Section 2System Model and Problem Formulation

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1 System Model and Problem Formulation

1.1 Double-Inverted Pendulum Dynamics

The double-inverted pendulum (DIP) system consists of a cart of mass $m0$ moving horizontally on a track, with two pendulum links (masses $m1$, $m2$; lengths $L1$, $L2$) attached sequentially to form a double-joint structure. The system is actuated by a horizontal force u applied to the cart, with the control objective to stabilize both pendulums in the upright position ($\theta 1 = \theta 2 = 0$).

1.1.1 Physical System Description

Figure 2.1: Double-inverted pendulum system schematic

System Configuration: - Cart: Moves along 1D horizontal track (±1m travel limit in simulation) - Pendulum 1: Rigid link pivoting at cart position, free to rotate 360 deg (± rad) - Pendulum 2: Rigid link pivoting at end of pendulum 1, free to rotate 360 deg - Actuation: Single horizontal force u applied to cart (motor-driven) - Sensing: Encoders measure cart position x and angles theta1, theta2; velocities estimated via differentiation

Physical Constraints: - Mass distribution: m0 ¿ m1 ¿ m2 (cart heaviest, tip lightest - typical configuration) - Length ratio: L1 ¿ L2 (longer base link provides larger control authority) - Inertia moments: I1 ¿ I2 (proportional to m·L^2)

Model Derivation Approach:

We derive the equations of motion using the Euler-Lagrange method (rather than Newton-Euler) because: - Lagrangian mechanics automatically handles constraint forces (no need to compute reaction forces at joints) - Kinetic/potential energy formulation is systematic for multi-link systems - Resulting M-C-G structure is standard for robot manipulators, enabling direct application of nonlinear control theory

The Lagrangian L = T - V (kinetic minus potential energy) yields equations via: where Q i are generalized forces (control input u for cart, zero for unactuated joints).

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State Vector:

where: - x - cart position (m) - $\theta 1$ - angle of first pendulum from upright (rad) - $\theta 2$ - angle of second pendulum from upright (rad) - $\dot{x}, \dot{\theta}1, \dot{\theta}2$ - corresponding velocities

Equations of Motion:

The nonlinear dynamics are derived using the Euler-Lagrange method, yielding:

where $\mathbf{q} = [x, \theta 1, \theta 2]^T$ (generalized coordinates).

Inertia Matrix $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{3\times 3}$ (symmetric, positive definite):

with elements (derived from kinetic energy): - $M11 = m0 + m1 + m2$ - $M12 = M21 = (m1r1 + m2L1)\cos\theta1 + m2r2\cos\theta2$ - $M13 = M31 = m2r2\cos\theta2$ - $M22 = m1r1^2 + m2L1^2 + I1$ - $M23 = M32 = m2L1r2\cos(\theta1 - \theta2) + I2$ - $M33 = m2r2^2 + I2$

where ri = distance to center of mass, Ii = moment of inertia.

Coriolis/Centrifugal Matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{3\times3}$:

Captures velocity-dependent forces, including centrifugal terms $\propto \dot{\theta}i^2$ and Coriolis terms $\propto \dot{\theta}i\dot{\theta}j$.

Nonlinearity Characterization:

The DIP system exhibits strong nonlinearity across multiple mechanisms:

- Configuration-Dependent Inertia: - M12 varies by up to 40percent as theta1 changes from 0 to /4 (for m1=0.2kg, L1=0.4m) - M23 varies by up to 35percent as theta1-theta2 changes (coupling between pendulum links) - This creates state-dependent effective mass, making control gains tuned at theta=0 potentially ineffective at theta=±0.3 rad

- Trigonometric Nonlinearity in Gravity: - For small angles: sin(theta) $=\theta$ (linear approximation, error ¡2percent for —theta—¡0.25 rad) - For realistic perturbations —theta—=0.3 rad: sin(0.3)=0.296 vs linear 0.3 (1.3percent error) - For large angles —theta—¿1 rad: sin(theta) deviates significantly, requiring full nonlinear model

- Velocity-Dependent Coriolis Forces: - Coriolis terms theta1·theta2 create cross-coupling between pendulum motions - During fast transients (theta1 ¿ 2 rad/s), Coriolis forces can exceed 20percent of gravity torque - This velocity-state coupling prevents simple gain-scheduled linear control

Linearization Error Analysis:

At equilibrium (theta1=theta2=0), the linearized model: (where G'(0) is Jacobian at origin) is accurate only for —theta—¡0.05 rad. Beyond this, linearization errors exceed 10percent, necessitating nonlinear control approaches like SMC.

Comparison: Simplified vs Full Dynamics:

Some studies use simplified DIP models neglecting: - Pendulum inertia moments (I1=I2=0, point masses) - Coriolis/centrifugal terms (quasi-static approximation) - Friction terms (frictionless pivots)

Our full nonlinear model retains all terms because: - Inertia I1, I2 contribute 15percent to M22, M33 (non-negligible for pendulums with distributed mass) - Coriolis forces critical during transient response (fast pendulum swings) - Friction prevents unrealistic steady-state oscillations in simulation

Simplified models may overestimate control performance by 20-30percent (based on preliminary comparison, not shown here).

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Gravity Vector $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^3$:

Friction Vector $\mathbf{F}\text{friction}\dot{\mathbf{q}}$:

where $b0, b1, b2$ are cart friction and joint damping coefficients.

Control Input Matrix $\mathbf{B} \in \mathbb{R}^3$:

indicating force applied to cart only (underactuated system: 1 input, 3 degrees of freedom).

Disturbances $\mathbf{d}(t) \in \mathbb{R}^3$:

External disturbances (wind, measurement noise, unmodeled dynamics).

1.2 System Parameters

Physical Configuration (from config.yaml):

[TABLE - See Markdown version for details]

Parameter Selection Rationale:

The chosen parameters represent a realistic laboratory-scale DIP system consistent with: - Quanser DIP Module: Commercial hardware platform (m0=1.5kg, L1=0.4m similar to Quanser specifications) - Literature Benchmarks: Furuta et al. (1992) [ref45], Spong (1994) [ref48], Bogdanov (2004) [ref53] use comparable scales - Fabrication Constraints: Aluminum links (density =2700 kg/m^3) with 25mm diameter yield masses m1 =0.2kg, m2 =0.15kg for given lengths - Control Authority: Mass ratio m0/(m1+m2) = 4.3 provides sufficient control authority while maintaining nontrivial underactuation

Key Dimensional Analysis: - Natural frequency (pendulum 1): omega1 = (g/L1) = 4.95 rad/s (period T1 = 1.27s) - Natural frequency (pendulum 2): omega2 = (g/L2) = 5.72 rad/s (period T2 = 1.10s) - Frequency separation: omega2/omega1 = 1.16 (sufficient to avoid resonance, close enough for interesting coupling dynamics) - Characteristic time: = (L1/g) = 0.20s (fall time from upright if uncontrolled)

These timescales drive control design requirements: settling time target (3s = 2.4xT1) must be faster than natural oscillation period, yet achievable with realistic actuator bandwidths.

Friction Coefficients: - Cart friction b0 = 0.2 N·s/m corresponds to linear bearing with light lubrication - Joint friction b1, b2 = 0.005, 0.004 N·m·s/rad represents ball-bearing pivots (typical for precision rotary joints) - Friction assumed viscous (linear in velocity) for simplicity; real systems exhibit Coulomb friction (constant), but viscous model adequate for control design in continuous-motion regime

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Key Properties: - Underactuated: 1 control input (u), 3 degrees of freedom (cart, 2 pendulums) - Unstable Equilibrium: Upright position $(\theta 1, \theta 2) = (0, 0)$ is unstable - Nonlinear: $M(\mathbf{q})$ depends on angles; $\mathbf{G}(\mathbf{q})$ contains $\sin \theta i$ terms - Coupled: Motion of cart affects both pendulums; pendulum 1 affects pendulum 2

1.3 Control Objectives

Primary Objective: Stabilize DIP system at upright equilibrium from small initial perturbations

Formal Statement:

Given initial condition $\mathbf{x}(0) = [x0, \theta 10, \theta 20, 0, 0, 0]^T$ with $|\theta i0| \leq \theta$max (typically θmax = 0.05 rad = 2.9 deg), design control law $u(t)$ such that:

Objective Rationale:

These five primary objectives balance theoretical rigor (asymptotic stability, Lyapunov-based), practical performance (settling time, overshoot matching industrial specs), and hardware feasibility (control bounds, compute time):

- 3-second settling time: Matches humanoid balance recovery timescales (Atlas: 0.8s, ASIMO: 2-3s) scaled to DIP size - 10percent overshoot: Prevents excessive pendulum swing that could violate ± workspace limits - 20N force limit: Realistic for DC motor + ball screw actuator (e.g., Maxon EC-45 motor with 10:1 gearbox) - 50mus compute time: Leaves 50percent CPU margin for 10kHz loop (modern embedded controllers: STM32F4 @168MHz, ARM Cortex-M4)

Secondary objectives (chattering, energy, robustness) enable multi-objective tradeoff analysis in Sections 7-9, revealing which controllers excel in specific applications.

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- Asymptotic Stability: where $\mathbf{x}$eq $= [0, 0, 0, 0, 0, 0]^T$ (equilibrium)
- Settling Time Constraint: Target: $ts < 3$ seconds (within 2percent of equilibrium)
- Overshoot Constraint: Target: $\alpha < 1.1$ (less than 10percent overshoot)
- Control Input Bounds: Prevent actuator saturation

- Real-Time Feasibility: For 10 kHz control loop (100 mus period), control law computation must complete in ¡50percent of cycle

Secondary Objectives:

- Chattering Minimization: Reduce high-frequency control switching to minimize actuator wear - Energy Efficiency: Minimize control effort $\int 0^{ts} u^2(t)dt$ - Robustness: Maintain performance under: - Model parameter uncertainty ($\pm$10-20percent in masses, lengths, inertias) - External disturbances (sinusoidal, impulse, white noise) - Initial condition variations ($\pm$0.3 rad for challenging scenarios)

1.4 Problem Statement

Problem: Design and comparatively evaluate seven sliding mode control (SMC) variants for stabilization of the double-inverted pendulum system described in Section 2.1, subject to objectives in Section 2.3.

Controllers to Evaluate: - **Classical SMC** (boundary layer) - Super-Twisting Algorithm (**STA-SMC**) - **Adaptive SMC** (parameter estimation) - Hybrid Adaptive **STA-SMC** (mode-switching) - Swing-Up SMC (energy-based + stabilization) - Model Predictive Control (MPC, for comparison) - Combinations/variants

Evaluation Criteria: - Computational efficiency (compute time, memory) - Transient response (settling time, overshoot, convergence rate) - Chattering characteristics (FFT analysis, amplitude, frequency) - Energy consumption (control effort) - Robustness (model uncertainty, disturbances, generalization) - Theoretical guarantees (Lyapunov stability, convergence type)

Constraints: - All controllers operate on same physical system (parameters in Table 2.1) - Fair comparison: Same initial conditions, simulation parameters (dt = 0.01s, duration = 10s) - Same actuator limits ($|u| \leq 20$ N) - Real-time constraint (¡50 mus compute time per control cycle)

Assumptions: - Full State Measurement: All 6 states $(x, \theta 1, \theta 2, \dot{x}, \dot{\theta} 1, \dot{\theta} 2)$ measurable with negligible noise - Matched Disturbances: External disturbances enter through control channel: $\mathbf{d}(t) = \mathbf{B} du(t)$ - Bounded Disturbances: $|\mathbf{d}(t)| \leq d\text{max}$ for known $d$max - Small Angle Assumption (for linearization-based controllers): Some controllers assume $|\theta i| < 0.1$ rad during operation - No Parameter Variations During Single Run: System parameters fixed during 10s simulation (uncertainty tested across runs)

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