This gives a quasi-sliding mode bound of

$$\delta_y = \frac{(\epsilon\tau + 2(d_1 + g_1) + l_1)}{(1 - q\tau)} = 0.185.$$

The resultant control is of the form

$$u(k) = \begin{bmatrix} -1.56 \\ 1.66 \end{bmatrix}^T y_k + 0.857u(k-1) - 0.182\,\mathrm{sgn}\left(\begin{bmatrix} -1.2 \\ 0.35 \end{bmatrix}^T y_k + 1.05u(k-1)\right).$$

The simulation results for an initial condition of $x(0) = [1\ 0]$, can be seen in Fig. 1. It can be noted here that though the actual QSM band is of much less width as compared to the bound δ_y. This is because the bound is calculated for the worst-case scenario of disturbance which may seldom persist for a long time in an uncertain system.

VI. Conclusion

A multirate output feedback based sliding mode controller for discrete-time linear time-invariant systems with uncertainty has been proposed in this note. The proposed control algorithm makes use of only the past input and output samples of the systems to realize a QSM motion in the system. Hence, it is more practical as compared to a state feedback based control algorithm. The proposed control algorithm was validated through a numerical example and the results were found to be encouraging.

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Robust Adaptive Output Control of Uncertain Nonlinear Plants With Unknown Backlash Nonlinearity

Jing Zhou, Chengjin Zhang, and Changyun Wen

***Abstract*—In this note, we consider a class of uncertain dynamic nonlinear systems preceded by unknown backlash nonlinearity. The control design is achieved by introducing a smooth inverse function of the backlash and using it in the controller design with backstepping technique. For the design and implementation of the controller, no knowledge is assumed on the unknown system parameters. It is shown that the proposed controller not only can guarantee stability, but also transient performance.**

***Index Terms*—Adaptive control, backstepping, backlash, nonlinear systems, stability.**

I. Introduction

Backlash exists in a wide range of physical systems and devices, such as biology optics, electro-magnetism, mechanical actuators, electronic relay circuits and other areas. Such nonlinearity is usually poorly known and often limits system performance. Control of systems with backlash nonlinearity is an important area of control system research and typically challenging. For backlash nonlinearity, several adaptive control schemes have recently been proposed, see for examples [1], [2], and [3]. In [4], [5], and [6] an inverse nonlinearity was constructed. In the controller design, the term multiplying the control and the uncertain parameters of the system and nonsmooth nonlinearity must be within known bounded intervals. Backlash compensation using neural network and fuzzy logic has also been used in feedback control systems [7]. The system states and uncertain weights must be within a known compact set. With these developed schemes, the transient performance is usually not guaranteed due to their design methods. In [8], variable structure control was proposed to stabilize the nonlinear plants by using a quasistatic and a dynamic description of the nonlinearity, where the parameters of the dead zone and backlash nonlinearities are bounded by known constants. In [9], a dynamic backlash model is defined to pattern a backlash nonlinearity rather than constructing an inverse model to mitigate the effects of the backlash. However, in [9], the term multiplying the control and the uncertain parameters of the system must be within known intervals and the "disturbance-like" term must be bounded with known bound. Projection was used to handle the "disturbance-like" term and unknown parameters. System stability was established and the tracking error was shown to converge to a residual. In [10], a state feedback backstepping design was developed to deal with the backlash nonlinearity, where the effect of backlash was treated as a bounded disturbance and an estimate was used to estimate its bound. The detailed characteristic of backlash was not considered in the controller design.

Manuscript received April, 21, 2006; revised September 21, 2006. Recommended by Associate Editor D. Dochain. This work was supported in part by the National Natural Science Foundation of China under Grant 60474007.
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Color versions of one or more of the figures in this note are available online at http://ieeexplore.ieee.org.
Digital Object Identifier 10.1109/TAC.2006.890473

In this note, we develop a simple output feedback adaptive control scheme for a class of nonlinear systems, in the presence of unknown backlash actuator nonlinearity. To further improve system performance, we take the system backlash into account in the controller design, instead of only considering its effect like bounded disturbances as in [9], [10]. An efficient smooth adaptive inverse was developed to compensate the effect of the backlash in controller design with backstepping approach. Furthermore, the over-parametrization problem is also solved by using the concept of tuning functions. A state observer is proposed for output feedback control. To avoid possible chattering caused by the sign function, we propose a smooth control law and the tracking error is ensured to approach a prescribed bound. In our design, the term multiplying the control and the system parameters are not assumed to be within known intervals. Besides showing global stability of the system, transient performance in terms of L_2 norm of the tracking error is derived to be an explicit function of design parameters and thus our scheme allows designers to obtain the closed loop behavior by tuning design parameters in an explicit way.

This note is organized as follows: Section II states the problem of this note and assumptions on the nonlinear systems. In Section III, filters are designed to estimate system states. Section IV presents the adaptive control design based on the backstepping technique and analyzes the stability and performance. Simulation results are presented in Section V. Finally, Section VI concludes this note.

II. Problem Statement

A. System Model

We consider the following class of nonlinear system as in [11], [12]:

$$\dot{x} = Ax + \psi(y) + \sum_{i=1}^{r} \theta_i \phi_i(y) + bu \tag{1}$$

$$y = e_1^T x \quad u = B(v) \tag{2}$$

where

$$A = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} \quad \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_r \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ b_m \\ \vdots \\ b_0 \end{bmatrix}$$
$$\psi(y) = \begin{bmatrix} \psi_1(y) \\ \vdots \\ \psi_n(y) \end{bmatrix} \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

where $x \in R^n$, $u \in R^1$, and $y \in R^1$ are the states, input and output of the system, respectively, $\theta \in R^r$ and $b \in R^n$ and are unknown constant vectors, $\phi_i(y) \in R^n$ and $\psi_i(y) \in R$ are known smooth functions. The actuator nonlinearity $u = B(v)$ is described as a backlash characteristic.

The control objective is to design an output feedback control law for $v(t)$ to ensure that all closed-loop signals are bounded and the plant output $y(t)$ tracks a given reference signal $y_r(t)$ under the following assumptions.

Assumption 1: The sign of b_m is known and the relative degree $\rho = n - m$ is fixed and known. The polynomial $B(s) = b_m s^m + \cdots + b_1 s + b_0$ is stable.

Assumption 2: The reference signal y_r and its $(n-1)$th-order derivatives are known and bounded.

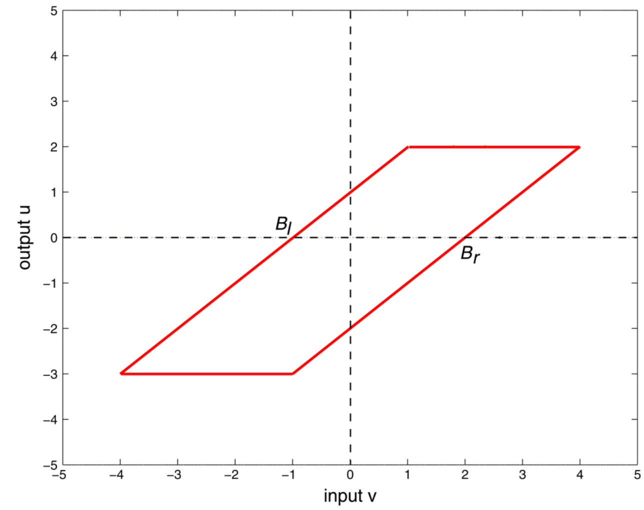

Fig. 1. Backlash.

B. Backlash Characteristic

Traditionally, a backlash nonlinearity in Fig. 1 can be described by

$$u(t) = B(v) = \begin{cases} m(v(t) - B_r), & \text{if } \dot{v}(t) > 0 \text{ and } u(t) = m(v(t) - B_r) \\ m(v(t) - B_l), & \text{if } \dot{v}(t) < 0 \text{ and } u(t) = m(v(t) - B_l) \\ u(t_-), & \text{otherwise} \end{cases} \tag{4}$$

where $m \geq m_0$ is the slope of the lines, with m_0 being a small positive constant, and $B_r > 0$, $B_l < 0$ are constant parameters, $u(t_-)$ means no change occurs in the output control signal $u(t)$.

The essence of compensating backlash effect is to employ a backlash inverse. However, the inverse of (4) is itself nonsmooth and may not be amenable to controller design.

In this note, we propose a new smooth inverse for the backlash as follows:

$$v = BI(u) = \frac{1}{m}u + B_r \chi_r(\dot{u}) + B_l \chi_l(\dot{u}) \tag{5}$$

where

$$\chi_r(\dot{u}) = \frac{e^{(k\dot{u})}}{e^{(k\dot{u})} + e^{(-k\dot{u})}} \tag{6}$$

$$\chi_l(\dot{u}) = \frac{e^{(-k\dot{u})}}{e^{(k\dot{u})} + e^{(-k\dot{u})}} \tag{7}$$

where k is a positive constant. χ_r and χ_l have the following properties:

$$\chi_r(\dot{u}) \to 1 \quad \text{as} \quad \dot{u} \to \infty$$
$$\chi_r(\dot{u}) \to 0 \quad \text{as} \quad \dot{u} \to -\infty \tag{8}$$
$$\chi_l(\dot{u}) \to 0 \quad \text{as} \quad \dot{u} \to \infty$$
$$\chi_l(\dot{u}) \to 1 \quad \text{as} \quad \dot{u} \to -\infty. \tag{9}$$

Note that the larger the value k, the closer χ_r to 1 and 0 when $\dot{u} \to \infty$ and $\dot{u} \to -\infty$. Also the larger the k, the closer χ_l to 0 and 1 when $\dot{u} \to \infty$ and $\dot{u} \to -\infty$. Such an inverse is shown in Fig. 2.

Remark 1: Note that we will obtain an efficient adaptive backlash inverse in (14) when the backlash parameters are unknown. In Section IV, we will take the system backlash into account in the controller

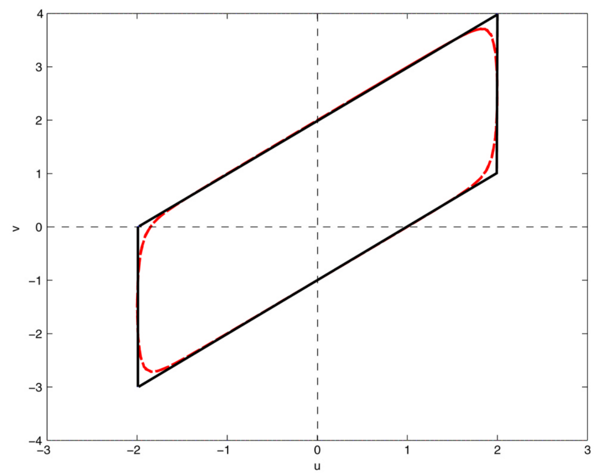

Fig. 2. Backlash inverse.

design, instead of only considering its effect like bounded disturbances in [9] and [10].

Remark 2: Note that the use of smooth functions $\chi_r(\dot{u})$ and $\chi_l(\dot{u})$ are continuous and differentiable. This is different from the inverse in [1], where the inverse indicator functions are nonsmooth. The latter case may cause chattering phenomenon in the recursive backstepping control.

To design an adaptive controller for the system, we reparameterize the backlash as in [1] as follows:

$$u(t) = \sigma_r(t)m(v(t) - B_r) + \sigma_l(t)m(v(t) - B_l) + \sigma_s(t)u_s \tag{10}$$

where u_s is a generic constant corresponding to the value at any active inner segment characterized by $(u_s/m) + B_l \leq v(t) \leq (u_s/m) + B_r$

$$\sigma_r(t) = \begin{cases} 1, & \text{if } \dot{u}(t) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$
$$\sigma_l(t) = \begin{cases} 1, & \text{if } \dot{u}(t) < 0 \\ 0, & \text{otherwise} \end{cases} \tag{12}$$
$$\sigma_s(t) = \begin{cases} 1, & \text{if } \dot{u}(t) = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

These functions satisfy that $\sigma_r(t) + \sigma_l(t) + \sigma_s(t) = 1$.

As parameters m, B_r, B_l are unknown and u is unavailable, the actual $v(t)$ is designed as

$$\begin{aligned} v(t) &= \widehat{BI}(u_d) \\ &= \frac{1}{\hat{m}}[u_d + \widehat{mB_r}\chi_r(\dot{u}_d) + \widehat{mB_l}\chi_l(\dot{u}_d)] \end{aligned} \tag{14}$$

where $\hat{m}$, $\widehat{mB_r}$ and $\widehat{mB_l}$ are estimates of m, mB_r and mB_l, $u_d(t)$ is the actual control input. Then, the corresponding control input $u_d(t)$ is given by

$$u_d(t) = \hat{m}v(t) - \widehat{mB_r}\chi_r(\dot{u}_d) - \widehat{mB_l}\chi_l(\dot{u}_d). \tag{15}$$

The resulting error between u and u_d is

$$u(t) - u_d(t) = \tilde{m}v - \widetilde{mB_r}\chi_r(\dot{u}_d) - \widetilde{mB_l}\chi_l(\dot{u}_d) + d_b(t) \tag{16}$$

where $d_b(t) = u_s\sigma_s - mv\sigma_s + mB_r(\chi_r(\dot{u}_d) - \sigma_r) + mB_l(\chi_l(\dot{u}_d) - \sigma_l)$.

Proposition: The unparameterizable part $d_b(t)$ of the backlash inverse control error $u(t) - u_d(t)$ is bounded for any $t \geq 0$.

Proof: There are three cases to be examined.

Case 1) $\sigma_r(t) = 1, \sigma_l(t) = \sigma_s(t) = 0$

$$d_b(t) = mB_r(\chi_r(\dot{u}_d) - 1) + mB_l\chi_l(\dot{u}_d) \tag{17}$$
$$\text{thus } |d_b(t)| \leq m(B_r - B_l). \tag{18}$$

Case 2) $\sigma_l(t) = 1, \sigma_r(t) = \sigma_s(t) = 0$

$$d_b(t) = mB_l(\chi_l(\dot{u}_d) - 1) + mB_r\chi_r(\dot{u}_d) \tag{19}$$
$$\text{Thus } |d_b(t)| \leq m(B_r - B_l). \tag{20}$$

Case 3) $\sigma_s(t) = 1, \sigma_r(t) = \sigma_l(t) = 0$

$$|d_b(t)| \leq m(B_r - B_l + B_s) \tag{21}$$

where B_s depends on the motion of $v(t)$ and $u(t)$ on the inner segment when $\sigma_s(t) = 1$ and B_s is in the internal (B_l, B_r). From these expressions, it is clear that $d_b(t)$ is bounded.

III. State Estimation Filters

We employ the filters similar to those in [13] as follows:

$$\hat{x}(t) = \xi_0 + \sum_{i=1}^{r}\theta_i\xi_i + \sum_{i=0}^{m} b_i\eta_i \tag{22}$$
$$\dot{\eta}_i = A_0\eta_i + e_{n-i}u, i = 0, 1, \ldots, m \tag{23}$$
$$\dot{\xi}_0 = A_0\xi_0 + ky + \psi(y) + \chi \tag{24}$$
$$\dot{\xi}_i = A_0\xi_i + \phi_i(y), i = 1, \ldots, r \tag{25}$$

where $k = [k_1, \ldots, k_n]^T$ such that all eigenvalues of $A_0 = A - ke_1^T$ are at some desired stable locations and χ is a design signal specified later. It can be shown that the state estimation error $\epsilon = x(t) - \hat{x}(t)$ satisfies $\dot{\epsilon} = A_0\epsilon - \chi$.

Note that the signal $u(t)$ is not available. Thus, the signal η in (23) needs to be reparameterized. Let p denote d/dt. With $\Delta(p) = \det(pI - A_0)$, we express $\eta(t)$ as

$$\begin{aligned} \eta_i(t) &= [\eta_{i1}(t), \eta_{i2}(t), \ldots, \eta_{in}(t)]^T \\ &= [q_{i1}(p), q_{i2}(p), \ldots, q_{in}(p)]^T \frac{1}{\Delta(p)}u(t), \\ &\qquad i = 0, \ldots, m \end{aligned} \tag{26}$$

for some known polynomials $q_{ij}(p), i = 0, \ldots, m, j = 1, \ldots, n$. Using (15) and (16), we have

$$\begin{aligned} u(t) &= mv - mB_r\chi_r(\dot{u}_d) - mB_l\chi_l(\dot{u}_d) + d_b(t) \\ &= \beta^T\hat{\omega}(t) + d_b(t) \end{aligned} \tag{27}$$

where

$$\beta = [m, mB_r, mB_l]^T \tag{28}$$
$$\hat{\omega}(t) = [v, -\chi_r(\dot{u}_d), -\chi_l(\dot{u}_d)]^T \tag{29}$$

With (26), we obtain

$$\eta_{ij}(t) = \beta^T \hat{\omega}_{ij}(t) + d_{ij}(t) \tag{30}$$

where

$$\hat{\omega}_{ij}(t) = \frac{q_{ij}(p) I_3}{\Delta(p)} \hat{\omega}(t) \quad d_{ij}(t) = \frac{q_{ij}(p)}{\Delta(p)} d_b(t) \tag{31}$$

and I_3 is a 3×3 identity matrix. Based on (30), $\hat{\omega}_i$ is available for controller design in place of u. Denoting the second component of ξ_i as $\xi_{i2}, i = 1, \ldots, r$, we have

$$\hat{x}_2 = \xi_{02} + \sum_{i=1}^{r} \theta_i \xi_{i2} + \sum_{i=0}^{m} b_i \beta^T \hat{\omega}_{i2}(t) + \sum_{i=0}^{m} b_i d_{i2}(t) \tag{32}$$
$$\hat{\omega}_{i2}(t) = \frac{(p^{m+1} + k_1 p^m) I_3}{p^n + k_1 p^{n-1} + \cdots + k_{n-1} p + k_n} \hat{\omega}(t). \tag{33}$$

IV. Design of Adaptive Controllers

As usual in backstepping approach, the following change of coordinates is made:

$$z_1 = y - y_r \quad z_i = \hat{\beta}^T \hat{\omega}_{m2}^{(i-2)} - \hat{e} y_r^{(i-1)} - \alpha_{i-1}, \quad i = 2, 3, \ldots, \rho \tag{34}$$

where $\hat{\beta}$ and $\hat{e}$ are estimates of β and $e = 1/b_m$, respectively, α_{i-1} is the virtual control at the ith step and will be determined in later discussion.

As in [10], we define functions $sg_i(z_i)$ and $f_i(z_i)$ as follows:

$$sg_i(z_i) = \begin{cases} \frac{z_i}{|z_i|} & |z_i| \geq \delta_i \\ \frac{z_i^{(2q+1)}}{(\delta_i^2 - z_i^2)^{\rho - i + 2} + |z_i|^{(2q+1)}} & |z_i| < \delta_i \end{cases}$$
$$f_i(z_i) = \begin{cases} 1 & |z_i| \geq \delta_i \\ 0 & |z_i| < \delta_i \end{cases} \tag{35}$$

where $\delta_i (i = 1, \ldots, \rho)$ is a positive design parameter and $q = \mathrm{round}\{(\rho - i + 2)/2\}$, where $\mathrm{round}\{x\}$ is the nearest integer to x. Clearly, $2q + 1 \geq \rho - i + 2$. It can be shown that $sg_i(z_i)$ is $(\rho - i + 1)$th order differentiable.

Remark 3: The standard adaptive backstepping design approach [11] results in difficulties in designing robust adaptive controllers with the capability of estimating the bound of disturbances for the case that relative degree $\rho > 1$. In this note, function sg_i is used to overcome the difficulties by counteracting the disturbance in each step and also avoid using sign function which is nondifferentiable and may cause the chattering.

Note that, the first and the last steps of the design are quite different from the approach in [10], due to output feedback and the use of estimated inverse backlash parameters in control design, and are elaborated in details. The results of other steps, i.e., step i, $i = 2, \ldots, \rho - 1$ are only presented without elaboration.

- *Step 1)* We start with the equation for the tracking error z_1 given (1) and (32) to obtain

$$\dot{z}_1 = \xi_{02} + \theta^T(\xi_{(2)} + \phi_1(y)) + b_m \tilde{\beta}^T \hat{\omega}_{m2}(t) - b_m \tilde{e} \dot{y}_r + b_m z_2 + b_m \alpha_1 + \sum_{i=0}^{m-1} b_i \beta^T \hat{\omega}_{i2}(t) + d(t) + \psi_1(y) + \epsilon_2 \tag{36}$$

where $\xi_{(2)} = [\xi_{12}, \ldots, \xi_{r2}]^T$ and $d(t) = \sum_{i=0}^{m} b_i d_{i2}(t)$. From proposition, there exists a positive constant D such that $|d(t)| \leq D$.

Remark 4: The unknown bound D of $d(t)$ will be estimated online and thus it is not assumed to be known in contrast with [1], [14], and [15]. With our proposed scheme, only one estimator will be used to estimate its bound in the backstepping design to overcome the overparametrization problem. This is in contrast to [13], where a number of estimators are used for the same variable D.

Now, we select the virtual control law α_1 as

$$\alpha_1 = \hat{e} \bar{\alpha}_1 \tag{37}$$
$$\bar{\alpha}_1 = -\left(c_1 + \frac{\hat{b}_m^2}{4}\right)(|z_1| - \delta_1)^n sg_1 - \xi_{02} - \hat{\Theta}^T \varphi(t) - \hat{D} sg_1 - (\delta_2 + 1)\sqrt{\hat{b}_m^2 + \delta_0} \cdot sg_1 \tag{38}$$

where δ_0 is a small positive real number, $\hat{b}_m, \hat{D}$ and $\hat{\Theta}$ are estimates of b_m, D and $\Theta^T = [\theta^T, b_0 \beta^T, \ldots, b_{m-1} \beta^T]$, and $\varphi(t) = [\xi_{(2)} + \phi_1(y), \hat{\omega}_{02}(t), \ldots, \hat{\omega}_{(m-1)2}(t)]^T$. Then the choice of (37) and (38) results in the following system:

$$\dot{z}_1 = -\left(c_1 + \frac{\hat{b}_m^2}{4}\right)(|z_1| - \delta_1)^\rho sg_1(z_1) + \tilde{\Theta}^T \varphi(t) + b_m z_2 - b_m(\bar{\alpha}_1 + \dot{y}_r)\tilde{e} - b_m \tilde{\beta}^T \hat{\omega}_{m2}(t) + d(t) - \hat{D} sg_1 + \epsilon_2 - (\delta_2 + 1)\sqrt{\hat{b}_m^2 + \delta_0} \cdot sg_1. \tag{39}$$

We define a positive–definite function V_1 as

$$V_1 = \frac{1}{\rho + 1}(|z_1| - \delta_1)^{\rho+1} f_1 + \frac{1}{2}|b_m| \tilde{\beta}^T \Gamma_\beta^{-1} \tilde{\beta} + \frac{1}{2} \tilde{\Theta}^T \Gamma_\Theta^{-1} \tilde{\Theta} + \frac{|b_m|}{2\gamma_e} \tilde{e}^2 + \frac{1}{2\gamma_d} \tilde{D}^2 + \frac{1}{2l_1} \epsilon^T P \epsilon \tag{40}$$

where $\tilde{\Theta} = \Theta - \hat{\Theta}, \tilde{\beta} = \beta - \hat{\beta}, \tilde{e} = e - \hat{e}, \tilde{D} = D - \hat{D}, \Gamma_\Theta, \Gamma_\beta$ are positive–definite matrices, γ_e, γ_d are positive constants, and $P = P^T > 0$ satisfies the equation $PA_0 + A_0^T P = -2I$. Let $\beta_i = \bar{e}_i^T \theta, i = 1, \ldots, 3$, where $\bar{e}_i \in R^3$ is an identity vector. We select the adaptive update laws as

$$\dot{\hat{\beta}}_i = \bar{e}_i^T \tau_\beta, \qquad i = 2, 3 \quad \dot{\hat{\beta}}_i = \mathrm{Proj}(\bar{e}_i^T \tau_\beta), \qquad i = 1 \tag{41}$$
$$\tau_\beta = -\mathrm{sign}(b_m) \Gamma_\beta \hat{\omega}_{m2}(t)(|z_1| - \delta_1)^\rho f_1 sg_1 \tag{42}$$
$$\dot{\hat{e}} = -\mathrm{sign}(b_m) \gamma_e (\bar{\alpha}_1 + \dot{y}_r)(|z_1| - \delta_1)^\rho f_1 sg_1 \tag{43}$$

where $\mathrm{Proj}(\cdot)$ is a smooth projection operation to ensure the estimate $\hat{m}(t) \geq m_0$. Such an operation can be found in [11].

Then, from (39) to (43) and using $\dot{\epsilon} = A_0\epsilon - \chi$ and the property $-\tilde{\beta}^T\Gamma_\beta^{-1}\mathrm{Proj}(\tau_\beta) \leq -\tilde{\beta}^T\Gamma_\beta^{-1}\tau_\beta$, we obtain the time derivative of V_1 as

$$\begin{aligned}\dot{V}_1 &= (|z_1|-\delta_1)^\rho f_1 sg_1\dot{z}_1 - |b_m|\tilde{\beta}^T\Gamma_\beta^{-1}\dot{\hat{\beta}} \\ &\quad - \tilde{\Theta}^T\Gamma_\Theta^{-1}\dot{\hat{\Theta}} - \frac{|b_m|}{\gamma_e}\tilde{e}\dot{\hat{e}} - \frac{1}{\gamma_d}\tilde{D}\dot{\hat{D}} + \frac{1}{l_1}\epsilon^T P\dot{\epsilon} \\ &\leq -\left(c_1 + \frac{\hat{b}_m^2}{4}\right)(|z_1|-\delta_1)^{2\rho} f_1 - |b_m|\tilde{\beta}^T \\ &\quad \times\left[\mathrm{sign}(b_m)\hat{\omega}_{m2}(|z_1|-\delta_1)^\rho f_1 sg_1 + \Gamma_\beta^{-1}\dot{\hat{\beta}}\right] \\ &\quad + \tilde{\Theta}^T\left[\varphi(|z_1|-\delta_1)^\rho f_1 sg_1 - \Gamma_\Theta^{-1}\dot{\hat{\Theta}}\right] \\ &\quad - |b_m|\tilde{e}\left[\mathrm{sign}(b_m)(\bar{\alpha}_1 + \dot{y}_r)(|z_1|-\delta_1)^\rho f_1 sg_1 + \frac{1}{\gamma_e}\dot{\hat{e}}\right] \\ &\quad + \tilde{D}\left[(|z_1|-\delta_1)^\rho f_1 - \frac{1}{\gamma_d}\dot{\hat{D}}\right] \\ &\quad + \epsilon^T\left[e_2(|z_1|-\delta_1)^\rho f_1 sg_1 - \frac{1}{l_1}P\chi\right] - \frac{1}{l_1}\epsilon^T\epsilon \\ &\quad + (|z_1|-\delta_1)^\rho f_1 sg_1\left[b_m z_2 - (\delta_2+1)\sqrt{\hat{b}_m^2 + \delta_0}sg_1\right] \\ &\leq -\left(c_1 + \frac{\hat{b}_m^2}{4}\right)(|z_1|-\delta_1)^{2\rho} f_1 + \tilde{\Theta}^T\left(\tau_{\Theta 1} - \Gamma_\Theta^{-1}\dot{\hat{\Theta}}\right) \\ &\quad + \tilde{D}\left(\tau_{D1} - \frac{1}{\gamma_d}\dot{\hat{D}}\right) + \epsilon^T\left(\tau_{\chi 1} - \frac{1}{l_1}P\chi\right) \\ &\quad - \frac{1}{l_1}\epsilon^T\epsilon + (|z_1|-\delta_1)^\rho f_1 sg_1\Big(b_m z_2 \\ &\quad - (\delta_2+1)\sqrt{\hat{b}_m^2+\delta_0}sg_1\Big)\end{aligned} \tag{44}$$

$$\begin{aligned}\tau_{\Theta 1} &= \phi(|z_1|-\delta_1)^\rho f_1 sg_1 \quad \tau_{\chi 1} = e_2(|z_1|-\delta_1)^\rho f_1 sg_1 \\ \tau_{D1} &= (|z_1|-\delta_1)^\rho f_1\end{aligned} \tag{45}$$

where $e_2 = [0,1,0,\ldots,0]^T \in R^n$.

- *Step* $i, (i = 2,\ldots,\rho)$ As detailed in [10], we choose

$$\begin{aligned}\alpha_i &= -(c_i+1)(|z_i|-\delta_i)^{\rho-i+1}sg_i - g_i - (\delta_{i+1}+1)sg_i \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial y}\hat{\Theta}^T\varphi + \frac{\partial\alpha_{i-1}}{\partial y}\hat{\vartheta}^T\hat{\omega}_{m2}(t) \\ &\quad + \sqrt{\left\|\frac{\partial\alpha_{i-1}}{\partial y}\right\|^2 + \delta_0}\cdot\hat{D}sg_i + \frac{\partial\alpha_{i-1}}{\partial\hat{\Theta}}\Gamma_\Theta\tau_{\Theta i} \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial\xi_0}l_1P^{-1}\tau_{\chi i} + \frac{\partial\alpha_{i-1}}{\partial\hat{\vartheta}}\Gamma_\vartheta\tau_{\vartheta i} \\ &\quad + \sum_{k=2}^{i-1}(|z_k|-\delta_k)^{\rho-k+1}f_k sg_k\left[-\frac{\partial\alpha_{k-1}}{\partial\hat{\Theta}}\frac{\partial\alpha_{i-1}}{\partial y}\varphi\right. \\ &\quad \left.- \frac{\partial\alpha_{k-1}}{\partial\xi_0}\frac{\partial\alpha_{i-1}}{\partial y}l_1P^{-1}e_2 - \frac{\partial\alpha_{k-1}}{\partial\hat{D}}\frac{\partial\alpha_{i-1}}{\partial y}sg_i\right] \\ &\quad - \sum_{k=3}^{i-1}(|z_k|-\delta_k)^{\rho-k+1}f_k sg_k\frac{\partial\alpha_{k-1}}{\partial\hat{\vartheta}}\frac{\partial\alpha_{i-1}}{\partial y}\hat{\omega}_{m2} \\ &\quad + \frac{\partial\alpha_{i-1}}{\partial\hat{D}}\frac{1}{\gamma_d}\tau_{Di}\end{aligned} \tag{46}$$

$$\dot{\hat{b}}_m = \gamma_b(|z_1|-\delta_1)^\rho f_1 sg_1 z_2 \tag{47}$$

$$\tau_{Di} = \tau_{Di-1} - \sqrt{\left\|\frac{\partial\alpha_{i-1}}{\partial y}\right\|^2 + \delta_0}\cdot(|z_i|-\delta_i)^{\rho-i+1}f_i \tag{48}$$

$$\tau_{\Theta i} = \tau_{\Theta i-1} - \frac{\partial\alpha_{i-1}}{\partial y}\varphi(|z_i|-\delta_i)^{\rho-i+1}f_i sg_i \tag{49}$$

$$\tau_{\chi i} = \tau_{\chi i-1} - \frac{\partial\alpha_{i-1}}{\partial y}(|z_i|-\delta_i)^{\rho-i+1}f_i sg_i e_2 \tag{50}$$

$$\tau_{\vartheta i} = \tau_{\vartheta i-1} - \frac{\partial\alpha_{i-1}}{\partial y}\hat{\omega}_{m2}(|z_i|-\delta_i)^{\rho-i+1}f_i sg_i \tag{51}$$

$$\begin{aligned}V_i &= \sum_{k=1}^{i}\frac{1}{\rho-k+2}(|z_k|-\delta_k)^{\rho-k+2}f_k \\ &\quad + \frac{1}{2}|b_m|\tilde{\beta}^T\Gamma_\beta^{-1}\tilde{\beta} + \frac{1}{2}\tilde{\Theta}^T\Gamma_\Theta\tilde{\Theta} + \frac{|b_m|}{2\gamma_e}\tilde{e}^2 \\ &\quad + \frac{1}{2}\tilde{\vartheta}^T\Gamma_\vartheta^{-1}\tilde{\vartheta} + \frac{1}{2\gamma_b}\tilde{b}_m^2 + \frac{1}{2l_1}\epsilon^T P\epsilon\end{aligned} \tag{52}$$

where $\hat{\vartheta}$ is an estimate of $\vartheta = b_m\beta$, $\tilde{\vartheta} = \vartheta - \hat{\vartheta}$, $\tilde{b}_m = b_m - \hat{b}_m$, g_i contains all known terms, γ_b is a positive constant, Γ_ϑ is a positive–definite matrix.

Step ρ: Using (15) and (33), we have

$$\begin{aligned}\hat{\beta}^T\hat{\omega}_{m2}^{(\rho-1)} &= \hat{\beta}^T\frac{(p^n + k_1p^{n-1})I_3}{p^n + k_1p^{n-1} + \cdots + k_{n-1}p + k_n}\hat{\omega}(t) \\ &= u_d(t) + \omega_0\end{aligned} \tag{53}$$

where ω_0 is given by

$$\omega_0 = -\frac{(k_2p^{n-2} + \cdots + k_{n-1}p + k_n)I_3}{p^n + k_1p^{n-1} + \cdots + k_{n-1}p + k_n}\hat{\omega}(t). \tag{54}$$

With this equation, the derivative of $z_n = -\hat{\theta}^T\hat{\omega}_{m2}^{(\rho-2)} - \hat{e}y_r^{(\rho-1)} - \alpha_{\rho-1}$ is

$$\begin{aligned}\dot{z}_\rho &= u_d + g_\rho - \frac{\partial\alpha_{\rho-1}}{\partial y}\Theta^T\varphi - \frac{\partial\alpha_{\rho-1}}{\partial y}\vartheta^T\hat{\omega}_{m2}(t) \\ &\quad - \frac{\partial\alpha_{\rho-1}}{\partial\hat{\Theta}}\dot{\hat{\Theta}} - \frac{\partial\alpha_{\rho-1}}{\partial\hat{D}}\dot{\hat{D}} - \frac{\partial\alpha_{\rho-1}}{\partial\hat{\vartheta}}\dot{\hat{\vartheta}} - \frac{\partial\alpha_{\rho-1}}{\partial\xi_0}\chi \\ &\quad - \frac{\partial\alpha_{\rho-1}}{\partial y}d(t) - \frac{\partial\alpha_{\rho-1}}{\partial y}\epsilon_2\end{aligned} \tag{55}$$

where β_ρ contains all known terms. Define a positive–definite Lyapunov function V_ρ as

$$V_\rho = V_{\rho-1} + \frac{1}{2}(|z_\rho|-\delta_\rho)^2 f_\rho + \frac{1}{2\gamma_d}\tilde{D}_\rho^2. \tag{56}$$

We choose the update laws for $\hat{a}$, $\hat{\Theta}$, $\hat{D}_\rho$

$$\dot{\hat{\Theta}} = \Gamma_\Theta\tau_{\Theta\rho} \quad \dot{\hat{\vartheta}} = \Gamma_\vartheta\tau_{\vartheta\rho} \quad \dot{\hat{D}} = \gamma_d\tau_{D\rho} \tag{57}$$

and the design signal χ as

$$\chi = l_1P^{-1}\tau_{\chi\rho}. \tag{58}$$

Finally, the control law is given by

$$u_d = \alpha_\rho \tag{59}$$

$$v(t) = \frac{1}{\hat{m}}[u_d + \widehat{mB}_r\chi_r(\dot{u}_d) + \widehat{mB}_l\chi_l(\dot{u}_d)]. \tag{60}$$

With this choice and similar steps in step 1) for $\dot{V}_1$, the derivative of V_n becomes

$$\begin{aligned}
\dot{V}_\rho \leq & -\sum_{i=1}^{\rho} c_i(|z_i|-\delta_i)^{2(\rho-i+1)} f_i + \tilde{\vartheta}^T\left(\tau_{\vartheta\rho} - \Gamma_\vartheta^{-1}\dot{\hat{\vartheta}}\right)\\
& + \frac{1}{\gamma_d}\tilde{D}(\gamma_d\tau_{D\rho} - \dot{\hat{D}}) + \epsilon^T\left(\tau_{\chi\rho} - \frac{1}{l_1}P\chi\right)\\
& + \sum_{k=2}^{\rho}(|z_k|-\delta_k)^{\rho-k+1} f_k\left[\frac{\partial\alpha_{k-1}}{\partial\hat{\vartheta}}(\Gamma_\vartheta\tau_{\vartheta\rho} - \dot{\hat{\vartheta}})\right.\\
& \left. + \frac{\partial\alpha_{k-1}}{\partial\hat{D}}(\gamma_d\tau_{D\rho} - \dot{\hat{D}}) + \frac{\partial\alpha_{k-1}}{\partial\xi_0}(l_1P^{-1}\tau_{\chi\rho} - \chi)\right]\\
& + \tilde{\Theta}^T\left(\tau_{\Theta\rho} - \Gamma_\Theta^{-1}\dot{\hat{\Theta}}\right) + \sum_{k=3}^{\rho}(|z_k|-\delta_k)^{\rho-k+1} f_k\\
& \times\frac{\partial\alpha_{k-1}}{\partial\hat{\Theta}}(\Gamma_\Theta\tau_{\Theta\rho} - \dot{\hat{\Theta}}) - \frac{1}{l_1}\epsilon^T\epsilon\\
= & -\sum_{i=1}^{\rho} c_i(|z_i|-\delta_i)^{2(\rho-i+1)} f_i - \frac{1}{l_1}\epsilon^T\epsilon. \qquad (61)
\end{aligned}$$

From (61), we get the following lemma.

Lemma 1: The adaptive controller designed previously ensures that $z_1, \ldots, z_\rho, \hat{\Theta}, \hat{e}, \hat{b}_m, \hat{\vartheta}, \hat{D}, \epsilon$ are all bounded.

With Lemma 1, all the signals in the closed-loop can be shown to be bounded and a bound can be established for the tracking error, as stated in the following theorem.

Theorem 1: Consider the system consisting of the parameter estimators given by (41), (43), (47) and (57), adaptive controllers designed using (59) with virtual control laws (37) and (46), and plant (1) with a backlash nonlinearity (4). The system is stable in the sense that all signals in the closed loop are bounded. Furthermore, the following hold.

- The tracking error converges $[\delta_1, -\delta_1]$ asymptotically, i.e.,

$$\lim_{t\to\infty}|y(t) - y_r(t)| = \delta_1. \qquad (62)$$

- The transient tracking error performance is given by

$$\begin{aligned}
&\|\, |y(t) - y_r(t)| - \delta_1 \,\|_2\\
&\leq \frac{1}{c_1^{1/2\rho}}\left(\frac{1}{2}\tilde{\Theta}(0)^T\Gamma_\Theta^{-1}\tilde{\Theta}(0) + \frac{|b_m|}{2\Gamma_\beta}\tilde{\beta}(0)^2 + \frac{1}{2\Gamma_\vartheta}\tilde{\vartheta}(0)^2\right.\\
&\quad\left. + \frac{|b_m|}{2\gamma_e}\tilde{e}(0)^2 + \frac{1}{2\gamma_d}\tilde{D}(0)^2 + \frac{1}{2\gamma_b}\tilde{b}_m(0)^2 + \frac{1}{2l_1}\epsilon(0)^2\right)^{1/2\rho}
\end{aligned} \qquad (63)$$

with $z_i(0) = 0, i = 1, \ldots, \rho$.

Proof: From Lemma 1, we have that $z_1, \ldots, z_\rho, \hat{\beta}, \hat{\Theta}, \hat{e}, \hat{b}_m, \hat{\vartheta}, \hat{D}, \epsilon$ are bounded. Following similar approaches to those in [13], we can obtain the boundedness of $\alpha_i, i = 1, \ldots, \rho$, χ and u_d, and so are $v = \widehat{B}I(u_d)$ and $u = DI(v)$. It follows that $\hat{\omega} \in L^\infty$. From (23), we have that η is bounded. Then $\hat{x}$ is bounded from (22) and finally $x(t) = \hat{x}(t) + \epsilon(t)$ is bounded from (22)–(23). Thus all signals in the closed-loop are bounded. The tracking error performance can be obtained from (61) following similar approaches to those in [10].

Remark 5: From Theorem 1 the following conclusions can be obtained.

- The transient performance depends on the initial estimate errors $\tilde{e}(0)$, $\tilde{\beta}(0)$, $\tilde{\Theta}(0)$, $\tilde{\vartheta}(0)$, $\tilde{D}(0)$, $\tilde{b}_m(0)$ and the explicit design parameters. The closer the initial estimates $\hat{e}(0)$, $\hat{\beta}(0)$, $\hat{\Theta}(0)$, $\hat{\vartheta}(0)$, $\hat{b}_m(0)$ and $\hat{D}(0)$ to the true values $e, \beta, \Theta, \vartheta, b_m$ and D, the better the transient performance.
- The bound for $\| y(t) - y_r(t) \|_2$ is an explicit function of design parameters and thus computable. We can decrease the effects of the initial error estimates on the transient performance by increasing the adaptation gains $c_1, \gamma_d, \gamma_e, \gamma_b$ and $\Gamma_\beta, \Gamma_\Theta, \Gamma_\vartheta$.
- The value of δ_1 can be chosen as small as possible according to the desired accuracy, since the output tracking error will converge to $[-\delta_1, \delta_1]$. Notes that δ_1 may influence the control input through the effects of sg_1, f_1 and their derivatives in the backstepping design.

V. Simulation Study

In this section, we illustrate the above methodology on the following nonlinear system:

$$\begin{aligned}
\dot{x}_1 &= x_2 \quad \dot{x}_2 = u + a\frac{1 - e^{-x_1(t)}}{1 + e^{-x_1(t)}}\\
y &= x_1 \quad u = B(v)
\end{aligned} \qquad (64)$$

where u represents the output of the backlash nonlinearity as in (4), parameter a and backlash parameters m, B_r, B_l are unknown, but $m \geq 0.1$. The actual parameter values are chosen as $a = 1$, $m = 1$, $B_r = 0.5$, $B_l = -0.8$ for simulation. The objective is to control the system output y to follow a desired trajectory $y_r(t) = 10\sin(2.5t)$. First, we choose the backlash inverse $v(t) = \widehat{B}I(u_d(t))$ as in (11) and the filters

$$\dot{\xi}_0 = A_0\xi_0 + ky + \chi \quad \dot{\xi}_1 = A_0\xi_1 + Y_1 \qquad (65)$$

$$\dot{\eta} = A_0\eta + e_2u \quad \hat{\omega}_2 = \frac{p + k_1}{p^2 + k_1p + k_2}I_3[\hat{\omega}] \qquad (66)$$

where

$$\begin{aligned}
Y_1 &= \left[0, \frac{1 - e^{-x_1(t)}}{1 + e^{-x_1(t)}}\right]^T \quad k = [k_1, k_2]^T = [1, 3]^T\\
A_0 &= \begin{bmatrix}-k_1 & 1\\ -k_2 & 0\end{bmatrix} = \begin{bmatrix}-1 & 1\\ -3 & 0\end{bmatrix}.
\end{aligned} \qquad (67)$$

Then, we apply our control design to the plant. In the simulations, we choose $c = 2, e_0 = 1, \delta_1 = 0.01, \gamma = 1, c_1 = c_2 = c, \Gamma_a = \gamma_d = \gamma, \Gamma_\theta = \gamma I_3$ and the initial parameters $\hat{a}(0) = 1.2, \hat{D}(0) = 0.4, \hat{\theta}(0) = [1, 0.4, -0.6]^T$. The initial state is chosen as $y(0) = 0.6$. The parameters and the initial states are the same as in [10]. For comparison, the scheme in [10] and our proposed scheme are both applied to the system. In [10], state feedback control is used and the effect of backlash is considered as a disturbance. The newly developed scheme studies output feedback control and a smooth backlash inverse is used to compensate the effect of backlash. The simulation results presented in the Figs. 3 and 4 are the tracking error and the controller output $v(t)$. Clearly, the simulation results verify our theoretical findings and show the effectiveness of our control scheme. The newly developed scheme gives better performance compared with [10].

VI. Conclusion

This note presents an output feedback backstepping adaptive controller design scheme for a class of uncertain nonlinear single-input–single-output system preceded by uncertain backlash actuator nonlinearity. We propose a new smooth adaptive inverse to compensate the effect of the unknown backlash. Such an inverse can avoid possible chattering phenomenon which may be caused by nonsmooth inverse. The inverse function is employed in the

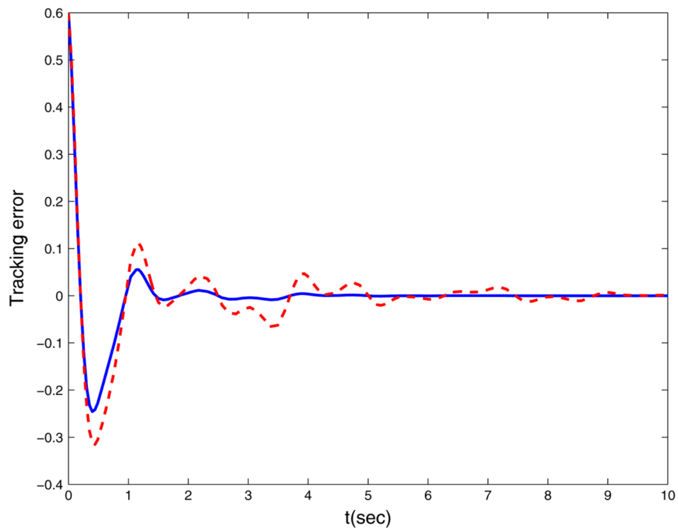

Fig. 3. Tracking error (solid line: proposed scheme; dash line: scheme in [10]).

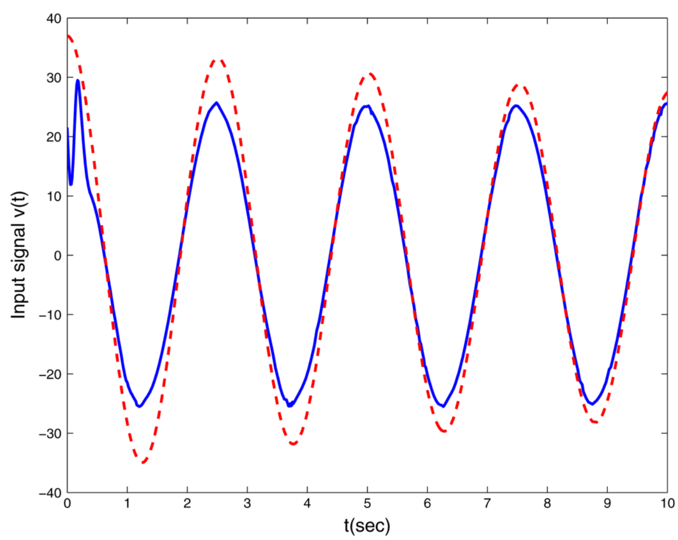

Fig. 4. Control signal $v(t)$ (solid line: proposed scheme; dash line: scheme in [10]).

backstepping controller design. The over-parametrization problem is solved by using the concept of tuning functions. For the design and implementation of the controller, no knowledge is assumed on the unknown system parameters. Besides showing stability, we also give an explicit bound on the L_2 performance of the tracking error in terms of design parameters. Simulation results illustrate the effectiveness of our proposed scheme.

We also conducted simulation studies to examine how control gains affect the control amplitude. It is observed that increasing control gains may result in larger control action. However, there is no result to ensure this observation theoretically and we feel that this is an interesting problem for further investigation.

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