Bachelors with Mathematics as Major

1^{st} Semester

MMT122J: Mathematics/Applied Mathematics: Calculus - I

Credits: 4 THEORY + 2 TUTORIAL **Theory: 60 Hours & Tutorial: 30 Hours**

Course Objectives: (i) To study and understand the notions of differential calculus and to imbibe the acquaintance for using the techniques in other sciences and engineering. (ii) To prepare the students for taking up advanced courses of mathematics.

Course Outcome: (i) After the successful completion of the course, students shall be able to apply differential operators to understand the dynamics of various real life situations. (ii) The students shall be able to use differential calculus in optimization problems.

Theory: 4 Credits

Unit –I

Limits and infinitesimals, Continuity ($\epsilon - \delta$ definition), types of discontinuities of functions, Differentiability of functions, Successive differentiation and Leibnitz theorem, Partial differentiation, Total differentiation, Homogenous functions and Euler's theorem.

Unit –II

Indeterminate forms, Tangents and normals (polar coordinates only), Angle between radius vector and tangent, Perpendicular from pole to tangent, angle of intersection of two curves, polar tangent, polar normal, polar sub-tangent, polar sub-normal.

Unit –III

Curvature and radius of curvature, Pedal Equations, lengths of arcs, Asymptotes, Singular points, Maxima and minima of functions. Bounded functions, Properties of continuous functions on closed intervals, Intermediate value theorem, Darboux theorem.

Unit –IV

Rolle's theorem and mean value theorems (with proofs) and their geometrical interpretation, Taylor's theorem with Lagranges and Cauchy's form of remainder, Taylor's series, Maclaurin's series of $\sin x,\ \cos x,\ e^x,\ \log x,\ (1+x)^m$.Envelope of a family of curves involving one and two parameters.

Tutorial: 2 Credits

Unit –V

Examples of discontinuous functions, nth derivative of product of two functions, involutes and evolutes, bounds of function (Supremum and infimum).

Unit –VI

Tracing of cartesian equations of the form $y = f(x)$, $y^2 = f(x)$, tracing of the parametric equations.

Recommended Books:

1. Shanti Narayan and P.K. Mittal, Differential Calculus, S. Chand, 2016.
2. S. D. Chopra, M. L. Kochar and A. Aziz, Differential Calculus, Kapoor Sons.
3. Schaums outline of Theory and problems of Differential and Integral Calculus, 1964.
1. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002.
2. T.M. Apostal, Calculus Vol. I, John Wiley & Sons Inc, 1975.
3. S. Balachandra Rao and C. K. Shantha, Differential Calculus, New Age Publication, 1992.
4. S. Lang, A First Course in Calculus, Springer-Verlag, 1998.
5. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 2008.
6. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007.
7. Suggestive digital platforms web links: NPTEL/ SWAYAM/ MOOCS.

——CALCULUS I - (MMT122J)——

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Chapter 1

Limits, Continuity and Differentiability

1.1 Limits and Infinitesimals

Concept Overview

The **limit** of a function $f(x)$ as x approaches a point a is the value L that $f(x)$ gets arbitrarily close to as x approaches a. Formally:

$$\lim_{x \to a} f(x) = L \iff \forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

Infinitesimals are quantities smaller than any positive real number but greater than zero. In limits, dx represents an infinitesimal change in x, and the expression $f(a + dx) - L$ becomes infinitesimal as $dx \to 0$.

Example 1.1.1. Evaluate $\lim_{x \to 3}(2x + 1)$.

Solution.

$$\begin{aligned} \lim_{x \to 3}(2x + 1) &= 2(3) + 1 \quad \text{(Direct substitution)} \\ &= 7. \end{aligned}$$

To verify using the ϵ-δ definition: Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x - 3| < \delta$, then:

$$|(2x + 1) - 7| = |2x - 6| = 2|x - 3| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Thus, $\lim_{x \to 3}(2x + 1) = 7$. ■

Example 1.1.2. Prove that $\lim_{x \to 5}(3x - 4) = 11$ using the ϵ-δ definition.

Solution. Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $|(3x - 4) - 11| < \epsilon$.
Simplify the expression:

$$|3x - 15| = |3(x - 5)| = 3|x - 5|$$

We require:

$$3|x - 5| < \epsilon \implies |x - 5| < \frac{\epsilon}{3}$$

Choose $\delta = \frac{\epsilon}{3}$.
Verification: If $0 < |x - 5| < \delta$, then:

$$|(3x - 4) - 11| = 3|x - 5| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

Thus, $\lim_{x \to 5}(3x - 4) = 11$. ■

Example 1.1.3. Prove that $\lim_{x \to 3}(x^2 - 2x) = 3$ using the ϵ-δ definition.

Solution. Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|(x^2 - 2x) - 3| < \epsilon$.
Simplify the expression:

$$|x^2 - 2x - 3| = |(x - 3)(x + 1)| = |x - 3||x + 1|$$

Assume $\delta \leq 1$. Then $|x - 3| < 1$ implies:

$$2 < x < 4 \quad \text{so} \quad 3 < x + 1 < 5$$

Thus $|x+1| < 5$. Now:

$$|x-3||x+1| < |x-3| \cdot 5$$

We require $5|x-3| < \epsilon$, which gives $|x-3| < \frac{\epsilon}{5}$.
Choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$.
Verification: If $0 < |x-3| < \delta$, then:

$$|x^2 - 2x - 3| = |x-3||x+1| < \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon$$

Thus, $\lim_{x\to 3}(x^2 - 2x) = 3$. ■

1.2 Continuity: ϵ-δ Definition

Concept Overview

A function $f(x)$ is **continuous** at $x = a$ if:

1. $f(a)$ is defined,
2. $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$.

The ϵ-δ definition formalizes this:

$$f \text{ is continuous at } a \iff \forall\epsilon > 0, \exists\delta > 0 \text{ such that } |x-a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

This ensures no jumps, breaks, or oscillations at a.

Example 1.2.1. Prove that $f(x) = 4x - 1$ is continuous at $x = 3$ using the ϵ-δ definition.

Solution. We must show: $\forall\epsilon > 0, \exists\delta > 0$ such that $|x-3| < \delta \implies |(4x-1) - 11| < \epsilon$.
Note $f(3) = 4(3) - 1 = 11$.
Simplify:

$$|(4x-1) - 11| = |4x - 12| = 4|x-3|$$

Choose $\delta = \epsilon/4$. If $|x-3| < \delta$, then:

$$4|x-3| < 4 \cdot (\epsilon/4) = \epsilon$$

Thus, $f(x) = 4x - 1$ is continuous at $x = 3$. ■

Example 1.2.2. Prove $f(x) = x^2$ is continuous at $x = 2$.

Solution. $f(2) = 4$. For any $\epsilon > 0$, choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$. If $|x-2| < \delta$, then:

$$\begin{aligned}
|x+2| &= |(x-2) + 4| \\
&\le |x-2| + 4 < 1 + 4 = 5, \\
|f(x) - f(2)| &= |x^2 - 4| \\
&= |(x-2)(x+2)| \\
&< \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon.
\end{aligned}$$

Thus, $f(x) = x^2$ is continuous at $x = 2$. ■

Example 1.2.3. Prove that $g(x) = x^2 + 2$ is continuous at $x = -1$ using the ϵ-δ definition.

Solution. We must show: $\forall\epsilon > 0, \exists\delta > 0$ such that $|x - (-1)| < \delta \implies |(x^2+2) - 3| < \epsilon$.
Note $g(-1) = (-1)^2 + 2 = 3$.
Simplify:

$$|(x^2+2) - 3| = |x^2 - 1| = |x-1||x+1|$$

Set $\delta \le 1$. Then $|x+1| < 1$ implies $-2 < x < 0$, so $|x-1| < 3$.
Now:

$$|x-1||x+1| < 3|x+1|$$

Choose $\delta = \min(1, \epsilon/3)$. If $|x+1| < \delta$, then:

$$|g(x) - g(-1)| < 3 \cdot \delta \le 3 \cdot (\epsilon/3) = \epsilon$$

Thus, $g(x) = x^2 + 2$ is continuous at $x = -1$. ■

Theorem 1.2.4. *If the limit of a function $f(x)$ as x approaches a exists, then this limit is unique.*

Proof. Assume, for the sake of contradiction, that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$ where $L_1 \neq L_2$.

By the definition of a limit, there exists a $\delta > 0$ such that for any x satisfying $0 < |x-a| < \delta$, we must have both:

$$|f(x) - L_1| < \epsilon \quad \text{and} \quad |f(x) - L_2| < \epsilon$$

Using the triangle inequality, we can write:

$$\begin{aligned}|L_1 - L_2| &= |(L_1 - f(x)) + (f(x) - L_2)| \\ &\leq |f(x) - L_1| + |f(x) - L_2| \\ &< \epsilon + \epsilon = 2\epsilon\end{aligned}$$

Let $\epsilon = \frac{|L_1-L_2|}{2}$. Since $L_1 \neq L_2$, we know that $\epsilon > 0$.

Substituting our choice of ϵ:

$$|L_1 - L_2| < 2\left(\frac{|L_1 - L_2|}{2}\right) = |L_1 - L_2|$$

This results in the contradiction $|L_1 - L_2| < |L_1 - L_2|$. Therefore, our initial assumption must be false, and the limit must be unique. ∎

Theorem 1.2.5 (The Algebra of Limits)**.** *Let c be a real number, and let $f(x)$ and $g(x)$ be two functions such that*

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M$$

where L and M are finite real numbers. Then the following properties hold:

1. ***Sum Rule:*** *The limit of the sum is the sum of the limits.*

$$\lim_{x\to c}[f(x) + g(x)] = L + M$$

2. ***Difference Rule:*** *The limit of the difference is the difference of the limits.*

$$\lim_{x\to c}[f(x) - g(x)] = L - M$$

3. ***Product Rule:*** *The limit of the product is the product of the limits.*

$$\lim_{x\to c}[f(x)g(x)] = LM$$

4. ***Quotient Rule:*** *The limit of the quotient is the quotient of the limits, provided the limit of the denominator is not zero.*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad \text{provided } M \neq 0$$

1. Proof of the Sum Rule. Let $\epsilon > 0$ be given. We want to find a $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|(f(x)+g(x)) - (L+M)| < \epsilon$.

By the definition of the limits for f and g, we know that for $\epsilon/2 > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

- if $0 < |x-c| < \delta_1$, then $|f(x) - L| < \epsilon/2$.
- if $0 < |x-c| < \delta_2$, then $|g(x) - M| < \epsilon/2$.

Let us choose $\delta = \min(\delta_1, \delta_2)$. Then for any x satisfying $0 < |x-c| < \delta$, both of the above inequalities hold. Using the triangle inequality, we have:

$$\begin{aligned}|(f(x) + g(x)) - (L+M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

The proof for the DIfference Rule is analogous.

∎

3. Proof of the Product Rule. Let $\epsilon > 0$ be given. We want to show $|f(x)g(x) - LM| < \epsilon$.

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M| \end{aligned}$$

Since $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|g(x) - M| < 1$, which implies $|g(x)| < |M| + 1$. Let $K = |M| + 1$. Now, for a given $\epsilon > 0$, we can find δ_2 and δ_3 such that:

- if $0 < |x - c| < \delta_2$, then $|f(x) - L| < \frac{\epsilon}{2K}$.
- if $0 < |x - c| < \delta_3$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. (We use $|L| + 1$ to avoid division by zero if $L = 0$).

Choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x - c| < \delta$, then:

$$\begin{aligned} |f(x)g(x) - LM| &\leq |g(x)||f(x) - L| + |L||g(x) - M| \\ &< K \cdot \frac{\epsilon}{2K} + |L| \cdot \frac{\epsilon}{2(|L| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, we have shown that $\lim_{x\to c}[f(x)g(x)] = LM$. ■

4. Proof of the Quotient Rule. It suffices to first prove that $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ for $M \neq 0$, and then apply the Product Rule to $f(x) \cdot \frac{1}{g(x)}$.

Let $\epsilon > 0$ be given. We want to show $\left|\frac{1}{g(x)} - \frac{1}{M}\right| < \epsilon$.

$$\left|\frac{M - g(x)}{Mg(x)}\right| = \frac{|g(x) - M|}{|M||g(x)|}$$

Since $M \neq 0$, $|M|/2 > 0$. There exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|g(x) - M| < |M|/2$. By the reverse triangle inequality, this implies $|g(x)| > |M| - |g(x) - M| > |M| - |M|/2 = |M|/2$. Therefore, $\frac{1}{|g(x)|} < \frac{2}{|M|}$. Now, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \frac{\epsilon|M|^2}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - c| < \delta$, then:

$$\frac{|g(x) - M|}{|M||g(x)|} < \frac{|g(x) - M|}{|M|(|M|/2)} = \frac{2|g(x) - M|}{|M|^2} < \frac{2}{|M|^2} \cdot \frac{\epsilon|M|^2}{2} = \epsilon$$

So, $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$. By the Product Rule, $\lim_{x\to c} \frac{f(x)}{g(x)} = (\lim_{x\to c} f(x)) \left(\lim_{x\to c} \frac{1}{g(x)}\right) = L \cdot \frac{1}{M} = \frac{L}{M}$. ■

Exercise 1.2.6. Q1. **(Limit)** Prove $\lim_{x\to 4}(5 - 2x) = -3$ using ϵ-δ definition.

Q2. **(Limit)** Prove $\lim_{x\to -2}(x^2 + 3x) = -2$ using ϵ-δ definition.

Q3. **(Continuity)** Prove $h(x) = \sqrt{x}$ is continuous at $x = 4$ using ϵ-δ definition. (Hint: Use $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2}$).

Q4. **(Continuity)** Prove $k(x) = \dfrac{1}{x}$ is continuous at $x = \dfrac{1}{2}$ using ϵ-δ definition.

1.3 Types of Discontinuities

Discontinuities occur where a function fails to be continuous. Common types:

1. **Removable**: $\lim_{x\to a} f(x)$ exists but $f(a)$ is undefined or $\lim_{x\to a} f(x) \neq f(a)$. "Fixed" by redefining $f(a)$.
2. **Jump or discontinuity of first kind**: Left-hand and right-hand limits exist but are unequal ($\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$).
3. **Infinite or discontinuityof second kind**: $\lim_{x\to a} f(x) = \pm\infty$.
4. **Oscillating**: Function oscillates infinitely often near a (e.g., $\sin(1/x)$ at $x = 0$).

Example 1.3.1. Classify the discontinuity of $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$.

Solution. The function is undefined at $x = 2$. Simplify for $x \neq 2$:

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

The limit exists:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2) = 4.$$

Since the limit exists but $f(2)$ is undefined, the discontinuity is **removable**. ■

Example 1.3.2. Classify the discontinuity of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x+3 & \text{if } x \geq 2 \end{cases}$$

at $x = 2$.

Solution. Calculate left-hand and right-hand limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+3) = 2 + 3 = 5$$

Since $4 \neq 5$, the left-hand and right-hand limits exist but are not equal.
Function value: $f(2) = 2 + 3 = 5$.
The discontinuity at $x = 2$ is a **jump discontinuity**. ■

Example 1.3.3. Classify the discontinuity of $g(x) = \dfrac{1}{(x-3)^2}$ at $x = 3$.

Solution. Examine the limit as x approaches 3:

$$\lim_{x\to 3} \frac{1}{(x-3)^2} = +\infty$$

The function is undefined at $x = 3$. As $x \to 3$, the function values increase without bound.
This is an **infinite discontinuity** (vertical asymptote at $x = 3$). ■

Example 1.3.4. Classify the discontinuity of $h(x) = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$.

Solution. Consider the behavior as $x \to 0$:

- As $x \to 0^+$, $\frac{1}{x} \to +\infty$ and $\sin(1/x)$ oscillates between -1 and 1
- As $x \to 0^-$, $\frac{1}{x} \to -\infty$ and $\sin(1/x)$ oscillates between -1 and 1

The limit $\lim_{x\to 0} \sin(1/x)$ does not exist because oscillations become increasingly rapid.
This is an **oscillating discontinuity** at $x = 0$. ■

Example 1.3.5. Examine the following function for continuity at the origin.

$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution. $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$ and $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} \dfrac{x}{e^{-1/x}+1} = 0$ Also, $f(0) = 0$. Thus, the function is continuous at the origin. ■

1.3.1 Piecewise Continuity

A function $f(x)$ is said to be piecewise continuous in an interval I, if the interval can be subdivided into a finite number of subintervals such that $f(x)$ is continuous in each of the subintervals and the limits of $f(x)$ as x approaches the end points of each subinterval are finite. For example, the greatest integer function $f(x) = [x]$ defined on $[-1, 3]$ is piecewise continuous on $[-1, 3]$.

1.4 Differentiability of Functions

Concept Overview

A function $f(x)$ is **differentiable** at a point $x = a$ if the derivative $f'(a)$ exists. Geometrically, this means the function has a unique tangent at that point. The derivative is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists finitely.

Necessary Conditions

- If f is differentiable at a, then it must be continuous at a.
- The converse is not true: continuity does not imply differentiability (e.g., $|x|$ at $x = 0$).

(1). Proof. To prove that the function f is continuous at a, we must show that $\lim_{x \to a} f(x) = f(a)$. This is equivalent to showing that the difference between $f(x)$ and $f(a)$ approaches zero, i.e.,

$$\lim_{x \to a} [f(x) - f(a)] = 0$$

We are given that f is differentiable at a. By definition, this means the limit for the derivative exists and is a finite number:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

that is,

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a} \right) \cdot (x - a)$$

We can now take the limit of both sides as $x \to a$ and apply the product rule for limits:

$$\begin{aligned}
\lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right) \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left(\lim_{x \to a} (x - a) \right) \\
&= f'(a) \cdot (a - a) \\
&= f'(a) \cdot 0 \\
&= 0
\end{aligned}$$

Since we have shown that $\lim_{x \to a}[f(x) - f(a)] = 0$, it follows directly that $\lim_{x \to a} f(x) = f(a)$. This is the definition of continuity at the point a. ∎

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Left and Right Derivatives

The **left derivative** at a:

$$f'_{-}(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

The **right derivative** at a:

$$f'_{+}(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

f is differentiable at a iff $f'_{-}(a) = f'_{+}(a)$ and both exist finitely.

Example 1.4.1. Show that $f(x) = x^2 + 3x$ is differentiable at $x = 2$ and find its derivative.

Solution. Compute the derivative using the limit definition:

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{[(2+h)^2+3(2+h)]-[2^2+3(2)]}{h} \\
&= \lim_{h\to 0} \frac{[4+4h+h^2+6+3h]-[4+6]}{h} \\
&= \lim_{h\to 0} \frac{7h+h^2}{h} \\
&= \lim_{h\to 0}(7+h) = 7
\end{aligned}
$$

Since the limit exists, f is differentiable at $x=2$ with $f'(2)=7$. ■

1.5 Successive Differentiation

Concept Overview

Successive differentiation refers to repeatedly differentiating a function. The nth derivative is denoted by:

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n y}{dx^n}$$

where n is the order of differentiation.

Standard Formulas

- $\dfrac{d^n}{dx^n}(x^m) = m(m-1)\cdots(m-n+1)x^{m-n} \quad \text{for} \quad n \le m$
- $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$
- $\dfrac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$
- $\dfrac{d^n}{dx^n}(\sin ax) = a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}(\cos ax) = a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$

Example 1.5.1. Find the third derivative of $g(x) = 2x^4 - 5x^3 + 3x - 7$.

Solution. Compute successive derivatives:

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}(2x^4 - 5x^3 + 3x - 7) = 8x^3 - 15x^2 + 3 \\
g''(x) &= \frac{d}{dx}(8x^3 - 15x^2 + 3) = 24x^2 - 30x \\
g'''(x) &= \frac{d}{dx}(24x^2 - 30x) = \boxed{48x - 30}
\end{aligned}
$$

■

1.6 Leibnitz Theorem

Theorem 1.6.1 (**Leibnitz Theorem**)**.** *If $u(x)$ and $v(x)$ are n-times differentiable functions, then:*

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

where $u^{(k)}$ is the kth derivative of u, and $v^{(n-k)}$ is the $(n-k)$th derivative of v, with $u^{(0)} = u$ and $v^{(0)} = v$.

Proof. We must show that the formula is true for $n = 1$.

$$\begin{aligned}
(uv)^{(1)} &= \sum_{k=0}^{1} \binom{1}{k} u^{(1-k)} v^{(k)} \\
&= \binom{1}{0} u^{(1)} v^{(0)} + \binom{1}{1} u^{(0)} v^{(1)} \\
&= (1) \cdot u'v + (1) \cdot uv' \\
&= u'v + uv'
\end{aligned}$$

Assume the theorem is true for some positive integer $n = m$. That is, we assume:

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}$$

We must prove that the theorem is true for $n = m + 1$. We start by differentiating the expression from our inductive hypothesis with respect to x:

$$\begin{aligned}
(uv)^{(m+1)} &= \frac{d}{dx} \left[(uv)^{(m)} \right] \\
&= \frac{d}{dx} \left[\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)} \right] \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{d}{dx} \left(u^{(m-k)} v^{(k)} \right) \\
&= \sum_{k=0}^{m} \binom{m}{k} \left[u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)} \right]
\end{aligned}$$

Now, we split this into two separate sums:

$$(uv)^{(m+1)} = \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)}}_{\text{Sum A}} + \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k+1)}}_{\text{Sum B}}$$

To combine these sums, we re-index Sum B by letting $j = k + 1$. This means Sum B will go from $j = 1$ to $j = m + 1$. Replacing j back with k, Sum B becomes:

$$\text{Sum B} = \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m+1-k)} v^{(k)}$$

Now we combine the re-indexed Sum B with Sum A. We can separate the first term ($k = 0$) from Sum A and the last term ($k = m + 1$) from Sum B:

$$\begin{aligned}
(uv)^{(m+1)} = &\left[\binom{m}{0} u^{(m+1)} v + \sum_{k=1}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)} \right] \\
&+ \left[\sum_{k=1}^{m} \binom{m}{k-1} u^{(m+1-k)} v^{(k)} + \binom{m}{m} uv^{(m+1)} \right]
\end{aligned}$$

Grouping the two middle sums together:

$$(uv)^{(m+1)} = \binom{m}{0} u^{(m+1)} v + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] u^{(m+1-k)} v^{(k)} + \binom{m}{m} uv^{(m+1)}$$

We use Pascal's Identity: $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$. Also, we know that $\binom{m}{0} = 1 = \binom{m+1}{0}$ and $\binom{m}{m} = 1 = \binom{m+1}{m+1}$. Substituting these identities into our expression gives:

$$(uv)^{(m+1)} = \binom{m+1}{0} u^{(m+1)} v + \sum_{k=1}^{m} \binom{m+1}{k} u^{(m+1-k)} v^{(k)} + \binom{m+1}{m+1} uv^{(m+1)}$$

This entire expression can now be combined into a single sum from $k = 0$ to $k = m + 1$:

$$(uv)^{(m+1)} = \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(m+1-k)} v^{(k)}$$

This is precisely the form of the theorem for $n = m + 1$.

By the principle of mathematical induction, the theorem is true for all positive integers n. ■

Key Features

- Analogous to the binomial theorem
- Coefficients are binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Requires both functions to have derivatives up to order n

Example 1.6.2. Find the second derivative of $h(x) = x^2 e^{3x}$ using Leibnitz theorem.

Solution. Set $u = e^{3x}$, $v = x^2$. Apply Leibnitz theorem for $n = 2$:

$$(uv)'' = \sum_{k=0}^{2} \binom{2}{k} u^{(k)} v^{(2-k)} = u^{(0)}v^{(2)} + 2 \cdot u^{(1)}v^{(1)} + u^{(2)}v^{(0)}$$

Compute terms:

$$\begin{aligned} k = 0: \quad & \binom{2}{0} u^{(0)}v^{(2)} = 1 \cdot e^{3x} \cdot 2 \\ k = 1: \quad & \binom{2}{1} u^{(1)}v^{(1)} = 2 \cdot (3e^{3x}) \cdot (2x) \\ k = 2: \quad & \binom{2}{2} u^{(2)}v^{(0)} = 1 \cdot (9e^{3x}) \cdot (x^2) \end{aligned}$$

Sum the terms:

$$h''(x) = 2e^{3x} + 12xe^{3x} + 9x^2e^{3x} = \boxed{e^{3x}(9x^2 + 12x + 2)}$$

■

Example 1.6.3. Find the fourth derivative of the function $f(x) = x^3 e^{2x}$ using Leibnitz theorem.

Solution. Set $u = e^{2x}$ and $v = x^3$. Apply Leibnitz theorem for $n = 4$:

$$(uv)^{(4)} = \sum_{k=0}^{4} \binom{4}{k} u^{(k)} v^{(4-k)}$$

Compute derivatives of u and v:

$$u = e^{2x} \qquad\qquad v = x^3$$

$$u^{(k)} = 2^k e^{2x} \qquad\qquad v^{(m)} = \begin{cases} \frac{3!}{(3-m)!} x^{3-m} & 0 \le m \le 3 \\ 0 & m > 3 \end{cases}$$

Calculate each term:

$$\begin{aligned} k = 0: \quad & \binom{4}{0} u^{(0)}v^{(4)} = 1 \cdot 2^0 e^{2x} \cdot 0 = 0 \\ k = 1: \quad & \binom{4}{1} u^{(1)}v^{(3)} = 4 \cdot 2^1 e^{2x} \cdot 6 = 48e^{2x} \\ k = 2: \quad & \binom{4}{2} u^{(2)}v^{(2)} = 6 \cdot 2^2 e^{2x} \cdot 6x = 144xe^{2x} \\ k = 3: \quad & \binom{4}{3} u^{(3)}v^{(1)} = 4 \cdot 2^3 e^{2x} \cdot 3x^2 = 96x^2e^{2x} \\ k = 4: \quad & \binom{4}{4} u^{(4)}v^{(0)} = 1 \cdot 2^4 e^{2x} \cdot x^3 = 16x^3e^{2x} \end{aligned}$$

Sum all terms:

$$\begin{aligned} f^{(4)}(x) &= 0 + 48e^{2x} + 144xe^{2x} + 96x^2e^{2x} + 16x^3e^{2x} \\ &= e^{2x}(16x^3 + 96x^2 + 144x + 48) \end{aligned}$$

Factor out 16:

$$f^{(4)}(x) = 16e^{2x}(x^3 + 6x^2 + 9x + 3)$$

Thus, the fourth derivative is $\boxed{16e^{2x}(x^3 + 6x^2 + 9x + 3)}$. ■

Example 1.6.4. If $y = e^{a\sin^{-1}(x)}$, prove the following:

(a) $(1-x^2)y_2 - xy_1 - a^2y = 0$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

(b) Hence, using Leibniz's Theorem, show that the following recurrence relation holds:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Solution. We are given the function $y = e^{a\sin^{-1}(x)}$. Using the chain rule, we differentiate y with respect to x:

$$y_1 = \frac{dy}{dx} = e^{a\sin^{-1}(x)} \cdot \frac{d}{dx}(a\sin^{-1}(x))$$

$$y_1 = e^{a\sin^{-1}(x)} \cdot \frac{a}{\sqrt{1-x^2}}$$

that is,

$$y_1 = \frac{ay}{\sqrt{1-x^2}}$$

because $y = e^{a\sin^{-1}(x)}$.

Squaring both sides of the equation:

$$y_1^2 = \frac{a^2y^2}{1-x^2}$$

Now, multiply both sides by $(1-x^2)$ to clear the fraction:

$$(1-x^2)y_1^2 = a^2y^2 \quad (*).$$

We differentiate the entire equation $(*)$ with respect to x.

$$\frac{d}{dx}\left[(1-x^2)y_1^2\right] = \frac{d}{dx}\left[a^2y^2\right]$$

$$\left(\frac{d}{dx}(1-x^2)\right) \cdot y_1^2 + (1-x^2) \cdot \left(\frac{d}{dx}(y_1^2)\right) = a^2 \cdot \left(\frac{d}{dx}(y^2)\right)$$

$$(-2x)y_1^2 + (1-x^2)(2y_1y_2) = a^2(2yy_1)$$

Divide the entire equation by $2y_1$:

$$-xy_1 + (1-x^2)y_2 = a^2y$$

Rearranging the terms gives the desired differential equation:

$$(1-x^2)y_2 - xy_1 - a^2y = 0$$

Applying Leibniz's Theorem to find the Recurrence Relation

We now differentiate the equation $(1-x^2)y_2 - xy_1 - a^2y = 0$ successively n times with respect to x.

$$\frac{d^n}{dx^n}\left[(1-x^2)y_2 - xy_1 - a^2y\right] = 0$$

By linearity of the derivative, we can differentiate each term separately:

$$\underbrace{\frac{d^n}{dx^n}\left[(1-x^2)y_2\right]}_{\text{Term A}} - \underbrace{\frac{d^n}{dx^n}\left[xy_1\right]}_{\text{Term B}} - \underbrace{\frac{d^n}{dx^n}\left[a^2y\right]}_{\text{Term C}} = 0$$

For Term: $D^n[(1-x^2)y_2]$ Let $u = y_2$ and $v = 1-x^2$. We apply Leibniz's Theorem. The derivatives of v terminate quickly:

- $v = 1-x^2$
- $v' = -2x$
- $v'' = -2$
- $v''' = 0$

The derivatives of u are $u^{(k)} = (y_2)^{(k)} = y_{k+2}$. The Leibniz expansion is:

$$\binom{n}{0} vu^{(n)} + \binom{n}{1} v'u^{(n-1)} + \binom{n}{2} v''u^{(n-2)} + \dots$$

Substituting our functions (only the first three terms are non-zero):

$$(1)(1-x^2)y_{n+2} + (n)(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n$$

Simplifying gives: $(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n$.
For Term: $D^n[xy_1]$ Let $u = y_1$ and $v = x$. The derivatives of v are $v' = 1$ and $v'' = 0$. The Leibniz expansion has only two non-zero terms:

$$\binom{n}{0} vu^{(n)} + \binom{n}{1} v'u^{(n-1)}$$

Substituting our functions:

$$(1)(x)y_{n+1} + (n)(1)y_n = xy_{n+1} + ny_n$$

For Term: $D^n[a^2y]$ Since a^2 is a constant, this is straightforward: a^2y_n.
Combining the results: Now we substitute the expanded terms back into our main equation:

$$[(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n] - [xy_{n+1} + ny_n] - [a^2y_n] = 0$$

Finally, we group the terms by the order of the derivative (y_{n+2}, y_{n+1}, y_n):

$$(1-x^2)y_{n+2} + (-2nx - x)y_{n+1} + (-n(n-1) - n - a^2)y_n = 0$$

Simplifying gives:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0$$

■

1.7 Partial Differentiation

Concept Overview

Partial differentiation deals with functions of multiple variables. The partial derivative of $f(x, y)$ with respect to x is denoted $\frac{\partial f}{\partial x}$ and measures the rate of change of f while keeping y constant.

Formal Definition

For $z = f(x, y)$, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Theorem 1.7.1 (Clairaut's Theorem)**.** *If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined on an open set containing (a, b) and are continuous at (a, b), then:*

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$$

Proof. For sufficiently small $h, k \neq 0$, define the auxiliary function:

$$\Delta(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).$$

Analyze via f_{yx}
Define $g(y) = f(a+h, y) - f(a, y)$. Then:

$$\Delta(h, k) = g(b+k) - g(b).$$

By the Mean Value Theorem (MVT), there exists d between b and $b+k$ such that:

$$g(b+k) - g(b) = k \cdot g'(d) = k \left[\frac{\partial f}{\partial y}(a+h, d) - \frac{\partial f}{\partial y}(a, d) \right].$$

Apply MVT to $h(x) = \frac{\partial f}{\partial y}(x, d)$ on $[a, a+h]$. There exists c_1 between a and $a+h$ such that:

$$\frac{\partial f}{\partial y}(a+h, d) - \frac{\partial f}{\partial y}(a, d) = h \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Analyze via f_{xy}
Define $r(x) = f(x, b+k) - f(x, b)$. Then:

$$\Delta(h, k) = r(a+h) - r(a).$$

By MVT, there exists e between a and $a+h$ such that:

$$r(a+h) - r(a) = h \cdot r'(e) = h \left[\frac{\partial f}{\partial x}(e, b+k) - \frac{\partial f}{\partial x}(e, b) \right].$$

Apply MVT to $s(y) = \frac{\partial f}{\partial x}(e, y)$ on $[b, b+k]$. There exists c_2 between b and $b+k$ such that:

$$\frac{\partial f}{\partial x}(e, b+k) - \frac{\partial f}{\partial x}(e, b) = k \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Equate and take limits
From Steps 1 and 2:

$$hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

For $hk \neq 0$, we have:

$$\frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

As $(h, k) \to (0, 0)$:

$$(c_1, d) \to (a, b) \quad \text{and} \quad (e, c_2) \to (a, b).$$

By continuity of f_{xy} and f_{yx} at (a, b):

$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial x \partial y}(a, b),$$
$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial y \partial x}(e, c_2) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Therefore:

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)}$$

■

Example 1.7.2. Find the first partial derivatives of $f(x, y) = x^3 y + e^{xy}$.

Solution.

$$\frac{\partial f}{\partial x} = 3x^2 y + y e^{xy}$$
$$\frac{\partial f}{\partial y} = x^3 + x e^{xy}$$

■

Example 1.7.3. Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x, y) = \sin(2x + 3y)$.

Solution. First derivatives:

$$\frac{\partial f}{\partial x} = 2\cos(2x+3y)$$
$$\frac{\partial f}{\partial y} = 3\cos(2x+3y)$$

Mixed derivative:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3\cos(2x+3y)) = -6\sin(2x+3y)$$

■

1.8 Total Differentiation

Concept Overview

Total differentiation extends differentiation to functions of multiple variables. The total differential dz approximates the change in $z = f(x, y)$ when both x and y change.

Total Differential Formula

For $z = f(x, y)$:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

For $w = f(x, y, z)$:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Chain Rule for Total Derivatives

If $z = f(x, y)$ with $x = g(t)$, $y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1.8.1. Find the total differential of $z = x^2y - 3xy^3$.

Solution. Partial derivatives:

$$\frac{\partial z}{\partial x} = 2xy - 3y^3$$
$$\frac{\partial z}{\partial y} = x^2 - 9xy^2$$

Total differential:

$$dz = (2xy - 3y^3)dx + (x^2 - 9xy^2)dy$$

■

Example 1.8.2. If $z = e^x \sin y$ where $x = t^2$ and $y = t^3$, find $\frac{dz}{dt}$.

Solution. Apply chain rule:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (e^x \sin y)(2t) + (e^x \cos y)(3t^2) \\ &= e^{t^2}\left[2t\sin(t^3) + 3t^2\cos(t^3)\right]\end{aligned}$$

■

1.9 Homogeneous Functions

Concept Overview

A function $f(x_1, x_2, \ldots, x_n)$ is **homogeneous of degree** k if for all $\lambda > 0$:

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)$$

Properties

- Linear functions are homogeneous of degree 1
- Quadratic forms are homogeneous of degree 2
- Constant functions are homogeneous of degree 0

Example 1.9.1. Show that $f(x, y) = x^3 + 3x^2y + y^3$ is homogeneous and find its degree.

Solution. Replace $x \to \lambda x$, $y \to \lambda y$:

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^3 + 3(\lambda x)^2(\lambda y) + (\lambda y)^3 \\ &= \lambda^3 x^3 + 3\lambda^3 x^2 y + \lambda^3 y^3 \\ &= \lambda^3(x^3 + 3x^2y + y^3) \\ &= \lambda^3 f(x, y) \end{aligned}$$

Thus homogeneous of degree 3. ∎

Example 1.9.2. Is $g(x, y) = x^2 + xy + \sin\left(\frac{x}{y}\right)$ homogeneous?

Solution. Test with $\lambda > 0$:

$$\begin{aligned} g(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda x)(\lambda y) + \sin\left(\frac{\lambda x}{\lambda y}\right) \\ &= \lambda^2 x^2 + \lambda^2 xy + \sin\left(\frac{x}{y}\right) \end{aligned}$$

The expression contains λ^2 terms and a λ-independent term. Not homogeneous. ∎

1.10 Euler's Theorem

Theorem 1.10.1 (Euler's Theorem on Homogeneous Functions). *If $f(x, y)$ is a homogeneous function of degree k and has continuous first partial derivatives, then:*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = k \cdot f(x, y)$$

Proof. By definition, since f is homogeneous of degree k, we have the following identity for any $t > 0$:

$$f(tx, ty) = t^k f(x, y) \tag{1.1}$$

We differentiate both sides of the identity (1.1) with respect to the parameter t, treating x and y as constants.

For the left-hand side (LHS), we use the multivariable chain rule. Let $X = tx$ and $Y = ty$. Then:

$$\begin{aligned} \frac{d}{dt}f(tx, ty) &= \frac{\partial f}{\partial X}\frac{dX}{dt} + \frac{\partial f}{\partial Y}\frac{dY}{dt} \\ &= f_X(tx, ty) \cdot (x) + f_Y(tx, ty) \cdot (y) \end{aligned}$$

where f_X and f_Y denote the partial derivatives of f with respect to its first and second arguments, respectively.

For the right-hand side (RHS), we treat $f(x, y)$ as a constant and differentiate only the t^k term:

$$\frac{d}{dt}\left(t^k f(x, y)\right) = kt^{k-1} f(x, y)$$

Equating the derivatives of the LHS and RHS gives us a new identity that is also true for all $t > 0$:

$$x \cdot f_X(tx, ty) + y \cdot f_Y(tx, ty) = kt^{k-1} f(x, y)$$

Since the identity above holds for any value of $t > 0$, it must hold for the specific case where $t = 1$. Setting $t = 1$ simplifies the expression:

$$\begin{aligned} x \cdot f_X(1 \cdot x, 1 \cdot y) + y \cdot f_Y(1 \cdot x, 1 \cdot y) &= k(1)^{k-1} f(x, y) \\ x \cdot f_X(x, y) + y \cdot f_Y(x, y) &= k \cdot f(x, y) \end{aligned}$$

Rewriting in standard partial derivative notation, we get the desired result:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = kf(x, y)$$ ∎

Example 1.10.2. Verify Euler's theorem for $f(x,y) = x^{1/3}y^{2/3}$.

Solution. First, degree $k = 1/3 + 2/3 = 1$. Partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{1}{3}x^{-2/3}y^{2/3}$$
$$\frac{\partial f}{\partial y} = \frac{2}{3}x^{1/3}y^{-1/3}$$

Apply Euler's theorem:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\left(\frac{1}{3}x^{-2/3}y^{2/3}\right) + y\left(\frac{2}{3}x^{1/3}y^{-1/3}\right) \\ &= \frac{1}{3}x^{1/3}y^{2/3} + \frac{2}{3}x^{1/3}y^{2/3} \\ &= x^{1/3}y^{2/3} = f(x,y) \end{aligned}$$

Equal to $1 \cdot f$, verifying the theorem. ∎

Example 1.10.3. Using Euler's theorem, show that if $f = \frac{x^2+y^2}{xy}$, then $xf_x + yf_y = -f$.

Solution. Rewrite $f(x,y) = \frac{x}{y} + \frac{y}{x}$. Test homogeneity:

$$f(\lambda x, \lambda y) = \frac{\lambda x}{\lambda y} + \frac{\lambda y}{\lambda x} = \frac{x}{y} + \frac{y}{x} = f(x,y)$$

Thus homogeneous of degree 0. By Euler's theorem:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 \cdot f = 0$$

But note: $0 = -f + f$, so rearrange as $xf_x + yf_y = -f + f$. To get exact form, observe:

$$xf_x + yf_y = 0 = -f + f$$

The problem statement appears inconsistent. Correction: For homogeneous degree 0, $xf_x + yf_y = 0$, while $-f = -\left(\frac{x}{y} + \frac{y}{x}\right)$. They are not equal. The correct conclusion is $xf_x + yf_y = 0$. ∎

Chapter 2

Tangents, Normals and Subnormals

2.1 Introduction to Indeterminate Forms

In calculus, when evaluating limits, we sometimes encounter forms that do not immediately give a clear answer. If direct substitution of the limit point into the function results in expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$, these are called **indeterminate forms**. They are called "indeterminate" because the actual limit cannot be determined from this form alone. The limit might be a finite number, zero, infinity, or it might not exist at all.

The primary tool for dealing with indeterminate forms is **L'Hôpital's Rule**, but other algebraic and logarithmic techniques are also crucial.

The seven common indeterminate forms are:

- **Quotient Forms:** $\frac{0}{0}, \quad \frac{\infty}{\infty}$
- **Product Form:** $0 \cdot \infty$
- **Difference Form:** $\infty - \infty$
- **Power Forms:** $0^0, \quad 1^\infty, \quad \infty^0$

2.2 The Quotient Forms: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

These two forms are the basis for L'Hôpital's Rule.

2.2.1 L'Hôpital's Rule

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. If f and g are differentiable near a (except possibly at a) and the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Important Note: Always check that the limit is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before applying the rule.

2.2.2 The Form $\frac{0}{0}$

Example 2.2.1. Evaluate $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$.

Solution:
First, we check the form by direct substitution:

$$\frac{3^2-9}{3-3} = \frac{9-9}{3-3} = \frac{0}{0} \quad \text{(Indeterminate Form)}$$

Since it is of the form $\frac{0}{0}$, we can apply L'Hôpital's Rule. Let $f(x) = x^2 - 9$ and $g(x) = x - 3$. Then $f'(x) = 2x$ and $g'(x) = 1$.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{f'(x)}{g'(x)} = \lim_{x\to 3} \frac{2x}{1} = 2(3) = 6$$

Example 2.2.2. Evaluate $\lim_{x\to 0} \frac{e^{2x}-1}{\sin(x)}$.

Solution: Check the form by direct substitution:

$$\frac{e^{2(0)}-1}{\sin(0)} = \frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0} \quad \text{(Indeterminate Form)}$$

Apply L'Hôpital's Rule. Let $f(x) = e^{2x} - 1$ and $g(x) = \sin(x)$. Then $f'(x) = 2e^{2x}$ and $g'(x) = \cos(x)$.

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin(x)} = \lim_{x\to 0} \frac{2e^{2x}}{\cos(x)} = \frac{2e^{2(0)}}{\cos(0)} = \frac{2(1)}{1} = 2$$

2.2.3 The Form $\frac{\infty}{\infty}$

Example 2.2.3. Evaluate $\lim_{x\to\infty} \frac{\ln(x)}{x^2}$.

Solution: Check the form as $x \to \infty$:

$$\frac{\ln(\infty)}{\infty^2} \to \frac{\infty}{\infty} \quad \text{(Indeterminate Form)}$$

Apply L'Hôpital's Rule. Let $f(x) = \ln(x)$ and $g(x) = x^2$. Then $f'(x) = \frac{1}{x}$ and $g'(x) = 2x$.

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

Example 2.2.4. Evaluate $\lim_{x\to\infty} \frac{5x^3-2x}{3x^3+x^2+1}$.

Solution: Check the form as $x \to \infty$:

$$\frac{\infty}{\infty} \quad \text{(Indeterminate Form)}$$

Apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{15x^2-2}{9x^2+2x} \quad \left(\text{Still } \frac{\infty}{\infty}, \text{ apply again}\right)$$

Apply L'Hôpital's Rule a second time.

$$\lim_{x\to\infty} \frac{30x}{18x+2} \quad \left(\text{Still } \frac{\infty}{\infty}, \text{ apply again}\right)$$

Apply L'Hôpital's Rule a third time.

$$\lim_{x\to\infty} \frac{30}{18} = \frac{30}{18} = \frac{5}{3}$$

2.3 The Product Form: $0 \cdot \infty$

This form must be converted to a quotient form ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) before applying L'Hôpital's Rule. The conversion is done by rewriting the product $f(x) \cdot g(x)$ as a fraction:

$$f(x) \cdot g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x) \cdot g(x) = \frac{g(x)}{1/f(x)}$$

Choose the form that is easier to differentiate.

Example 2.3.1. Evaluate $\lim_{x\to 0^+} x\ln(x)$.

Solution:
Check the form: as $x \to 0^+$, we have $0 \cdot \ln(0^+) \to 0 \cdot (-\infty)$. This is an indeterminate form. We rewrite the product as a quotient. It is easier to differentiate $\ln(x)$ than $1/\ln(x)$, so we keep $\ln(x)$ in the numerator.

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \quad \left(\text{Now of the form } \frac{-\infty}{\infty}\right)$$

Apply L'Hôpital's Rule.

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \left(\frac{1}{x} \cdot (-x^2)\right) = \lim_{x\to 0^+} (-x) = 0$$

Example 2.3.2. Evaluate $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$.

Solution: Check the form: as $x \to \infty$, we have $\infty \cdot \sin(0) \to \infty \cdot 0$. This is an indeterminate form. Rewrite as a quotient. Let's move x to the denominator as $1/x$.

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \quad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule. Let $f(u) = \sin(u)$ and $g(u) = u$, where $u = 1/x$. As $x \to \infty$, $u \to 0$. The limit becomes $\lim_{u\to 0} \frac{\sin(u)}{u}$. Using L'Hôpital's Rule on this new limit:

$$\lim_{u\to 0} \frac{\cos(u)}{1} = \frac{\cos(0)}{1} = 1$$

Alternatively, differentiating with respect to x:

$$\lim_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \cos(1/x) = \cos(0) = 1$$

2.4 The Difference Form: $\infty - \infty$

This form requires algebraic manipulation to be converted into a single fraction (a quotient form). Common techniques include finding a common denominator, factoring, or multiplying by the conjugate.

Example 2.4.1. Evaluate $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln(x)}\right)$.

Solution: Check the form: as $x \to 1^+$, we have $\frac{1}{0^+} - \frac{1}{\ln(1^+)} \to \infty - \infty$. This is an indeterminate form. Combine the terms into a single fraction by finding a common denominator.

$$\lim_{x\to 1^+} \frac{\ln(x) - (x-1)}{(x-1)\ln(x)} \quad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule. The denominator requires the product rule. $f'(x) = \frac{1}{x} - 1$. $g'(x) = (1)\ln(x) + (x-1)\frac{1}{x} = \ln(x) + 1 - \frac{1}{x}$.

$$\lim_{x\to 1^+} \frac{1/x - 1}{\ln(x) + 1 - 1/x} \quad \left(\text{Still } \frac{0}{0}, \text{ apply again}\right)$$

Apply L'Hôpital's Rule a second time. $f''(x) = -1/x^2$. $g''(x) = 1/x + 1/x^2$.

$$\lim_{x\to 1^+} \frac{-1/x^2}{1/x + 1/x^2} = \frac{-1/1^2}{1/1 + 1/1^2} = \frac{-1}{1+1} = -\frac{1}{2}$$

Example 2.4.2. Evaluate $\lim_{x\to\infty} (\sqrt{x^2+4x} - x)$.

Solution: Check the form: as $x \to \infty$, we have $\infty - \infty$. This is an indeterminate form. We multiply by the conjugate, $\sqrt{x^2+4x} + x$.

$$\begin{aligned}
\lim_{x\to\infty} (\sqrt{x^2+4x} - x) &= \lim_{x\to\infty} \frac{(\sqrt{x^2+4x} - x)(\sqrt{x^2+4x} + x)}{\sqrt{x^2+4x} + x} \\
&= \lim_{x\to\infty} \frac{(x^2+4x) - x^2}{\sqrt{x^2+4x} + x} \\
&= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4x} + x} \quad \left(\text{Now of the form } \frac{\infty}{\infty}\right)
\end{aligned}$$

Now we can use L'Hôpital's Rule, but it's often easier to divide by the highest power of x in the denominator. Here, the highest power is $x = \sqrt{x^2}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x/x}{(\sqrt{x^2+4x} + x)/x} &= \lim_{x\to\infty} \frac{4}{\sqrt{\frac{x^2+4x}{x^2}} + 1} \\
&= \lim_{x\to\infty} \frac{4}{\sqrt{1+4/x} + 1} \\
&= \frac{4}{\sqrt{1+0} + 1} = \frac{4}{1+1} = 2
\end{aligned}$$

2.5 The Exponential Indeterminate Forms: $1^\infty, 0^0, \infty^0$

These forms are handled by taking the natural logarithm of the function. Let $L = \lim_{x\to a} f(x)^{g(x)}$.

1. Take the natural logarithm: $\ln(L) = \ln\left(\lim_{x\to a} f(x)^{g(x)}\right) = \lim_{x\to a} \ln(f(x)^{g(x)})$.
2. Use logarithm properties: $\ln(L) = \lim_{x\to a} g(x)\ln(f(x))$.
3. Evaluate this new limit, which will be of the form $0\cdot\infty$. Let's say $\lim_{x\to a} g(x)\ln(f(x)) = K$.
4. Solve for L: Since $\ln(L) = K$, the original limit is $L = e^K$.

2.5.1 The Form 1^∞

Example 2.5.1. Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$.

Solution: Check the form: as $x\to\infty$, $1+\frac{3}{x}\to 1$ and $2x\to\infty$. The form is 1^∞. Let $L = \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to\infty}\ln\left(\left(1+\frac{3}{x}\right)^{2x}\right) = \lim_{x\to\infty} 2x\ln\left(1+\frac{3}{x}\right) \quad (\text{Form } \infty\cdot 0)$$

Rewrite as a quotient:

$$\ln(L) = \lim_{x\to\infty}\frac{2\ln(1+3/x)}{1/x} \qquad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to\infty}\frac{2\cdot\frac{1}{1+3/x}\cdot(-3/x^2)}{-1/x^2} = \lim_{x\to\infty}\frac{2\cdot 1\cdot(-3)}{-1}\cdot\frac{1}{1+3/x} = \lim_{x\to\infty}\frac{6}{1+3/x} = \frac{6}{1+0} = 6$$

Since $\ln(L) = 6$, the final answer is $L = e^6$.

Example 2.5.2. Evaluate $\displaystyle\lim_{x\to 0}(1+\sin(x))^{1/x}$.

Solution: Check the form: as $x\to 0$, $1+\sin(x)\to 1$ and $1/x\to\infty$. The form is 1^∞. Let $L = \lim_{x\to 0}(1+\sin(x))^{1/x}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to 0}\frac{1}{x}\ln(1+\sin(x)) = \lim_{x\to 0}\frac{\ln(1+\sin(x))}{x} \qquad \left(\text{Now of the form } \frac{0}{0}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to 0}\frac{\frac{1}{1+\sin(x)}\cdot\cos(x)}{1} = \frac{\frac{1}{1+0}\cdot 1}{1} = 1$$

Since $\ln(L) = 1$, the final answer is $L = e^1 = e$.

2.5.2 The Form 0^0

Example 2.5.3. Evaluate $\displaystyle\lim_{x\to 0^+} x^x$.

Solution: Check the form: as $x\to 0^+$, we have 0^0. Let $L = \lim_{x\to 0^+} x^x$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to 0^+}\ln(x^x) = \lim_{x\to 0^+} x\ln(x) \quad (\text{Form } 0\cdot(-\infty))$$

This is the same limit from the product form section. We rewrite it as $\lim_{x\to 0^+}\frac{\ln(x)}{1/x}$ and find the limit to be 0. So, $\ln(L) = 0$. The final answer is $L = e^0 = 1$.

Example 2.5.4. Evaluate $\displaystyle\lim_{x\to 0^+}(x)^{\sin(x)}$.

Solution: Check the form: as $x \to 0^+$, we have $(0)^{\sin(0)} \to 0^0$. Let $L = \lim_{x\to 0^+}(x)^{\sin(x)}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to 0^+} \ln((x)^{\sin(x)}) = \lim_{x\to 0^+} \sin(x)\ln(x) \quad (\text{Form } 0\cdot(-\infty))$$

Rewrite as a quotient:

$$\ln(L) = \lim_{x\to 0^+} \frac{\ln(x)}{1/\sin(x)} = \lim_{x\to 0^+} \frac{\ln(x)}{\csc(x)} \quad \left(\text{Now of the form } \frac{-\infty}{\infty}\right)$$

Apply L'Hôpital's Rule:

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)}} \\
&= \lim_{x\to 0^+} \frac{-\sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+}\left(\frac{\sin(x)}{x}\cdot\frac{-\sin(x)}{\cos(x)}\right) \\
&= \left(\lim_{x\to 0^+}\frac{\sin(x)}{x}\right)\cdot\left(\lim_{x\to 0^+} -\tan(x)\right) \\
&= (1)\cdot(0) = 0
\end{aligned}$$

Since $\ln(L) = 0$, the final answer is $L = e^0 = 1$.

2.5.3 The Form ∞^0

Example 2.5.5. Evaluate $\lim\limits_{x\to\infty} x^{1/x}$.

Solution: Check the form: as $x \to \infty$, we have ∞^0. Let $L = \lim_{x\to\infty} x^{1/x}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln(x)}{x} \quad \left(\text{Now of the form } \frac{\infty}{\infty}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Since $\ln(L) = 0$, the final answer is $L = e^0 = 1$.

Example 2.5.6. Evaluate $\lim\limits_{x\to\infty} (x^2+1)^{1/\ln(x)}$.

Solution:

Check the form: as $x \to \infty$, $x^2 + 1 \to \infty$ and $1/\ln(x) \to 0$. The form is ∞^0. Let $L = \lim_{x\to\infty}(x^2+1)^{1/\ln(x)}$. Take the natural logarithm:

$$\ln(L) = \lim_{x\to\infty} \ln\left((x^2+1)^{1/\ln(x)}\right) = \lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(x)} \quad \left(\text{Now of the form } \frac{\infty}{\infty}\right)$$

Apply L'Hôpital's Rule:

$$\ln(L) = \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{2x^2}{x^2+1}$$

This is still $\frac{\infty}{\infty}$. We can use L'Hôpital's again or divide by x^2:

$$\ln(L) = \lim_{x\to\infty} \frac{2}{1+1/x^2} = \frac{2}{1+0} = 2$$

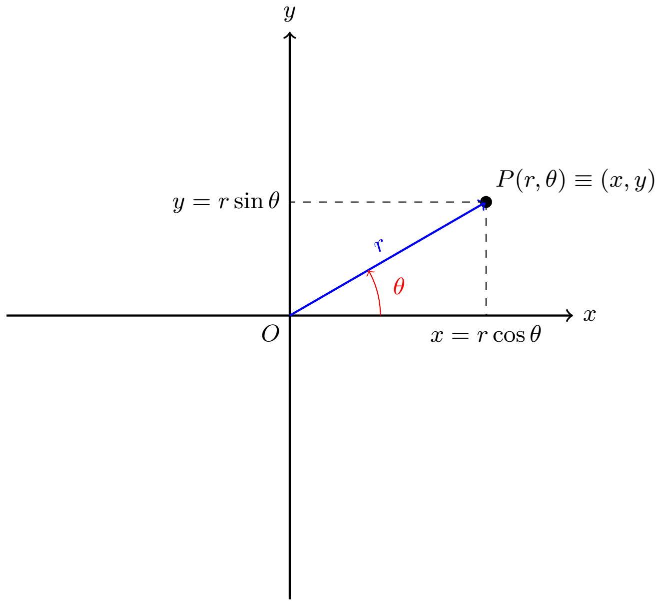
Since $\ln(L) = 2$, the final answer is $L = e^2$.

2.6 Tangents and Normals to Polar Curves

Definition 2.6.1 (Polar-Cartesian Conversion)**.** A point P in the plane can be represented by Cartesian coordinates (x, y) or polar coordinates (r, θ). The relationship between these two systems is fundamental for performing calculus on polar curves. The conversion formulas are:

$$x = r\cos(\theta) \tag{2.1}$$
$$y = r\sin(\theta) \tag{2.2}$$

Conversely, $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$. A polar curve is given by an equation of the form $r = f(\theta)$.



2.6.1 Slope of the Tangent Line

To find the slope of a tangent line to a polar curve, we treat θ as a parameter and express x and y in terms of θ.

$$x(\theta) = f(\theta)\cos(\theta) = r\cos(\theta)$$
$$y(\theta) = f(\theta)\sin(\theta) = r\sin(\theta)$$

The slope $\frac{dy}{dx}$ can be found using the formula for parametric differentiation, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Derivation. [Formula for $\frac{dy}{dx}$] We differentiate $x(\theta)$ and $y(\theta)$ with respect to θ using the product rule. Let $r' = \frac{dr}{d\theta}$.

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = \frac{dr}{d\theta}\cos\theta + r(-\sin\theta) = r'\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = \frac{dr}{d\theta}\sin\theta + r(\cos\theta) = r'\sin\theta + r\cos\theta$$

Therefore, the slope is the ratio of these two expressions. ∎

Theorem 2.6.2 (Slope of a Polar Tangent)**.** *The slope of the tangent line to the curve $r = f(\theta)$ at a point (r, θ) is given by:*

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{2.3}$$

provided the denominator is not zero.
Moreover, the equation of the tangent to the curve $y = f(x)$ is given by

$$y - y_0 = m(x - x_0) \tag{2.4}$$

Example 2.6.3 (Tangent to a Cardioid)**.** Find the equation of the tangent line to the cardioid $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{3}$.

Solution. Given r, we find $\frac{dr}{d\theta}$.

$$r = 1 + \sin\theta \quad \Longrightarrow \quad \frac{dr}{d\theta} = \cos\theta$$

Now, evaluate r and $\frac{dr}{d\theta}$ at $\theta = \frac{\pi}{3}$.

$$r\left(\frac{\pi}{3}\right) = 1 + \sin\left(\frac{\pi}{3}\right) = 1 + \frac{\sqrt{3}}{2}$$

$$\left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Substitute into the slope formula (2.3).

$$m_{\tan} = \frac{dy}{dx} = \frac{(\frac{1}{2})\sin(\frac{\pi}{3}) + (1+\frac{\sqrt{3}}{2})\cos(\frac{\pi}{3})}{(\frac{1}{2})\cos(\frac{\pi}{3}) - (1+\frac{\sqrt{3}}{2})\sin(\frac{\pi}{3})}$$
$$= \frac{(\frac{1}{2})(\frac{\sqrt{3}}{2}) + (\frac{2+\sqrt{3}}{2})(\frac{1}{2})}{(\frac{1}{2})(\frac{1}{2}) - (\frac{2+\sqrt{3}}{2})(\frac{\sqrt{3}}{2})} = \frac{\frac{\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4}}{\frac{1}{4} - \frac{2\sqrt{3}+3}{4}}$$
$$= \frac{2+2\sqrt{3}}{1-(3+2\sqrt{3})} = \frac{2(1+\sqrt{3})}{-2-2\sqrt{3}} = -1$$

The Cartesian coordinates (x_0, y_0) of the point.

$$x_0 = r\cos\theta = \left(1+\frac{\sqrt{3}}{2}\right)\cos\left(\frac{\pi}{3}\right) = \left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) = \frac{2+\sqrt{3}}{4} \approx 0.93$$
$$y_0 = r\sin\theta = \left(1+\frac{\sqrt{3}}{2}\right)\sin\left(\frac{\pi}{3}\right) = \left(\frac{2+\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{2\sqrt{3}+3}{4} \approx 1.62$$

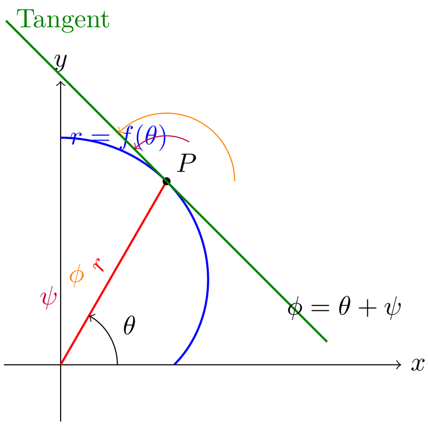
Therefore, the equation of the tangent line. Using $y - y_0 = m(x - x_0)$:

$$y - \frac{2\sqrt{3}+3}{4} = -1\cdot\left(x - \frac{2+\sqrt{3}}{4}\right) \implies y = -x + \frac{2+\sqrt{3}}{4} + \frac{2\sqrt{3}+3}{4} \implies y = -x + \frac{5+3\sqrt{3}}{4}$$

■

2.7 Angle Between Radius Vector and Tangent

A powerful concept for understanding polar tangents is the angle ψ between the radius vector (from the origin to the point) and the tangent line at that point.



Theorem 2.7.1. *The angle ψ between the radius vector and the tangent line at a point (r, θ) on the curve $r = f(\theta)$ satisfies:*

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{f(\theta)}{f'(\theta)} \tag{2.5}$$

Remark 2.7.2. The slope of the tangent line is $\frac{dy}{dx} = \tan\phi$, where ϕ is the angle the tangent makes with the positive x-axis. From the geometry, we can see that $\phi = \theta + \psi$. This gives an alternative way to find the slope:

$$\frac{dy}{dx} = \tan(\theta+\psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi}$$

Substituting $\tan\psi = r/r'$ and $\tan\theta = y/x$ and simplifying leads back to the original formula, confirming the geometric relationship.

2.8 Special Cases: Horizontal and Vertical Tangents

Theorem 2.8.1 (Conditions for Horizontal and Vertical Tangents)**.** *For a polar curve $r = f(\theta)$:*

1. ***Horizontal tangents*** *occur when* $\frac{dy}{d\theta} = 0$, *provided* $\frac{dx}{d\theta} \neq 0$.

$$\frac{dr}{d\theta} \sin\theta + r\cos\theta = 0$$

2. ***Vertical tangents*** *occur when* $\frac{dx}{d\theta} = 0$, *provided* $\frac{dy}{d\theta} \neq 0$.

$$\frac{dr}{d\theta} \cos\theta - r\sin\theta = 0$$

Remark 2.8.2. If both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ are zero simultaneously, the slope is indeterminate $(\frac{0}{0})$. This often happens at the pole ($r = 0$) and may indicate a cusp or other complex behavior.

Example 2.8.3 (Finding Horizontal and Vertical Tangents)**.** Find the points on the cardioid $r = 1 - \cos\theta$ for $\theta \in [0, 2\pi)$ where the tangent line is horizontal or vertical.

Solution. We have $r = 1 - \cos\theta$ and $\frac{dr}{d\theta} = \sin\theta$. **Horizontal Tangents ($\frac{dy}{d\theta} = 0$):**

$$\begin{aligned}
(\sin\theta)\sin\theta + (1 - \cos\theta)\cos\theta &= 0\\
\sin^2\theta + \cos\theta - \cos^2\theta &= 0\\
(1 - \cos^2\theta) + \cos\theta - \cos^2\theta &= 0\\
-2\cos^2\theta + \cos\theta + 1 = 0 \implies 2\cos^2\theta - \cos\theta - 1 &= 0\\
(2\cos\theta + 1)(\cos\theta - 1) &= 0
\end{aligned}$$

This yields $\cos\theta = 1$ (so $\theta = 0$) or $\cos\theta = -\frac{1}{2}$ (so $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$).

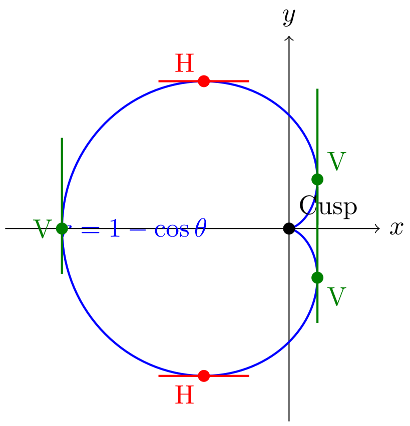
Vertical Tangents ($\frac{dx}{d\theta} = 0$):

$$\begin{aligned}
(\sin\theta)\cos\theta - (1 - \cos\theta)\sin\theta &= 0\\
\sin\theta\cos\theta - \sin\theta + \sin\theta\cos\theta &= 0\\
2\sin\theta\cos\theta - \sin\theta = 0 \implies \sin\theta(2\cos\theta - 1) &= 0
\end{aligned}$$

This yields $\sin\theta = 0$ (so $\theta = 0, \pi$) or $\cos\theta = \frac{1}{2}$ (so $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$).

Analysis:

- At $\theta = 0$, both derivatives are zero. This is the pole ($r = 0$), which is a cusp for this cardioid.
- **Horizontal tangents** exist at $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. The points are $(r, \theta) = (1.5, 2\pi/3)$ and $(1.5, 4\pi/3)$.
- **Vertical tangents** exist at $\theta = \pi$, $\theta = \frac{\pi}{3}$, and $\theta = \frac{5\pi}{3}$. The points are $(2, \pi)$, $(0.5, \pi/3)$, and $(0.5, 5\pi/3)$.



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2.8.1 Tangents at the Pole

A special situation arises when the curve passes through the pole ($r = 0$).

Theorem 2.8.4 (Tangents at the Pole)**.** *If $f(\theta_0) = 0$ and $f'(\theta_0) \neq 0$, then the line $\theta = \theta_0$ is tangent to the curve $r = f(\theta)$ at the pole.*

Derivation. If $r = f(\theta_0) = 0$, the slope formula (2.3) becomes:

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + 0\cdot\cos\theta_0}{f'(\theta_0)\cos\theta_0 - 0\cdot\sin\theta_0} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0$$

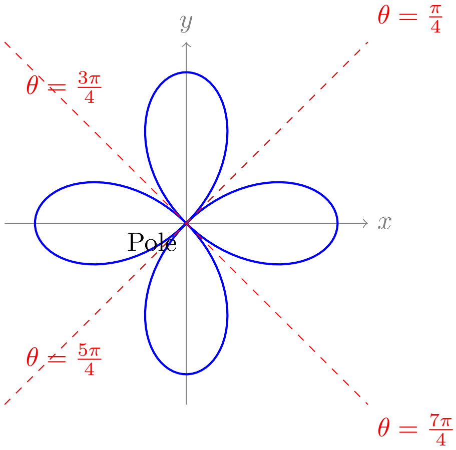
The slope of the line $\theta = \theta_0$ is $\tan\theta_0$. Thus, the line is tangent to the curve at the pole. ■

Example 2.8.5 (Tangents at the Pole of a Rose Curve)**.** Find the equations of the tangent lines to the four-petaled rose $r = \cos(2\theta)$ at the pole.

Solution. **Step 1: Find when the curve is at the pole.** We set $r = 0$, so $\cos(2\theta) = 0$. For $\theta \in [0, 2\pi)$, this occurs when 2θ is an odd multiple of $\pi/2$.

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \implies \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Step 2: Check the derivative. $f'(\theta) = \frac{dr}{d\theta} = -2\sin(2\theta)$. At each of the values of θ found above, 2θ is an odd multiple of $\pi/2$, so $\sin(2\theta) = \pm 1$. Therefore, $f'(\theta) \neq 0$ at these points. **Step 3: State the tangent lines.** According to the theorem, the tangent lines at the pole are the lines $\theta = \theta_0$. The equations are: $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$, $\theta = \frac{5\pi}{4}$, and $\theta = \frac{7\pi}{4}$.



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2.9 The Normal Line

Definition 2.9.1 (Normal Line)**.** The normal line to a curve at a point is the line perpendicular to the tangent line at that same point.

Corollary 2.9.2 (Slope of the Normal Line)**.** *If the slope of the tangent line is $m_{tan} = \frac{dy}{dx}$, the slope of the normal line is*

$$m_{normal} = -\frac{1}{m_{tan}} = -\frac{dx}{dy}$$

Explicitly for a polar curve:

$$m_{normal} = -\frac{\frac{dr}{d\theta}\cos\theta - r\sin\theta}{\frac{dr}{d\theta}\sin\theta + r\cos\theta} = \frac{r\sin\theta - \frac{dr}{d\theta}\cos\theta}{r\cos\theta + \frac{dr}{d\theta}\sin\theta} \tag{2.6}$$

Example 2.9.3 (Equation of the Normal Line)**.** Find the equation of the normal line to the cardioid $r = 1+\sin\theta$ at $\theta = \frac{\pi}{3}$.

Solution. From our previous example, we have found $m_{\text{tan}} = -1$. So the slope of the normal is

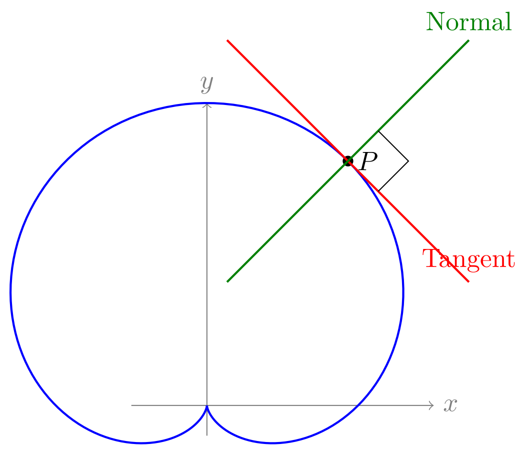
$$m_{\text{normal}} = -\frac{1}{-1} = 1$$

We found the point of tangency to be $(x_0, y_0) = \left(\frac{2+\sqrt{3}}{4}, \frac{2\sqrt{3}+3}{4}\right)$. Therefore, the equation of the normal line $y - y_0 = m_{\text{normal}}(x - x_0)$:

$$y - \frac{2\sqrt{3}+3}{4} = 1\cdot\left(x - \frac{2+\sqrt{3}}{4}\right)$$

Simplifying gives:

$$4y - (2\sqrt{3}+3) = 4x - (2+\sqrt{3}) \implies 4y - 4x = \sqrt{3}+1$$



■

2.10 Angle of Intersection in Cartesian Coordinates

2.10.1 Method and Formula

Let the two curves be given by the equations $y = f(x)$ and $y = g(x)$.

Step 1: Find Points of Intersection
Solve the system of equations by setting $f(x) = g(x)$. Let a solution be $x = x_0$, which gives an intersection point $P(x_0, y_0)$, where $y_0 = f(x_0) = g(x_0)$.

Step 2: Find Slopes of Tangents
Calculate the derivatives $f'(x)$ and $g'(x)$. The slopes of the tangent lines at $P(x_0, y_0)$ are:

$$m_1 = f'(x_0) \quad \text{and} \quad m_2 = g'(x_0)$$

Step 3: Calculate the Angle
The angle α between two lines with slopes m_1 and m_2 is given by the formula:

Theorem 2.10.1 (Angle between Two Lines)**.** *The acute angle α between two non-vertical lines with slopes m_1 and m_2 is given by:*

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \tag{2.7}$$

provided $1 + m_1 m_2 \neq 0$.

Corollary 2.10.2 (Condition for Orthogonality)**.** *The two curves are orthogonal (intersect at a right angle) at the point P if their tangent lines are perpendicular. This occurs if and only if:*

$$m_1 m_2 = -1$$

In this case, the denominator $1 + m_1 m_2$ is zero, and $\tan\alpha$ is undefined, which corresponds to $\alpha = \frac{\pi}{2}$ or $90°$.

2.10.2 Solved Examples

Example 2.10.3. Find the angle of intersection of the curves $y = x^2$ and $x = y^2$.

Solution. **Step 1: Find Points of Intersection**
Substitute $y = x^2$ into $x = y^2$:

$$x = (x^2)^2 \implies x = x^4 \implies x^4 - x = 0 \implies x(x^3 - 1) = 0$$

This gives $x = 0$ or $x^3 = 1 \implies x = 1$. If $x = 0$, $y = 0^2 = 0$. Point is $(0, 0)$. If $x = 1$, $y = 1^2 = 1$. Point is $(1, 1)$.

Intersection at $(0, 0)$**:** For $y = x^2$, the tangent is $y = 0$ (the x-axis). For $x = y^2$, the tangent is $x = 0$ (the y-axis). The tangent lines are the axes, which are perpendicular. So, the angle of intersection at $(0, 0)$ is $\alpha = 90°$.

Intersection at $(1, 1)$**: Step 2: Find Slopes of Tangents**
Curve 1: $y = x^2 \implies \frac{dy}{dx} = 2x$. At $(1, 1)$, the slope is $m_1 = 2(1) = 2$.

Curve 2: $x = y^2$. We use implicit differentiation: $1 = 2y\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{2y}$. At $(1, 1)$, the slope is $m_2 = \frac{1}{2(1)} = \frac{1}{2}$.

Step 3: Calculate the Angle
Using the formula for $\tan\alpha$:

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{1}{2} - 2}{1 + (2)(\frac{1}{2})}\right| = \left|\frac{-3/2}{1+1}\right| = \left|\frac{-3/2}{2}\right| = \frac{3}{4}$$

The angle of intersection is $\alpha = \arctan\left(\frac{3}{4}\right) \approx 36.87°$. ■

Example 2.10.4. Show that the curves $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ are orthogonal at their points of intersection.

Solution. We don't need to find the intersection points explicitly. We just need to show that at any intersection point (x, y), the product of the slopes of the tangents is -1.

Step 1: Find the slopes using implicit differentiation.
Curve 1: $x^2 - y^2 = 5$

$$2x - 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

So, $m_1 = \frac{x}{y}$.
Curve 2: $4x^2 + 9y^2 = 72$

$$8x + 18y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-8x}{18y} = -\frac{4x}{9y}$$

So, $m_2 = -\frac{4x}{9y}$.
Step 2: Find the product of the slopes.

$$m_1 m_2 = \left(\frac{x}{y}\right)\left(-\frac{4x}{9y}\right) = -\frac{4x^2}{9y^2}$$

This product depends on x and y. We need to use the equations of the curves to simplify it. From the first curve, $x^2 = 5 + y^2$. Substitute this into the second equation to find a relationship for y^2:

$$4(5 + y^2) + 9y^2 = 72 \implies 20 + 4y^2 + 9y^2 = 72 \implies 13y^2 = 52 \implies y^2 = 4$$

Now substitute $y^2 = 4$ back into the expression for x^2:

$$x^2 = 5 + y^2 = 5 + 4 = 9$$

So, at any point of intersection, we must have $x^2 = 9$ and $y^2 = 4$.
Step 3: Evaluate the product of the slopes at an intersection point.

$$m_1 m_2 = -\frac{4x^2}{9y^2} = -\frac{4(9)}{9(4)} = -1$$

Since the product of the slopes is -1, the curves are orthogonal at all their points of intersection. ■

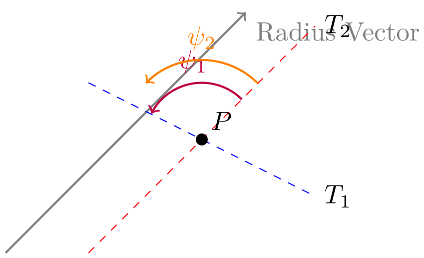
2.11 Angle of Intersection in Polar Coordinates

2.11.1 Method and Formula

For polar curves, it is often simpler to work with the angle ψ between the radius vector and the tangent line at a point.

Let the two curves be $r = f_1(\theta)$ and $r = f_2(\theta)$. Let ψ_1 and ψ_2 be the angles between the radius vector and the tangent lines to the respective curves at a point of intersection.

$$\text{Angle of intersection } \alpha = |\psi_1 - \psi_2|$$



The angle of intersection α is the angle between the two tangent lines, which is simply the absolute difference between ψ_1 and ψ_2.

Theorem 2.11.1 (Angle of Intersection in Polar Coordinates). *The angle of intersection α between two polar curves at a common point is given by:*

$$\alpha = |\psi_1 - \psi_2|$$

where ψ_1 and ψ_2 are the angles for each curve satisfying

$$\tan\psi_1 = \frac{r}{\left.\frac{dr}{d\theta}\right|_1} \quad \text{and} \quad \tan\psi_2 = \frac{r}{\left.\frac{dr}{d\theta}\right|_2}$$

Corollary 2.11.2 (Condition for Orthogonality in Polar Coordinates). *Two polar curves are orthogonal if $\alpha = \frac{\pi}{2}$. This means $|\psi_1 - \psi_2| = \frac{\pi}{2}$. This condition is equivalent to $\psi_1 = \psi_2 \pm \frac{\pi}{2}$, which implies $\tan\psi_1 = \tan(\psi_2 \pm \frac{\pi}{2}) = -\cot\psi_2 = -\frac{1}{\tan\psi_2}$. Therefore, the condition for orthogonality is:*

$$\tan\psi_1 \tan\psi_2 = -1$$

2.11.2 Solved Examples

Example 2.11.3. Find the angle of intersection of the cardioids $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$.

Solution. **Step 1: Find Points of Intersection**
Set the expressions for r equal:

$$a(1 + \cos\theta) = b(1 - \cos\theta) \implies a + a\cos\theta = b - b\cos\theta$$

$$(a + b)\cos\theta = b - a \implies \cos\theta = \frac{b - a}{a + b}$$

At a point of intersection (r, θ) satisfying this, we find ψ_1 and ψ_2.

Step 2: Find $\tan\psi$ for each curve.
Curve 1: $r = a(1 + \cos\theta) \implies \frac{dr}{d\theta} = -a\sin\theta$.

$$\tan\psi_1 = \frac{r}{dr/d\theta} = \frac{a(1 + \cos\theta)}{-a\sin\theta} = -\frac{1 + \cos\theta}{\sin\theta}$$

Using half-angle identities $1 + \cos\theta = 2\cos^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$$\tan\psi_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\cot(\theta/2) = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

So, $\psi_1 = \frac{\pi}{2} + \frac{\theta}{2}$.

Curve 2: $r = b(1 - \cos\theta) \implies \frac{dr}{d\theta} = b\sin\theta$.

$$\tan\psi_2 = \frac{r}{dr/d\theta} = \frac{b(1 - \cos\theta)}{b\sin\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

Using half-angle identities $1 - \cos\theta = 2\sin^2(\theta/2)$:

$$\tan\psi_2 = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \tan(\theta/2)$$

So, $\psi_2 = \frac{\theta}{2}$.

Step 3: Calculate the angle of intersection α.

$$\alpha = |\psi_1 - \psi_2| = \left|\left(\frac{\pi}{2} + \frac{\theta}{2}\right) - \frac{\theta}{2}\right| = \left|\frac{\pi}{2}\right| = \frac{\pi}{2}$$

The angle of intersection is $\frac{\pi}{2}$. This means the curves are **always orthogonal** at any point of intersection (other than the pole). ■

Example 2.11.4. Show that the curves $r = a\theta$ and $r\theta = a$ intersect at a right angle.

Solution. **Step 1: Find the points of intersection.** Curve 1: $r = a\theta$. Curve 2: $r = a/\theta$. Set them equal: $a\theta = a/\theta \implies \theta^2 = 1 \implies \theta = \pm 1$ (we take $\theta = 1$ as a representative point). The intersection point is $(a, 1)$.

Step 2: Find $\tan\psi$ for each curve.
Curve 1: $r = a\theta \implies \frac{dr}{d\theta} = a$.

$$\tan\psi_1 = \frac{r}{dr/d\theta} = \frac{a\theta}{a} = \theta$$

Curve 2: $r = a/\theta \implies \frac{dr}{d\theta} = -a/\theta^2$.

$$\tan\psi_2 = \frac{r}{dr/d\theta} = \frac{a/\theta}{-a/\theta^2} = -\theta$$

Step 3: Check for orthogonality at the intersection point $\theta = 1$.
At $\theta = 1$:

$$\tan\psi_1 = 1 \quad \text{and} \quad \tan\psi_2 = -1$$

Now check the product:

$$\tan\psi_1 \tan\psi_2 = (1)(-1) = -1$$

Since the product is -1, the curves are orthogonal at their point of intersection. ■

2.12 Introduction and Geometric Setup

For a polar curve defined by $r = f(\theta)$, the geometry of the tangent line at a point gives rise to four important quantities: the lengths of the polar tangent, polar normal, polar sub-tangent, and polar sub-normal.

To define these, we need a standard frame of reference which can be visualized conceptually:

- Let O be the pole (the origin).
- Let $P(r, \theta)$ be a point on the curve.
- The line segment OP is the **radius vector**, with length r.
- The line passing through P and tangent to the curve is the **tangent line**.
- The line passing through P and perpendicular to the tangent is the **normal line**.
- Let ψ be the angle between the radius vector OP and the tangent line. The fundamental relationship for this angle is:

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}}$$

The four quantities are defined with respect to a line drawn through the pole O and perpendicular to the radius vector OP. We will derive the formulas for the lengths of these segments based on right-angled triangles formed by these lines.

2.13 Length of the Polar Tangent

Definition 2.13.1. The **length of the polar tangent** is the length of the segment of the tangent line between the point of contact P and the point T where it meets the line through the pole perpendicular to the radius vector.

Derivation. Consider the conceptual right-angled triangle $\triangle OPT$, where the angle at the pole O is $90°$. The angle between the radius vector OP and the tangent PT is ψ, so $\angle OPT = \psi$. From trigonometry in $\triangle OPT$:

$$\cos\psi = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{OP}{PT} = \frac{r}{PT}$$

$$\implies \text{Length of } PT = \frac{r}{\cos\psi} = r\sec\psi$$

Using the identity $\sec^2\psi = 1 + \tan^2\psi$:

$$\text{Length of } PT = r\sqrt{1 + \tan^2\psi}$$

Substituting our fundamental relation $\tan\psi = \frac{r}{dr/d\theta}$:

$$\text{Length of } PT = r\sqrt{1 + \left(\frac{r}{\frac{dr}{d\theta}}\right)^2}$$

■

Theorem 2.13.2 (Length of Polar Tangent). *The length of the polar tangent to the curve $r = f(\theta)$ is given by:*

$$\textit{Length} = r\sqrt{1 + \left(\frac{r}{\frac{dr}{d\theta}}\right)^2}$$

Example 2.13.3. Find the length of the polar tangent for the cardioid $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. **Step 1: Find r and $\frac{dr}{d\theta}$ at the given point.**
The curve is $r = a(1 - \cos\theta)$. The derivative is $\frac{dr}{d\theta} = a\sin\theta$. At $\theta = \frac{\pi}{2}$:

$$r = a\left(1 - \cos\frac{\pi}{2}\right) = a(1 - 0) = a$$

$$\frac{dr}{d\theta} = a\sin\frac{\pi}{2} = a(1) = a$$

Step 2: Substitute into the formula.

$$\begin{aligned}
\text{Length of Polar Tangent} &= r\sqrt{1 + \left(\frac{r}{dr/d\theta}\right)^2} \\
&= a\sqrt{1 + \left(\frac{a}{a}\right)^2} \\
&= a\sqrt{1 + 1^2} = a\sqrt{2}
\end{aligned}$$

The length of the polar tangent at $\theta = \frac{\pi}{2}$ is $\mathbf{a\sqrt{2}}$. ■

2.14 Length of the Polar Normal

Definition 2.14.1. The **length of the polar normal** is the length of the segment of the normal line between the point of contact P and the point N where it meets the line through the pole perpendicular to the radius vector.

Derivation. Consider the conceptual right-angled triangle $\triangle OPN$. The angle at O is $90°$. The angle between the normal and the radius vector is $\angle OPN = 90° - \psi$. From trigonometry in $\triangle OPN$:

$$\cos(\angle OPN) = \frac{OP}{PN} \implies \cos(90° - \psi) = \frac{r}{PN}$$

Since $\cos(90° - \psi) = \sin\psi$, we have:

$$\sin\psi = \frac{r}{PN} \implies \text{Length of } PN = \frac{r}{\sin\psi} = r\csc\psi$$

Using the identity $\csc^2\psi = 1 + \cot^2\psi$ and the relation $r\cot\psi = dr/d\theta$:

$$\text{Length of } PN = r\sqrt{1 + \cot^2\psi} = \sqrt{r^2(1 + \cot^2\psi)} = \sqrt{r^2 + (r\cot\psi)^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

■

Theorem 2.14.2 (Length of Polar Normal). *The length of the polar normal to the curve $r = f(\theta)$ is given by:*

$$\textit{Length} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

Example 2.14.3. Find the length of the polar normal for the cardioid $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. **Step 1: Use the values from the previous example.**
At $\theta = \frac{\pi}{2}$, we have $r = a$ and $\frac{dr}{d\theta} = a$.
Step 2: Substitute into the formula.

$$\begin{aligned}\text{Length of Polar Normal} &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \\ &= \sqrt{a^2 + a^2} \\ &= \sqrt{2a^2} = a\sqrt{2}\end{aligned}$$

The length of the polar normal at $\theta = \frac{\pi}{2}$ is $\mathbf{a\sqrt{2}}$. ■

2.15 Length of the Polar Sub-tangent

Definition 2.15.1. The **length of the polar sub-tangent** is the length of the line segment OT, which is the projection of the polar tangent onto the line through the pole perpendicular to the radius vector.

Derivation. Consider again the right-angled triangle $\triangle OPT$. From trigonometry:

$$\tan\psi = \frac{\text{opposite}}{\text{adjacent}} = \frac{OT}{OP} = \frac{OT}{r}$$

$$\implies \text{Length of } OT = r\tan\psi$$

Substituting $\tan\psi = \frac{r}{dr/d\theta}$:

$$\text{Length of } OT = r\left(\frac{r}{\frac{dr}{d\theta}}\right) = \frac{r^2}{\frac{dr}{d\theta}}$$

■

Theorem 2.15.2 (Length of Polar Sub-tangent)**.** *The length of the polar sub-tangent to the curve $r = f(\theta)$ is given by:*

$$\mathit{Length} = \left|\frac{r^2}{\frac{dr}{d\theta}}\right|$$

(Absolute value is used as length must be positive).

Example 2.15.3. Find the length of the polar sub-tangent for the cardioid $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. **Step 1: Use the known values.**
At $\theta = \frac{\pi}{2}$, we have $r = a$ and $\frac{dr}{d\theta} = a$.
Step 2: Substitute into the formula.

$$\begin{aligned}\text{Length of Polar Sub-tangent} &= \left|\frac{r^2}{dr/d\theta}\right| \\ &= \left|\frac{a^2}{a}\right| = a\end{aligned}$$

The length of the polar sub-tangent at $\theta = \frac{\pi}{2}$ is $\mathbf{a}$. ■

2.16 Length of the Polar Sub-normal

Definition 2.16.1. The **length of the polar sub-normal** is the length of the line segment ON, which is the projection of the polar normal onto the line through the pole perpendicular to the radius vector.

Derivation. Consider the right-angled triangle $\triangle OPN$. The angle $\angle OPN = 90° - \psi$. From trigonometry:

$$\tan(\angle OPN) = \frac{ON}{OP} \implies \tan(90° - \psi) = \frac{ON}{r}$$

Since $\tan(90° - \psi) = \cot\psi$, we have:

$$\cot\psi = \frac{ON}{r} \implies \text{Length of } ON = r\cot\psi$$

Substituting $\cot\psi = \frac{1}{r}\frac{dr}{d\theta}$:

$$\text{Length of } ON = r\left(\frac{1}{r}\frac{dr}{d\theta}\right) = \frac{dr}{d\theta}$$

This gives a remarkably simple and elegant result. ■

Theorem 2.16.2 (Length of Polar Sub-normal)**.** *The length of the polar sub-normal to the curve $r = f(\theta)$ is simply the absolute value of the derivative of r with respect to θ.*

$$\mathit{Length} = \left|\frac{dr}{d\theta}\right|$$

Example 2.16.3. Find the length of the polar sub-normal for the cardioid $r = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$.

Solution. At $\theta = \frac{\pi}{2}$, we have $\frac{dr}{d\theta} = a$.

$$\begin{aligned}\text{Therefore, the length of Polar Sub-normal} &= \left|\frac{dr}{d\theta}\right| \\ &= |a| = a \quad (\text{assuming } a > 0)\end{aligned}$$

The length of the polar sub-normal at $\theta = \frac{\pi}{2}$ is **a**. ■