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Adaptive sliding controller synthesis for non-linear systems

J.-J. E. SLOTINE† and J. A. COETSEE†

Classical 'sliding mode control', as investigated mostly in Soviet literature, features excellent robustness properties in relation to parametric uncertainty, but presents several important drawbacks that severely limit its practical applicability. These drawbacks, including large control authority and control chattering, were remedied by Slotine and Sastry (1983) and Slotine (1984) by replacing control switching at a fixed sliding surface by a smooth control interpolation in a boundary layer neighbouring a time-varying sliding surface. This avoids the excitation of high-frequency unmodelled dynamics, and leads to an explicit trade-off between model uncertainty and controller tracking performance. The present paper examines how to further improve performance by effectively coupling on-line parameter estimation to sliding controller design. The boundary layer concept leads to a compact measure of the quality of parameter estimation, and provides a consistent rule on when to stop adaptation. The approach is demonstrated by simulations.

1. Introduction

The notion of a sliding surface (Filippov 1960) and the associated 'sliding mode control' theory has been investigated mostly in Soviet literature (see Utkin 1977 for a review), where it has been used to stabilize a class of non-linear systems. Although it theoretically features excellent robustness properties in the face of *parametric* uncertainty, classical sliding mode control presents several important drawbacks that severely limit its practical applicability. In particular, it involves large control authority and control *chattering*, as discussed in detail by Slotine and Sastry (1983). Chattering is generally highly undesirable in practice (with a few exceptions, such as the control of electric DC motors), since it implies extremely high control activity. Further, it may excite high-frequency dynamics neglected in the course of modelling (such as resonant structural modes, neglected actuator time-delays, or sampling effects). These problems are remedied by Slotine and Sastry (1983) and Slotine (1984) by replacing chattering control by a smooth control interpolation in a *boundary layer* neighbouring a *time-varying* sliding surface. Slotine (1984) shows how to monitor the boundary layer width so as not to excite high-frequency unmodelled dynamics, and quantifies the corresponding trade-off between modelling effort and controller tracking performance. The methodology has been applied to the control of high-performance robots (Slotine 1985) and remotely operated underwater vehicles (Yoerger and Slotine 1985).

This paper examines how further to improve tracking performance by effectively coupling on-line parameter estimation to sliding controller design. The algebraic distance of the current state to the boundary layer is found to represent a natural and compact measure of the quality of parameter adaptation, and is thus selected as the error signal in the estimation scheme. This leads, in particular, to a consistent rule on when to stop adaptation, and avoids the undesirable long-term drift often found in the

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adaptive control literature. The approach is demonstrated in simulations, in the presence of high-frequency unmodelled dynamics.

The layout of the paper is as follows: § 2 summarizes the control methodology of Slotine and Sastry (1983) and Slotine (1984). The inclusion and effective monitoring of on-line estimation schemes are detailed in § 3. Simulation results are presented in § 4. Brief concluding remarks are offered in § 5.

2. Robust sliding control of non-linear systems

In this section, we summarize the methodology of Slotine and Sastry (1983) and Slotine (1984). For notational simplicity, the concepts are presented for systems with a single control input.

2.1. *Sliding surfaces*

Consider the dynamic system:

$$x^{(n)}(t) = f(\mathbf{X};\, t) + b(\mathbf{X};\, t)u(t) + d(t) \tag{1}$$

where $u(t)$ is the control input, x is the output of interest, and $\mathbf{X} = [x \ \ \dot{x} \ \ \ldots \ \ x^{(n-1)}]^{\mathrm{T}}$ is the state. In (1), the function $f(\mathbf{X};\, t)$ (in general non-linear) is not known exactly, but the *extent of the imprecision* $|\Delta f|$ *on* $f(\mathbf{X};\, t)$ *is upper bounded by a known continuous function of* X *and* t; similarly, control gain $b(\mathbf{X}, t)$ is not known exactly, but is of constant sign (say positive), and is bounded by known, continuous functions of $\mathbf{X}$ and t. Both $f(\mathbf{X};\, t)$ and $b(\mathbf{X};\, t)$ are assumed to be continuous in $\mathbf{X}$. Disturbance $d(t)$ is unknown but bounded in absolute value by a known continuous function of time. The control problem is to get the state $\mathbf{X}$ to *track a specific state* $\mathbf{X}_{\mathrm{d}} = [x_{\mathrm{d}} \ \ \dot{x}_{\mathrm{d}} \ \ \ldots \ \ x_{\mathrm{d}}^{(n-1)}]^T$ *in the presence of model imprecision on* $f(\mathbf{X};\, t)$ *and* $b(\mathbf{X};\, t)$, *and of disturbances* $d(t)$. For this to be achievable using a *finite* control u, we must assume:

$$\tilde{\mathbf{X}}|_{t=0} = 0 \tag{2}$$

where

$$\tilde{\mathbf{X}} := \mathbf{X} - \mathbf{X}_d = [\tilde{x} \ \ \dot{\tilde{x}} \ \ \ldots \ \ \tilde{x}^{(n-1)}]^{\mathrm{T}}$$

is the tracking error vector (this assumption will be discussed further later.) We define a *time-varying sliding surface* S(t) in the state-space $\mathbb{R}^n$ as $s(\mathbf{X};\, t) = 0$ with

$$s(\mathbf{X};\, t) := \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}, \quad \lambda > 0 \tag{3}$$

where λ is a positive constant (design parameter λ will be interpreted later as the desired control bandwidth). Given initial condition (2), *the problem of tracking* $\mathbf{X} \equiv \mathbf{X}_{\mathrm{d}}$ *is equivalent to that of remaining on the surface* S(t) *for all* $t > 0$—indeed $s \equiv 0$ represents a linear differential equation whose unique solution is $\tilde{x} \equiv 0$, given initial condition (2). Now a sufficient condition for such positive invariance of $S(t)$ is to choose the control law u of (1) such that outside $S(t)$

$$\frac{1}{2}\frac{d}{dt} s^2(\mathbf{X};\, t) \leqslant -\eta |s| \tag{4}$$

where η is a positive constant. Inequality (4) constrains trajectories to point towards the surface $S(t)$, as illustrated in Fig. 1, and is referred to as the *sliding condition.*

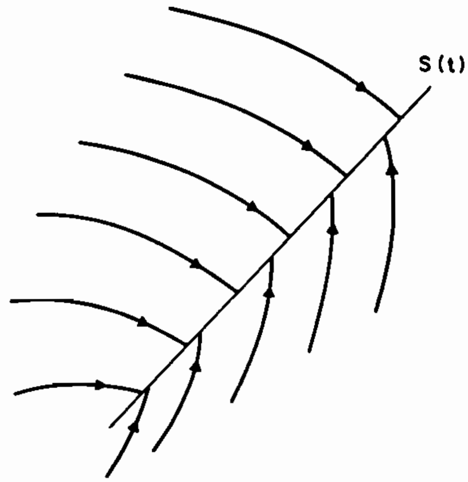


Figure 1. Illustrating the sliding condition.

The idea behind (3) and (4) is to pick-up a well-behaved function of the tracking error, s, according to (3)and then select the feedback control law u in (1) such that s^2 remains a Lyapunov function of the closed-loop system despite the presence of model imprecision and disturbances. Further, satisfying (3) guarantees that if Condition (2) is not exactly verified, i.e. if $\mathbf{X}|_{t=0}$ is actually off $\mathbf{X}_d|_{t=0}$, the surface $S(t)$ will nonetheless be reached in a finite time inferior to $|s(\mathbf{X}(0); 0)|/\eta$, while Definition (3) then implies that $\tilde{\mathbf{X}} \to 0$ as $t \to \infty$.

The controller design procedure consists of two steps. First, as illustrated in § 2.2, a feedback control law u is selected so as to verify Sliding Condition (4). However, in order to account for the presence of modelling imprecision and disturbances, such a control law is *discontinuous across* $S(t)$, which leads to control *chattering*. Now chattering is undesirable in practice, since it involves high control activity and, further, may excite high-frequency dynamics neglected in the course of modelling (such as unmodelled structural modes, neglected time-delays etc). Thus, in a second step detailed in § 3.3 *discontinuous control law* u *is suitably smoothed to achieve an optimal trade-off between control bandwidth and tracking precision*: while the first step accounts for parametric uncertainty, the second step achieves robustness to high-frequency unmodelled dynamics.

2.2. *Perfect tracking using switched control laws*

Given the bounds on uncertainties on $f(\mathbf{X}; t)$ and $b(\mathbf{X}; t)$, and on disturbances $d(t)$, constructing a control law to verify Sliding Condition (4) is straightforward, as we now illustrate by simple examples.

Example 1

Consider the scalar second-order system

$$\ddot{x} = f + u \tag{5}$$

where u is the control input and the dynamics f (possibly non-linear or time-varying) are not known exactly, but estimated as $\hat{f}$. Dynamics f need not depend only on x or $\dot{x}$, but may more generally be a function of any *measured* variables external to System (5). The estimation error on f is assumed to be *bounded* by some known function F:

$$|\hat{f}-f| \leqslant F \tag{6}$$

In order to have the system track $x(t) \equiv x_d(t)$, we define a sliding surface $s=0$ according to (3), namely:

$$s := \left(\frac{d}{dt}+\lambda\right)\tilde{x} = \dot{\tilde{x}} + \lambda\tilde{x}$$

We then have:

$$\dot{s} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}} = f + u - \ddot{x}_d + \lambda\dot{\tilde{x}} \tag{7}$$

The best approximation $\hat{u}$ of a continuous control law that would achieve $\dot{s}=0$ is thus:

$$\hat{u} = -\hat{f} + \ddot{x}_d - \lambda\dot{\tilde{x}} \tag{8}$$

In order to satisfy Sliding Condition (4) despite uncertainty on the dynamics f, we add to $\hat{u}$ a term *discontinuous* across the surface $s=0$:

$$u := \hat{u} - k \operatorname{sgn}(s) \tag{9}$$

By choosing k in (9) to be large enough, we can now guarantee that (4) is verified. Indeed, we have from (7)–(9)

$$\frac{1}{2}\frac{d}{dt}s^2 = \dot{s}.s = (f-\hat{f}-k.\operatorname{sgn}(s))s = (f-\hat{f})s - k|s|$$

so that, letting

$$k := F + \eta \tag{10}$$

we get from (6)

$$\frac{1}{2}\frac{d}{dt}s^2 \leqslant -\eta|s|$$

as desired. Note from (10) that the control discontinuity, k, across the surface $s=0$, increases with the extent of parametric uncertainty.

Example 2

An equivalent result would be obtained by using integral control, i.e. formally letting

$$\int_0^t \tilde{x}\, d\tau$$

be the variable of interest. System (5) is third order relative to this variable, and (3) gives:

$$s := \left(\frac{d}{dt}+\lambda\right)^2\left(\int_0^t \tilde{x}\, d\tau\right) = \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2\int_0^t \tilde{x}\, d\tau$$

We then obtain, instead of (8):

$$\hat{u} = -\hat{f} + \ddot{x}_d - 2\lambda\dot{\tilde{x}} - \lambda^2\tilde{x}$$

with (9), (10) formally unchanged. Note that

$$\int_0^t \tilde{x}\, d\tau$$

can be replaced by

$$\int^t \tilde{x}\, d\tau$$

i.e. the integral can be defined to within a constant. The constant can be chosen to obtain $s(t=0)=0$ regardless of $\mathbf{X}_d(t=0)$, by letting

$$s := \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2 \int_0^t \tilde{x}\, d\tau - \dot{\tilde{x}}(0) - 2\lambda\tilde{x}(0)$$

Example 3

Assume now that (5) be replaced by

$$\ddot{x} = f + bu \tag{11}$$

where (possibly non-linear or time-varying) control gain b is known only to within a certain *gain margin* β:

$$\beta^{-1} \leqslant \frac{\hat{b}}{b} \leqslant \beta \tag{12}$$

where $\hat{b}$ is the available estimate of gain b. With s and $\hat{u}$ defined as in Example 1, one can then easily show that the control law:

$$u := (\hat{b})^{-1}[\hat{u} - k \operatorname{sgn}(s)]$$

where

$$k := \beta(F+\eta) + (\beta - 1)|\hat{u}|$$

satisfies the sliding condition. Note that control discontinuity has been increased in order to account for uncertainty in control gain b.

Example 4

Still for the system (11), assume that we have

$$\beta_{\min} \leqslant \frac{\hat{b}}{b} \leqslant \beta_{\max}$$

instead of (12). Uncertainty in control gain b can be put back in the form (12) by letting

$$\beta := (\beta_{\max}/\beta_{\min})^{1/2}$$

and replacing estimated gains $\hat{b}$ by $(\beta_{\max}/\beta_{\min})^{-1/2}\hat{b}$. Similarly, knowing that

$$b_{\min} \leqslant b \leqslant b_{\max}$$

leads to

$$\hat{b} := (b_{\min} \cdot b_{\max})^{1/2}; \quad \beta = (b_{\max}/b_{\min})^{1/2}$$

2.3. *Continuous control laws to approximate switched control*

As remarked in the preceding examples, control laws that satisfy Sliding Condition (4) are discontinuous across the surface $S(t)$, thus leading to control *chattering*. Chattering is in general highly undesirable in practice, since it involves extremely high control activity, and, further, may excite high-frequency dynamics neglected in the course of modelling. We can remedy this situation by smoothing out the control discontinuity in a thin *boundary layer* neighbouring the switching surface:

$$B(t) = \{\mathbf{X}, |s(\mathbf{X}; t)| \leqslant \Phi\}; \quad \Phi > 0$$

where Φ is the boundary layer *thickness*, and $\varepsilon := \Phi/\lambda^{n-1}$ is the boundary layer *width* (Fig. 2). Such continuous approximation is similar in spirit to those of Corless and Leitmann (1981) or Gutman and Palmor (1982). Control smoothing is achieved by choosing control law u *outside* $B(t)$ as before (i.e. satisfying Sliding Condition (4)), which guarantees boundary layer attractiveness and hence positive invariance—all trajectories starting inside $B(t = 0)$ remain inside $B(t)$ for all $t \geqslant 0$—and then interpolating u inside $B(t)$—for instance, replacing in the expression of u the term sgn s by s/Φ inside $B(t)$ as illustrated in Fig. 3. As proved by Slotine (1983), this operation leads to *tracking to within a guaranteed precision* ε, and more generally guarantees that for all trajectories starting inside $B(t = 0)$

$$|\tilde{x}^{(i)}(t)| \leqslant (2\lambda)^i \varepsilon \quad i = 0, \ldots, n-1 \tag{13}$$

Further, as shown by Slotine (1984), *the smoothing of control discontinuity inside* B(t) *essentially assigns a lowpass filter structure to the local dynamics of the variable* s, thus eliminating chattering. Recognizing this filter structure allows us to tune up the control law so as to achieve a trade-off between tracking precision and robustness to unmodelled dynamics. Boundary layer thickness Φ is made to be *time varying*, and is monitored so as always to exploit the maximum control bandwidth available. The development of Slotine (1984) is summarized below for the case $\beta = 1$ (no gain margin), and then generalized.

Consider again System (1) with $b = \hat{b} = 1$. In order to maintain attractiveness of the boundary layer now that Φ is allowed to vary with time, we must actually modify Sliding Condition (4) into

$$|s| \geqslant \Phi \Rightarrow \frac{1}{2}\frac{d}{dt}s^2 \leqslant (\dot{\Phi} - \eta)|s| \tag{14}$$

The additional term $\dot{\Phi}|s|$ in (14) reflects the fact that the boundary layer attraction condition is more stringent during boundary layer contraction ($\dot{\Phi} < 0$) and less stringent during boundary layer expansion ($\dot{\Phi} > 0$). In order to satisfy (14), the quantity $-\dot{\Phi}$ is added to control discontinuity gain $k(\mathbf{X}; t)$, i.e. in our smoothed implementation the term $k(\mathbf{X};\ t)$ sgn (s) obtained from switched control law u is actually replaced by $\bar{k}(\mathbf{X};\ t)$ sat (s/Φ), where:

$$\bar{k}(\mathbf{X}; t) := k(\mathbf{X}; t) - \dot{\Phi} \tag{15}$$

Accordingly, control law u becomes:

$$u = \hat{u} - \bar{k}\,\mathrm{sat}\,(s/\Phi)$$

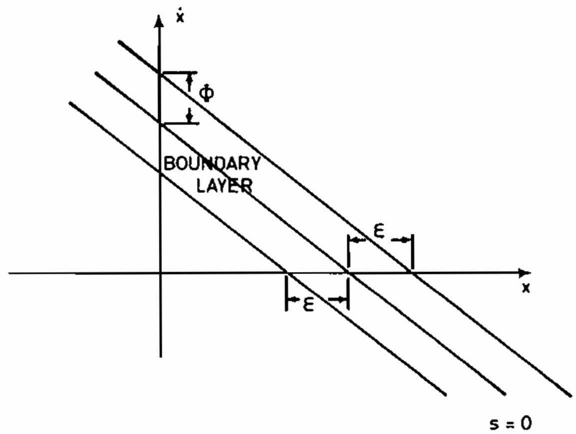


Figure 2. Showing the construction of the boundary layer in the case where $n = 2$.

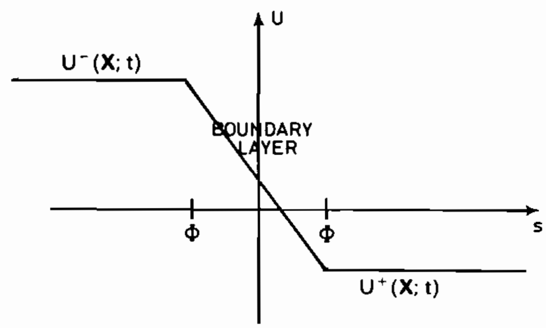


Figure 3. Showing a sample interpolation of the control law in the boundary layer.

Consider now the system trajectories *inside the boundary layer*, where they lie by construction. They can be expressed directly in terms of the variable s, as:

$$\dot{s} = -\bar{k}(\mathbf{X};\, t)s/\Phi + (\Delta f(\mathbf{X};\, t) + d(t)) \tag{16}$$

Now since $\bar{k}$ and Δf are continuous in $\mathbf{X}$, we can exploit (13) to rewrite (16) in the form:

$$\dot{s} = -\bar{k}(\mathbf{X}_d;\, t)s/\Phi + (\Delta f(\mathbf{X}_d;\, t) + d(t) + O(\varepsilon)) \tag{17}$$

We see from (17) that *the variable* s (which is a measure of the algebraic distance to the surface $S(t)$) *is the output of a stable first-order filter*, whose dynamics depend only on the desired state $\mathbf{X}_d(t)$ and perhaps explicitly on time, and whose inputs are (to first order) 'perturbations', namely disturbance $d(t)$ and uncertainty $\Delta f(\mathbf{X}_d;\, t)$. Chattering is thus indeed eliminated, as long as high-frequency unmodelled dynamics are not excited. The dynamic structure of the closed-loop system is summarized in Fig. 4: perturbations are filtered according to (17) to give s, which in turn provides tracking error $\tilde{x}$ by further lowpass filtering, according to Definition (3). Control u is a function of $\tilde{\mathbf{X}}$ and $\mathbf{X}_d$. Now, since λ is the break-frequency of filter (3), it must be chosen to be 'small' with respect to high-frequency unmodelled dynamics (such as unmodelled structural modes or neglected time delays). Further, we can now tune boundary layer thickness Φ so that (17) also represents a first-order filter of bandwidth λ. It suffices to let

$$\frac{\bar{k}(\mathbf{X}_d;\, t)}{\Phi} := \lambda \tag{18}$$

which can be written from (15) as

$$\dot{\Phi} + \lambda\Phi = k(\mathbf{X}_d;\, t) \tag{19}$$

Equation (19) defines the desired time-history of boundary layer thickness Φ, and in the light of Fig. 4 will be referred to as the *balance condition*. Intuitively, it amounts to tune up the closed-loop system so that it mimics an nth order critically damped system. Further, Definition (15) can then be rewritten as:

$$\bar{k}(\mathbf{X};\, t) := k(\mathbf{X};\, t) - k(\mathbf{X}_d;\, t) + \lambda\Phi \tag{20}$$

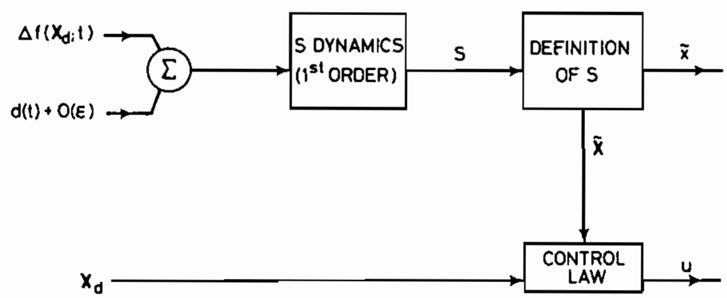


Figure 4. Dynamic structure of the closed-loop system.

In the case where $\beta \neq 1$, one can show (Slotine 1984) that (19), (20) become (with $\beta_d = \beta(\mathbf{X}_d;\, t)$)

$$k(\mathbf{X}_d;\, t) \geqslant \frac{\lambda\Phi}{\beta_d} \Rightarrow \dot{\Phi} + \lambda\Phi = \beta_d k(\mathbf{X}_d;\, t) \tag{21}$$

$$k(\mathbf{X}_d;\, t) \leqslant \frac{\lambda\Phi}{\beta_d} \Rightarrow \dot{\Phi} + \frac{\lambda\Phi}{\beta_d^2} = k(\mathbf{X}_d;\, t)/\beta_d \tag{22}$$

$$\bar{k}(\mathbf{X};\, t) := k(\mathbf{X};\, t) - k(\mathbf{X}_d;\, t) + \lambda\Phi/\beta_d \tag{23}$$

with initial condition $\Phi(0)$ defined as:

$$\Phi(0) := \beta_d k(\mathbf{X}_d(0);\, 0)/\lambda \tag{24}$$

Note that the balance conditions (21), (22) imply that Φ, and thus $\tilde{\mathbf{X}}$, are bounded for bounded $\mathbf{X}_d$.

The balance conditions have a simple and intuitive physical interpretation: neglecting time constants of order $1/\lambda$, they imply

$$\lambda^n \varepsilon \approx \beta_d k(\mathbf{X}_d;\, t) \tag{25}$$

that is

(bandwidth)^n × (tracking precision)

≈ (parametric uncertainty measured along the desired trajectory)

Such a trade-off is quite similar to the situation encountered in linear time-invariant systems, but applies here to the more general non-linear system structure (1). Also, remark that the *s-trajectory*, i.e. the variation of s/Φ with time, is a compact descriptor of the closed-loop system behaviour: control activity directly depends on s/Φ, while, by Definition (3), tracking error $\tilde{x}$ is merely a filtered version of s. Further, the *s-trajectory represents a time-varying measure of the validity of the assumptions on model uncertainty*. Similarly, boundary layer thickness Φ describes the evolution of dynamic model uncertainty with time.

3. Adaptive sliding control

As summarized in the previous section, sliding control achieves effective tracking control of a class of non-linear systems in the presence of disturbances and parameter variations, while avoiding the excitation of high-frequency dynamics. Further, the balance conditions quantify the relationship between maximum tracking error and the extent of parametric uncertainty. In order further to improve performance when large parametric uncertainties are present, we now introduce an *adaptive* sliding controller, where uncertain parameters are estimated on-line. This represents a natural extension of the previous development, and is similar in spirit to Narendra *et al.* (1980) or Corless and Leitmann (1983). An important feature of the methodology, however, is that it provides a consistent rule for ceasing adaptation. The sliding controller itself is designed as if the parameters to be estimated were known exactly, i.e. as if adaptation were successfully and exactly complete. Since this is initially not the case, the system first wanders outside the boundary layer, and the information thus generated is used to improve the parameter estimates. Conversely, the scheme recognizes that once in the boundary layer no advantage is gained by further adaptation, since even if the parameters of concern were known exactly, no improvement in tracking performance could be guaranteed formally.

For clarity, we first detail a simplified version in § 3.1 before presenting the general case in § 3.2. The robustness issues linked to the presence of high-frequency unmodelled dynamics are discussed in § 3.3.

3.1. *Basic adaptive sliding controller*

The basic form of the proposed adaptive sliding controller is applicable to systems of the type

$$x^{(n)} + \sum_{i=1}^{r} a_i Y_i = bu + d \tag{26}$$

where Y_i are known continuous, possibly non-linear functions of the state variables (or of any measured variables 'external' to System (26)), parameters a_i and control gain b are unknown but constant, and $d = d(t)$ is a bounded disturbance. For notational simplicity, the theory is first developed using a simple example and then generalized.

Consider the non-linear system:

$$\ddot{x} + a_1\dot{x} + a_2 x^2 = bu + d \tag{27}$$

where a_1, a_2 and b are unknown but constant, and

$$|d(t)| \leqslant D$$

As in § 2, we want $x(t)$ to track a desired trajectory $x_d(t)$. We define the sliding surface

$s=0$ as

$$s=\left(\frac{d}{dt}+\lambda\right)^2\left(\int_0^t \tilde{x}(\tau)\,d\tau\right)=\dot{\tilde{x}}+2\lambda\tilde{x}+\lambda^2\int_0^t \tilde{x}(\tau)\,d\tau$$

where $\tilde{x}(t)=x(t)-x_d(t)$ is the tracking error, and again we smooth out the control discontinuities inside a boundary layer:

$$B(t)=\{\mathbf{X}:|s(\mathbf{X};t)|\leqslant\Phi\};\quad \Phi>0$$

In this section, we let $\dot{\Phi}\equiv 0$ since residual dynamic uncertainty would be time invariant if adaptation were perfect, as it would be owing only to $d(t)$. To derive a control law that ensures convergence to the boundary layer, we first define a Lyapunov function candidate $V(t)$ as

$$V(t):=\frac{1}{2}\left[s_\Delta^2+b\left(\left(\hat{h}_1-\frac{a_1}{b}\right)^2+\left(\hat{h}_2-\frac{a_2}{b}\right)^2+(\hat{b}^{-1}-b^{-1})^2\right)\right] \tag{28}$$

where

$$s_\Delta:=s-\Phi\ \mathrm{sat}\ (s/\Phi) \tag{29}$$

is a measure of the algebraic distance of the current state to the boundary layer. The $\hat{h}_j$ in (28) can be thought of as the estimates of the (a_j/b), while $\hat{b}$ is the estimate of b. We then select control law u as

$$u:=\hat{h}_1\dot{x}+\hat{h}_2x^2-\hat{b}^{-1}[u^*+(D+\eta)\ \mathrm{sat}\ (s/\Phi)] \tag{30}$$

where:

$$u^*:=-\ddot{x}_d+2\lambda\dot{\tilde{x}}+\lambda^2\tilde{x} \tag{31}$$

and $\hat{h}_1$, $\hat{h}_2$ and $\hat{b}$ are estimated on-line according to some rule yet to be defined. Given the interpretations of $\hat{h}_1$, $\hat{h}_2$ and $\hat{b}$, we see that u of (30) is defined as if adaptation was complete and exact.

Noting that $\dot{s}_\Delta=\dot{s}$ outside the boundary layer, while $s_\Delta=0$ inside the boundary layer, we get:

$$\dot{V}(t)=(b\hat{h}_1-a_1)(\dot{x}s_\Delta+\dot{\hat{h}}_1)+(b\hat{h}_2-a_2)(x^2s_\Delta+\dot{\hat{h}}_2)$$
$$-(b\hat{b}^{-1}-1)u^*s_\Delta-b\hat{b}^{-1}(D+\eta)s_\Delta\ \mathrm{sat}\ (s/\Phi)+d.s_\Delta+(b\hat{b}^{-1}-1)\dot{\hat{b}}^{-1}$$

where $\dot{\hat{b}}^{-1}$ is the time-derivative of $\hat{b}^{-1}$. Selecting the adaptation laws to be:

$$\dot{\hat{h}}_1:=-\dot{x}s_\Delta \tag{32}$$

$$\dot{\hat{h}}_2:=-x^2s_\Delta \tag{33}$$

$$\dot{\hat{b}}^{-1}:=[u^*+(D+\eta)\ \mathrm{sat}\ (s/\Phi)]s_\Delta \tag{34}$$

leads to

$$\dot{V}(t)=(-(D+\eta)\ \mathrm{sat}\ (s/\Phi)+d)s_\Delta \tag{35}$$

so that

$$\dot{V}(t)\leqslant-\eta|s_\Delta| \tag{36}$$

outside the boundary layer. Given Definition (28), Inequality (36) can be thought of as a 'dual control' effect, in the sense that we trade a reduction in parametric uncertainty for an increase in the tracking error. Further, we see from (32)–(34) that adaptation ceases as soon as we reach the boundary layer. This avoids the undesirable long term drift found in many adaptive schemes, and provides a consistent rule on when to stop adaptation. Indeed, disturbance $d(t)$ can drive the system anywhere in the boundary layer without providing any information about the residual adaptation error. Finally, Definition (28) implies that $\dot{V} = 0$ inside the boundary layer, which shows that (36) is valid everywhere and thus guarantees further that trajectories eventually converge to the boundary layer.

The Lyapunov function V of (28) can be written as

$$V(t) = \tfrac{1}{2}(s_\Delta^2 + r_\Delta^2) \tag{37}$$

with

$$r_\Delta^2 = b\left[\left(\hat{h}_1 - \frac{a_1}{b}\right)^2 + \left(\hat{h}_2 - \frac{a_2}{b}\right)^2 + (\hat{b}^{-1} - b^{-1})^2\right] \tag{38}$$

so that r_Δ represents a composite measure of parameter error. With this interpretation, we see that, instead of Definition (38), we can assign weights to the components of the error according to the relative importance we attribute to each term, i.e. define r_Δ according to

$$r_\Delta^2 := b\left[\left(\frac{\hat{h}_1 - a_1/b}{h_{1n}}\right)^2 + \left(\frac{\hat{h}_2 - a_2/b}{h_{2n}}\right)^2 + \left(\frac{\hat{b}^{-1} - b^{-1}}{b_n}\right)^2\right] \tag{39}$$

instead of (38). The estimation rules (32)–(34) are then modified into

$$\dot{\hat{h}}_1 = -h_{1n}^2 \dot{x} s_\Delta \tag{40}$$

$$\dot{\hat{h}}_2 := -h_{2n}^2 x^2 s_\Delta \tag{41}$$

$$\dot{\hat{b}}^{-1} = b_n^2[u^* + (D + \eta)\,\text{sat}\,(s/\Phi)]s_\Delta \tag{42}$$

The preceding development can immediately be extended to systems of the form (26). We then use the sliding surface:

$$s = \left(\frac{d}{dt} + \lambda\right)^n \left(\int_0^t \tilde{x}\, d\tau\right) \tag{43}$$

the control law:

$$u = \sum_{i=1}^{r} \hat{h}_i Y_i - \hat{b}^{-1}[u^* + (D + \eta)\,\text{sat}\,(s/\Phi)]$$

with

$$u^* = -x_d^{(n)} + \sum_{i=1}^{n} \binom{n}{i} \lambda^i \tilde{x}^{(n-i)} \tag{44}$$

the Lyapunov function:

$$V(t) = \frac{1}{2}\left[s_\Delta^2 + b\left(\sum_{i=0}^{r}\left(\frac{\hat{h}_i - a_i/b}{h_{in}}\right)^2 + \left(\frac{\hat{b}^{-1} - b^{-1}}{b_n}\right)^2\right)\right] \tag{45}$$

and the estimation rules:

$$\dot{\hat{h}}_i = -h_{in}^2 Y_i s_\Delta \quad i = 0, \ldots, r \tag{46}$$

$$\dot{\hat{b}}^{-1} = b_n^2(u^* + (D + \eta)\,\text{sat}\,(s/\Phi)) \tag{47}$$

Further, it is often the case that an explicit upper bound on b is available. Adaptation on $\hat{b}$ can then be stopped if the estimate obtained from (47) becomes larger than the upper bound on b. Indeed, one can easily see that $b\hat{b}^{-1} \leqslant 1$ then implies that (36) remains verified even if adaptation on $\hat{b}$ is stopped, which is consistent with the physical intuition that overestimating control gain b cannot improve robustness to parametric uncertainty. Adaptation on $\hat{b}$ is resumed as soon as the right-hand side of (47) becomes positive. Similarly, adaptation on h_i can be stopped if a known upper (lower) bound is reached, and then resumed as soon as the right-hand side of (46) becomes negative (positive).

3.2. *Hybrid adaptive sliding control*

The controller of § 3.1 assumed that all system parameters were unknown but constant. We now extend the methodology to the case where the process may have some 'fast' time-varying parameters with known upper bounds. The controller we derive here is also applicable to the case where some parameters are composed of an unknown mean value with a time-varying but bounded part.

The development is again most conveniently presented in a simple example. Consider a slightly modified version of System (27) namely:

$$\ddot{x} + a_1\dot{x} + a_2(t)x^2 = (b_s \cdot b_f)u + d$$

where parameters a_1 and b_s are unknown but constant, 'fast' parameter $a_2(t)$ is bounded according to:

$$|a_2(t)| \leqslant \Delta a_2$$

and $b_f = b_f(t)$ is time-varying and estimated as $\hat{b}_f(t)$, where

$$\frac{1}{\beta_f} \leqslant \frac{\hat{b}_f(t)}{b_f(t)} \leqslant \beta_f; \quad \beta_f = \beta \geqslant 1 \tag{48}$$

The control problem is again to track a desired trajectory $x_d(t)$. Using the same sliding surface as in Subsection 3.1, we now make the boundary layer width Φ time-varying, since even if adaptation were perfect there would remain a time-varying dynamic uncertainty owing to the 'fast' parameters. We then define a Lyapunov function candidate $V(t)$ as:

$$V(t) = \frac{1}{2}\left[\gamma(t)s_\Delta^2 + b_s\left(\left(\hat{h}_1 - \frac{a_1}{b_s}\right)^2 \hat{h}_1 + (\hat{b}_s^{-1} - b_s^{-1})^2\right)\right] \tag{49}$$

Note that we use in (49) a time-varying coefficient $\gamma(t) > 0$, which will later be exploited to modulate the adaptation rate so as not to excite high-frequency unmodelled dynamics. Differentiating (49) yields:

$$\dot{V} = \frac{1}{2}\dot{\gamma}s_\Delta^2 + \gamma s_\Delta \dot{s}_\Delta + b_s\left[\left(\hat{h}_1 - \frac{a_1}{b_s}\right)\dot{\hat{h}}_1 + (\hat{b}_s^{-1} - b_s^{-1})(\dot{\hat{b}}_s^{-1})\right] \tag{50}$$

where again s_Δ of (29) measures the algebraic distance to the boundary layer, but now with

$$\dot{s} = \dot{s} - \dot{\Phi}\,\mathrm{sgn}\,(s) \quad \text{outside the boundary layer}$$

$$\dot{s}_\Delta = 0 \quad \text{inside the boundary layer}$$

Note that the attraction condition (14) of § 2.3 can be written simply in terms of s_Δ as

$$\frac{1}{2}\frac{d}{dt} s_\Delta^2 \leqslant -\eta |s_\Delta|$$

We define the control law u again as if adaptation were complete and exact, namely:

$$u = (\hat{b}_f)^{-1} \hat{h}_1 \dot{x} - (b_f \hat{b}_s)^{-1} [u^* + \bar{k}(\mathbf{X}) \operatorname{sat}(s/\Phi) + \beta \gamma' s_\Delta] \tag{51}$$

with (similarly to Example 3 of § 2)

$$k(\mathbf{X}) := \beta_f(|\hat{h}_1 \dot{x} \hat{b}_s - u^*|(1 - \beta_f^{-1}) + \Delta a_2 x^2 + D + \eta)$$

and $\bar{k}(\mathbf{X})$, Φ defined according to Balance Conditions (21)–(24). The term $(\gamma' s_\Delta)$ is introduced in the expression (51) of control law u in order to compensate for the term $(\dot{\gamma} s_\Delta^2/2)$ in (50) when $\dot{\gamma} > 0$, and thus must verify

$$\gamma' \geqslant \max(0, \dot{\gamma}/2\gamma) \tag{52}$$

We then select the adaptation law as

$$\dot{\hat{h}}_1 = -\gamma(t) \dot{x} s_\Delta \tag{53}$$

$$\dot{\hat{b}}_s^{-1} = \gamma(t)[u^* + (\Delta a_2 x^2 + D + \eta - \Phi) \operatorname{sat}(s/\Phi) + \gamma' s_\Delta] s_\Delta \tag{54}$$

Conceptually, the estimate of h_1 is generated by (53) as if the system had no fast parameter variations, while the estimate of b_s is generated by (54) as if b_f were equal to one. Expressions (53), (54) yield, after some tedious but straightforward manipulation:

$$\dot{V} \leqslant -\gamma \eta |s_\Delta| \tag{55}$$

and thus guarantee eventual convergence to the boundary layer, as long as $\gamma(t)$ is chosen uniformly bounded away from zero. Note that (55) assumes $\hat{b}_s^{-1} > 0$, and thus requires one actually to stop adaptation on $\hat{b}_s$ when some known upper bound of b_s is reached, as suggested in § 3.1; adaptation on $\hat{b}_s$ is resumed as soon as the right-hand side of (54) becomes positive. Further, similarly to § 3.1, Expressions (53), (54) imply that adaptation is stopped in the boundary layer. This is not surprising, since disturbance $d(t)$ and fast parameter errors can drive the system anywhere in the boundary layer without providing any information about the residual adaptation error.

The preceding development can immediately be generalized to systems of the form

$$x^{(n)} + \sum_{i=1}^{r} (a_{si} + a_{fi}) Y_i = (b_s \cdot b_f) u + d$$

where $a_{fi} = a_{fi}(t)$. Defining the sliding surface as in (43), the Lyapunov function

$$V = \frac{1}{2}\left[\gamma(t) s_\Delta^2 + b_s \left(\sum_{i=0}^{r} \left(\frac{\hat{h}_i - a_i/b_s}{h_{in}} \right)^2 + \left(\frac{\hat{b}_s^{-1} - b_s^{-1}}{b_n} \right)^2 \right)\right]$$

leads to

$$\dot{\hat{h}}_i = -\gamma(t) h_{in}^2 Y_i s_\Delta \quad i = 0, \ldots, r \tag{56}$$

$$\dot{\hat{b}}_s^{-1} = \gamma(t) b_n^2 \left[u^* + \left(\sum_{j=0}^{r} \Delta a_{sj} |Y_j| + D + \eta - \Phi \right) \operatorname{sat}(s/\Phi) + \gamma' s_\Delta \right] s_\Delta \tag{57}$$

where $\hat{h}_i$ is the estimate of a_{si}/b_s, and u^* is given by (44). The sliding controller itself is designed again as if b_s and the a_{si} were exactly estimated.

Remark 1

Time-varying parameters can often be regarded naturally as the sum of a constant or slowly-varying mean value and a bounded time-varying part, and thus can be put in the form $(a_{si} + a_{fi}(t))$. Further, the above development can be used to couple selectively an estimation scheme to an existing sliding controller: it suffices to delete from control gain $k(\mathbf{X}; t)$ the contribution of the uncertainty on the terms to be estimated, and then use for each such term an estimation scheme of the form (56). Similarly, one can progressively refine controller performance by sequentially introducing new terms to be estimated.

Remark 2

As in Example 2 of § 2.2, the integral term in the expression of s can be defined such that $s(t = 0) = 0$ regardless of initial conditions. Along similar lines, one may stop integration (i.e. maintain the integral term constant in the expression of s) as long as the system is outside the boundary layer. This leads generally to smoother parameter convergence and further alleviates problems of integrator windup.

3.3. *Robustness issues*

The adaptive controller developed above explicitly guarantees robustness to bounded parametric uncertainties and disturbances. This section discusses the issue of robustness to high-frequency unmodelled dynamics. The development mainly deals with the case of no adaptation on the control gain b, i.e. in the notations of § 3.2, the case $b_s = \hat{b}_s = 1$ (which also implies $h_i = a_{si}$ for all i). In other words, uncertainty on b is entirely accounted for by the known (perhaps time-varying) gain margin $\beta = \beta_f$ of (48). Further, since pure sliding control is applied in the boundary layer, we should only be concerned with robustness issues outside the boundary layer, i.e. those arising from the adaptation process itself.

The approach taken here consists in modulating the adaptation rate parameter $\gamma(t)$ of (49) so as not to excite high-frequency dynamics neglected in the course of modelling. Indeed, it is intuitively clear that larger $\gamma(t)$ lead to faster adaptation (from (49), (55)), but also increase the frequency content of control input u. Consider then at each instant the time-derivative $\dot{u}$ of the control input. It can be written:

$$\dot{u} = (\dot{u})_0 + (\dot{u})_{\text{adapt}}$$

where $(\dot{u})_0$ is the value of $\dot{u}$ that would be obtained if no adaptation were performed at instant t, and accordingly $(\dot{u})_{\text{adapt}}$ is, from (56):

$$(\dot{u})_{\text{adapt}} = -\hat{b}^{-1}\gamma(t) \sum_{i=1}^{r} h_{in}^2 Y_i^2 \leqslant 0$$

The dynamics of the closed-loop system outside the boundary layer can thus be written in terms of s_Δ as:

$$\dot{s}_\Delta + (b\hat{b}^{-1})\gamma' s_\Delta + b\hat{b}^{-1} \int_0^t \left(\gamma \sum_{i=1}^{r} h_{in}^2 Y_{id}^2\right) s_\Delta \, d\tau$$
$$= \sum_{i=1}^{r} \left(\int_0^t (b\hat{b}^{-1}\hat{h}_i - a_{si}) \, \dot{Y}_{id} \, d\tau + (b\hat{b}^{-1}\hat{a}_{fi} - a_{fi}) Y_{id} \right)$$
$$- (b\hat{b}^{-1})\beta^{-1}\lambda\Phi \operatorname{sgn}(s) + O(\tilde{\mathbf{X}}) \tag{58}$$

where $Y_{id} = Y_i(\mathbf{X}_d)$ is the value of Y_i along the desired trajectory. This suggests updating the adaptation rate $\gamma(t)$ according to

$$\dot{\gamma} + \lambda\gamma = \lambda^3 \left(\beta \sum_{i=1}^{r} h_{in}^2 Y_{id}^2 + \eta^2 \right)^{-1} \tag{59}$$

and then choosing γ' of (52) according to

$$\gamma' := \max\,(2\lambda,\ \dot{\gamma}/2\gamma) \tag{60}$$

so that (58) mimics a second-order low-pass filter structure, whose input reflects current dynamic uncertainty along the desired trajectory, and whose output is the integral of s_Δ. The small term η^2 is introduced in (59) in order to keep γ bounded.

Although clearly heuristic, expressions (59), (60) were found to be adequate in extensive simulations. Of course, one may want in practice to be more conservative and reduce the right-hand side of Update Law (59) by a constant factor, while modifying γ' of (60) accordingly. In the case where adaptation on control gain b is introduced, according to (57), the problem is complicated by the fact that underestimating b may excite unmodelled dynamics *inside* the boundary layer, since the gain margin $\beta = \beta_f$ used in the balance conditions (21), (22) does not then adequately reflect uncertainty in the control gain. This effect can often be avoided in practice by using a large initial gain estimate $\hat{b}_s(0)$ and a small adaptation weight b_n, as illustrated in the next section.

4. Simulation results

In this section, we illustrate the above methodology on a simple non-linear time-varying system. Simulation results confirm that the adaptive sliding controller achieves good tracking after an initial adaptation period, and is robust with respect to high-frequency unmodelled dynamics.

Consider the system

$$\ddot{x} + (a_{1s} + a_{1f})\dot{x} + a_2 \sin(x) = (b_s b_f) u_n + d(t) \tag{61}$$

which can be thought of as an inverted pendulum on the presence of viscous friction and disturbances. The mass properties of the pendulum and the amount of viscous friction are unknown. The actual parameter values are:

$$b_s = 1; \quad a_2 = -100; \quad a_{1s} = 2$$

while the actual disturbance terms are

$$b_f = 1 + \tfrac{1}{3}\sin(2t); \quad d(t) = 2\sin(2t); \quad a_{1f} = \sin(2t)$$

Further, high-frequency 'unmodelled' dynamics are included in the simulations in the sense that u_n of (61) is actually obtained from control law u through a poorly damped second-order filter:

$$\ddot{u}_n + (2\xi\omega)\dot{u}_n + \omega^2 u_n = \omega^2 u$$

with $\xi = 0{\cdot}3$ and $\omega = 100$ rad/s.

The control input based on the plant model is given by:

$$u = \hat{b}_f^{-1}[\hat{h}_1 \dot{x} + \hat{h}_2 \sin(x)] - (\hat{b}_s \hat{b}_f)^{-1}[u^* + \bar{k}(\mathbf{X})\,\mathrm{sat}(s/\Phi) + \beta\gamma' s_\Delta]$$

where s is obtained from (43), namely

$$s = \dot{\tilde{x}} + 2\lambda\tilde{x} + \lambda^2 \int_0^t \tilde{x}(\tau)\, d\tau$$

and accordingly

$$u^* := -\ddot{x}_d + 2\lambda\dot{\tilde{x}} + \lambda^2\tilde{x}$$

Expression (60) defines γ', while $\Phi(t)$ and $\bar{k}(\mathbf{X})$ are obtained from (21)–(24), with

$$k(\mathbf{X}) = \beta(|\hat{b}_s(\hat{h}_1\dot{x} + \hat{h}_2 \sin(x)) - u^*|(1-\beta^{-1}) + \Delta a_1|\dot{x}| + D + \eta)$$

Assuming that initially we know only that for all $t \geqslant 0$

$$|d(t)| \leqslant 2; \quad |a_{1f}(t)| \leqslant 1$$

$$\tfrac{2}{3} \leqslant b_f(t) \leqslant \tfrac{4}{3}$$

we let

$$\Delta a_1 = 1; \quad D = 2; \quad \eta = 0{\cdot}5$$

and, similarly to Example 4 of § 2:

$$\hat{b}_f := 2\sqrt{2}/3; \quad \beta = \sqrt{2}$$

Further, the desired control bandwidth λ is set to 15 rad/s, given the frequency range of unmodelled dynamics.

Parameter estimates $\hat{h}_j$, both initialized at zero, are updated according to Correlation Integrals (56), namely

$$\dot{\hat{h}}_1 = -\gamma h_{1n}\dot{x}s_\Delta \tag{62}$$

$$\dot{\hat{h}}_2 = -\gamma h_{2n} \sin(x)s_\Delta \tag{63}$$

However, adaptation in friction term $\hat{h}_1$ is stopped if the estimate becomes negative, and then resumed as soon as the right-hand side of (62) becomes positive. Further, we assume that $b_{sM} = 10$ is a known upper bound of b_s, and initialize $\hat{b}_s$ to this value. Gain estimate $\hat{b}_s$ is then updated according to (57) namely

$$\dot{\hat{b}}_s^{-1} = b_n^2\gamma(t)[u^* + (\Delta a_1|\dot{x}| + D + \eta - \Phi) \operatorname{sat}(s/\Phi) + \gamma' s_\Delta]s_\Delta \tag{64}$$

Again, adaptation on $\hat{b}_s$ is stopped if the estimate exceeds b_{sM}, and then resumed as soon as the right-hand side of (64) becomes positive. Adaptation weights h_{1n}, h_{2n} are both set to 1, while b_n is set to 0·01. Adaptation rate $\gamma(t)$ is obtained from (59):

$$\dot{\gamma} + \lambda\gamma = \lambda^3[\beta((h_{1n}\dot{x}_{id})^2 + (h_{2n} \sin(x_d)^2) + \eta^2]^{-1}$$

with initially

$$\gamma(0) = \lambda^2[\beta((h_{1n}\dot{x}_d(0))^2 + (h_{2n} \sin(x_d(0)))^2) + \eta^2]^{-1}$$

The system, initially at rest, is required to track the reference trajectory $x_d(t)$ of Fig. 5, which consists of a square wave filtered through a third-order low-pass filter of bandwidth λ. Tracking error, plotted in Fig. 6, remains within 4% of the maximum value of $x_d(t)$ after about 0·5 s. Unmodelled dynamics are not visibly excited. The corresponding control law u is plotted in Fig. 7. As predicted, the parameter estimates (Fig. 8) do not reach their true values precisely, but are adequate to ensure that s remains inside the boundary layer after the adaptation period, as shown by the s-trajectory of Fig. 9—actually, as already noticed in Fig. 6, the bulk of parameter adaptation is completed in about 0·5 s.

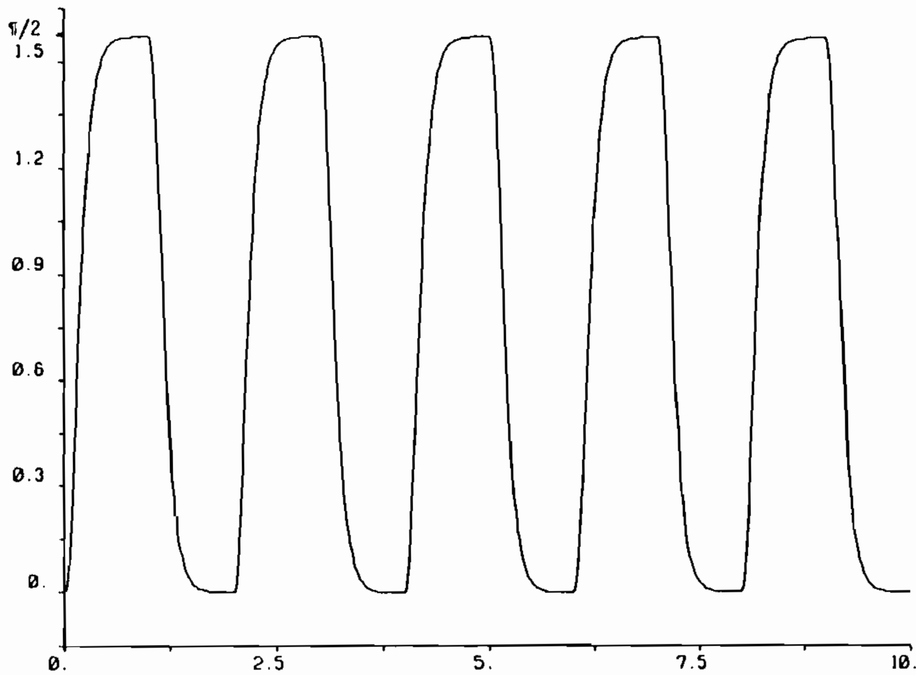


Figure 5. Desired trajectory x_d.

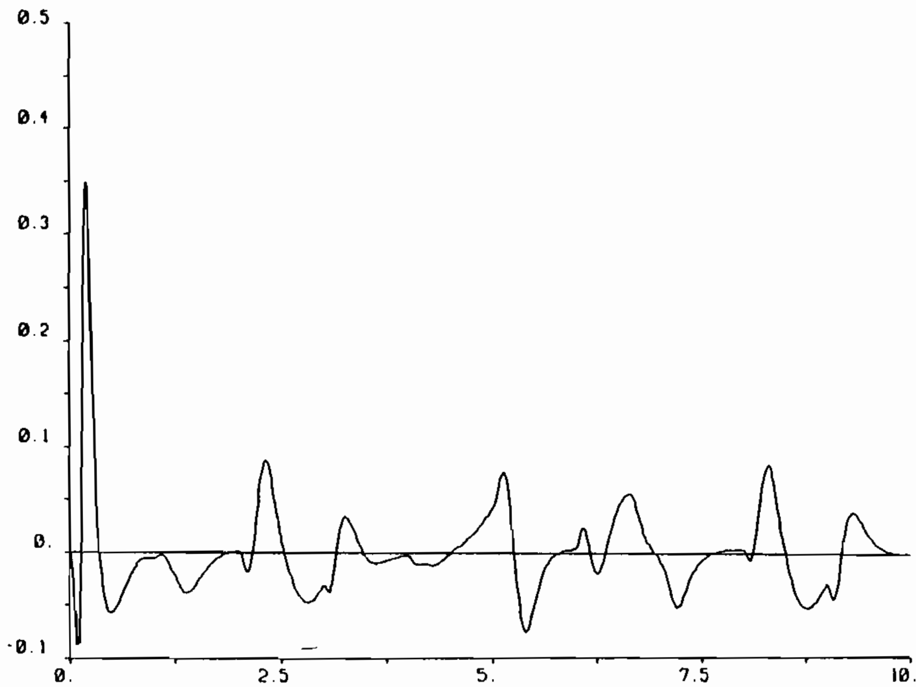


Figure 6. Tracking error $\tilde{x}$.

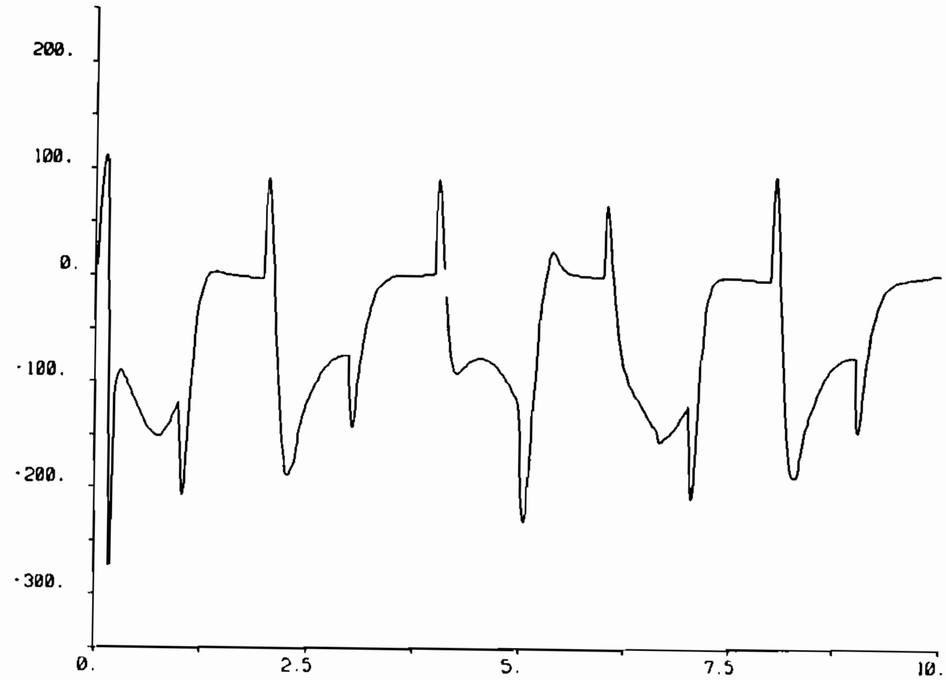


Figure 7. Control law u.

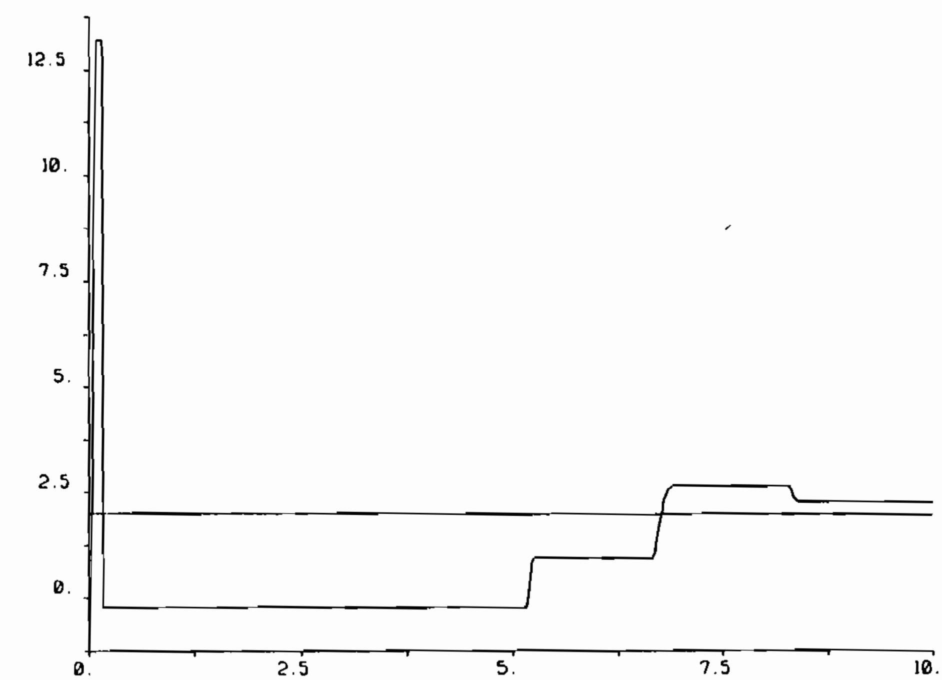
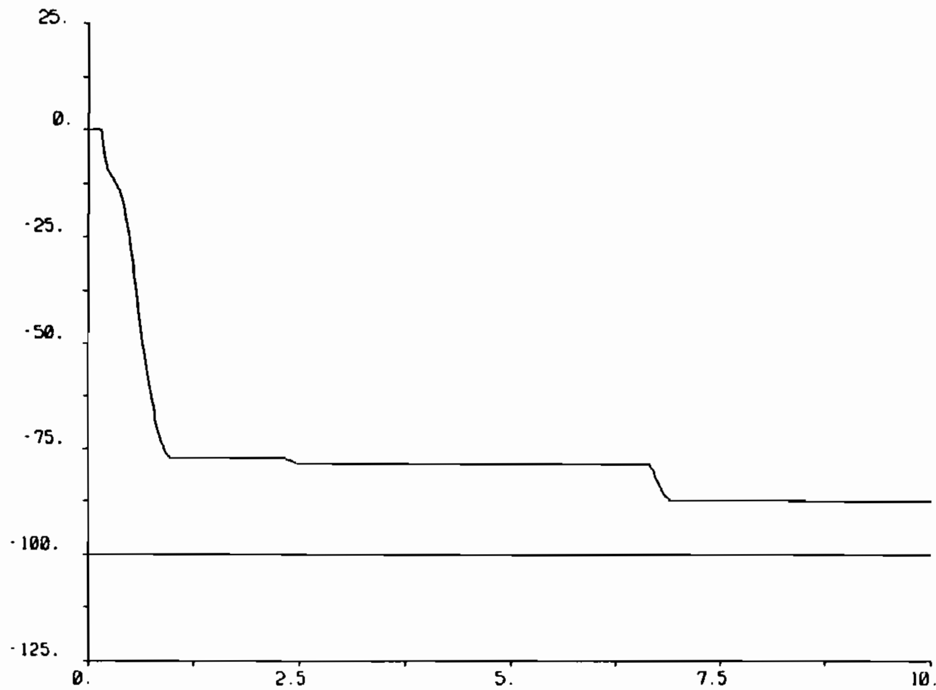
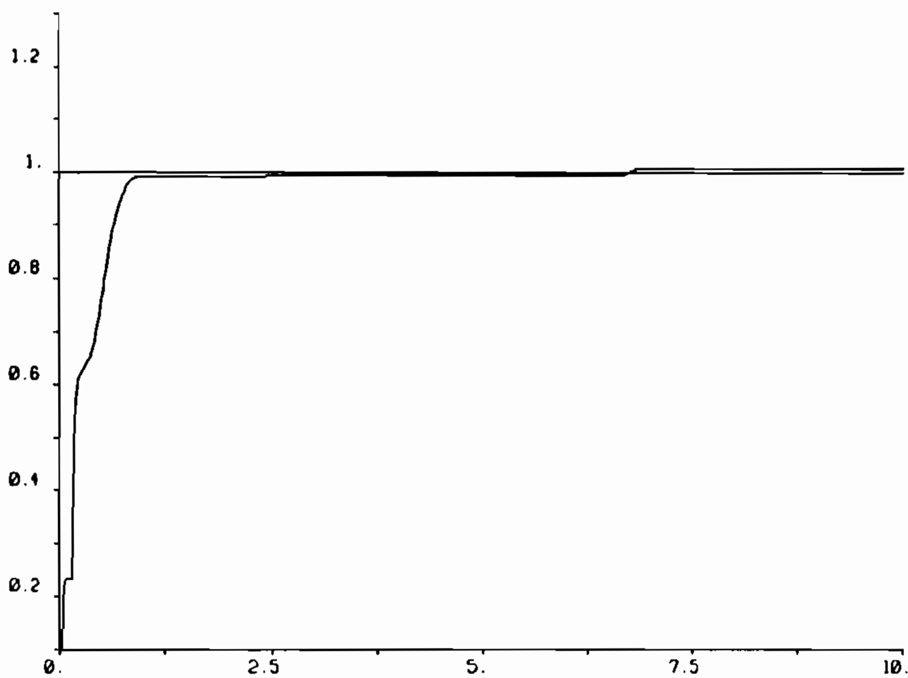


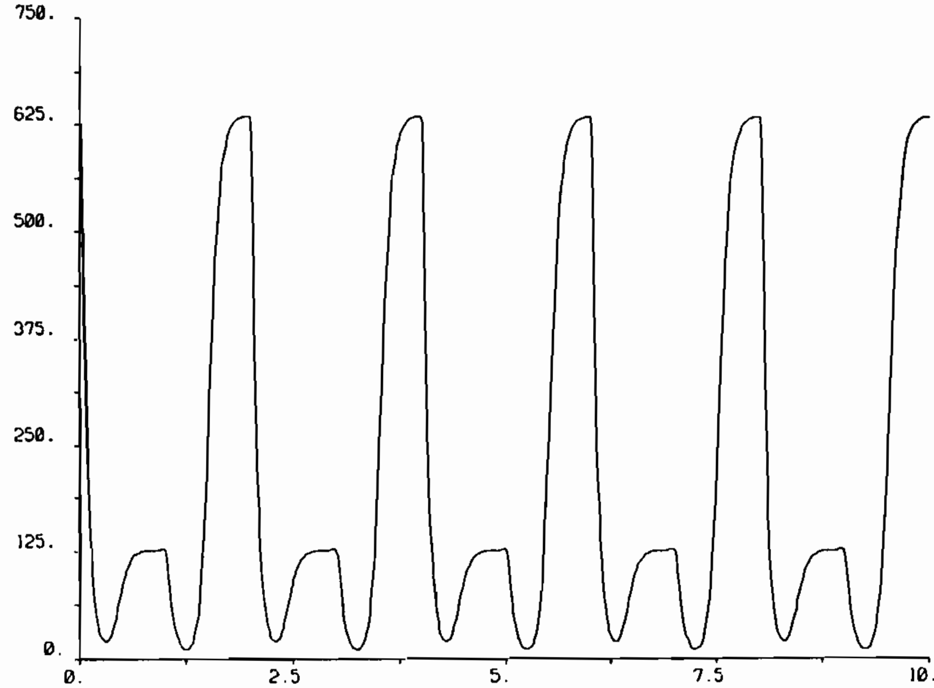
Figure 8(a).



8(*b*)



8(*c*)



(*d*)

Figure 8. Parameter estimates: (*a*) $\hat{h}_1(t)$ and (*b*) $\hat{h}_2(t)$; (c) control gain estimate $\hat{b}_s^{-1}(t)$; and (*d*) adaptation rate $\gamma(t)$.

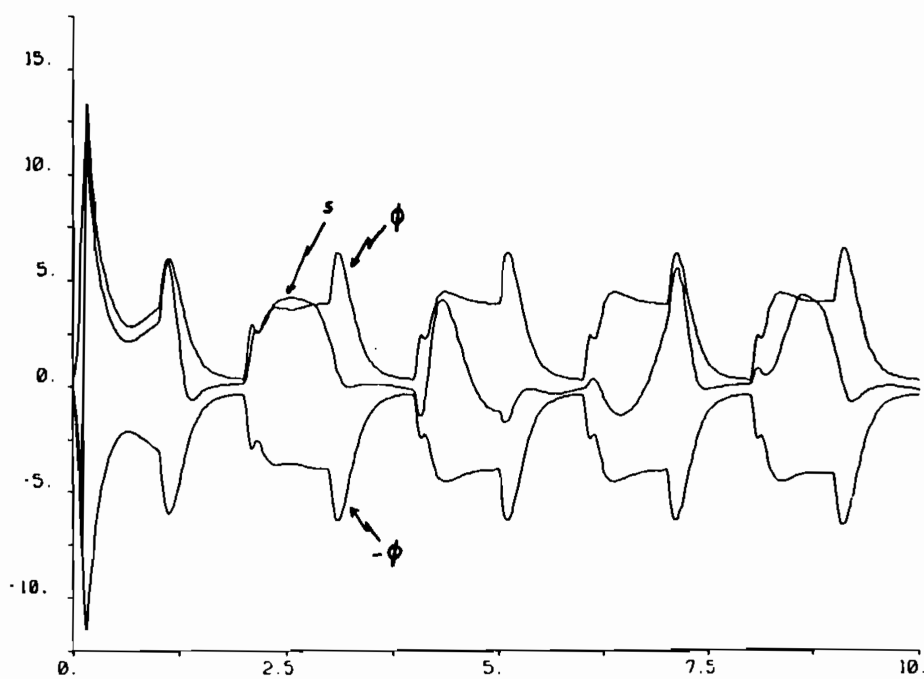


Figure 9. *S*-trajectory.

5. Concluding remarks

As can be seen from the development of § 3 and the above simulation results, sliding control and parameter estimation are tightly coupled in the sense that the adaptation scheme fully exploits the closed-loop dynamic structure created by the sliding controller and, in particular, the boundary layer concept. This provides significant advantages over standard adaptive schemes, both in terms of tracking performance in face of bounded disturbances, and in avoiding long-term parameter drift. From a practical point of view, one of the most promising aspects of the scheme is that it can be added *incrementally* to an existing sliding controller, as mentioned in § 3.2. In other words, the designer can progressively refine controller tracking performance by identifying the most significant parameter errors and sequentially extending adaptation to them. The performance improvement to be expected at each step is predicted explicitly by the balance conditions.

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