Bachelors with Mathematics as Major

1st Semester

MMT122J: Mathematics/Applied Mathematics: Calculus - I

Credits: 4 THEORY + 2 TUTORIAL **Theory: 60 Hours & Tutorial: 30 Hours**

Course Objectives: (i) To study and understand the notions of differential calculus and to imbibe the acquaintance for using the techniques in other sciences and engineering. (ii) To prepare the students for taking up advanced courses of mathematics.

Course Outcome: (i) After the successful completion of the course, students shall be able to apply differential operators to understand the dynamics of various real life situations. (ii) The students shall be able to use differential calculus in optimization problems.

Theory: 4 Credits

Unit –I

Limits and infinitesimals, Continuity ($\epsilon - \delta$ definition), types of discontinuities of functions, Differentiability of functions, Successive differentiation and Leibnitz theorem, Partial differentiation, Total differentiation, Homogenous functions and Euler's theorem.

Unit –II

Indeterminate forms, Tangents and normals (polar coordinates only), Angle between radius vector and tangent, Perpendicular from pole to tangent, angle of intersection of two curves, polar tangent, polar normal, polar sub-tangent, polar sub-normal.

Unit –III

Curvature and radius of curvature, Pedal Equations, lengths of arcs, Asymptotes, Singular points, Maxima and minima of functions. Bounded functions, Properties of continuous functions on closed intervals, Intermediate value theorem, Darboux theorem.

Unit –IV

Rolle's theorem and mean value theorems (with proofs) and their geometrical interpretation, Taylor's theorem with Lagranges and Cauchy's form of remainder, Taylor's series, Maclaurin's series of $\sin x$, $\cos x$, e^x, $\log x$, $(1+x)^m$.Envelope of a family of curves involving one and two parameters.

Tutorial: 2 Credits

Unit –V

Examples of discontinuous functions, nth derivative of product of two functions, involutes and evolutes, bounds of function (Supremum and infimum).

Unit –VI

Tracing of cartesian equations of the form $y = f(x)$, $y^2 = f(x)$, tracing of the parametric equations.

Recommended Books:

1. Shanti Narayan and P.K. Mittal, Differential Calculus, S. Chand, 2016.
2. S. D. Chopra, M. L. Kochar and A. Aziz, Differential Calculus, Kapoor Sons.
3. Schaums outline of Theory and problems of Differential and Integral Calculus, 1964.
1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002.
2. T.M. Apostal, Calculus Vol. I, John Wiley & Sons Inc, 1975.
3. S. Balachandra Rao and C. K. Shantha, Differential Calculus, New Age Publication, 1992.
4. S. Lang, A First Course in Calculus, Springer-Verlag, 1998.
5. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 2008.
6. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007.
7. Suggestive digital platforms web links: NPTEL/ SWAYAM/ MOOCS.

——CALCULUS I - (MMT122J)—–

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Chapter 1

Limits, Continuity and Differentiability

1.1 Limits and Infinitesimals

Concept Overview

The **limit** of a function $f(x)$ as x approaches a point a is the value L that $f(x)$ gets arbitrarily close to as x approaches a. Formally:

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Infinitesimals are quantities smaller than any positive real number but greater than zero. In limits, dx represents an infinitesimal change in x, and the expression $f(a + dx) - L$ becomes infinitesimal as $dx \to 0$.

Example 1.1.1. Evaluate $\lim_{x \to 3}(2x + 1)$.

Solution.

$$\begin{aligned} \lim_{x \to 3}(2x + 1) &= 2(3) + 1 \quad \text{(Direct substitution)} \\ &= 7. \end{aligned}$$

To verify using the ϵ-δ definition: Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x - 3| < \delta$, then:

$$|(2x + 1) - 7| = |2x - 6| = 2|x - 3| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Thus, $\lim_{x \to 3}(2x + 1) = 7$. ■

Example 1.1.2. Prove that $\lim_{x \to 5}(3x - 4) = 11$ using the ϵ-δ definition.

Solution. Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $|(3x - 4) - 11| < \epsilon$.
Simplify the expression:

$$|3x - 15| = |3(x - 5)| = 3|x - 5|$$

We require:

$$3|x - 5| < \epsilon \implies |x - 5| < \frac{\epsilon}{3}$$

Choose $\delta = \frac{\epsilon}{3}$.
Verification: If $0 < |x - 5| < \delta$, then:

$$|(3x - 4) - 11| = 3|x - 5| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

Thus, $\lim_{x \to 5}(3x - 4) = 11$. ■

Example 1.1.3. Prove that $\lim_{x \to 3}(x^2 - 2x) = 3$ using the ϵ-δ definition.

Solution. Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|(x^2 - 2x) - 3| < \epsilon$.
Simplify the expression:

$$|x^2 - 2x - 3| = |(x - 3)(x + 1)| = |x - 3||x + 1|$$

Assume $\delta \le 1$. Then $|x - 3| < 1$ implies:

$$2 < x < 4 \quad \text{so} \quad 3 < x + 1 < 5$$

Thus $|x+1| < 5$. Now:

$$|x-3||x+1| < |x-3| \cdot 5$$

We require $5|x-3| < \epsilon$, which gives $|x-3| < \frac{\epsilon}{5}$.
Choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$.
Verification: If $0 < |x-3| < \delta$, then:

$$|x^2 - 2x - 3| = |x-3||x+1| < \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon$$

Thus, $\lim_{x \to 3}(x^2 - 2x) = 3$. ■

1.2 Continuity: ϵ-δ Definition

Concept Overview

A function $f(x)$ is **continuous** at $x = a$ if:

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$.

The ϵ-δ definition formalizes this:

$$f \text{ is continuous at } a \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |x-a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

This ensures no jumps, breaks, or oscillations at a.

Example 1.2.1. Prove that $f(x) = 4x - 1$ is continuous at $x = 3$ using the ϵ-δ definition.

Solution. We must show: $\forall \epsilon > 0, \exists \delta > 0$ such that $|x-3| < \delta \implies |(4x-1) - 11| < \epsilon$.
Note $f(3) = 4(3) - 1 = 11$.
Simplify:

$$|(4x-1) - 11| = |4x - 12| = 4|x-3|$$

Choose $\delta = \epsilon/4$. If $|x-3| < \delta$, then:

$$4|x-3| < 4 \cdot (\epsilon/4) = \epsilon$$

Thus, $f(x) = 4x - 1$ is continuous at $x = 3$. ■

Example 1.2.2. Prove $f(x) = x^2$ is continuous at $x = 2$.

Solution. $f(2) = 4$. For any $\epsilon > 0$, choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$. If $|x-2| < \delta$, then:

$$\begin{aligned} |x+2| &= |(x-2) + 4| \\ &\le |x-2| + 4 < 1 + 4 = 5, \\ |f(x) - f(2)| &= |x^2 - 4| \\ &= |(x-2)(x+2)| \\ &< \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon. \end{aligned}$$

Thus, $f(x) = x^2$ is continuous at $x = 2$. ■

Example 1.2.3. Prove that $g(x) = x^2 + 2$ is continuous at $x = -1$ using the ϵ-δ definition.

Solution. We must show: $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - (-1)| < \delta \implies |(x^2+2) - 3| < \epsilon$.
Note $g(-1) = (-1)^2 + 2 = 3$.
Simplify:

$$|(x^2+2) - 3| = |x^2 - 1| = |x-1||x+1|$$

Set $\delta \le 1$. Then $|x+1| < 1$ implies $-2 < x < 0$, so $|x-1| < 3$.
Now:

$$|x-1||x+1| < 3|x+1|$$

Choose $\delta = \min(1, \epsilon/3)$. If $|x+1| < \delta$, then:

$$|g(x) - g(-1)| < 3 \cdot \delta \le 3 \cdot (\epsilon/3) = \epsilon$$

Thus, $g(x) = x^2 + 2$ is continuous at $x = -1$. ■

Theorem 1.2.4. *If the limit of a function $f(x)$ as x approaches a exists, then this limit is unique.*

Proof. Assume, for the sake of contradiction, that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ where $L_1 \neq L_2$.

By the definition of a limit, there exists a $\delta > 0$ such that for any x satisfying $0 < |x-a| < \delta$, we must have both:

$$|f(x) - L_1| < \epsilon \quad \text{and} \quad |f(x) - L_2| < \epsilon$$

Using the triangle inequality, we can write:

$$\begin{aligned} |L_1 - L_2| &= |(L_1 - f(x)) + (f(x) - L_2)| \\ &\leq |f(x) - L_1| + |f(x) - L_2| \\ &< \epsilon + \epsilon = 2\epsilon \end{aligned}$$

Let $\epsilon = \frac{|L_1 - L_2|}{2}$. Since $L_1 \neq L_2$, we know that $\epsilon > 0$.

Substituting our choice of ϵ:

$$|L_1 - L_2| < 2\left(\frac{|L_1 - L_2|}{2}\right) = |L_1 - L_2|$$

This results in the contradiction $|L_1 - L_2| < |L_1 - L_2|$. Therefore, our initial assumption must be false, and the limit must be unique. ∎

Theorem 1.2.5 (The Algebra of Limits)**.** *Let c be a real number, and let $f(x)$ and $g(x)$ be two functions such that*

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = M$$

where L and M are finite real numbers. Then the following properties hold:

1. ***Sum Rule:*** *The limit of the sum is the sum of the limits.*

$$\lim_{x\to c}[f(x) + g(x)] = L + M$$

2. ***Difference Rule:*** *The limit of the difference is the difference of the limits.*

$$\lim_{x\to c}[f(x) - g(x)] = L - M$$

3. ***Product Rule:*** *The limit of the product is the product of the limits.*

$$\lim_{x\to c}[f(x)g(x)] = LM$$

4. ***Quotient Rule:*** *The limit of the quotient is the quotient of the limits, provided the limit of the denominator is not zero.*

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad \textit{provided } M \neq 0$$

1. Proof of the Sum Rule. Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|(f(x)+g(x)) - (L+M)| < \epsilon$.

By the definition of the limits for f and g, we know that for $\epsilon/2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

- if $0 < |x-c| < \delta_1$, then $|f(x) - L| < \epsilon/2$.
- if $0 < |x-c| < \delta_2$, then $|g(x) - M| < \epsilon/2$.

Let us choose $\delta = \min(\delta_1, \delta_2)$. Then for any x satisfying $0 < |x-c| < \delta$, both of the above inequalities hold. Using the triangle inequality, we have:

$$\begin{aligned} |(f(x)+g(x)) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

The proof for the DIfference Rule is analogous.

∎

3. Proof of the Product Rule. Let $\epsilon > 0$ be given. We want to show $|f(x)g(x) - LM| < \epsilon$.

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M|\end{aligned}$$

Since $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|g(x) - M| < 1$, which implies $|g(x)| < |M| + 1$. Let $K = |M| + 1$. Now, for a given $\epsilon > 0$, we can find δ_2 and δ_3 such that:

- if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \frac{\epsilon}{2K}$.
- if $0 < |x - c| < \delta_3$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. (We use $|L| + 1$ to avoid division by zero if $L = 0$).

Choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x - c| < \delta$, then:

$$\begin{aligned}|f(x)g(x) - LM| &\leq |g(x)||f(x) - L| + |L||g(x) - M| \\ &< K \cdot \frac{\epsilon}{2K} + |L| \cdot \frac{\epsilon}{2(|L| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Thus, we have shown that $\lim_{x\to c}[f(x)g(x)] = LM$. ■

4. Proof of the Quotient Rule. It suffices to first prove that $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ for $M \neq 0$, and then apply the Product Rule to $f(x) \cdot \frac{1}{g(x)}$.

Let $\epsilon > 0$ be given. We want to show $\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon$.

$$\left|\frac{M - g(x)}{Mg(x)}\right| = \frac{|g(x) - M|}{|M||g(x)|}$$

Since $M \neq 0$, $|M|/2 > 0$. There exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|g(x) - M| < |M|/2$. By the reverse triangle inequality, this implies $|g(x)| > |M| - |g(x) - M| > |M| - |M|/2 = |M|/2$. Therefore, $\frac{1}{|g(x)|} < \frac{2}{|M|}$. Now, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \frac{\epsilon|M|^2}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - c| < \delta$, then:

$$\frac{|g(x) - M|}{|M||g(x)|} < \frac{|g(x) - M|}{|M|(|M|/2)} = \frac{2|g(x) - M|}{|M|^2} < \frac{2}{|M|^2} \cdot \frac{\epsilon|M|^2}{2} = \epsilon$$

So, $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$. By the Product Rule, $\lim_{x\to c} \frac{f(x)}{g(x)} = (\lim_{x\to c} f(x)) \left(\lim_{x\to c} \frac{1}{g(x)}\right) = L \cdot \frac{1}{M} = \frac{L}{M}$. ■

Exercise 1.2.6. Q1. **(Limit)** Prove $\lim_{x\to 4}(5 - 2x) = -3$ using ϵ-δ definition.

Q2. **(Limit)** Prove $\lim_{x\to -2}(x^2 + 3x) = -2$ using ϵ-δ definition.

Q3. **(Continuity)** Prove $h(x) = \sqrt{x}$ is continuous at $x = 4$ using ϵ-δ definition. (Hint: Use $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2}$).

Q4. **(Continuity)** Prove $k(x) = \dfrac{1}{x}$ is continuous at $x = \dfrac{1}{2}$ using ϵ-δ definition.

1.3 Types of Discontinuities

Discontinuities occur where a function fails to be continuous. Common types:

1. **Removable**: $\lim_{x\to a} f(x)$ exists but $f(a)$ is undefined or $\lim_{x\to a} f(x) \neq f(a)$. "Fixed" by redefining $f(a)$.
2. **Jump or discontinuity of first kind**: Left-hand and right-hand limits exist but are unequal ($\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$).
3. **Infinite or discontinuityof second kind**: $\lim_{x\to a} f(x) = \pm\infty$.
4. **Oscillating**: Function oscillates infinitely often near a (e.g., $\sin(1/x)$ at $x = 0$).

Example 1.3.1. Classify the discontinuity of $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$.

Solution. The function is undefined at $x = 2$. Simplify for $x \neq 2$:

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

The limit exists:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2) = 4.$$

Since the limit exists but $f(2)$ is undefined, the discontinuity is **removable**. ■

Example 1.3.2. Classify the discontinuity of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x+3 & \text{if } x \geq 2 \end{cases}$$

at $x = 2$.

Solution. Calculate left-hand and right-hand limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+3) = 2+3 = 5$$

Since $4 \neq 5$, the left-hand and right-hand limits exist but are not equal.
Function value: $f(2) = 2 + 3 = 5$.
The discontinuity at $x = 2$ is a **jump discontinuity**. ■

Example 1.3.3. Classify the discontinuity of $g(x) = \dfrac{1}{(x-3)^2}$ at $x = 3$.

Solution. Examine the limit as x approaches 3:

$$\lim_{x\to 3} \frac{1}{(x-3)^2} = +\infty$$

The function is undefined at $x = 3$. As $x \to 3$, the function values increase without bound.
This is an **infinite discontinuity** (vertical asymptote at $x = 3$). ■

Example 1.3.4. Classify the discontinuity of $h(x) = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$.

Solution. Consider the behavior as $x \to 0$:

- As $x \to 0^+$, $\frac{1}{x} \to +\infty$ and $\sin(1/x)$ oscillates between -1 and 1
- As $x \to 0^-$, $\frac{1}{x} \to -\infty$ and $\sin(1/x)$ oscillates between -1 and 1

The limit $\lim_{x\to 0} \sin(1/x)$ does not exist because oscillations become increasingly rapid.
This is an **oscillating discontinuity** at $x = 0$. ■

Example 1.3.5. Examine the following function for continuity at the origin.

$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$ and $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0-} \dfrac{x}{e^{-1/x}+1} = 0$ Also, $f(0) = 0$. Thus, the function is continuous at the origin. ■

1.3.1 Piecewise Continuity

A function $f(x)$ is said to be piecewise continuous in an interval I, if the interval can be subdivided into a finite number of subintervals such that $f(x)$ is continuous in each of the subintervals and the limits of $f(x)$ as x approaches the end points of each subinterval are finite. For example, the greatest integer function $f(x) = [x]$ defined on $[-1, 3]$ is piecewise continuous on $[-1, 3]$.

1.4 Differentiability of Functions

Concept Overview

A function $f(x)$ is **differentiable** at a point $x = a$ if the derivative $f'(a)$ exists. Geometrically, this means the function has a unique tangent at that point. The derivative is defined as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists finitely.

Necessary Conditions

- If f is differentiable at a, then it must be continuous at a.
- The converse is not true: continuity does not imply differentiability (e.g., $|x|$ at $x = 0$).

(1). Proof. To prove that the function f is continuous at a, we must show that $\lim_{x\to a} f(x) = f(a)$. This is equivalent to showing that the difference between $f(x)$ and $f(a)$ approaches zero, i.e.,

$$\lim_{x\to a} [f(x) - f(a)] = 0$$

We are given that f is differentiable at a. By definition, this means the limit for the derivative exists and is a finite number:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

that is,

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a}\right) \cdot (x - a)$$

We can now take the limit of both sides as $x \to a$ and apply the product rule for limits:

$$\begin{aligned}
\lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) \\
&= \left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \left(\lim_{x\to a} (x - a)\right) \\
&= f'(a) \cdot (a - a) \\
&= f'(a) \cdot 0 \\
&= 0
\end{aligned}$$

Since we have shown that $\lim_{x\to a}[f(x) - f(a)] = 0$, it follows directly that $\lim_{x\to a} f(x) = f(a)$. This is the definition of continuity at the point a. ■

■

Left and Right Derivatives

The **left derivative** at a:

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}$$

The **right derivative** at a:

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$$

f is differentiable at a iff $f'_-(a) = f'_+(a)$ and both exist finitely.

Example 1.4.1. Show that $f(x) = x^2 + 3x$ is differentiable at $x = 2$ and find its derivative.

Solution. Compute the derivative using the limit definition:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{[(2+h)^2+3(2+h)]-[2^2+3(2)]}{h} \\ &= \lim_{h\to 0} \frac{[4+4h+h^2+6+3h]-[4+6]}{h} \\ &= \lim_{h\to 0} \frac{7h+h^2}{h} \\ &= \lim_{h\to 0}(7+h) = 7 \end{aligned}$$

Since the limit exists, f is differentiable at $x=2$ with $f'(2)=7$. ■

1.5 Successive Differentiation

Concept Overview

Successive differentiation refers to repeatedly differentiating a function. The nth derivative is denoted by:

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n y}{dx^n}$$

where n is the order of differentiation.

Standard Formulas

- $\frac{d^n}{dx^n}(x^m) = m(m-1)\cdots(m-n+1)x^{m-n} \quad \text{for} \quad n \le m$
- $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$
- $\frac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$
- $\frac{d^n}{dx^n}(\sin ax) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$
- $\frac{d^n}{dx^n}(\cos ax) = a^n \cos\left(ax + \frac{n\pi}{2}\right)$

Example 1.5.1. Find the third derivative of $g(x) = 2x^4 - 5x^3 + 3x - 7$.

Solution. Compute successive derivatives:

$$\begin{aligned} g'(x) &= \frac{d}{dx}(2x^4-5x^3+3x-7) = 8x^3-15x^2+3 \\ g''(x) &= \frac{d}{dx}(8x^3-15x^2+3) = 24x^2-30x \\ g'''(x) &= \frac{d}{dx}(24x^2-30x) = \boxed{48x-30} \end{aligned}$$

■

1.6 Leibnitz Theorem

Theorem 1.6.1 (Leibnitz Theorem). *If $u(x)$ and $v(x)$ are n-times differentiable functions, then:*

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

where $u^{(k)}$ is the kth derivative of u, and $v^{(n-k)}$ is the $(n-k)$th derivative of v, with $u^{(0)} = u$ and $v^{(0)} = v$.

Proof. We must show that the formula is true for $n = 1$.

$$\begin{aligned}
(uv)^{(1)} &= \sum_{k=0}^{1} \binom{1}{k} u^{(1-k)} v^{(k)} \\
&= \binom{1}{0} u^{(1)} v^{(0)} + \binom{1}{1} u^{(0)} v^{(1)} \\
&= (1) \cdot u'v + (1) \cdot uv' \\
&= u'v + uv'
\end{aligned}$$

Assume the theorem is true for some positive integer $n = m$. That is, we assume:

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}$$

We must prove that the theorem is true for $n = m + 1$. We start by differentiating the expression from our inductive hypothesis with respect to x:

$$\begin{aligned}
(uv)^{(m+1)} &= \frac{d}{dx}\left[(uv)^{(m)}\right] \\
&= \frac{d}{dx}\left[\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}\right] \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{d}{dx}\left(u^{(m-k)} v^{(k)}\right) \\
&= \sum_{k=0}^{m} \binom{m}{k} \left[u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)}\right]
\end{aligned}$$

Now, we split this into two separate sums:

$$(uv)^{(m+1)} = \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)}}_{\text{Sum A}} + \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k+1)}}_{\text{Sum B}}$$

To combine these sums, we re-index Sum B by letting $j = k + 1$. This means Sum B will go from $j = 1$ to $j = m + 1$. Replacing j back with k, Sum B becomes:

$$\text{Sum B} = \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m+1-k)} v^{(k)}$$

Now we combine the re-indexed Sum B with Sum A. We can separate the first term ($k = 0$) from Sum A and the last term ($k = m + 1$) from Sum B:

$$\begin{aligned}
(uv)^{(m+1)} = &\left[\binom{m}{0} u^{(m+1)} v + \sum_{k=1}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)}\right] \\
&+ \left[\sum_{k=1}^{m} \binom{m}{k-1} u^{(m+1-k)} v^{(k)} + \binom{m}{m} uv^{(m+1)}\right]
\end{aligned}$$

Grouping the two middle sums together:

$$(uv)^{(m+1)} = \binom{m}{0} u^{(m+1)} v + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right] u^{(m+1-k)} v^{(k)} + \binom{m}{m} uv^{(m+1)}$$

We use Pascal's Identity: $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$. Also, we know that $\binom{m}{0} = 1 = \binom{m+1}{0}$ and $\binom{m}{m} = 1 = \binom{m+1}{m+1}$. Substituting these identities into our expression gives:

$$(uv)^{(m+1)} = \binom{m+1}{0} u^{(m+1)} v + \sum_{k=1}^{m} \binom{m+1}{k} u^{(m+1-k)} v^{(k)} + \binom{m+1}{m+1} uv^{(m+1)}$$

This entire expression can now be combined into a single sum from $k = 0$ to $k = m + 1$:

$$(uv)^{(m+1)} = \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(m+1-k)} v^{(k)}$$

This is precisely the form of the theorem for $n = m + 1$.
By the principle of mathematical induction, the theorem is true for all positive integers n. ∎

Key Features

- Analogous to the binomial theorem
- Coefficients are binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Requires both functions to have derivatives up to order n

Example 1.6.2. Find the second derivative of $h(x) = x^2 e^{3x}$ using Leibnitz theorem.

Solution. Set $u = e^{3x}$, $v = x^2$. Apply Leibnitz theorem for $n = 2$:

$$(uv)'' = \sum_{k=0}^{2} \binom{2}{k} u^{(k)} v^{(2-k)} = u^{(0)}v^{(2)} + 2 \cdot u^{(1)}v^{(1)} + u^{(2)}v^{(0)}$$

Compute terms:

$$\begin{aligned} k = 0 : &\quad \binom{2}{0} u^{(0)}v^{(2)} = 1 \cdot e^{3x} \cdot 2 \\ k = 1 : &\quad \binom{2}{1} u^{(1)}v^{(1)} = 2 \cdot (3e^{3x}) \cdot (2x) \\ k = 2 : &\quad \binom{2}{2} u^{(2)}v^{(0)} = 1 \cdot (9e^{3x}) \cdot (x^2) \end{aligned}$$

Sum the terms:

$$h''(x) = 2e^{3x} + 12xe^{3x} + 9x^2e^{3x} = \boxed{e^{3x}(9x^2 + 12x + 2)}$$

■

Example 1.6.3. Find the fourth derivative of the function $f(x) = x^3 e^{2x}$ using Leibnitz theorem.

Solution. Set $u = e^{2x}$ and $v = x^3$. Apply Leibnitz theorem for $n = 4$:

$$(uv)^{(4)} = \sum_{k=0}^{4} \binom{4}{k} u^{(k)} v^{(4-k)}$$

Compute derivatives of u and v:

$$u = e^{2x} \qquad\qquad v = x^3$$

$$u^{(k)} = 2^k e^{2x} \qquad\qquad v^{(m)} = \begin{cases} \frac{3!}{(3-m)!} x^{3-m} & 0 \le m \le 3 \\ 0 & m > 3 \end{cases}$$

Calculate each term:

$$\begin{aligned} k = 0 : &\quad \binom{4}{0} u^{(0)}v^{(4)} = 1 \cdot 2^0 e^{2x} \cdot 0 = 0 \\ k = 1 : &\quad \binom{4}{1} u^{(1)}v^{(3)} = 4 \cdot 2^1 e^{2x} \cdot 6 = 48e^{2x} \\ k = 2 : &\quad \binom{4}{2} u^{(2)}v^{(2)} = 6 \cdot 2^2 e^{2x} \cdot 6x = 144xe^{2x} \\ k = 3 : &\quad \binom{4}{3} u^{(3)}v^{(1)} = 4 \cdot 2^3 e^{2x} \cdot 3x^2 = 96x^2e^{2x} \\ k = 4 : &\quad \binom{4}{4} u^{(4)}v^{(0)} = 1 \cdot 2^4 e^{2x} \cdot x^3 = 16x^3e^{2x} \end{aligned}$$

Sum all terms:

$$\begin{aligned} f^{(4)}(x) &= 0 + 48e^{2x} + 144xe^{2x} + 96x^2e^{2x} + 16x^3e^{2x} \\ &= e^{2x}(16x^3 + 96x^2 + 144x + 48) \end{aligned}$$

Factor out 16:

$$f^{(4)}(x) = 16e^{2x}(x^3 + 6x^2 + 9x + 3)$$

Thus, the fourth derivative is $\boxed{16e^{2x}(x^3 + 6x^2 + 9x + 3)}$. ■

Example 1.6.4. If $y = e^{a\sin^{-1}(x)}$, prove the following:

(a) $(1-x^2)y_2 - xy_1 - a^2y = 0$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

(b) Hence, using Leibniz's Theorem, show that the following recurrence relation holds:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Solution. We are given the function $y = e^{a\sin^{-1}(x)}$. Using the chain rule, we differentiate y with respect to x:

$$y_1 = \frac{dy}{dx} = e^{a\sin^{-1}(x)} \cdot \frac{d}{dx}(a\sin^{-1}(x))$$

$$y_1 = e^{a\sin^{-1}(x)} \cdot \frac{a}{\sqrt{1-x^2}}$$

that is,

$$y_1 = \frac{ay}{\sqrt{1-x^2}}$$

because $y = e^{a\sin^{-1}(x)}$.

Squaring both sides of the equation:

$$y_1^2 = \frac{a^2y^2}{1-x^2}$$

Now, multiply both sides by $(1-x^2)$ to clear the fraction:

$$(1-x^2)y_1^2 = a^2y^2 \quad (*).$$

We differentiate the entire equation $(*)$ with respect to x.

$$\frac{d}{dx}\left[(1-x^2)y_1^2\right] = \frac{d}{dx}\left[a^2y^2\right]$$

$$\left(\frac{d}{dx}(1-x^2)\right) \cdot y_1^2 + (1-x^2)\cdot\left(\frac{d}{dx}(y_1^2)\right) = a^2 \cdot \left(\frac{d}{dx}(y^2)\right)$$

$$(-2x)y_1^2 + (1-x^2)(2y_1y_2) = a^2(2yy_1)$$

Divide the entire equation by $2y_1$:

$$-xy_1 + (1-x^2)y_2 = a^2y$$

Rearranging the terms gives the desired differential equation:

$$(1-x^2)y_2 - xy_1 - a^2y = 0$$

Applying Leibniz's Theorem to find the Recurrence Relation

We now differentiate the equation $(1-x^2)y_2 - xy_1 - a^2y = 0$ successively n times with respect to x.

$$\frac{d^n}{dx^n}\left[(1-x^2)y_2 - xy_1 - a^2y\right] = 0$$

By linearity of the derivative, we can differentiate each term separately:

$$\underbrace{\frac{d^n}{dx^n}\left[(1-x^2)y_2\right]}_{\text{Term A}} - \underbrace{\frac{d^n}{dx^n}\left[xy_1\right]}_{\text{Term B}} - \underbrace{\frac{d^n}{dx^n}\left[a^2y\right]}_{\text{Term C}} = 0$$

For Term: $D^n[(1-x^2)y_2]$ Let $u = y_2$ and $v = 1-x^2$. We apply Leibniz's Theorem. The derivatives of v terminate quickly:

- $v = 1-x^2$
- $v' = -2x$
- $v'' = -2$
- $v''' = 0$

The derivatives of u are $u^{(k)} = (y_2)^{(k)} = y_{k+2}$. The Leibniz expansion is:

$$\binom{n}{0} v u^{(n)} + \binom{n}{1} v' u^{(n-1)} + \binom{n}{2} v'' u^{(n-2)} + \dots$$

Substituting our functions (only the first three terms are non-zero):

$$(1)(1-x^2)y_{n+2} + (n)(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n$$

Simplifying gives: $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$.
For Term: $D^n[xy_1]$ Let $u = y_1$ and $v = x$. The derivatives of v are $v' = 1$ and $v'' = 0$. The Leibniz expansion has only two non-zero terms:

$$\binom{n}{0} v u^{(n)} + \binom{n}{1} v' u^{(n-1)}$$

Substituting our functions:

$$(1)(x)y_{n+1} + (n)(1)y_n = xy_{n+1} + ny_n$$

For Term: $D^n[a^2 y]$ Since a^2 is a constant, this is straightforward: $a^2 y_n$.
Combining the results: Now we substitute the expanded terms back into our main equation:

$$[(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n] - [xy_{n+1} + ny_n] - [a^2 y_n] = 0$$

Finally, we group the terms by the order of the derivative (y_{n+2}, y_{n+1}, y_n):

$$(1-x^2)y_{n+2} + (-2nx - x)y_{n+1} + (-n(n-1) - n - a^2)y_n = 0$$

Simplifying gives:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$$

■

1.7 Partial Differentiation

Concept Overview

Partial differentiation deals with functions of multiple variables. The partial derivative of $f(x, y)$ with respect to x is denoted $\frac{\partial f}{\partial x}$ and measures the rate of change of f while keeping y constant.

Formal Definition

For $z = f(x, y)$, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Theorem 1.7.1 (Clairaut's Theorem). *If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined on an open set containing (a, b) and are continuous at (a, b), then:*

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$$

Proof. For sufficiently small $h, k \neq 0$, define the auxiliary function:

$$\Delta(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).$$

Analyze via f_{yx}
Define $g(y) = f(a+h, y) - f(a, y)$. Then:

$$\Delta(h, k) = g(b+k) - g(b).$$

By the Mean Value Theorem (MVT), there exists d between b and $b+k$ such that:

$$g(b+k) - g(b) = k \cdot g'(d) = k \left[\frac{\partial f}{\partial y}(a+h, d) - \frac{\partial f}{\partial y}(a, d) \right].$$

Apply MVT to $h(x) = \frac{\partial f}{\partial y}(x, d)$ on $[a, a+h]$. There exists c_1 between a and $a+h$ such that:

$$\frac{\partial f}{\partial y}(a+h, d) - \frac{\partial f}{\partial y}(a, d) = h \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Analyze via f_{xy}
Define $r(x) = f(x, b+k) - f(x, b)$. Then:

$$\Delta(h, k) = r(a+h) - r(a).$$

By MVT, there exists e between a and $a+h$ such that:

$$r(a+h) - r(a) = h \cdot r'(e) = h \left[\frac{\partial f}{\partial x}(e, b+k) - \frac{\partial f}{\partial x}(e, b) \right].$$

Apply MVT to $s(y) = \frac{\partial f}{\partial x}(e, y)$ on $[b, b+k]$. There exists c_2 between b and $b+k$ such that:

$$\frac{\partial f}{\partial x}(e, b+k) - \frac{\partial f}{\partial x}(e, b) = k \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Equate and take limits
From Steps 1 and 2:

$$hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

For $hk \neq 0$, we have:

$$\frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

As $(h, k) \to (0, 0)$:

$$(c_1, d) \to (a, b) \quad \text{and} \quad (e, c_2) \to (a, b).$$

By continuity of f_{xy} and f_{yx} at (a, b):

$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial x \partial y}(a, b),$$
$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial y \partial x}(e, c_2) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Therefore:

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)}$$

■

Example 1.7.2. Find the first partial derivatives of $f(x, y) = x^3 y + e^{xy}$.

Solution.

$$\frac{\partial f}{\partial x} = 3x^2 y + y e^{xy}$$
$$\frac{\partial f}{\partial y} = x^3 + x e^{xy}$$

■

Example 1.7.3. Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x, y) = \sin(2x + 3y)$.

Solution. First derivatives:

$$\frac{\partial f}{\partial x} = 2\cos(2x+3y)$$
$$\frac{\partial f}{\partial y} = 3\cos(2x+3y)$$

Mixed derivative:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(3\cos(2x+3y)\right) = -6\sin(2x+3y)$$

■

1.8 Total Differentiation

Concept Overview

Total differentiation extends differentiation to functions of multiple variables. The total differential dz approximates the change in $z = f(x, y)$ when both x and y change.

Total Differential Formula

For $z = f(x, y)$:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

For $w = f(x, y, z)$:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Chain Rule for Total Derivatives

If $z = f(x, y)$ with $x = g(t)$, $y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1.8.1. Find the total differential of $z = x^2y - 3xy^3$.

Solution. Partial derivatives:

$$\frac{\partial z}{\partial x} = 2xy - 3y^3$$
$$\frac{\partial z}{\partial y} = x^2 - 9xy^2$$

Total differential:

$$dz = (2xy - 3y^3)dx + (x^2 - 9xy^2)dy$$

■

Example 1.8.2. If $z = e^x \sin y$ where $x = t^2$ and $y = t^3$, find $\frac{dz}{dt}$.

Solution. Apply chain rule:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (e^x \sin y)(2t) + (e^x \cos y)(3t^2) \\ &= e^{t^2}\left[2t\sin(t^3) + 3t^2\cos(t^3)\right]\end{aligned}$$

■

1.9 Homogeneous Functions

Concept Overview

A function $f(x_1, x_2, \ldots, x_n)$ is **homogeneous of degree** k if for all $\lambda > 0$:

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)$$

Properties

- Linear functions are homogeneous of degree 1
- Quadratic forms are homogeneous of degree 2
- Constant functions are homogeneous of degree 0

Example 1.9.1. Show that $f(x, y) = x^3 + 3x^2y + y^3$ is homogeneous and find its degree.

Solution. Replace $x \to \lambda x$, $y \to \lambda y$:

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^3 + 3(\lambda x)^2(\lambda y) + (\lambda y)^3 \\ &= \lambda^3 x^3 + 3\lambda^3 x^2 y + \lambda^3 y^3 \\ &= \lambda^3(x^3 + 3x^2y + y^3) \\ &= \lambda^3 f(x, y) \end{aligned}$$

Thus homogeneous of degree 3. ■

Example 1.9.2. Is $g(x, y) = x^2 + xy + \sin\left(\frac{x}{y}\right)$ homogeneous?

Solution. Test with $\lambda > 0$:

$$\begin{aligned} g(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda x)(\lambda y) + \sin\left(\frac{\lambda x}{\lambda y}\right) \\ &= \lambda^2 x^2 + \lambda^2 xy + \sin\left(\frac{x}{y}\right) \end{aligned}$$

The expression contains λ^2 terms and a λ-independent term. Not homogeneous. ■

1.10 Euler's Theorem

Theorem 1.10.1 (Euler's Theorem on Homogeneous Functions)**.** *If $f(x, y)$ is a homogeneous function of degree k and has continuous first partial derivatives, then:*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = k \cdot f(x, y)$$

Proof. By definition, since f is homogeneous of degree k, we have the following identity for any $t > 0$:

$$f(tx, ty) = t^k f(x, y) \tag{1.1}$$

We differentiate both sides of the identity (1.1) with respect to the parameter t, treating x and y as constants.

For the left-hand side (LHS), we use the multivariable chain rule. Let $X = tx$ and $Y = ty$. Then:

$$\begin{aligned} \frac{d}{dt} f(tx, ty) &= \frac{\partial f}{\partial X}\frac{dX}{dt} + \frac{\partial f}{\partial Y}\frac{dY}{dt} \\ &= f_X(tx, ty) \cdot (x) + f_Y(tx, ty) \cdot (y) \end{aligned}$$

where f_X and f_Y denote the partial derivatives of f with respect to its first and second arguments, respectively.

For the right-hand side (RHS), we treat $f(x, y)$ as a constant and differentiate only the t^k term:

$$\frac{d}{dt}\left(t^k f(x, y)\right) = kt^{k-1} f(x, y)$$

Equating the derivatives of the LHS and RHS gives us a new identity that is also true for all $t > 0$:

$$x \cdot f_X(tx, ty) + y \cdot f_Y(tx, ty) = kt^{k-1} f(x, y)$$

Since the identity above holds for any value of $t > 0$, it must hold for the specific case where $t = 1$. Setting $t = 1$ simplifies the expression:

$$\begin{aligned} x \cdot f_X(1 \cdot x, 1 \cdot y) + y \cdot f_Y(1 \cdot x, 1 \cdot y) &= k(1)^{k-1} f(x, y) \\ x \cdot f_X(x, y) + y \cdot f_Y(x, y) &= k \cdot f(x, y) \end{aligned}$$

Rewriting in standard partial derivative notation, we get the desired result:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = kf(x, y)$$

■

Example 1.10.2. Verify Euler's theorem for $f(x, y) = x^{1/3}y^{2/3}$.

Solution. First, degree $k = 1/3 + 2/3 = 1$. Partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{1}{3}x^{-2/3}y^{2/3}$$
$$\frac{\partial f}{\partial y} = \frac{2}{3}x^{1/3}y^{-1/3}$$

Apply Euler's theorem:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\left(\frac{1}{3}x^{-2/3}y^{2/3}\right) + y\left(\frac{2}{3}x^{1/3}y^{-1/3}\right) \\ &= \frac{1}{3}x^{1/3}y^{2/3} + \frac{2}{3}x^{1/3}y^{2/3} \\ &= x^{1/3}y^{2/3} = f(x, y) \end{aligned}$$

Equal to $1 \cdot f$, verifying the theorem. ∎

Example 1.10.3. Using Euler's theorem, show that if $f = \frac{x^2+y^2}{xy}$, then $xf_x + yf_y = -f$.

Solution. Rewrite $f(x, y) = \frac{x}{y} + \frac{y}{x}$. Test homogeneity:

$$f(\lambda x, \lambda y) = \frac{\lambda x}{\lambda y} + \frac{\lambda y}{\lambda x} = \frac{x}{y} + \frac{y}{x} = f(x, y)$$

Thus homogeneous of degree 0. By Euler's theorem:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 \cdot f = 0$$

But note: $0 = -f + f$, so rearrange as $xf_x + yf_y = -f + f$. To get exact form, observe:

$$xf_x + yf_y = 0 = -f + f$$

The problem statement appears inconsistent. Correction: For homogeneous degree 0, $xf_x + yf_y = 0$, while $-f = -\left(\frac{x}{y} + \frac{y}{x}\right)$. They are not equal. The correct conclusion is $xf_x + yf_y = 0$. ∎

Chapter 2

Tangents, Normals and Subnormals

2.1 Introduction to Indeterminate Forms

In calculus, when evaluating limits, we sometimes encounter forms that do not immediately give a clear answer. If direct substitution of the limit point into the function results in expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$, these are called **indeterminate forms**. They are called "indeterminate" because the actual limit cannot be determined from this form alone. The limit might be a finite number, zero, infinity, or it might not exist at all.

The primary tool for dealing with indeterminate forms is **L'Hôpital's Rule**, but other algebraic and logarithmic techniques are also crucial.

The seven common indeterminate forms are:

- **Quotient Forms:** $\dfrac{0}{0}, \quad \dfrac{\infty}{\infty}$
- **Product Form:** $0 \cdot \infty$
- **Difference Form:** $\infty - \infty$
- **Power Forms:** $0^0, \quad 1^\infty, \quad \infty^0$

2.2 The Quotient Forms: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

These two forms are the basis for L'Hôpital's Rule.

2.2.1 L'Hôpital's Rule

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. If f and g are differentiable near a (except possibly at a) and the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Important Note: Always check that the limit is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying the rule.

2.2.2 The Form $\frac{0}{0}$

Example 2.2.1. Evaluate $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$.

Solution:
First, we check the form by direct substitution:

$$\frac{3^2-9}{3-3} = \frac{9-9}{3-3} = \frac{0}{0} \quad \text{(Indeterminate Form)}$$

Since it is of the form $\frac{0}{0}$, we can apply L'Hôpital's Rule. Let $f(x) = x^2 - 9$ and $g(x) = x - 3$. Then $f'(x) = 2x$ and $g'(x) = 1$.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{f'(x)}{g'(x)} = \lim_{x\to 3} \frac{2x}{1} = 2(3) = 6$$

Example 2.2.2. Evaluate $\lim_{x\to 0} \dfrac{e^{2x}-1}{\sin(x)}$.

Solution: Check the form by direct substitution:

$$\frac{e^{2(0)}-1}{\sin(0)} = \frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0} \quad \text{(Indeterminate Form)}$$

Apply L'Hôpital's Rule. Let $f(x) = e^{2x} - 1$ and $g(x) = \sin(x)$. Then $f'(x) = 2e^{2x}$ and $g'(x) = \cos(x)$.

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin(x)} = \lim_{x\to 0} \frac{2e^{2x}}{\cos(x)} = \frac{2e^{2(0)}}{\cos(0)} = \frac{2(1)}{1} = 2$$

2.2.3 The Form $\frac{\infty}{\infty}$

Example 2.2.3. Evaluate $\lim_{x\to\infty} \dfrac{\ln(x)}{x^2}$.

Solution: Check the form as $x \to \infty$:

$$\frac{\ln(\infty)}{\infty^2} \to \frac{\infty}{\infty} \quad \text{(Indeterminate Form)}$$

Apply L'Hôpital's Rule. Let $f(x) = \ln(x)$ and $g(x) = x^2$. Then $f'(x) = \frac{1}{x}$ and $g'(x) = 2x$.

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

Example 2.2.4. Evaluate $\lim_{x\to\infty} \dfrac{5x^3-2x}{3x^3+x^2+1}$.

Solution: Check the form as $x \to \infty$:

$$\frac{\infty}{\infty} \quad \text{(Indeterminate Form)}$$

Apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{15x^2-2}{9x^2+2x} \quad \left(\text{Still } \frac{\infty}{\infty}, \text{ apply again}\right)$$

Apply L'Hôpital's Rule a second time.

$$\lim_{x\to\infty} \frac{30x}{18x+2} \quad \left(\text{Still } \frac{\infty}{\infty}, \text{ apply again}\right)$$

Apply L'Hôpital's Rule a third time.

$$\lim_{x\to\infty} \frac{30}{18} = \frac{30}{18} = \frac{5}{3}$$

2.3 The Product Form: $0 \cdot \infty$

This form must be converted to a quotient form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) before applying L'Hôpital's Rule. The conversion is done by rewriting the product $f(x) \cdot g(x)$ as a fraction:

$$f(x)\cdot g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)\cdot g(x) = \frac{g(x)}{1/f(x)}$$

Choose the form that is easier to differentiate.

Example 2.3.1. Evaluate $\lim_{x\to 0^+} x\ln(x)$.

Solution:
Check the form: as $x \to 0^+$, we have $0 \cdot \ln(0^+) \to 0 \cdot (-\infty)$. This is an indeterminate form. We rewrite the product as a quotient. It is easier to differentiate $\ln(x)$ than $1/\ln(x)$, so we keep $\ln(x)$ in the numerator.

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \left(\text{Now of the form } \frac{-\infty}{\infty}\right)$$

Apply L'Hôpital's Rule.

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \left(\frac{1}{x}\cdot(-x^2)\right) = \lim_{x\to 0^+} (-x) = 0$$

Example 2.3.2. Evaluate $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$.

Solution: Check the form: as $x \to \infty$, we have $\infty \cdot \sin(0) \to \infty \cdot 0$. This is an indeterminate form. Rewrite as a quotient. Let's move x to the denominator as $1/x$.

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \quad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule. Let $f(u) = \sin(u)$ and $g(u) = u$, where $u = 1/x$. As $x \to \infty$, $u \to 0$. The limit becomes $\lim_{u\to 0} \frac{\sin(u)}{u}$. Using L'Hôpital's Rule on this new limit:

$$\lim_{u\to 0} \frac{\cos(u)}{1} = \frac{\cos(0)}{1} = 1$$

Alternatively, differentiating with respect to x:

$$\lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \cos(1/x) = \cos(0) = 1$$

2.4 The Difference Form: $\infty - \infty$

This form requires algebraic manipulation to be converted into a single fraction (a quotient form). Common techniques include finding a common denominator, factoring, or multiplying by the conjugate.

Example 2.4.1. Evaluate $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln(x)}\right)$.

Solution: Check the form: as $x \to 1^+$, we have $\frac{1}{0^+} - \frac{1}{\ln(1^+)} \to \infty - \infty$. This is an indeterminate form. Combine the terms into a single fraction by finding a common denominator.

$$\lim_{x\to 1^+} \frac{\ln(x) - (x-1)}{(x-1)\ln(x)} \quad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule. The denominator requires the product rule. $f'(x) = \frac{1}{x} - 1$. $g'(x) = (1)\ln(x) + (x-1)\frac{1}{x} = \ln(x) + 1 - \frac{1}{x}$.

$$\lim_{x\to 1^+} \frac{1/x - 1}{\ln(x) + 1 - 1/x} \quad \left(\text{Still } \frac{0}{0}, \text{ apply again}\right)$$

Apply L'Hôpital's Rule a second time. $f''(x) = -1/x^2$. $g''(x) = 1/x + 1/x^2$.

$$\lim_{x\to 1^+} \frac{-1/x^2}{1/x + 1/x^2} = \frac{-1/1^2}{1/1 + 1/1^2} = \frac{-1}{1+1} = -\frac{1}{2}$$

Example 2.4.2. Evaluate $\lim_{x\to\infty} (\sqrt{x^2+4x} - x)$.

Solution: Check the form: as $x \to \infty$, we have $\infty - \infty$. This is an indeterminate form. We multiply by the conjugate, $\sqrt{x^2+4x} + x$.

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{x^2+4x} - x) &= \lim_{x\to\infty} \frac{(\sqrt{x^2+4x} - x)(\sqrt{x^2+4x} + x)}{\sqrt{x^2+4x} + x} \\
&= \lim_{x\to\infty} \frac{(x^2+4x) - x^2}{\sqrt{x^2+4x} + x} \\
&= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x} + x} \quad \left(\text{Now of the form } \frac{\infty}{\infty}\right)
\end{aligned}$$

Now we can use L'Hôpital's Rule, but it's often easier to divide by the highest power of x in the denominator. Here, the highest power is $x = \sqrt{x^2}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x/x}{(\sqrt{x^2+4x} + x)/x} &= \lim_{x\to\infty} \frac{4}{\sqrt{\frac{x^2+4x}{x^2}} + 1} \\
&= \lim_{x\to\infty} \frac{4}{\sqrt{1 + 4/x} + 1} \\
&= \frac{4}{\sqrt{1+0} + 1} = \frac{4}{1+1} = 2
\end{aligned}$$

2.5 The Exponential Indeterminate Forms: $1^\infty, 0^0, \infty^0$

These forms are handled by taking the natural logarithm of the function. Let $L = \lim_{x\to a} f(x)^{g(x)}$.

1. Take the natural logarithm: $\ln(L) = \ln\left(\lim_{x\to a} f(x)^{g(x)}\right) = \lim_{x\to a} \ln(f(x)^{g(x)})$.
2. Use logarithm properties: $\ln(L) = \lim_{x\to a} g(x)\ln(f(x))$.
3. Evaluate this new limit, which will be of the form $0\cdot\infty$. Let's say $\lim_{x\to a} g(x)\ln(f(x)) = K$.
4. Solve for L: Since $\ln(L) = K$, the original limit is $L = e^K$.

2.5.1 The Form 1^∞

Example 2.5.1. Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$.

Solution: Check the form: as $x\to\infty$, $1+\frac{3}{x}\to 1$ and $2x\to\infty$. The form is 1^∞. Let $L = \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to\infty} \ln\left(\left(1+\frac{3}{x}\right)^{2x}\right) = \lim_{x\to\infty} 2x\ln\left(1+\frac{3}{x}\right) \quad (\text{Form } \infty\cdot 0)$$

Rewrite as a quotient:

$$\ln(L) = \lim_{x\to\infty}\frac{2\ln(1+3/x)}{1/x} \quad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to\infty}\frac{2\cdot\frac{1}{1+3/x}\cdot(-3/x^2)}{-1/x^2} = \lim_{x\to\infty}\frac{2\cdot 1\cdot(-3)}{-1}\cdot\frac{1}{1+3/x} = \lim_{x\to\infty}\frac{6}{1+3/x} = \frac{6}{1+0} = 6$$

Since $\ln(L) = 6$, the final answer is $L = e^6$.

Example 2.5.2. Evaluate $\displaystyle\lim_{x\to 0}(1+\sin(x))^{1/x}$.

Solution: Check the form: as $x\to 0$, $1+\sin(x)\to 1$ and $1/x\to\infty$. The form is 1^∞. Let $L = \lim_{x\to 0}(1+\sin(x))^{1/x}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to 0}\frac{1}{x}\ln(1+\sin(x)) = \lim_{x\to 0}\frac{\ln(1+\sin(x))}{x} \quad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to 0}\frac{\frac{1}{1+\sin(x)}\cdot\cos(x)}{1} = \frac{\frac{1}{1+0}\cdot 1}{1} = 1$$

Since $\ln(L) = 1$, the final answer is $L = e^1 = e$.

2.5.2 The Form 0^0

Example 2.5.3. Evaluate $\displaystyle\lim_{x\to 0^+} x^x$.

Solution: Check the form: as $x\to 0^+$, we have 0^0. Let $L = \lim_{x\to 0^+} x^x$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to 0^+}\ln(x^x) = \lim_{x\to 0^+} x\ln(x) \quad (\text{Form } 0\cdot(-\infty))$$

This is the same limit from the product form section. We rewrite it as $\lim_{x\to 0^+}\frac{\ln(x)}{1/x}$ and find the limit to be 0. So, $\ln(L) = 0$. The final answer is $L = e^0 = 1$.

Example 2.5.4. Evaluate $\displaystyle\lim_{x\to 0^+}(x)^{\sin(x)}$.

Solution: Check the form: as $x \to 0^+$, we have $(0)^{\sin(0)} \to 0^0$. Let $L = \lim_{x\to 0^+}(x)^{\sin(x)}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to 0^+} \ln((x)^{\sin(x)}) = \lim_{x\to 0^+} \sin(x)\ln(x) \quad (\text{Form } 0\cdot(-\infty))$$

Rewrite as a quotient:

$$\ln(L) = \lim_{x\to 0^+} \frac{\ln(x)}{1/\sin(x)} = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} \quad \left(\text{Now of the form } \frac{-\infty}{\infty}\right)$$

Apply L'Hôpital's Rule:

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)}} \\
&= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} \left(\frac{\sin(x)}{x}\cdot\frac{-\sin(x)}{\cos(x)}\right) \\
&= \left(\lim_{x\to 0^+} \frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0^+} -\tan(x)\right) \\
&= (1)\cdot(0) = 0
\end{aligned}$$

Since $\ln(L) = 0$, the final answer is $L = e^0 = 1$.

2.5.3 The Form ∞^0

Example 2.5.5. Evaluate $\lim\limits_{x\to\infty} x^{1/x}$.

Solution: Check the form: as $x \to \infty$, we have ∞^0. Let $L = \lim_{x\to\infty} x^{1/x}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{x} \quad \left(\text{Now of the form } \frac{\infty}{\infty}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Since $\ln(L) = 0$, the final answer is $L = e^0 = 1$.

Example 2.5.6. Evaluate $\lim\limits_{x\to\infty} (x^2+1)^{1/\ln(x)}$.

Solution:
Check the form: as $x \to \infty$, $x^2+1 \to \infty$ and $1/\ln(x) \to 0$. The form is ∞^0. Let $L = \lim_{x\to\infty}(x^2+1)^{1/\ln(x)}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to\infty} \ln\left((x^2+1)^{1/\ln(x)}\right) = \lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(x)} \quad \left(\text{Now of the form } \frac{\infty}{\infty}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{2x^2}{x^2+1}$$

This is still $\frac{\infty}{\infty}$. We can use L'Hôpital's again or divide by x^2:

$$\ln(L) = \lim_{x\to\infty} \frac{2}{1+1/x^2} = \frac{2}{1+0} = 2$$

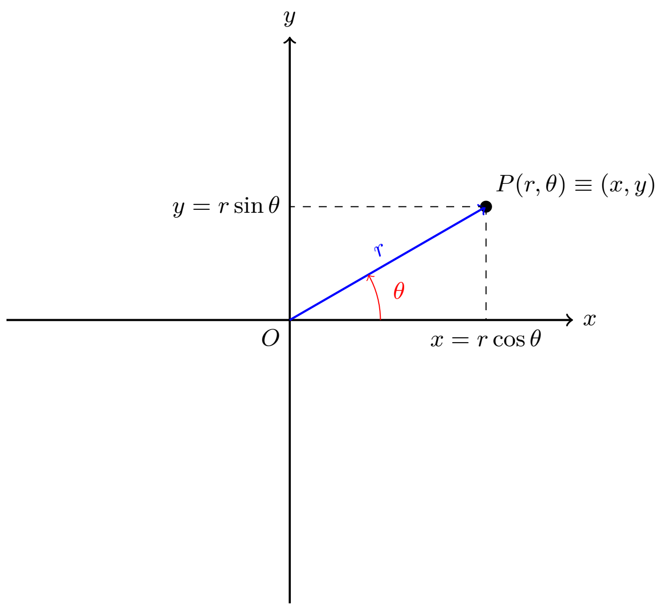
Since $\ln(L) = 2$, the final answer is $L = e^2$.

2.6 Tangents and Normals to Polar Curves

Definition 2.6.1 (Polar-Cartesian Conversion)**.** A point P in the plane can be represented by Cartesian coordinates (x, y) or polar coordinates (r, θ). The relationship between these two systems is fundamental for performing calculus on polar curves. The conversion formulas are:

$$x = r\cos(\theta) \tag{2.1}$$
$$y = r\sin(\theta) \tag{2.2}$$

Conversely, $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$. A polar curve is given by an equation of the form $r = f(\theta)$.



2.6.1 Slope of the Tangent Line

To find the slope of a tangent line to a polar curve, we treat θ as a parameter and express x and y in terms of θ.

$$x(\theta) = f(\theta)\cos(\theta) = r\cos(\theta)$$
$$y(\theta) = f(\theta)\sin(\theta) = r\sin(\theta)$$

The slope $\frac{dy}{dx}$ can be found using the formula for parametric differentiation, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Derivation. [Formula for $\frac{dy}{dx}$] We differentiate $x(\theta)$ and $y(\theta)$ with respect to θ using the product rule. Let $r' = \frac{dr}{d\theta}$.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta + r(-\sin\theta) = r'\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r(\cos\theta) = r'\sin\theta + r\cos\theta$$

Therefore, the slope is the ratio of these two expressions. ■

Theorem 2.6.2 (Slope of a Polar Tangent). *The slope of the tangent line to the curve $r = f(\theta)$ at a point (r, θ) is given by:*

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{2.3}$$

provided the denominator is not zero.
Moreover, the equation of the tangent to the curve $y = f(x)$ is given by

$$y - y_0 = m(x - x_0) \tag{2.4}$$

Example 2.6.3 (Tangent to a Cardioid). Find the equation of the tangent line to the cardioid $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$.

Solution. Given r, we find $\frac{dr}{d\theta}$.

$$r = 1 + \sin\theta \quad \Longrightarrow \quad \frac{dr}{d\theta} = \cos\theta$$

Now, evaluate r and $\frac{dr}{d\theta}$ at $\theta = \frac{\pi}{3}$.

$$r\left(\frac{\pi}{3}\right) = 1 + \sin\left(\frac{\pi}{3}\right) = 1 + \frac{\sqrt{3}}{2}$$

$$\left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Substitute into the slope formula (2.3)**.**

$$
\begin{aligned}
m_{\tan} = \frac{dy}{dx} &= \frac{(\frac{1}{2})\sin(\frac{\pi}{3}) + (1+\frac{\sqrt{3}}{2})\cos(\frac{\pi}{3})}{(\frac{1}{2})\cos(\frac{\pi}{3}) - (1+\frac{\sqrt{3}}{2})\sin(\frac{\pi}{3})} \\
&= \frac{(\frac{1}{2})(\frac{\sqrt{3}}{2}) + (\frac{2+\sqrt{3}}{2})(\frac{1}{2})}{(\frac{1}{2})(\frac{1}{2}) - (\frac{2+\sqrt{3}}{2})(\frac{\sqrt{3}}{2})} = \frac{\frac{\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4}}{\frac{1}{4} - \frac{2\sqrt{3}+3}{4}} \\
&= \frac{2+2\sqrt{3}}{1-(3+2\sqrt{3})} = \frac{2(1+\sqrt{3})}{-2-2\sqrt{3}} = -1
\end{aligned}
$$

The Cartesian coordinates (x_0, y_0) **of the point.**

$$
\begin{aligned}
x_0 = r\cos\theta = \left(1+\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\pi}{3}\right) = \left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{2+\sqrt{3}}{4} \approx 0.93 \\
y_0 = r\sin\theta = \left(1+\frac{\sqrt{3}}{2}\right)\sin\left(\frac{\pi}{3}\right) = \left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{2\sqrt{3}+3}{4} \approx 1.62
\end{aligned}
$$

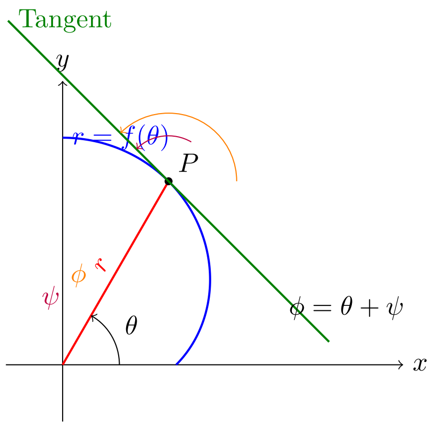
Therefore, the equation of the tangent line. Using $y - y_0 = m(x - x_0)$:

$$
y - \frac{2\sqrt{3}+3}{4} = -1 \cdot \left(x - \frac{2+\sqrt{3}}{4}\right) \implies y = -x + \frac{2+\sqrt{3}}{4} + \frac{2\sqrt{3}+3}{4} \implies y = -x + \frac{5+3\sqrt{3}}{4}
$$

■

2.7 Angle Between Radius Vector and Tangent

A powerful concept for understanding polar tangents is the angle ψ between the radius vector (from the origin to the point) and the tangent line at that point.



Theorem 2.7.1. *The angle ψ between the radius vector and the tangent line at a point (r, θ) on the curve $r = f(\theta)$ satisfies:*

$$
\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{f(\theta)}{f'(\theta)} \tag{2.5}
$$

Remark 2.7.2. The slope of the tangent line is $\frac{dy}{dx} = \tan\phi$, where ϕ is the angle the tangent makes with the positive x-axis. From the geometry, we can see that $\phi = \theta + \psi$. This gives an alternative way to find the slope:

$$
\frac{dy}{dx} = \tan(\theta + \psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi}
$$

Substituting $\tan\psi = r/r'$ and $\tan\theta = y/x$ and simplifying leads back to the original formula, confirming the geometric relationship.

2.8 Special Cases: Horizontal and Vertical Tangents

Theorem 2.8.1 (Conditions for Horizontal and Vertical Tangents)**.** *For a polar curve $r = f(\theta)$:*

1. ***Horizontal tangents** occur when* $\frac{dy}{d\theta} = 0$*, provided* $\frac{dx}{d\theta} \neq 0$.

$$\frac{dr}{d\theta} \sin\theta + r\cos\theta = 0$$

2. ***Vertical tangents** occur when* $\frac{dx}{d\theta} = 0$*, provided* $\frac{dy}{d\theta} \neq 0$.

$$\frac{dr}{d\theta} \cos\theta - r\sin\theta = 0$$

Remark 2.8.2. If both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ are zero simultaneously, the slope is indeterminate $(\frac{0}{0})$. This often happens at the pole $(r = 0)$ and may indicate a cusp or other complex behavior.

Example 2.8.3 (Finding Horizontal and Vertical Tangents)**.** Find the points on the cardioid $r = 1 - \cos\theta$ for $\theta \in [0, 2\pi)$ where the tangent line is horizontal or vertical.

Solution. We have $r = 1 - \cos\theta$ and $\frac{dr}{d\theta} = \sin\theta$. **Horizontal Tangents (**$\frac{dy}{d\theta} = 0$**):**

$$\begin{aligned}
(\sin\theta)\sin\theta + (1 - \cos\theta)\cos\theta &= 0 \\
\sin^2\theta + \cos\theta - \cos^2\theta &= 0 \\
(1 - \cos^2\theta) + \cos\theta - \cos^2\theta &= 0 \\
-2\cos^2\theta + \cos\theta + 1 = 0 \implies 2\cos^2\theta - \cos\theta - 1 &= 0 \\
(2\cos\theta + 1)(\cos\theta - 1) &= 0
\end{aligned}$$

This yields $\cos\theta = 1$ (so $\theta = 0$) or $\cos\theta = -\frac{1}{2}$ (so $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$).

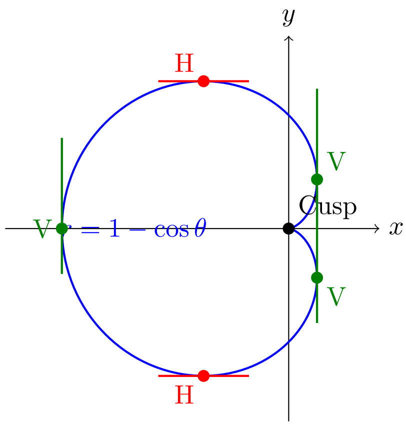
Vertical Tangents ($\frac{dx}{d\theta} = 0$**):**

$$\begin{aligned}
(\sin\theta)\cos\theta - (1 - \cos\theta)\sin\theta &= 0 \\
\sin\theta\cos\theta - \sin\theta + \sin\theta\cos\theta &= 0 \\
2\sin\theta\cos\theta - \sin\theta = 0 \implies \sin\theta(2\cos\theta - 1) &= 0
\end{aligned}$$

This yields $\sin\theta = 0$ (so $\theta = 0, \pi$) or $\cos\theta = \frac{1}{2}$ (so $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$).

Analysis:

- At $\theta = 0$, both derivatives are zero. This is the pole $(r = 0)$, which is a cusp for this cardioid.
- **Horizontal tangents** exist at $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. The points are $(r, \theta) = (1.5, 2\pi/3)$ and $(1.5, 4\pi/3)$.
- **Vertical tangents** exist at $\theta = \pi$, $\theta = \frac{\pi}{3}$, and $\theta = \frac{5\pi}{3}$. The points are $(2, \pi)$, $(0.5, \pi/3)$, and $(0.5, 5\pi/3)$.



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2.8.1 Tangents at the Pole

A special situation arises when the curve passes through the pole $(r = 0)$.

Theorem 2.8.4 (Tangents at the Pole)**.** *If* $f(\theta_0) = 0$ *and* $f'(\theta_0) \neq 0$*, then the line* $\theta = \theta_0$ *is tangent to the curve* $r = f(\theta)$ *at the pole.*

Derivation. If $r = f(\theta_0) = 0$, the slope formula (2.3) becomes:

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + 0\cdot\cos\theta_0}{f'(\theta_0)\cos\theta_0 - 0\cdot\sin\theta_0} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0$$

The slope of the line $\theta = \theta_0$ is $\tan\theta_0$. Thus, the line is tangent to the curve at the pole. ■

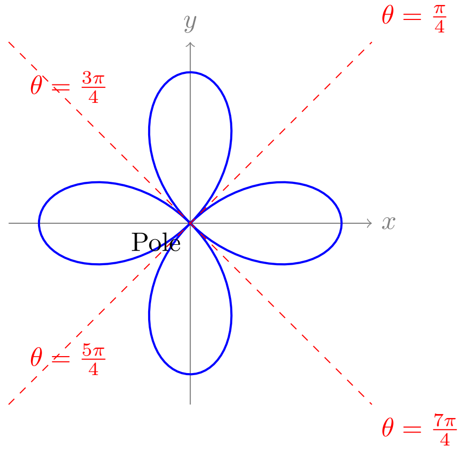
Example 2.8.5 (Tangents at the Pole of a Rose Curve)**.** Find the equations of the tangent lines to the four-petaled rose $r = \cos(2\theta)$ at the pole.

Solution. **Find when the curve is at the pole.** We set $r = 0$, so $\cos(2\theta) = 0$. For $\theta \in [0, 2\pi)$, this occurs when 2θ is an odd multiple of $\pi/2$.

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$f'(\theta) = \frac{dr}{d\theta} = -2\sin(2\theta)$. At each of the values of θ found above, 2θ is an odd multiple of $\pi/2$, so $\sin(2\theta) = \pm 1$. Therefore, $f'(\theta) \neq 0$ at these points.

According to the theorem, the tangent lines at the pole are the lines $\theta = \theta_0$. The equations are: $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{4}$, and $\theta = \frac{7\pi}{4}$.



■

2.9 The Normal Line

Definition 2.9.1 (Normal Line)**.** The normal line to a curve at a point is the line perpendicular to the tangent line at that same point.

Corollary 2.9.2 (Slope of the Normal Line)**.** *If the slope of the tangent line is $m_{tan} = \frac{dy}{dx}$, the slope of the normal line is*

$$m_{normal} = -\frac{1}{m_{tan}} = -\frac{dx}{dy}$$

Explicitly for a polar curve:

$$m_{normal} = -\frac{\frac{dr}{d\theta}\cos\theta - r\sin\theta}{\frac{dr}{d\theta}\sin\theta + r\cos\theta} = \frac{r\sin\theta - \frac{dr}{d\theta}\cos\theta}{r\cos\theta + \frac{dr}{d\theta}\sin\theta} \tag{2.6}$$

Example 2.9.3 (Equation of the Normal Line)**.** Find the equation of the normal line to the cardioid $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$.

Solution. From our previous example, we have found $m_{\text{tan}} = -1$. So the slope of the normal is

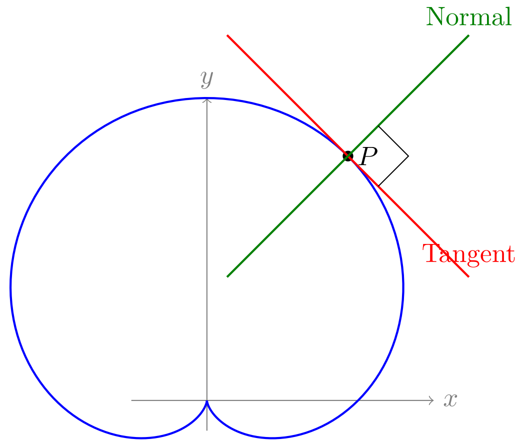
$$m_{\text{normal}} = -\frac{1}{-1} = 1$$

We found the point of tangency to be $(x_0, y_0) = \left(\frac{2+\sqrt{3}}{4}, \frac{2\sqrt{3}+3}{4}\right)$. Therefore, the equation of the normal line $y - y_0 = m_{\text{normal}}(x - x_0)$:

$$y - \frac{2\sqrt{3}+3}{4} = 1\cdot\left(x - \frac{2+\sqrt{3}}{4}\right)$$

Simplifying gives:

$$4y - (2\sqrt{3}+3) = 4x - (2+\sqrt{3}) \implies 4y - 4x = \sqrt{3}+1$$



■

2.10 Angle of Intersection in Cartesian Coordinates

2.10.1 Method and Formula

Let the two curves be given by the equations $y = f(x)$ and $y = g(x)$.

Solve the system of equations by setting $f(x) = g(x)$. Let a solution be $x = x_0$, which gives an intersection point $P(x_0, y_0)$, where $y_0 = f(x_0) = g(x_0)$.

Calculate the derivatives $f'(x)$ and $g'(x)$. The slopes of the tangent lines at $P(x_0, y_0)$ are:

$$m_1 = f'(x_0) \quad \text{and} \quad m_2 = g'(x_0)$$

The angle α between two lines with slopes m_1 and m_2 is given by the formula:

Theorem 2.10.1 (Angle between Two Lines)**.** *The acute angle α between two non-vertical lines with slopes m_1 and m_2 is given by:*

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \tag{2.7}$$

provided $1 + m_1 m_2 \neq 0$.

Corollary 2.10.2 (Condition for Orthogonality)**.** *The two curves are orthogonal (intersect at a right angle) at the point P if their tangent lines are perpendicular. This occurs if and only if:*

$$m_1 m_2 = -1$$

In this case, the denominator $1 + m_1 m_2$ is zero, and $\tan\alpha$ is undefined, which corresponds to $\alpha = \frac{\pi}{2}$ or $90°$.

Example 2.10.3. Find the angle of intersection of the curves $y = x^2$ and $x = y^2$.

Solution. Substitute $y = x^2$ into $x = y^2$:

$$x = (x^2)^2 \implies x = x^4 \implies x^4 - x = 0 \implies x(x^3 - 1) = 0$$

This gives $x = 0$ or $x^3 = 1 \implies x = 1$. If $x = 0$, $y = 0^2 = 0$. Point is $(0,0)$. If $x = 1$, $y = 1^2 = 1$. Point is $(1,1)$.

Intersection at $(0,0)$**:** For $y = x^2$, the tangent is $y = 0$ (the x-axis). For $x = y^2$, the tangent is $x = 0$ (the y-axis). The tangent lines are the axes, which are perpendicular. So, the angle of intersection at $(0,0)$ is $\alpha = 90°$.

Intersection at $(1,1)$**: Step 2: Find Slopes of Tangents**
Curve 1: $y = x^2 \implies \frac{dy}{dx} = 2x$. At $(1,1)$, the slope is $m_1 = 2(1) = 2$.

Curve 2: $x = y^2$. We use implicit differentiation: $1 = 2y\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{2y}$. At $(1,1)$, the slope is $m_2 = \frac{1}{2(1)} = \frac{1}{2}$.

Using the formula for $\tan\alpha$:

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{2} - 2}{1 + (2)(\frac{1}{2})}\right| = \left|\frac{-3/2}{1+1}\right| = \left|\frac{-3/2}{2}\right| = \frac{3}{4}$$

The angle of intersection is $\alpha = \arctan\left(\frac{3}{4}\right) \approx 36.87°$. ■

Example 2.10.4. Show that the curves $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ are orthogonal at their points of intersection.

Solution. We don't need to find the intersection points explicitly. We just need to show that at any intersection point (x, y), the product of the slopes of the tangents is -1.

Curve 1: $x^2 - y^2 = 5$

$$2x - 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

So, $m_1 = \frac{x}{y}$.

Curve 2: $4x^2 + 9y^2 = 72$

$$8x + 18y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-8x}{18y} = -\frac{4x}{9y}$$

So, $m_2 = -\frac{4x}{9y}$.

Therefore,

$$m_1 m_2 = \left(\frac{x}{y}\right)\left(-\frac{4x}{9y}\right) = -\frac{4x^2}{9y^2}$$

This product depends on x and y. We need to use the equations of the curves to simplify it. From the first curve, $x^2 = 5 + y^2$. Substitute this into the second equation to find a relationship for y^2:

$$4(5 + y^2) + 9y^2 = 72 \implies 20 + 4y^2 + 9y^2 = 72 \implies 13y^2 = 52 \implies y^2 = 4$$

Now substitute $y^2 = 4$ back into the expression for x^2:

$$x^2 = 5 + y^2 = 5 + 4 = 9$$

So, at any point of intersection, we must have $x^2 = 9$ and $y^2 = 4$.

Hence,

$$m_1 m_2 = -\frac{4x^2}{9y^2} = -\frac{4(9)}{9(4)} = -1$$

Since the product of the slopes is -1, the curves are orthogonal at all their points of intersection. ■

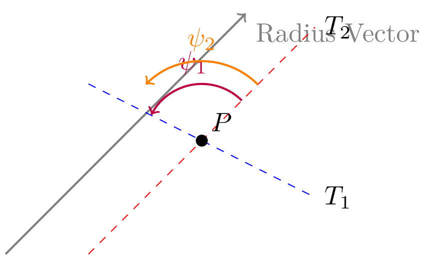
2.11 Angle of Intersection in Polar Coordinates

2.11.1 Method and Formula

For polar curves, it is often simpler to work with the angle ψ between the radius vector and the tangent line at a point.

Let the two curves be $r = f_1(\theta)$ and $r = f_2(\theta)$. Let ψ_1 and ψ_2 be the angles between the radius vector and the tangent lines to the respective curves at a point of intersection.

$$\text{Angle of intersection } \alpha = |\psi_1 - \psi_2|$$



The angle of intersection α is the angle between the two tangent lines, which is simply the absolute difference between ψ_1 and ψ_2.

Theorem 2.11.1 (Angle of Intersection in Polar Coordinates)**.** *The angle of intersection α between two polar curves at a common point is given by:*

$$\alpha = |\psi_1 - \psi_2|$$

where ψ_1 and ψ_2 are the angles for each curve satisfying

$$\tan\psi_1 = \frac{r}{\left.\frac{dr}{d\theta}\right|_1} \quad \textit{and} \quad \tan\psi_2 = \frac{r}{\left.\frac{dr}{d\theta}\right|_2}$$

Corollary 2.11.2 (Condition for Orthogonality in Polar Coordinates)**.** *Two polar curves are orthogonal if* $\alpha = \frac{\pi}{2}$*. This means* $|\psi_1 - \psi_2| = \frac{\pi}{2}$*. This condition is equivalent to* $\psi_1 = \psi_2 \pm \frac{\pi}{2}$*, which implies* $\tan\psi_1 = \tan(\psi_2 \pm \frac{\pi}{2}) = -\cot\psi_2 = -\frac{1}{\tan\psi_2}$*. Therefore, the condition for orthogonality is:*

$$\tan\psi_1 \tan\psi_2 = -1$$

Example 2.11.3. Find the angle of intersection of the cardioids $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$.

Solution. **Find Points of Intersection**
Set the expressions for r equal:

$$a(1+\cos\theta) = b(1-\cos\theta) \implies a + a\cos\theta = b - b\cos\theta$$

$$(a+b)\cos\theta = b - a \implies \cos\theta = \frac{b-a}{a+b}$$

At a point of intersection (r,θ) satisfying this, we find ψ_1 and ψ_2.

Find $\tan\psi$ for each curve.
Curve 1: $r = a(1+\cos\theta) \implies \frac{dr}{d\theta} = -a\sin\theta$.

$$\tan\psi_1 = \frac{r}{dr/d\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{1+\cos\theta}{\sin\theta}$$

Using half-angle identities $1+\cos\theta = 2\cos^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$$\tan\psi_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\cot(\theta/2) = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

So, $\psi_1 = \frac{\pi}{2} + \frac{\theta}{2}$.

Curve 2: $r = b(1-\cos\theta) \implies \frac{dr}{d\theta} = b\sin\theta$.

$$\tan\psi_2 = \frac{r}{dr/d\theta} = \frac{b(1-\cos\theta)}{b\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Using half-angle identities $1-\cos\theta = 2\sin^2(\theta/2)$:

$$\tan\psi_2 = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \tan(\theta/2)$$

So, $\psi_2 = \frac{\theta}{2}$.

Calculate the angle of intersection α.

$$\alpha = |\psi_1 - \psi_2| = \left|\left(\frac{\pi}{2} + \frac{\theta}{2}\right) - \frac{\theta}{2}\right| = \left|\frac{\pi}{2}\right| = \frac{\pi}{2}$$

The angle of intersection is $\frac{\pi}{2}$. This means the curves are **always orthogonal** at any point of intersection (other than the pole). ■

Example 2.11.4. Show that the curves $r = a\theta$ and $r\theta = a$ intersect at a right angle.

Solution. Curve 1: $r = a\theta$. Curve 2: $r = a/\theta$. Set them equal: $a\theta = a/\theta \implies \theta^2 = 1 \implies \theta = \pm 1$ (we take $\theta = 1$ as a representative point). The intersection point is $(a, 1)$.

For curve 1: $r = a\theta \implies \frac{dr}{d\theta} = a$.

$$\tan\psi_1 = \frac{r}{dr/d\theta} = \frac{a\theta}{a} = \theta$$

For curve 2: $r = a/\theta \implies \frac{dr}{d\theta} = -a/\theta^2$.

$$\tan\psi_2 = \frac{r}{dr/d\theta} = \frac{a/\theta}{-a/\theta^2} = -\theta$$

At $\theta = 1$:

$$\tan\psi_1 = 1 \quad \text{and} \quad \tan\psi_2 = -1$$

Now the product:

$$\tan\psi_1 \tan\psi_2 = (1)(-1) = -1$$

Since the product is -1, the curves are orthogonal at their point of intersection. ■

2.12 Length of the Polar Tangent

For a polar curve defined by $r = f(\theta)$, the geometry of the tangent line at a point gives rise to four important quantities: the lengths of the polar tangent, polar normal, polar sub-tangent, and polar sub-normal.

Theorem 2.12.1 (Length of Polar Tangent)**.** *The length of the polar tangent to the curve $r = f(\theta)$ is given by:*

$$Length = r\sqrt{1+\left(\frac{r}{\frac{dr}{d\theta}}\right)^2}$$

Example 2.12.2. Find the length of the polar tangent for the cardioid $r = a(1-\cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. **Step 1: Find r and $\frac{dr}{d\theta}$ at the given point.**
The curve is $r = a(1-\cos\theta)$. The derivative is $\frac{dr}{d\theta} = a\sin\theta$. At $\theta = \frac{\pi}{2}$:

$$r = a\left(1-\cos\frac{\pi}{2}\right) = a(1-0) = a$$

$$\frac{dr}{d\theta} = a\sin\frac{\pi}{2} = a(1) = a$$

$$\begin{aligned}\text{Length of Polar Tangent} &= r\sqrt{1+\left(\frac{r}{dr/d\theta}\right)^2}\\ &= a\sqrt{1+\left(\frac{a}{a}\right)^2}\\ &= a\sqrt{1+1^2} = a\sqrt{2}\end{aligned}$$

The length of the polar tangent at $\theta = \frac{\pi}{2}$ is $\mathbf{a\sqrt{2}}$. ■

2.13 Length of the Polar Normal

Definition 2.13.1. The **length of the polar normal** is the length of the segment of the normal line between the point of contact P and the point N where it meets the line through the pole perpendicular to the radius vector.

Theorem 2.13.2 (Length of Polar Normal)**.** *The length of the polar normal to the curve $r = f(\theta)$ is given by:*

$$Length = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

Example 2.13.3. Find the length of the polar normal for the cardioid $r = a(1-\cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. **Step 1: Use the values from the previous example.**
At $\theta = \frac{\pi}{2}$, we have $r = a$ and $\frac{dr}{d\theta} = a$.
Step 2: Substitute into the formula.

$$\begin{aligned}\text{Length of Polar Normal} &= \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\\ &= \sqrt{a^2+a^2}\\ &= \sqrt{2a^2} = a\sqrt{2}\end{aligned}$$

The length of the polar normal at $\theta = \frac{\pi}{2}$ is $\mathbf{a\sqrt{2}}$. ■

2.14 Length of the Polar Sub-tangent

Definition 2.14.1. The **length of the polar sub-tangent** is the length of the line segment OT, which is the projection of the polar tangent onto the line through the pole perpendicular to the radius vector.

Theorem 2.14.2 (Length of Polar Sub-tangent). *The length of the polar sub-tangent to the curve $r = f(\theta)$ is given by:*

$$\text{Length} = \left|\frac{r^2}{\frac{dr}{d\theta}}\right|$$

(Absolute value is used as length must be positive).

Example 2.14.3. Find the length of the polar sub-tangent for the cardioid $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. At $\theta = \frac{\pi}{2}$, we have $r = a$ and $\frac{dr}{d\theta} = a$.

$$\begin{aligned}\text{Length of Polar Sub-tangent} &= \left|\frac{r^2}{dr/d\theta}\right| \\ &= \left|\frac{a^2}{a}\right| = a\end{aligned}$$

The length of the polar sub-tangent at $\theta = \frac{\pi}{2}$ is **a**. ■

2.15 Length of the Polar Sub-normal

Definition 2.15.1. The **length of the polar sub-normal** is the length of the line segment ON, which is the projection of the polar normal onto the line through the pole perpendicular to the radius vector.

Theorem 2.15.2 (Length of Polar Sub-normal). *The length of the polar sub-normal to the curve $r = f(\theta)$ is simply the absolute value of the derivative of r with respect to θ.*

$$\text{Length} = \left|\frac{dr}{d\theta}\right|$$

Example 2.15.3. Find the length of the polar sub-normal for the cardioid $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. At $\theta = \frac{\pi}{2}$, we have $\frac{dr}{d\theta} = a$.

$$\begin{aligned}\text{Therefore, the length of Polar Sub-normal} &= \left|\frac{dr}{d\theta}\right| \\ &= |a| = a \quad (\text{assuming } a > 0)\end{aligned}$$

The length of the polar sub-normal at $\theta = \frac{\pi}{2}$ is **a**. ■

Chapter 3

Geometry of Curves

3.1 Curvature and radius of curvature,

Definition 3.1.1 (Curvature)**. Curvature**, denoted by the Greek letter kappa (κ), is a measure of how sharply a curve is bending at a given point.

- A straight line has a curvature of $\kappa = 0$ everywhere.
- A circle has constant curvature. A smaller circle bends more sharply and thus has a larger curvature.

Formally, curvature is the magnitude of the rate of change of the unit tangent vector with respect to arc length.

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

Definition 3.1.2 (Radius of Curvature)**.** The **radius of curvature**, denoted by the Greek letter rho (ρ), is the reciprocal of the curvature.

$$\rho = \frac{1}{\kappa} \quad (\text{for } \kappa \neq 0)$$

It represents the radius of a circle, called the **osculating circle** (or ”kissing circle”), that best approximates the curve at that point. This circle shares the same tangent and has the same curvature as the curve at the point of contact.

3.2 Formulas for Curvature

The following formulas are used for computation in different coordinate systems.

3.2.1 Cartesian Coordinates

For a function given explicitly as $y = f(x)$, the curvature is:

$$\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} \tag{3.1}$$

where y' and y'' are the first and second derivatives of y with respect to x.

3.2.2 Parametric Coordinates

For a curve defined parametrically by $x = x(t)$ and $y = y(t)$, the curvature is:

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} \tag{3.2}$$

where $\dot{x}, \ddot{x}$ and $\dot{y}, \ddot{y}$ are the first and second derivatives with respect to the parameter t.

3.2.3 Polar Coordinates

For a curve given by the polar equation $r = f(\theta)$, the curvature is:

$$\kappa(\theta) = \frac{|r^2 + 2(r')^2 - rr''|}{[r^2 + (r')^2]^{3/2}} \tag{3.3}$$

where r' and r'' are the first and second derivatives of r with respect to θ.

Example 3.2.1 (Cartesian Form)**.** Find the curvature and radius of curvature of the parabola $y = x^2$ at the vertex $(0, 0)$.

Solution. We have $y = x^2$.

$$y' = 2x \implies y'(0) = 0$$
$$y'' = 2$$

Plugging into the Cartesian formula:

$$\kappa(0) = \frac{|2|}{[1 + (0)^2]^{3/2}} = \frac{2}{1} = 2$$

The curvature is $\kappa = 2$. The radius of curvature is $\rho = \frac{1}{\kappa} = \frac{1}{2}$. ■

Example 3.2.2 (Parametric Form)**.** Find the curvature and radius of curvature for the circle defined by $x(t) = a\cos(t)$ and $y(t) = a\sin(t)$, where $a > 0$.

Solution. First, we find the derivatives with respect to t:

$$\dot{x} = -a\sin(t) \qquad \ddot{x} = -a\cos(t)$$
$$\dot{y} = a\cos(t) \qquad \ddot{y} = -a\sin(t)$$

Numerator of the formula:

$$|\dot{x}\ddot{y} - \dot{y}\ddot{x}| = |(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)| = |a^2\sin^2 t + a^2\cos^2 t| = |a^2| = a^2$$

Denominator of the formula:

$$[\dot{x}^2 + \dot{y}^2]^{3/2} = [(-a\sin t)^2 + (a\cos t)^2]^{3/2} = [a^2(\sin^2 t + \cos^2 t)]^{3/2} = [a^2]^{3/2} = a^3$$

The curvature is:

$$\kappa(t) = \frac{a^2}{a^3} = \frac{1}{a}$$

The curvature is constant, $\kappa = 1/a$. The radius of curvature is $\rho = a$. ■

Example 3.2.3 (Polar Form)**.** Find the curvature and radius of curvature of the cardioid $r = a(1 - \cos\theta)$ at the point where $\theta = \pi$.

Solution. We have $r = a(1 - \cos\theta)$.

$$r' = a\sin\theta$$
$$r'' = a\cos\theta$$

At $\theta = \pi$:

$$r = a(1 - \cos\pi) = a(1 - (-1)) = 2a$$
$$r' = a\sin\pi = 0$$
$$r'' = a\cos\pi = -a$$

Plugging into the polar formula:

$$\begin{aligned}
\kappa(\pi) &= \frac{|r^2 + 2(r')^2 - rr''|}{[r^2 + (r')^2]^{3/2}} \\
&= \frac{|(2a)^2 + 2(0)^2 - (2a)(-a)|}{[(2a)^2 + (0)^2]^{3/2}} \\
&= \frac{|4a^2 + 2a^2|}{[4a^2]^{3/2}} = \frac{6a^2}{(2a)^3} = \frac{6a^2}{8a^3} = \frac{3}{4a}
\end{aligned}$$

The curvature is $\kappa = \frac{3}{4a}$. The radius of curvature is $\rho = \frac{4a}{3}$. ■

3.3 Pedal Equations

Definition 3.3.1 (Pedal Equation)**.** The **pedal equation** of a curve is a relation between the length of the radius vector, r, and the length of the perpendicular from the pole (origin) to the tangent at that point, denoted by p. The equation is of the form $f(p, r) = 0$.

Theorem 3.3.2 (Derivation Formula)**.** *Let the equation of the curve be given in polar form, $r = f(\theta)$. Then the pedal equation is given by the formula:*

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \tag{3.4}$$

To find the pedal equation, one must eliminate θ from this formula using the original curve's equation.

Example 3.3.3. Find the pedal equation of the cardioid $r = a(1 - \cos\theta)$.

Solution. First, find the derivative: $\frac{dr}{d\theta} = a\sin\theta$. Substitute into the formula:

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(a\sin\theta)^2 = \frac{1}{r^2} + \frac{a^2\sin^2\theta}{r^4}$$

From the curve's equation, $\cos\theta = 1 - \frac{r}{a}$. We also know $\sin^2\theta = 1 - \cos^2\theta$.

$$\begin{aligned} \sin^2\theta = 1 - \left(1 - \frac{r}{a}\right)^2 = 1 - \left(1 - \frac{2r}{a} + \frac{r^2}{a^2}\right) = \frac{2r}{a} - \frac{r^2}{a^2} = \frac{r(2a - r)}{a^2} \\ \implies \frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}\left(\frac{r(2a - r)}{a^2}\right) = \frac{1}{r^2} + \frac{2a - r}{r^3} \\ = \frac{r + (2a - r)}{r^3} = \frac{2a}{r^3} \end{aligned}$$

The pedal equation is $r^3 = 2ap^2$. ■

3.4 Lengths of Arcs

Definition 3.4.1 (Arc Length)**.** The **length of an arc** of a curve is the distance measured along the curve between two points. It is found by integrating a differential element of length, ds.

3.4.1 Formulas for Arc Length

The differential arc length ds and the total arc length S from a point A to a point B are given by:

1. **Cartesian Form:** For $y = f(x)$ from $x = a$ to $x = b$.

$$ds = \sqrt{1 + (y')^2}\,dx \quad \implies \quad S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

2. **Parametric Form:** For $x = x(t), y = y(t)$ from $t = t_1$ to $t = t_2$.

$$ds = \sqrt{(\dot{x})^2 + (\dot{y})^2}\,dt \quad \implies \quad S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

3. **Polar Form:** For $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$.

$$ds = \sqrt{r^2 + (r')^2}\,d\theta \quad \implies \quad S = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

Example 3.4.2. Find the total length of the cardioid $r = a(1 + \cos\theta)$.

Solution. The curve is symmetric about the initial line, so we can find the length from $\theta = 0$ to $\theta = \pi$ and double it. First, find the derivative: $\frac{dr}{d\theta} = -a\sin\theta$. Now, set up the expression under the square root:

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= a^2(1+\cos\theta)^2 + (-a\sin\theta)^2 \\ &= a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta) \\ &= a^2(2 + 2\cos\theta) = 2a^2(1+\cos\theta) \\ &= 2a^2(2\cos^2(\theta/2)) = 4a^2\cos^2(\theta/2) \end{aligned}$$

The integrand is $\sqrt{4a^2\cos^2(\theta/2)} = 2a|\cos(\theta/2)|$. For $\theta \in [0,\pi]$, $\cos(\theta/2) \geq 0$. The total length S is:

$$\begin{aligned} S &= 2\int_0^\pi 2a\cos(\theta/2)\,d\theta = 4a\left[\frac{\sin(\theta/2)}{1/2}\right]_0^\pi \\ &= 8a[\sin(\theta/2)]_0^\pi = 8a(\sin(\pi/2) - \sin(0)) = 8a(1-0) = 8a \end{aligned}$$

The total length of the cardioid is $8a$. ■

Example 3.4.3. Find the length of one arch of the cycloid given by $x(t) = a(t - \sin t)$ and $y(t) = a(1-\cos t)$, where one arch corresponds to $t \in [0, 2\pi]$.

Solution. First, find the derivatives with respect to t:

$$\frac{dx}{dt} = a(1-\cos t) \quad \text{and} \quad \frac{dy}{dt} = a\sin t$$

Next, set up the expression under the square root:

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= a^2(1-\cos t)^2 + (a\sin t)^2 \\ &= a^2(1 - 2\cos t + \cos^2 t + \sin^2 t) \\ &= a^2(2 - 2\cos t) = 2a^2(1-\cos t) \\ &= 2a^2(2\sin^2(t/2)) = 4a^2\sin^2(t/2) \end{aligned}$$

The integrand becomes $\sqrt{4a^2\sin^2(t/2)} = 2a|\sin(t/2)|$. For $t \in [0, 2\pi]$, $\sin(t/2) \geq 0$. The arc length is:

$$\begin{aligned} S &= \int_0^{2\pi} 2a\sin(t/2)\,dt = 2a\left[\frac{-\cos(t/2)}{1/2}\right]_0^{2\pi} \\ &= -4a[\cos(t/2)]_0^{2\pi} \\ &= -4a(\cos(\pi) - \cos(0)) = -4a(-1-1) = 8a \end{aligned}$$

The length of one arch of the cycloid is $8a$. ■

Example 3.4.4. Find the total length of the astroid given by $x(t) = a\cos^3 t$ and $y(t) = a\sin^3 t$.

Solution. The curve is symmetric. We will find the length of the arc in the first quadrant (from $t = 0$ to $t = \pi/2$) and multiply by 4. First, find the derivatives:

$$\frac{dx}{dt} = -3a\cos^2 t\sin t \quad \text{and} \quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

Next, set up the expression under the square root:

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2 \\ &= 9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t \\ &= 9a^2\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) \\ &= 9a^2\cos^2 t\sin^2 t \end{aligned}$$

The integrand is $\sqrt{9a^2\cos^2 t\sin^2 t} = 3a|\cos t\sin t|$. For $t \in [0, \pi/2]$, this is positive. The total length S is:

$$\begin{aligned} S &= 4\int_0^{\pi/2} 3a\cos t\sin t\,dt = 12a\int_0^{\pi/2}\sin t\cos t\,dt \\ &= 12a\left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} \\ &= 6a[\sin^2 t]_0^{\pi/2} = 6a(\sin^2(\pi/2) - \sin^2(0)) = 6a(1-0) = 6a \end{aligned}$$

The total length of the astroid is $6a$. ■

3.5 Arc Length of Polar Curves

The arc length S of a curve defined by the polar equation $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by:

$$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 3.5.1. Find the length of the spiral $r = e^{a\theta}$ from $\theta = 0$ to $\theta = 2\pi$.

Solution. First, find the derivative with respect to θ:

$$\frac{dr}{d\theta} = ae^{a\theta}$$

Next, set up the expression under the square root:

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (e^{a\theta})^2 + (ae^{a\theta})^2 \\ &= e^{2a\theta} + a^2e^{2a\theta} \\ &= e^{2a\theta}(1 + a^2) \end{aligned}$$

The integrand becomes $\sqrt{e^{2a\theta}(1 + a^2)} = e^{a\theta}\sqrt{1 + a^2}$. The arc length is:

$$\begin{aligned} S &= \int_0^{2\pi} e^{a\theta}\sqrt{1 + a^2}\, d\theta = \sqrt{1 + a^2}\int_0^{2\pi} e^{a\theta}\, d\theta \\ &= \sqrt{1 + a^2}\left[\frac{e^{a\theta}}{a}\right]_0^{2\pi} \\ &= \frac{\sqrt{1 + a^2}}{a}[e^{a\theta}]_0^{2\pi} = \frac{\sqrt{1 + a^2}}{a}(e^{2a\pi} - e^0) \end{aligned}$$

The arc length is $\frac{\sqrt{1+a^2}}{a}(e^{2a\pi} - 1)$. ■

Example 3.5.2. Find the length of the circumference of the circle $r = 2a\cos\theta$.

Solution. The circle is traced once as θ goes from $-\pi/2$ to $\pi/2$. First, find the derivative:

$$\frac{dr}{d\theta} = -2a\sin\theta$$

Next, set up the expression under the square root:

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (2a\cos\theta)^2 + (-2a\sin\theta)^2 \\ &= 4a^2\cos^2\theta + 4a^2\sin^2\theta \\ &= 4a^2(\cos^2\theta + \sin^2\theta) = 4a^2 \end{aligned}$$

The integrand is $\sqrt{4a^2} = 2a$. The arc length is:

$$\begin{aligned} S &= \int_{-\pi/2}^{\pi/2} 2a\, d\theta = 2a[\theta]_{-\pi/2}^{\pi/2} \\ &= 2a\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2a(\pi) = 2\pi a \end{aligned}$$

This confirms the circumference of a circle with radius a is $2\pi a$. ■

Exercise Find the pedal equation of $r = 2a\sin\theta$.

3.6 Asymptotes

Definition 3.6.1 (Asymptote)**.** An **asymptote** of a curve is a straight line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tend to infinity.

There are three types of asymptotes:

1. **Vertical Asymptotes:** Occur at $x = c$ if $\lim_{x\to c} |f(x)| = \infty$.
2. **Horizontal Asymptotes:** Occur at $y = L$ if $\lim_{x\to\pm\infty} f(x) = L$.
3. **Oblique (or Slant) Asymptotes:** A line $y = mx + c$ is an oblique asymptote if:

$$m = \lim_{x\to\pm\infty} \frac{f(x)}{x} \quad \text{and} \quad c = \lim_{x\to\pm\infty} (f(x) - mx)$$

   Both limits must exist and m must be non-zero.

3.6.1 Working Rule for Algebraic Curves

For an algebraic curve $F(x, y) = 0$ of degree n, a common method is:

1. **Asymptotes parallel to axes:** Find the coefficient of the highest power of y and equate it to zero. If it yields real linear factors $x = c_i$, these are vertical asymptotes. Do the same for the highest power of x to find horizontal asymptotes.
2. **Oblique asymptotes:** Let $y = mx + c$. In the highest degree terms of the equation, substitute $x = 1$ and $y = m$ to get $\phi_n(m)$. Solve $\phi_n(m) = 0$ for real roots m_i. For each non-repeated root m, find c using:

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

   where $\phi_{n-1}(m)$ and $\phi_n'(m)$ are found from the $(n-1)$ degree terms and the derivative of $\phi_n(m)$, respectively.

Example 3.6.2. Find all the asymptotes of the curve $x^3 + y^3 = 3axy$.

Solution. The equation is $x^3 + y^3 - 3axy = 0$. This is the Folium of Descartes. The degree is $n = 3$. **1. Asymptotes parallel to axes:** The coefficient of the highest power of x (x^3) is 1 (a constant). No horizontal asymptotes. The coefficient of the highest power of y (y^3) is 1 (a constant). No vertical asymptotes.

2. Oblique asymptotes ($y = mx + c$): Find the functions $\phi_n(m)$:

$$\phi_3(m) = 1 \cdot m^3 + 1^3 = m^3 + 1 \quad (\text{from } y^3 + x^3)$$

$$\phi_2(m) = -3a(1)(m) = -3am \quad (\text{from } -3axy)$$

Solve $\phi_3(m) = 0$:

$$m^3 + 1 = 0 \implies (m + 1)(m^2 - m + 1) = 0$$

The only real root is $m = -1$. Now find the derivative $\phi_3'(m) = 3m^2$. Calculate c for $m = -1$:

$$c = -\frac{\phi_2(-1)}{\phi_3'(-1)} = -\frac{-3a(-1)}{3(-1)^2} = -\frac{3a}{3} = -a$$

The only asymptote is the line $y = mx + c \implies y = -x - a$, or $x + y + a = 0$. ■

Example 3.6.3. Find all the asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$.

Solution. The equation is of degree $n = 3$. We can rewrite it as:

$$(x^2y - xy^2) + (xy + y^2) + (x - y) = 0$$

Step 1: Asymptotes parallel to the y-axis (Vertical)
The highest power of y is y^2. The coefficient is $(-x + 1)$. Equating to zero: $-x + 1 = 0 \implies x = 1$. So, $x = 1$ is a vertical asymptote.

Step 2: Asymptotes parallel to the x-axis (Horizontal)
The highest power of x is x^2. The coefficient is y. Equating to zero: $y = 0$. So, $y = 0$ is a horizontal asymptote.

Step 3: Oblique Asymptotes ($y = mx + c$)
Find the functions $\phi_n(m)$ by substituting $x = 1, y = m$:

- From degree 3 terms ($x^2y - xy^2$): $\phi_3(m) = (1)^2(m) - (1)(m^2) = m - m^2$.
- From degree 2 terms ($xy + y^2$): $\phi_2(m) = (1)(m) + (m^2) = m + m^2$.

Solve $\phi_3(m) = 0$:

$$m - m^2 = 0 \implies m(1 - m) = 0$$

This gives two real, non-repeated roots: $m = 0$ and $m = 1$.

Case 1: $m = 0$

This corresponds to a horizontal asymptote. We have already found it ($y = 0$), but we can verify it with the formula for c.

$$\phi_3'(m) = 1 - 2m \implies \phi_3'(0) = 1$$

$$c = -\frac{\phi_2(0)}{\phi_3'(0)} = -\frac{0 + 0^2}{1} = 0$$

This gives the asymptote $y = 0x + 0 \implies y = 0$. This confirms our earlier result.

Case 2: $m = 1$

Find the corresponding c.

$$\phi_3'(1) = 1 - 2(1) = -1$$

$$c = -\frac{\phi_2(1)}{\phi_3'(1)} = -\frac{1 + 1^2}{-1} = -\frac{2}{-1} = 2$$

This gives the asymptote $y = 1x + 2 \implies y = x + 2$.

Conclusion:

The asymptotes of the curve are:

1. $x = 1$
2. $y = 0$
3. $y = x + 2$

■

3.7 Singular Points

Definition 3.7.1 (Singular Point)**.** A **singular point** on an algebraic curve $f(x, y) = 0$ is a point where the curve behaves in an extraordinary way. Mathematically, it is a point (x_0, y_0) on the curve where both partial derivatives are simultaneously zero:

$$f(x_0, y_0) = 0, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = 0, \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = 0$$

At a singular point, the tangent to the curve is not uniquely defined.

3.7.1 Classification of Singular Points (Double Points)

A common type of singular point is a **double point**, where two branches of the curve pass. We classify double points by examining the quantity:

$$\Delta = \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)$$

evaluated at the singular point.

1. If $\Delta > 0$, the point is a **node**. The curve has two distinct real tangents at this point.
2. If $\Delta = 0$, the point is a **cusp**. The curve has two coincident real tangents.
3. If $\Delta < 0$, the point is a **conjugate point** or isolated point. The tangents are imaginary.

Example 3.7.2. Find and classify the singular points on the curve $y^2 = x^2(x + 1)$.

Solution. Let $f(x, y) = y^2 - x^3 - x^2 = 0$. Find partial derivatives:

$$\frac{\partial f}{\partial x} = -3x^2 - 2x \qquad \frac{\partial f}{\partial y} = 2y$$

Set them to zero: $2y = 0 \implies y = 0$. And $-3x^2 - 2x = -x(3x + 2) = 0 \implies x = 0$ or $x = -2/3$. We check which of these potential points, $(0, 0)$ and $(-2/3, 0)$, are on the curve.

- For $(0, 0)$: $0^2 - 0^3 - 0^2 = 0$. This point is on the curve. It is a singular point.
- For $(-2/3, 0)$: $0^2 - (-2/3)^3 - (-2/3)^2 = 8/27 - 4/9 = -4/27 \neq 0$. Not on the curve.

The only singular point is $(0, 0)$. To classify it, find second derivatives at $(0, 0)$:

$$\frac{\partial^2 f}{\partial x^2} = -6x - 2 \implies -2 \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

Calculate $\Delta = (0)^2 - (-2)(2) = 4$. Since $\Delta > 0$, the point $(0, 0)$ is a **node**. ■

Example 3.7.3. Find and classify the singular points on the curve $y^2 = x^3$.

Solution. Let the curve be $f(x, y) = y^2 - x^3 = 0$.

$$\frac{\partial f}{\partial x} = -3x^2 \implies -3x^2 = 0 \implies x = 0$$

$$\frac{\partial f}{\partial y} = 2y \implies 2y = 0 \implies y = 0$$

The only potential singular point is $(0, 0)$. This point satisfies the original equation ($0^2 - 0^3 = 0$), so it is a singular point.

The second-order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = -6x \qquad \frac{\partial^2 f}{\partial y^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

Evaluate these at the singular point $(0, 0)$:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = -6(0) = 0$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2$$

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = 0$$

Therefore,

$$\Delta = (0)^2 - (0)(2) = 0$$

Since $\Delta = 0$, the singular point $(0, 0)$ is a **cusp**. ■

3.8 Maxima and Minima of Functions

Definition 3.8.1 (Local Extrema)**.** A function $f(x)$ has a **local maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in some open interval containing c. A function $f(x)$ has a **local minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in some open interval containing c.

3.8.1 Finding Local Extrema

1. **Find Critical Points:** These are points where the first derivative is zero or undefined, i.e., $f'(x) = 0$ or $f'(x)$ does not exist.
2. **First Derivative Test:**
   - If $f'(x)$ changes from positive to negative at a critical point c, then f has a local maximum at c.
   - If $f'(x)$ changes from negative to positive at a critical point c, then f has a local minimum at c.
   - If $f'(x)$ does not change sign, there is no local extremum at c.
3. **Second Derivative Test:** Let c be a critical point such that $f'(c) = 0$.
   - If $f''(c) < 0$, then f has a local maximum at c.
   - If $f''(c) > 0$, then f has a local minimum at c.
   - If $f''(c) = 0$, the test is inconclusive. Use the First Derivative Test.

Example 3.8.2. Find the local maxima and minima for the function $f(x) = x^3 - 6x^2 + 5$.

Solution.

$$f'(x) = 3x^2 - 12x$$

Set $f'(x) = 0 \implies 3x(x-4) = 0$. The critical points are $x = 0$ and $x = 4$.

We use the Second Derivative Test.

$$f''(x) = 6x - 12$$

- At $x = 0$: $f''(0) = 6(0) - 12 = -12 < 0$. So there is a **local maximum** at $x = 0$. The maximum value is $f(0) = 0^3 - 6(0)^2 + 5 = 5$.
- At $x = 4$: $f''(4) = 6(4) - 12 = 12 > 0$. So there is a **local minimum** at $x = 4$. The minimum value is $f(4) = 4^3 - 6(4)^2 + 5 = 64 - 96 + 5 = -27$.

■

Example 3.8.3. Find the local extrema of $f(x) = x + \frac{1}{x}$.

Solution.

$$f'(x) = 1 - \frac{1}{x^2}$$

Set $f'(x) = 0 \implies 1 - \frac{1}{x^2} = 0 \implies x^2 = 1 \implies x = \pm 1$. The derivative is undefined at $x = 0$, but $x = 0$ is not in the domain of $f(x)$. The critical points are $x = 1$ and $x = -1$.

We use the Second Derivative Test.

$$f''(x) = \frac{2}{x^3}$$

- At $x = 1$: $f''(1) = \frac{2}{1^3} = 2 > 0$. So there is a **local minimum** at $x = 1$. The minimum value is $f(1) = 1 + \frac{1}{1} = 2$.
- At $x = -1$: $f''(-1) = \frac{2}{(-1)^3} = -2 < 0$. So there is a **local maximum** at $x = -1$. The maximum value is $f(-1) = -1 + \frac{1}{-1} = -2$.

■

3.9 Bounded Functions

Definition 3.9.1 (Bounded Function)**.** A function f is said to be **bounded** on an interval I if there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. This means the function's range is contained within the finite interval $[-M, M]$.

- f is **bounded above** if there is a number K such that $f(x) \leq K$ for all $x \in I$.
- f is **bounded below** if there is a number k such that $f(x) \geq k$ for all $x \in I$.

A function is bounded if and only if it is bounded both above and below.

Example 3.9.2 (Bounded Function)**.** The function $f(x) = \sin(x)$ is bounded on $\mathbb{R}$ because for all x, we have $-1 \leq \sin(x) \leq 1$. We can choose $M = 1$, since $|\sin(x)| \leq 1$.

Example 3.9.3 (Unbounded Function)**.** The function $f(x) = \frac{1}{x}$ is unbounded on the interval $(0, 1]$. As x approaches 0 from the right, $f(x)$ approaches ∞, so there is no upper bound M that can contain all function values.

3.10 Properties of Continuous Functions on Closed Intervals

Continuous functions defined on a closed and bounded interval $[a, b]$ exhibit special and powerful properties that are not guaranteed on open intervals.

Theorem 3.10.1 (The Boundedness Theorem)**.** *If a function f is continuous on a closed interval $[a, b]$, then f is bounded on $[a, b]$.*

Proof. We prove by contradiction. Assume f is not bounded on $[a, b]$. This means for any natural number n, there exists a point $x_n \in [a, b]$ such that $|f(x_n)| > n$.

This creates a sequence $\{x_n\}$ in the closed and bounded interval $[a, b]$. By the Bolzano-Weierstrass theorem, this sequence must have a convergent subsequence, let's call it $\{x_{n_k}\}$, that converges to some point $c \in [a, b]$.

Since f is continuous at c, we must have $\lim_{k\to\infty} f(x_{n_k}) = f(c)$. This implies the sequence $\{f(x_{n_k})\}$ is convergent and therefore must be bounded.

However, by our initial construction, $|f(x_{n_k})| > n_k$. As $k \to \infty$, $n_k \to \infty$, which means the sequence $\{f(x_{n_k})\}$ is unbounded.

This is a contradiction. Therefore, our initial assumption must be false, and f must be bounded on $[a, b]$. ■

Example 3.10.2. $f(x) = x^2$ is continuous on $[-2, 3]$. The theorem guarantees it is bounded. Indeed, its values range from $f(0) = 0$ to $f(3) = 9$, so it is bounded by $M = 9$.

Theorem 3.10.3 (The Extreme Value Theorem (EVT))**.** *If a function f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value and an absolute minimum value on $[a, b]$. This means there exist numbers $c, d \in [a, b]$ such that $f(c) \le f(x) \le f(d)$ for all $x \in [a, b]$.*

Proof. We will prove the existence of the maximum; the proof for the minimum is analogous.

By the Boundedness Theorem, the set of values $S = \{f(x) : x \in [a, b]\}$ is bounded above. Therefore, by the completeness axiom of real numbers, S has a least upper bound (supremum). Let $M = \sup S$.

We must show that there exists a point $d \in [a, b]$ such that $f(d) = M$. By the definition of a supremum, for any natural number n, there exists an element $y_n \in S$ such that $M - \frac{1}{n} < y_n \le M$. For each y_n, there is a corresponding $x_n \in [a, b]$ such that $f(x_n) = y_n$. This creates a sequence $\{x_n\}$ in $[a, b]$.

By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ that converges to some point $d \in [a, b]$.

Since f is continuous at d, we have $\lim_{k\to\infty} f(x_{n_k}) = f(d)$. Also, from our construction, we have $M - \frac{1}{n_k} < f(x_{n_k}) \le M$. As $k \to \infty$, $\frac{1}{n_k} \to 0$. By the Squeeze Theorem, $\lim_{k\to\infty} f(x_{n_k}) = M$.

Equating the two limits, we get $f(d) = M$. Thus, the function attains its maximum value at d. ■

Example 3.10.4. $f(x) = x^3 - 3x$ is continuous on $[0, 2]$. The theorem guarantees it attains its absolute max and min. By checking critical points and endpoints, we find the absolute minimum is $f(1) = -2$ and the absolute maximum is $f(2) = 2$.

3.11 The Intermediate Value Theorem (IVT)

This theorem formalizes the intuitive idea that a continuous function cannot "skip" values.

Theorem 3.11.1 (Intermediate Value Theorem)**.** ***Statement:*** *If a function f is continuous on the closed interval $[a, b]$, and N is any number between $f(a)$ and $f(b)$ (where $f(a) \ne f(b)$), then there exists at least one number c in the open interval (a, b) such that $f(c) = N$.*

Proof. Assume without loss of generality that $f(a) < N < f(b)$. Consider the set $S = \{x \in [a, b] : f(x) \le N\}$.

The set S is non-empty since $a \in S$. The set S is also bounded above by b. By the completeness axiom, S has a least upper bound (supremum). Let $c = \sup S$. Since $a \in S$ and b is an upper bound, we have $a \le c \le b$.

We will show that $f(c) = N$. By the definition of a supremum, for any natural number n, there exists $x_n \in S$ such that $c - \frac{1}{n} < x_n \le c$. This implies that $\lim_{n\to\infty} x_n = c$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(c)$. As each $x_n \in S$, we have $f(x_n) \le N$. Taking the limit, we get $f(c) \le N$.

Now, since $f(b) > N$, $c \ne b$, so $c < b$. Consider the sequence $z_n = c + \frac{1}{n}$ for sufficiently large n so that $z_n < b$. None of these z_n can be in S because c is the supremum. Thus, $f(z_n) > N$ for all such n. As $n \to \infty$, $z_n \to c$. By continuity, $\lim_{n\to\infty} f(z_n) = f(c)$. From $f(z_n) > N$, we conclude $f(c) \ge N$.

Combining the two inequalities, $f(c) \le N$ and $f(c) \ge N$, we must have $f(c) = N$. Since $f(a) < N$ and $f(b) > N$, we know c cannot be a or b, so $c \in (a, b)$. ■

Example 3.11.2. Show that the equation $x^3 + 2x - 1 = 0$ has a root in the interval $[0, 1]$.

Solution. Let $f(x) = x^3 + 2x - 1$. The function f is a polynomial, so it is continuous everywhere, including on $[0, 1]$. We evaluate the function at the endpoints:

- $f(0) = 0^3 + 2(0) - 1 = -1$.
- $f(1) = 1^3 + 2(1) - 1 = 2$.

Let $N = 0$. Since $f(0) < 0 < f(1)$, the IVT guarantees that there must be a number $c \in (0, 1)$ such that $f(c) = 0$. This c is a root of the equation. ■

3.12 Darboux's Theorem

Darboux's Theorem is like an Intermediate Value Theorem, but for derivatives. It states that the derivative of a differentiable function, even if the derivative itself is not continuous, must still satisfy the intermediate value property.

Theorem 3.12.1 (Darboux's Theorem)**.** ***Statement:*** *If a function f is differentiable on the closed interval $[a, b]$, and k is any number between $f'(a)$ and $f'(b)$, then there exists at least one number c in the open interval (a, b) such that $f'(c) = k$.*

Theorem 3.12.2 (Darboux's Theorem)**.** *If a function f is differentiable on $[a, b]$, and k is any number between $f'(a)$ and $f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = k$.*

Proof. Assume without loss of generality that $f'(a) < k < f'(b)$. Define a new function $g(x) = f(x) - kx$. This function is differentiable on $[a, b]$ because $f(x)$ and kx are.

The derivative of $g(x)$ is $g'(x) = f'(x) - k$. At the endpoints, we have:

- $g'(a) = f'(a) - k < 0$.
- $g'(b) = f'(b) - k > 0$.

Since $g(x)$ is differentiable on the closed interval $[a, b]$, it must also be continuous on $[a, b]$. By the Extreme Value Theorem, $g(x)$ must attain an absolute minimum value at some point $c \in [a, b]$.

Because $g'(a) < 0$, the function is decreasing at $x = a$, so the minimum cannot occur at a. Because $g'(b) > 0$, the function is increasing at $x = b$, so the minimum cannot occur at b.

Therefore, the minimum must occur at an interior point $c \in (a, b)$. At an interior minimum, the derivative must be zero. So, $g'(c) = 0$. By definition of $g'(x)$, this means:

$$f'(c) - k = 0 \implies f'(c) = k$$

Thus, we have found a point $c \in (a, b)$ where the derivative equals k. ■

Example 3.12.3. Consider the function $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Its derivative is $f'(x) = 2x \sin(1/x) - \cos(1/x)$ for $x \neq 0$, and $f'(0) = 0$. The derivative $f'(x)$ is not continuous at $x = 0$. Let's apply Darboux's Theorem on $[0, 2/\pi]$.

Solution. The function f is differentiable on $[0, 2/\pi]$. We evaluate the derivative at the endpoints:

- $f'(0) = 0$.
- $f'(2/\pi) = 2(2/\pi) \sin(\pi/2) - \cos(\pi/2) = 4/\pi \approx 1.27$.

Let's choose an intermediate value $k = 1/2$. Since $0 < 1/2 < 4/\pi$, Darboux's Theorem guarantees that there exists a $c \in (0, 2/\pi)$ such that $f'(c) = 1/2$, even though $f'(x)$ oscillates wildly and is not continuous at the endpoint $x = 0$. ■

Chapter 4

Mean Value Theorems and their Applications

4.1 Rolle's Theorem

Theorem 4.1.1 (Rolle's Theorem)**.** *Let a function f be:*

1. *continuous on the closed interval* $[a, b]$*,*
2. *differentiable on the open interval* (a, b)*, and*
3. *such that* $f(a) = f(b)$*.*

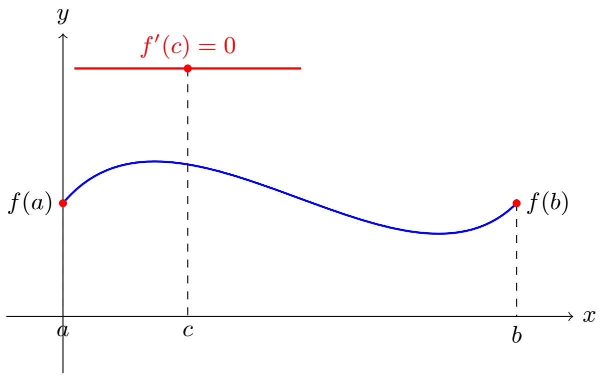
Then there exists at least one number c in the open interval (a, b) such that $f'(c) = 0$.

Proof. Since f is continuous on the closed interval $[a, b]$, by the Extreme Value Theorem, it must attain an absolute maximum and an absolute minimum on $[a, b]$.

Case 1: The maximum and minimum values are equal. This implies that $f(x)$ is a constant function on $[a, b]$. For a constant function, the derivative is zero everywhere, so $f'(c) = 0$ for any $c \in (a, b)$.

Case 2: The maximum and minimum values are different. Since $f(a) = f(b)$, at least one of the extreme values (the maximum or the minimum) must occur at an interior point $c \in (a, b)$. At an interior extremum of a differentiable function, the derivative must be zero. Therefore, $f'(c) = 0$. ∎

Interpretation 4.1.2. Geometrically, Rolle's Theorem states that if a smooth curve starts and ends at the same height, there must be at least one point between them where the tangent line is horizontal.



Example 4.1.3. Verify Rolle's Theorem for the function $f(x) = x^2 - 4x + 3$ on the interval $[1, 3]$.

Solution. First, we check the three conditions of Rolle's Theorem.

1. **Continuity:** $f(x)$ is a polynomial, so it is continuous on $[1, 3]$.
2. **Differentiability:** $f'(x) = 2x - 4$, which exists for all $x \in (1, 3)$.
3. **Equal Endpoints:** $f(1) = 1^2 - 4(1) + 3 = 0$. And $f(3) = 3^2 - 4(3) + 3 = 9 - 12 + 3 = 0$. So, $f(1) = f(3)$.

All conditions are satisfied. The theorem guarantees a point $c \in (1, 3)$ where $f'(c) = 0$. To find c, we set the derivative to zero:

$$f'(c) = 2c - 4 = 0 \implies 2c = 4 \implies c = 2$$

Since $c = 2$ is in the open interval $(1, 3)$, Rolle's Theorem is verified. ∎

4.2 Lagrange's Mean Value Theorem (MVT)

This is a generalization of Rolle's Theorem.

Theorem 4.2.1 (Lagrange's Mean Value Theorem)**.** *Let a function f be:*

1. *continuous on the closed interval $[a, b]$, and*
2. *differentiable on the open interval (a, b).*

Then there exists at least one number c in the open interval (a, b) such that:

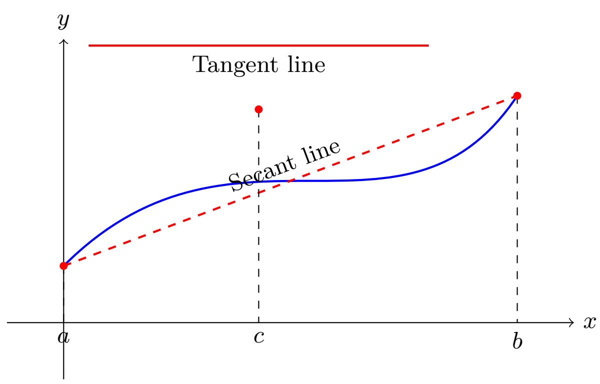
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define an auxiliary function $g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right] x$. This function $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b) because $f(x)$ and the linear term are. Now, check the values of $g(x)$ at the endpoints:

$$g(a) = f(a) - \frac{f(b) - f(a)}{b - a} a$$
$$g(b) = f(b) - \frac{f(b) - f(a)}{b - a} b$$
$$g(b) - g(a) = (f(b) - f(a)) - \frac{f(b) - f(a)}{b - a}(b - a) = 0$$

So, $g(a) = g(b)$. The function $g(x)$ satisfies all conditions of Rolle's Theorem. Therefore, there exists a $c \in (a, b)$ such that $g'(c) = 0$. The derivative of $g(x)$ is $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. At $x = c$, we have $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. This implies $f'(c) = \frac{f(b)-f(a)}{b-a}$. ■

Interpretation 4.2.2. Geometrically, the MVT states that for any smooth curve, there is at least one point where the instantaneous rate of change (slope of the tangent line) is equal to the average rate of change over the interval (slope of the secant line connecting the endpoints).



Example 4.2.3. Verify the Mean Value Theorem for the function $f(x) = \sqrt{x}$ on the interval $[1, 4]$.

Solution. First, we check the two conditions of the MVT.

1. **Continuity:** $f(x) = \sqrt{x}$ is continuous on $[1, 4]$.
2. **Differentiability:** $f'(x) = \frac{1}{2\sqrt{x}}$, which exists for all $x \in (1, 4)$.

All conditions are satisfied. The theorem guarantees a point $c \in (1, 4)$ such that $f'(c) = \frac{f(4)-f(1)}{4-1}$. First, calculate the slope of the secant line:

$$\frac{f(4) - f(1)}{4 - 1} = \frac{\sqrt{4} - \sqrt{1}}{3} = \frac{2 - 1}{3} = \frac{1}{3}$$

Now, we set the derivative equal to this value to find c:

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{1}{3} \implies 2\sqrt{c} = 3 \implies \sqrt{c} = \frac{3}{2} \implies c = \frac{9}{4}$$

Since $c = 9/4 = 2.25$ is in the open interval $(1, 4)$, the Mean Value Theorem is verified. ■

4.3 Cauchy's Mean Value Theorem (CMVT)

This is a further generalization of the MVT and is crucial for proving L'Hôpital's Rule.

Theorem 4.3.1 (Cauchy's Mean Value Theorem)**.** *Let two functions f and g be:*

1. *continuous on the closed interval $[a, b]$,*
2. *differentiable on the open interval (a, b), and*
3. *such that $g'(x) \neq 0$ for all $x \in (a, b)$.*

Then there exists at least one number c in the open interval (a, b) such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Note that since $g'(x) \neq 0$ on (a, b), by Rolle's Theorem, we must have $g(b) \neq g(a)$, so the denominator on the right is not zero. Define an auxiliary function $h(x) = f(x) - \left[\frac{f(b)-f(a)}{g(b)-g(a)}\right] g(x)$. This function $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Let's check the values of $h(x)$ at the endpoints. Let $K = \frac{f(b)-f(a)}{g(b)-g(a)}$.

$$\begin{aligned}
h(a) &= f(a) - K \cdot g(a) \\
h(b) &= f(b) - K \cdot g(b) \\
h(b) - h(a) &= (f(b) - f(a)) - K(g(b) - g(a)) \\
&= (f(b) - f(a)) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(b) - g(a)) = 0
\end{aligned}$$

So, $h(a) = h(b)$. The function $h(x)$ satisfies all conditions of Rolle's Theorem. Therefore, there exists a $c \in (a, b)$ such that $h'(c) = 0$. The derivative of $h(x)$ is $h'(x) = f'(x) - K \cdot g'(x)$. At $x = c$, we have $h'(c) = f'(c) - K \cdot g'(c) = 0$. This implies $f'(c) = K \cdot g'(c)$, and since $g'(c) \neq 0$, we can write:

$$\frac{f'(c)}{g'(c)} = K = \frac{f(b) - f(a)}{g(b) - g(a)}$$

∎

Interpretation 4.3.2. The CMVT relates the ratio of the derivatives of two functions at an interior point to the ratio of the changes in the functions over the whole interval. It can be seen as a parametric version of the MVT. If we consider a curve defined parametrically by $x = g(t)$ and $y = f(t)$, then $\frac{f'(c)}{g'(c)}$ is the slope of the tangent at $t = c$, while $\frac{f(b)-f(a)}{g(b)-g(a)}$ is the slope of the secant line connecting the endpoints of the parametric curve. The theorem guarantees these slopes are equal for some $t = c$.

Example 4.3.3. Verify Cauchy's Mean Value Theorem for the functions $f(x) = x^2$ and $g(x) = x^3$ on the interval $[1, 2]$.

Solution. First, we check the three conditions of the CMVT.

1. **Continuity:** $f(x)$ and $g(x)$ are polynomials, so they are continuous on $[1, 2]$.
2. **Differentiability:** $f'(x) = 2x$ and $g'(x) = 3x^2$ exist for all $x \in (1, 2)$.
3. **Non-zero derivative:** $g'(x) = 3x^2$ is not zero for any $x \in (1, 2)$.

All conditions are satisfied. The theorem guarantees a point $c \in (1, 2)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$. First, calculate the ratio of the differences:

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{2^2 - 1^2}{2^3 - 1^3} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

Next, find the ratio of the derivatives:

$$\frac{f'(c)}{g'(c)} = \frac{2c}{3c^2} = \frac{2}{3c}$$

Now, we set these two expressions equal to find c:

$$\frac{2}{3c} = \frac{3}{7} \implies 9c = 14 \implies c = \frac{14}{9}$$

Since $c = 14/9 \approx 1.556$ is in the open interval $(1, 2)$, Cauchy's Mean Value Theorem is verified. ∎

4.4 Introduction to Taylor's Theorem

Taylor's theorem is a powerful result that allows us to approximate a sufficiently differentiable function by a polynomial, called the **Taylor polynomial**. The theorem also provides a precise formula for the error, or **remainder**, in this approximation.

Let f be a function such that its first n derivatives exist in a neighborhood of a point a. The Taylor polynomial of degree n for f at a is:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's Theorem states that $f(x) = P_n(x) + R_n(x)$, where $R_n(x)$ is the remainder term. The specific form of $R_n(x)$ depends on which version of the theorem is used.

4.4.1 Taylor's Theorem with Lagrange's Form of Remainder

This is the most common form of the remainder and is a direct generalization of the Mean Value Theorem.

Theorem 4.4.1 (Taylor's Theorem with Lagrange's Remainder)**.** *Let f be a function such that its first n derivatives are continuous on a closed interval $[a, x]$ and its $(n+1)$-th derivative, $f^{(n+1)}$, exists on the open interval (a, x). Then there exists at least one number $c \in (a, x)$ such that:*

$$f(x) = P_n(x) + R_n(x)$$

where the remainder $R_n(x)$ is given by:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Proof. Define an auxiliary function $F(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n$. The derivative of $F(t)$ with respect to t is:

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

This is found by applying the product rule to each term, which results in a telescoping cancellation.

Now define another function $G(t) = F(t) - \left(\frac{x-t}{x-a}\right)^{n+1} F(a)$. This function is continuous on $[a, x]$ and differentiable on (a, x). We check the values at the endpoints:

- $G(x) = F(x) - \left(\frac{x-x}{x-a}\right)^{n+1} F(a) = 0 - 0 = 0$. (Note that $F(x) = 0$ by definition).
- $G(a) = F(a) - \left(\frac{x-a}{x-a}\right)^{n+1} F(a) = F(a) - F(a) = 0$.

Since $G(a) = G(x) = 0$, by Rolle's Theorem, there exists a $c \in (a, x)$ such that $G'(c) = 0$. Differentiating $G(t)$:

$$G'(t) = F'(t) + (n+1)\frac{(x-t)^n}{(x-a)^{n+1}}F(a)$$

At $t = c$, we have $G'(c) = 0$, so $F'(c) = -(n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a)$. Substituting the expression for $F'(c)$:

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n = -(n+1)\frac{(x-c)^n}{(x-a)^{n+1}}F(a)$$

Solving for $F(a)$ (and noting $x \neq c$):

$$F(a) = \frac{f^{(n+1)}(c)}{n!}\frac{(x-a)^{n+1}}{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

By definition, $F(a)$ is the remainder $R_n(x) = f(x) - P_n(x)$. This completes the proof. ∎

Example 4.4.2. Use the second-degree Taylor polynomial for $f(x) = \ln(1+x)$ centered at $a = 0$ to approximate $\ln(1.1)$. Then use Lagrange's remainder to find an upper bound for the error in this approximation.

Solution. The derivatives of $f(x) = \ln(1+x)$ at $a = 0$ are $f(0) = 0$, $f'(0) = 1$, and $f''(0) = -1$. The second-degree Taylor polynomial is therefore $P_2(x) = x - \frac{x^2}{2}$. To approximate $\ln(1.1)$, we set $x = 0.1$:

$$\ln(1.1) \approx P_2(0.1) = 0.1 - \frac{(0.1)^2}{2} = 0.095$$

The error in this approximation is given by the Lagrange remainder term $R_2(x) = \frac{f'''(c)}{3!}x^3$ for some c between 0 and x. The third derivative is $f'''(x) = \frac{2}{(1+x)^3}$. For $x = 0.1$, the error is:

$$R_2(0.1) = \frac{1}{3!} \cdot \frac{2}{(1+c)^3} \cdot (0.1)^3 = \frac{0.001}{3(1+c)^3} \quad \text{for some } c \in (0, 0.1)$$

To find an upper bound for the magnitude of this error, we must maximize the expression. This occurs when the denominator is minimized, which is when $c = 0$.

$$|R_2(0.1)| \leq \frac{0.001}{3(1+0)^3} = \frac{0.001}{3} \approx 0.000333$$

The error in our approximation is less than 0.000334. ■

4.4.2 Taylor's Theorem with Cauchy's Form of Remainder

This form of the remainder is often used in theoretical contexts, such as proving the convergence of Taylor series.

Theorem 4.4.3 (Taylor's Theorem with Cauchy's Remainder)**.** *Let f be a function satisfying the same conditions as for Lagrange's form. Then there exists at least one number $c \in (a, x)$ such that the remainder $R_n(x)$ is given by:*

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

Proof. We use the same auxiliary function $F(t)$ as in the previous proof, where $F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$. Now, consider the simple function $\phi(t) = x - t$. Applying Cauchy's Mean Value Theorem to the functions $F(t)$ and $\phi(t)$ on the interval $[a, x]$, there exists a $c \in (a, x)$ such that:

$$\frac{F(x) - F(a)}{\phi(x) - \phi(a)} = \frac{F'(c)}{\phi'(c)}$$

We evaluate each piece:

- $F(x) = 0$
- $\phi(x) = x - x = 0$
- $\phi(a) = x - a$
- $\phi'(t) = -1$, so $\phi'(c) = -1$

Substituting these into the CMVT equation:

$$\frac{0 - F(a)}{0 - (x-a)} = \frac{F'(c)}{-1} \implies \frac{-F(a)}{-(x-a)} = -F'(c)$$

$$\frac{F(a)}{x-a} = \frac{f^{(n+1)}(c)}{n!}(x-c)^n$$

Solving for $F(a)$, which is the remainder $R_n(x)$:

$$R_n(x) = F(a) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$$

This completes the proof. ■

Example 4.4.4. Consider the same approximation as before: $f(x) = \ln(1+x)$ approximated by $P_2(x) = x - x^2/2$ at $x = 0.1$. Use Cauchy's remainder to find an upper bound for the error.

Solution. The remainder $R_n(x)$ in Cauchy's form is $R_n(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$. For our problem, $n=2$, $a=0$, $x=0.1$, and $f'''(x) = \frac{2}{(1+x)^3}$. The error is:

$$R_2(0.1) = \frac{f'''(c)}{2!}(0.1-c)^2(0.1-0) = \frac{2/(1+c)^3}{2}(0.1-c)^2(0.1)$$

$$R_2(0.1) = \frac{(0.1-c)^2}{(1+c)^3}(0.1) \quad \text{for some } c \in (0, 0.1)$$

To bound this error, we can find an upper bound for the fractional term $g(c) = \frac{(0.1-c)^2}{(1+c)^3}$ on the interval $c \in (0, 0.1)$. The numerator $(0.1-c)^2$ is maximized when c is at its minimum ($c=0$), and the denominator $(1+c)^3$ is also minimized at $c=0$. Therefore, the entire fraction is bounded by its value at $c=0$:

$$\frac{(0.1-c)^2}{(1+c)^3} \le \frac{(0.1-0)^2}{(1+0)^3} = 0.01$$

The error is thus bounded by:

$$|R_2(0.1)| \le (0.01) \cdot (0.1) = 0.001$$

This bound is correct, though less tight than the one found using Lagrange's form. ■

Taylor's Series and Maclaurin's Series

Definition 4.4.5 (Taylor's Series)**.** If a function $f(x)$ has derivatives of all orders at a point $x=a$, then the **Taylor series** for $f(x)$ centered at a is the power series given by:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Definition 4.4.6 (Maclaurin's Series)**.** A **Maclaurin series** is a special case of a Taylor series centered at the point $a=0$.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

This is the most common type of Taylor series used for elementary functions.

4.4.3 Standard Maclaurin Series Expansions

Below are the standard Maclaurin series for several fundamental functions, along with their intervals of convergence.

Exponential Function: e^x

For $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ for all n. At $x=0$, $f^{(n)}(0) = e^0 = 1$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Sine Function: $\sin x$

For $f(x) = \sin x$, the derivatives at $x=0$ follow a pattern: $0, 1, 0, -1, \dots$ This results in a series with only odd powers of x.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Cosine Function: $\cos x$

For $f(x) = \cos x$, the derivatives at $x=0$ follow a pattern: $1, 0, -1, 0, \dots$ This results in a series with only even powers of x.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Logarithmic Function: $\log(1+x)$

The Maclaurin series for $\log x$ itself does not exist because $\log(0)$ is undefined. We find the series for $f(x) = \log(1+x)$ instead. The derivatives at $x=0$ are $f(0)=0$, $f'(0)=1$, $f''(0)=-1$, $f'''(0)=2$, etc.

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Binomial Series: $(1+x)^m$

For $f(x) = (1+x)^m$, where m is any real number, the derivatives are $f^{(n)}(x) = m(m-1)\dots(m-n+1)(1+x)^{m-n}$. At $x=0$, this gives $f^{(n)}(0) = m(m-1)\dots(m-n+1)$. This leads to the generalized binomial coefficient $\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}$.

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \dots$$

Interval of Convergence: $(-1,1)$. The convergence at the endpoints $x=\pm 1$ depends on the value of m. If m is a non-negative integer, the series is finite (a polynomial) and converges everywhere.

4.5 Envelope of a One-Parameter Family of Curves

Definition 4.5.1 (Envelope)**.** An **envelope** of a one-parameter family of curves is a curve that is tangent to each member of the family at some point. The family of curves is represented by an equation of the form $F(x,y,\alpha)=0$, where α is the parameter.

4.5.1 Method for Finding the Envelope

To find the envelope, we treat the parameter α as a variable and solve the system of equations formed by the family and its partial derivative with respect to the parameter.

1. The family of curves: $F(x,y,\alpha)=0$.
2. The partial derivative: $\frac{\partial F}{\partial \alpha}(x,y,\alpha) = 0$.

The equation of the envelope is found by eliminating the parameter α from these two equations.

Example 4.5.2. Find the envelope of the family of straight lines $y = mx + \frac{a}{m}$, where m is the parameter.

Solution. First, represent the family of lines in the form $F(x,y,m)=0$:

$$F(x,y,m) \equiv y - mx - \frac{a}{m} = 0$$

Next, differentiate partially with respect to the parameter m:

$$\frac{\partial F}{\partial m} = -x - a\left(-\frac{1}{m^2}\right) = -x + \frac{a}{m^2} = 0$$

From the partial derivative equation, we solve for m:

$$-x + \frac{a}{m^2} = 0 \implies m^2 = \frac{a}{x} \implies m = \pm\sqrt{\frac{a}{x}}$$

Now, we eliminate m by substituting it back into the original family equation. It is easier to substitute for m and $1/m$: From $y = mx + a/m$, we can write $y = m(x + a/m^2)$. From the derivative, we know $x = a/m^2$, so:

$$y = m(x+x) = 2mx$$

Now we solve for m to get $m = y/(2x)$. Substitute this into the relation $m^2 = a/x$:

$$\left(\frac{y}{2x}\right)^2 = \frac{a}{x} \implies \frac{y^2}{4x^2} = \frac{a}{x}$$

Multiplying by $4x^2$ (assuming $x \neq 0$) gives the envelope equation:

$$y^2 = 4ax$$

The envelope is a parabola. ■

4.6 Envelope of a Two-Parameter Family of Curves

Definition 4.6.1. A two-parameter family of curves is given by an equation of the form $F(x, y, \alpha, \beta) = 0$. For an envelope to exist, the two parameters must be related by a constraint equation, $g(\alpha, \beta) = 0$.

4.6.1 Method for Finding the Envelope

The envelope is found by eliminating the parameters α and β from a system of three equations. This method is related to Lagrange multipliers.

1. The family of curves: $F(x, y, \alpha, \beta) = 0$.
2. The constraint equation: $g(\alpha, \beta) = 0$.
3. The condition of proportionality of partial derivatives:

$$\frac{\partial F/\partial\alpha}{\partial g/\partial\alpha} = \frac{\partial F/\partial\beta}{\partial g/\partial\beta}$$

Example 4.6.2. Find the envelope of the family of lines $\frac{x}{\alpha} + \frac{y}{\beta} = 1$, where the parameters α and β are related by the constraint $\alpha + \beta = c$, for a constant c.

Solution. The family of curves and the constraint are:

$$F(x, y, \alpha, \beta) \equiv \frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0 \quad \text{and} \quad g(\alpha, \beta) \equiv \alpha + \beta - c = 0$$

We find the four necessary partial derivatives:

$$\frac{\partial F}{\partial\alpha} = -\frac{x}{\alpha^2} \qquad \frac{\partial F}{\partial\beta} = -\frac{y}{\beta^2}$$
$$\frac{\partial g}{\partial\alpha} = 1 \qquad \frac{\partial g}{\partial\beta} = 1$$

Now, set up the proportionality equation:

$$\frac{-x/\alpha^2}{1} = \frac{-y/\beta^2}{1} \implies \frac{x}{\alpha^2} = \frac{y}{\beta^2}$$

This implies $\frac{\sqrt{x}}{\alpha} = \frac{\sqrt{y}}{\beta}$. Let this common ratio be k.

$$\frac{\sqrt{x}}{\alpha} = k \implies \alpha = \frac{\sqrt{x}}{k} \quad \text{and} \quad \frac{\sqrt{y}}{\beta} = k \implies \beta = \frac{\sqrt{y}}{k}$$

Substitute these expressions for α and β into the constraint $\alpha + \beta = c$:

$$\frac{\sqrt{x}}{k} + \frac{\sqrt{y}}{k} = c \implies \frac{\sqrt{x} + \sqrt{y}}{k} = c \implies k = \frac{\sqrt{x} + \sqrt{y}}{c}$$

Finally, substitute α and β into the original family equation $\frac{x}{\alpha} + \frac{y}{\beta} = 1$:

$$x\left(\frac{k}{\sqrt{x}}\right) + y\left(\frac{k}{\sqrt{y}}\right) = 1 \implies k\sqrt{x} + k\sqrt{y} = 1 \implies k(\sqrt{x} + \sqrt{y}) = 1$$

Now substitute the expression for k we found:

$$\left(\frac{\sqrt{x} + \sqrt{y}}{c}\right)(\sqrt{x} + \sqrt{y}) = 1 \implies (\sqrt{x} + \sqrt{y})^2 = c$$

The equation of the envelope is $\sqrt{x} + \sqrt{y} = \sqrt{c}$. This is also a parabola. ■

Visualizing an Envelope

The concept of an envelope can be understood visually. Consider a family of curves, such as the set of straight lines given by the equation $y = mx + \frac{1}{m}$. Each value of the parameter m defines a different line. The **envelope** is a new curve that is tangent to every one of these lines at some point.

In the figure below, several lines from the family are drawn in blue. As you can see, they collectively outline a distinct shape. This shape, shown as a thick red curve, is the envelope. The envelope itself is not part of the original family, but it is "carved out" by the family of lines. For this specific family of lines, the envelope is the parabola $y^2 = 4x$.