TRACKING CONTROL OF NON-LINEAR SYSTEMS
USING SLIDING SURFACES
WITH APPLICATION TO ROBOT MANIPULATORS

by

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ABSTRACT

We develop a methodology of feedback control to achieve accurate tracking in a class of non-linear, time-varying systems in the presence of disturbances and parameter variations. The methodology uses in its idealized form piecewise continuous feedback control, resulting in the state trajectory 'sliding' along a time-varying sliding surface in the state space. This idealized control law achieves perfect tracking; however, non-idealities in its implementation result in the generation of an undesirable high frequency component in the state trajectory. To rectify this, we show how continuous control laws may be used to approximate the discontinuous control law to obtain robust tracking to within a prescribed accuracy and decrease the extent of high frequency signal.

The method is applied to the control of a two-link manipulator handling variable loads in a flexible manufacturing system environment.

Section 1. Introduction

   We present a methodology of feedback control to achieve accurate
tracking for a class of non-linear time-varying systems in the presence of
disturbances and parameter variations. The methodology uses in its idealized
form piecewise continuous feedback control laws, resulting in the state
trajectory 'sliding' along a discontinuity or sliding surface in the
state space. The idealized form of the methodology results in perfect
tracking of the required signals; however certain non-idealities associated
with its implementation cause the trajectory to 'chatter' along the sliding
surface, resulting in the generation of an undesirable high frequency
component in the state trajectory. Not only is the high frequency component
undesirable in itself, but also it may excite high-frequency dynamics
associated with the control system which have been neglected in the course
of modelling. To rectify this situation, we show how continuous control
laws, which approximate in a suitable sense the discontinuous control law,
may be used to obtain tracking to within a prescribed accuracy which is
robust to disturbance signals and parameter variations. At the same time
the continuous control laws decrease the extent of unwanted high-frequency
signals.

   The basic concept we use is that of sliding mode control. This has
been studied in great detail in the Soviet literature (see [7], [8] and refer-
ences contained therein), where it has been used to robustly stabilize a class
of non-linear systems. The basic mathematical idea comes from Fillipov [1]:
consider a piecewise continuous differential equation, with the right hand side
discontinuous across a hypersurface. If the trajectories of the differential
equation off the discontinuity surface point towards the discontinuity surface,
it is intuitively plausible that trajectories that start on the discontinuity

surface stay on (slide along) the discontinuity surface - the 'sliding surface'. This, in turn, imposes certain constraints on their dynamics. Further, even if the right hand side is perturbed, these contraints on the dynamics on the sliding surface remain the same, so long as the perturbed equation has trajectories pointing towards the sliding surface (and of course, the sliding surface is itself not perturbed). By a suitable choice of sliding surface, piece-wise continuous control law and class of non-linear systems under investigation, we obtain instances in which the dynamics of the state trajectory on the sliding surface are completely specified by the constraint that it stay on the sliding surface. These dynamics are in turn insensitive to parameter variations in the dynamics of the sliding surface for the same reasons as those noted above. The shortcomings of the methodology developed so far in the literature are as follows:

(i)   There is a 'reaching' phase in which the trajectories starting from a given initial condition off the sliding surface tend towards the sliding surface. The trajectories in this phase are sensitive to parameter variations. Further, convergence to the sliding surface may only be asymptotic, so that the benefits of sliding mode control cannot be realized. The literature [10, 12] suggests alleviating these difficulties by the use of high gain feedback to speed-up the reaching phase. This has the usual drawbacks associated with high gain feedback - extreme sensitivity to unmodelled dynamics, actuator saturation, etc. The problem is compounded in the multi-input case, when the 'hierarchical control' methodology of [7] is applied. In this method, one starts with a nested chain of sliding surfaces and derives control laws for a particular sliding surface on the

assumption that the trajectory actually lies in the intersection of all preceding sliding surfaces. Since convergence to each sliding surface is only asymptotic, this may be an invalid assumption.

(ii) Unavoidable small imperfections in switching between control laws at the discontinuity surface result in the trajectory chattering rather than sliding along the switching surface. This will in turn excite high-frequency unmodelled dynamics in the plant.

We remove these drawbacks by developing and using the concept of a time-varying sliding surface in the state space. We also use time-varying surfaces to discuss application of our methodology to tracking rather than stabilization problems. Further, by approximating the discontinuous control law by a continuous one, we trade off accuracy in tracking against the generation of high frequency chattering in the state trajectory. The layout of the paper is as follows:

In Section 2, we review results on the dynamics of systems with switches. We use basic results of Fillipov [1] on solution concepts for discontinuous differential equations to define a solution concept for piecewise continuous dynamical systems with the surface of discontinuity varying with time (time-varying sliding surfaces).

Section 3 illustrates in the simple instance of time-varying linear systems our methodology of sliding mode control for robust tracking of specified signals.

In Section 4, we extend the previous results to a class of non-linear, time-varying systems. Using discontinuous control, we obtain perfect

tracking in the presence of disturbances and parameter variations. Section 5 modifies the framework of Section 4 to obtain continuous control laws which approximate the discontinuous control laws of Section 4. We trade off tracking accuracy for a smaller component of high-frequency signal.

Section 6 describes the applications of our methodology to the control of a two-link manipulator. We believe our methodology has important applications to the problems of controlling robots handling variable loads in a flexible manufacturing system environment. In Section 7 we briefly indicate areas of further research.

Section 2. Dynamics of Systems with Switches

The simplest kind of controllers are on-off controllers. They attracted the attention of control engineers when a number of process controllers for stabilizing non-linear processes were successfully implemented. On-off controllers have since been studied by optimal control theorists in connection with the concepts of relaxed control and bang-bang control (see Flügge-Lotz [2]). After this time, control of systems using switches has been investigated largely in the Soviet literature (the book of Utkin [7] has several references) with few exceptions ([5], [10], [11], [12]).

The basic mathematical problem in studying the dynamics of systems with switched control laws is that they represent differential equations with discontinuous right hand sides. The conventional existence-uniqueness theory for ordinary differential equations is then no longer valid. To illustrate, consider the following example of a discontinuous differential equation on $\mathbb{R}^n$:

Let S be a manifold of dimension $(n-1)$, defined by $\{x: s(x) = 0\}$, where s is a function from $\mathbb{R}^n$ to $\mathbb{R}$. S represents the switching boundary. Let the dynamics be defined by

$$\dot{x} = f_+(x) \quad \text{for } \{x: s(x) > 0\} =: G_+ \tag{2.1}$$

$$\dot{x} = f_-(x) \quad \text{for } \{x: s(x) < 0\} =: G_- \tag{2.2}$$

where f_+ and f_- are smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Note that in general f_+ and f_- do not match on S so that the dynamics are discontinuous at S. In Figure 1, we show some of the possible phase portraits associated with the discontinuity. In Figure 1(a), the trajectories of f_+ and f_- both point towards the discontinuity surface S. Intuitively, imperfections in the
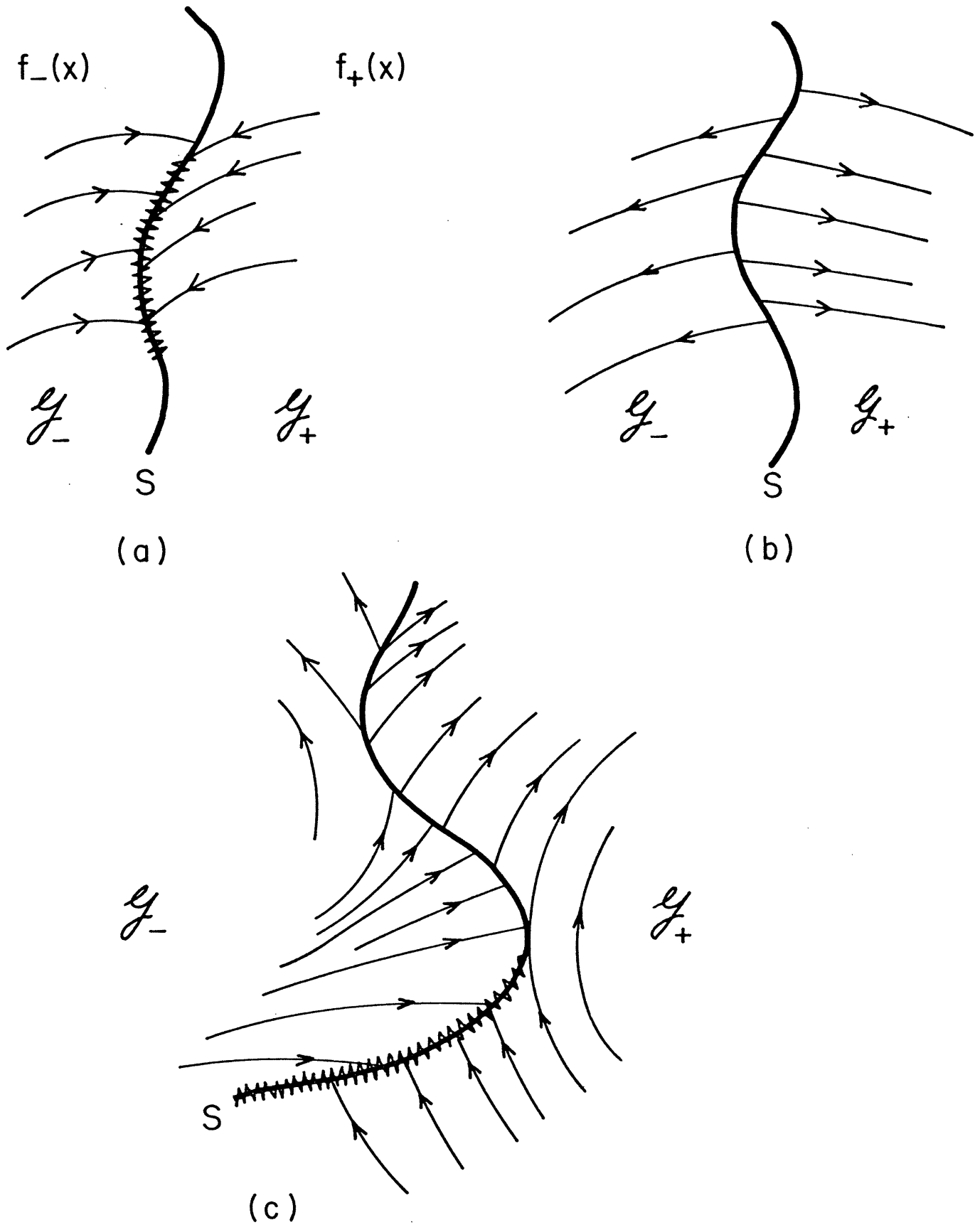
Fig. 1: Showing possible flows near a switching surface.

switching mechanism should then cause the state trajectory to cross S infinitely many times or 'chatter' along the surface (as suggested by the jagged line in the figure). In Figure 1(b), the trajectories of f_+ and f_- both point away from S. It would seem that initial conditions at S would follow either trajectories of f_+ or f_- (which one, specifically, appears to be ambiguous) and be repelled from S. In Figure 1(c), we have a combination of circumstances represented in Figures 1(a) and 1(b), as well as a region of S in which f_- points towards S and f_+ away from it.

One way of <u>regularizing</u> the system description (2.1), (2.2) consistent with intuition is to assume that (2.1), (2.2) are the degenerate limit (as $\Delta \downarrow 0$) of the hysteretic switching mechanism shown in Figure 2. The variable y represents the switching variable - when y=+1, the dynamics are described by $f_+(x)$ and when y=-1 they are described by $f_-(x)$. Applying this regularization to the instance of Figure 1(a) yields the phase portraits shown in Figure 3 for successively smaller values of Δ. Note the increase in the frequency of crossing S as $\Delta \downarrow 0$-chattering. Other forms of regularization for (2.1), (2.2) represent various imperfections in the switching mechanisms - e.g. time delays associated with switching, neglected 'fast' dynamics associated with the switching mechanism.

Consistent with the foregoing intuition, Fillipov [1] proposed the following definition for the dynamics of (2.1), (2.2) which we abbreviate as

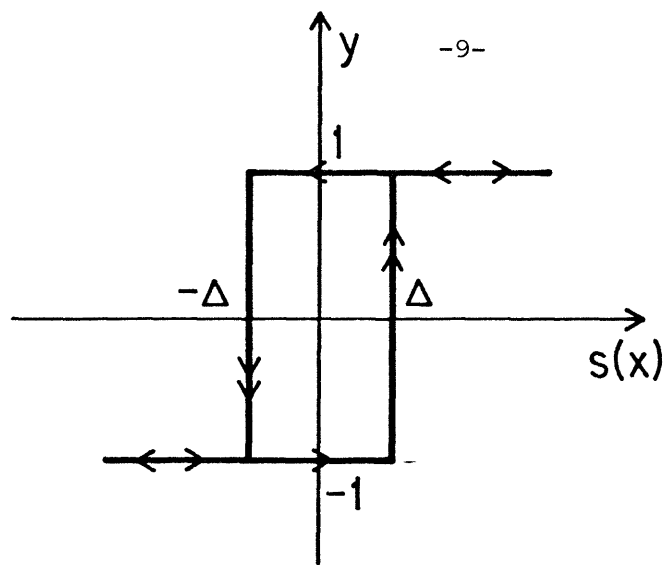$$\dot{x} = f(x) \tag{2.3}$$

with the understanding that

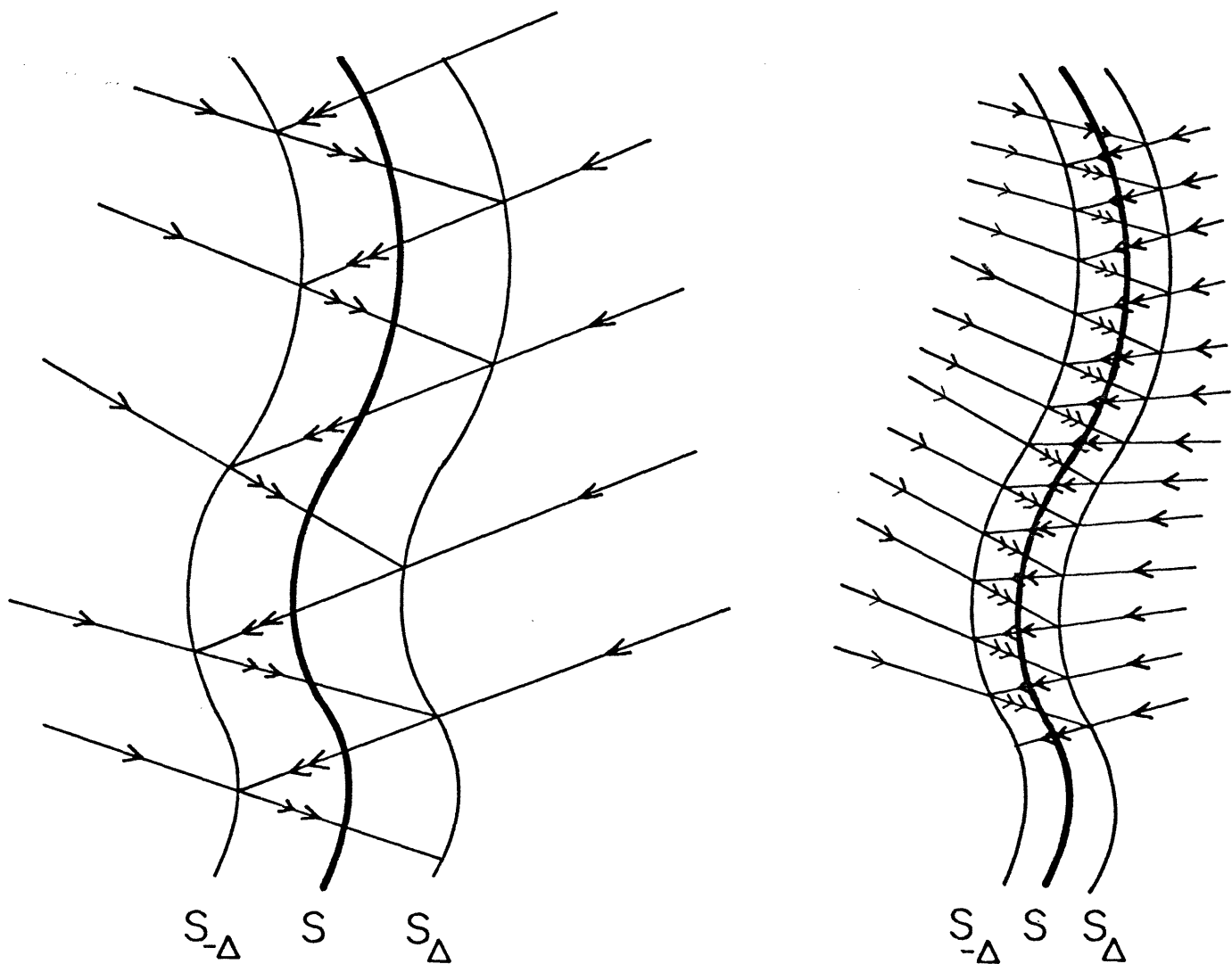Figure 2: Illustrating Hysteretic Switching

Figure 3: Showing the Effects of the Regularization for two values of Δ.

$$f(x) = f_+(x) \quad \text{for } x \in G_+ \text{, and}$$

$$f(x) = f_-(x) \quad \text{for } x \in G_- \text{.}$$

<u>Definition</u>   (Solution Concept for discontinuous differential equations)

An absolutely continuous function $x(t): [0,T] \to \mathbb{R}^n$ is a solution of (2.3) if for almost all $t \in [0,T]$

$$\frac{dx}{dt} \in \bigcap_{\delta > 0} \bigcap_{N} \text{Conv } f(B(x(t),\delta) - N) \tag{2.4}$$

where $B(x(t),\delta)$ is a ball of radius δ centered at $x(t)$, and the intersection is taken over all sets N of zero measure   (Conv refers to the convex hull of a set).

<u>Remarks</u>   (1) The definition (2.4) allows us to exclude sets of zero measure, such as S, on which $f(x)$ is not defined.

(2)   The definition (2.4) is quite general - it includes more general classes of discontinuous differential equations than those with a piece-wise continuous right hand side, which are the systems of interest to us.

We now study the application of the definition to our system (2.3):

Denote by $\lambda_+(x)$ (or $\lambda_-(x)$) the rate of change of $s(x)$ along the trajectory of $f_+(x)$ (or $f_-(x)$), i.e.

$$\lambda_+(x) = \frac{\partial}{\partial x} s(x) \cdot f_+(x) \quad \text{for } x \in G_+$$

$$\lambda_-(x) = \frac{\partial}{\partial x} s(x) \cdot f_-(x) \quad \text{for } x \in G_-$$

Since $s(x)$, $f_+(x)$, $f_-(x)$ are all smooth functions of x, both

$$\lambda_+(x^*) = \lim_{x \to x^*} \lambda(x)$$

and

$$\lambda_-(x^*) = \lim_{x \to x^*} \lambda(x)$$

can be defined for $x^* \in S$.

Then, in the instance that $\lambda_+(x^*) < 0$ and $\lambda_-(x^*) > 0$ (the situation of Figure 1(a)), it may be shown that for $x^* \in S$, definition (2.4) yields (Lemma 3 of [1]) that

$$\dot{x}^* = f_0(x^*) \qquad (2.5)$$

$$\text{where } f_0(x^*) = -\frac{\lambda_-(x^*)}{\lambda_+(x^*) - \lambda_-(x^*)} f_+(x^*) + \frac{\lambda_+(x^*)}{\lambda_+(x^*) - \lambda_-(x^*)} f_-(x^*) \qquad (2.6)$$

Note that $\frac{\partial}{\partial x} s(x^*) \cdot f_0(x^*) = 0$ (see the construction of Figure 4) so that the trajectory slides along S once it hits S (this is referred to as the sliding mode). This is consistent with the intuition of the regularization of Figure 3 which suggests that in the limit that $\Delta \downarrow 0$, the chattering becomes infinitely rapid and of infinitesimally small amplitude - $f_0(x^*)$ then is the resultant averaging of the chattering. Note also that $f_0(x^*)$ is a convex combination of $f_+(x^*)$ and $f_-(x^*)$.

Further, $\lambda_+(x^*) < 0$ implies that $\lambda_+(x) < 0$ for $x \in G_+ \cap B(x^*, \delta)$, where $B(x^*, \delta)$ is a δ neighbourhood of x^* - similarly for $\lambda_-(x^*) < 0$. The conditions

$$\lambda_+(x) < 0 \quad \text{for } x \in B(x^*, \delta) \cap G_+$$

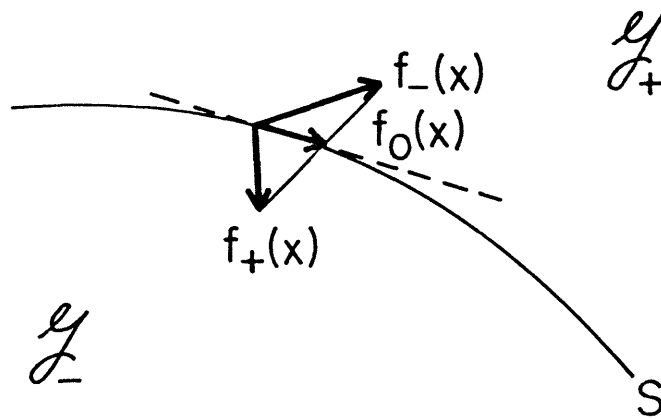$$\lambda_-(x) > 0 \quad \text{for } x \in B(x^*, \delta) \cap G_-$$

may be combined as

Fig. 4: Illustrating the construction of $f_0(x)$ by Fillipov's method

$$\frac{d}{dt} s^2(x) < 0 \quad \text{for } x \in B(x^*, \delta) - S \tag{2.7}$$

with the understanding that $\frac{d}{dt} s^2(x)$ is evaluated along the trajectories of $f_+(x)$ in G_+, and along those of $f_-(x)$ in G_-. The condition (2.7) is referred to as the <u>local sliding condition</u>, since it is sufficient to guarantee that trajectories originating from initial conditions close to S converge to S and then slide along S.

If in fact we have that $s(x)$ is a proper function and

$$\frac{d}{dt} s^2(x) < -\psi(|s|) \quad \text{for } x \in \mathbb{R}^n - S \tag{2.8}$$

where ψ is some function of Class K (see e.g. Vidyasagar [9]), then all inital conditions lying off S will be attracted to S <u>(global sliding condition)</u> and then slide along S. Of course, the convergence to S in either case may only be asymptotic, so that the chattering behavior indicated in Figure 1(a) may not be observed in finite time. Equations (2.7) or (2.8) however guarantee that trajectories originating on S will remain on S.

Further, in the instance that $\lambda_+(x^*) > 0$ and $\lambda_-(x^*) > 0$, Lemma 9 of [1] establishes that the trajectory of definition (2.1) has only x^* in common with S and goes from G_- to G_+ through x^*. Similar conclusions hold for the case when $\lambda_-(x^*)$, $\lambda_+(x^*) < 0$.

Fillipov proves existence and continuability of solutions theorems for his solution concept (Theorems 4 and 5). For the uniqueness of solutions, some further conditions are required. For our case of a piecewise continuous differential equation, Fillipov's Theorem 14 states that so long as at least one of the two inequalities

$$\lambda_-(x^*) > 0 \ , \qquad \lambda_+(x^*) < 0 \qquad\qquad (2.9)$$

is satisfied at each point $x^* \in S$, the system (2.3) has a unique solution (in the sense of Definition(2.4.)) for a given initial condition. Further, the solution depends continuously on initial conditions.

Remark: The requirement that one of the two inequalities of (2.9) hold rules out the ambiguous situation of Figure 1(b), for instance.

The preceding development was for the stationary case, i.e., s, f_+, f_- were not explicitly functions of time. For the case when s, f_+ and f_- are functions of x and t, it may be seen that the development generalizes as follows: define the sliding surface M_0 in (x,t) space as

$$M_0 = \{(x;t) : s(x;t) = 0\} \subset \mathbb{R}^{n+1} \ .$$

Define, also

$$\lambda_+(x;t) = \frac{\partial}{\partial t} s(x;t) + \frac{\partial}{\partial x} s(x;t) \cdot f_+(x;t)$$

$$\lambda_-(x;t) = \frac{\partial}{\partial t} s(x;t) + \frac{\partial}{\partial x} s(x;t) \cdot f_-(x;t)$$

In the instance that $\lambda_+(x^*;t) < 0$ and $\lambda_-(x^*; t) > 0$, formulae completely analogous to (2.5), (2.6) may be obtained. One way of observing this is to note that the time-varying case can be converted to the form studied earlier by augmenting the state space with the t-variable and augmenting the dynamics with $\dot{t}=1$. We may then state that the (x,t) trajectory slides along the manifold M_0 once it reaches M_0.

As before, the uniqueness theorem is also valid so long as at least one of the two inequalities

$$\lambda_-(x^*; t^*) > 0, \quad \lambda_+(x^*; t^*) < 0 \tag{2.10}$$

is satisfied for each $(x^*; t^*) \in M_0$. As before, $\lambda_+(x^*, t^*) < 0$ implies

that $\lambda_+(x; t) < 0$ for $(x; t) \in B((x^*; t^*), \delta) \cap M_+$ where $M_+ = \{(x; t) : s(x; t) > 0\}$.

Also, $\lambda_-(x^*; t^*) > 0$ implies that $\lambda_+(x; t) > 0$ for $(x; t) \in G((x^*; t^*), \delta) \cap M_-$

where $M_- = \{(x; t) : s(x; t) < 0\}$, and the conditions for sliding along the

surface M_0, namely,

$$\lambda_-(x; t) > 0 \quad \text{for } (x; t) \in B((x^*; t^*), \delta) \cap M_+$$

$$\lambda_+(x; t) < 0 \quad \text{for } (x; t) \in B((x^*; t^*), \delta) \cap M_-$$

may be combined as

$$\frac{d}{dt} s^2(x; t) < 0 \quad \text{for } (x; t) \in B((x^*; t^*), \delta) - M_0 \tag{2.11}$$

with the understanding that $\frac{d}{dt} s^2(x; t)$ is evaluated along trajectories of

$f_+(x; t)$ in M_+, and along those of $f_-(x; t)$ in M_-. (2.11) is the local sliding

condition.

If we have that $\dot{s}(x, t)$ is a proper function and

$$\frac{d}{dt} s^2(x; t) \leq -\psi(|s|) \quad \text{for } (x; t) \in \mathbb{R}^n \times \mathbb{R}_+ - M_0 \tag{2.12}$$

for some function ψ of class K, then all initial conditions lying off

M_0 will be attracted to M_0 and slide along M_0. As before, the convergence

to M_0 may only be asymptotic so that the sliding mode is not observed

in finite time.

By a minor abuse of notation, we shall denote by $S(t)$ sections of the

manifold M_0 in the state space $\mathbb{R}^n$, i.e.,

$$S(t) = \{x \in \mathbb{R}^n : s(x; t) = 0\} \tag{2.13}$$

(2.13) has the interpretation of a time-varying sliding surface in the state space. Then, we may rewrite the local sliding condition (2.7) for the time-varying case as

$$\frac{d}{dt} s^2(x;t) < 0 \quad \text{for } x \in B(x^*, \delta) - S(t) \tag{2.14}$$

and the global sliding condition (2.12) for the time-varying case as

$$\frac{d}{dt} s^2(x;t) < -\psi(|s(x;t)|) \quad \text{for } x \in \mathbb{R}^n - S(t) \tag{2.15}$$

at time t.

In this section, we assumed that f^+ and f^- were smooth functions (smooth means C^r for some r). However, as it may be seen from [1], all of our conclusions hold when the functions f^+ and f^- are merely Lipschitz continuous.

Section 3. Sliding Mode Control for a Class of Single-Input Linear Time
Varying Systems

We illustrate some of the robustness and parameter insensitivity properties of discontinuous or sliding mode control for the case of an nth order linear time-varying control system with a single input. Specifically, consider:

$$x_1^{(n)} + a_{n-1}(t) x_1^{(n-1)} + a_{n-2}(t) x_1^{(n-2)} + \ldots + a_0(t) x_1 = u \qquad (3.1)$$

The control problem to be solved is to get $x_1(t)$ to track a specified trajectory $x_{d1}(t)$: a given smooth function from $\mathbb{R}_+$ to $\mathbb{R}$. Some conditions need to be imposed on $x_{d1}(t)$ to match the initial conditions of (3.1): precisely, define the vectors $x(t) = [x_1(t), \dot{x}_1(t), \ldots, x_1^{(n-1)}(t)]^T \in \mathbb{R}^n$ and $x_d(t) = [x_{d1}(t), \dot{x}_{d1}(t), \ldots, x_{d1}^{(n-1)}(t)]^T \in \mathbb{R}^n$. For simplicity, we denote $x_1^{(k)}(t)$ and $x_{d1}^{(k)}(t)$ by $x_{k+1}(t)$ and $x_{d,k+1}(t)$ respectively. We define the tracking error $\tilde{x}(t)$ as

$$\tilde{x}(t) = x(t) - x_d(t) = [\tilde{x}_1(t), \ldots, \tilde{x}_n(t)]^T$$

Then, we assume that the tracking error is zero at time zero:

$$\tilde{x}(0) = 0 \qquad (3.2)$$

Further, equations (3.1) can be written in controllable canonical form as

$$\dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \mathbf{O} \\ & & & 0 & 1 \\ \mathbf{O} & & & & \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u =: A(t)x + Bu \qquad (3.3)$$

Now, we <u>assume</u> that the $n^{\underline{th}}$ derivative of x_{dl} is bounded by a constant v:

$$\left| x_{d,n+1}(t) \right| \leq v . \qquad \forall\, t \in \mathbb{R} \qquad\qquad (3.4)$$

and define the time-varying sliding surface $S(t)$ by

$$s(x;t) = C\, \tilde{x}(t) = 0 \qquad\qquad (3.5)$$

where C is a row vector of the form $[c_1,\ldots,\, c_{n-1},\, 1]$. If the control $u(t)$ could be chosen so as to keep the trajectory on $s(x;t) = 0$ we would have from (3.5) that

$$x_1^{(n-1)} + \sum_{i=0}^{n-2} c_{i+1} x_1^{(i)} = x_{dl}^{(n-1)} + \sum_{i=0}^{n-2} c_{i+1} x_{dl}^{(i)} \qquad\qquad (3.6)$$

Since the initial conditions on $x_1^{(i)}$ match those on the $x_{dl}^{(i)}$ we would then have from standard uniqueness results for ordinary differential equations that

$$x(t) \equiv x_d(t) \qquad \forall t \in \mathbb{R}_+ .$$

Thus, it remains only to choose control $u(t)$ so as to cause the $(x;t)$ trajectory to slide along the surface specified by (3.5), i.e. a control u that satisfies condition (2.12) with

$$S(t) : = \{x : C\tilde{x} = 0\}$$

We will choose control u of the form

$$u = \beta^T(x) \cdot x + \sum_{i=1}^{n-1} k_i(x;t)\,\tilde{x}_{i+1} - k_n\, \mathrm{sgn}\, s$$

$$= [\beta_1(x),\ldots,\beta_n(x)]x + \sum_{i=1}^{n-1} k_i(x;t)\tilde{x}_{i+1} - k_n\, \mathrm{sgn}\, s \;^{\dagger} \qquad\qquad (3.7)$$

_____

$\dagger$ sgn s is defined as:

$$\mathrm{sgn}\, s = 1 \;\text{ for } s > 0; \qquad \mathrm{sgn}\, s = -1 \;\text{ for } s < 0$$

with $k_i(x)$, $i=1,\ldots,n$ suitably selected. Using (3.7) we obtain

$$\frac{1}{2}\frac{d}{dt}s^2(x;t) = \sum_{i=1}^{n}(\beta_i(x) - a_{i-1}(t))x_i \cdot s + \sum_{i=1}^{n-1}(c_i + k_i(x;t))\tilde{x}_{i+1} \cdot s$$

$$-s \cdot x_{d,n+1} - k_n|s| \tag{3.8}$$

To get (3.8) to satisfy (3.12), we use

$$\beta_i(x) := \beta_i^+ \leq a_{i-1}(t) \quad \text{for } x_i \cdot s > 0 \text{ and all } t, i=1,\ldots,n \tag{3.9}$$

$$\beta_i(x) := \beta_i^- \geq a_{i-1}(t) \quad \text{for } x_i \cdot s < 0 \text{ and all } t, i=1,\ldots,n \tag{3.10}$$

$$k_i(x;t) := k_i^+ \leq -c_i \quad \text{for } \tilde{x}_{i+1} \cdot s > 0 \text{ and all } t, i=1,\ldots,n-1 \tag{3.11}$$

$$k_i(x;t) := k_i^- \geq -c_i \quad \text{for } \tilde{x}_{i+1} \cdot s < 0 \text{ and all } t, i=1,\ldots,n-1 \tag{3.12}$$

and
$$k_n > v \tag{3.13}$$

with the understanding that when $c_i = 0$ for some i, we will discard the corresponding term $k_i(x;t)\tilde{x}_{i+1}$ in the control law. Note that the control law defined by (3.7), (3.9)-(3.13) has discontinuities at

$$x_i = 0 \quad i=1,\ldots,n \quad ; \quad \tilde{x}_j = 0 \quad j=2,\ldots,n$$

and

$$s(x;t) := C\tilde{x} = 0$$

It is easy to verify, however, that the (possibly time-varying) discontinuity surfaces $\{x:\tilde{x}_j = 0\}$, and $\{x:x_i = 0\}$ are not sliding surfaces since for each x* on any one of these surfaces, we have

$$\lambda_+(x^*;t^*) = \lambda_-(x^*,t^*) \neq 0 \quad \text{for all } t^* \tag{3.14}$$

so that trajectories may be continued through them. Further, from (3.8), we have an equation of the form (2.15), namely:

$$\frac{1}{2}\frac{d}{dt}s^2(x;t) < -(k_n-v)\,|s(x;t)| \tag{3.15}$$

for all t and $x \in \{x : s\neq0, \, x_i\neq0, \, \tilde{x}_j\neq0; \, i=1,\ldots,n; \, j=2,\ldots,n\}$. From equations (3.14), (3.15) we may conclude that $\{x:s(x;t) = 0\}$ is a sliding surface, that is, all trajectories starting off the sliding surface, converge to it. Further, all state trajectories starting on the surface stay on it for all future time. Thus the feedback control defined by equations (3.7), (3.9)-(3.13) yields $x(t) = x_d(t)$.

We now exhibit the parameter insensitivity of the sliding mode control law. Assume that $a_i(t)$ is not known exactly - rather, only bounds on its magnitude α_i, γ_i are known, i.e.,

$$\alpha_i < a_i(t) < \gamma_i \quad i=0,\ldots,n-1 \tag{3.16}$$

Then (3.9) is satisfied for all t if

$$\left.\begin{array}{l} \beta_i^+ \leq \alpha_{i-1} \\ \\ \text{and} \\ \\ \beta_i^- \geq \gamma_{i-1} \end{array}\right\} \quad \text{for } i=1,\ldots,n$$

and the resultant control law (3.7), (3.9)-(3.13) yields $x(t) = x_d(t)$.

Robustness of the control law to disturbances follows along similar lines: consider an additive disturbance vector of the form $d(x;t) = [0,\ldots,0,d_1(x,t)]^T$ where

$$\left| d_1(x;t) \right| \leq \sum_{i=1}^{n} \delta_i \left| x_i \right| + \delta_0$$

The form of the disturbance follows from the fact that equations (3.3) are a state space realization of (3.1). Then the control law of the form (3.8) will yield $x(t) = x_d(t)$ so long as

$$\left. \begin{array}{l} \beta_i^+ \leq \alpha_{i-1} - \delta_i \\[2em] \beta_i^- \geq \gamma_{i-1} + \delta_i \end{array} \right\} \quad i=1,\ldots,n \qquad (3.17)$$

and

$$k_n > v + \delta_0. \qquad (3.18)$$

Conditions (3.11), (3.12) on $k_1, \ldots, k_{n-1}$ do not need to be modified to reject this disturbance. We remark here, that from (3.17) we have that

$$\beta_i^- - \beta_i^+ \geq \gamma_{i-1} - \alpha_{i-1} + 2\delta_i \qquad i=1,\ldots,n$$

As expected, the minimum discontinuity in the control u (measured by $\beta_i^- - \beta_i^+$ for $i=1,\ldots,n$, $k_i^+ - k_i^-$ for $j=1,\ldots,n-1$, and $2k_n$) required to reject disturbances and parameter variation increases with the strength of the disturbance to be rejected and the range of parameter variation in the dynamics of the system.

We next comment on the choice of $C \in \mathbb{R}^n$ in the definition of the sliding surface in (3.5). The choice of initial condition $x(0) = x_d(0)$ guarantees perfect tracking $x(t) = x_d(t)$ for all future time. In practice, however, equation (3.2) is not satisfied exactly, i.e., $x(0)$ is not equal to $x_d(0)$. If the offset in initial condition causes the trajectory at t=0 to lie off the sliding surface $s(x;0) = 0$ our control

law causes it to tend towards the sliding surface. On the other hand,
if the offset in initial condition results merely in an offset between
the desired trajectory and actual trajectory with $s(x;0) = 0$, the offset
will be reduced to zero asymptotically i.e. $\tilde{x}(t) \to 0$ asymptotically,
provided equation (3.6) is stable i.e. the polynomial $z^{n-1} + \sum_{i=0}^{n-2} c_{i+1} z^i$
is Hurwitz. We refer to the sliding surface of (3.5) as <u>stable</u> in this
case. Consider for example, with n=2, the sliding surface (3.6) with
$c_0 = 1/\tau$:

$$\dot{x}_1 + \frac{1}{\tau} x_1 = \dot{x}_{d1} + \frac{1}{\tau} x_{d1} . \qquad (3.18)$$

Further assume $x_1(0) = x_{d1}(0) + \varepsilon$. Then (3.18) yields that

$$x_1(t) = x_{d1}(t) + \varepsilon e^{-t/\tau}$$

so that $x_1(t) \to x_{d1}(t)$ asymptotically, provided that $\tau > 0$ (the larger τ is,
the faster the convergence).

The preceding development illustrates the philosophy of our approach.
The state vector $x(t)$ is constrained to follow the desired trajectory
by suitable choice of $s(x(t); t)$. The discontinuity in the control
law across s is chosen so as to make $s(x;t) = 0$ a sliding surface
in the presence of both parameter variations and disturbances. Next,
we generalise this philosophy to a class of non-linear, multi-input
time varying systems.

Section 4.   Robust Sliding Mode Control of a Class of Non-Linear Systems

Consider the class of non-linear, time varying systems shown in Figure 5 and described by the equations

$$\theta_j^{(n_j)} = f_j(\theta_1, \theta_2, \ldots, \theta_p; t) + u_j \qquad\qquad j = 1, \ldots, p \qquad (4.1)$$

where for $i = 1, \ldots, p$

$$\theta_i = [\theta_i, \dot{\theta}_i, \ldots, \theta_i^{(n_i-1)}]^T$$

By way of notation, define

$$\theta = [\theta_1^T, \ldots, \theta_p^T]^T$$

$$|\theta_j| = [|\theta_j|, |\dot{\theta}_j|, \ldots, |\theta_j^{(n_j-1)}|]^T \quad \text{and}$$

$$|\theta| = [|\theta_1|^T, \ldots, |\theta_p|^T]^T$$

We <u>assume</u> that the functions f_j are polynomially bounded, i.e. there exist polynomials $F_j(|\theta|; t)$ such that for $j = 1, \ldots, p$

$$|f_j(\theta; t)| \leq F_j(|\theta|; t) \qquad\qquad (4.2)$$

Without loss of generality, the coefficients of $F_j(\theta, t)$ may be chosen to be smooth, positive functions of time.

We want to design a control law that makes each $\theta_j(t)$ track a desired trajectory $\theta_{dj}(t)$.

Let $\theta_{dj}(t)$ be the n_j vector of $\theta_{dj}(t)$ and its first (n_j-1) derivatives, and define the tracking error $\tilde{\theta}_j(t) = \theta_j(t) - \theta_{dj}(t) = [\tilde{\theta}_j, \tilde{\theta}_j^{(1)}, \ldots, \tilde{\theta}_j^{(n_j-1)}]^T$.

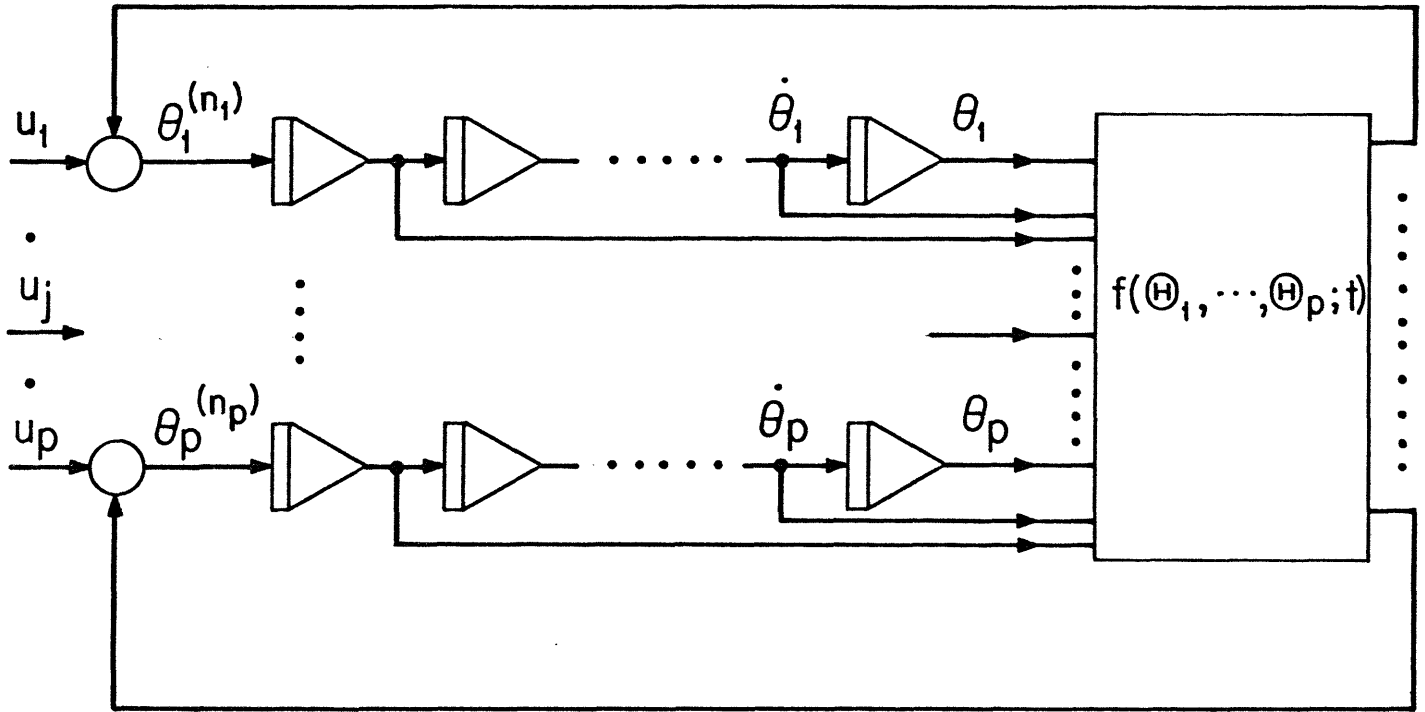Define a set of sliding surfaces $S_j(t)$ in the θ_j space by:

Fig. 5.   Showing the Class of Non-Linear Systems Considered in Section 4.

$$S_j(t) = \{\theta_j : s_j(\theta_j; t) = 0\} \tag{4.3}$$

where for $j = 1, \ldots, p$

$$s_j(\theta_j; t) = C_j(\theta_j - \theta_{dj}) = C_j \tilde{\theta}_j \tag{4.4}$$

In (4.4) C_j is a constant row vector of the form $[c_{j1}, \ldots, c_{jn_j-1}, 1]$ such that the surface defined by (4.4) is stable in the sense of section 3, i.e.

such that the polynomial $z^{n_j-1} + \sum_{i=0}^{n_j-2} c_{ji+1} z^i$ is Hurwitz. If we can

maintain $s_j(\theta_j; t) = 0$, then assuming that $\tilde{\theta}_j(0) = 0$, we have $\tilde{\theta}_j(t) = 0$ for all positive time.

We again assume that $\theta_{dj}^{(n_j)}$ is bounded by a known function of time $v_j(t)$:

$$\left| \theta_{dj}^{(n_j)}(t) \right| \leq v_j(t) \qquad\qquad \forall \, t \geq 0 \tag{4.5}$$

Now $F_j(\theta; t)$ in (4.2) is a polynomial in θ. Hence a representative term (say the kth) is of the form

$$F_{jk} = \alpha_{jk}(t) \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)} \tag{4.6}$$

where $\alpha_{jk}(t)$ is a positive (not necessarily bounded) function of time. In (4.6) above, $m(i,\ell,j,k)$ is the power of $\theta_i^{(\ell)}$ in F_{jk}. F_j is the summation over k of terms of the form (4.6).

We choose from (4.6) a control law of the form:

$$u_j = \sum_k u_{jk}(\theta; t) + \sum_{i=1}^{n_j-1} K_{ji}(\theta; t) \, \tilde{\theta}_j^{(i)} - K_{jn_j}(\theta; t) \, \text{sgn} \, s_j(\theta; t) \tag{4.7}$$

where

$$u_{jk}(\theta; t) = \beta_{jk}(\theta; t) \cdot \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)} \tag{4.8}$$

and, as in section 3, with the convention that we discard the terms $K_{ji}(\theta; t) \, \tilde{\theta}_j^{(i)}$ in u_j for those i for which $c_{ji} = 0$.

A small calculation shows that the sliding condition (2.12) is satisfied for each $s_j(\theta_j;t)$, $j = 1,\ldots,p$ if we choose $\beta_{jk}(\theta;t)$, $\kappa_{ji}(\theta;t)$ according to the following rule:

$$s_j \cdot \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)} < 0 \Rightarrow \beta_{jk}(\theta;t) = \beta_{jk}^{-}(t) \geq \alpha_{jk}(t) \qquad (4.9)$$

$$s_j \cdot \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)} > 0 \Rightarrow \beta_{jk}(\theta;t) = \beta_{jk}^{+}(t) \leq -\alpha_{jk}(t) \qquad (4.10)$$

$$s_j \cdot \tilde{\theta}_j^{(i)} < 0 \Rightarrow \kappa_{ji}(\theta;t) = \kappa_{ji}^{-}(t) \geq -c_{ji} \qquad (4.11)$$

$$s_j \cdot \tilde{\theta}_j^{(i)} > 0 \Rightarrow \kappa_{ji}(\theta;t) = \kappa_{ji}^{+}(t) \leq -c_{ji} \qquad (4.12)$$

and

$$\kappa_{jn_j}(\theta;t) > v_j(t) \quad \text{uniformly in t.} \qquad (4.13)$$

The conditions (4.9), (4.10) are easier to verify than they appear: since we only need determine the sign of powers of $\theta_i^{(\ell)}$ we replace $m(i,\ell,j,k)$ by 0 or 1, according to whether $m(i,\ell,j,k)$ is even or odd, respectively. Moreover, we need only to know the sign of the product

$$s_j \cdot \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)}$$

to determine the control law, so that its numerical value need not be explicitly calculated.

As in section 3, the control law $u_j(\theta,t)$ is not only discontinuous across $\{\theta:s_j = 0\}$, but also across $\{\theta:\theta_i^{(\ell)} = 0\}$ for those i,ℓ for which some $m(i,\ell,j,k)$ is odd,[†] and across the $\{\theta:\tilde{\theta}_j^{(i)} = 0\}$. However, surfaces of the last two varieties may again be shown not to be sliding surfaces, i.e. trajectories may be continued uniquely through them.

_____

[†] This could be avoided by choosing $\beta_{jk}^{-}(t) = -\beta_{jk}^{+}(t) \geq \alpha_{jk}(t)$. Equations (4.8) – (4.10) can then be written

$$u_{jk}(\theta;t) = \beta_{jk}(\theta;t) \cdot \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)}$$

$$s_j < 0 \Rightarrow \beta_{jk}(\theta;t) = \beta_{jk}^{-}(t) \quad ; \quad s_j > 0 \Rightarrow \beta_{jk}(\theta;t) = \beta_{jk}^{+}(t)$$

The approach though complicated notationally is simple in spirit, as the example below shows. Though the control problem is a multi-input problem, it is in effect treated as p single-input problems: the jth sliding surface $s_j(\theta_j;t)$ depends only on θ_j (it involves no contraints on the θ_k for $k \neq j$). Also, in the choice of u_j the terms in θ_k for $k \neq j$ are treated as disturbances as the example shown below explicates.

Example

Consider the system described by the equations

$$\ddot{\theta}_1 = 3\theta_1 + \theta_2^2 + 2\theta_1 \cdot \dot{\theta}_2 \cos \theta_2 + u_1 \tag{4.14}$$

$$\ddot{\theta}_2 = \theta_1^3 - \cos \theta_1 \cdot \theta_2 + u_2 \tag{4.15}$$

The problem to be addressed is to get θ_1, θ_2 to track the parabolas $2t^2$ and t^2 respectively. The sliding surfaces s_1 and s_2 are chosen with this objective in mind as

$$s_1(\theta_1,t) = \dot{\theta}_1 + \theta_1 - 2t(t+2) = 0 \tag{4.16}$$

and $$s_2(\theta_2,t) = \dot{\theta}_2 + \theta_2 - t(t+2) = 0 \tag{4.17}$$

Note that (4.16) and (4.17) are the differential equations governing the two parabolas. Consider first the choice of u_1, of the form (4.7), namely:

$$u_1 = \beta_{11}\theta_1 + \beta_{12}\theta_2^2 + \beta_{13}\theta_1\dot{\theta}_2 + \kappa_{11}(\dot{\theta}_1 - 4t) - \kappa_{12} \text{ sgn } s_1$$

then, we have

$$\frac{1}{2}\frac{d}{dt}s_1^2 = s_1\theta_1(\beta_{11} + 3) + s_1\theta_2^2(\beta_{12} + 1) + s_1\theta_1\dot{\theta}_2(\beta_{13} + 2\cos\theta_2)$$

$$+ s_1(\dot{\theta}_1 - 4t)(\kappa_{11} + 1) - \kappa_{12}|s_1| - 4s_1$$

In accordance with the prescription suggested above, we choose the β_{jk} as follows:

$$s_1 \cdot \theta_1 > 0 \Rightarrow \beta_{11} \leq -3 \qquad\qquad s_1 \theta_1 < 0 \Rightarrow \beta_{11} \geq -3$$

$$s_1 > 0 \Rightarrow \beta_{12} \leq -1 \qquad\qquad s_1 < 0 \Rightarrow \beta_{12} \geq -1$$

$$s_1 \theta_1 \dot{\theta}_2 > 0 \Rightarrow \beta_{13} \leq -2 \qquad\qquad s_1 \theta_1 \dot{\theta}_2 < 0 \Rightarrow \beta_{13} \geq +2$$

$$s_1 \cdot (\dot{\theta}_1 - 4t) > 0 \Rightarrow \kappa_{11} \leq -1 \qquad\qquad s_1 \cdot (\dot{\theta}_1 - 4t) < 0 \Rightarrow \kappa_{11} \geq -1$$

and
$$\kappa_{12} > 4$$

For the choice of u_2, consider again the form

$$u_2 = \beta_{21} \theta_1^3 + \beta_{22}\theta_2 + \kappa_{21}(\dot{\theta}_2 - 2t) - \kappa_{22} \, \mathrm{sgn} \, s_2$$

Then, we have

$$\frac{1}{2} \frac{d}{dt} s_2^2 = \theta_1^3 s_2 (\beta_{21} + 1) + s_2 \theta_2 (\beta_{22} - \cos \theta_1) + s_2 (\dot{\theta}_2 - 2t)(\kappa_{21} + 1)$$

$$- \kappa_{22} |s_2| - 2s_2$$

The β_{jk} are now chosen as follows:

$$s_2 \theta_1 > 0 \Rightarrow \beta_{21} \leq -1 \qquad\qquad s_2 \theta_1 < 0 \Rightarrow \beta_{21} \geq -1$$

$$s_2 \theta_2 > 0 \Rightarrow \beta_{22} \leq -1 \qquad\qquad s_2 \theta_2 < 0 \Rightarrow \beta_{22} \geq 1$$

$$s_2 \cdot (\dot{\theta}_2 - 2t) > 0 \Rightarrow \kappa_{21} \leq -1 \qquad\qquad s_2 \cdot (\dot{\theta}_2 - 2t) < 0 \Rightarrow \kappa_{21} \geq -1$$

and
$$\kappa_{22} > 2 \qquad\qquad\qquad\qquad \blacksquare$$

By a minor modification of the foregoing procedure, it may be extended to the control of systems of the form

$$\theta_j^{(n_j)} = f_j (\theta_1, \theta_2, \ldots, \theta_p; t) + b_j (\theta; t) u_j \qquad\qquad j = 1, \ldots, p$$

so long as the $b_j(\theta,t)$ are of constant sign, say positive, and bounded as follows

$$0 < \chi_j(t) \leq b_j(\theta;t) \leq \psi_j(t)$$

The right hand sides of equation (4.9) - (4.12) are then replaced by

$$\beta_{jk}(\theta;t) = \beta_{jk}^-(t) \geq \alpha_{jk}(t)/\chi_j(t)$$

$$\beta_{jk}(\theta;t) = \beta_{jk}^+(t) \geq \alpha_{jk}(t)/\chi_j(t)$$

and

$$\kappa_{ji}(\theta;t) = \kappa_{ji}^-(t) \geq \text{Max}(-c_{ji}/\chi_j(t), - c_{ji}/\psi_j(t))$$

$$\kappa_{ji}(\theta;t) = \kappa_{ji}^+(t) \leq \text{Min}(-c_{ji}/\chi_j(t), - c_{ji}/\psi_j(t))$$

respectively.

As in Section 3, the effect of time variation of parameters in the right hand side of (4.1) and of disturbances $d_j(\theta;t)$ in each of the equations (4.1) can be nullified by suitable choice of $u_j(\theta;t)$. Consider, for instance, insensitivity to disturbances. Let the disturbances $d_j(\theta;t)$ in the right hand side of (4.1) satisfy

$$\left| d_j(\theta;t) \right| \leq \delta_{jo}(t) + \sum_k \delta_{jk}(t) \prod_{i=1}^{p} \prod_{\ell=0}^{n_i-1} \left| \theta_i^{(\ell)} \right|^{m(i,\ell,j,k)} \tag{4.18}$$

with the $\delta_{jk}(t)$ positive functions of time. The sliding condition (2.12) is then satisfied if the right hand sides of equations (4.9), (4.10), (4.13) are modified as follows:

$$\beta_{jk}^-(t) \geq \alpha_{jk}(t) + \delta_{jk}(t) \tag{4.19}$$

$$\beta_{jk}^+(t) \leq -\alpha_{jk}(t) - \delta_{jk}(t) \tag{4.20}$$

$$\kappa_{jn_j}(\theta;t) > v_j(t) + \delta_{jo}(t) \quad \text{uniformly in t} \tag{4.21}$$

respectively. Equations (4.11), (4.12) are not modified.

Of course, if there are terms of the form $(\theta_i^{(\ell)})^{m(i,\ell,j,k)}$ included in (4.18) which are not present in (4.6), then corresponding terms in u_{jk} need to be included in order to satisfy (4.19), (4.20).

Consider the application of this procedure to the system of the example, with equation (4.14) replaced by

$$\ddot{\theta}_1 = 3\theta_1 + \theta_2^2 + 2\theta_1\theta_2\cos\theta_2 + u_1 + d_1(\theta_1) \qquad (4.22)$$

with d_1 satisfying

$$|d_1(\theta_1)| < \delta_{10} + \delta_{11}|\theta_1| . \qquad (4.23)$$

The terms β_{11} and κ_{12} in the control law need to be modified in accordance with (4.19) - (4.21) to

$$s_1 \cdot \theta_1 > 0 \Longrightarrow \beta_{11} < -3 - \delta_{11} \qquad\qquad s_1 \cdot \theta_1 < 0 \Longrightarrow \beta_{11} > -3 + \delta_{11}$$

and $\kappa_{12} > 4 + \delta_{10}$

in order to retain the tracking in the presence of the disturbance d_1.

Note, as before, that the magnitude of the discontinuity in the control law is proportional to the magnitude of the disturbance. Also, once the trajectory is on a sliding surface $S_j(t)$, its dynamics are governed by

$$\tilde{\theta}_j^{(n_j-1)} + \sum_{i=0}^{n_j-2} c_{ji+1}\,\tilde{\theta}_j^{(i)} = 0 \qquad (4.24)$$

which does not contain the disturbance term.

The choice of C_j such that the surface defined in (4.24) is stable proves particularly useful in Section 5, where we replace the idealized discontinuous control law of this section by a continuous control law which approximates u in a suitable sense.

We remark here that the development of this section using the poly-nomial bounds of equations (4.2), (4.18) can be generalized when $f_j(\theta;t)$, $d_j(\theta;t)$ are bounded by other classes of functions. For instance, if the disturbance d_1 in (4.22) is a function of θ_1 and θ_2 and satisfies (instead of (4.23))

$$\left| d_1(\theta_1, \theta_2) \right| < \delta_{10} + \delta_{11}\left| \theta_1 \right| + \exp \theta_2$$

then we modify the control law u_1 to contain in addition a term of the form $\beta_{14} \exp \theta_2$ where

$$s_1 > 0 \Longrightarrow \beta_{14} \leq -1 \qquad \text{and} \qquad s_1 < 0 \Longrightarrow \beta_{14} \geq 1 \quad .$$

Section 5. <u>Continuous Control Laws to Approximate Sliding Mode Control</u>

The usage of discontinuous or switched control to generate robust

control laws is not without a price. In practice, imperfections such

as delays in switching, hysteresis in switching, will cause the trajectory

to <u>chatter</u> along the sliding surface as was illustrated in Section 2.

This will of course be accompanied by a rapidly (time)-varying control

law. Chattering is undesirable both in itself and in the fact that it

represents a 'high frequency' signal component in the state trajectory,

which may excite unmodelled 'high-frequency' dynamics. Thus, while

sliding mode control provides control laws which are robust to parameter

variations and disturbance inputs, they are, in themselves, not robust to

the usual modelling approximations (i e., neglect of dynamics which lie

outside the frequency range of interest) that go into control system design.

We seek in this section to remedy this situation by replacing the dis-

continuous or switched feedback laws of the previous section by continuous

control laws which will preserve the disturbance rejection properties of

sliding mode control, and in addition not generate undesirable high

frequency signals.

The basic idea is simple: it consists of 'smoothing' out the dis-

continuity in the control law at the switching surface, i.e., find,

in the notation of Section 4, a continuous control law $u_j(\theta;t)$ whose

terms are continuous functions of θ inside a small boundary

layer neighboring the switching surface . The boundary layer then

plays the role of a smudged switching surface i.e., trajectories starting

outside the boundary layer converge to it and further, the positive

invariance of the boundary layer is robust to parametric variations in the dynamics outside. The penalty paid for smudging the sliding surface is that the dynamics of the state trajectory inside the boundary layer are only an approximation to the desired dynamics on the sliding surface. The advantage of the scheme is that the state trajectory does not chatter very rapidly close to the sliding surface - in fact the wider the boundary layer, the lower the chatter, but the lower the tracking accuracy.

To carry out the preceding program we use again the class of stable sliding surfaces considered in section 4, with the $s_j(\theta_j;t)$ of the form:

$$s_j(\theta_j;t) = c_j\tilde{\theta}_j(t) \tag{5.1}$$

To define the boundary layer about the sliding surface of (5.1), define

$$s_j^-(\theta_j;t) = s_j(\theta_j;t) + c_{j1}\varepsilon_j \tag{5.2}$$

and

$$s_j^+(\theta_j;t) = s_j(\theta_j;t) - c_{j1}\varepsilon_j \tag{5.3}$$

Figure 6 shows the relative position of the surfaces $s_j=0$, $s_j^-=0$ and $s_j^+=0$ for the case that $n_j=2$. Note that $c_{j1}>0$ since (4.2) is Hurwitz. The boundary layer $B_j(t)$ is defined by

$$B_j(t) = \{\theta : s_j^-(\theta_j;t) > 0 \text{ and } s_j^+(\theta_j;t) < 0\} \tag{5.4}$$

It is immediate from (5.2) and (5.3) that

$$\frac{d}{dt}s_j^-(\theta_j;t) = \frac{d}{dt}s_j(\theta_j;t) = \frac{d}{dt}s_j^+(\theta_j;t) \quad .$$
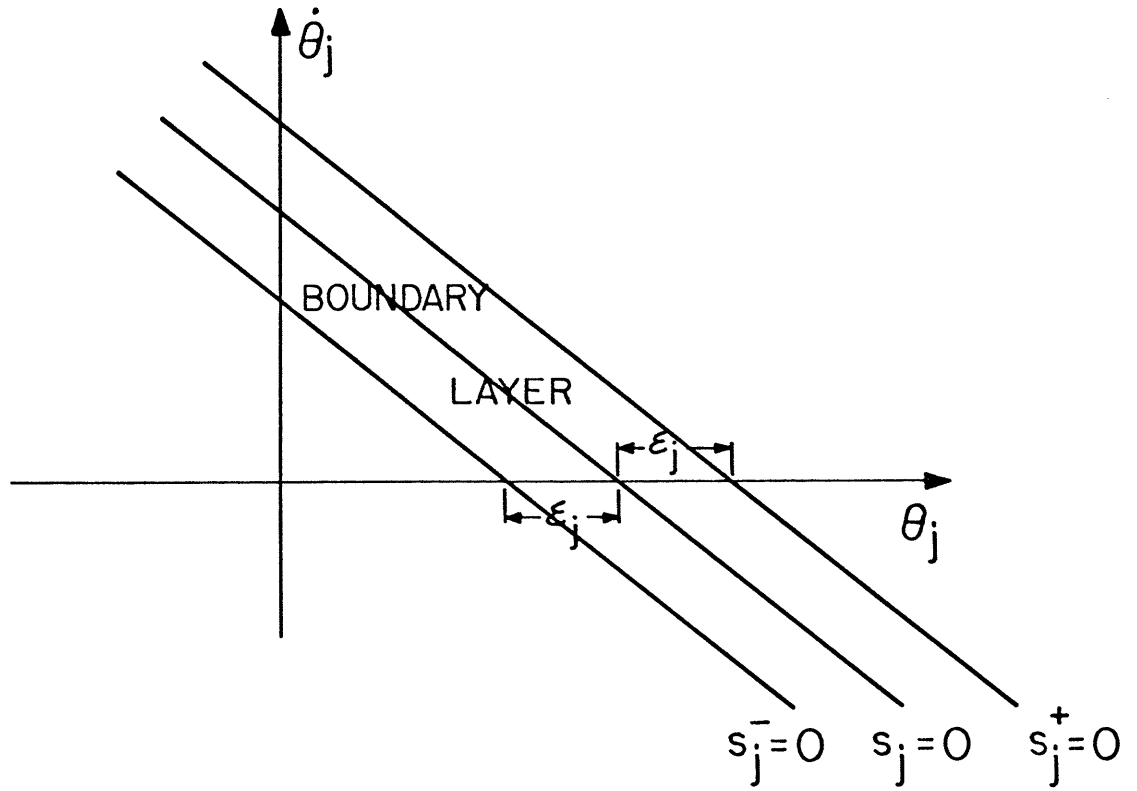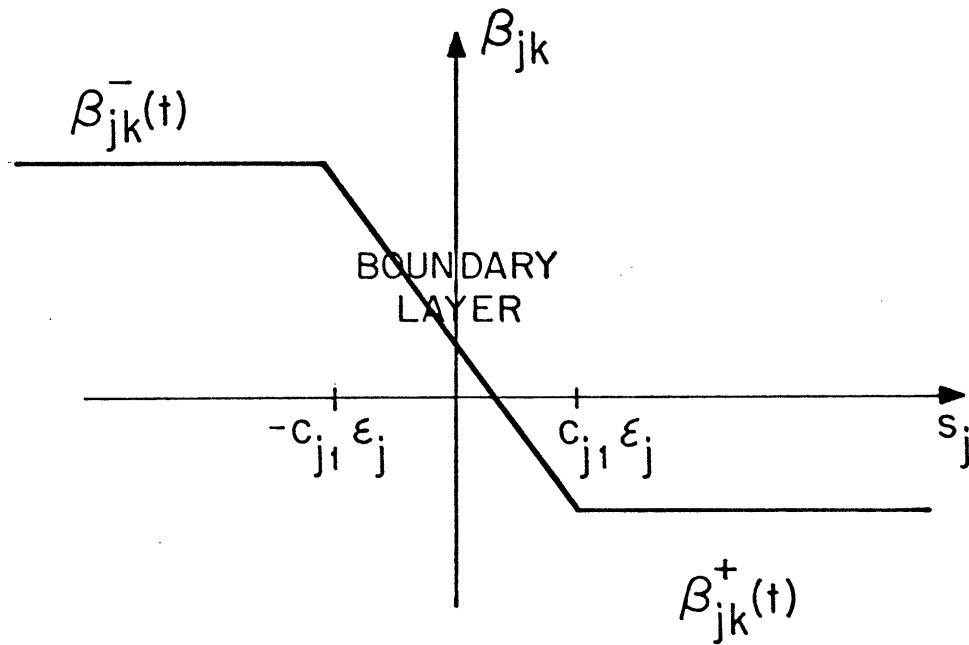
Figure 6: Showing the Construction of the Boundary Layer

Figure 7: Showing a Sample Interpolation between $\beta_{jk}^{-}(\Theta,t)$ and $\beta_{jk}^{+}(\Theta,t)$ in the Boundary Layer

We choose control law $u_j(\theta;t)$ as given by (4.7)-(4.13) for

$\theta \in \{\theta : s_j^-(\theta_j;t) < 0\}$ or $\theta \in \{\theta:s_j^+(\theta_j;t) > 0\}$, i.e. outside $B_j(t)$. This

guarantees that

$$\frac{d}{dt} s_j^-(\theta_j;t) > 0 \quad \text{for } \theta \in \{\theta:s_j^-(\theta_j;t) \leq 0\} =: S_j^-(t) \tag{5.5}$$

and

$$\frac{d}{dt} s_j^+(\theta_j;t) < 0 \quad \text{for } \theta \in \{\theta:s_j^+(\theta_j;t) \geq 0\} =: S_j^+(t) \tag{5.6}$$

(5.5) and (5.6) establish (by the same arguments as in Section 2) that

trajectories starting outside $B_j(t)$ tend towards $B_j(t)$, and further

trajectories starting inside $B_j(t)$, stay in it for all future

time. It only remains to specify $u_j(\theta;t)$ to be a continuous function of

θ inside $B_j(t)$. We claim that any continuous interpolation between

$u_j(\theta;t)$ defined on $S_j^-(t)$ and $u_j(\theta;t)$ defined on $S_j^+(t)$ will suffice

for our purposes (at least one such interpolation exists, by Urysohn's

lemma [4]). Figure 7 illustrates a sample interpolation for one of the

$\beta_{jk}(\theta;t)$ of (4.9), (4.10) in the case that $\prod\limits_{i=1}^{p} \prod\limits_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)} > 0$

at time $t^\dagger$. Similar interpolations are to be performed for the $\kappa_{ji}(\theta;t)$.

We now show that with the preceding choice of the control law

$u_j(\theta;t)$, $\theta_j(t)$ tracks $\theta_{dj}(t)$ to within a small error linearly proportional

to ε_j (in particular, the error goes to zero when ε_j does). Note that

with the preceding choice of continuous control law the trajectory

$\theta(t)$ satisfies a regular differential equation. Further, if at t=0,

$\theta(0) \in B(0)$, then $\theta(t) \in B(t)$ for all time. Hence

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$\dagger$

For the case $\prod\limits_{i=1}^{p} \prod\limits_{\ell=0}^{n_i-1} (\theta_i^{(\ell)})^{m(i,\ell,j,k)} < 0$, replace s_j by $-s_j$ in Fig.7.

$$s_j(\theta_j;t) = c_{j1}\Delta(t) \qquad \forall\ t \geq 0 \qquad\qquad (5.7)$$

where $\Delta(t)$ is some function satisfying

$$|\Delta(t)| \leq \epsilon_j$$

Using the form of $s_j(\theta_j;t)$ given by (5.1) it is then possible to bound $||\theta_j(t) - \theta_{jd}(t)||$, using elementary linear algebra. It is easy to verify for instance, that if $s_j(\theta_j;t)$ is chosen to be of the form

$$s_j(\theta_j;t) = (\frac{d}{dt} + \lambda_j)^{n_j-1}(\theta_j-\theta_{dj})$$

then with $\theta_j(0) = \theta_{dj}(0)$ the tracking accuracy is

$$|\theta_j(t) - \theta_{dj}(t)| \leq \epsilon_j \qquad \forall\ t \geq 0. \qquad\qquad (5.8)$$

In the instance that $\theta_j(0)$ does not exactly match $\theta_{dj}(0)$, (5.8) is modified to

$$|\theta_j(t) - \theta_{dj}(t)| \leq \epsilon_j + P(t)||\tilde{\theta}_j(0)||\exp{-\lambda t} \qquad \forall t \geq 0$$

with $P(t)$ a polynomial in t.

An examination of (4.9)-(4.13) shows that the control law $u_j(\theta;t)$ of equation (4.7) is discontinuous not only across the surfaces $\{\theta:s_j=0\}$, but also across the surfaces given by $\{\theta:\theta_i^{(\ell)}=0\}$ for those i,ℓ for which some $m(i,\ell,j,k)$ in (4.6) is odd, and across $\{\theta:\tilde{\theta}_j^{(i)} = 0\}$. However, as noted in section 4, surfaces of the last two categories are not sliding surfaces (in particular our solution concept calls for a unique extension of trajectories through them). Hence, we need not replace the discontinuous control law at these surfaces by a continuous one - and, of course, no high frequency chattering is generated at these surfaces by switching imperfections.

We now illustrate the application of the methodology of Sections 4 and 5 to the control of a two-link manipulator.

Section 6. Application: Sliding Mode Control of a Two-Link Manipulator

The accurate, high-speed tracking of desired trajectories is the control-challenge in the development of modern industrial robots and manipulators. Typically, the equations of motion of these robots are highly non-linear and coupled. Also, the development of flexible manufacturing systems calls for robustness of performance with regard to the variation of the load, task or real time trajectory specification, as well as other 'disturbances'. Given these complexities, no viable control methodology has as yet been proposed for these problems.

In general, the dynamics of industrial robots can be described by equations of the form (4.1), with bounds of the form (4.2) arising from the presence of sine and cosine terms (as shown in the sequel). System parameters undergo variations because of variations in the loads in a flexible manufacturing system environment, variations in the ambients, imprecise modelling and the like. We describe the application of our procedure to the robust, sliding mode control of a two-link manipulator. In Section 4, we showed how a multiple (p) input control problem was decomposed into a set of decoupled single-input problems. By this token, we see that the complexity of our design procedure for a more sophisticated manipulator involving more than two links is not significantly increased. By design, our sliding mode feedback controller is robust to certain variations in parameter values, an improvement over the 'non-linear decoupling techniques' proposed by Freund [3], on-line computational schemes proposed by Luh, et al [6], [13], and the linearization techniques of Melouah, et al. [14].

Consider the two-link manipulator (in the horizontal plane) of Figure 8, with rigid links of nominally equal length ℓ and mass m (both m and ℓ will be assumed normalized to 1). Choosing as state variables the angle θ_1 (made by the first link with the x-axis), and the angle θ_2 made by the second link with respect to the first; the angular velocities $\dot{\theta}_1$, $\dot{\theta}_2$; and, as inputs, torques T_1 and T_2 applied to the two joints, we get for the dynamics of the manipulator (see e.g. [15]):

$$\ddot{\theta}_1 = [2/3 \ T_1' - (2/3 + \cos \theta_2)T_2']/(16/9 - \cos^2\theta_2) \qquad (6.1)$$
$$\ddot{\theta}_2 = [-(2/3 + \cos \theta_2)T_1' + 2(5/3 + \cos \theta_2)T_2']/(16/9 - \cos^2\theta_2) \qquad (6.2)$$

where

$$T_1' = 2T_1 + \sin \theta_2 \cdot \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2)$$

and

$$T_2' = 2T_2 - \sin \theta_2 \dot{\theta}_1^2$$

Adopting the notation of Section 4, we have

$$\Theta_j = [\theta_j, \dot{\theta}_j]^T \qquad \text{for } j = 1,2$$

Define u_1 and u_2 to be

$$u_1 = 4/3 \ T_1 - (4/3 + 2 \cos \theta_2)T_2 \qquad (6.3)$$

$$u_2 = -(4/3 + 2 \cos \theta_2)T_1 + (20/3 + 4 \cos \theta_2)T_2 \qquad (6.4)$$

The idea behind (6.3) and (6.4) is that they are 'invertible' i.e. they can be solved to obtain T_1 and T_2 as functions of u_1 and u_2. With the definitions of u_1 and u_2 as in (6.3) and (6.4), equations (6.1), (6.2) can be written as
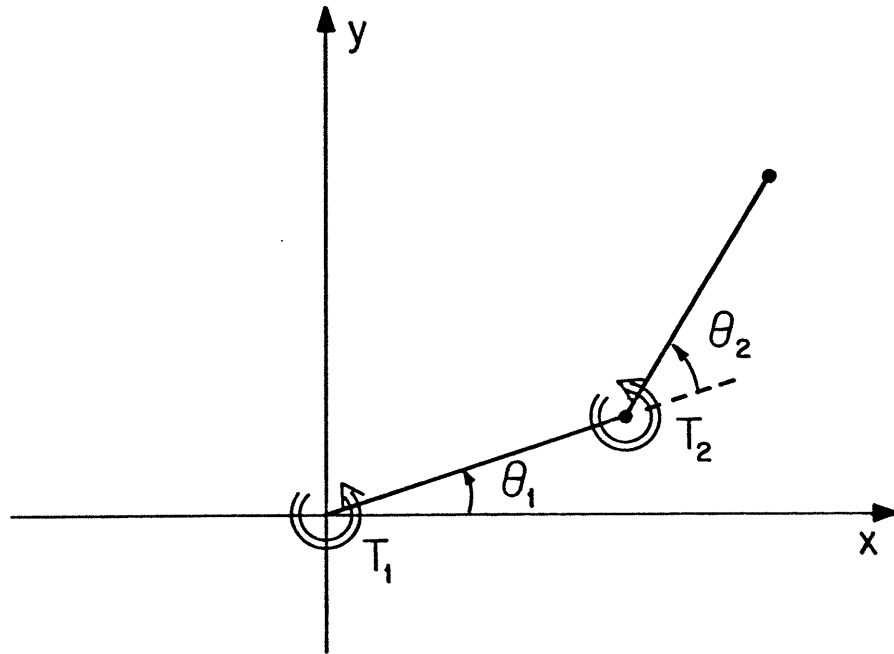
Figure 8: A Two-Link Manipulator

$$\ddot{\theta}_1 = [2/3 \cdot \sin\theta_2 \cdot \dot{\theta}_2 \cdot (2\dot{\theta}_1 + \dot{\theta}_2) - (2/3 + \cos\theta_2)\sin\theta_2 \dot{\theta}_1^2 + u_1]/(16/9 - \cos^2\theta_2)$$

(6.5)

$$\ddot{\theta}_2 = [-(2/3 + \cos\theta_2)\sin\theta_2 \cdot \dot{\theta}_2 \cdot (2\dot{\theta}_1 + \dot{\theta}_2) - 2(5/3 + \cos\theta_2)\sin\theta_2 \dot{\theta}_1^2 + u_2](16/9 - \cos^2\theta_2)$$

(6.6)

The aim of the design is to get $\theta_j(t)$ to track a desired trajectory $\theta_{dj}(t)$ (for j=1,2). Accordingly, we choose

$$s_j(\theta_j;t) = (\dot{\theta}_j - \dot{\theta}_{dj}) + 5(\theta_j - \theta_{dj}) \quad \text{for } j=1,2 \qquad (6.7)$$

and we assume as in (4.5) that $\ddot{\theta}_{dj}$ is bounded, specifically that:

$$|\ddot{\theta}_{dj}(t)| < 1.75 \qquad \text{rad./sec.}^2 \qquad (6.8)$$

(6.8) is the only <u>a priori</u> information required regarding θ_{dj}.
As in Section 4, we choose u_1 and u_2 to be of the form

$$u_1 = \beta_{11}\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2) + \beta_{12}\dot{\theta}_1^2 + \kappa_{11}(\dot{\theta}_1 - \dot{\theta}_{d1}) - \kappa_{12}\,\text{sgn}\,s_1 \qquad (6.9)$$

$$u_2 = \beta_{21}\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2) + \beta_{22}\dot{\theta}_1^2 + \kappa_{21}(\dot{\theta}_2 - \dot{\theta}_{d2}) - \kappa_{22}\,\text{sgn}\,s_2 \qquad (6.10)$$

We choose to keep the terms $\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2)$ grouped in (6.9), (6.10), since they appear in this form in the system description (6.1), (6.2). The surface $\{\theta: 2\dot{\theta}_1 + \dot{\theta}_2 = 0\}$ is not a sliding surface in the following development since trajectories can be uniquely continued through it. Again, proceeding as in Section 4, we chose for the control law:

$$\beta_{11}^- = -\beta_{11}^+ = .7$$

$$\beta_{12}^- = \beta_{21}^- = -\beta_{12}^+ = -\beta_{21}^+ = 1.2$$

$$\beta^-_{22} = -\beta^+_{22} = 4.4$$

$$\kappa^-_{j1} = -3.8 \quad ; \quad \kappa^+_{j1} = -9 \quad ; \quad \kappa_{j2} = 3.15 \quad \text{for } j=1,2$$

The preceding manipulations yield a discontinuous control law. Following the development of Section 5, we obtain a continuous control law inside $B_j(t)$ given by

$$\kappa_{j2} \left| s_j / 5\varepsilon_j \right|$$

in place of $\kappa_{j2} \text{ sgn } s_j$, and similar linear interpolations for the β_{jk} and κ_{j1}. Both ε_1 and ε_2 are chosen to be 1°.

Given the values of u_1 and u_2 in (6.3) and (6.4), we solve for T_1 and T_2, the real control inputs for the manipulator.

A digital simulation of the preceding control scheme was performed using a sampling rate of 50 Hz. The simulation also added random measurement noise (uniformly distributed on the intervals [0, 0.25] degrees for angles, and [0,0.5] degrees per second for the angular velocities) to study experimentally the robustness of our proposed scheme to noise, a topic that remains to be studied analytically. The motion of the manipulator was simulated with the aid of a fourth order Adams-Bashforth algorithm (with fixed step size of 6.67 milliseconds). Plots of the simulated trajectory of the manipulator are presented in Figure 9 (the rate of the plotter was 150 points per second). The manipulator was initially idle at $\theta_1 = -90°$, $\theta_2 = 170°$ and was required to track
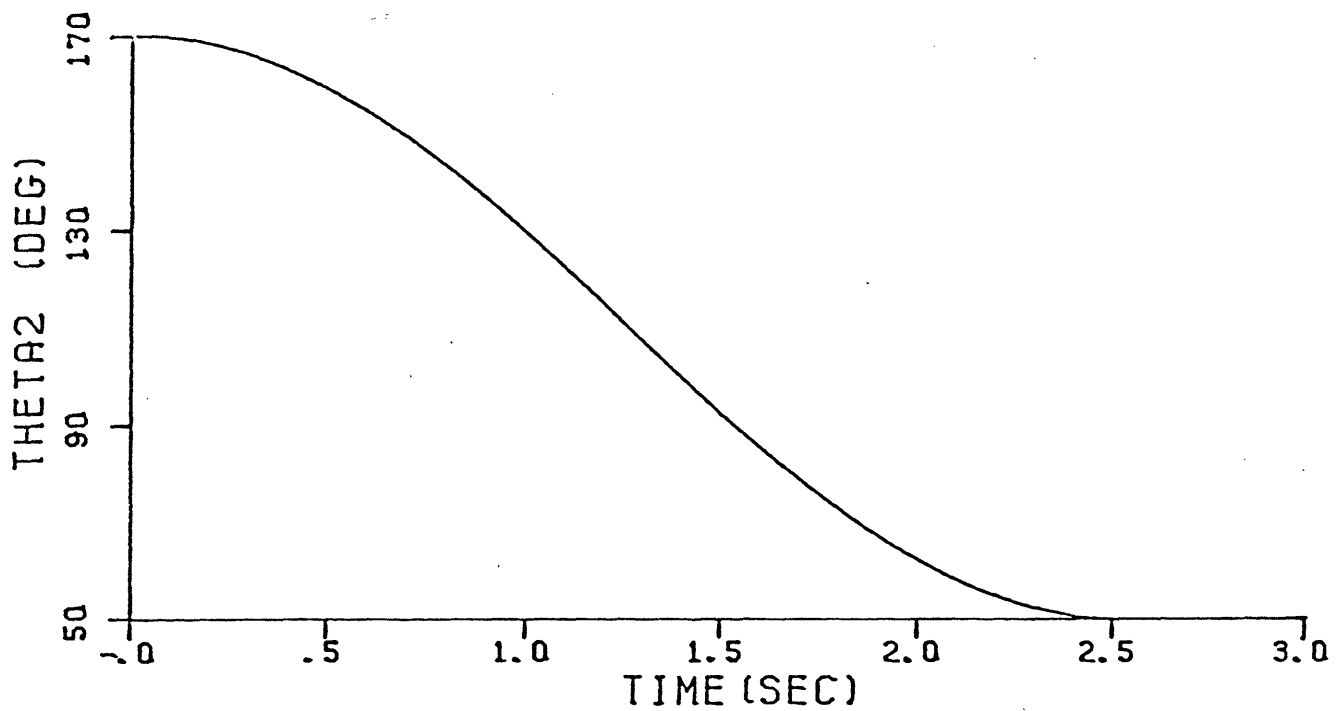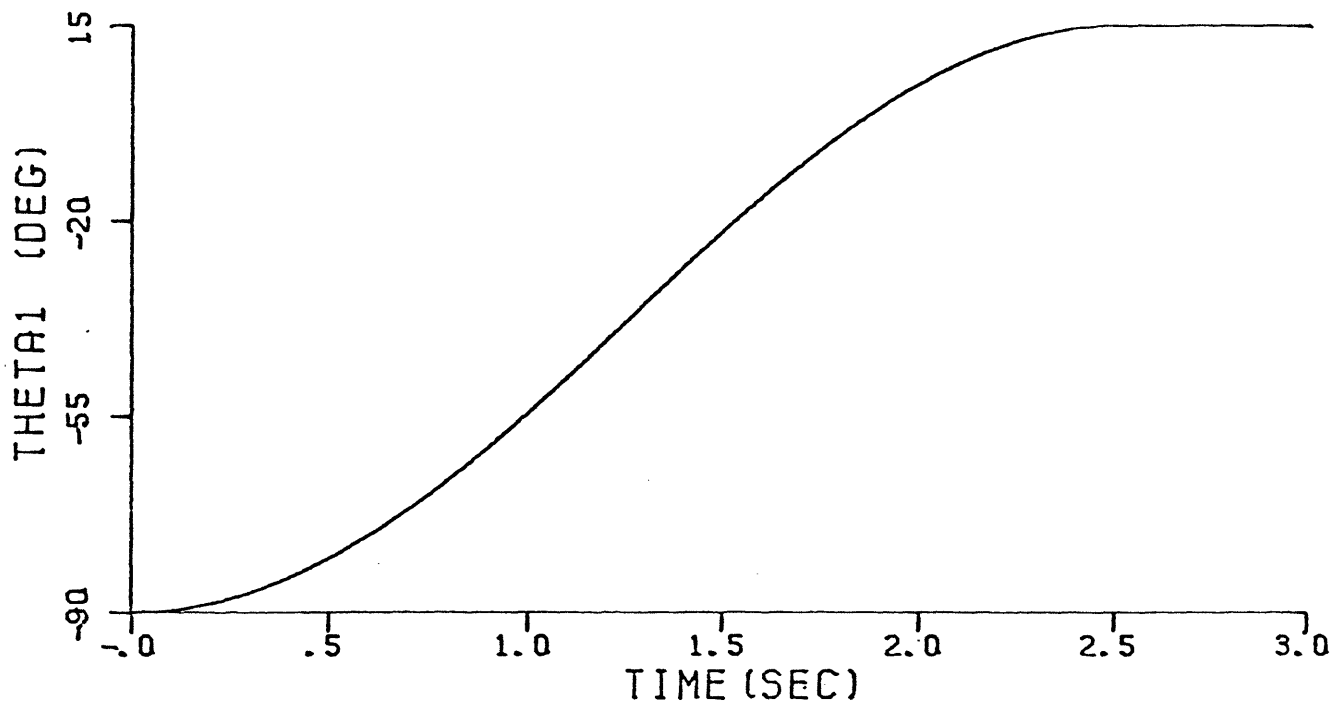
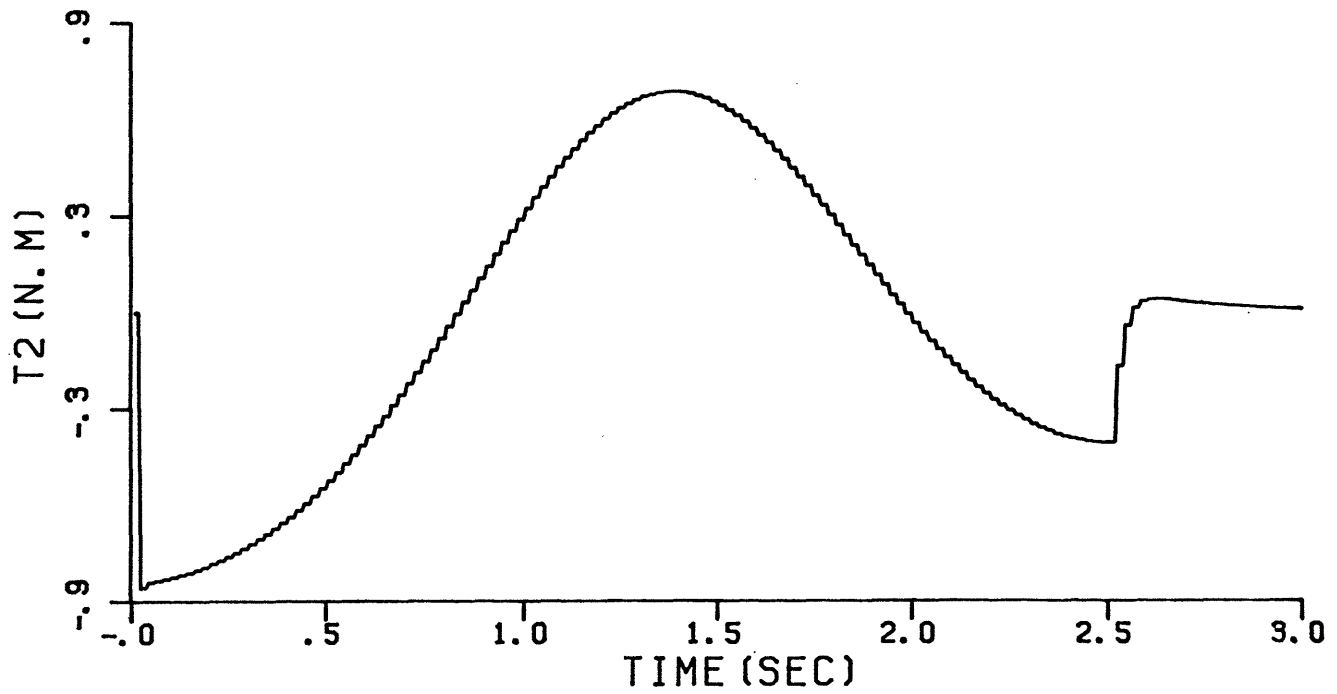Fig. 9.a: Trajectories of the angles θ_1, θ_2 of the manipulator
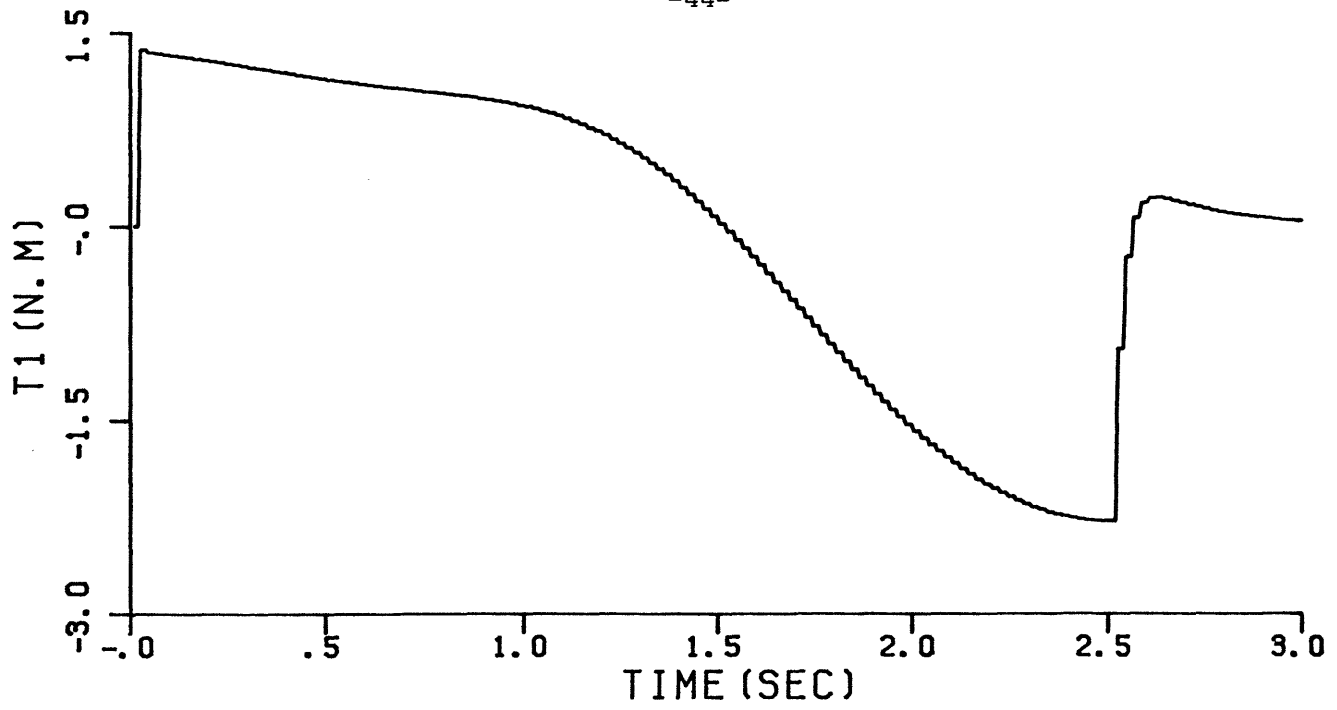
Fig. 9.b   Control Torques T_1 and T_2 .

$$\theta_{d1}(t) = -90° + 52.5° \ (1-\cos 1.26 \ t) \qquad \text{for } t \leq 2.5$$

$$= 15° \qquad \text{for } t > 2.5$$

$$\theta_{d2}(t) = 170° - 60° \ (1-\cos 1.26 \ t) \qquad \text{for } t \leq 2.5$$

$$= 50° \qquad \text{for } t > 2.5$$

which satisfy equation (6.8) with the angles $\theta_{d1}(t)$, $\theta_{d2}(t)$ measured in radians. The computational delay is one sampling time.

The simulation results show tracking to within an error of .7° in θ_1 and θ_2. Note that $\ddot{\theta}_{d1}$, $\ddot{\theta}_{d2}$ and hence T_1, T_2 are discontinuous at t=2.5.

To show the robustness of our scheme to parameter variation, we demonstrate how a modification of our control law results in tracking in the face of varying load μ (between 0 and 0.25) at the tip of the load arm. The system equations are then modified to (see e.g. [15]):

$$\ddot{\theta}_1[2(5/3+\cos \theta_2) + 4\mu(1+\cos \theta_2)] + \ddot{\theta}_2[2/3 + \cos \theta_2 + 2\mu(1+\cos \theta_2)]$$

$$= 2T_1 + \sin \theta_2 \cdot \dot{\theta}_2 \cdot (2\dot{\theta}_1+\dot{\theta}_2)(1+2\mu) \qquad (6.11)$$

$$\ddot{\theta}_2[2/3 + \cos \theta_2 + 2\mu(1+\cos\theta_2)] + \ddot{\theta}_2[2/3 + 2\mu] = 2T_2 - \sin \theta_2\dot{\theta}_1^2(1+2\mu)$$

$$(6.12)$$

To keep (6.7) a sliding surface for μ belonging to [0,0.25], we choose the control law of (6.9) and (6.10) with

$$\beta_{11}^- = -\beta_{11}^+ = 1.2$$

$$\beta_{12}^- = \beta_{21}^- = -\beta_{12}^+ = -\beta_{21}^+ = 2.1$$

$$\beta_{22}^- = -\beta_{22}^+ = 6.4$$

$$\kappa_{j1}^- = -2.4 \ ; \ \kappa_{j1}^+ = -15.2 \quad \text{for } j=1,2$$

The modification of the terms K_{12} and K_{22} in the control law is more involved. The disturbance term in u_1, u_2 of (6.3), (6.4) caused by the presence of μ in (6.11), (6.12) includes terms in T_1 and T_2. This leads us to choose the k_{j2} to contain terms in T_1 and T_2:

$$K_{12} = 5.5 + |T_2|/2 \tag{6.13}$$

$$K_{22} = 5.5 + |T_1|/2 + |T_2| \tag{6.14}$$

From an inspection of the values of T_1 and T_2 in simulations, we found that their variation was small and that (6.13), (6.14) could be replaced by constant K_{12}, K_{22} using conservative bounds on $|T_1|$ and $|T_2|$.

Plots of the simulated trajectory of the manipulator with this modified control law, tracking the same θ_{dj} as before, are presented in Figure 10 (for the no-load, $\mu=0$. case) and Figure 11 (for the full load, $\mu=0.25$ case). The idealized control laws are approximated by continuous control laws exactly as before and yield tracking precision of $0.9°$ for the no-load case and $1.9°$ for the full-load case, with $\varepsilon_1 = \varepsilon_2 = 2.5°$. The tracking error may be decreased by decreasing the sampling time and the $\varepsilon_j's$.

Young [11] has proposed the use of classical sliding surface methodology to stabilize a two dimensional manipulator, and has suggested extensions to tracking. Our approach is explicitly for the purpose of tracking and does not involve a 'reaching' phase to the sliding surface. By decoupling a multiple-input problem into several single-input problems, we avoid the problems associated with reaching a 'hierarchy' of sliding surfaces. By smudging the control across the discontinuity surface we mitigate the extent of the chattering.
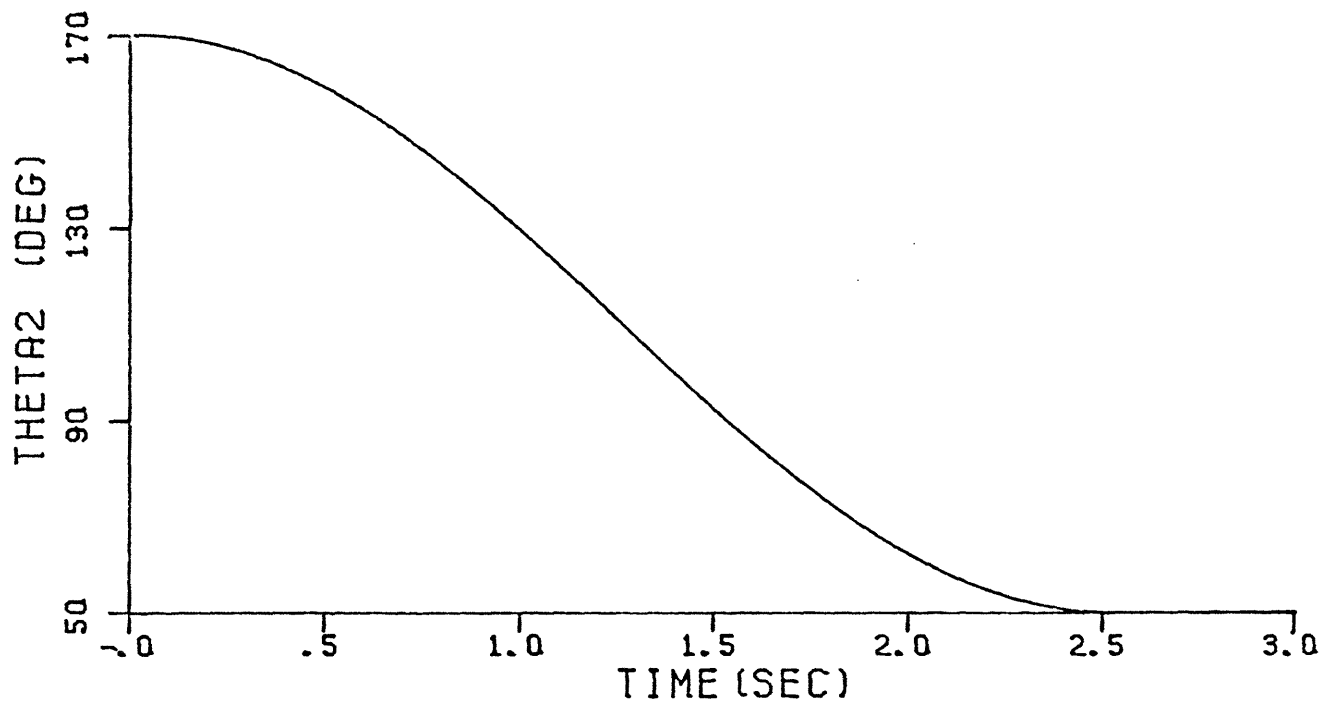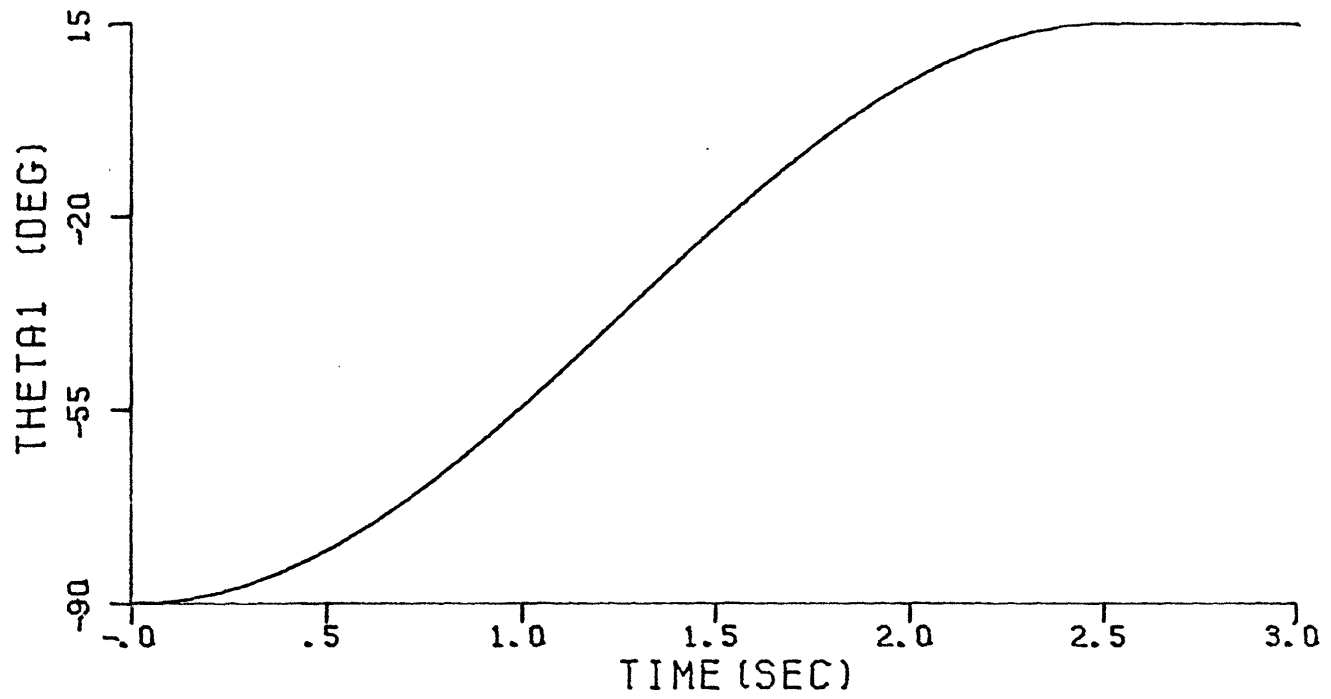
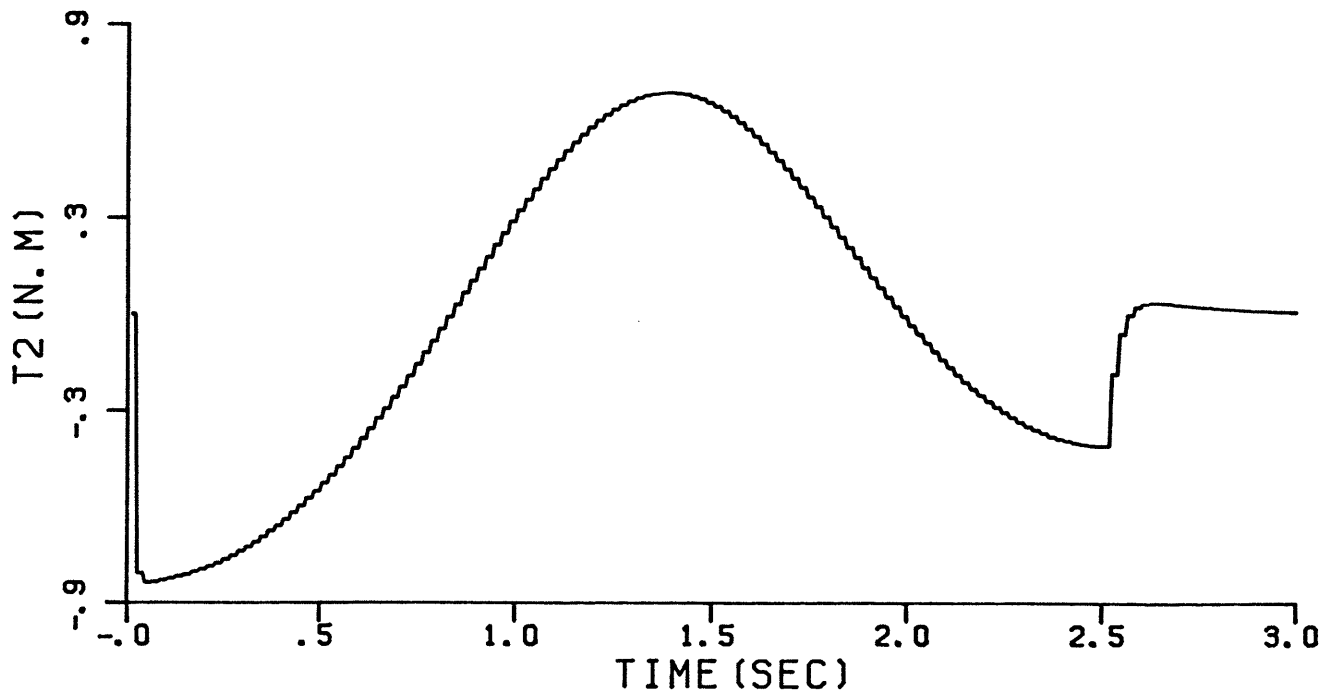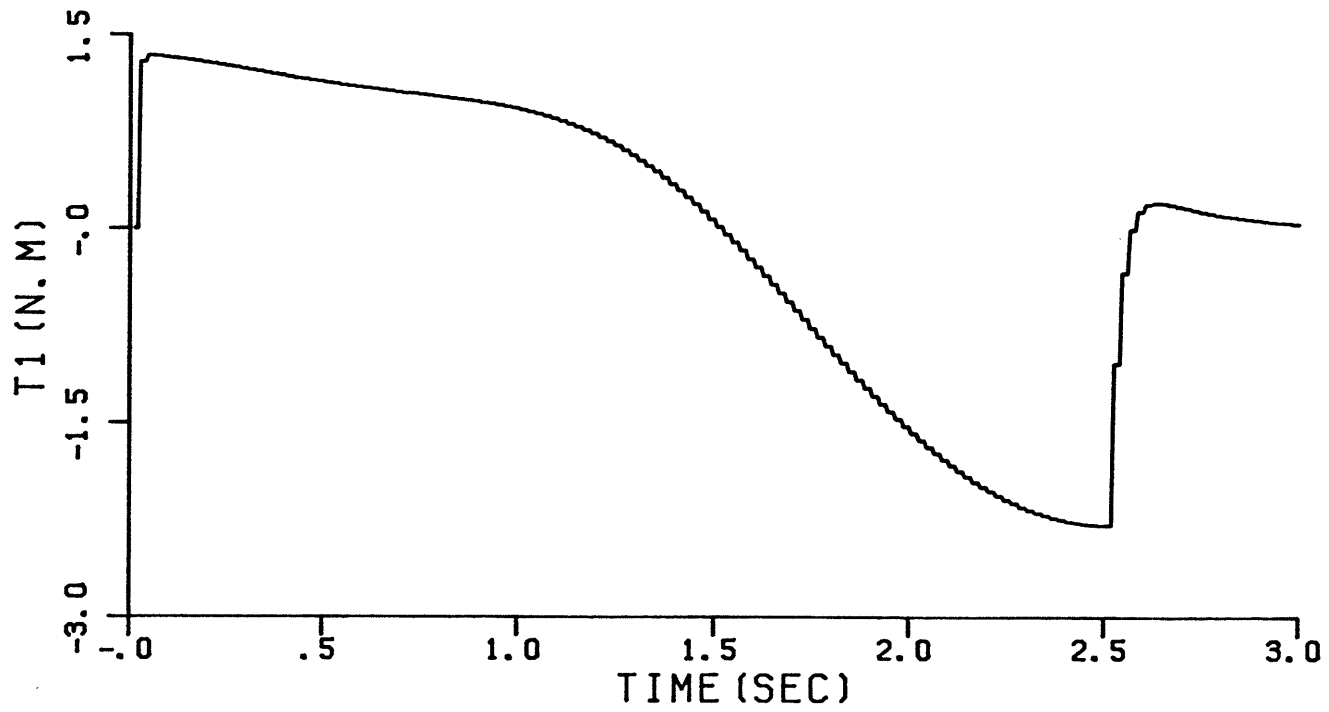Fig. 10a: Trajectories of the angles θ_1, θ_2 of the manipulator under no load

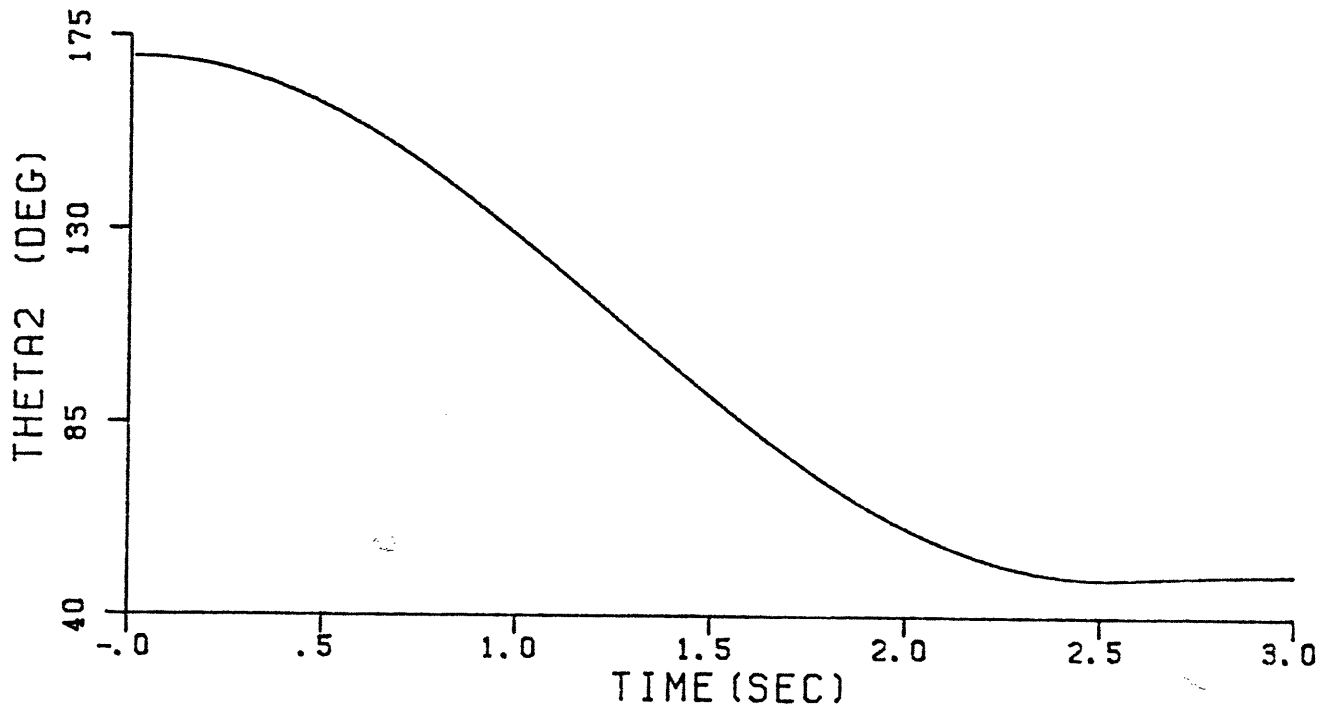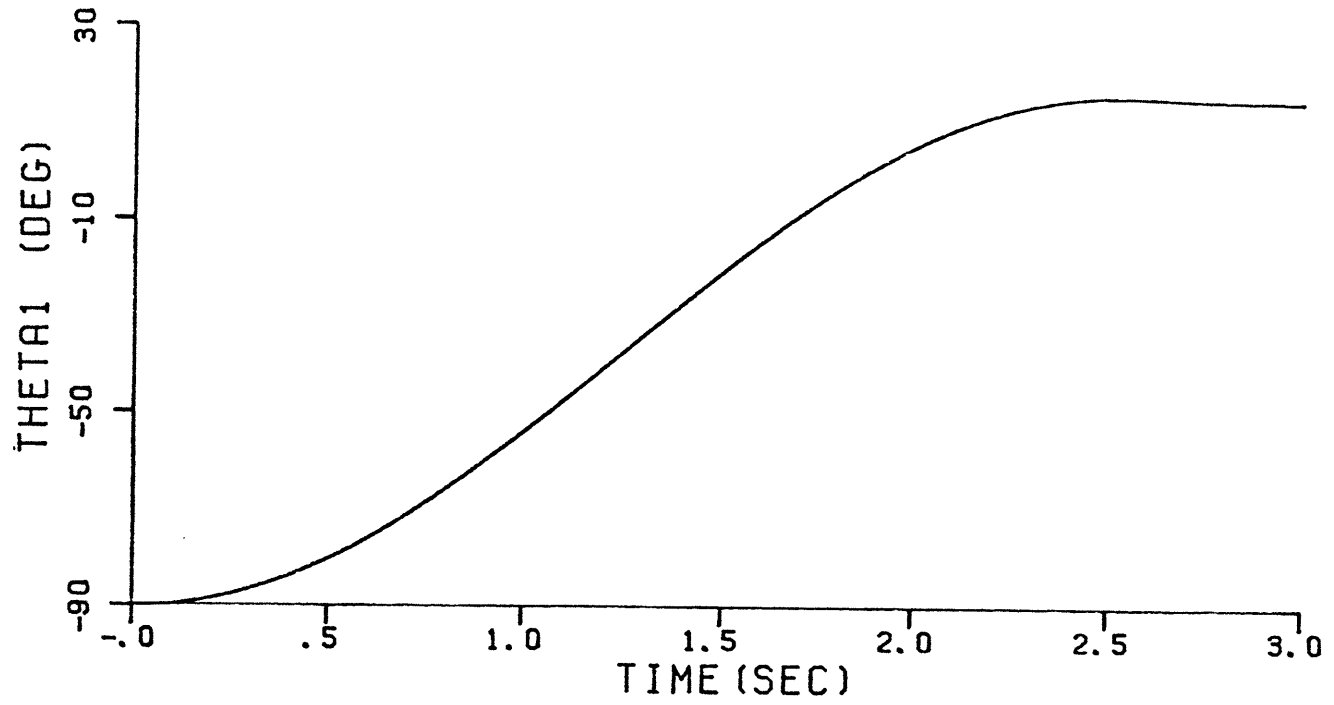Fig. 10.b: Control Torques T_1 and T_2 under no load

Fig. 11.a: Trajectories of the angles θ_1, θ_2 of the manipulator under full load

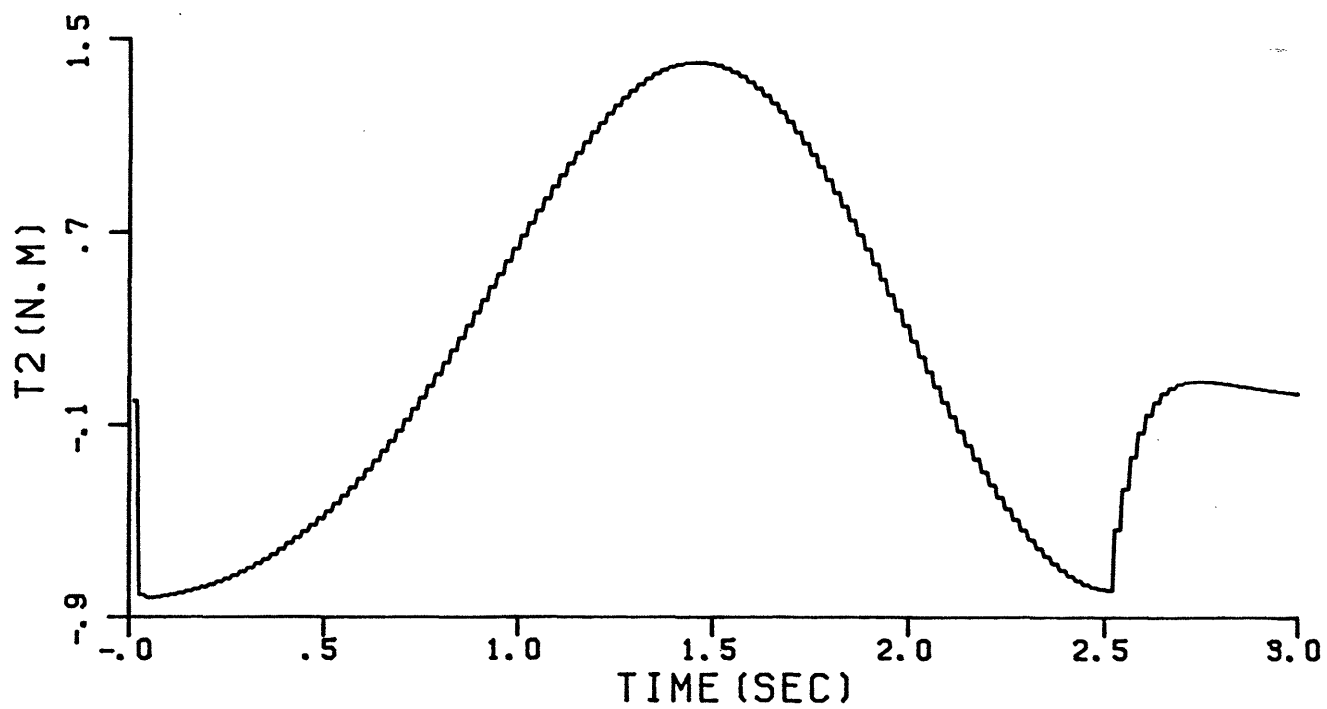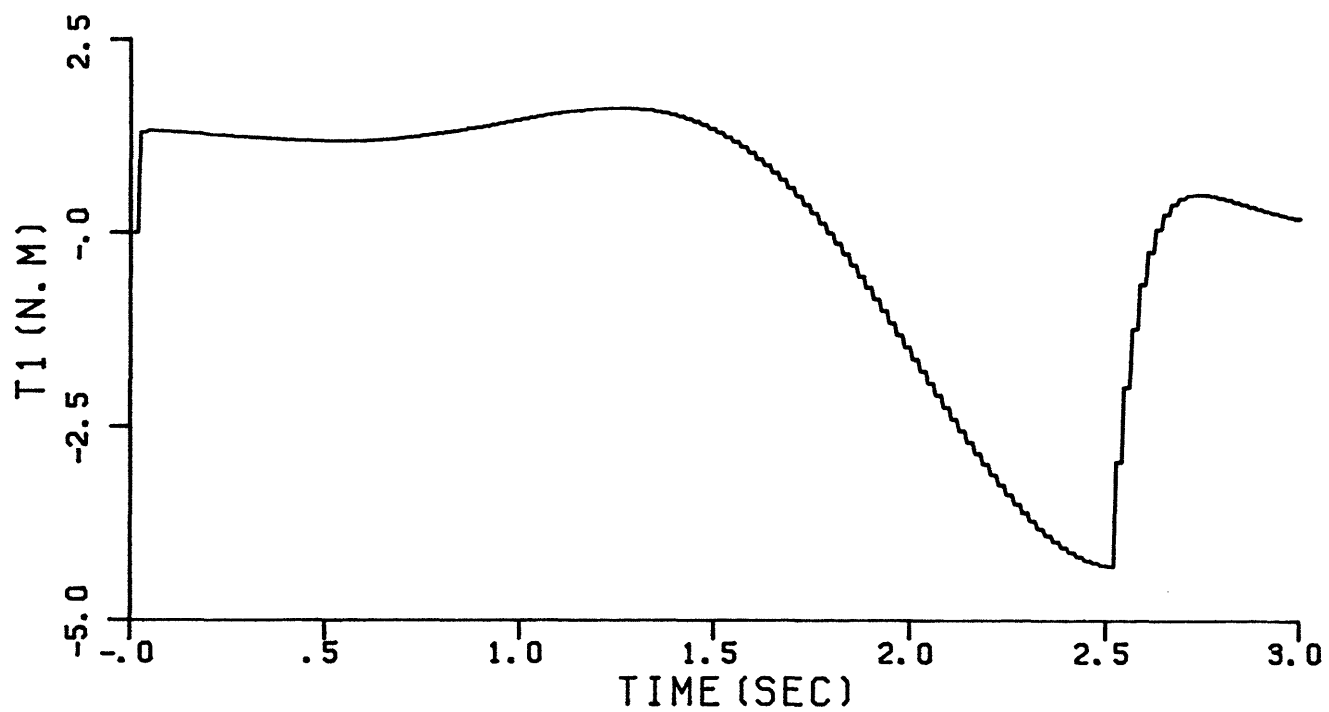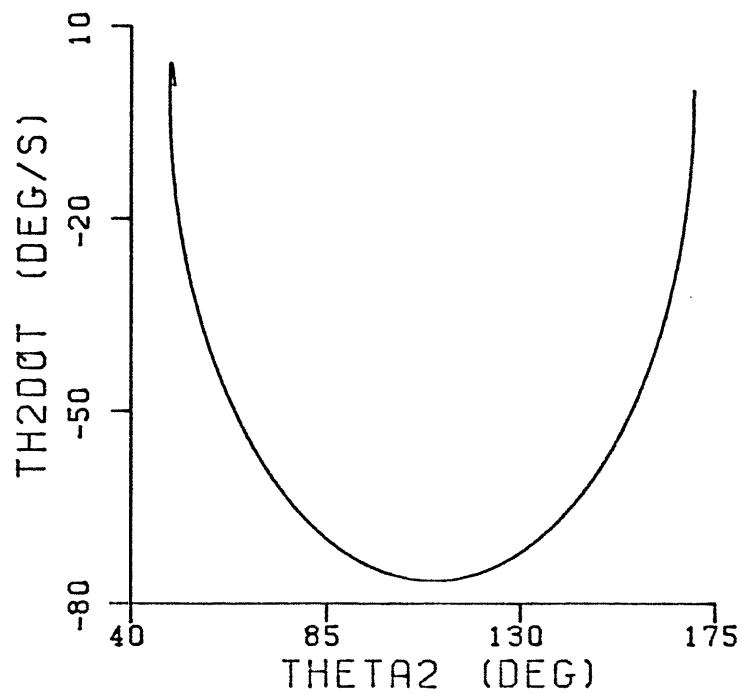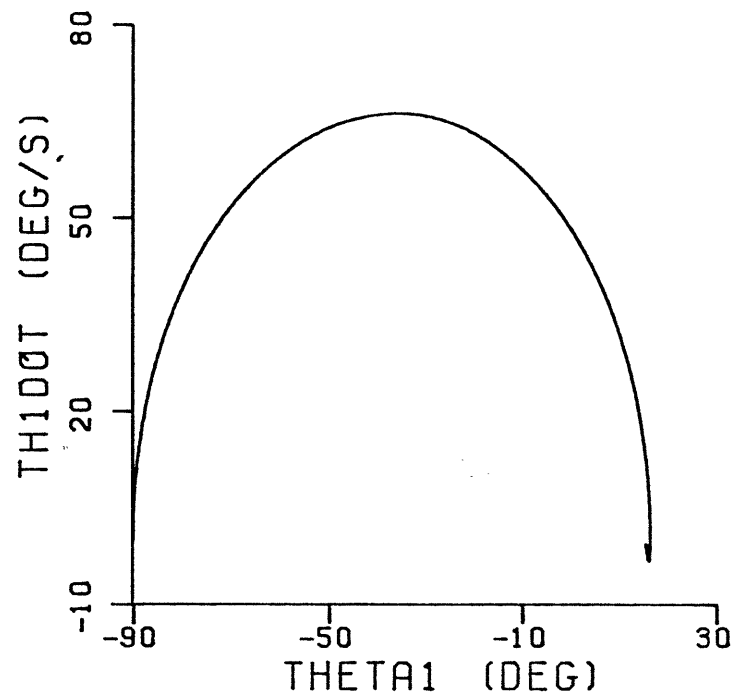Fig. 11.b. Control Torques T_1 and T_2 under full load

Fig. 11.c: Phase portraits for θ_1 and θ_2 (TH1DOT=$\dot{\theta}_1$; TH2DOT=$\dot{\theta}_2$) under full load, showing no noticeable chattering

Section 7. Areas of Further Research

Certainly the present paper is only a step in developing the
sliding-mode control methodology for the robust control of a class of non-
linear time-varying systems. The methodology needs to be extended to more
general classes of non-linear systems than those discussed in Section 4.
In its present form, the feedback control law uses full state feedback -
the case of output feedback (with observers) remains to be investigated.
In a related context, the effects of measurement noise and process noise
on the sliding mode control methodology have yet to be studied. The
continuous control laws of Section 5 were derived in order to trade off
the generation of undesirable high frequency signal against tracking
accuracy. The precise nature of this trade-off needs to be quantified.
Finally, the use of sampled-data control to implement the sliding mode
control presents new problems in the analysis of the resultant hybrid
scheme. While sampled-data control was in fact used successfully in the
example of Section 6, we believe that further research needs to be
done in this direction of implementation.

Finally, we have used the currently important area of manipulator
control as the trial area for our methodology. We are now in the process
of implementing sliding mode control laws on different kinds of manipulators
and simulating their performance. Given the inherent non-linearities
involved in all but Cartesian manipulators, we feel that our methodology
is particularly suited for this application.

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