2025-11-01

E003: Plant Models and Dynamics

The Physics Behind the Double Broomstick

Part 1 · Duration: 30-35 minutes

Beginner-Friendly Visual Study Guide

Repository: https://github.com/theSadeQ/dip-smc-pso
Documentation: academic/paper/presentations/podcasts/episodes/

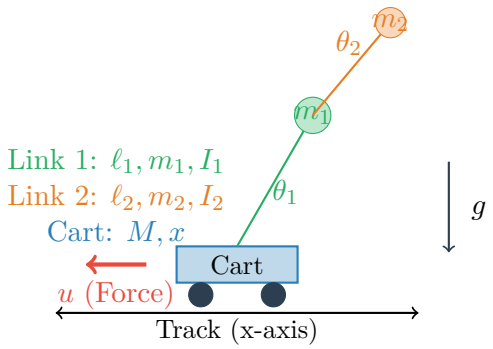
Learning Objective: Understand DIP physics, Lagrangian mechanics approach, and the three model variants for different speed/accuracy tradeoffs

What is the Plant?

Key Concept

The "plant" is the physical system you're trying to control - the body that control decisions act upon. **Key Question:** "If I push the cart with force F right now, how will angles and velocities change in the next millisecond?"

Physical System Description



Components:

- **Cart**: 1.5 kg (like a laptop), slides on track with friction
- **Pendulum 1**: 0.2 kg, 40 cm (wooden ruler weight/length)
- **Pendulum 2**: 0.15 kg, 30 cm (slightly lighter/shorter)

Three Coordinates:

enumi$x$ - Cart position (left/right)

0. enumiθ_1 - First pendulum angle (from vertical)
0. enumiθ_2 - Second pendulum angle (from vertical)

Pro Tip

Based on actual control lab rigs - not arbitrary!

Lagrangian Mechanics: The Elegant Shortcut

Why NOT Use Newton's $F = ma$ Directly?

Common Pitfall

Newton's approach would require:

0. enumiFree-body diagrams for cart + both pendulums
0. enumiAll internal forces (pin forces, constraint forces)
0. enumiCoupled force balance equations
0. enumiSolving for dozen+ variables you don't care about!

Result: Nightmare of algebra with unknown internal forces

The Lagrangian Shortcut: Ignore Internal Glue

Key Concept

Key Insight (1700s): You don't need internal constraint forces if you focus on energy!
Recipe:

0. enumiCalculate total kinetic energy (motion)
0. enumiCalculate total potential energy (gravity)
0. enumiForm Lagrangian: $\mathcal{L} = T - V$
0. enumiApply Euler-Lagrange equations (systematic calculus)
0. enumiGet equations of motion - NO internal forces!

The Beautiful Result

</> Matrix Equation of Motion

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q,\dot{q})\dot{q} + \mathbf{G}(q) = \mathbf{B}u + d$$

Physical Meaning:

0. $\mathbf{M}(q)$ - Mass matrix: "How mass/inertia distributed at these angles?"

- $\mathbf{C}(q,\dot{q})$ - Spinning forces: Coriolis & centrifugal effects
- $\mathbf{G}(q)$ - Gravity: "How hard does gravity pull at these angles?"
- $\mathbf{B}$ - Input distribution: "Force only pushes cart directly"
- u - Control force, d - Disturbances

Mass Matrix Structure: Action and Reaction

Diagonal Elements

"Self-inertia":

- M_{11} = Total mass (cart + pendulums)
- M_{22} = Rotational inertia of Pendulum 1
- M_{33} = Rotational inertia of Pendulum 2

Off-Diagonal Elements

Coupling terms:

- M_{12} = "Cart acceleration → Pendulum 1 torque"
- Depend on $\cos(\theta_i)$
- Strongest when vertical

Key Concept

Key Property - Symmetry: $M_{12} = M_{21}$, $M_{13} = M_{31}$, $M_{23} = M_{32}$
This is Newton's Third Law: **Action = Reaction**
Effect of Link 1 on Link 2 = Effect of Link 2 on Link 1 (opposite directions)

Coriolis & Centrifugal Terms

Example

Coriolis Force: Apparent force due to rotation

- Example: Walk straight on spinning merry-go-round ⇒ path curves
- Term: $-c_{12}\sin(\theta_1 - \theta_2)\dot{\theta}_2$ couples pendulum velocities

Centrifugal Force: Outward push from rotation

- Example: Car turn ⇒ pushed against door
- Term: $-c_1\sin(\theta_1)\dot{\theta}_1^2$ pushes cart sideways

Gravity Vector

$$\mathbf{G}(q) = \begin{bmatrix} 0 \\ -g_1 g \sin(\theta_1) \\ -g_2 g \sin(\theta_2) \end{bmatrix}$$

- No gravity on cart (horizontal motion)
- Pendulum torques: $g \approx 9.81$ m/s^2
- **Sign:** Upright ($\theta = 0$) is unstable - gravity torque pushes away!

Singularities: When Math Locks Up

Physical Analogy:

Your arm fully extended with locked elbow - can't push further. Geometry "locks up."

Same happens with pendulums in specific configurations.

Condition Number (κ):
Health score for mass matrix:

- κ = 1-100: Healthy
- $\kappa > 10^6$: Sick
- $\kappa \to \infty$: Singular

Common Pitfall

Singularity Occurs: Both pendulums horizontal (lying flat)
How We Handle:

enumiCheck κ before inverting $\mathbf{M}$

0. enumiIf $\kappa > 10^8$: Switch to pseudoinverse with regularization

0. enumiPrevents division-by-near-zero crashes

In Practice: Near upright, $\kappa = 10\text{-}100$ (safe). Controller avoids dangerous configs.

Three Model Variants: The Team

Key Concept

Trade-off: **Speed** vs. **Accuracy**
Different tasks need different balances. Meet the three teammates!

Model 1: The Sprinter (Simplified Linear)

Personality: Fast, agile, assumes near-perfect
Assumptions:

0. $\sin(\theta) \approx \theta$ (small angles)
- $\cos(\theta) \approx 1$
- Constant mass matrix
- Ignores coupled effects

Superpowers:

- 10-100x faster than full model
- Perfect for PSO (1500 sims)
- Great for teaching

Kryptonite:

- ⚠ Can't handle angles > 10ř
- ⚠ Lies about nonlinear effects
- ⚠ Swing-up? Garbage results!

When to Use:

- Initial prototyping
- PSO gain optimization
- Educational demos
- Near-upright operation

Model 2: The Simulator (Full Nonlinear)

Personality: Slow, heavy, brutally honest
Reality Check:

- Every trig term computed exactly
- All Coriolis effects
- All coupling captured
- Angle-dependent matrices

Superpowers:

- Truth: Full operating range
- Accurate: Hanging $\rightarrow$ upright
- Publishable benchmark results

Kryptonite:

- ⚠ 10-100x slower
- ⚠ Overkill for simple tasks
- ⚠ Sledgehammer to crack a nut

When to Use:

- Final validation
- Swing-up control
- Research benchmarks
- Hardware deployment prep

Model 3: The Efficient Pro (Low-Rank POD)

Personality: Smart compromise
Clever Trick: Proper Orthogonal Decomposition

enumiRun Simulator 1000x times

0. enumiCollect data snapshots
0. enumiSVD: Find important patterns
0. enumiKeep signal, discard noise

Superpowers:

0. 10-50x speedup
- Preserves key dynamics
- Real-time capable (HIL)

Kryptonite:

- ⚠ Needs training (run Simulator first)
- ⚠ Can miss rare edge cases
- ⚠ Not drop-in replacement

When to Use:

- Monte Carlo (1000 sims)
- Parameter sweeps
- Sensitivity analysis
- HIL testing

Model Accuracy Comparison (MT-6 Benchmark)

Validation Test Results

Setup: $\theta_1 = 10$ř, $\theta_2 = 5$ř, Classical SMC, 10 seconds

Model	Settling [s]	Overshoot [°]	RMS Error [°]	Speed [sims/s]
Simplified	2.31	4.2	0.12	450
Full Nonlinear	2.58	5.1	0.15	8
Low-Rank (k=10)	2.54	4.9	0.14	95

Observations:

- Simplified: Optimistic (underestimates settling time)
- Full: Conservative (most realistic)
- Low-Rank: Sweet spot (2% error, 12x speedup)

Pro Tip

The Bottom Line:

- Quick prototyping? **Sprinter**
- Final validation? **Simulator**
- Massive data crunching? **Efficient Pro**

Like having hammer, precision screwdriver, and power drill - use the right tool!

Numerical Integration Methods

Three Integrators Available

Euler (1st Order)

Algorithm:

```
lstnumberstate_dot = f(state, u)
lstnumberstate_new = state + state_dot * dt
```

Pros: Simplest, fastest **Cons:** Inaccurate (large errors)
Use: Educational only, NOT for research

RK4 (4th Order)

Algorithm: Four slope evaluations per step

```
lstnumberk1 = f(state, u)
lstnumberk2 = f(state + 0.5*dt*k1, u)
lstnumberk3 = f(state + 0.5*dt*k2, u)
lstnumberk4 = f(state + dt*k3, u)
lstnumberstate_new = state + (dt/6) * (k1 + 2*k2 + 2*k3 + k4)
```

Pros: Good balance **Cons:** Fixed step size
Use: Standard choice with $dt = 0.001s$ (1 kHz)

RK45 (Adaptive)

Algorithm: SciPy's `solve_ivp` with automatic step sizing
Error Control: `rtol=1e-6`, `atol=1e-9`
Pros: Most accurate, adaptive **Cons:** Slower
Use: High-accuracy validation, research benchmarks

Quick Reference: Key Equations

Equation of Motion

$$\mathbf{M}(q)\ddot{q} + \mathbf{C}(q,\dot{q})\dot{q} + \mathbf{G}(q) = \mathbf{B}u$$

Solve for acceleration: $\ddot{q} = \mathbf{M}^{-1}\left[\mathbf{B}u - \mathbf{C}\dot{q} - \mathbf{G}\right]$

State Vector

$$x = [x, \theta_1, \theta_2, \dot{x}, \dot{\theta}_1, \dot{\theta}_2]^T$$

Derivative: $\dot{x} = [\dot{x}, \dot{\theta}_1, \dot{\theta}_2, \ddot{x}, \ddot{\theta}_1, \ddot{\theta}_2]^T$

Condition Number Check

$$\kappa(\mathbf{M}) = \frac{\sigma_{\max}}{\sigma_{\min}}$$

If $\kappa > 10^8$: Use pseudoinverse with regularization

Implementation Workflow

Quick Summary

Full Nonlinear Model (Python):

enumiUnpack state: Extract $q = [x, \theta_1, \theta_2]$ and $\dot{q} = [\dot{x}, \dot{\theta}_1, \dot{\theta}_2]$

0. enumiBuild mass matrix: $\mathbf{M}(q)$ based on current angles (trig functions)
0. enumiCalculate Coriolis: $\mathbf{C}(q, \dot{q})$ with velocity coupling
0. enumiCalculate gravity: $\mathbf{G}(q)$ angle-dependent torques
0. enumiApply control: Force u enters through $\mathbf{B}$
0. enumiSolve for $\ddot{q}$: Invert $\mathbf{M}$ (check κ first!)
0. enumiReturn derivative: Pack $[\dot{q}, \ddot{q}]$ for integrator

NumPy makes it simple: `accel = np.linalg.solve(M, B*u - C@vel - G)`

Configuration Parameters

Physical:

0. Cart mass: $M = 1.5$ kg
- Pendulum 1: $m_1 = 0.2$ kg, $L_1 = 0.4$ m
- Pendulum 2: $m_2 = 0.15$ kg, $L_2 = 0.3$ m
- Gravity: $g = 9.81$ m/s^2

Numerical:

- Time step: $dt = 0.001$ s (1 kHz)
- Integrator: RK4 (standard)
- Singularity threshold: $\kappa > 10^8$
- Duration: 10 s (typical)

What's Next?

Key Concept

E004: PSO Optimization - How to automatically tune controller gains using particle swarm intelligence
E005: Simulation Engine - The runner, vectorized simulators, and how we achieve 100x speedups
Remember: Now you understand the physics. Next, we optimize the control to make it work perfectly!