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Zero-divisor graphs of unitary *R*-modules over commutative rings

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ABSTRACT

Let R be a commutative ring with unity $1 \neq 0$ and let M be a unitary $R-$module. In this paper, we derive some completeness conditions on the zero divisor graphs of modules over commutative rings. It is shown that the weak zero divisor graph of a simple $R-$module is complete if and only if R is a field. We investigate the zero divisor graphs in finitely generated $R-$modules. We find the diameter, the girth, the clique number and the vertex degrees of the zero-divisor graphs of the rings of integer modulo n as $\mathbb{Z}-$modules.

KEYWORDS
Module; ring; zero divisor graph of module; complete graph; diameter; clique

AMS SUBJECT CLASSIFICATION
13A99; 05C12; 05C25; 05C78

1. Introduction

A simple graph G consists of a vertex set $V(G) \neq \emptyset$ and an edge set $E(G)$ of unordered pairs of distinct vertices. The cardinality of $V(G)$ is called the *order* of G and the cardinality of $E(G)$ is its *size*. A graph G is connected if and only if there exists a path between every pair of vertices u and v. A graph on n vertices such that every pair of distinct vertices is joined by an edge is called a *complete* graph, denoted by K_n. A complete subgraph of G of largest order is called a maximal clique of G and its order is called the *clique number* of G, denoted by $cl(G)$. The number of edges incident on a vertex v is called the degree of v and is denoted by d_v or $d(v)$. A vertex of degree 1 is called a pendent vertex. In a connected graph G, the distance between two vertices u and v is the length of the shortest path between u and v. The *diameter* of a graph G is defined as $diam(G) = max\{(d(u,v)|\ u,v \in V(G))\}$, where $d(u,\ v)$ denotes the distance between vertices u and v of G. For more definitions and terminology of graph theory, we refer to [9].

Throughout, R shall denote a commutative ring with unity $1 \neq 0$. Let $Z(R)$ be the set of zero-divisors of R. The concept of the zero-divisor graph of a commutative ring was first introduced by Beck [4]. The zero-divisor graph $\Gamma(R)$ associated to a ring R has its vertices as elements of $Z^*(R) = Z(R)\backslash\{0\}$ and two vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$.

We denote a unitary $R-$module by M, unless otherwise stated. For an $R-$module M and $x \in M$, the set $[x : M] = \{r \in R : rM \subseteq Rx\}$, is clearly an ideal of R and an annihilator of the factor module M/Rx. The annihilator of M denoted by $ann(M)$ is $[0 : M]$. The concept of the zero-divisor graph has been extended to modules over rings, see for instance, [5, 10, 11]. Further, Ghalandarzadeh and Rad [6] extended the notion of the zero-divisor graph to the torsion graph associated with a module M over a ring R, whose vertices are the non-zero torsion elements of M such that two distinct vertices a and b are adjacent if and only if $(a : M)(b : M)M = 0$. The idea was extended to other graph structures, like the zero-divisor graphs of idealizations with respect to the prime modules [2], the L-total graph of an L-module [1], etc, to mention a few.

For any set X, let $|X|$ denote the cardinality of X and X^* denote the set of the non-zero elements of X. We denote an empty set by $\emptyset$ and the complement of X shall be denoted by X^c. We denote the ring of integers by $\mathbb{Z}$, the ring of integer modulo n by $\mathbb{Z}_n$ and the finite field with q elements by $\mathbb{F}_q$. For more definitions and terminology of module and ring theory, we refer to [3, 7].

The rest of the paper is organized as follows. In Section 2, we include some completeness conditions of the zero-divisor graph of the unitary $R-$modules. For instance, it is shown that the zero divisor graph of $M \oplus M$ is complete for every simple module M. In Section 3, we investigate some graph parameters of the zero-divisor graphs of the modules like the diameter, the girth, the clique number and the vertex degrees.

2. Graphs associated with modules over commutative rings

Throughout, we treat M as a unitary $R-$module. Let $N \subseteq M$. We define the *annihilator* of N by $(0 : N) = \{r \in R|\text{for all } m \in M,\ rm = 0\}$. For $m \in M$, we denote the annihilator of the factor module M/Rm by m_M. Thus, $m_M := \{r \in R|rM \subseteq mR\}$. Let z be an element in M. The

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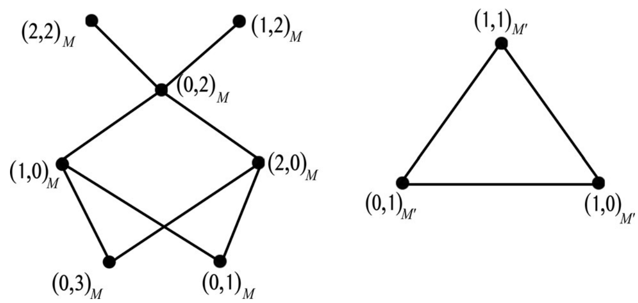

Figure 1. $\Gamma_z(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and $\Gamma_w(\mathbb{Z}_2 \times \mathbb{Z}_2) = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

following definition is due to Behboodi [5]. An element $z \in M$ is a

(a) *weak zero divisor*, if either $z = 0$ or $z_M z'_M M = 0$ for some $0 \neq z' \in M$ with $z'_M \subset R$;
(b) *zero divisor*, if either $z = 0$ or $0 \neq z_M$ and $z_M z'_M M = 0$ for some $0 \neq z' \in M$ with $0 \neq z'_M \subset R$;
(c) *strong zero divisor*, if either $z = 0$ or $(0:_R M) \subset z_M$ and $z_M z'_M M = 0$ for some $0 \neq z' \in M$ with $0 \neq z'_M \subset R$.

For any $R-$module M, we write $Z_w(M)$, $Z(M)$ and $Z_s(M)$, respectively, for the set of non-zero weak zero divisors, non-zero zero divisors and non-zero strong zero divisors. Clearly, $Z_s(M) \subseteq Z(M) \subseteq Z_w(M)$ and all of these sets coincide with the set of zero divisors of R when $M = R$. Behboodi [5] associated three simple graphs, denoted by $\Gamma_w(M)$, $\Gamma(M)$ and $\Gamma_s(M)$, called the *weak zero-divisor graph, zero-divisor graph* and *strong zero-divisor graph*, to an $R-$module M with vertex sets defined as $Z_w(M)$, $Z(M)$ and $Z_s(M)$, respectively. Two distinct vertices z_M and z'_M being adjacent if and only if $z_M z'_M M = 0$. From the definition, clearly $\Gamma_w(M) \subseteq \Gamma(M) \subseteq \Gamma_s(M)$ as induced subgraphs.

Behboodi [5] showed that for any $R-$module M, either $\Gamma_w(M) = \Gamma(M)$ or $\Gamma(M) = \Gamma_s(M)$ and also, $\Gamma_s(M)$ is always connected with diameter at most 3. Moreover, if $\Gamma_s(M)$ is not a tree, then the girth of $\Gamma_s(M)$ is at most 4. Further, characterized the $R-$modules M for which $\Gamma_w(M) = \Gamma(M) = \Gamma_s(M)$ and showed that such a property is only enjoyed by the multiplication modules. Whenever, $\Gamma_w(M) = \Gamma(M) = \Gamma_s(M)$, we shall write $\Gamma_z(M)$ with vertex set $Z_z(M)$. Behboodi showed that the weak zero-divisor graph of a module M is finite if and only if either M is finite or prime multiplication-like module.

Example 2.1. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_3 \times \mathbb{Z}_4$. Then M consists of 12 elements as an $R-$module. As M is a multiplication-like module, we have $\Gamma_w(M) = \Gamma(M) = \Gamma_s(M)$. Also, we have $Z_z(M) = \{(0,1)_M, (0,2)_M, (0,3)_M, (1,2)_M, (2,0)_M, (2,2)_M, (1,0)_M\}$. Further, it can be verified that $(0,1)_M = 3\mathbb{Z} = (0,3)_M$, $(0,2)_M = 6\mathbb{Z}$, $(1,2)_M = 2\mathbb{Z} = (2,2)_M$, $(2,0)_M = 4\mathbb{Z} = (1,0)_M$. Now, let $M' = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $Z_s(M') = \emptyset$, $Z(M') = Z_w(M') = M' - \{(0,0)\}$. For any $z, z' \in M' - \{(0,0)\}$, we have $z_{M'} z'_{M'} M' = 0$. The zero divisor graphs of M and M' are given in Figure 1.

If X is a subset of a module M over a ring R, then the intersection of all submodules of M containing X is called the *submodule generated by* X (or spanned by X). If X is finite, and X generates the module M, then M is said to be *finitely generated*. If $X = \emptyset$, then X clearly generates the zero module. If X consists of a single element, say, $X = \{a\}$, then the submodule generated by X is called the *cyclic (sub)-module* generated by a. Finally, if $\{M_i | i \in I\}$ is a family of submodules of M, then the submodule generated by $X = \cup_{i \in I} M_i$ is called the sum of the modules M_i. If the index set I is finite, the sum of $M_1, M_2, \ldots, M_n$ is denoted by $M_1 + M_2 + \ldots + M_n$. A non zero module M is said to be simple if it has no submodules other than (0) and M.

The following theorem provides a condition for the adjacency of two distinct vertices in the zero divisor graph of a finitely generated module.

Theorem 2.2. *Let $M_1, M_2, \ldots, M_k, \ldots$ be a sequence of finitely generated simple $R-$modules and let $M = \oplus_{i \in \mathbb{N}} M_i$. Then $x_M y_M M = 0$ if and only if xR and yR are disjoint $R-$modules.*

Proof. Let $x_M y_M M = 0$. Assume to the contrary and let $0 \neq z \in Rx \cap Ry$. Then, the submodule generated by z is given as $< z >= M_k \subseteq Rx \cap Ry$. So there exist subsets $\mathbb{N}_1$ and $\mathbb{N}_2$ of $\mathbb{N}$ such that $Rx = \oplus_{i \in \mathbb{N}_1} M_i$ and $Ry = \oplus_{j \in \mathbb{N}_2} M_j$. Therefore, we can write $M = Rx \oplus (\oplus_{j \in \mathbb{N}_2} M_j) = Ry \oplus (\oplus_{i \in \mathbb{N}_1} M_i)$. In this notation, we have

$$y_M = y_{Ry \oplus (\oplus_{i \in \mathbb{N}_1} M_i)} = \left(0 : \bigoplus_{i \in \mathbb{N}_1} M'_i\right) = \bigcap_{i \in \mathbb{N}_1} (0 : M_i) \quad \text{and } Rx \cong \bigoplus_{i \in \mathbb{N}_2^c} M'_i.$$

Thus, we have $(0 : Rx) = (0 : x) = (0 : \oplus_{i \in \mathbb{N}_2^c} M_i) = \cap_{i \in \mathbb{N}_2^c} (0 : M_i)$. Since $x_M y_M M = 0$, we have $y_M \subseteq (0 : x)$. This implies that

$$\bigcap_{i \in \mathbb{N}_1} (0 : M_i) \subseteq \left(0 : \bigoplus_{i \in \mathbb{N}_2^c M_i}\right).$$

Now, since each M_t, $t \in \mathbb{N}$ is simple and $M_{t_1} \not\cong M_{t_2}$ for all $t_1 \neq t_2$, we conclude that $(0 : M_{t_1})$ and $(0 : M_{t_2})$ are coprime. Therefore, we can write

$$\bigcap_{i \in \mathbb{N}_1} (0 : M_i) = \bigotimes_{i \in \mathbb{N}_1} (0 : M_i) \subseteq \bigcap_{i \in \mathbb{N}_2^c} (0 : M_i) \subseteq (0 : M_t) \text{for all } t \in \mathbb{N}_2^c. \tag{2.1}$$

This implies that for every $p \in \mathbb{N}_2^c$, there exists $q \in \mathbb{N}_1$ such that $(0 : M_q) \subseteq (0 : M_p)$. Therefore, $(0 : M_q) = (0 : M_p)$ and so $M_q = M_p$. Finally, $M_k \subseteq Rx = \oplus_{i \in \mathbb{N}_2^c}$. So there exists $s \in \mathbb{N}_2^c$ such that $M_k = M_s$. As in Equation (2.1), there exists $t' \in \mathbb{N}_2^c$ such that $M_k = M_s = M'_t$. Thus, $M_k \subseteq Ry \cap (\oplus_{i \in \mathbb{N}_1} M_i) = (0)$. This implies that $z = 0$, which is contradicts the hypothesis. On the other hand, since $x_M y_M M \subseteq Rx \cap Ry$, we conclude that $Rx \cap Ry = (0)$, which implies that $x_M y_M M = 0$. □

The following lemma will be used in the sequel.

Lemma 2.3. [Proposition 5.3.4, [8]] *An $R-$module M is simple if and only if $M \cong R/\mathfrak{a}$ for some maximal ideal $\mathfrak{a}$ in R.*

An $R-$module M is said to be *decomposable* if there exist two non-zero submodules M_1 and M_2 such that $M = M_1 \oplus M_2$ and *indecomposable* if it is not a direct sum of two non-zero submodules. The following theorem shows that the zero divisor graph of a simple R-module is complete.

Theorem 2.4. If M is a simple $R-$module, then $\Gamma_w(M \oplus M)$ is complete.

Proof. Let M be a simple $R-$module, and let $\mathcal{M} = M \oplus M$. By definition, $(0, m)_{\mathcal{M}} = (0 : M)$ for every $0 \neq m \in M$. Therefore, for each $(m', m'') \in \mathcal{M}$, we have $(m', m'')_{\mathcal{M}} (0, m)_{\mathcal{M}} \mathcal{M} = 0$. Similarly, $(m', m'')_{\mathcal{M}} (m, 0)_{\mathcal{M}} \mathcal{M} = 0$. Now, for each $m_1, m_2 \in M - \{0\}$, we have $(0 : m_1) = (0 : m_2) = (0 : M)$. By Lemma 2.3, we see that $(0 : M)$ is a maximal ideal of R, which is contained in $(m_1, m_2)_{\mathcal{M}}$. Now, if $(m_1, m_2)_{\mathcal{M}} \subseteq (0 : M)$, we are done. Otherwise, $1 \in (m_1, m_2)_{\mathcal{M}}$, which gives $(m_1, 0)1 \in (m_1, m_2)R$. Therefore, there exists $r \in R$ such that $m_1 = m_1 r$ and $m_2 r = 0$. Thus, $r \in (0 : m_2) = (0 : m_1)$, which implies that $m_1 r = 0$. Therefore $m_1 = 0$, a contradiction. Thus we have $(0 : M) = (m_1, m_2)_{\mathcal{M}}$, and so $\Gamma_w(M)$ is complete. □

Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ ($n-$copies of $\mathbb{Z}_p$). Then $(z_1, z_2, \ldots, z_n)_M = p\mathbb{Z}$ if some $z_i = 0$ and some $z_j \neq 0$ for some $1 \leq i \neq j \leq n$ and $(1, 1, \ldots, 1)_M = \mathbb{Z}$. Thus, the strong zero divisor graph of M is empty and that $\Gamma(M) = \Gamma_w(M) \cong K_{p^n - 1}$.

As seen above, $\Gamma_w(\mathbb{Z}_p \times \mathbb{Z}_p)$ is complete when $M = \mathbb{Z}_p \times \mathbb{Z}_p$ is considered as a $\mathbb{Z}-$module. However, the same does not hold true in general for all non-simple modules M when the ring R is chosen arbitrarily. The following theorem restricts the choice for the ring R for a module M to have a complete zero divisor graph.

Theorem 2.5. *Let M be an $R-$module which is not simple. Then $\Gamma_w(M)$ is complete if and only if R is a field.*

Proof. As M is not simple, there exists an $R-$submodule M' such that $(0) \subsetneq M' \subsetneq M$. Let $0 \neq y \in x_M$ for some $x \in M$. Then $yM \subseteq xR$. This implies that $M \subseteq y^{-1}xR \subseteq xR$, which is a contradiction. Therefore, $x_M = 0$. Thus, for all $x, y \in M - \{0\}$, we have $x_M y_M M = 0$. Conversely. assume that $x_M y_M M = 0$ for all $x_M, y_M \in Z_w(M)$. Let N be a proper ideal of R. Consider $M = \frac{R}{N} \oplus R$ and let $m = (\hat{r}_1, r_1)$, $m' = (\hat{r}'_1, r'_1)$, where $\hat{r}_1, \hat{r}'_1 \in \frac{R}{N}$ and $r_1, r'_1 \in R$. Choose $r \in R - \{0, 1\}$. As $N \subset R$, we have $(0, r)_M \supseteq NrR$. Because $(0, r_1)_M (0, r'_1)_M M = 0$ for every $r_1, r'_1 \in R$, we have $rN = 0$ for every $r \in R - \{0, 1\}$. This also implies that $(1 - r)N = 0$ and hence $N = rN = (0)$. □

Let M be a $\mathbb{Z}$module. Let z be a non-zero weak zero divisor in M. Then $z_M = n\mathbb{Z}$ for some $n \in \mathbb{N}$. It is trivial to see that $|n\mathbb{Z} \cap m\mathbb{Z}| \geq 2$ for all $m \in \mathbb{N}$. Also, $Z_w(M) = \{0\}$ if and only if M is a simple $\mathbb{Z}-$module. Since every finite module M is a finite abelian group, so we have the following proposition.

Proposition 2.6. *A vertex in a weak zero divisor graph of a finite $\mathbb{Z}-$module M represents an essential ideal if and only if M is a non-simple finite group.*

Let $M = \mathbb{Z}^{\times n}$ ($n-$ copies of $\mathbb{Z}$) be a $\mathbb{Z}-$module. Then it is easy to see that each non-zero element of M is a weak zero divisor and that for all $z, z' \in Z_w(M)$, we have $z_M z'_M M = 0$. Therefore, $\Gamma_w(M) = K_{|M|-1}$. Now, the submodules generated by the non-zero weak zero divisors of M are the lines with integral coordinates in the hyperplane $\mathbb{R}^{\times n}$ intersecting at the origin only. It follows that for every non-zero weak zero divisor m of M the ideal $\{z \in Z | zM \subseteq mZ\}$ is not an essential ideal. This shows that Proposition 2.6 is not true for infinite modules.

Theorem 2.7. *Let R be an integral domain and M an $R-$module. If there exists an element $m \in M$ such that $(0 : m) = 0$, then $\Gamma_w(M \oplus R)$ is complete.*

Proof. Choose $0 \neq r \in R$ and $m \in M$. Let $z \in (r, m)_M$. Then $(0, m)z = (r, m)r'$ for some $r' \in R$. This gives $r'r = 0$, which implies that $r' = 0$, since R is a domain. This further implies that $z = 0$ because $mz = r'r$ and $(0 : m) = 0$. Therefore, for each $0 \neq r \in R$ and $m \in M$, we have $(r, m)_M = 0$. Further, let $0 \neq m' \in M$ and choose $z' \in (0, m')_M$. Then $(1, 0)z' = (0, m')r''$ for some $r'' \in R$. This gives $z' = 0$ and so $(0, m')_M = 0$. Therefore, $\Gamma_w(M \oplus R)$ is complete. □

Definition 2.8. Let R be a ring and M be an $R-$module. If for every non-zero submodule N of M and an ideal A of R with $NA = 0$ implies $MA = 0$, we say that M is a prime module. This is equivalent to saying that $(0 : M) = (0 : N)$ for every non-zero submodule N of M. It is immediate that $(0 : M)$ is a prime ideal, and it is called the affiliated prime of M. Also, if each submodule of M is of the form AM for some ideal A of R, then we say that M is a multiplication module. Moreover, if a multiplication module M satisfies $(0 : M) \subset (0 : M/N)$ for every non-zero submodule N of M, then M is called a multiplication-like module.

Theorem 2.9. *Let M be a multiplication module over a ring R. Then the zero divisor graph of M is empty if and only if M is a prime multiplication-like module.*

Proof. Since every multiplication module is a multiplication-like module, therefore it suffices to prove the result for multiplication-like modules. Assume that M is not a prime multiplication-like module. We will show that $\Gamma_w(M)$ is non-empty. As M is not a prime multiplication-like module, we have $(0 : M)$ is not a prime ideal. Thus, there exist ideals $\mathfrak{a}$ and $\mathfrak{b}$ which properly contain $(0 : M)$ and satisfy $\mathfrak{a}M \neq 0$, $\mathfrak{b}M \neq 0$ and $\mathfrak{a}\mathfrak{b}M = 0$. Thus, we can find $0 \neq a \in \mathfrak{a}M$ and $0 \neq b \in \mathfrak{b}M$ such that $a_M \subseteq aR \subseteq \mathfrak{a}M$ and $b_M \subseteq bR \subseteq \mathfrak{b}M$. Then, we have $a_M b_M \subseteq \mathfrak{a}\mathfrak{b}M = 0$. Therefore, $\Gamma(M) \neq 0$.

On the other hand, if M is a prime multiplication-like module, then $(0 : M) \subseteq m_M$ for every non-zero $m \in M$, that is, for each non-zero $m', m'' \in M$, we have $m'_M m''_M M \neq 0$. Therefore, $\Gamma_w(M) = \phi$. □

3. Graph parameters of zero divisor graphs of modules

In the following theorem, we compute the clique number of the zero divisor graph of a multiplication module. Noting that the weak zero divisor graph, the zero divisor graph and the strong zero divisor graph all coincide in case of multiplication-like modules, we write $\Gamma_z(M)$ to denote the zero divisor graph of such modules.

Theorem 3.1. *Let M be an $R-$module, where $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^t}$ for $t \in \mathbb{N}$ and a prime p. Then the clique number of $\Gamma_z(M)$ is equal to $p^{\frac{t}{2}} - 1$ or $p^{\frac{t-1}{2}}$ according as t is even or odd.*

Proof. It follows immediately that every vertex of $\Gamma_z(M)$ is of the form rp for some $r \in R$. We divide the vertex set of $\Gamma_z(M)$ into disjoint subsets $\mathscr{Z}_1, \mathscr{Z}_2, ..., \mathscr{Z}_{t-1}$, where $\mathscr{Z}_k = \{rp^k | p \nmid r\}$. It is not difficult to see that the cardinality of $\mathscr{Z}_k$ as a subset of $V(\Gamma_z(M))$ is equal to $(p-1)p^{n-k-1}$, $1 \leq k \leq t-1$. Let $u = rp^{k_1}$ and $u' = r'p^{k_2}$ be two vertices of $\Gamma_z(M)$. Then $u_M u'_M M = 0$ if and only if $k_1 + k_2 \geq n$. Thus, for all $v, v' \in M_s$, we have $v_M v'_M M = 0$ for all integers $s \geq \lceil \frac{t}{2} \rceil$. Now, assume that t is even. Then $w_M w'_M M \neq 0$ for all $w \in \mathscr{Z}'_s$, $1 \leq s' \leq \frac{t}{2}$ and $w' \in \mathscr{Z}_{\frac{t}{2}}$. Also, when t is odd, no two vertices are adjacent inside $\mathscr{Z}_{\lfloor \frac{t}{2} \rfloor}$ and every vertex of $\mathscr{Z}_{\lfloor \frac{t}{2} \rfloor}$ is adjacent to every vertex of $\mathscr{Z}_{\lceil \frac{t}{2} \rceil}$. Therefore, it follows that $cl(\Gamma_z(M)) = p^{\lfloor \frac{t}{2} \rfloor} - 1$, when t is even and is equal to $p^{\lfloor \frac{t}{2} \rfloor}$, when t is odd. □

The girth of a graph G is defined as the length (or order) of the smallest cycle contained in G, and is denoted by $gr(G)$. If G has no cycle, then $gr(G) = \infty$. The following theorem characterizes the diameter, the smallest (d_δ) and the largest (d_Δ) vertex degree and the girth of the zero divisor graph of a $\mathbb{Z}-$module M.

Theorem 3.2. *Let p be a prime integer and $t \in \mathbb{N} - \{1\}$. Then for an $R-$module M, where $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^t}$, the following statements hold, unless $t = p = 2$.*

(1) $diam(\Gamma_z(M)) = 2$
(2) $d_\delta(\Gamma_z(M)) = 1$ *and* $d_\Delta(\Gamma_z(M)) = p^{t-1} - 2$.
(3) $gr(\Gamma_z(M)) = \infty$ *if and only if* t=4, 8, 9, *otherwise* $gr(\Gamma_z(M)) = 3$.

Proof. As in the proof of Theorem 3.1, we define $\mathscr{Z}_k = \{rp^k | p \nmid r\}$. Then $\mathscr{Z}_1, \mathscr{Z}_2, ..., \mathscr{Z}_{t-1}$ gives a partition of the vertex set of $\Gamma_z(M)$ and $\mathscr{Z}_k = (p-1)p^{t-k-1}$, $1 \leq k \leq t-1$. Now, two elements $m = r_1 p^k$ and $m' = r_2 p^l$ of M satisfy $m_M m'_M M = 0$ if and only if $k + l \geq t$. Therefore, it follows instantly that every vertex $u \in V(\Gamma_z(M))$ is adjacent to every vertex contained in $\mathscr{Z}_{t-1}$. Thus, $diam(\Gamma_z(M)) = 2$. Further, let $n \in \mathscr{Z}_1$ and choose $n' \in V(\Gamma_z(M))$. Then $n_M n'_M M = 0$ if and only if $n' \in \mathscr{Z}_{t-1}$. Thus, it follows that $\delta(\Gamma_z(M)) = p - 1$ and that $\Delta(\Gamma_z(M)) = p^{t-1} - 2$. This proves (1) and (2).

(3) From the previous paragraph, we see that for all $a \in \mathscr{Z}_{t-1}$, $d(a, b) = 1$ for all vertices $b \neq a$ of $\Gamma_z(M)$. Also, if $c \in \mathscr{Z}_1$, then $d(c, e) = 1$ only if $e \in \mathscr{Z}_{t-1}$. Thus, the set of elements in $\mathscr{Z}_{t-1}$ form the center of $\Gamma_z(M)$. Now, assume that $\Gamma_z(M)$ is a tree. Then $\Gamma_z(M)$ has either one or two centres. Therefore, $|\mathscr{Z}_{t-1}|$ must be either 1 or 2, so that $p = 2$ or 3. Now, either $z_1 p^2 - 3z_2 p^2 - z_3 p^3 - z_1 p^2$ or $z'_2 p^2 - z'_1 p - 2z'_3 p^2 - z'_1 p$ form a triangle in $\Gamma_z(M)$ for all $z_1, z_2, z_3, z'_1, z'_2, z'_3 \in R$ and $p^t \notin \{4, 8, 9\}$. Therefore, the result follows. □

Corollary 3.3. *$gr(\Gamma_z(M)) = \infty$ if and only if $\Gamma_z(M)$ is a star graph, where $M = \mathbb{Z}_{p^t}$, $t \in \mathbb{N}$, is considered a $\mathbb{Z}-$module.*

Let $M = \mathbb{Z}_p \times \mathbb{Z}_q$, where p and q are distinct primes, be a $\mathbb{Z}-$module. Then, it can be easily verified that $(z_1, z_2)_M = p\mathbb{Z}$, if $z_1 = 0 \neq z_2$; $(z_1, z_2)_M = q\mathbb{Z}$, if $z_2 = 0 \neq z_1$, and $(z_1, z_2)_M = \mathbb{Z}$, if $z_1 \neq 0 \neq z_2$. Thus, $\Gamma_z(M)$ is complete bipartite. While one expects that if $p_1, p_2, ..., p_t$, $t \geq 3$, are distinct primes, and $M = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_t}$ then $\Gamma_z(M)$ is complete $t-$ partite, but this is not a case. However, $\Gamma_z(M)$ contains the so expected $t-$ partite graph as a subgraph as can be seen in $\Gamma_z(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$ in which, $(0, 1, 1)_M = (0, 1, 2)_M = (0, 1, 3)_M = (0, 1, 4)_M = (0, 2, 1)_M = (0, 2, 2)_M = (0, 2, 3)_M = (0, 2, 4)_M = 2\mathbb{Z}$, $(1, 0, 1)_M = (1, 0, 2)_M = (1, 0, 3)_M = (1, 0, 4)_M = 3\mathbb{Z}$, $(1, 1, 0)_M = (1, 2, 0)_M = 5\mathbb{Z}$, $(0, 0, 1)_M = (0, 0, 2)_M = (0, 0, 3)_M = (0, 0, 4)_M = 6\mathbb{Z}$, $(0, 1, 0)_M = (0, 2, 0)_M = 10\mathbb{Z}$ and $(1, 0, 0)_M = 15\mathbb{Z}$, so that $\deg((0, 1, 1)_M) = 1$, $\deg((1, 0, 1)_M) = 2$, $\deg((1, 1, 0)_M) = 4$, $\deg((0, 0, 1)_M) = 5$ $\deg((0, 1, 0)_M) = 9$, $\deg((1, 0, 0)_M) = 14$, thus containing six different vertex degrees. However, a complete $t-$ partite graph can possess at most t distinct vertex degrees. Therefore, $\Gamma_z(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$ is not complete $t-$ partite, but we see that the subsets $\mathscr{V}_1 = \{(0, 0, 1)_M, (0, 0, 2)_M, (0, 0, 3)_M, (0, 0, 4)_M\}$, $\mathscr{V}_2 = \{(0, 1, 0)_M, (0, 2, 0)_M\}$ and $\mathscr{V}_3 = \{(1, 0, 0)_M\}$ of the vertex set of $\Gamma_z(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5)$ induce a complete $3-$ partite subgraph.

If p, q, r are distinct primes and $M = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ be a $\mathbb{Z}-$module, then it is always possible to partition the vertex of $\Gamma_z(M)$ into six disjoint sets, say, $\mathscr{V}_i$, $1 \leq i \leq 6$, where $\mathscr{V}'_i s$ are defined in the following way. Let u_p, u_q, u_r denote the arbitrary non-zero elements in $\mathbb{Z}_p$, $\mathbb{Z}_q$ and $\mathbb{Z}_r$, respectively. Then $\mathscr{V}_1 = \{(u_p, 0, 0)_M\}$, $\mathscr{V}_2 = \{(0, u_q, 0)_M\}$, $\mathscr{V}_3 = \{(0, 0, u_r)_M\}$, $\mathscr{V}_4 = \{(u_p, u_q, 0)_M\}$, $\mathscr{V}_5 = \{(u_p, 0, u_r)_M\}$ and $\mathscr{V}_6 = \{(0, u_q, u_r)_M\}$. Let $x^{(i)} \in \mathscr{V}_i$. Then it is an easy exercise to verify that $x^{(1)}_M = qr\mathbb{Z}$, $x^{(2)}_M = pr\mathbb{Z}$, $x^{(3)}_M = pq\mathbb{Z}$, $x^{(4)}_M = r\mathbb{Z}$, $x^{(5)}_M = q\mathbb{Z}$ and $x^{(6)}_M = p\mathbb{Z}$. Thus, it can be easily seen that $\deg(x^{(i)}_M)$, for $1 \leq i \leq 6$ is an element of the ordered set $\{p-1, q-1, r-1, pq-1, pr-1, qr-1\}$. Moreover, the clique number of $\Gamma_z(M)$ is 3 and the sets $\mathscr{V}_4$, $\mathscr{V}_5$ and $\mathscr{V}_6$ induce a complete $3-$ partite subgraph. In fact, this can be generalized to the following theorem.

Theorem 3.4. Let M be an $R-$module, where $M = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \mathbb{Z}_{p_t}$ and $R = \mathbb{Z}$, then

(1) $cl(\Gamma_z(M)) = 3$.
(2) $\deg((x_1, x_2, ..., x_t)_M) = \frac{p_1 p_2 ... p_t}{\Pi p_i} - 1$, where i runs over the indices of $x'_i s$ in $(x_1, x_2, ..., x_t)_M$ which are equal to 0.
(3) The set of vertices $\mathscr{V}_i = \{(x_1, x_2, ..., x_{i-1}, 0, x_{i+1}, ..., x_t)_M | x_i \in \mathbb{Z}_{p_i}\}$ induces a complete $t-$ partite subgraph in $\Gamma_z(M)$.

Theorem 3.5. *Let M and N be two $R-$modules such that the sum of their annihilators equals R. Then the following statements hold.*

1. If $\Gamma_w(M) = \Gamma_w(N) = \emptyset$, then $cl(\Gamma(M \oplus N)) = 2$.
2. If $\Gamma_w(M) = \emptyset$ and $\Gamma_w(N) \neq \emptyset$, then $cl(\Gamma_w(M \oplus N)) = cl\ (\Gamma_w(N)) + 1$.
3. If $\Gamma_w(M) \neq \emptyset$ and $\Gamma_w(N) \neq \emptyset$, then $cl(\Gamma_w(M \oplus N)) = cl(\Gamma_w(M)) + cl(\Gamma(N)) + \eta_1 \eta_2$, where η_1, η_2 denotes the number of elements η in cliques of $\Gamma_w(M)$ and $\Gamma_w(N)$, respectively, whose square is 0.

Proof. For any $R-$module M, let $Z_w(M)$ and $Z^*_w(M)$ denote the set of weak zero divisors and non-zero weak zero divisors of M. Let M and N be two $R-$modules, $m_1, m_2 \in M$ and $n_1, n_2 \in N$ and $M \bigoplus N = \mathcal{M}$.

1. Assume that $Z_w(M) = Z_w(N) = \{0\}$. Then for each $m_1, m_2 \in M$ and $n_1, n_2 \in N$, we have $(m_1, 0)_{\mathcal{M}}(m_2, 0)_{\mathcal{M}}\mathcal{M} \neq 0$, $(0, n_1)_{\mathcal{M}}(0, n_2)_{\mathcal{M}}\mathcal{M} \neq 0$ and $(m_1, 0)_{\mathcal{M}}(0, n_1)_{\mathcal{M}}\mathcal{M} = 0$. Thus, $cl(\Gamma(M \bigoplus N)) = 2$.
2. As $\Gamma_w(N) \neq \emptyset$, we have $Z_w(N) \neq (0)$. Let $\mathscr{K} = \{k_1, k_2, ..., k_t\}$ be an induced maximal clique in $\Gamma_w(N)$. Then, for each $n \in Z_w^*(N) - \mathscr{K}$, there exists some $k_i \in \mathscr{K}$ such that $n_N k_{iN} N \neq 0$. Now, for each $(m, 0), (0, n) \in M \bigoplus N$, we have $(m, 0)_{\mathcal{M}}(0, n)_{\mathcal{M}}\mathcal{M} = 0$ and for all $m' \in M - \{0\}$ and $n' \in N$, we have $(m', 0)_{\mathcal{M}}(m', n')_{\mathcal{M}}\mathcal{M} \neq 0$. Thus, the vertices of the form $(m, 0)$ contribute 1 to the clique number and the fact that $\Gamma_w(M) = \emptyset$, we conclude that $cl(\Gamma_w(M \bigoplus N)) = 1 + cl(\Gamma_w(N))$.
3. Let $Z_w(M) \neq \{0\}$ and $Z_w(N) \neq \{0\}$. Let $\mathscr{K}_M = \{k_1, k_2, ..., k_{t_1}\}$ and $\mathscr{K}_N = \{l_1, l_2, ..., l_{t_2}\}$ be the induced maximal complete subgraphs of $\Gamma_w(M)$ and $\Gamma_w(N)$. Then, for each $m \in Z_w^*(M) - \mathscr{K}_M$ and $n \in Z_w^*(N) - \mathscr{K}_N$, we can find $k_i \in \mathscr{K}_M$, $1 \leq i \leq t_1$ and $k_j \in \mathscr{K}_N$, $1 \leq j \leq t_2$ which satisfy $(m, 0)(k_i, 0)M \neq 0$ and $(0, n)_{\mathcal{M}}(0, k_j)_{\mathcal{M}}\mathcal{M} \neq 0$. Also, for every $m_1 \in M - \{0\}$ and $n_1 \in N - \{0\}$, there exist no $m'_1 \in M - Z_w^*(M)$ and $n'_1 \in N - Z_w^*(N)$ for which $(m_1, n_1)_{\mathcal{M}}(m'_1, n'_1)_{\mathcal{M}}\mathcal{M} = 0$ holds true. Even if m_1 (or equivalently m'_1) is chosen from M, then a similar statement holds true if $n_{1_N} n_{1_N} N \neq 0$ (or equivalently $n'_{1_N} n'_{1_N} N \neq 0$). A similar argument is valid for n_1, n'_1 if chosen from N. Thus, such vertices do not contribute to the clique number. Now, for all $m' \in M$ and $n' \in N$, we have $(m', 0)_{\mathcal{M}}(0, n')_{\mathcal{M}}\mathcal{M} = 0$, but however such a vertex, say $(m'', 0)$, contributes to the clique if and only if $m'' \in \mathscr{K}_M$ and $m''_M m''_M M = 0$. This argument adds each vertex $k \in \mathscr{K}_M$ and $k' \in \mathscr{K}_N$ which satisfy $k_M k_M M = 0$ and $k'_N k'_N N = 0$ to the clique. Therefore, the clique number of $\Gamma_w(M \bigoplus N)$ is equal to $cl(\Gamma_w(M)) + cl(\Gamma_w(N)) + \eta_1\eta_2$, where η_1 and η_2 are the number of vertices $k^{(1)} \in \mathscr{K}_M$ and $k^{(2)} \in \mathscr{K}_N$, respectively, which satisfy $k_M^{(1)} k_M^{(1)} M = 0$ and $k_N^{(2)} k_N^{(2)} N = 0$ □

Theorem 3.6. *Let R be a finite integral domain and M be an $R-$module which is not simple. Then $cl(\Gamma_w(M)) = |M| - 1$.*

Proof. The proof follows by Theorem 2.5. □

Theorem 3.7. *Let $M = \mathbb{Z}_{p_1^{a_1} p_2^{a_2} ... p_t^{a_t}}$ be a $\mathbb{Z}-$module, where $p_1, p_2, ..., p_t$ are distinct primes. Then the clique number of $\Gamma_z(M)$ is equal to t, if $a_1 = a_2 = \cdots = a_t = 1$. In case $a_i = 2b_i$, then the clique number of $\Gamma_z(M)$ is $p_1^{b_1} p_2^{b_2} ... p_t^{b_t} - 1$.*

Proof. Let $n = p_1 p_2 ... p_t$. Define $n_i = \frac{n}{p_i}$, $1 \leq i \leq t$, and choose $z_i \in \mathbb{Z}n_i$. Then $z_i M z_j MM = 0$ for all $i \neq j$. Therefore, contains a clique of order t. Moreover, if $m \in \mathbb{Z}n_i n_j$, then $x_i M m_M M \neq 0$ for all $x_i \in M$. Hence, $cl(\Gamma_z(M)) = t$. Now, let $m = p_1^{b_1} p_2^{b_2} ... p_t^{b_t}$. Then $m \in M$ and we have $m_M m_M M = 0$. Consider the submodules $\mathbb{Z}m$, $\mathbb{Z}2m$, ..., $\mathbb{Z}(m-1)m$, then we have $z_i M z_j MM = 0$ for all $z_i \in \mathbb{Z}_i$ and $z_j \in \mathbb{Z}_j$, where $i, j = \{m, 2m, ..., (m-1)m\}$. Moreover, let $\mathbb{Z}k$ be a submodule of M, where $k \notin \{m, 2m, ..., (m-1)m\}$. Then k is of the form $p_1^{c_1} p_2^{c_2} ... p_t^{c_t}$, where some $c_i < b_i$. Without loss of generality, let $c_1 < b_1$ and let $z \in \mathbb{Z}k$, then we get $z_M z'_M M \neq 0$ where z' is an element of $\mathbb{Z}m$. Therefore, the clique number is equal to $p_1^{b_1} p_2^{b_2} ... p_t^{b_t} - 1$. □

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