Strassen's Matrix Multiplication Algo

- If $A = (a_{ij})$ and $B = (b_{ij})$ $n \times n$ Square matrices. then product $C = A \cdot B$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \quad \text{for } i,j = 1,2,3 \ldots n$$

SQUARE-MATRIX-MULTIPLY(A, B)

1. $n = A.rows$
2. let C be a new $n \times n$ matrix
3. for $i = 1$ to n
4.   for $j = 1$ to n
5. $C_{ij} = 0$
6.   for $k = 1$ to n
7. $C_{ij} = C_{ij} + a_{ik} \cdot b_{kj}$
8. return C

- Above procedure take $\Theta(n^3)$ time.

Divide and Conquer Algo

- For Simplicity, n is an exact power of 2.

- In each divide step, we divide $n \times n$ matrixes into 4 $n/2 \times n/2$ matrices

- Let we partition each of A, B and C into 4 $n/2 \times n/2$ matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

So equation $C = A \cdot B$ as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

(AJAY RAWAT)

$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$

$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$

$C_{21} = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$

$C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22}$

SQUARE - MATRIX - MULTIPLY - RECURSIVE (A, B)

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1.  n = A.rows
2.  let C be a new n x n matrix
3.  if n == 1
4.        c11 = a11 . b11
5.  else partition A, B and C
6.        C11 = SMMR(A11, B11) + SMMR(A12 . B21)
7.        C12 = SMMR(A11, B12) + SMMR(A12, B22)
8.        C21 = SMMR(A21, B11) + SMMR(A22, B21)
9.        C22 = SMMR(A21 . B12) + SMMR(A22 . B22)
10. return C.
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- $T(1) = \Theta(1)$ (line 4 only one scalar multiplication)

- $8T(n/2)$ (line 6-9, 8 recursive calls, each recursive call multiplies two $n/2 \times n/2$ matrices.)

- $\Theta(n^2)$ for ~~each~~ four matrix addition (each ~~entri~~ matrices contain $n^2/4$ entries)

So $T(n) = \Theta(1) + 8T(n/2) + \Theta(n^2)$

$= 8T(n/2) + \Theta(n^2)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 8T(n/2) + \Theta(n^2) & \text{if } n > 1 \end{cases}$$

with the help of master Theorem $T(n) = \Theta(n^3)$

- So Simple divide and Conquer approach is no faster than Straight forward

Strassen's method

- Instead of Performing eight recursive multiplication of $n/2 * n/2$ matrices it perform only Seven.

- Cost of eliminating one matrix mul will be Several new additions of $n/2 * n/2$ matrices.

Steps

1. Divide the input matrices A and B and output matrix C into $n/2 \times n/2$ Submatrices (takes $\Theta(1)$ time)

2. Create 10 matrices $S_1, S_2 \ldots S_{10}$ of $n/2 \times n/2$. ($\Theta(n^2)$ time)

3. Using Submatrices recursively compute Seven matrix products $P_1, P_2 \ldots P_7$. Each P_i matrix is $n/2 \times n/2$.

4. Compute the desired matrices $C_{11}, C_{12}, C_{21}, C_{22}$ of result matrix C. ($\Theta(n^2)$ time)

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 7T(n/2) + \Theta(n^2) & \text{if } n>1 \end{cases}$$

Using master method $T(n) = \Theta(n^{\log 7}) = \Theta(n^{2.81})$

$\log 7$ value lies in between $2.80 - 2.81$.

$S_1 = B_{12} - B_{22}$
$S_2 = A_{11} + A_{12}$
$S_3 = A_{21} + A_{22}$
$S_4 = B_{21} - B_{11}$
$S_5 = A_{11} + A_{22}$
$S_6 = B_{11} + B_{22}$
$S_7 = A_{12} - A_{22}$
$S_8 = B_{21} + B_{22}$
$S_9 = A_{11} - A_{21}$
$S_{10} = B_{11} + B_{12}$

Since we add or Sub $n/2 \times n/2$ matrices 10 times it takes $\Theta(n^2)$ times

$P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22}$
$P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22}$
$P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11}$
$P_4 = A_{22} \cdot S_4 = A_{22} \cdot B_{21} - A_{22} \cdot B_{11}$
$P_5 = S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}$
$P_6 = S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22}$
$P_7 = S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}$

$C_{11} = P_5 + P_4 - P_2 + P_6$ Putting values

$= A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}$
$- A_{22} \cdot B_{11} + A_{22} \cdot B_{21}$
$- A_{11} \cdot B_{22} - A_{12} \cdot B_{22}$
$- A_{22} \cdot B_{22} - A_{22} \cdot B_{21} + A_{12} \cdot B_{22} + A_{12} \cdot B_{21}$

$A_{11} \cdot B_{11} + A_{12} \cdot B_{21}$

Similarly $C_{12} = P_1 + P_2 = A_{11} \cdot B_{12} + A_{12} \cdot B_{22}$
$C_{21} = P_3 + P_4 = A_{21} \cdot B_{11} + A_{22} \cdot B_{21}$
$C_{22} = P_5 + P_1 - P_3 - P_7 = A_{22} \cdot B_{22} + A_{21} \cdot B_{12}$