| Bachelors with Mathematics as Major<br>$1^{st}$ Semester<br>MMT122J: Mathematics/Applied Mathematics: Calculus - I<br>Credits: 4 THEORY + 2 TUTORIAL     Theory: 60 Hours & Tutorial: 30 Hours |
|---|
| **Course Objectives:** (i) To study and understand the notions of differential calculus and to imbibe the acquaintance for using the techniques in other sciences and engineering. (ii) To prepare the students for taking up advanced courses of mathematics.<br>**Course Outcome:** (i) After the successful completion of the course, students shall be able to apply differential operators to understand the dynamics of various real life situations. (ii) The students shall be able to use differential calculus in optimization problems. |
| **Theory: 4 Credits** |
| **Unit –I**<br>Limits and infinitesimals, Continuity ($\epsilon - \delta$ definition), types of discontinuities of functions, Differentiability of functions, Successive differentiation and Leibnitz theorem, Partial differentiation, Total differentiation, Homogenous functions and Euler's theorem. |
| **Unit –II**<br>Indeterminate forms, Tangents and normals (polar coordinates only), Angle between radius vector and tangent, Perpendicular from pole to tangent, angle of intersection of two curves, polar tangent, polar normal, polar sub-tangent, polar sub-normal. |
| **Unit –III**<br>Curvature and radius of curvature, Pedal Equations, lengths of arcs, Asymptotes, Singular points, Maxima and minima of functions. Bounded functions, Properties of continuous functions on closed intervals, Intermediate value theorem, Darboux theorem. |
| **Unit –IV**<br>Rolle's theorem and mean value theorems (with proofs) and their geometrical interpretation, Taylor's theorem with Lagranges and Cauchy's form of remainder, Taylor's series, Maclaurin's series of $\sin x$, $\cos x$, $e^x$, $\log x$, $(1+x)^m$.Envelope of a family of curves involving one and two parameters. |
| **Tutorial: 2 Credits** |
| **Unit –V**<br>Examples of discontinuous functions, nth derivative of product of two functions, involutes and evolutes, bounds of function (Supremum and infimum). |
| **Unit –VI**<br>Tracing of cartesian equations of the form $y = f(x)$, $y^2 = f(x)$, tracing of the parametric equations. |
| **Recommended Books:**<br>1. Shanti Narayan and P.K. Mittal, Differential Calculus, S. Chand, 2016.<br>2. S. D. Chopra, M. L. Kochar and A. Aziz, Differential Calculus, Kapoor Sons.<br>3. Schaums outline of Theory and problems of Differential and Integral Calculus, 1964.<br>1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002.<br>2. T.M. Apostal, Calculus Vol. I, John Wiley & Sons Inc, 1975.<br>3. S. Balachandra Rao and C. K. Shantha, Differential Calculus, New Age Publication, 1992.<br>4. S. Lang, A First Course in Calculus, Springer-Verlag, 1998.<br>5. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 2008.<br>6. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007.<br>7. Suggestive digital platforms web links: NPTEL/ SWAYAM/ MOOCS. |

# ————-CALCULUS I - (MMT122J)—————

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# Chapter 1

# Limits, Continuity and Differentiability

## 1.1 Limits and Infinitesimals

### Concept Overview

The **limit** of a function $f(x)$ as $x$ approaches a point $a$ is the value $L$ that $f(x)$ gets arbitrarily close to as $x$ approaches $a$. Formally:

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

**Infinitesimals** are quantities smaller than any positive real number but greater than zero. In limits, $dx$ represents an infinitesimal change in $x$, and the expression $f(a + dx) - L$ becomes infinitesimal as $dx \to 0$.

**Example 1.1.1.** Evaluate $\lim_{x\to 3}(2x + 1)$.

*Solution.*

$$\begin{aligned}\lim_{x\to 3}(2x+1) &= 2(3) + 1 \quad \text{(Direct substitution)}\\ &= 7.\end{aligned}$$

To verify using the $\epsilon$-$\delta$ definition: Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x - 3| < \delta$, then:

$$|(2x + 1) - 7| = |2x - 6| = 2|x - 3| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Thus, $\lim_{x\to 3}(2x + 1) = 7$. ■

**Example 1.1.2.** Prove that $\lim_{x\to 5}(3x - 4) = 11$ using the $\epsilon$-$\delta$ definition.

*Solution.* Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $|(3x - 4) - 11| < \epsilon$.
Simplify the expression:

$$|3x - 15| = |3(x - 5)| = 3|x - 5|$$

We require:

$$3|x - 5| < \epsilon \implies |x - 5| < \frac{\epsilon}{3}$$

Choose $\delta = \frac{\epsilon}{3}$.
Verification: If $0 < |x - 5| < \delta$, then:

$$|(3x - 4) - 11| = 3|x - 5| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

Thus, $\lim_{x\to 5}(3x - 4) = 11$. ■

**Example 1.1.3.** Prove that $\lim_{x\to 3}(x^2 - 2x) = 3$ using the $\epsilon$-$\delta$ definition.

*Solution.* Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|(x^2 - 2x) - 3| < \epsilon$.
Simplify the expression:

$$|x^2 - 2x - 3| = |(x - 3)(x + 1)| = |x - 3||x + 1|$$

Assume $\delta \le 1$. Then $|x - 3| < 1$ implies:

$$2 < x < 4 \quad \text{so} \quad 3 < x + 1 < 5$$

Thus $|x + 1| < 5$. Now:

$$|x - 3||x + 1| < |x - 3| \cdot 5$$

We require $5|x - 3| < \epsilon$, which gives $|x - 3| < \frac{\epsilon}{5}$.
Choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$.
Verification: If $0 < |x - 3| < \delta$, then:

$$|x^2 - 2x - 3| = |x - 3||x + 1| < \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon$$

Thus, $\lim_{x\to 3}(x^2 - 2x) = 3$. ■

## 1.2 Continuity: $\epsilon$-$\delta$ Definition

### Concept Overview

A function $f(x)$ is **continuous** at $x = a$ if:

1. $f(a)$ is defined,
2. $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$.

The $\epsilon$-$\delta$ definition formalizes this:

$$f \text{ is continuous at } a \iff \forall\epsilon > 0, \exists\delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

This ensures no jumps, breaks, or oscillations at $a$.

**Example 1.2.1.** Prove that $f(x) = 4x - 1$ is continuous at $x = 3$ using the $\epsilon$-$\delta$ definition.

*Solution.* We must show: $\forall\epsilon > 0, \exists\delta > 0$ such that $|x - 3| < \delta \implies |(4x - 1) - 11| < \epsilon$.
Note $f(3) = 4(3) - 1 = 11$.
Simplify:

$$|(4x - 1) - 11| = |4x - 12| = 4|x - 3|$$

Choose $\delta = \epsilon/4$. If $|x - 3| < \delta$, then:

$$4|x - 3| < 4 \cdot (\epsilon/4) = \epsilon$$

Thus, $f(x) = 4x - 1$ is continuous at $x = 3$. ■

**Example 1.2.2.** Prove $f(x) = x^2$ is continuous at $x = 2$.

*Solution.* $f(2) = 4$. For any $\epsilon > 0$, choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$. If $|x - 2| < \delta$, then:

$$\begin{aligned}
|x + 2| &= |(x - 2) + 4| \\
&\le |x - 2| + 4 < 1 + 4 = 5, \\
|f(x) - f(2)| &= |x^2 - 4| \\
&= |(x - 2)(x + 2)| \\
&< \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon.
\end{aligned}$$

Thus, $f(x) = x^2$ is continuous at $x = 2$. ■

**Example 1.2.3.** Prove that $g(x) = x^2 + 2$ is continuous at $x = -1$ using the $\epsilon$-$\delta$ definition.

*Solution.* We must show: $\forall\epsilon > 0, \exists\delta > 0$ such that $|x - (-1)| < \delta \implies |(x^2 + 2) - 3| < \epsilon$.
Note $g(-1) = (-1)^2 + 2 = 3$.
Simplify:

$$|(x^2 + 2) - 3| = |x^2 - 1| = |x - 1||x + 1|$$

Set $\delta \le 1$. Then $|x + 1| < 1$ implies $-2 < x < 0$, so $|x - 1| < 3$.
Now:

$$|x - 1||x + 1| < 3|x + 1|$$

Choose $\delta = \min(1, \epsilon/3)$. If $|x + 1| < \delta$, then:

$$|g(x) - g(-1)| < 3 \cdot \delta \le 3 \cdot (\epsilon/3) = \epsilon$$

Thus, $g(x) = x^2 + 2$ is continuous at $x = -1$. ■

**Theorem 1.2.4.** *If the limit of a function $f(x)$ as $x$ approaches $a$ exists, then this limit is unique.*

*Proof.* Assume, for the sake of contradiction, that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ where $L_1 \neq L_2$.
By the definition of a limit, there exists a $\delta > 0$ such that for any $x$ satisfying $0 < |x - a| < \delta$, we must have both:

$$|f(x) - L_1| < \epsilon \quad \text{and} \quad |f(x) - L_2| < \epsilon$$

Using the triangle inequality, we can write:

$$\begin{aligned}
|L_1 - L_2| &= |(L_1 - f(x)) + (f(x) - L_2)| \\
&\le |f(x) - L_1| + |f(x) - L_2| \\
&< \epsilon + \epsilon = 2\epsilon
\end{aligned}$$

Let $\epsilon = \frac{|L_1 - L_2|}{2}$. Since $L_1 \neq L_2$, we know that $\epsilon > 0$.
Substituting our choice of $\epsilon$:

$$|L_1 - L_2| < 2\left(\frac{|L_1 - L_2|}{2}\right) = |L_1 - L_2|$$

This results in the contradiction $|L_1 - L_2| < |L_1 - L_2|$. Therefore, our initial assumption must be false, and the limit must be unique. ■

**Theorem 1.2.5** (The Algebra of Limits)**.** *Let $c$ be a real number, and let $f(x)$ and $g(x)$ be two functions such that*

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

*where $L$ and $M$ are finite real numbers. Then the following properties hold:*

1. ***Sum Rule:*** *The limit of the sum is the sum of the limits.*
$$\lim_{x \to c} [f(x) + g(x)] = L + M$$

2. ***Difference Rule:*** *The limit of the difference is the difference of the limits.*
$$\lim_{x \to c} [f(x) - g(x)] = L - M$$

3. ***Product Rule:*** *The limit of the product is the product of the limits.*
$$\lim_{x \to c} [f(x)g(x)] = LM$$

4. ***Quotient Rule:*** *The limit of the quotient is the quotient of the limits, provided the limit of the denominator is not zero.*
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{provided } M \neq 0$$

*1. Proof of the Sum Rule.* Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|(f(x) + g(x)) - (L + M)| < \epsilon$.

By the definition of the limits for $f$ and $g$, we know that for $\epsilon/2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

- if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \epsilon/2$.
- if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \epsilon/2$.

Let us choose $\delta = \min(\delta_1, \delta_2)$. Then for any $x$ satisfying $0 < |x - c| < \delta$, both of the above inequalities hold. Using the triangle inequality, we have:

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

The proof for the DIfference Rule is analogous. ■

*3. Proof of the Product Rule.* Let $\epsilon > 0$ be given. We want to show $|f(x)g(x) - LM| < \epsilon$.

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M| \end{aligned}$$

Since $\lim_{x \to c} g(x) = M$, there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|g(x) - M| < 1$, which implies $|g(x)| < |M| + 1$. Let $K = |M| + 1$. Now, for a given $\epsilon > 0$, we can find $\delta_2$ and $\delta_3$ such that:

- if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \frac{\epsilon}{2K}$.
- if $0 < |x - c| < \delta_3$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. (We use $|L| + 1$ to avoid division by zero if $L = 0$).

Choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x - c| < \delta$, then:

$$\begin{aligned} |f(x)g(x) - LM| &\leq |g(x)||f(x) - L| + |L||g(x) - M| \\ &< K \cdot \frac{\epsilon}{2K} + |L| \cdot \frac{\epsilon}{2(|L| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, we have shown that $\lim_{x \to c}[f(x)g(x)] = LM$. ■

*4. Proof of the Quotient Rule.* It suffices to first prove that $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ for $M \neq 0$, and then apply the Product Rule to $f(x) \cdot \frac{1}{g(x)}$.

Let $\epsilon > 0$ be given. We want to show $\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon$.

$$\left|\frac{M - g(x)}{Mg(x)}\right| = \frac{|g(x) - M|}{|M||g(x)|}$$

Since $M \neq 0$, $|M|/2 > 0$. There exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|g(x) - M| < |M|/2$. By the reverse triangle inequality, this implies $|g(x)| > |M| - |g(x) - M| > |M| - |M|/2 = |M|/2$. Therefore, $\frac{1}{|g(x)|} < \frac{2}{|M|}$. Now, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \frac{\epsilon|M|^2}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - c| < \delta$, then:

$$\frac{|g(x) - M|}{|M||g(x)|} < \frac{|g(x) - M|}{|M|(|M|/2)} = \frac{2|g(x) - M|}{|M|^2} < \frac{2}{|M|^2} \cdot \frac{\epsilon|M|^2}{2} = \epsilon$$

So, $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$. By the Product Rule, $\lim_{x\to c} \frac{f(x)}{g(x)} = (\lim_{x\to c} f(x)) \left(\lim_{x\to c} \frac{1}{g(x)}\right) = L \cdot \frac{1}{M} = \frac{L}{M}$. ■

**Exercise 1.2.6.** Q1. **(Limit)** Prove $\lim_{x\to 4}(5 - 2x) = -3$ using $\epsilon$-$\delta$ definition.

Q2. **(Limit)** Prove $\lim_{x\to -2}(x^2 + 3x) = -2$ using $\epsilon$-$\delta$ definition.

Q3. **(Continuity)** Prove $h(x) = \sqrt{x}$ is continuous at $x = 4$ using $\epsilon$-$\delta$ definition. (Hint: Use $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2}$).

Q4. **(Continuity)** Prove $k(x) = \dfrac{1}{x}$ is continuous at $x = \dfrac{1}{2}$ using $\epsilon$-$\delta$ definition.

## 1.3 Types of Discontinuities

Discontinuities occur where a function fails to be continuous. Common types:

1. **Removable**: $\lim_{x\to a} f(x)$ exists but $f(a)$ is undefined or $\lim_{x\to a} f(x) \neq f(a)$. "Fixed" by redefining $f(a)$.
2. **Jump or discontinuity of first kind**: Left-hand and right-hand limits exist but are unequal ($\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$).
3. **Infinite or discontinuityof second kind**: $\lim_{x\to a} f(x) = \pm\infty$ (vertical asymptote).
4. **Oscillating**: Function oscillates infinitely often near $a$ (e.g., $\sin(1/x)$ at $x = 0$).

**Example 1.3.1.** Classify the discontinuity of $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$.

*Solution.* The function is undefined at $x = 2$. Simplify for $x \neq 2$:

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

The limit exists:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x + 2) = 4.$$

Since the limit exists but $f(2)$ is undefined, the discontinuity is **removable**. ■

**Example 1.3.2.** Classify the discontinuity of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x + 3 & \text{if } x \geq 2 \end{cases}$$

at $x = 2$.

*Solution.* Calculate left-hand and right-hand limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x + 3) = 2 + 3 = 5$$

Since $4 \neq 5$, the left-hand and right-hand limits exist but are not equal.
Function value: $f(2) = 2 + 3 = 5$.
The discontinuity at $x = 2$ is a **jump discontinuity**. ■

**Example 1.3.3.** Classify the discontinuity of $g(x) = \dfrac{1}{(x-3)^2}$ at $x = 3$.

*Solution.* Examine the limit as $x$ approaches 3:

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty$$

The function is undefined at $x = 3$. As $x \to 3$, the function values increase without bound.
This is an **infinite discontinuity** (vertical asymptote at $x = 3$). ■

**Example 1.3.4.** Classify the discontinuity of $h(x) = \sin\left(\frac{1}{x}\right)$ at $x = 0$.

*Solution.* Consider the behavior as $x \to 0$:

- As $x \to 0^+$, $\frac{1}{x} \to +\infty$ and $\sin(1/x)$ oscillates between $-1$ and 1
- As $x \to 0^-$, $\frac{1}{x} \to -\infty$ and $\sin(1/x)$ oscillates between $-1$ and 1

The limit $\lim_{x\to 0} \sin(1/x)$ does not exist because oscillations become increasingly rapid.
This is an **oscillating discontinuity** at $x = 0$. ■

**Example 1.3.5.** Examine the following function for continuity at the origin.

$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

*Solution.* $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$ and $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0-} \dfrac{x}{e^{-1/x}+1} = 0$ Also, $f(0) = 0$. Thus, the function is continuous at the origin. ■

### 1.3.1 Piecewise Continuity

A function $f(x)$ is said to be piecewise continuous in an interval $I$, if the interval can be subdivided into a finite number of subintervals such that $f(x)$ is continuous in each of the subintervals and the limits of $f(x)$ as $x$ approaches the end points of each subinterval are finite. For example, the greatest integer function $f(x) = [x]$ defined on $[-1, 3]$ is piecewise continuous on $[-1, 3]$.

## 1.4 Differentiability of Functions

### Concept Overview

A function $f(x)$ is **differentiable** at a point $x = a$ if the derivative $f'(a)$ exists. Geometrically, this means the function has a unique tangent at that point. The derivative is defined as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists finitely.

### Necessary Conditions

- If $f$ is differentiable at $a$, then it must be continuous at $a$.
- The converse is not true: continuity does not imply differentiability (e.g., $|x|$ at $x = 0$).

*(1). Proof.* To prove that the function $f$ is continuous at $a$, we must show that $\lim_{x\to a} f(x) = f(a)$. This is equivalent to showing that the difference between $f(x)$ and $f(a)$ approaches zero, i.e.,

$$\lim_{x\to a} [f(x) - f(a)] = 0$$

We are given that $f$ is differentiable at $a$. By definition, this means the limit for the derivative exists and is a finite number:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

that is,

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a}\right) \cdot (x - a)$$

We can now take the limit of both sides as $x \to a$ and apply the product rule for limits:

$$\begin{aligned}
\lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \left( \frac{f(x) - f(a)}{x - a} \cdot (x - a) \right) \\
&= \left( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left( \lim_{x \to a} (x - a) \right) \\
&= f'(a) \cdot (a - a) \\
&= f'(a) \cdot 0 \\
&= 0
\end{aligned}$$

Since we have shown that $\lim_{x \to a} [f(x) - f(a)] = 0$, it follows directly that $\lim_{x \to a} f(x) = f(a)$. This is the definition of continuity at the point $a$. ■

■

### Left and Right Derivatives

The **left derivative** at $a$:

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

The **right derivative** at $a$:

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$f$ is differentiable at $a$ iff $f'_-(a) = f'_+(a)$ and both exist finitely.

**Example 1.4.1.** Show that $f(x) = x^2 + 3x$ is differentiable at $x = 2$ and find its derivative.

*Solution.* Compute the derivative using the limit definition:

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{[(2+h)^2 + 3(2+h)] - [2^2 + 3(2)]}{h} \\
&= \lim_{h \to 0} \frac{[4 + 4h + h^2 + 6 + 3h] - [4 + 6]}{h} \\
&= \lim_{h \to 0} \frac{7h + h^2}{h} \\
&= \lim_{h \to 0} (7 + h) = 7
\end{aligned}$$

Since the limit exists, $f$ is differentiable at $x = 2$ with $f'(2) = 7$. ■

## 1.5 Successive Differentiation

### Concept Overview

**Successive differentiation** refers to repeatedly differentiating a function. The $n$th derivative is denoted by:

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n y}{dx^n}$$

where $n$ is the order of differentiation.

### Standard Formulas

- $\dfrac{d^n}{dx^n}(x^m) = m(m-1)\cdots(m-n+1)x^{m-n} \quad \text{for} \quad n \leq m$
- $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$
- $\dfrac{d^n}{dx^n}(\ln x) = (-1)^{n-1} \dfrac{(n-1)!}{x^n}$
- $\dfrac{d^n}{dx^n}(\sin ax) = a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}(\cos ax) = a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$

**Example 1.5.1.** Find the third derivative of $g(x) = 2x^4 - 5x^3 + 3x - 7$.

*Solution.* Compute successive derivatives:

$$\begin{aligned} g'(x) &= \frac{d}{dx}(2x^4 - 5x^3 + 3x - 7) = 8x^3 - 15x^2 + 3 \\ g''(x) &= \frac{d}{dx}(8x^3 - 15x^2 + 3) = 24x^2 - 30x \\ g'''(x) &= \frac{d}{dx}(24x^2 - 30x) = \boxed{48x - 30} \end{aligned}$$

■

## 1.6 Leibnitz Theorem

**Theorem 1.6.1 (Leibnitz Theorem).** *If $u(x)$ and $v(x)$ are $n$-times differentiable functions, then:*

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

*where $u^{(k)}$ is the $k$th derivative of $u$, and $v^{(n-k)}$ is the $(n-k)$th derivative of $v$, with $u^{(0)} = u$ and $v^{(0)} = v$.*

*Proof.* We must show that the formula is true for $n = 1$.

$$\begin{aligned} (uv)^{(1)} &= \sum_{k=0}^{1} \binom{1}{k} u^{(1-k)} v^{(k)} \\ &= \binom{1}{0} u^{(1)} v^{(0)} + \binom{1}{1} u^{(0)} v^{(1)} \\ &= (1) \cdot u'v + (1) \cdot uv' \\ &= u'v + uv' \end{aligned}$$

Assume the theorem is true for some positive integer $n = m$. That is, we assume:

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}$$

We must prove that the theorem is true for $n = m + 1$. We start by differentiating the expression from our inductive hypothesis with respect to $x$:

$$\begin{aligned} (uv)^{(m+1)} &= \frac{d}{dx}\left[(uv)^{(m)}\right] \\ &= \frac{d}{dx}\left[\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}\right] \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{d}{dx}\left(u^{(m-k)} v^{(k)}\right) \\ &= \sum_{k=0}^{m} \binom{m}{k} \left[u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)}\right] \end{aligned}$$

Now, we split this into two separate sums:

$$(uv)^{(m+1)} = \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)}}_{\text{Sum A}} + \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k+1)}}_{\text{Sum B}}$$

To combine these sums, we re-index Sum B by letting $j = k + 1$. This means Sum B will go from $j = 1$ to $j = m + 1$. Replacing $j$ back with $k$, Sum B becomes:

$$\text{Sum B} = \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m+1-k)} v^{(k)}$$

Now we combine the re-indexed Sum B with Sum A. We can separate the first term ($k = 0$) from Sum A and the last term ($k = m+1$) from Sum B:

$$(uv)^{(m+1)} = \left[\binom{m}{0}u^{(m+1)}v + \sum_{k=1}^{m}\binom{m}{k}u^{(m+1-k)}v^{(k)}\right] + \left[\sum_{k=1}^{m}\binom{m}{k-1}u^{(m+1-k)}v^{(k)} + \binom{m}{m}uv^{(m+1)}\right]$$

Grouping the two middle sums together:

$$(uv)^{(m+1)} = \binom{m}{0}u^{(m+1)}v + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right]u^{(m+1-k)}v^{(k)} + \binom{m}{m}uv^{(m+1)}$$

We use Pascal's Identity: $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$. Also, we know that $\binom{m}{0} = 1 = \binom{m+1}{0}$ and $\binom{m}{m} = 1 = \binom{m+1}{m+1}$. Substituting these identities into our expression gives:

$$(uv)^{(m+1)} = \binom{m+1}{0}u^{(m+1)}v + \sum_{k=1}^{m}\binom{m+1}{k}u^{(m+1-k)}v^{(k)} + \binom{m+1}{m+1}uv^{(m+1)}$$

This entire expression can now be combined into a single sum from $k = 0$ to $k = m+1$:

$$(uv)^{(m+1)} = \sum_{k=0}^{m+1}\binom{m+1}{k}u^{(m+1-k)}v^{(k)}$$

This is precisely the form of the theorem for $n = m+1$.
By the principle of mathematical induction, the theorem is true for all positive integers $n$. ■

## Key Features

- Analogous to the binomial theorem
- Coefficients are binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Requires both functions to have derivatives up to order $n$

**Example 1.6.2.** Find the second derivative of $h(x) = x^2e^{3x}$ using Leibnitz theorem.

*Solution.* Set $u = e^{3x}$, $v = x^2$. Apply Leibnitz theorem for $n = 2$:

$$(uv)'' = \sum_{k=0}^{2}\binom{2}{k}u^{(k)}v^{(2-k)} = u^{(0)}v^{(2)} + 2\cdot u^{(1)}v^{(1)} + u^{(2)}v^{(0)}$$

Compute terms:

$$\begin{aligned} k=0: &\quad \binom{2}{0}u^{(0)}v^{(2)} = 1\cdot e^{3x}\cdot 2 \\ k=1: &\quad \binom{2}{1}u^{(1)}v^{(1)} = 2\cdot(3e^{3x})\cdot(2x) \\ k=2: &\quad \binom{2}{2}u^{(2)}v^{(0)} = 1\cdot(9e^{3x})\cdot(x^2) \end{aligned}$$

Sum the terms:

$$h''(x) = 2e^{3x} + 12xe^{3x} + 9x^2e^{3x} = \boxed{e^{3x}(9x^2 + 12x + 2)}$$

■

**Example 1.6.3.** Find the fourth derivative of the function $f(x) = x^3e^{2x}$ using Leibnitz theorem.

*Solution.* Set $u = e^{2x}$ and $v = x^3$. Apply Leibnitz theorem for $n = 4$:

$$(uv)^{(4)} = \sum_{k=0}^{4}\binom{4}{k}u^{(k)}v^{(4-k)}$$

Compute derivatives of $u$ and $v$:

$$u = e^{2x} \qquad v = x^3$$

$$u^{(k)} = 2^k e^{2x} \qquad v^{(m)} = \begin{cases} \frac{3!}{(3-m)!}x^{3-m} & 0 \le m \le 3 \\ 0 & m > 3 \end{cases}$$

Calculate each term:

$$\begin{aligned}
k = 0: &\quad \binom{4}{0}u^{(0)}v^{(4)} = 1 \cdot 2^0 e^{2x} \cdot 0 = 0 \\
k = 1: &\quad \binom{4}{1}u^{(1)}v^{(3)} = 4 \cdot 2^1 e^{2x} \cdot 6 = 48e^{2x} \\
k = 2: &\quad \binom{4}{2}u^{(2)}v^{(2)} = 6 \cdot 2^2 e^{2x} \cdot 6x = 144xe^{2x} \\
k = 3: &\quad \binom{4}{3}u^{(3)}v^{(1)} = 4 \cdot 2^3 e^{2x} \cdot 3x^2 = 96x^2e^{2x} \\
k = 4: &\quad \binom{4}{4}u^{(4)}v^{(0)} = 1 \cdot 2^4 e^{2x} \cdot x^3 = 16x^3e^{2x}
\end{aligned}$$

Sum all terms:

$$\begin{aligned}
f^{(4)}(x) &= 0 + 48e^{2x} + 144xe^{2x} + 96x^2e^{2x} + 16x^3e^{2x} \\
&= e^{2x}(16x^3 + 96x^2 + 144x + 48)
\end{aligned}$$

Factor out 16:

$$f^{(4)}(x) = 16e^{2x}(x^3 + 6x^2 + 9x + 3)$$

Thus, the fourth derivative is $\boxed{16e^{2x}(x^3 + 6x^2 + 9x + 3)}$. ■

**Example 1.6.4.** If $y = e^{a\sin^{-1}(x)}$, prove the following:

(a) $(1 - x^2)y_2 - xy_1 - a^2y = 0$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

(b) Hence, using Leibniz's Theorem, show that the following recurrence relation holds:

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0$$

*Solution.* We are given the function $y = e^{a\sin^{-1}(x)}$. Using the chain rule, we differentiate $y$ with respect to $x$:

$$y_1 = \frac{dy}{dx} = e^{a\sin^{-1}(x)} \cdot \frac{d}{dx}(a\sin^{-1}(x))$$

$$y_1 = e^{a\sin^{-1}(x)} \cdot \frac{a}{\sqrt{1 - x^2}}$$

that is,

$$y_1 = \frac{ay}{\sqrt{1 - x^2}}$$

because $y = e^{a\sin^{-1}(x)}$.

Squaring both sides of the equation:

$$y_1^2 = \frac{a^2y^2}{1 - x^2}$$

Now, multiply both sides by $(1 - x^2)$ to clear the fraction:

$$(1 - x^2)y_1^2 = a^2y^2 \quad (*).$$

We differentiate the entire equation $(*)$ with respect to $x$.

$$\frac{d}{dx}\left[(1 - x^2)y_1^2\right] = \frac{d}{dx}\left[a^2y^2\right]$$

$$\left(\frac{d}{dx}(1 - x^2)\right) \cdot y_1^2 + (1 - x^2) \cdot \left(\frac{d}{dx}(y_1^2)\right) = a^2 \cdot \left(\frac{d}{dx}(y^2)\right)$$

$$(-2x)y_1^2 + (1 - x^2)(2y_1y_2) = a^2(2yy_1)$$

Divide the entire equation by $2y_1$:

$$-xy_1 + (1 - x^2)y_2 = a^2y$$

Rearranging the terms gives the desired differential equation:

$$(1 - x^2)y_2 - xy_1 - a^2y = 0$$

### Applying Leibniz's Theorem to find the Recurrence Relation

We now differentiate the equation $(1-x^2)y_2 - xy_1 - a^2y = 0$ successively $n$ times with respect to $x$.

$$\frac{d^n}{dx^n}\left[(1-x^2)y_2 - xy_1 - a^2y\right] = 0$$

By linearity of the derivative, we can differentiate each term separately:

$$\underbrace{\frac{d^n}{dx^n}\left[(1-x^2)y_2\right]}_{\text{Term A}} - \underbrace{\frac{d^n}{dx^n}\left[xy_1\right]}_{\text{Term B}} - \underbrace{\frac{d^n}{dx^n}\left[a^2y\right]}_{\text{Term C}} = 0$$

**For Term:** $D^n[(1-x^2)y_2]$ Let $u = y_2$ and $v = 1-x^2$. We apply Leibniz's Theorem. The derivatives of $v$ terminate quickly:

- $v = 1 - x^2$
- $v' = -2x$
- $v'' = -2$
- $v''' = 0$

The derivatives of $u$ are $u^{(k)} = (y_2)^{(k)} = y_{k+2}$. The Leibniz expansion is:

$$\binom{n}{0}vu^{(n)} + \binom{n}{1}v'u^{(n-1)} + \binom{n}{2}v''u^{(n-2)} + \dots$$

Substituting our functions (only the first three terms are non-zero):

$$(1)(1-x^2)y_{n+2} + (n)(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n$$

Simplifying gives: $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$.
**For Term:** $D^n[xy_1]$ Let $u = y_1$ and $v = x$. The derivatives of $v$ are $v' = 1$ and $v'' = 0$. The Leibniz expansion has only two non-zero terms:

$$\binom{n}{0}vu^{(n)} + \binom{n}{1}v'u^{(n-1)}$$

Substituting our functions:

$$(1)(x)y_{n+1} + (n)(1)y_n = xy_{n+1} + ny_n$$

**For Term:** $D^n[a^2y]$ Since $a^2$ is a constant, this is straightforward: $a^2y_n$.
**Combining the results:** Now we substitute the expanded terms back into our main equation:

$$[(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n] - [xy_{n+1} + ny_n] - [a^2y_n] = 0$$

Finally, we group the terms by the order of the derivative $(y_{n+2}, y_{n+1}, y_n)$:

$$(1-x^2)y_{n+2} + (-2nx - x)y_{n+1} + (-n(n-1) - n - a^2)y_n = 0$$

Simplifying gives:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$$

■

## 1.7 Partial Differentiation

### Concept Overview

**Partial differentiation** deals with functions of multiple variables. The partial derivative of $f(x, y)$ with respect to $x$ is denoted $\frac{\partial f}{\partial x}$ and measures the rate of change of $f$ while keeping $y$ constant.

## Formal Definition

For $z = f(x, y)$, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

**Theorem 1.7.1** (Clairaut's Theorem)**.** *If $f(x, y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are defined on an open set containing $(a, b)$ and are continuous at $(a, b)$, then:*

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$$

*Proof.* For sufficiently small $h, k \neq 0$, define the auxiliary function:

$$\Delta(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b).$$

**Analyze via $f_{yx}$**
Define $g(y) = f(a + h, y) - f(a, y)$. Then:

$$\Delta(h, k) = g(b + k) - g(b).$$

By the Mean Value Theorem (MVT), there exists $d$ between $b$ and $b + k$ such that:

$$g(b + k) - g(b) = k \cdot g'(d) = k \left[\frac{\partial f}{\partial y}(a + h, d) - \frac{\partial f}{\partial y}(a, d)\right].$$

Apply MVT to $h(x) = \frac{\partial f}{\partial y}(x, d)$ on $[a, a + h]$. There exists $c_1$ between $a$ and $a + h$ such that:

$$\frac{\partial f}{\partial y}(a + h, d) - \frac{\partial f}{\partial y}(a, d) = h \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

**Analyze via $f_{xy}$**
Define $r(x) = f(x, b + k) - f(x, b)$. Then:

$$\Delta(h, k) = r(a + h) - r(a).$$

By MVT, there exists $e$ between $a$ and $a + h$ such that:

$$r(a + h) - r(a) = h \cdot r'(e) = h \left[\frac{\partial f}{\partial x}(e, b + k) - \frac{\partial f}{\partial x}(e, b)\right].$$

Apply MVT to $s(y) = \frac{\partial f}{\partial x}(e, y)$ on $[b, b + k]$. There exists $c_2$ between $b$ and $b + k$ such that:

$$\frac{\partial f}{\partial x}(e, b + k) - \frac{\partial f}{\partial x}(e, b) = k \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

**Equate and take limits**
From Steps 1 and 2:

$$hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

For $hk \neq 0$, we have:

$$\frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

As $(h, k) \to (0, 0)$:

$$(c_1, d) \to (a, b) \quad \text{and} \quad (e, c_2) \to (a, b).$$

By continuity of $f_{xy}$ and $f_{yx}$ at $(a, b)$:

$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial x \partial y}(a, b),$$
$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial y \partial x}(e, c_2) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Therefore:

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)}$$

■

**Example 1.7.2.** Find the first partial derivatives of $f(x, y) = x^3y + e^{xy}$.

*Solution.*

$$\frac{\partial f}{\partial x} = 3x^2y + ye^{xy}$$
$$\frac{\partial f}{\partial y} = x^3 + xe^{xy}$$

■

**Example 1.7.3.** Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x, y) = \sin(2x + 3y)$.

*Solution.* First derivatives:

$$\frac{\partial f}{\partial x} = 2\cos(2x + 3y)$$
$$\frac{\partial f}{\partial y} = 3\cos(2x + 3y)$$

Mixed derivative:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(3\cos(2x + 3y)\right) = -6\sin(2x + 3y)$$

■

## 1.8 Total Differentiation

### Concept Overview

**Total differentiation** extends differentiation to functions of multiple variables. The total differential $dz$ approximates the change in $z = f(x, y)$ when both $x$ and $y$ change.

### Total Differential Formula

For $z = f(x, y)$:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

For $w = f(x, y, z)$:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

### Chain Rule for Total Derivatives

If $z = f(x, y)$ with $x = g(t)$, $y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

**Example 1.8.1.** Find the total differential of $z = x^2y - 3xy^3$.

*Solution.* Partial derivatives:

$$\frac{\partial z}{\partial x} = 2xy - 3y^3$$
$$\frac{\partial z}{\partial y} = x^2 - 9xy^2$$

Total differential:

$$dz = (2xy - 3y^3)dx + (x^2 - 9xy^2)dy$$

■

**Example 1.8.2.** If $z = e^x \sin y$ where $x = t^2$ and $y = t^3$, find $\frac{dz}{dt}$.

*Solution.* Apply chain rule:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (e^x \sin y)(2t) + (e^x \cos y)(3t^2) \\ &= e^{t^2}\left[2t\sin(t^3) + 3t^2\cos(t^3)\right]\end{aligned}$$

■

## 1.9 Homogeneous Functions

### Concept Overview

A function $f(x_1, x_2, \ldots, x_n)$ is **homogeneous of degree** $k$ if for all $\lambda > 0$:

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)$$

### Properties

- Linear functions are homogeneous of degree 1
- Quadratic forms are homogeneous of degree 2
- Constant functions are homogeneous of degree 0

**Example 1.9.1.** Show that $f(x, y) = x^3 + 3x^2y + y^3$ is homogeneous and find its degree.

*Solution.* Replace $x \to \lambda x$, $y \to \lambda y$:

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^3 + 3(\lambda x)^2(\lambda y) + (\lambda y)^3 \\ &= \lambda^3 x^3 + 3\lambda^3 x^2 y + \lambda^3 y^3 \\ &= \lambda^3(x^3 + 3x^2y + y^3) \\ &= \lambda^3 f(x, y) \end{aligned}$$

Thus homogeneous of degree 3. ■

**Example 1.9.2.** Is $g(x, y) = x^2 + xy + \sin\left(\frac{x}{y}\right)$ homogeneous?

*Solution.* Test with $\lambda > 0$:

$$\begin{aligned} g(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda x)(\lambda y) + \sin\left(\frac{\lambda x}{\lambda y}\right) \\ &= \lambda^2 x^2 + \lambda^2 xy + \sin\left(\frac{x}{y}\right) \end{aligned}$$

The expression contains $\lambda^2$ terms and a $\lambda$-independent term. Not homogeneous. ■

## 1.10 Euler's Theorem

**Theorem 1.10.1** (Euler's Theorem on Homogeneous Functions)**.** *If $f(x, y)$ is a homogeneous function of degree $k$ and has continuous first partial derivatives, then:*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = k \cdot f(x, y)$$

*Proof.* By definition, since $f$ is homogeneous of degree $k$, we have the following identity for any $t > 0$:

$$f(tx, ty) = t^k f(x, y) \tag{1.1}$$

We differentiate both sides of the identity (1.1) with respect to the parameter $t$, treating $x$ and $y$ as constants. For the left-hand side (LHS), we use the multivariable chain rule. Let $X = tx$ and $Y = ty$. Then:

$$\begin{aligned} \frac{d}{dt} f(tx, ty) &= \frac{\partial f}{\partial X}\frac{dX}{dt} + \frac{\partial f}{\partial Y}\frac{dY}{dt} \\ &= f_X(tx, ty) \cdot (x) + f_Y(tx, ty) \cdot (y) \end{aligned}$$

where $f_X$ and $f_Y$ denote the partial derivatives of $f$ with respect to its first and second arguments, respectively.

For the right-hand side (RHS), we treat $f(x, y)$ as a constant and differentiate only the $t^k$ term:

$$\frac{d}{dt}\left(t^k f(x, y)\right) = kt^{k-1} f(x, y)$$

Equating the derivatives of the LHS and RHS gives us a new identity that is also true for all $t > 0$:

$$x \cdot f_X(tx, ty) + y \cdot f_Y(tx, ty) = kt^{k-1} f(x, y)$$

Since the identity above holds for any value of $t > 0$, it must hold for the specific case where $t = 1$. Setting $t = 1$ simplifies the expression:

$$\begin{aligned} x \cdot f_X(1 \cdot x, 1 \cdot y) + y \cdot f_Y(1 \cdot x, 1 \cdot y) &= k(1)^{k-1} f(x, y) \\ x \cdot f_X(x, y) + y \cdot f_Y(x, y) &= k \cdot f(x, y) \end{aligned}$$

Rewriting in standard partial derivative notation, we get the desired result:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = kf(x, y)$$

■

**Example 1.10.2.** Verify Euler's theorem for $f(x, y) = x^{1/3}y^{2/3}$.

*Solution.* First, degree $k = 1/3 + 2/3 = 1$. Partial derivatives:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{3}x^{-2/3}y^{2/3} \\ \frac{\partial f}{\partial y} &= \frac{2}{3}x^{1/3}y^{-1/3} \end{aligned}$$

Apply Euler's theorem:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\left(\frac{1}{3}x^{-2/3}y^{2/3}\right) + y\left(\frac{2}{3}x^{1/3}y^{-1/3}\right) \\ &= \frac{1}{3}x^{1/3}y^{2/3} + \frac{2}{3}x^{1/3}y^{2/3} \\ &= x^{1/3}y^{2/3} = f(x, y) \end{aligned}$$

Equal to $1 \cdot f$, verifying the theorem. ■

**Example 1.10.3.** Using Euler's theorem, show that if $f = \frac{x^2+y^2}{xy}$, then $xf_x + yf_y = -f$.

*Solution.* Rewrite $f(x, y) = \frac{x}{y} + \frac{y}{x}$. Test homogeneity:

$$f(\lambda x, \lambda y) = \frac{\lambda x}{\lambda y} + \frac{\lambda y}{\lambda x} = \frac{x}{y} + \frac{y}{x} = f(x, y)$$

Thus homogeneous of degree 0. By Euler's theorem:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 \cdot f = 0$$

But note: $0 = -f + f$, so rearrange as $xf_x + yf_y = -f + f$. To get exact form, observe:

$$xf_x + yf_y = 0 = -f + f$$

The problem statement appears inconsistent. Correction: For homogeneous degree 0, $xf_x + yf_y = 0$, while $-f = -\left(\frac{x}{y} + \frac{y}{x}\right)$. They are not equal. The correct conclusion is $xf_x + yf_y = 0$. ■