PURDUE UNIVERSITY · CS 52000
COMPUTATIONAL OPTIMIZATION

WEEKLY REPORT 1
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Overview

For this week, our main focus is on understanding the Primal-Dual Interior-Point Method and start setting up the components and steps of the implementation of this method. We conceived the outline of the project.

Outline

- Abstract of the Project
- Introduction to Primal-Dual Interior-Point Method
  - Topic
  - Reason why we do research
- Interior-point method LP Solver Explanation
- Test and Result of Solver
- Discussion
- Reference with APA style

Linear Programming

The project focusing on solving Linear Programming problems, which has the standard form (primal problem) of:

$$\min c^T x \text{ subject to } Ax = b, x \geq 0$$

with the dual problem of:

$$\min b^T \lambda \text{ subject to } A^T \lambda + s = c, s \geq 0$$

This 2 problems have the solutions described by the following KKT conditions:

$$\begin{aligned} A^T\lambda + s &= c \\ Ax &= b \\ x_i s_i &= 0, i = 1, 2, 3, \ldots, n \\ (x, s) &\geq 0 \end{aligned}$$

Primal-Dual Basic Ideas

Primal-Dual Interior-Point Method derives first restates the KKT optimality conditions as:

$$\begin{aligned} F(x, \lambda, s) = \begin{bmatrix} A^T + s - c \\ Ax - b \\ XSe \end{bmatrix} &= 0, \\ (x, s) &\geq 0, \\ X &= \text{diag}(x_1, x_2, \ldots, x_n) \\ S &= \text{diag}(s_1, s_2, \ldots, s_n) \end{aligned}$$

We focused on reading chapter 11 of the book this week. We have a clearer understanding of the implementation of the Primal-dual Interior-point method.

Code Outline

Base on the the books and the lecture notes, we have conducted a basic outline for our project code so that we can follow it and implement it in the later weeks, moreover, revise part of the structure base on the difficulties we may encounter. This section describes

Pre-processing

According to Chapter 11 in *Primal-Dual Interior-Point Method*, we notice that the input problem may have special cases that the code need to handle before entering the solver.
The matrix A from the original input problem may contains unknown amount of zero-rows and zero-columns. These zero-rows and zero-columns can be remove straight-forward by the following function:

```
function removeZeroRows(A::Matrix)
    non_zero_rows = find(sum(abs.(A), dims=2) .> 0)
    return A[non_zero_rows, :]
end

function removeZeroCols(A::Matrix)
    non_zero_cols = find(sum(abs.(A), dims=1) .> 0)
    return A[:, non_zero_cols]
end
```

Furthermore, the size of matrix A can be reduced by 1 when merging two columns/rows of A are scalar multipliers of one another. When we tried to implement the code here, we met some difficulties where we are not sure that if we merged these columns/rows from A, should we modify b accordingly.

Row singleton operation is also useful for size reduction of A. When the i^{th} row of A contains a single non-zero element A_{ij}, we can set $x_j = b_i/A_{ij}$ and eliminate row i and row j from the problem.

```
function solve_with_single_nonzero(A::Matrix, b::Vector)
    m, n = size(A)
    x = Vector{Union{Float64, Missing}}(missing, n)

    for i in 1:m
        nonzero_indices = findall(!iszero, A[i, :])

        if length(nonzero_indices) == 1
            j = nonzero_indices[1]
            x[j] = b[i] / A[i, j]
        end
    end

    return x
end
```

After the pre-solving part, our code need to convert the problem to the standard form. The initial problem looks like:

$$\min\ c^T x \text{ subject to } Ax = b, x > low, x < high$$

and we need to convert is to the standard form:

$$\min\ c'^T x' \text{ subject to } A'x' = b', x' \geq 0$$

The detailed code need to be decided in the upcoming week.

Starting Point and Termination

Based on chapter 5 and chapter 6, we know that a good starting point should satisfy two conditions.

- The point should be well centered, so that the pairwise products $x_i^0 s_i^0$ are similar for all $i = 1, 2, ..., n$.
- The point should not be too infeasible. (i.e. the ratio $||(r_b^0, r_c^0)||/\mu_0$ of infeasibility to duality measure should not be too large)

The starting point is defined as

$$(x^0, \lambda^0, s^0) = (\tilde{x} + \tilde{\delta}_x e, \tilde{\lambda}, \tilde{s} + \tilde{\delta}_s e)$$

where the scalars $\tilde{\delta}_x$ and $\tilde{\delta}_s$ are calculated to satisfy the two conditions.

Primal-dual Interior-point method never find and exact solution of the Linear program, so we need a condition for termination.

We will have a small positive number, $\epsilon = 10^{-8}$, which we will consider it as tolerance. When we satisfy these three conditions, the solver will be terminated.

$$\frac{||r_b||}{1 + ||b||} = \frac{||Ax - b||}{1 + ||b||} \leq \epsilon$$

$$\frac{||r_c||}{1 + ||c||} = \frac{||A^T\lambda + s - c||}{1 + ||c||} \leq \epsilon$$

$$\frac{|c^T x - b^T y|}{1 + |c^T x|} \leq \epsilon$$