Chapter 1

Limits. Continuity and Differentiability

1.1 Limits and Infinitesimals

Concept Overview

The **limit** of a function $f(x)$ as x approaches a point a is the value L that $f(x)$ gets arbitrarily close to as x approaches a. Formally:

$$\lim_{x\to a} f(x) = L \iff \forall\epsilon > 0, \exists\delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon.$$

Infinitesimals are quantities smaller than any positive real number but greater than zero. In limits, dx represents an infinitesimal change in x, and the expression $f(a + dx) - L$ becomes infinitesimal as $dx \to 0$.

Example 1.1.1. Evaluate $\lim_{x\to 3}(2x + 1)$.

Solution.

$$\begin{aligned}\lim_{x\to 3}(2x+1) &= 2(3) + 1 \quad \text{(Direct substitution)}\\ &= 7.\end{aligned}$$

To verify using the ϵ-δ definition: Given $\epsilon > 0$, choose $\delta = \epsilon/2$. If $0 < |x - 3| < \delta$, then:

$$|(2x + 1) - 7| = |2x - 6| = 2|x - 3| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Thus, $\lim_{x\to 3}(2x + 1) = 7$. ■

Example 1.1.2. Prove that $\lim_{x\to 5}(3x - 4) = 11$ using the ϵ-δ definition.

Solution. Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 5| < \delta$, then $|(3x - 4) - 11| < \epsilon$.
Simplify the expression:

$$|3x - 15| = |3(x - 5)| = 3|x - 5|$$

We require:

$$3|x - 5| < \epsilon \implies |x - 5| < \frac{\epsilon}{3}$$

Choose $\delta = \frac{\epsilon}{3}$.
Verification: If $0 < |x - 5| < \delta$, then:

$$|(3x - 4) - 11| = 3|x - 5| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

Thus, $\lim_{x\to 5}(3x - 4) = 11$. ■

Example 1.1.3. Prove that $\lim_{x\to 3}(x^2 - 2x) = 3$ using the ϵ-δ definition.

Solution. Given $\epsilon > 0$, we must find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|(x^2 - 2x) - 3| < \epsilon$.
Simplify the expression:

$$|x^2 - 2x - 3| = |(x - 3)(x + 1)| = |x - 3||x + 1|$$

Assume $\delta \leq 1$. Then $|x - 3| < 1$ implies:

$$2 < x < 4 \quad \text{so} \quad 3 < x + 1 < 5$$

Thus $|x + 1| < 5$. Now:

$$|x - 3||x + 1| < |x - 3| \cdot 5$$

We require $5|x - 3| < \epsilon$, which gives $|x - 3| < \frac{\epsilon}{5}$.
Choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$.
Verification: If $0 < |x - 3| < \delta$, then:

$$|x^2 - 2x - 3| = |x - 3||x + 1| < \delta \cdot 5 \leq \frac{\epsilon}{5} \cdot 5 = \epsilon$$

Thus, $\lim_{x\to 3}(x^2 - 2x) = 3$. ■

1.2 Continuity: ϵ-δ Definition

Concept Overview

A function $f(x)$ is **continuous** at $x = a$ if:

1. $f(a)$ is defined,
2. $\lim_{x\to a} f(x)$ exists,
3. $\lim_{x\to a} f(x) = f(a)$.

The ϵ-δ definition formalizes this:

$$f \text{ is continuous at } a \iff \forall\epsilon > 0, \exists\delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$

This ensures no jumps, breaks, or oscillations at a.

Example 1.2.1. Prove that $f(x) = 4x - 1$ is continuous at $x = 3$ using the ϵ-δ definition.

Solution. We must show: $\forall\epsilon > 0, \exists\delta > 0$ such that $|x - 3| < \delta \implies |(4x - 1) - 11| < \epsilon$.
Note $f(3) = 4(3) - 1 = 11$.
Simplify:

$$|(4x - 1) - 11| = |4x - 12| = 4|x - 3|$$

Choose $\delta = \epsilon/4$. If $|x - 3| < \delta$, then:

$$4|x - 3| < 4 \cdot (\epsilon/4) = \epsilon$$

Thus, $f(x) = 4x - 1$ is continuous at $x = 3$. ■

Example 1.2.2. Prove $f(x) = x^2$ is continuous at $x = 2$.

Solution. $f(2) = 4$. For any $\epsilon > 0$, choose $\delta = \min\left(1, \frac{\epsilon}{5}\right)$. If $|x - 2| < \delta$, then:

$$\begin{aligned}
|x + 2| &= |(x - 2) + 4| \\
&\le |x - 2| + 4 < 1 + 4 = 5, \\
|f(x) - f(2)| &= |x^2 - 4| \\
&= |(x - 2)(x + 2)| \\
&< \delta \cdot 5 \le \frac{\epsilon}{5} \cdot 5 = \epsilon.
\end{aligned}$$

Thus, $f(x) = x^2$ is continuous at $x = 2$. ■

Example 1.2.3. Prove that $g(x) = x^2 + 2$ is continuous at $x = -1$ using the ϵ-δ definition.

Solution. We must show: $\forall\epsilon > 0, \exists\delta > 0$ such that $|x - (-1)| < \delta \implies |(x^2 + 2) - 3| < \epsilon$.
Note $g(-1) = (-1)^2 + 2 = 3$.
Simplify:

$$|(x^2 + 2) - 3| = |x^2 - 1| = |x - 1||x + 1|$$

Set $\delta \le 1$. Then $|x + 1| < 1$ implies $-2 < x < 0$, so $|x - 1| < 3$.
Now:

$$|x - 1||x + 1| < 3|x + 1|$$

Choose $\delta = \min(1, \epsilon/3)$. If $|x + 1| < \delta$, then:

$$|g(x) - g(-1)| < 3 \cdot \delta \le 3 \cdot (\epsilon/3) = \epsilon$$

Thus, $g(x) = x^2 + 2$ is continuous at $x = -1$. ■

Exercise 1.2.4.

1. **(Limit)** Prove $\lim_{x\to 4}(5 - 2x) = -3$ using ϵ-δ definition.
2. **(Limit)** Prove $\lim_{x\to -2}(x^2 + 3x) = -2$ using ϵ-δ definition.
3. **(Continuity)** Prove $h(x) = \sqrt{x}$ is continuous at $x = 4$ using ϵ-δ definition. (Hint: Use $|\sqrt{x} - 2| = \frac{|x-4|}{\sqrt{x}+2}$).
4. **(Continuity)** Prove $k(x) = \dfrac{1}{x}$ is continuous at $x = \dfrac{1}{2}$ using ϵ-δ definition.

1.3 Types of Discontinuities

Discontinuities occur where a function fails to be continuous. Common types:

1. **Removable**: $\lim_{x\to a} f(x)$ exists but $f(a)$ is undefined or $\lim_{x\to a} f(x) \neq f(a)$. "Fixed" by redefining $f(a)$.
2. **Jump**: Left-hand and right-hand limits exist but are unequal ($\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$).
3. **Infinite**: $\lim_{x\to a} f(x) = \pm\infty$ (vertical asymptote).
4. **Oscillating**: Function oscillates infinitely often near a (e.g., $\sin(1/x)$ at $x = 0$).

Example 1.3.1. Classify the discontinuity of $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$.

Solution. The function is undefined at $x = 2$. Simplify for $x \neq 2$:

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

The limit exists:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+2) = 4.$$

Since the limit exists but $f(2)$ is undefined, the discontinuity is **removable**. ■

Example 1.3.2. Classify the discontinuity of the function

$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x+3 & \text{if } x \geq 2 \end{cases}$$

at $x = 2$.

Solution. Calculate left-hand and right-hand limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 2^2 = 4$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+3) = 2+3 = 5$$

Since $4 \neq 5$, the left-hand and right-hand limits exist but are not equal.
Function value: $f(2) = 2 + 3 = 5$.
The discontinuity at $x = 2$ is a **jump discontinuity**. ■

Example 1.3.3. Classify the discontinuity of $g(x) = \dfrac{1}{(x-3)^2}$ at $x = 3$.

Solution. Examine the limit as x approaches 3:

$$\lim_{x\to 3} \frac{1}{(x-3)^2} = +\infty$$

The function is undefined at $x = 3$. As $x \to 3$, the function values increase without bound.
This is an **infinite discontinuity** (vertical asymptote at $x = 3$). ■

Example 1.3.4. Classify the discontinuity of $h(x) = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$.

Solution. Consider the behavior as $x \to 0$:

- As $x \to 0^+$, $\frac{1}{x} \to +\infty$ and $\sin(1/x)$ oscillates between -1 and 1
- As $x \to 0^-$, $\frac{1}{x} \to -\infty$ and $\sin(1/x)$ oscillates between -1 and 1

The limit $\lim_{x\to 0} \sin(1/x)$ does not exist because oscillations become increasingly rapid.
This is an **oscillating discontinuity** at $x = 0$. ■

1.4 Differentiability of Functions

Concept Overview

A function $f(x)$ is **differentiable** at a point $x = a$ if the derivative $f'(a)$ exists. Geometrically, this means the function has a unique tangent at that point. The derivative is defined as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists finitely.

Necessary Conditions

- If f is differentiable at a, then it must be continuous at a.
- The converse is not true: continuity does not imply differentiability (e.g., $|x|$ at $x = 0$).

(1). Proof. To prove that the function f is continuous at a, we must show that $\lim_{x\to a} f(x) = f(a)$. This is equivalent to showing that the difference between $f(x)$ and $f(a)$ approaches zero, i.e.,

$$\lim_{x\to a}[f(x) - f(a)] = 0$$

We are given that f is differentiable at a. By definition, this means the limit for the derivative exists and is a finite number:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

that is,

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a}\right) \cdot (x - a)$$

We can now take the limit of both sides as $x \to a$ and apply the product rule for limits:

$$\begin{aligned}
\lim_{x\to a}[f(x) - f(a)] &= \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a} \cdot (x - a)\right) \\
&= \left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right) \cdot \left(\lim_{x\to a}(x - a)\right) \\
&= f'(a) \cdot (a - a) \\
&= f'(a) \cdot 0 \\
&= 0
\end{aligned}$$

Since we have shown that $\lim_{x\to a}[f(x) - f(a)] = 0$, it follows directly that $\lim_{x\to a} f(x) = f(a)$. This is the definition of continuity at the point a. ■

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Left and Right Derivatives

The **left derivative** at a:

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a + h) - f(a)}{h}$$

The **right derivative** at a:

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a + h) - f(a)}{h}$$

f is differentiable at a iff $f'_-(a) = f'_+(a)$ and both exist finitely.

Example 1.4.1. Show that $f(x) = x^2 + 3x$ is differentiable at $x = 2$ and find its derivative.

Solution. Compute the derivative using the limit definition:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2 + h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{[(2 + h)^2 + 3(2 + h)] - [2^2 + 3(2)]}{h} \\
&= \lim_{h\to 0} \frac{[4 + 4h + h^2 + 6 + 3h] - [4 + 6]}{h} \\
&= \lim_{h\to 0} \frac{7h + h^2}{h} \\
&= \lim_{h\to 0}(7 + h) = 7
\end{aligned}$$

Since the limit exists, f is differentiable at $x = 2$ with $f'(2) = 7$. ■

1.5 Successive Differentiation

Concept Overview

Successive differentiation refers to repeatedly differentiating a function. The nth derivative is denoted by:

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n y}{dx^n}$$

where n is the order of differentiation.

Standard Formulas

- $\dfrac{d^n}{dx^n}(x^m) = m(m-1)\cdots(m-n+1)x^{m-n} \quad \text{for} \quad n \leq m$
- $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$
- $\dfrac{d^n}{dx^n}(\ln x) = (-1)^{n-1}\dfrac{(n-1)!}{x^n}$
- $\dfrac{d^n}{dx^n}(\sin ax) = a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}(\cos ax) = a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$

Example 1.5.1. Find the third derivative of $g(x) = 2x^4 - 5x^3 + 3x - 7$.

Solution. Compute successive derivatives:

$$\begin{aligned}
g'(x) &= \frac{d}{dx}(2x^4 - 5x^3 + 3x - 7) = 8x^3 - 15x^2 + 3 \\
g''(x) &= \frac{d}{dx}(8x^3 - 15x^2 + 3) = 24x^2 - 30x \\
g'''(x) &= \frac{d}{dx}(24x^2 - 30x) = \boxed{48x - 30}
\end{aligned}$$

■

1.6 Leibnitz Theorem

Theorem 1.6.1 (Leibnitz Theorem). *If $u(x)$ and $v(x)$ are n-times differentiable functions, then:*

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}$$

where $u^{(k)}$ is the kth derivative of u, and $v^{(n-k)}$ is the $(n-k)$th derivative of v, with $u^{(0)} = u$ and $v^{(0)} = v$.

Proof. We must show that the formula is true for $n = 1$.

$$\begin{aligned}
(uv)^{(1)} &= \sum_{k=0}^{1} \binom{1}{k} u^{(1-k)} v^{(k)} \\
&= \binom{1}{0} u^{(1)} v^{(0)} + \binom{1}{1} u^{(0)} v^{(1)} \\
&= (1) \cdot u'v + (1) \cdot uv' \\
&= u'v + uv'
\end{aligned}$$

Assume the theorem is true for some positive integer $n = m$. That is, we assume:

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}$$

We must prove that the theorem is true for $n = m + 1$. We start by differentiating the expression from our inductive hypothesis with respect to x:

$$\begin{aligned}
(uv)^{(m+1)} &= \frac{d}{dx}\left[(uv)^{(m)}\right] \\
&= \frac{d}{dx}\left[\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}\right] \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{d}{dx}\left(u^{(m-k)} v^{(k)}\right) \\
&= \sum_{k=0}^{m} \binom{m}{k} \left[u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)}\right]
\end{aligned}$$

Now, we split this into two separate sums:

$$(uv)^{(m+1)} = \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)}}_{\text{Sum A}} + \underbrace{\sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k+1)}}_{\text{Sum B}}$$

To combine these sums, we re-index Sum B by letting $j = k + 1$. This means Sum B will go from $j = 1$ to $j = m + 1$. Replacing j back with k, Sum B becomes:

$$\text{Sum B} = \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m+1-k)} v^{(k)}$$

Now we combine the re-indexed Sum B with Sum A. We can separate the first term ($k = 0$) from Sum A and the last term ($k = m + 1$) from Sum B:

$$\begin{aligned}(uv)^{(m+1)} &= \left[\binom{m}{0} u^{(m+1)} v + \sum_{k=1}^{m} \binom{m}{k} u^{(m+1-k)} v^{(k)}\right] \\ &\quad + \left[\sum_{k=1}^{m} \binom{m}{k-1} u^{(m+1-k)} v^{(k)} + \binom{m}{m} uv^{(m+1)}\right]\end{aligned}$$

Grouping the two middle sums together:

$$(uv)^{(m+1)} = \binom{m}{0} u^{(m+1)} v + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1}\right] u^{(m+1-k)} v^{(k)} + \binom{m}{m} uv^{(m+1)}$$

We use Pascal's Identity: $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$. Also, we know that $\binom{m}{0} = 1 = \binom{m+1}{0}$ and $\binom{m}{m} = 1 = \binom{m+1}{m+1}$. Substituting these identities into our expression gives:

$$(uv)^{(m+1)} = \binom{m+1}{0} u^{(m+1)} v + \sum_{k=1}^{m} \binom{m+1}{k} u^{(m+1-k)} v^{(k)} + \binom{m+1}{m+1} uv^{(m+1)}$$

This entire expression can now be combined into a single sum from $k = 0$ to $k = m + 1$:

$$(uv)^{(m+1)} = \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(m+1-k)} v^{(k)}$$

This is precisely the form of the theorem for $n = m + 1$.
By the principle of mathematical induction, the theorem is true for all positive integers n. ■

Key Features

- Analogous to the binomial theorem
- Coefficients are binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Requires both functions to have derivatives up to order n

Example 1.6.2. Find the second derivative of $h(x) = x^2 e^{3x}$ using Leibnitz theorem.

Solution. Set $u = e^{3x}$, $v = x^2$. Apply Leibnitz theorem for $n = 2$:

$$(uv)'' = \sum_{k=0}^{2} \binom{2}{k} u^{(k)} v^{(2-k)} = u^{(0)} v^{(2)} + 2 \cdot u^{(1)} v^{(1)} + u^{(2)} v^{(0)}$$

Compute terms:

$$\begin{aligned} k = 0 : &\quad \binom{2}{0} u^{(0)} v^{(2)} = 1 \cdot e^{3x} \cdot 2 \\ k = 1 : &\quad \binom{2}{1} u^{(1)} v^{(1)} = 2 \cdot (3e^{3x}) \cdot (2x) \\ k = 2 : &\quad \binom{2}{2} u^{(2)} v^{(0)} = 1 \cdot (9e^{3x}) \cdot (x^2) \end{aligned}$$

Sum the terms:

$$h''(x) = 2e^{3x} + 12xe^{3x} + 9x^2 e^{3x} = \boxed{e^{3x}(9x^2 + 12x + 2)}$$

■

Example 1.6.3. Find the fourth derivative of the function $f(x) = x^3 e^{2x}$ using Leibnitz theorem.

Solution. Set $u = e^{2x}$ and $v = x^3$. Apply Leibnitz theorem for $n = 4$:

$$(uv)^{(4)} = \sum_{k=0}^{4} \binom{4}{k} u^{(k)} v^{(4-k)}$$

Compute derivatives of u and v:

$$u = e^{2x} \qquad\qquad v = x^3$$

$$u^{(k)} = 2^k e^{2x} \qquad\qquad v^{(m)} = \begin{cases} \frac{3!}{(3-m)!} x^{3-m} & 0 \leq m \leq 3 \\ 0 & m > 3 \end{cases}$$

Calculate each term:

$$\begin{aligned}
k = 0 :&\quad \binom{4}{0} u^{(0)} v^{(4)} = 1 \cdot 2^0 e^{2x} \cdot 0 = 0 \\
k = 1 :&\quad \binom{4}{1} u^{(1)} v^{(3)} = 4 \cdot 2^1 e^{2x} \cdot 6 = 48e^{2x} \\
k = 2 :&\quad \binom{4}{2} u^{(2)} v^{(2)} = 6 \cdot 2^2 e^{2x} \cdot 6x = 144xe^{2x} \\
k = 3 :&\quad \binom{4}{3} u^{(3)} v^{(1)} = 4 \cdot 2^3 e^{2x} \cdot 3x^2 = 96x^2 e^{2x} \\
k = 4 :&\quad \binom{4}{4} u^{(4)} v^{(0)} = 1 \cdot 2^4 e^{2x} \cdot x^3 = 16x^3 e^{2x}
\end{aligned}$$

Sum all terms:

$$\begin{aligned}
f^{(4)}(x) &= 0 + 48e^{2x} + 144xe^{2x} + 96x^2 e^{2x} + 16x^3 e^{2x} \\
&= e^{2x}(16x^3 + 96x^2 + 144x + 48)
\end{aligned}$$

Factor out 16:

$$f^{(4)}(x) = 16e^{2x}(x^3 + 6x^2 + 9x + 3)$$

Thus, the fourth derivative is $\boxed{16e^{2x}(x^3 + 6x^2 + 9x + 3)}$. ■

Example 1.6.4. If $y = e^{a \sin^{-1}(x)}$, prove the following:

(a) $(1 - x^2)y_2 - xy_1 - a^2 y = 0$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$.

(b) Hence, using Leibniz's Theorem, show that the following recurrence relation holds:

$$(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0$$

Solution. We are given the function $y = e^{a \sin^{-1}(x)}$. Using the chain rule, we differentiate y with respect to x:

$$y_1 = \frac{dy}{dx} = e^{a \sin^{-1}(x)} \cdot \frac{d}{dx}(a \sin^{-1}(x))$$

$$y_1 = e^{a \sin^{-1}(x)} \cdot \frac{a}{\sqrt{1 - x^2}}$$

that is,

$$y_1 = \frac{ay}{\sqrt{1 - x^2}}$$

because $y = e^{a \sin^{-1}(x)}$.

Squaring both sides of the equation:

$$y_1^2 = \frac{a^2 y^2}{1 - x^2}$$

Now, multiply both sides by $(1 - x^2)$ to clear the fraction:

$$(1 - x^2)y_1^2 = a^2 y^2 \quad (*).$$

We differentiate the entire equation $(*)$ with respect to x.

$$\frac{d}{dx}\left[(1 - x^2)y_1^2\right] = \frac{d}{dx}\left[a^2 y^2\right]$$

$$\left(\frac{d}{dx}(1-x^2)\right)\cdot y_1^2+(1-x^2)\cdot\left(\frac{d}{dx}(y_1^2)\right)=a^2\cdot\left(\frac{d}{dx}(y^2)\right)$$
$$(-2x)y_1^2+(1-x^2)(2y_1y_2)=a^2(2yy_1)$$

Divide the entire equation by $2y_1$:

$$-xy_1+(1-x^2)y_2=a^2y$$

Rearranging the terms gives the desired differential equation:

$$(1-x^2)y_2-xy_1-a^2y=0$$

Applying Leibniz's Theorem to find the Recurrence Relation

We now differentiate the equation $(1-x^2)y_2-xy_1-a^2y=0$ successively n times with respect to x.

$$\frac{d^n}{dx^n}\left[(1-x^2)y_2-xy_1-a^2y\right]=0$$

By linearity of the derivative, we can differentiate each term separately:

$$\underbrace{\frac{d^n}{dx^n}\left[(1-x^2)y_2\right]}_{\text{Term A}}-\underbrace{\frac{d^n}{dx^n}\left[xy_1\right]}_{\text{Term B}}-\underbrace{\frac{d^n}{dx^n}\left[a^2y\right]}_{\text{Term C}}=0$$

For Term: $D^n[(1-x^2)y_2]$ Let $u=y_2$ and $v=1-x^2$. We apply Leibniz's Theorem. The derivatives of v terminate quickly:

- $v=1-x^2$
- $v'=-2x$
- $v''=-2$
- $v'''=0$

The derivatives of u are $u^{(k)}=(y_2)^{(k)}=y_{k+2}$. The Leibniz expansion is:

$$\binom{n}{0}vu^{(n)}+\binom{n}{1}v'u^{(n-1)}+\binom{n}{2}v''u^{(n-2)}+\dots$$

Substituting our functions (only the first three terms are non-zero):

$$(1)(1-x^2)y_{n+2}+(n)(-2x)y_{n+1}+\frac{n(n-1)}{2}(-2)y_n$$

Simplifying gives: $(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n$.
For Term: $D^n[xy_1]$ Let $u=y_1$ and $v=x$. The derivatives of v are $v'=1$ and $v''=0$. The Leibniz expansion has only two non-zero terms:

$$\binom{n}{0}vu^{(n)}+\binom{n}{1}v'u^{(n-1)}$$

Substituting our functions:

$$(1)(x)y_{n+1}+(n)(1)y_n=xy_{n+1}+ny_n$$

For Term: $D^n[a^2y]$ Since a^2 is a constant, this is straightforward: a^2y_n.
Combining the results: Now we substitute the expanded terms back into our main equation:

$$[(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n]-[xy_{n+1}+ny_n]-[a^2y_n]=0$$

Finally, we group the terms by the order of the derivative (y_{n+2},y_{n+1},y_n):

$$(1-x^2)y_{n+2}+(-2nx-x)y_{n+1}+(-n(n-1)-n-a^2)y_n=0$$

Simplifying gives:

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$$

■

1.7 Partial Differentiation

Concept Overview

Partial differentiation deals with functions of multiple variables. The partial derivative of $f(x,y)$ with respect to x is denoted $\frac{\partial f}{\partial x}$ and measures the rate of change of f while keeping y constant.

Formal Definition

For $z = f(x, y)$, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \quad \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Theorem 1.7.1 (Clairaut's Theorem). *If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined on an open set containing (a, b) and are continuous at (a, b), then:*

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$$

Proof. For sufficiently small $h, k \neq 0$, define the auxiliary function:

$$\Delta(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).$$

Analyze via f_{yx}
Define $g(y) = f(a+h, y) - f(a, y)$. Then:

$$\Delta(h, k) = g(b+k) - g(b).$$

By the Mean Value Theorem (MVT), there exists d between b and $b+k$ such that:

$$g(b+k) - g(b) = k \cdot g'(d) = k \left[\frac{\partial f}{\partial y}(a+h, d) - \frac{\partial f}{\partial y}(a, d) \right].$$

Apply MVT to $h(x) = \frac{\partial f}{\partial y}(x, d)$ on $[a, a+h]$. There exists c_1 between a and $a+h$ such that:

$$\frac{\partial f}{\partial y}(a+h, d) - \frac{\partial f}{\partial y}(a, d) = h \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d).$$

Analyze via f_{xy}
Define $r(x) = f(x, b+k) - f(x, b)$. Then:

$$\Delta(h, k) = r(a+h) - r(a).$$

By MVT, there exists e between a and $a+h$ such that:

$$r(a+h) - r(a) = h \cdot r'(e) = h \left[\frac{\partial f}{\partial x}(e, b+k) - \frac{\partial f}{\partial x}(e, b) \right].$$

Apply MVT to $s(y) = \frac{\partial f}{\partial x}(e, y)$ on $[b, b+k]$. There exists c_2 between b and $b+k$ such that:

$$\frac{\partial f}{\partial x}(e, b+k) - \frac{\partial f}{\partial x}(e, b) = k \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Thus:

$$\Delta(h, k) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

Equate and take limits
From Steps 1 and 2:

$$hk \cdot \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = hk \cdot \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

For $hk \neq 0$, we have:

$$\frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial y \partial x}(e, c_2).$$

As $(h, k) \to (0, 0)$:

$$(c_1, d) \to (a, b) \quad \text{and} \quad (e, c_2) \to (a, b).$$

By continuity of f_{xy} and f_{yx} at (a, b):

$$\lim_{(h,k) \to (0,0)} \frac{\partial^2 f}{\partial x \partial y}(c_1, d) = \frac{\partial^2 f}{\partial x \partial y}(a, b),$$
$$\lim_{(h,k) \to (0,0)} \frac{\partial^2 f}{\partial y \partial x}(e, c_2) = \frac{\partial^2 f}{\partial y \partial x}(a, b).$$

Therefore:

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)}$$

∎

Example 1.7.2. Find the first partial derivatives of $f(x, y) = x^3y + e^{xy}$.

Solution.

$$\frac{\partial f}{\partial x} = 3x^2y + ye^{xy}$$
$$\frac{\partial f}{\partial y} = x^3 + xe^{xy}$$

■

Example 1.7.3. Find $\frac{\partial^2 f}{\partial x \partial y}$ for $f(x, y) = \sin(2x + 3y)$.

Solution. First derivatives:

$$\frac{\partial f}{\partial x} = 2\cos(2x + 3y)$$
$$\frac{\partial f}{\partial y} = 3\cos(2x + 3y)$$

Mixed derivative:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3\cos(2x + 3y)) = -6\sin(2x + 3y)$$

■

1.8 Total Differentiation

Concept Overview

Total differentiation extends differentiation to functions of multiple variables. The total differential dz approximates the change in $z = f(x, y)$ when both x and y change.

Total Differential Formula

For $z = f(x, y)$:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

For $w = f(x, y, z)$:

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Chain Rule for Total Derivatives

If $z = f(x, y)$ with $x = g(t)$, $y = h(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1.8.1. Find the total differential of $z = x^2y - 3xy^3$.

Solution. Partial derivatives:

$$\frac{\partial z}{\partial x} = 2xy - 3y^3$$
$$\frac{\partial z}{\partial y} = x^2 - 9xy^2$$

Total differential:

$$dz = (2xy - 3y^3)dx + (x^2 - 9xy^2)dy$$

■

Example 1.8.2. If $z = e^x \sin y$ where $x = t^2$ and $y = t^3$, find $\frac{dz}{dt}$.

Solution. Apply chain rule:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (e^x \sin y)(2t) + (e^x \cos y)(3t^2) \\ &= e^{t^2}\left[2t\sin(t^3) + 3t^2\cos(t^3)\right]\end{aligned}$$

■

1.9 Homogeneous Functions

Concept Overview

A function $f(x_1, x_2, \ldots, x_n)$ is **homogeneous of degree** k if for all $\lambda > 0$:

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n)$$

Properties

- Linear functions are homogeneous of degree 1
- Quadratic forms are homogeneous of degree 2
- Constant functions are homogeneous of degree 0

Example 1.9.1. Show that $f(x, y) = x^3 + 3x^2y + y^3$ is homogeneous and find its degree.

Solution. Replace $x \to \lambda x$, $y \to \lambda y$:

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^3 + 3(\lambda x)^2(\lambda y) + (\lambda y)^3 \\ &= \lambda^3 x^3 + 3\lambda^3 x^2 y + \lambda^3 y^3 \\ &= \lambda^3(x^3 + 3x^2y + y^3) \\ &= \lambda^3 f(x, y) \end{aligned}$$

Thus homogeneous of degree 3. ■

Example 1.9.2. Is $g(x, y) = x^2 + xy + \sin\left(\frac{x}{y}\right)$ homogeneous?

Solution. Test with $\lambda > 0$:

$$\begin{aligned} g(\lambda x, \lambda y) &= (\lambda x)^2 + (\lambda x)(\lambda y) + \sin\left(\frac{\lambda x}{\lambda y}\right) \\ &= \lambda^2 x^2 + \lambda^2 xy + \sin\left(\frac{x}{y}\right) \end{aligned}$$

The expression contains λ^2 terms and a λ-independent term. Not homogeneous. ■

1.10 Euler's Theorem

Theorem 1.10.1 (Euler's Theorem on Homogeneous Functions)**.** *If $f(x, y)$ is a homogeneous function of degree k and has continuous first partial derivatives, then:*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = k \cdot f(x, y)$$

Proof. By definition, since f is homogeneous of degree k, we have the following identity for any $t > 0$:

$$f(tx, ty) = t^k f(x, y) \tag{1.1}$$

We differentiate both sides of the identity (1.1) with respect to the parameter t, treating x and y as constants. For the left-hand side (LHS), we use the multivariable chain rule. Let $X = tx$ and $Y = ty$. Then:

$$\begin{aligned} \frac{d}{dt}f(tx, ty) &= \frac{\partial f}{\partial X}\frac{dX}{dt} + \frac{\partial f}{\partial Y}\frac{dY}{dt} \\ &= f_X(tx, ty) \cdot (x) + f_Y(tx, ty) \cdot (y) \end{aligned}$$

where f_X and f_Y denote the partial derivatives of f with respect to its first and second arguments, respectively.

For the right-hand side (RHS), we treat $f(x, y)$ as a constant and differentiate only the t^k term:

$$\frac{d}{dt}\left(t^k f(x, y)\right) = kt^{k-1} f(x, y)$$

Equating the derivatives of the LHS and RHS gives us a new identity that is also true for all $t > 0$:

$$x \cdot f_X(tx, ty) + y \cdot f_Y(tx, ty) = kt^{k-1} f(x, y)$$

Since the identity above holds for any value of $t > 0$, it must hold for the specific case where $t = 1$. Setting $t = 1$ simplifies the expression:

$$x \cdot f_X(1 \cdot x, 1 \cdot y) + y \cdot f_Y(1 \cdot x, 1 \cdot y) = k(1)^{k-1} f(x, y)$$
$$x \cdot f_X(x, y) + y \cdot f_Y(x, y) = k \cdot f(x, y)$$

Rewriting in standard partial derivative notation, we get the desired result:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = kf(x, y)$$ ■

Example 1.10.2. Verify Euler's theorem for $f(x, y) = x^{1/3}y^{2/3}$.

Solution. First, degree $k = 1/3 + 2/3 = 1$. Partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{1}{3}x^{-2/3}y^{2/3}$$
$$\frac{\partial f}{\partial y} = \frac{2}{3}x^{1/3}y^{-1/3}$$

Apply Euler's theorem:

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\left(\frac{1}{3}x^{-2/3}y^{2/3}\right) + y\left(\frac{2}{3}x^{1/3}y^{-1/3}\right) \\ &= \frac{1}{3}x^{1/3}y^{2/3} + \frac{2}{3}x^{1/3}y^{2/3} \\ &= x^{1/3}y^{2/3} = f(x, y) \end{aligned}$$

Equal to $1 \cdot f$, verifying the theorem. ■

Example 1.10.3. Using Euler's theorem, show that if $f = \frac{x^2+y^2}{xy}$, then $xf_x + yf_y = -f$.

Solution. Rewrite $f(x, y) = \frac{x}{y} + \frac{y}{x}$. Test homogeneity:

$$f(\lambda x, \lambda y) = \frac{\lambda x}{\lambda y} + \frac{\lambda y}{\lambda x} = \frac{x}{y} + \frac{y}{x} = f(x, y)$$

Thus homogeneous of degree 0. By Euler's theorem:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0 \cdot f = 0$$

But note: $0 = -f + f$, so rearrange as $xf_x + yf_y = -f + f$. To get exact form, observe:

$$xf_x + yf_y = 0 = -f + f$$

The problem statement appears inconsistent. Correction: For homogeneous degree 0, $xf_x + yf_y = 0$, while $-f = -\left(\frac{x}{y} + \frac{y}{x}\right)$. They are not equal. The correct conclusion is $xf_x + yf_y = 0$. ■