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## Survey Paper

# Variable Structure Systems with Sliding Modes

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**Abstract**—Variable structure systems consist of a set of continuous subsystems together with suitable switching logic. Advantageous properties result from changing structures according to this switching logic. Design and analysis for this class of systems are surveyed in this paper.

### I. INTRODUCTION

THE basic philosophy of the variable structure approach is simply explained by contrasting it with the linear state regulator design for the single-input system

$$\dot{x} = Ax + bu.$$

In the linear state regulator design, the structure of the state feedback is fixed as

$$u = k^T x$$

where the constant parameters are chosen according to various design procedures, such as eigenvalue placement or quadratic minimization. In variable structure systems the control is allowed to change its structure, that is, to switch at any instant from one to another member of a set of possible continuous functions of the state. The variable structure design problem is then to select the parameters of each of the structures and to define the switching logic.

The idea of changing a structure is a natural one, and early utilization of this approach can be found in the papers published about 20 years ago [18], [19], [26], [48], [50], [59], [61], [65], [66], [75], [88], [90]. A reward for introducing this additional complexity is the possibility to combine useful properties of each of the structures. Moreover, a variable structure system can possess new properties not present in any of the structures used. For instance, an asymptotically stable system may consist of two structures neither of which is asymptotically stable. This possibility is illustrated by some early examples, which stimulated interest in variable structure systems (VSS). Although very simple, two such examples are quoted here because they present the advantages of changing structures during a control phase.

In the first example we consider a second-order system

$$\ddot{x} = -\Psi x$$

having two structures defined by  $\Psi = \alpha_1^2$  and  $\Psi = \alpha_2^2$ , where  $\alpha_1^2 > \alpha_2^2$ . The phase portrait consists of families of ellipses [Fig. 1(a), (b)] and hence, neither structure is asymptotically stable. However, asymptotic stability is achieved if the structure of the system is changed on the coordinate axes, that is, if the switching logic is

$$\Psi = \begin{cases} \alpha_1^2, & \text{if } x\dot{x} > 0 \\ \alpha_2^2, & \text{if } x\dot{x} < 0. \end{cases}$$

The resulting phase portrait is shown in Fig. 1(c).

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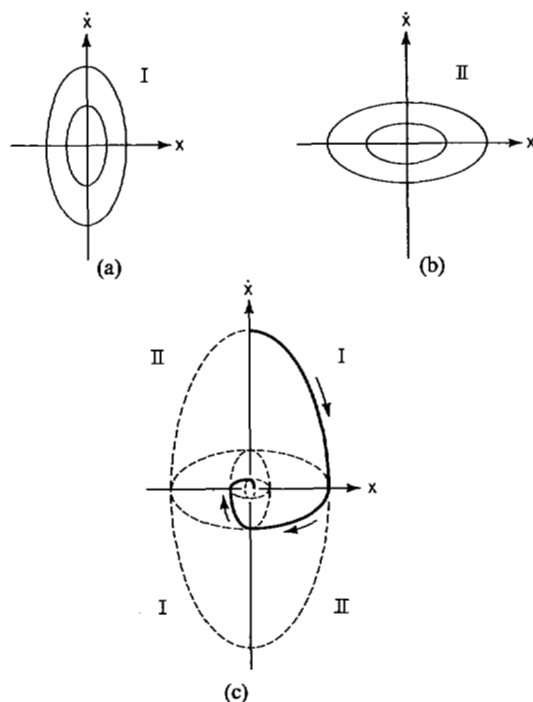


Fig. 1. Asymptotically stable VSS consisting of two stable structures.

In the second example, the system

$$\ddot{x} - \xi \dot{x} + \Psi x = 0, \quad \xi > 0$$

is considered, where the linear structure corresponds to negative and positive feedback when  $\Psi$  is equal to either  $\alpha > 0$  or to  $-\alpha$ . Both structures are unstable [Fig. 2(a), (b)]. Note that the only motion converging to the origin is along the stable eigenvector of the structure with  $\Psi = -\alpha$ . If the switching occurs on this line and on  $x=0$  with the switching law

$$\Psi = \begin{cases} \alpha, & \text{if } xs > 0 \\ -\alpha, & \text{if } xs < 0, \end{cases} \quad \text{where } s = cx + \dot{x},$$

$$c = \lambda = -\frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} + \alpha}$$

then the resulting VSS will be asymptotically stable.

In the above examples, new system properties are obtained by composing a desired trajectory from the parts of trajectories of different structures. An even more fundamental aspect of VSS is the possibility to obtain trajectories not inherent in any of the structures. These trajectories describe a new type of motion—the so-called sliding mode.

To show how such motion occurs let us reconsider the second example using  $0 < c < \lambda$  instead of  $c = \lambda$  (Fig. 3). The phase trajectories are directed towards the switching line  $s = cx + \dot{x} = 0$  and hence once on this line the state must remain on it. The motion along a line which is not a trajectory of any of the structures is called the sliding mode.

The equation

$$\dot{x} + cx = 0$$

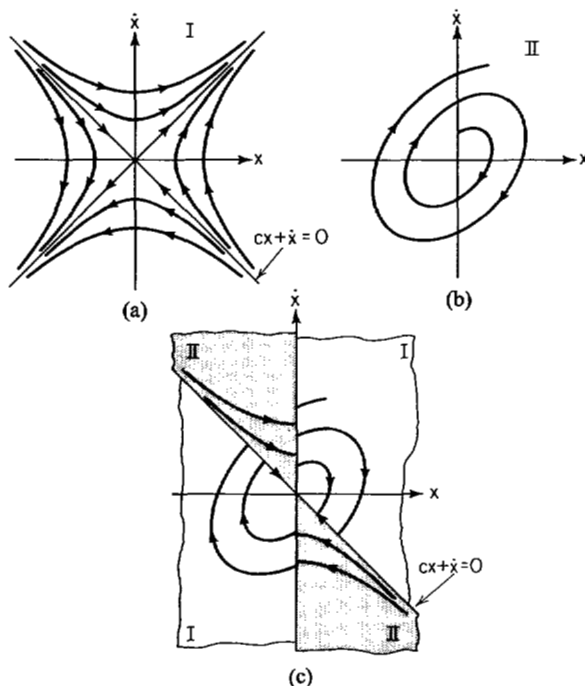


Fig. 2. Asymptotically stable VSS consisting of two unstable structures.

determines the behavior of the system in the sliding mode. It is useful to note that this behavior depends on the parameter  $c$ . This invariance with respect to plant parameters and disturbances is of extreme importance when controlling time-varying plants or treating disturbance rejection problems.

In Section II the properties revealed in these simple examples are utilized to design VSS in phase canonic form. Methods developed to analyze VSS as differential equations with discontinuous right-hand sides are surveyed in Section III. In Section IV, general design principles for multiinput variable structure systems are discussed.

## II. VSS IN PHASE CANONIC FORM

In this section we consider the design of variable structure controllers for zeroing the output  $y = x_1$  of the system

$$\dot{x}_i = x_{i+1}, \quad i = 1, \dots, n-1$$

$$\dot{x}_n = -\sum_{i=1}^n a_i x_i + f(t) + u \quad (1)$$

where  $u$  is control,  $f(t)$  is a disturbance,  $a_i$  are constants or time-varying parameters,  $f(t)$ ,  $a_i$  may be unknown.

Suppose that  $u$  as a function of the state vector  $x$  undergoes discontinuities on some plane  $s=0$ , where

$$s = \sum_{i=1}^n c_i x_i, \quad c_i = \text{const}, \quad c_n = 1. \quad (2)$$

Then the velocity vector undergoes discontinuities in the same plane. As in the second-order example of Section I, if the trajectories are directed towards the plane, an  $s=0$

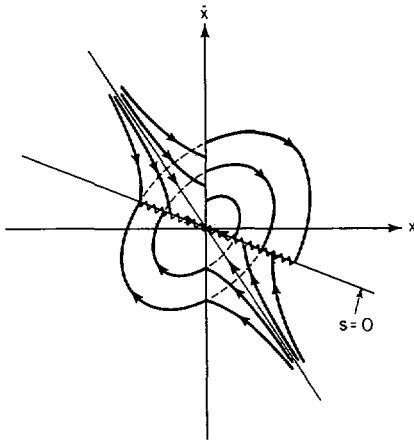


Fig. 3. Sliding mode in a second-order VSS.

sliding mode will appear in this plane. The pair of inequalities

$$\lim_{s \rightarrow -0} \dot{s} > 0 \text{ and } \lim_{s \rightarrow +0} \dot{s} < 0 \quad (3)$$

are a sufficient condition for sliding mode to exist [6]. To prove the invariance of the sliding mode with respect to the plant parameters  $a_i$  and the disturbance  $f(t)$ , we solve the equation  $s=0$  for the variable  $x_n$  and substitute into (1). The resulting equations of the sliding mode

$$\begin{aligned} \dot{x}_1 &= x_{i+1}, \quad i=1, \dots, n-2 \\ \dot{x}_{n-1} &= -\sum_{i=1}^{n-1} c_i x_i \end{aligned} \quad (4)$$

depend only on parameters  $c_i$  [28]. Implications and uses of this result have been the subject of many early works summarized in [25], [40].

A design procedure based on the invariance property can be outlined as follows. First, the desired sliding mode is formed by a choice of the parameters  $c_i$ . Second, a discontinuous control is found which guarantees the existence of sliding modes at every point of the plane  $s=0$ . Such a plane will be referred to as a sliding plane. Third, the control must steer the state to the sliding plane. This approach is now applied to several control problems.

#### Time-Invariant Plants

Let the parameters of the plant  $a_i$  be constant and  $f(t)=0$ . The problem is to force the state to zero. Analogous to what was done with the second-order VSS, the control  $u$  is chosen as a piecewise linear function of  $x$  with discontinuous coefficients

$$u = -\sum_{i=1}^k \Psi_i x_i - \delta_0 \text{sgns}, \quad 1 \leq k \leq n-1 \quad (5)$$

$$\Psi = \begin{cases} \alpha_i, & \text{if } x_i s > 0 \\ \beta_i, & \text{if } x_i s < 0 \end{cases} \quad \text{sgns} = \begin{cases} +1, & \text{if } s > 0 \\ -1, & \text{if } s < 0 \end{cases}$$

$\alpha_i, \beta_i, \delta_0 = \text{const},$   
 $\delta_0$  is small positive scalar.

A necessary and sufficient condition for a sliding plane to exist [9], [37], [40] is

$$\begin{aligned} \alpha_i &\geq c_{i-1} - a_i - c_i c_{n-1} + c_i a_n \\ \beta_i &\leq c_{i-1} - a_i - c_i c_{n-1} + c_i a_n, \quad i=1, \dots, k, c_0=0 \quad (6) \\ \frac{c_{i-1} - a_i}{c_i} &= c_{n-1} - a_n, \quad i=k+1, \dots, n-1. \end{aligned}$$

Hence, coefficients  $c_i$  needed for the design of a desired sliding mode (4) cannot be chosen freely. The inequalities in (6) may be satisfied by a proper choice of  $\alpha_i$  and  $\beta_i$  but the equalities represent  $(n-k-1)$  constraints for  $(n-1)$  coefficients  $c_i$ . These constraints vanish only for  $k=n-1$ .

For  $k < n-1$  a class of linear plants has been found [5], [38], [51], [40] for which a sliding plane with stable motion exists. These plants are characterized by the following theorems.

**Theorem 1:** Let  $\lambda_1, \dots, \lambda_n$  be eigenvalues of the system (1) with

$$u = \sum_{i=1}^k \Omega_i x_i, \quad \Omega_i = c_{i-1} - a_i - c_i c_{n-1} + c_i a_n. \quad (7)$$

The sliding mode in a sliding plane is asymptotically stable in systems (1), (5) if and only if  $\text{Re} \lambda_i < 0$ ,  $i=1, \dots, n-1$ ; one of the eigenvalues  $\lambda_n$  is equal to  $c_{n-1} - c_n$  and may be arbitrary.

**Theorem 2:** For the sliding mode in a sliding plane to be asymptotically stable it is sufficient that for the  $(n-k+1)$ th-order VSS

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i=k, \dots, n-1 \\ \dot{x}_n &= -\sum_{i=k}^n a_i x_i - \Psi'_k x_k - \delta'_0 \text{sgns}' \end{aligned} \quad (8)$$

$$\Psi'_k = \begin{cases} \alpha'_k, & \text{if } x_k s' > 0 \\ \beta'_k, & \text{if } x_k s' < 0, \end{cases} \quad s' = \sum_{i=k}^n c'_i x_i, \alpha'_k, \beta'_k, c'_i, \delta'_0 = \text{const}$$

a sliding plane  $s'=0$  with asymptotically stable sliding mode exists or, equivalently, that the condition of Theorem 1 for the truncated system (8) is fulfilled.

Theorems 1 and 2 show that the motion in a sliding plane may be stable even when none of the  $2^k$  employed structures is stable. This is particularly clear for  $k=n-2$ . In this case, (8) is a third-order VSS, whose control is a piecewise linear function of only one variable  $x_{n-2}$ . There exists  $\Omega_{n-2}$  such that the characteristic equation of this truncated system with  $\Psi'_{n-2} = \Omega_{n-2}$  is

$$\lambda^3 + a_n \lambda^2 + a_{n-1} \lambda + (a_{n-2} + \Omega_{n-2}) = 0$$

which has only one eigenvalue with nonnegative real part. It follows from Theorem 1 that the motion in a sliding plane for (8) is asymptotically stable. On the other hand, Theorem 2 insures the asymptotic stability of the sliding mode in a sliding plane for the original system (1), (5) when  $k=n-2$ . If either  $a_{n-1}$  or  $a_n$  is negative none of the available structures is stable.

Asymptotic stability as determined by Theorems 1 and 2 is usually not the only requirement to be met by the desired sliding mode. In [52] the response time is reduced by placing the eigenvalues as far to the left as possible providing a sliding plane exists. Under the same existence constraint sliding plane parameters minimizing a quadratic performance index are found [14]. Time-optimal VSS of the second order were considered in [73].

Finally, it is necessary to guarantee that the designed sliding plane is reached from all initial states. Various "reaching conditions" have been proposed throughout the development of the VSS theory. The following necessary conditions were formulated in [37].

**Theorem 3:** For the state to reach  $s=0$  defined by (2) with  $c_i > 0$ , it is necessary that all real eigenvalues of the systems (1), (5) with  $\Psi_i = \alpha_i$  ( $i=1, \dots, k$ ) be nonnegative.

Note that  $c_i > 0$  is needed since the sliding mode governed by (4) must be asymptotically stable. Note also that (1), (5) with  $\Psi_i = \alpha_i$  is permitted to be unstable due to complex eigenvalues with positive real parts. It was shown in [12] and [43] that the condition of Theorem 3 is also sufficient for second- and third-order VSS, respectively. Two theorems concerning sufficient conditions are given below.

**Theorem 4 [6]:** For the state to reach a sliding plane it is sufficient that

$$c_{n-1} - a_n \leq 0.$$

**Theorem 5 [13]:** For the state to reach the plane  $s=0$  with  $c_i > 0$  it is sufficient that

$$k = n-1, \alpha_i \geq -a_i, \beta_i \leq -a_i, \quad (i=1, \dots, n-1)$$

and that the system (1), (5) with  $\Psi_i = \alpha_i$  ( $i=1, \dots, n-1$ ) has no nonnegative eigenvalues.

The condition of Theorem 4 is easily verified but it is valid only for positive  $a_n$  since  $c_{n-1} > 0$ . The most effective reachability result is the following theorem.

**Theorem 6 [15], [16]:** The necessary condition of Theorem 3 is also sufficient if the plane  $s=0$  is a sliding plane and the sliding mode is asymptotically stable.

An estimate of the time needed to reach the plane  $s=0$  is given in [23].

To summarize, the design procedure begins with the determination of a continuous control  $u = -\sum_{i=1}^k \Omega_i x_i$  placing  $(n-1)$  eigenvalues in desired locations. Then the control (5) assures the existence of the sliding plane with the sliding mode determined by these  $(n-1)$  eigenvalues [82]. Reaching conditions are met by increasing  $\alpha_i$ . This will eliminate nonnegative eigenvalues when  $\Psi_i = \alpha_i$  and thus satisfy Theorem 6.

For instance in the third-order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

with  $u = -\Omega_1 x_1$ ,  $\Omega_1 < 0$  two of three eigenvalues have negative real parts. This means that variable structure control  $u = -\Psi_1 x_1$  assure the existence of a sliding plane with asymptotically stable motion. Indeed, the condition

(6) and asymptotic stability of the system (4) are met if

$$s = c_1 x_1 + c_2 x_2 + x_3, \quad c_1 = c_2^2, \quad \alpha_1 > -c_1 c_2, \\ \beta_1 < -c_1 c_2, \quad c_1 > 0, \quad c_2 > 0.$$

The reaching condition of Theorem 6 is fulfilled if  $\alpha_1 > 0$ . Note that none of the controls which are linear functions of  $x_1$  (even  $x_1$  and  $x_2$ ) reduces the state to zero.

The above design principle constrains choice of parameter  $c_i$  (6) and hence it restricts the variety of sliding modes. VSS without sliding planes are difficult to analyze and their properties are established only for a few special cases [6], [10]. They are also studied with the help of approximation methods [74], [92].

When the system equation depends not only on control but also on its time-derivatives, control discontinuities result in discontinuous trajectories. Design principles have been modified for this class of linear plants [31], [33], [70]. Additional possibilities appear if the coefficients  $\alpha_i, \beta_i, c_i$  in control (5) can be made infinitely large, since then the order of a sliding mode equation may be reduced [6]–[8]. VSS have been applied to linear plants with pure delay [4], [27], [89], nonlinear plants with multiple stable and unstable equilibrium states [30], and plants with state variable constraints [3], [40], [60].

In VSS discussed here the control is a function of the output and its derivatives. In practice pure derivatives are not available. VSS with estimators have been studied [35], [77], and variable structure estimators (filters with piecewise constant parameters) have been developed [24], [36], [39], [78].

### Time-Varying Plants

The variable structure approach to plants with varying parameters  $a_i(t)$  is based on the invariance of the sliding mode with respect to parameter variations [11], [29], [34], [42], [44]. Parameters  $a_i(t)$  are assumed to be inaccessible for measurement and are arbitrarily varying in some known ranges:

$$a_{i\min} \leq a_i(t) \leq a_{i\max}, \quad i=1, \dots, n.$$

The problem is to steer to zero the state of the system without disturbance ( $f(t)=0$ ).

Again the control (5) is to guarantee both the existence of a sliding plane with asymptotically stable sliding mode and the reaching conditions. The equality constraints for  $c_i$  in (6) cannot be satisfied when parameters  $a_i$  are unknown. It has already been mentioned that these constraints vanish if  $k=n-1$ . Then the  $(n-1)$  inequalities in (6) can always be satisfied for any  $c_i$  and bounded parameters  $a_i(t)$  with

$$\alpha_i \geq \sup_t [c_{i-1} - a_i(t) - c_i c_{n-1} + c_i a_{n-1}(t)]$$

$$\beta_i \leq \inf_t [c_{i-1} - a_i(t) - c_i c_{n-1} - c_i a_{n-1}(t)], \quad i=1, \dots, n-1.$$

When the control (5) is a piecewise linear function of  $(n-1)$  variables the inequalities (9) are necessary and sufficient conditions for a sliding plane to exist [42]. Then there are no constraints on  $c_i$  and any desirable sliding mode (4) can be designed.

The reaching conditions for time-varying VSS have been obtained as a result of generalization of the theorems for time-invariant systems. The reaching condition for time-varying second-order systems [12] is that the reaching condition for time-invariant systems be satisfied for every fixed set of parameters from the range of their variation. Theorem 4 holds if  $c_{n-1} - a_n(t) \leq 0$  holds for any  $a_n(t)$  [84]. The sufficient conditions for time-varying VSS of arbitrary order given in [13] are

$$c_i > 0, \alpha_i \geq -a_i(t), \beta_i \leq -a_i(t), a_n(t) \geq 0.$$

An implication is that the coefficients  $a_n(t)$  is either greater than  $c_{n-1}$  (which is always positive) or positive. For the VSS with an arbitrary  $a_n(t)$  reaching conditions for third- and higher order systems are formulated in [10] and [40] and [41], respectively. According to these results there exists a positive scalar  $\alpha_0$  such that the state reaches the switching plane  $s=0$  if  $\alpha_1 \geq \alpha_0$  and  $\beta_1 \leq -\alpha_0$ . Thus, for VSS with control (5) and  $k=n-1$  the existence of a sliding plane which is reached can always be guaranteed.

When the plant parameters  $a_i(t)$  are not equal to their extreme values, the coefficients  $c_i$  may be varied without violation of the conditions (9). As the coefficients  $c_i$  determine the behavior of the system in sliding mode these degrees of freedom may be utilized to improve the dynamic characteristics of the motion in sliding mode. The design methods for adaptive VSS based on this idea are developed in [40] and [69].

### Rejection of Disturbances

Let us now consider the case when  $f(t) \neq 0$  in (1) and is not accessible for measurement. Let  $x_1$  be the difference between some reference input and the system output which is to be steered to zero together with its  $n-1$  derivatives. The idea is to introduce a sliding mode for disturbance rejection. This should be possible since the sliding mode is described by the homogeneous differential equation (4) whose solution tends to zero, if all the  $c_i$  satisfy Hurwitz conditions. It is easily seen that the control (5) is not suitable for this task, since at  $x=0$  it is close to zero,  $u=\delta$ . To introduce an additional control term counteracting the disturbance  $f(t)$  we note from the system diagram in Fig. 4 that the output  $y$  of the servomechanism (SM) should coincide with  $-f(t)$ . (Without loss of generality it is assumed that the reference input  $g(t)=0$ .) If  $u$  is a piecewise linear function not only of  $x$  but also of the state variables of SM, that is, if

$$u = - \sum_{i=1}^k \Psi_i x_i - \delta - \sum_{i=0}^{m-1} \Psi_i^y y^{(i)} \quad (10)$$

$$\Psi_i^y = \begin{cases} \alpha_i^y, & \text{if } y^{(i)} s > 0 \\ -\beta_i^y, & \text{if } y^{(i)} s < 0, \end{cases}$$

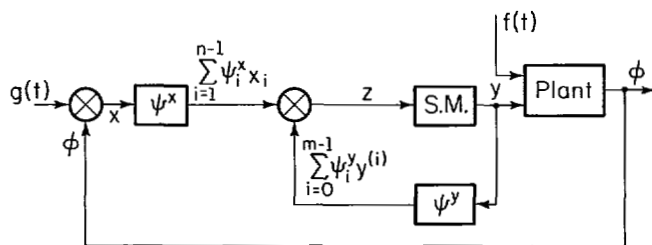


Fig. 4. Disturbance rejection in VSS.

then in the desired state  $x_i=0$  the control is of the same order of magnitude as  $f(t)$  and it is able to maintain the sliding mode. Here  $\Psi_i, \delta, s$  are the same functions as in (5), and  $m$  is the order of the differential equation for SM. It should be noted that the output of SM is usually available for measurement.

The conditions of the existence of a sliding plane, the reaching conditions for time-invariant and time-varying VSS and different types of disturbances have been established in [44], [67], and [79]–[81]. In these references it is shown that the control (11) is able to reject the disturbances and solve a tracking problem with zero error for disturbances from the class

$$\left| \frac{d^m f}{dt^m} \right| \leq B \sum_{i=0}^{m-1} \left| \frac{df}{dt^i} \right|, \quad B = \text{const} \quad (11)$$

which encompasses exponential, harmonic functions, polynomials, etc. The implementation of the control law (10) does not require knowledge of the disturbance  $f(t)$  nor the plant parameters. Only the parameter  $B$  and ranges of plant parameter variations are needed. To clarify the nature of this type of VSS suppose that the transfer function of SM has  $m$  poles at the origin. Then its behavior for  $x_i=0$  is governed by

$$y^{(m)} = - \sum_{i=0}^{m-1} \Psi_i^y y^{(i)} - \delta. \quad (12)$$

When a sliding mode occurs all the coefficients  $\Psi_i^y$  oscillate at high frequency and to find the solution of (12) these coefficients should be replaced by average values  $\bar{\Psi}_i^y$  of  $\Psi_i^y$  such that  $-\beta_i^y \leq \bar{\Psi}_i^y(t) \leq \alpha_i^y$ .<sup>1</sup> If  $\alpha_i^y = B$  and  $\beta_i^y = -B$  then the output of SM belongs to the class (11) and therefore SM is able to generate the function coinciding with  $f(t)$ . This will occur automatically if sliding mode exists. Similar design principles assuming that the parameters of the disturbance model are known using continuous control can be found in [21] and [52].

When disturbances can be measured, the control is a piecewise linear function of the state and disturbances [22], [45], [46], [71]. An important property of these "combined" VSS is that they are much less sensitive to the inaccuracy of the measurement of disturbances than disturbances rejection schemes based on a compensation principle [58].

<sup>1</sup>This consideration is a qualitative one and the physical meaning of an average value of control in a sliding mode will be discussed in the next section.

Rejection of disturbances has much in common with the control of interacting systems. To provide decoupling, interactions are considered as disturbances and rejected by the above methods. Moreover, a desirable interaction between subsystem outputs may be established [68] by letting the equation of a sliding plane for each subsystem depend on the state variables of the others.

The design methods described in Section III have been worked out only for single-input systems. Another limitation for these methods results from the necessity to have high-order derivatives of the output. Since all real differentiators have denominators in transfer functions, the system representation (1) may be an inadmissible idealization.

### III. SLIDING MODES IN DISCONTINUOUS DYNAMIC SYSTEMS

We now consider VSS of a general type described by

$$\dot{x} = f(x, t, u) \quad (13)$$

where the  $x \in R^n$ ,  $u \in R^m$ ,  $f \in R^n$ . Each component of control vector may be equal to one of the set of continuous functions of  $x$  and  $t$ . Analogous to what was done for VSS with scalar control each component of control is assumed to undergo discontinuity on an appropriate surface in the state space,

$$u_i = \begin{cases} u_i^+(x, t), & \text{if } s_i(x) > 0 \\ u_i^-(x, t), & \text{if } s_i(x) < 0 \end{cases} \quad i = 1, \dots, m. \quad (14)$$

The design problem consists of choosing the continuous functions  $u_i^+$ ,  $u_i^-$  and  $m$ -dimensional vector  $s$  ( $s \in R^m$ ) with the functions  $s_i(x)$  as components.

Suppose that the state of the system (13), (14) has reached the intersection of the discontinuity surfaces. In certain cases the state would inevitably return to this manifold should any small deviation occur. As is seen in Fig. 5 such a situation can arise even in the case when sliding mode does not exist on each of the surfaces taken separately. Having reached the intersection the state will stay in this manifold. Such a motion along the intersection of a number of discontinuity surfaces will also be referred to as sliding mode. A sliding mode of this type is a new phenomenon and an interesting subject for investigation. From a practical point of view we are interested in utilization of this motion for the design of control systems. The methods for the study of sliding modes in discontinuous systems and the design of systems with discontinuous controls will be considered in the remaining part of the paper.

The two problems treated in this section are the derivation of the differential equations of sliding mode and conditions for the existence of sliding mode. (For  $m=1$  the inequalities (3) provide existence conditions and the equations (4) describe the sliding mode when the discontinuity surface is chosen in the space of phase variables.)

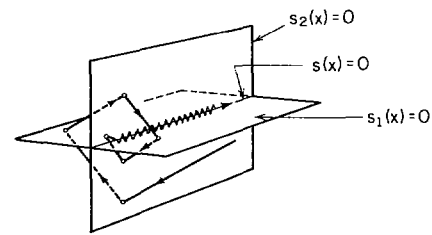


Fig. 5. Sliding mode in the intersection of discontinuity surfaces.

#### Differential Equations of Sliding Mode

The differential equations (13) and (14) do not formally satisfy the classical theorems on the existence and uniqueness of the solutions, since they have discontinuous right-hand sides. Moreover, the right-hand sides usually are not defined on the discontinuity surfaces. A series of works simply postulate the equations of sliding mode [64], [72], [82], [85]–[87].

These mathematical difficulties arise because nonidealities of real systems are neglected in their models. To treat the sliding mode carefully, various small nonidealities of time-delay, hysteresis, and other types should be taken into consideration. These nonidealities determine the real sliding mode, that is the motion in the vicinity of the discontinuity surfaces. Loosely speaking, as the nonidealities tend to zero this motion tends to the ideal sliding mode. If different types of nonidealities and different limiting processes result in the same sliding mode equations, then it is reasonable to consider them as correct descriptions of the ideal sliding mode. In the opposite case, we have to admit that (13) and (14) describing the system outside the discontinuity surfaces do not allow an unambiguous formulation of the equations for ideal sliding modes on these surfaces. Such limiting processes will always be used as the criterion of the validity of sliding mode equations.<sup>2</sup>

A formal technique named the *equivalent control method* [83] will be used for finding equations of ideal sliding modes. In this technique a time derivative of the vector  $s(x)$  along the system trajectory of (13) is set equal to zero, and the resulting algebraic system is solved for the control vector. This “equivalent control” (if it exists) is substituted into the original system. The resulting equations are the equations of ideal sliding mode. From a geometric point of view the above method means finding a continuous control which directs the velocity vector along the intersection of the discontinuity surfaces.

This formal technique is now substantiated for systems linear with respect to control,

$$\dot{x} = f(x, t) + B(x, t)u \quad (15)$$

where  $u$  is determined by (14). The equivalent control method results in the sliding mode equations for the system (14), (15),

$$\dot{x} = f - B(GB)^{-1}Gf, \quad (16)$$

<sup>2</sup>This approach was applied to the system with scalar control and nonidealities of hysteresis or time-delay type in [1], [2], [17], and [63].

where  $m \times n$ -dimensional matrix  $G = \partial s / \partial x$  and  $\det(GB)$  is assumed to be different from zero. Introduction of nonidealities into an ideal model would result in the appearance of real sliding modes. For example in the scalar control case nonidealities of time-delay or hysteresis type lead to oscillations in some vicinity of a switching surface [1], [2], [17], [63]. The amplitude and frequency of these oscillations depend on the value of the time-delay and hysteresis loop width.

In the vector case when an ideal sliding mode occurs in an  $(n-m)$ -dimensional manifold  $s=0$ , these nonidealities allow the trajectories in state space to occur in some  $\Delta$ -vicinity of this manifold. Let the behavior of the system with nonidealities be described by the equation

$$\dot{x} = f(x, t) + B(x, t)\tilde{u}. \quad (17)$$

Suppose that all the nonidealities are taken into account by the control  $\tilde{u}$ . Their nature is not specified,  $\tilde{u}$  is only known to guarantee the motion in a domain  $\|s\| \leq \Delta$  and the existence of the solution of (17). For this real sliding mode the following theorem holds.

**Theorem 7 [83]:** If for any finite interval of time the solution  $x(t)$  of (17) is such that  $\|s\| \leq \Delta$ , then for this interval,  $\lim_{\Delta \rightarrow 0} x(t) = x^*(t)$ , where  $x^*(t)$  is the solution of (15) resulting from the equivalent control method.<sup>3</sup>

In [83] a class of the systems is given for which this theorem is true for an infinite interval of time. Theorem 7 serves as a substantiation of the validity of equivalent control method for the systems which are linear with respect to discontinuous control.

For singular cases when  $\det(GB)=0$  the equivalent control is either not unique or does not exist. The first situation may result in both unique equations of sliding mode or in a variety of these equations depending on the types of nonidealities and limiting processes. When equivalent control does not exist sliding modes cannot appear, that is, the state leaves the intersection of discontinuity surfaces.

For the systems (13) which are nonlinear with respect to control, even if an equivalent control exists and is unique, in general, the differential equations for sliding modes are not unique and depend on the types of nonidealities and limiting processes. All these facts are established in [83], where also the reasons for ambiguity are shown and the systems of the type (13) with unique equations for sliding modes are delineated.

Sliding modes in discontinuous systems are analyzed with the help of an auxiliary continuous equivalent control. It is of interest to show the physical meaning of this function. In sliding modes each component of control may be considered as a function consisting of a low frequency, or average, component, and a high frequency component. The behavior of the system primarily depends on the average rather than on the high frequency component. Since the equivalent control does not have a high-

frequency component, it is reasonable to expect that the original control without a high-frequency component is to be close to the equivalent control. In [83] this suggestion has been verified using first-order low-pass filters whose time-constants tend to zero slower than the nonidealities. The proof consists in showing that if the filters input is  $\tilde{u}$ , their output tends to  $u_{eq}$ .

### Existence of a Sliding Mode

The condition (3) for the existence of a sliding mode in discontinuous systems with scalar control was obtained from the evident geometric consideration: the velocity vectors should be directed towards the discontinuity surface in its small vicinity. As is seen in Fig. 5, the sliding mode may exist in the vector case ( $m > 1$ ) even if it does not occur in each of the discontinuity surfaces. This example also shows that the conditions for sliding modes to exist are closely linked with the convergence of the state to the manifold  $s=0$  or to the origin in the  $(n-m)$ -dimensional subspace  $(s_1, \dots, s_m)$ . Therefore, in the vector case it is reasonable to use stability theory to formulate the concept "sliding mode" and the conditions for its existence. For the systems of the type (14), (15) this approach leads to the following results which are systematically considered in [84].

**Definition:** A domain  $S$  in the manifold  $s=0$  is a sliding mode domain if for each  $\epsilon > 0$ , a  $\delta > 0$  exists such that any motion starting in the  $n$ -dimensional  $\delta$ -vicinity of  $S$  may leave the  $n$ -dimensional  $\epsilon$ -vicinity of  $S$  only through the  $n$ -dimensional  $\epsilon$ -vicinity of the boundaries of  $S$  (Fig. 6).

**Theorem 8:** For the  $(n-m)$ -dimensional domain  $S$  to be the domain of sliding mode, it is sufficient that in some  $n$ -dimensional domain  $\Omega$ ,  $S \subset \Omega$ , there exists a continuously differentiable function  $v(x, s, t)$ , satisfying the following conditions.

1)  $v$  is positive definite with respect to  $s$  and for any  $x \in S$  and  $t$

$$\inf_{\|s\|=R} v = h_R, \quad \sup_{\|s\|=R} v = H_R,$$

$$h_R \neq 0 \text{ if } R \neq 0, \quad (h_R, H_R \text{ depend only on } R).$$

2) Time derivative of  $v$  for (15) has negative supremum on small enough spheres  $\|s\|=R$  with removed points on the discontinuity surfaces where this derivative does not exist.

The equation of the motion projected on subspace  $(s_1, \dots, s_m)$ ,

$$\dot{s} = Gf + GBu, \quad (18)$$

should be considered in using Theorem 8, which is an analog of a Lyapunov stability theorem. The domain  $S$  is the set of  $x$  for which the origin in subspace  $(s_1, \dots, s_m)$  is an asymptotically stable equilibrium point for the dynamic system (14), (18). Unfortunately there are no standard methods to find the function  $v$  (as there are no methods to generate Lyapunov functions for arbitrary

<sup>3</sup>The assumptions about functions  $f$ ,  $B$ ,  $s$ , and initial conditions  $x(t_0)$  and  $x^*(t_0)$  are given in [83].



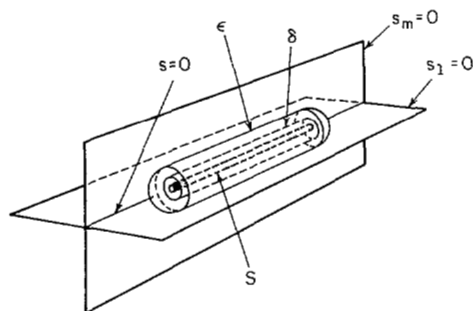


Fig. 6. Definition of a domain of sliding mode.

nonlinear systems). Therefore, special cases have been classified when the domain of the sliding mode may be derived. These cases are as follows.

1) There exist matrices  $W(x, t)$  and  $L(x, t)$  such that  $L = -WGB$ ,  $W^T = W$ ,  $W > 0$ ,  $L = \|l_{ij}\|$ ,  $l_{ii} > \sum_{j=1}^m |l_{ij}|$ ,  $i = 1, \dots, m$ .

2) Matrix  $GB$  in (18) is symmetric.

3) Matrix  $GB$  is diagonal.

4) Matrix  $GB$  has a dominating diagonal (for each row the absolute value of the diagonal element is greater than the sum of absolute values of the rest of the elements).

5) The systems with so-called hierarchy of controls, that is when  $|u_i| > \sum_{j=i+1}^m |d_{ij}u_j|$ ,  $d_{ij}$  some functions of  $x$  and  $t$ .

For the first two cases the domain of the sliding mode may be found with the help of Lyapunov functions which are quadratic forms of  $s$  with coefficients depending on  $x$  and  $t$ . Cases 3 and 4 may be decoupled into  $m$  scalar cases. For case 5, hierarchy of controls implies that one of the components of control assures the existence of sliding mode on the first discontinuity surface, the second on the intersection of the first two surfaces, etc., up to the intersection of all  $m$  discontinuity surfaces. The analysis of the cases 3, 4, and 5 is based on the inequalities (3) which are the conditions for the existence of a sliding mode for the systems with scalar control.

#### IV. DESIGN OF MULTIINPUT VSS

In this section we treat control systems linear with respect to control (14), (15). The design idea is similar to the basic approach in the single-input case.

First, a sliding mode is designed to have some prescribed properties. Second, it is guaranteed to exist at any point of the intersection  $s=0$  of the discontinuity surfaces, which is then referred to as a sliding manifold. Third, it is guaranteed that the state reaches a sliding manifold.

As it follows from (16) the motion in a sliding mode depends on the  $m \times n$  elements of matrix  $G$ . That means that desirable properties for this motion may be obtained by a proper choice of discontinuity surfaces. It should be noted that the system order is thus reduced from  $n$  to  $n-m$  due to the  $m$ -dimensional equation  $s=0$ , allowing the elimination of  $m$  state variables.

Although the existence conditions discussed in Section III are unknown for an arbitrary matrix  $GB$  in (18), the

problem of desirable characteristics of the sliding mode and the problem of the existence of a sliding manifold may be decoupled, as the following theorem shows.

**Theorem 9 [84]:** The equation of the sliding mode is invariant with respect to nonsingular transformations

$$s^* = H_s(x, t)s, \quad u^* = H_u(x, t)u, \quad \det H_s \neq 0, \quad \det H_u \neq 0. \quad (19)$$

This theorem means that the sliding mode is governed by the same equation (16) if the components of the control vector undergo discontinuities on new surfaces  $s_i^*=0$  or the components of the new control vector  $u^*$  undergo discontinuities on already chosen surfaces  $s_i=0$ .

At the same time, the equation of the motion projected on subspaces  $(s_1^*, \dots, s_m^*)$  in the first case and on  $(s_1, \dots, s_m)$  in the second case do depend on  $H_s$  or  $H_u$ , respectively. In general, for any matrix  $G$  the matrices  $H_s$  or  $H_u$  can be chosen such that this  $m$ th-order equation is reduced to one of the cases (listed in Section III) when the existence conditions can be obtained.

The methods for study of the existence conditions may be applied to analyze "reaching conditions." These two problems are closely related because the first one means asymptotic stability in small, and the second in large.

We now consider the properties of multiinput VSS resulting from the above design procedure [84]. Let the control system be represented by

$$\dot{x} = Ax + Df(t) + Bu \quad (20)$$

where  $x \in R^n$ ,  $u \in R^m$ ,  $A, D, B$ -constant or time-varying matrices, and  $f(t) \in R^l$  is the disturbance. The components of  $u$  undergo discontinuities on  $m$  planes  $s_i=0$ , that is

$$s = Cx, \quad s \in R^m, \quad C = \text{const}. \quad (21)$$

For time-invariant VSS when  $f=0$  the control is a piecewise linear function of some of the state variables

$$u = -\Psi x^k - \delta, \quad (x^k)^T = (x_1, \dots, x_k), \quad \Psi_{ij} = \begin{cases} \alpha_{ij}, & \text{if } s_i x_j > 0, i = 1, \dots, m \\ \beta_{ij}, & \text{if } s_i x_j < 0, j = 1, \dots, k, \end{cases} \quad \alpha_{ij}, \beta_{ij} = \text{const} \quad (22)$$

$\delta^T = (\delta_1, \dots, \delta_m)$ ,  $\delta_i = \delta_{0i} \text{sgn } s_i$ ,  $\delta_{0i}$  are small positive scalars.

To assure the existence of a sliding manifold with asymptotically stable sliding motion it is necessary and sufficient that the system (20)–(22) with  $\Psi_{ij} = \Omega_{ij}$ , where  $\Omega_{ij}$  some constant values between  $\alpha_{ij}$  and  $\beta_{ij}$ , has  $n-m$  eigenvalues with negative real parts.

The design procedure consists of finding  $R = \text{const}$  such that  $u = Rx^k$  places  $n-m$  eigenvalues at desired locations. Then the parameters of control in (22) are chosen to result in the sliding mode determined by these  $(n-m)$  eigenvalues.

Different types of reaching conditions depending on the matrices  $H_s$  and  $H_u$  in (19) are given in [84].



In general, the motion in sliding mode depends not only on matrix  $C$  in (21), but also on the parameters of the system (20) and disturbances. If the influence of disturbances and parameter variations can be reduced enough by a proper choice of discontinuity surfaces (or matrix  $C$ ), then it is reasonable to design VSS such that the sliding mode in the intersection  $s=0$  always arises. This problem may be solved with the help of a hierarchy of controls when only the ranges of the parameter variations and the class of disturbances (11) are known. The sensitivity of sliding mode with respect to plant parameters is studied in [62].

The invariancy conditions of a sliding mode are studied in [20] for the system

$$\dot{x} = Ax + h(x, t) + Bu$$

where  $h(x, t) \in R^n$ ,  $h(x, t)$  depends on state variables, disturbances, and time-varying parameters. The equations of the sliding mode on manifold  $s=0$  do not depend on  $h(x, t)$  if

$$\text{rank}(B, h) = \text{rank } B. \quad (23)$$

The conditions of invariancy with respect to plant parameter variations and disturbances, and the conditions of so called selective invariancy (when one set of state variables do not depend on another set of variables) follow from (23), [20]. Note that the equations in phase canonic form satisfy condition (23).

The next set of problems considered in the VSS theory pertains to self-optimizing systems and nonlinear programming [54], [55]. In such problems we search for the extremum of some function

$$y = f(x), \quad y \in R^1, \quad x \in R^n, \quad x \in \Omega. \quad (24)$$

For  $f(x)$  to be maximized, the variable  $y$  is compared with a monotonously increasing reference input  $g(t)$ . Discontinuous control reduces the difference between  $y$  and  $g$  to zero in sliding mode. As a result  $y(t)$  tracks  $g(t)$  and reaches the maximum point. If the constraints are active, an auxiliary discontinuous control guarantees the existence of sliding mode on the boundary of the admissible domain which leads to the search of maximum point in this domain. The above method does not need the measurement of the gradients of the function to be minimized nor those of the constraint functions.

Application of the technique of Section III substantiates the convergence of the gradient procedures for piecewise differentiable penalty functions [56]. These functions with finite penalty factors are known to have maximum point coinciding with the solution of a nonlinear programming problem [91].

Identification methods based on the models with variable structure are described in [57], [84]. It is assumed that the order of the differential equation representing the plant is known and its right-hand side may be expanded into finite series with unknown coefficients. If the state of the model with variable structure tracks the state of the

plant in a sliding mode, then the average values of discontinuous coefficients of the model depend on the plant parameters. These average values (which are equal to the equivalent control) can be measured employing first-order filters (see Section III) and then used for calculation of the unknown parameters.

In conclusion we list some results of practical application of VSS. Control methods for chemical processes are considered in [30]. A universal system of control devices for a wide range of processes in steel, power, nonferrous, chemical and food industries is described in [33]. Applications of VSS to automation of a power station and flight are given in [47] and [76], respectively.

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