Review of Mathematics Foundation - Part 2

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CSCI 566 Deep Learning and Its Applications

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Outline

- **Probability and Statistics**: basic concepts
- **Convex Optimization**: convex, concave, basic algorithms

Probability and Statistics

Probability

Sample Space: set of all possible outcomes or realizations.
Example: Toss a coin twice; the sample space is
$\Omega = \{HH, HT, TH, TT\}$.

Event: A subset of sample space
Example: the event that at least one toss is a head is
$A = \{HH, HT, TH\}$.

Probability: We assign a real number P(A) to each event A, called the probability of A.

Probability Axioms: The probability P must satisfy three axioms:

1. $P(A) \geq 0$ for every A;
2. $P(\Omega) = 1$;
3. If $A_1, A_2, \ldots$ are disjoints, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Random Variable

Definition: A random variable is a measurable function that maps a probability space into a measurable space, i.e. $X : \Omega \to R$, that assigns a real number $X(\omega)$ to each outcome ω.

Example: if $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ and our outcomes are samples (x, y) from the unit disk, then these are some examples of random variables: $X(\omega) = x$, $Y(\omega) = y$, $Z(\omega) = x + y$.

Data and Statistics The data are specific realizations of random variables; A statistics is just any function of the data or random variables.

Distribution Function

Definition: Suppose X is a random variable, x is a specific value of it, *Cumulative distribution function (CDF)* is the function $F : R \rightarrow [0, 1]$, where $F(x) = P(X \leq x)$.

If X is discrete $\Rightarrow$ *probability mass function*: $f(x) = P(X = x)$.
If X is continuous $\Rightarrow$ *probability density function* for X if there exists a function f such that $f(x) \geq 0$ for all x, $\int_{-\infty}^{\infty} f(x)dx = 1$ and for every $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

If $F(x)$ is differentiable everywhere, $f(x) = F'(x)$.

Expectation

Expected Values

- Discrete random variable X, $E[g(X)] = \sum_{x \in \mathcal{X}} g(x)f(x)$;
- Continuous random variable X, $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)$

Mean and Variance $\mu = E[X]$ is the mean; $var[X] = E[(X-\mu)^2]$ is the variance.
We also have $var[X] = E[X^2] - \mu^2$.

Common Distributions

Discrete variable	Probability function	Mean	Variance
Uniform $X \sim U[1, \dots, N]$	$1/N$	$\frac{N+1}{2}$	
Binomial $X \sim Bin(n, p)$	$\binom{x}{n} p^x (1-p)^{(n-x)}$	np	
Geometric $X \sim Geom(p)$	$(1-p)^{x-1} p$	$1/p$	
Poisson $X \sim Poisson(\lambda)$	$\frac{e^{-\lambda} \lambda^x}{x!}$	λ	
Continuous variable	Probability density function	Mean	Variance
Uniform $X \sim U(a, b)$	1/ (b-a)	(a + b)/2	
Gaussian $X \sim N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$	μ	
Gamma $X \sim \Gamma(\alpha, \beta)$ $(x \geq 0)$	$\frac{1}{\Gamma(\alpha)\beta^a} x^{a-1} e^{-x/\beta}$	$\frac{\alpha}{\beta}$	
Exponential $X \sim exponen(\beta)$	$\frac{1}{\beta} e^{-\frac{x}{\beta}}$	β	

Multivariate Distributions

Definition:

$$F_{X,Y}(x,y) := P(X \leq x, Y \leq y),$$

and

$$f_{X,Y}(x,y) := \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y},$$

Marginal Distribution of X (Discrete case):

$$f_X(x) = P(X = x) = \sum_y P(X = x, Y = y) = \sum_y f_{X,Y}(x,y)$$

or $f_X(x) = \int_y f_{X,Y}(x,y)dy$ for continuous variable.
Conditional probability of X given $Y = y$ is

$$f_{X|Y}(x|y) = P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Bayes Rule

Law of total Probability: X takes values $x_1, \ldots, x_n$ and y is a value of Y, we have

$$f_Y(y) = \sum_j f_{Y|X}(y|x_j) f_X(x_j)$$

Bayes Rule:
(Simple Form)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

(Discrete Random Variables)

$$f_{X|Y}(x_i|y) = \frac{f_{Y|X}(y|x_i) f_X(x_i)}{\sum_j f_{Y|X}(y|x_j) f_X(x_j)}$$

(Continuous Random Variables)

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{\int_x f_{Y|X}(y|x) f_X(x) dx}$$

Independence

Independent Variables X and Y are *independent* if and only if:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

or $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all values x and y.

IID variables: *Independent and identically distributed* (IID) random variables are drawn from the same distribution and are all mutually independent.

If $X_1, \ldots, X_n$ are independent, we have

$$E[\prod_{i=1}^{n} X_i] = \prod_{i=1}^{n} E[X_i], \quad var[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i^2 var[X_i]$$

Linearity of Expectation: Even if $X_1, \ldots, X_n$ are not independent,

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i].$$

Correlation

Covariance

$$cov(X, Y) = E[(X - \mu_x)(Y - \mu_y)],$$

Correlation coefficients

$$corr(X, Y) = Cov(X, Y)/\sigma_x \sigma_y$$

- Independence $\Rightarrow$ Uncorrelated ($corr(X, Y) = 0$).

However, the reverse is generally not true.
The important special case: multi-variate Gaussian distribution.

Exponential family

Definition A family of pdf or pmfs is called an exponential family if

$$f(x|\theta) = h(x)c(\theta)\exp(\sum_{i=1}^{k} w_i(\theta)t_i(x))$$

Natural parameterization Form: For $k = 1$, we have

$$f(x|\eta) = h(x)\exp(\eta t(x) - A(\eta)),$$

where $A(\eta) = log \int h(x)\exp(\eta t(x))dx$ and:

- $t(x)$ is a *sufficient statistics* of the distribution,
- η is called the *natural parameter*,
- $A(\eta)$ is a *normalization factor*, or *log-partition function*.

Properties:

$$E[t(x)] = A'(\eta), \;\; Var[t(x)] = A''(\eta).$$

Examples: Gaussian, Exponential, Poisson, Bionomial distributions. Note that uniform distribution is NOT.

Statistics

Suppose $X_1, \ldots, X_n$ are random variables:

Sample Mean:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

Sample Variance:

$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - \bar{X})^2.$$

If X_i are iid:

$$\begin{aligned} E[\bar{X}] &= E[X_i] = \mu, \\ Var(\bar{X}) &= \sigma^2/N, \\ E[S^2] &= \sigma^2 \end{aligned}$$

Point Estimation

Definition The *point estimator* $\hat{\theta}_N$ is a function of samples $X_1, \ldots, X_N$ that approximates a parameter θ of the distribution of X_i.

Sample Bias: The bias of an estimator is

$$bias(\hat{\theta}_N) = E_\theta[\hat{\theta}_N] - \theta$$

An estimator is *unbiased estimator* if $E_\theta[\hat{\theta}_N] = \theta$

Standard error The standard deviation (i.e. the square-root of variance) of $\hat{\theta}_N$ is called the *standard error*

$$se(\hat{\theta}_N) = \sqrt{Var(\hat{\theta}_N)}.$$

Optimization

Definition: Optimization refers to choosing the best element from some set of available alternatives. A general form is as follows:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & h_i(x) = 0, i = 1, \ldots, p. \end{aligned} \tag{1}$$

Difficulties:

1. Local or global optimimum?
2. Difficulty to find a feasible point,
3. Stopping criteria,
4. Poor convergence rate,
5. numerical issues

Convex Optimization

Convex Functions: if for any two points x_1 and x_2 in its domain X and any $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

A function f is said to be *concave* if $-f$ is convex.
Convex Set a set S is convex if and only if for any $x_1, x_2 \in S$, $tx_1 + (1-t)x_2 \in S$ for any $t \in [0, 1]$,
Convex Optimization is minimization (maximization) of a convex (concave) function over a convex set.
Examples: Linear Programming (LP), Quadratic Programming (QP), and Semi-Definite Programming (SDP).
Popular convex optimization algorithms:

- Gradient descent
- Conjugate gradient
- Newton's method
- Quasi-Newton method
- Subgradient method