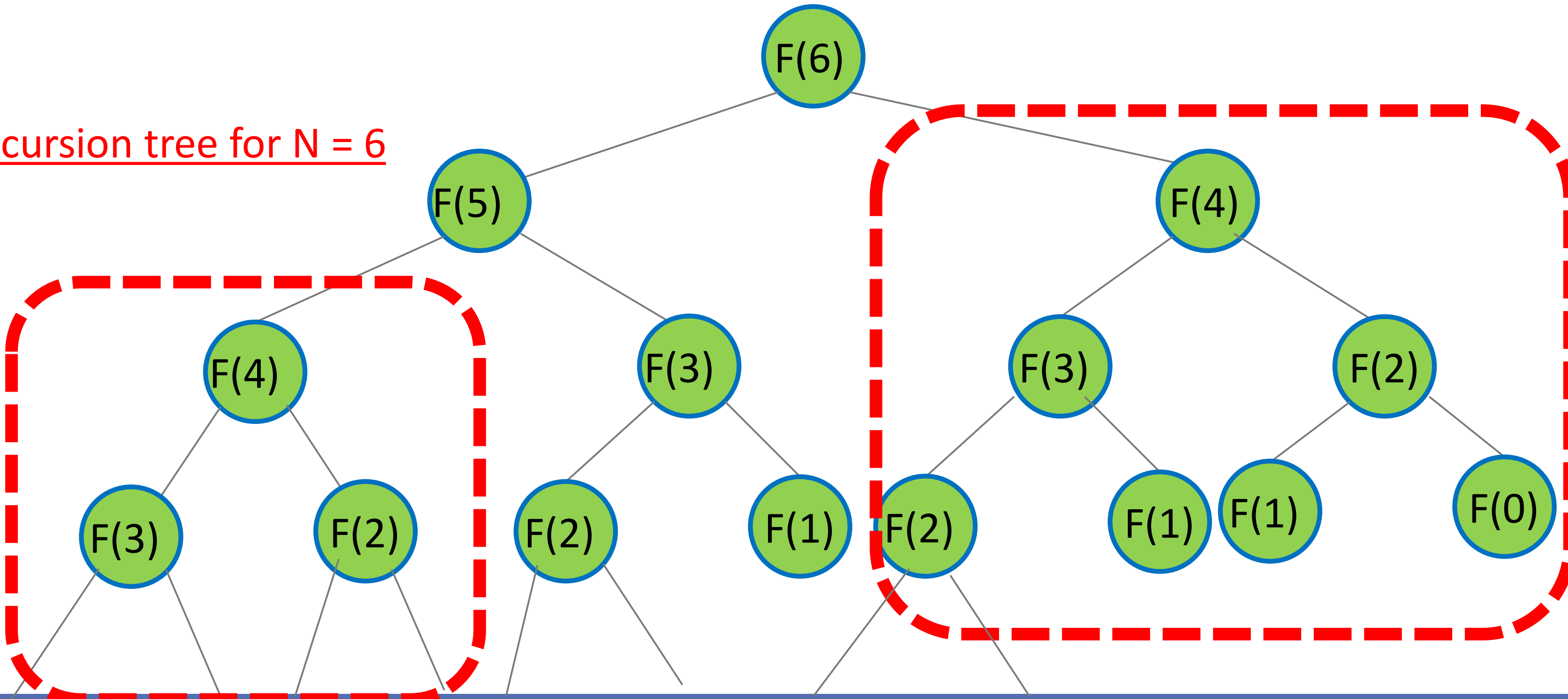
Time Complexity

$T(1) = b$ // b and c are constants

$T(N) = T(N-1) + T(N-2) + c$

$= O(2^N)$

Recursion tree for N = 6



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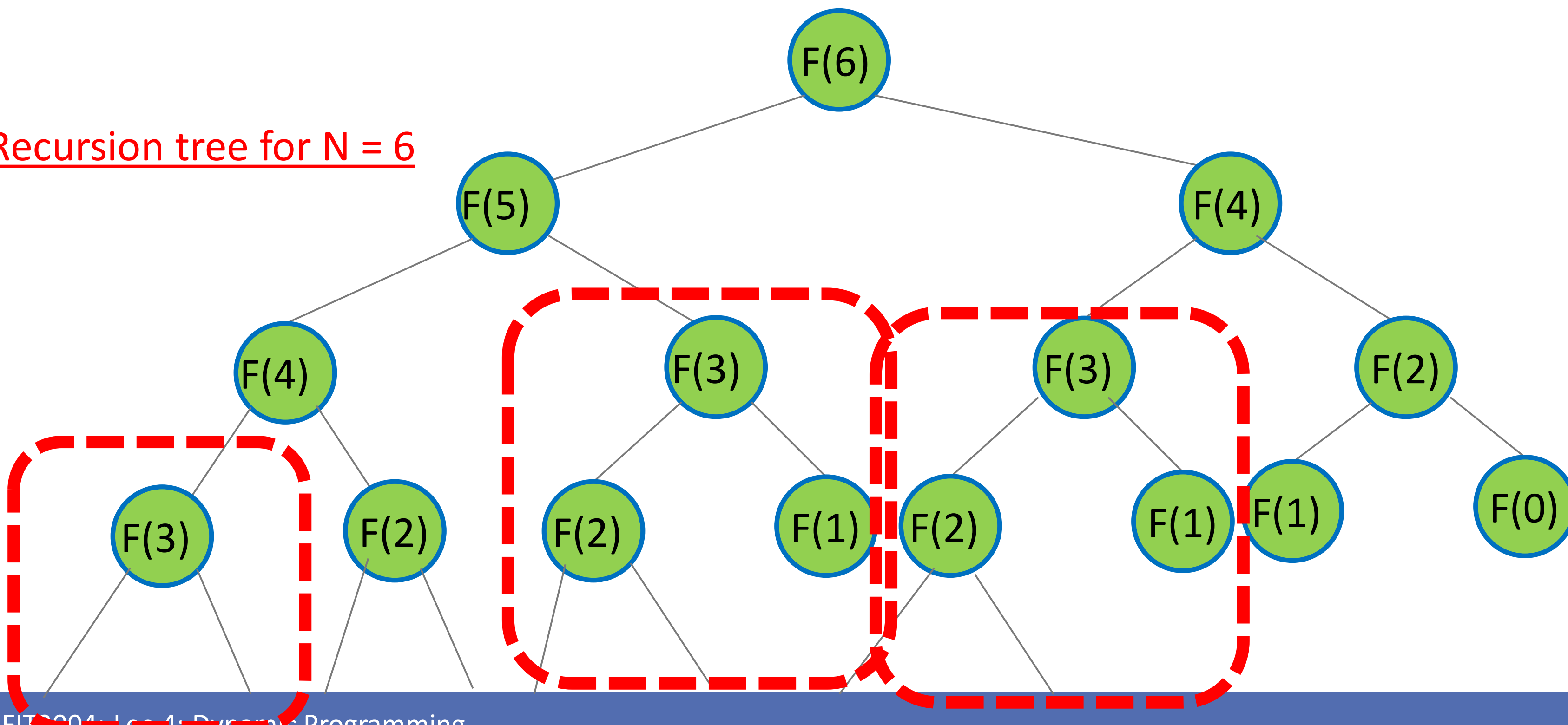


N-th Fibonacci Number

```
fib(N)
    if N == 0 or N == 1
        return N
    else
        return fib(N - 1) + fib(N - 2)
```

Time Complexity

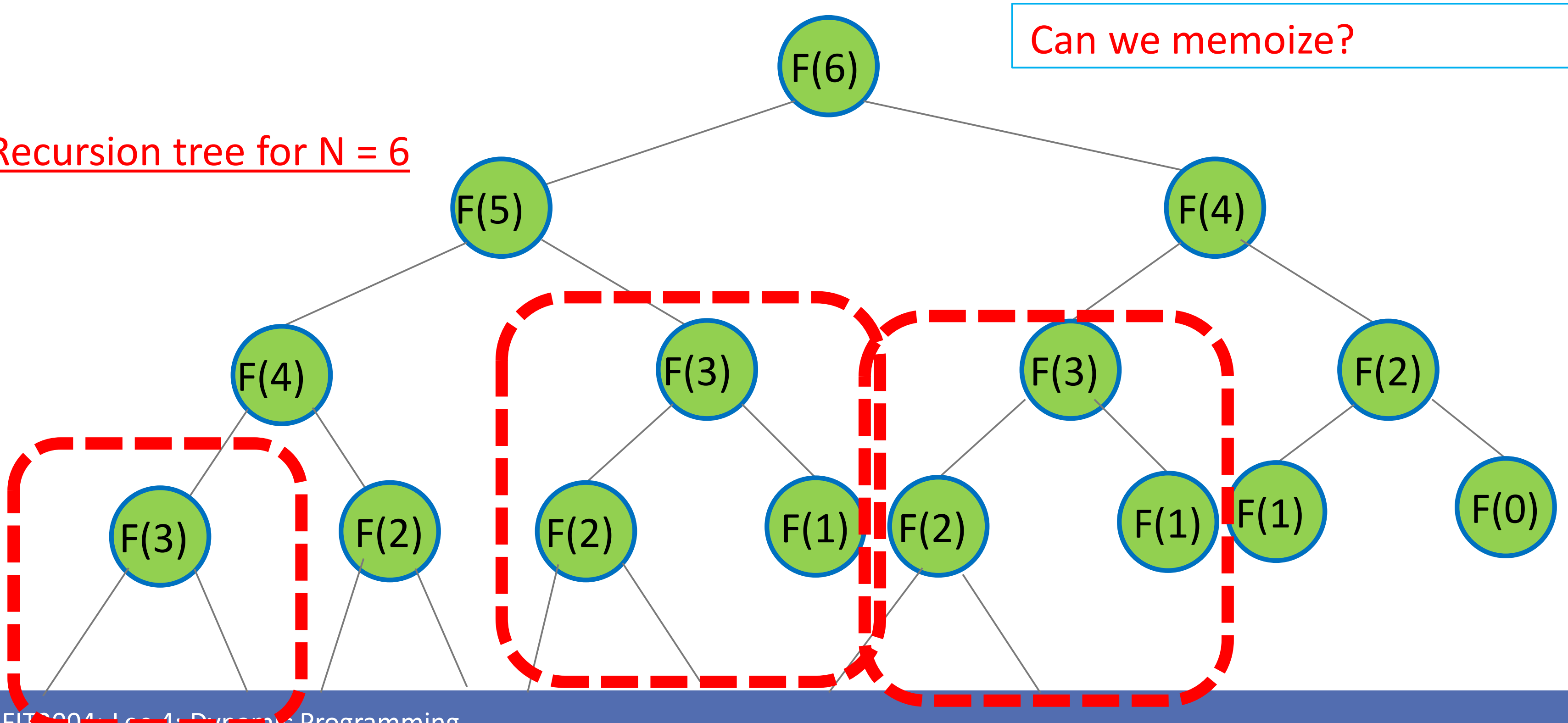
$T(1) = b$ // b and c are constants

$T(N) = T(N-1) + T(N-2) + c$

$= O(2^N)$

Can we memoize?

Recursion tree for N = 6



Fibonacci with Memoization: Version 1

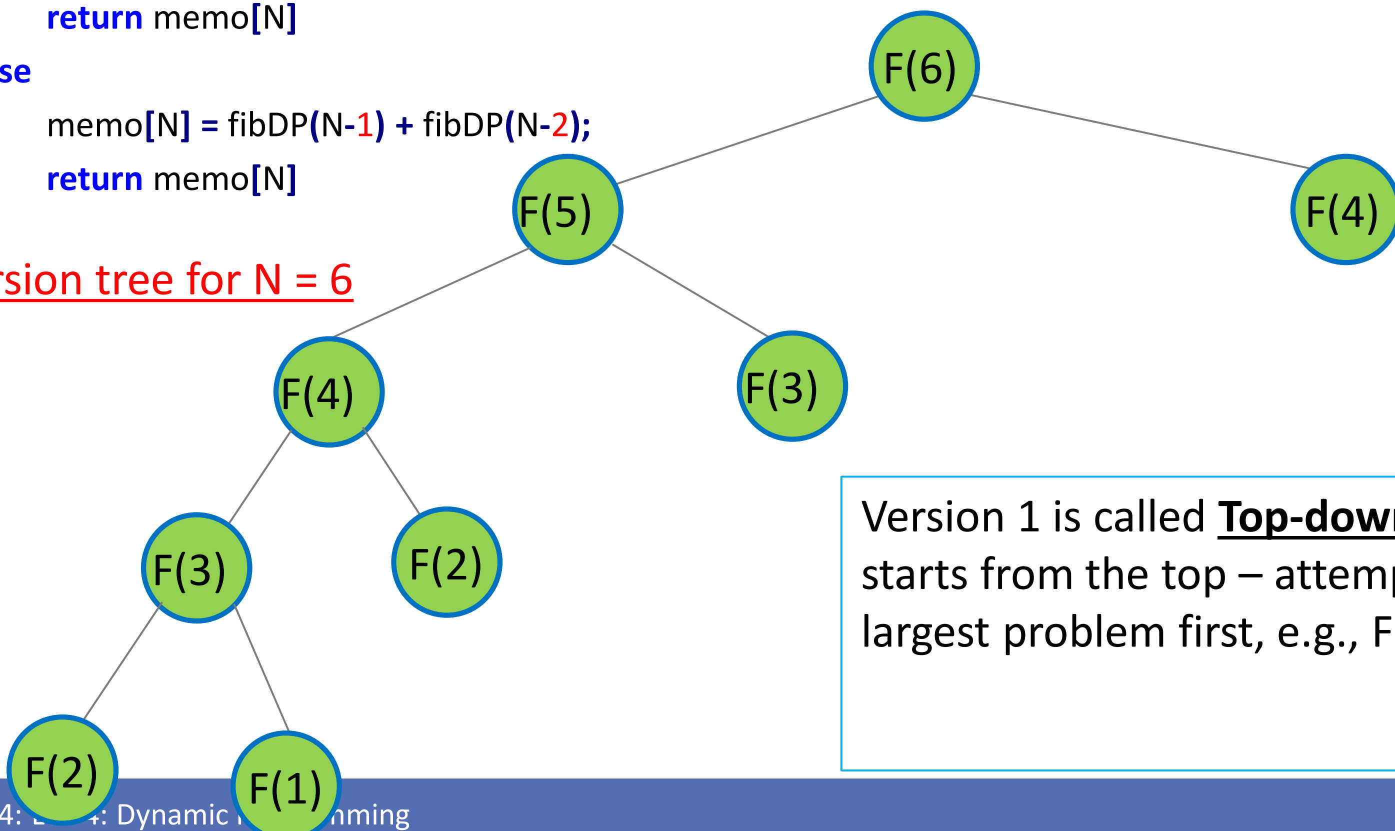
```
memo[0] = 0  // 0th Fibonacci number
memo[1] = 1  // 1st Fibonacci number
for i=2 to i=N:
    memo[i] = -1
fibDP(N)
    if memo[N] != -1
        return memo[N]
    else
        memo[N] = fibDP(N-1) + fibDP(N-2);
        return memo[N]
```

Time Complexity

calls fibDP() roughly 2*N times

So the complexity is O(N)

Recursion tree for N = 6



Version 1 is called **Top-down** because it starts from the top – attempting the largest problem first, e.g., F(6)

Fibonacci with Memoization: Version 2

```
memo[0] = 0  // 0th Fibonacci number
memo[1] = 1  // 1st Fibonacci number
for i=2 to i=N:
      memo[i] = memo[i-1] + memo[i-2]
```

Time Complexity

O(N)

0	1	1	2	4	5	8	13	21	34

Version 2 is called **Bottom-up** because it starts from the bottom – solving the smallest problem first, e.g., F(0), F(1), and so on

Dynamic Programming Strategy

1. Assume you already know the solutions of all sub-problems and have memoized these solutions (overlapping subproblems)
   - E.g., Assume you know Fib(i) for every i < n
2. Observe how you can solve the original problem using memoized solutions (optimal substructure)
   - E.g., Fib(n) = Fib(n-1) + Fib(n-2)
3. Solve the original problem by building upon solutions to the sub-problems
   - E.g., Fib(0), Fib(1), Fib(2), …, Fib(n)

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Coins Change Problem

Problem: A country uses N coins with denominations {a1, a2, …, aN}. Given a value V, find the minimum number of coins that add up to V.

Example: Suppose the coins are {1, 5, 10, 50} and the value V is 110. The minimum number of coins required to make 110 is 3 (two 50 coins, and one 10 coin).

Greedy solution does not always work.
E.g., Coins = {1, 5, 6, 9}

The minimum number of coins to make 12 is 2 (i.e., two 6 coins).

What is the minimum number of coins to make 13?

Coins Change Problem

Problem: A country uses N coins with denominations {a1, a2, …, aN}. Given a value V, find the minimum number of coins that add up to V.

Overlapping subproblems: What shall we store in the memo array?

Quiz time!
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Coins Change Problem

Problem: A country uses N coins with denominations {a1, a2, …, aN}. Given a value V, find the minimum number of coins that add up to V.

Overlapping subproblems: What shall we store in the memo array?

We want to know the minimum number of coins which add up to V, so lets try

MinCoins[v] = {The fewest coins required to make exactly $v}

Note: your first guess at what to put in the memo array may not be right, so try the most obvious thing and then play around if you can't make it work

Coins Change Problem

Problem: A country uses N coins with denominations {a1, a2, …, aN}. Given a value V, find the minimum number of coins that add up to V.

Overlapping subproblems:

MinCoins[v] = {The fewest coins required to make exactly $v}

Optimal substructure: To find the optimal substructure, we first deal with the base case(s). In this case, to make $0 requires 0 coins, so MinCoins[0] = 0

Assume we have optimal solutions for all v < V (stored in MinCoins[0..V-1]

How could we determine MinCoins[V]?

Quiz time!
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Coins Change Problem - Example

Coins: [9, 5, 6, 1]

V: 12

MinCoins:

0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	4	1	1	2	3	1	2	2	

What options do we have to try and make $12?

We have to use a coin!

Lets try using the 9…

After choosing the 9, what would be the optimal thing to do?

Look at MinCoins[12-9] since now we need to make the other $3, and we already know the best way to do that

Repeat this idea for the other coins and see which is best

Coins Change Problem - Example

Coins: [9, 5, 6, 1]

V: 12

MinCoins:

0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	4	1	1	2	3	1	2	2	

MinCoins[12] =

1+min(MinCoins[12-9], MinCoins[12-5], MinCoins[12-6], MinCoins[12-1]

= 1 + Min(3, 2, 1, 2)

= 2

In general, MinCoins[*v*] = 1 + min(MinCoins[*v*-*c*] for all c in coins, where *c* <= *v*)

Note also that if the value is less than every coin, then it cannot be made. This can be ignored if there is a $1 coin

Coins Change Problem - Example

Coins: [9, 5, 6, 1]

V: 12

MinCoins:

0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	4	1	1	2	3	1	2	2	

In general, MinCoins[*v*] = 1 + min(MinCoins[*v*-*c*] for all c in coins, where *c* <= *v*)

Why is that restriction on *c* necessary?

Note also that if *v* is less than every coin, then it cannot be made. This can be ignored if there is a $1 coin

Coins Change Problem - Example

Coins: [9, 5, 6, 1]

V: 12

MinCoins:

0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	3	4	1	1	2	3	1	2	2	

$$\text{MinCoins[v]} = \begin{cases} 0 & \text{if } v = 0, \\ \infty & \text{if } v < c[i] \text{ for all } i, \\ \min\limits_{\substack{1 \le i \le n \\ c[i] \le v}} (1 + \text{MinCoins[v - c[i]]}) & \text{otherwise} \end{cases}$$

Coins Change Problem - Example

Overlapping subproblems:

MinCoins[v] = {The fewest coins required to make exactly $v}

Optimal substructure:

$$\mathrm{MinCoins[v]} = \begin{cases} 0 & \text{if } v = 0, \\ \infty & \text{if } v < c[i] \text{ for all } i, \\ \min\limits_{\substack{1 \le i \le n \\ c[i] \le v}} (1 + \mathrm{MinCoins[v - c[i]]}) & \text{otherwise} \end{cases}$$

Coins Change – Implementation (bottom up)

With DP, you can generally implement straight from the recurrence to code

$$\text{MinCoins[v]} = \begin{cases} 0 & \text{if } v = 0, \\ \infty & \text{if } v < c[i] \text{ for all } i, \\ \min\limits_{\substack{1 \le i \le n \\ c[i] \le v}} (1 + \text{MinCoins[v - c[i]]}) & \text{otherwise} \end{cases}$$

Coin_change(c[1..n], V)
    min_coins[0..v] = infinity (note we start from 0 index here)
    min_coins[0] = 0 *(from recurrence)*
    for each value 1 to V
        if v less than every value in c[1..n] *(from recurrence)*
            do nothing
        else
            option = [all values of MinCoins[v-c[i]], provided v >= c[i]] *(from recurrence)*
            set min_coins[value] to min(options)+1
    return min_coins[V]

Coins Change – Implementation (top down)

```
//Assume that the coin_change function has appropriately initialised memo to an
//array of nulls, and called our auxiliary function
Coin_change_aux(c[1..n], V)
    if V = 0, return 0
    if memo[v] = null
        min_coins = infinity
        for i in 1 to n
            if c[i] <= V
                min_coins = min(min_coins, 1 + coin_change_aux(c, V-c[i]))
        memo[v] = min_coins
    return memo[v]
```

This recursive call just returns memo[V-c[i]] instantly if we have already calculated it (because of this "if")

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Unbounded Knapsack Problem

Problem: Given a capacity C and a set of items with their weights and values, you need to pick items such that their total weight is at most C and their total value is maximized. What is the maximum value you can take? In unbounded knapsack, you can pick an item as many times as you want.

Example: What is the maximum value for the example given below given capacity is 12 kg?

Answer: $780 (take two Bs and two Ds)
Greedy solution does not always work.

Item	A	B	C	D
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

DP Solution for Unbounded Knapsack

Problem: Given a capacity C and a set of items with their weights and values, you need to pick items such that their total weight is at most C and their total value is maximized. What is the maximum value you can take? In unbounded knapsack, you can pick an item as many times as you want.

- We want the most value given that we are under a given weight.
- Overlapping subproblems: Memo[i] = Most value with capacity at most i
- If we know optimal solutions to all subproblems, how can we build an optimal solution to a larger problem?

- Similar logic to coin change: We need to choose an item
- For each possible item choice, find out how much value we could get (using subproblems) and then take the best one

DP Solution for Unbounded Knapsack

- Similar logic to coin change: We need to choose an item
- For each possible item choice, find out how much value we could get (using subproblems) and then take the best one
- If we take item 1, then we have 3kg left
- The best we can do with 3kg is memo[3] = $120
- So one option for memo[12] would be value[1] + memo[12-weight[1]]

Item	1	2	3	4
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

Memo

1	2	3	4	5	6	7	8	9	10	11	12
40	80	120	160	350	390	430	470	550	700	740	

DP Solution for Unbounded Knapsack

Memo[12] could be:

- value[1] + memo[12-weight[1]] = 550 + 120 = 670
- Value[2] + memo[12-weight[2]] = 350 + 430 = 780
- Value[3] + memo[12-weight[3]] = 390 + 180 = 570
- Value[4] + memo[12-weight[4]] = 740 + 40 = 780
- Choose the best!

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Value	$550	$350	$180	$40

Memo

1	2	3	4	5	6	7	8	9	10	11	12
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Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

	1	2	3	4	5	6	7	8	9	10	11	12
Memo	40	80	120	160	350	390	430	470	550	700	740	780

DP Solution for Unbounded Knapsack

Lets write our recurrence

- What is our base case?
- With no capacity, we cannot take any items
- Also note, as before, that if an item is heavier than the capacity we have left, we cannot take it
- Otherwise, we want the maximum over all values ($1 <= i <= n$, v_i) of items that we could take ($w_i <= c$)
- But also taking into account the optimal value we could fit into the rest of our knapsack, one we took that item (MaxValue[c-w_i])

Quiz time!
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DP Solution for Unbounded Knapsack

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- But also taking into account the optimal value we could fit into the rest of our knapsack, one we took that item (MaxValue[c-w_i])

$$\text{MaxValue}[c] = \begin{cases} 0 & \text{if } c < w_i \text{ for all } i, \\ \max\limits_{\substack{1 \le i \le n \\ w_i \le c}} (v_i + \text{MaxValue}[c - w_i]) & \text{otherwise.} \end{cases}$$

DP Solution for Unbounded Knapsack

Overlapping subproblems: Memo[i] = Most value with capacity at most i

Optimal substructure:

$$\mathrm{MaxValue}[c] = \begin{cases} 0 & \text{if } c < w_i \text{ for all } i, \\ \max\limits_{\substack{1 \le i \le n \\ w_i \le c}} (v_i + \mathrm{MaxValue}[c - w_i]) & \text{otherwise.} \end{cases}$$

Bottom-up Solution

```
// Construct Memo[ ] starting from 1 until C in a way similar to previous slide .
Initialize Memo[ ] to contain 0 for all indices
for c = 1 to C
   maxValue = 0
   for i=1 to N
      if Weight[ i ] <= c
         thisValue = Value[i]  + Memo[c - Weight[ i ] ]
         if thisValue > maxValue
            maxValue = thisValue
   Memo[c] = maxValue
```

Time Complexity:

O(NC)

Space Complexity:

O(C + N)

E.g., Fill Memo[13]

Item	1	2	3	4
Weight	9kg	5kg	6kg	1kg
Value	$550	$350	$180	$40

Memo													
	40	80	120	160	350	390	430	470	550	700	740	780	
	1	2	3	4	5	6	7	8	9	10	11	12	13

Top-down Solution

```
Initialize Memo[ ] to contain -1 for all indices // -1 indicates solution for this index has not
yet been computed
Memo[0] = 0
function knapsack(Capacity)
   if Memo[ Capacity ] != -1:
     return Memo[Capacity]
   else:
     maxValue = 0
     for i=1 to N
       if Weight[ i ] <= Capacity
          thisValue = Value[i]  + knapsack(Capacity - Weight[ i ] )
          if thisValue > maxValue
             maxValue = thisValue
      Memo[Capacity] = maxValue
      return Memo[Capacity]
```

Bottom up solution:

Values[i] + Memo[Capacity – Weights[i]]

Top Down vs Bottom Up

- Top-down **may** save some computations (E.g., some smaller subproblems may not needed to be solved)
- Space saving trick may be applied for bottom-up to reduce space complexity
- You may find one easier to think about
- In some cases, the solution cannot be written bottom-up without some silly contortions

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0/1 Knapsack Problem

Same as unbounded knapsack except that each item can only be picked at most once.

Example: What is the maximum value for the example given below given capacity is 11 kg?

Answer: $590 (B and D)

Greedy solution may not always work.

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

0/1 Knapsack Problem

Same as unbounded knapsack except that each item can only be picked at most once.

Different from unbounded: If we pick an item X, giving us a remaining capacity R, we have to somehow make sure that X is not part of the optimal solution to our new subproblem of size R

Idea: Lets have two axes on which we think about subproblems.

- Capacity
- Which items are part of the subproblem

0/1 Knapsack Problem

Problem: What is the solution for 0/1 knapsack for items {A,B,C,D} where capacity = 11.

Assume that we have computed solutions for every capacity<=11 considering the items {A,B,C} (see table below).

What is the solution for capacity=11 and set {A,B,C,D} ?

- **Case 1:** the knapsack must **NOT** contain D
  - Solution for 0/1 knapsack with set {A,B,C} and capacity 11.

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

	1	2	3	4	5	6	7	8	9	10	11
{A,B,C}	40	40	40	40	350	390	390	390	390	390	580

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Assume that we have computed solutions for every capacity<=11 considering the items {A,B,C} (see table below).

What is the solution for capacity=11 and set {A,B,C,D} ?

- **Case 1:** the knapsack must **NOT** contain D
  - Solution for 0/1 knapsack with set {A,B,C} and capacity 11 = 580
- **Case 2:** the knapsack **must** contain D
  - The value of item D + solution for 0/1 knapsack with set {A,B,C} and capacity 11-9=2
  - This gives a value of 550+40

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

	1	2	3	4	5	6	7	8	9	10	11
{A,B,C}	40	40	40	40	350	390	390	390	390	390	580

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- Case 1: the knapsack must **NOT** contain D
  - Solution for 0/1 knapsack with set {A,B,C} and capacity 11 = 580
- Case 2: the knapsack **must** contain D
  - The value of item D + solution for 0/1 knapsack with set {A,B,C} and capacity 11-9=2
  - This gives a value of 550+40
- **Solution = max(Case1, Case2)**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

	1	2	3	4	5	6	7	8	9	10	11
{A,B,C}	40	40	40	40	350	390	390	390	390	390	580

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590		

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Value	$230	$40	$350	$550

max(

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
4	D	40	40	40	40	350	390	390	390	550	590		

i

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Item	A	B	C	D
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max(580

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
i 4	D	40	40	40	40	350	390	390	390	550	590		

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(580, 550 + 40)

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
i 4	D	40	40	40	40	350	390	390	390	550	590		

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(580, 550 + 40)

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
i 4	D	40	40	40	40	350	390	390	390	550	590	590	

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Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(620

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 … i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	$230	$40	$350	$550

max(620, 550 + 40)

		1	2	3	4	5	6	7	8	9	10	11	12
0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
1	A	0	0	0	0	0	230	230	230	230	230	230	230
2	B	40	40	40	40	40	230	270	270	270	270	270	270
3	C	40	40	40	40	350	390	390	390	390	390	580	620
i 4	D	40	40	40	40	350	390	390	390	550	590	590	

0/1 Knapsack Problem

Assume we know the optimal solutions for every subproblem and results are stored in Memo[ ][ ]

Memo[i][c] contains the solution of knapsack for **Set[1 ... i]** and capacity **c**

Item	A	B	C	D
Weight	6kg	1kg	5kg	9kg
Value	\$230	\$40	\$350	\$550

max(620, 550 + 40)

			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	620

0/1 Knapsack Problem

Complexity:

- We need to fill the grid
- Filling each cell is O(1) since it is the max of 2 numbers, each of which can be computer in a constant number of lookups
- Therefore, the time and space complexity are both O(NC) where N is the number of items and C is the capacity of the knapsack

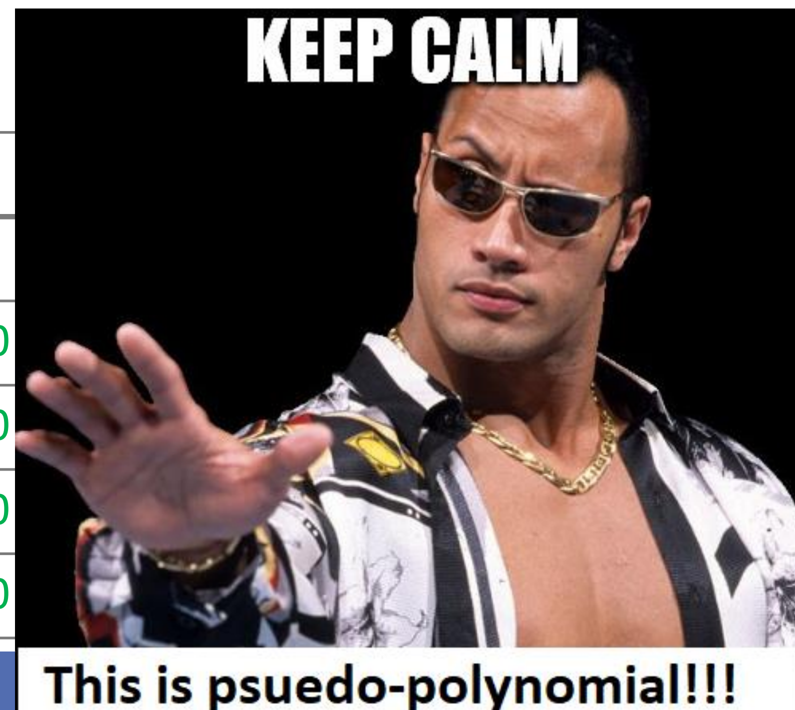
			1	2	3	4	5	6	7	8	9	10	11	12
	0	Φ	0	0	0	0	0	0	0	0	0	0	0	0
	1	A	0	0	0	0	0	230	230	230	230	230	230	230
	2	B	40	40	40	40	40	230	270	270	270	270	270	270
	3	C	40	40	40	40	350	390	390	390	390	390	580	620
i	4	D	40	40	40	40	350	390	390	390	550	590	590	620

0/1 Knapsack Problem

Complexity:

…, each of which can be computer in a

…NC) where N is the number of items and C is



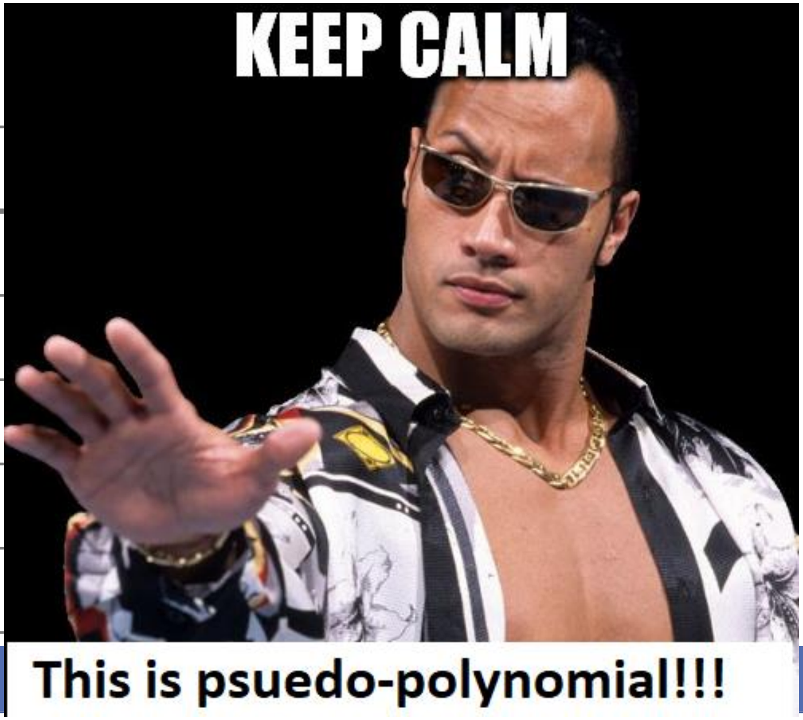
	1	A	0	0	0	0	0	230
	2	B	40	40	40	40	40	230
	3	C	40	40	40	40	350	390
i	4	D	40	40	40	40	350	390

0/1 Knapsack Problem

Complexity:

- Think about how many bits it takes to specify input
- N is the number of items. The N items require O(N) bits to describe
- C is the capacity. C can be described with log(C) bits
- Instead of C, lets talk about B, the number of bits to specify C
- $\log_2 C = B \Rightarrow C = 2^B$
- Now we can say our algorithm runs in O(CN) = O($2^B N$), which is not polynomial in the size of the input (as expected for an NP-complete problem)

		1	2	3	4	5	6
0	Φ	0	0	0	0	0	0
1	A	0	0	0	0	0	230
2	B	40	40	40	40	40	230
3	C	40	40	40	40	350	390
i 4	D	40	40	40	40	350	390



Reducing Space Complexity

- While generating each row, we only need to look at values from the previous row
- So all values from the earlier rows may be discarded
- Reduces space complexity to O(C) (or O(2^B) as we saw)

Note: Space saving not possible for top-down dynamic programming (since we don't know the order we solve subproblems)

		1	2	3	4	5	6	7	8	9	10	11	12
3	C	40	40	40	40	350	390	390	390	390	390	580	620
4	D	40	40	40	40	350	390	390	390	550	590		

Outline

1. Introduction to Dynamic Programming
2. Coins Change
3. Unbounded Knapsack
4. 0/1 Knapsack
5. Edit Distance
6. Constructing Optimal Solution

Edit Distance

- The words computer and commuter are very similar, and a change of just one letter, p → m, will change the first word into the second.
- The word sport can be changed into sort by the deletion of p, or equivalently, sort can be changed into sport by the insertion of p'.
- Notion of editing provides a simple and handy formalisation to compare two strings.
- The goal is to convert the first string (i.e., sequence) into the second through a series of edit operations
- The permitted edit operations are:
  1. insertion of a symbol into a sequence.
  2. deletion of a symbol from a sequence.
  3. substitution or replacement of one symbol with another in a sequence.

Edit Distance

Edit distance between two sequences

- Edit distance is the minimum number of edit operations required to convert one sequence into another

For example:

- Edit distance between computer and commuter is 1
- Edit distance between sport and sort is 1.
- Edit distance between shine and sings is ?
- Edit distance between dnasgivethis and dentsgnawstrims is ?

Some Applications of Edit Distance

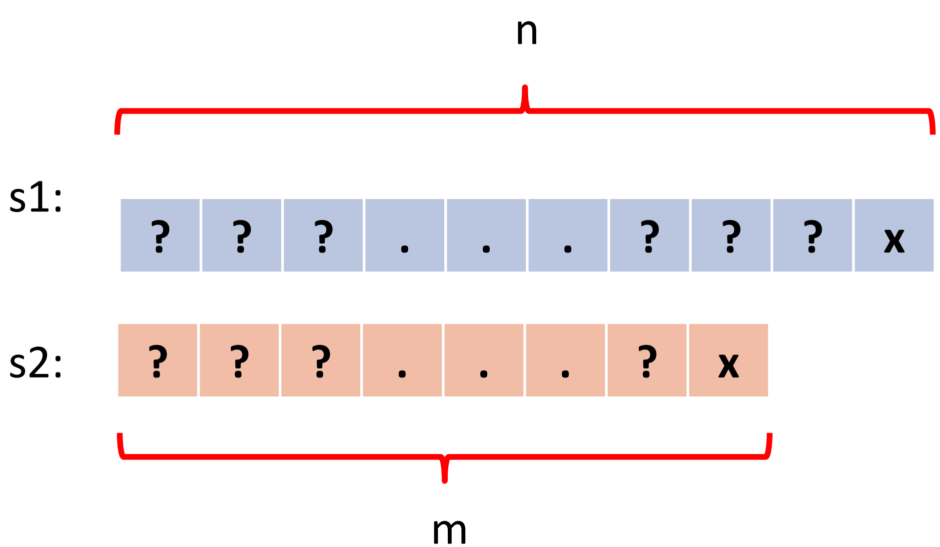
- Natural Language Processing
  - Auto-correction
  - Query suggestions
- BioInformatics
  - DNA/Protein sequence alignment

Computing Edit Distance

We want to convert s1 to s2 containing n and m letters, respectively

To gain an intuition for this problem, lets look at some situations we might run into

This is a good technique in general, try playing around with the problem and see what happens



How much does it cost to turn s1 into s2 if the last characters are the same?

We can leave the last character, and just convert the front part of one string into the front part of the other

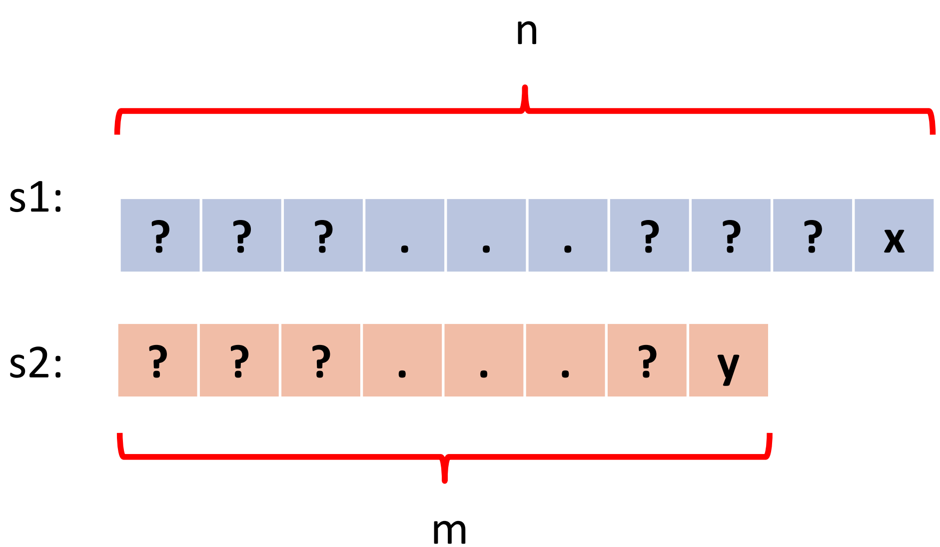
edit(s1[1..n], s2[1..m])
=edit(s1[1..n-1], s2[1..m-1])

Computing Edit Distance

We want to convert s1 to s2 containing n and m letters, respectively

To gain an intuition for this problem, lets look at some situations we might run into

This is a good technique in general, try playing around with the problem and see what happens



How much does it cost to turn s1 into s2 if the last characters are different?

We have some options

Computing Edit Distance

- Remember! We can assume that we have solved ALL subproblems already. In other words, we know
- Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m BUT NOT when i = n AND j = m (since this is the exact problem we are trying to solve)
- Alternatively, we could think about it visually
- In this table, cell[i][j] is the cost of turning s1[1..i] into s2[1..j]

	1	2	3	.	.	.	m
1							
2							
3							
.							
.							
.							
n							

Known:

Unknown:

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m
BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1							
2							
3							
.							
.							
.							
n							

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j])for all i<=n, j<=m

BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1							
2							
3							
.							
.							
.							
n							

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Going up:

Subproblem: edit(s1[1..n-1], s2[1..m])

- First delete s1[n]
- Then turn s1[1..n-1] into s2[1..m]

Total cost:

cost(delete) + edit(s1[1..n-1], s2[1..m])

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m
BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1							
2							
3							
.							
.							
.							
n							

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Going left:

Subproblem: edit(s1[1..n], s2[1..m-1])

- First turn s1[1..n] into s2[1..m-1]
- First insert s2[m] at the end of s2

Total cost:

edit(s1[1..n], s2[1..m-1]) + cost(insert)

Computing Edit Distance

We know:

Edit(s1[1..i], s2[1..j]) for all i<=n, j<=m
BUT NOT i = n AND j = m

Equivalently, we know all the blue cells

	1	2	3	.	.	.	m
1							
2							
3							
.							
.							
.							
n							

From the bottom right corner, there are three things we can do:

- Go up
- Go left
- Go left and up

Going left:
Subproblem: edit(s1[1..n-1], s2[1..m-1])

- Replace s1[n] with s2[m]
- Turn s1[1..n-1] into s2[1..m-1]

Total cost:
edit(s1[1..n-1], s2[1..m-1]) + cost(replace)

Computing Edit Distance

Base cases?

- When one string is empty, the cost is just the length of the other string
- we would have to insert each character in the other string, starting from nothing
- So edit(s1[], s2[1..j]) = j
- edit(s1[1..i], s2[]) = i

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ & \\ & \end{cases}$$

Computing Edit Distance

If the last characters are the same:

- edit(s1[1..n-1], s2[1..m-1])

If the last characters are different, three options:

- cost(delete) + edit(s1[1..n-1], s2[1..m])
- edit(s1[1..n], s2[1..m-1]) + cost(insert)
- edit(s1[1..n-1], s2[1..m-1]) + cost(replace)

We want the minimum cost, so our optimal substructure will be (if all costs are 1)

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

Computing Edit Distance

Overlapping subproblems: Dist[I,j] = cost of operations to turn S[1...i] into S[1…j]

Optimal substructure:

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j=0, \\ j & \text{if } i=0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j] + 1 \\ \text{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ						
S						
I						
N						
G						
S						

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S						
I						
N						
G						
S						

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j=0, \\ j & \text{if } i=0, \\ \min\begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j] + 1 \\ \text{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1					
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1					
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0				
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0				
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1			
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1			
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2		
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2					
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1				
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1,j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1,j] + 1 \\ \text{Dist}[i,j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1			
N	3					
G	4					
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1			
N	3					
G	4		Now	you	Try!	
S	5					

Example

$$\text{Dist}[i,j] = \begin{cases} i & \text{if } j = 0, \\ j & \text{if } i = 0, \\ \min \begin{cases} \text{Dist}[i-1, j-1] + 1_{S_1[i] \neq S_2[j]} \\ \text{Dist}[i-1, j] + 1 \\ \text{Dist}[i, j-1] + 1 \end{cases} & \text{otherwise} \end{cases}$$

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Outline

1. Introduction to Dynamic Programming
2. Coins Change
3. Unbounded Knapsack
4. 0/1 Knapsack
5. Edit Distance
6. Constructing Optimal Solution

Constructing optimal solutions

- The algorithms we have seen determine optimal values, e.g.,
  - Minimum number of coins
  - Maximum value of knapsack
  - Edit distance
- How do we construct optimal solution that gives the optimal value, e.g.,
  - The coins to give the change
  - The items to put in knapsack
  - Converting one string to the other
- There may be multiple optimal solutions and our goal is to return just one solution!
- Two strategies can be used.
  1. Create an additional array recording decision at each step
  2. Backtracking

Decision Array

- Make a second array of the same size
- Each time you fill in a cell of the memo array, record your decision in the decision array
- Remember, going right (or coming from the left) is insert
- Going down (or coming from up) is delete
- Going down and right (or coming from up and left) is replace OR do nothing

	Φ	S	H	I	N	E
Φ	0					
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S				
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1				
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H			
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2			
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I		
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2	3		
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S						
I						
N						
G						
S						

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S						
I						
N						
G						
S						

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S					
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1					
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing				
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0				
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H			
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1			
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I		
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2		
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I					
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2					
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I				
N	Delete N					
G	Delete G					
S	Delete S					

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1				
N	3					
G	4					
S	5					

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	Choice?	Choice?	Do nothing	Insert E
G	Delete G	Delete G	Choice?	Choice?	Delete G	replace G, E
S	Delete S	Delete S	Choice?	Choice?	Delete S	Choice?

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Decision Array

	Φ	S	H	I	N	E
Φ	null	Insert S	Insert H	Insert I	Insert N	Insert E
S	Delete S	Do nothing	Insert H	Insert I	Insert N	Insert E
I	Delete I	Delete I	replace I,H	Do nothing	Insert N	Insert E
N	Delete N	Delete N	replace N,H	Delete N	Do nothing	Insert E
G	Delete G	Delete G	Delete G	Delete G	Delete G	replace G, E
S	Delete S	Delete S	Delete S	Delete S	Delete S	Delete S

Sequence of operations:
Delete S (from position 5)
replace G with E (at position 4)
insert H (at position 2)

- SINGS
- SING
- SINE
- SHINE

Backtracking

- Start in bottom right
- Are the letters the same?

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Start in bottom right
- Are the letters the same?
- No

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Start in bottom right
- Are the letters the same?
- No
- From recurrence…
- We know that our current value (3) was obtained from any of the three previous cells by adding 1

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Start in bottom right
- Are the letters the same?
- No
- From recurrence…
- We know that our current value (3) was obtained from any of the three previous cells by adding 1
- So our options are up or up-and-left
- Choose one arbitrarily

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue from our new cell (but remember the path)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue from our new cell (but remember the path)
- Letters are different

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue from our new cell (but remember the path)
- Letters are different
- Must have come from the cell above

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Letter are the same
- From our recurrence, we know
  - IF our value is the same as the up-and-left cell, then we could have came from there
  - IF our value is one more than the up cell or the left cell, then we could have come from there
- In this case, we came from up-and-left

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Letter are the same
- From our recurrence, we know
  - IF our value is the same as the up-and-left cell, then we could have came from there
  - IF our value is one more than the up cell or the left cell, then we could have come from there
- In this case, we came from up-and-left

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4]

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing

	Φ	**S**	**H**	**I**	**N**	**E**
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing, insert(2, H)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing, insert(2, H), nothing

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking

- Continue in this way
- When you reach the top left cell, you are done
- So the sequence of operations is
- Replace(5, E), delete[4], nothing, nothing, insert(2, H), nothing
- SINGS
- SINGE (replace position 5 with E)
- SINE   (delete G in position 4)
- SHINE (insert H at position 2)

	Φ	S	H	I	N	E
Φ	0	1	2	3	4	5
S	1	0	1	2	3	4
I	2	1	1	1	2	3
N	3	2	2	2	1	2
G	4	3	3	3	2	2
S	5	4	4	4	3	3

Backtracking Vs Decision array?

- Space usage
  - Backtracking requires less space as it does not require creating an additional array
  - However, space **complexity** is the same
- Efficiency
  - Backtracking requires to **determine** what decision was made which costs additional computation
  - However, time **complexity** is the same
- Note the space saving tricks discussed for 0/1 knapsack and edit distance can only be used when solution is not to be constructed
  - e.g., all rows are needed for backtracking, and all rows must be stored for 2D-decision array

Concluding Remarks

Dynamic Programming Strategy

- Assume you already know the optimal solutions for all subproblems and have memoized these solutions
- Observe how you can solve the original problem using this memoization
- Iteratively solve the sub-problems and memoize

Things to do (this list is not exhaustive)

- Practice, practice, practice
  - http://www.geeksforgeeks.org/tag/dynamic-programming/
  - https://www.topcoder.com/community/data-science/data-science-tutorials/dynamic-programming-from-novice-to-advanced/
  - http://weaklearner.com/problems/search/dp
- Revise hash tables and binary search tree

Coming Up Next

- Hashing, Binary Search Tree, AVL Tree