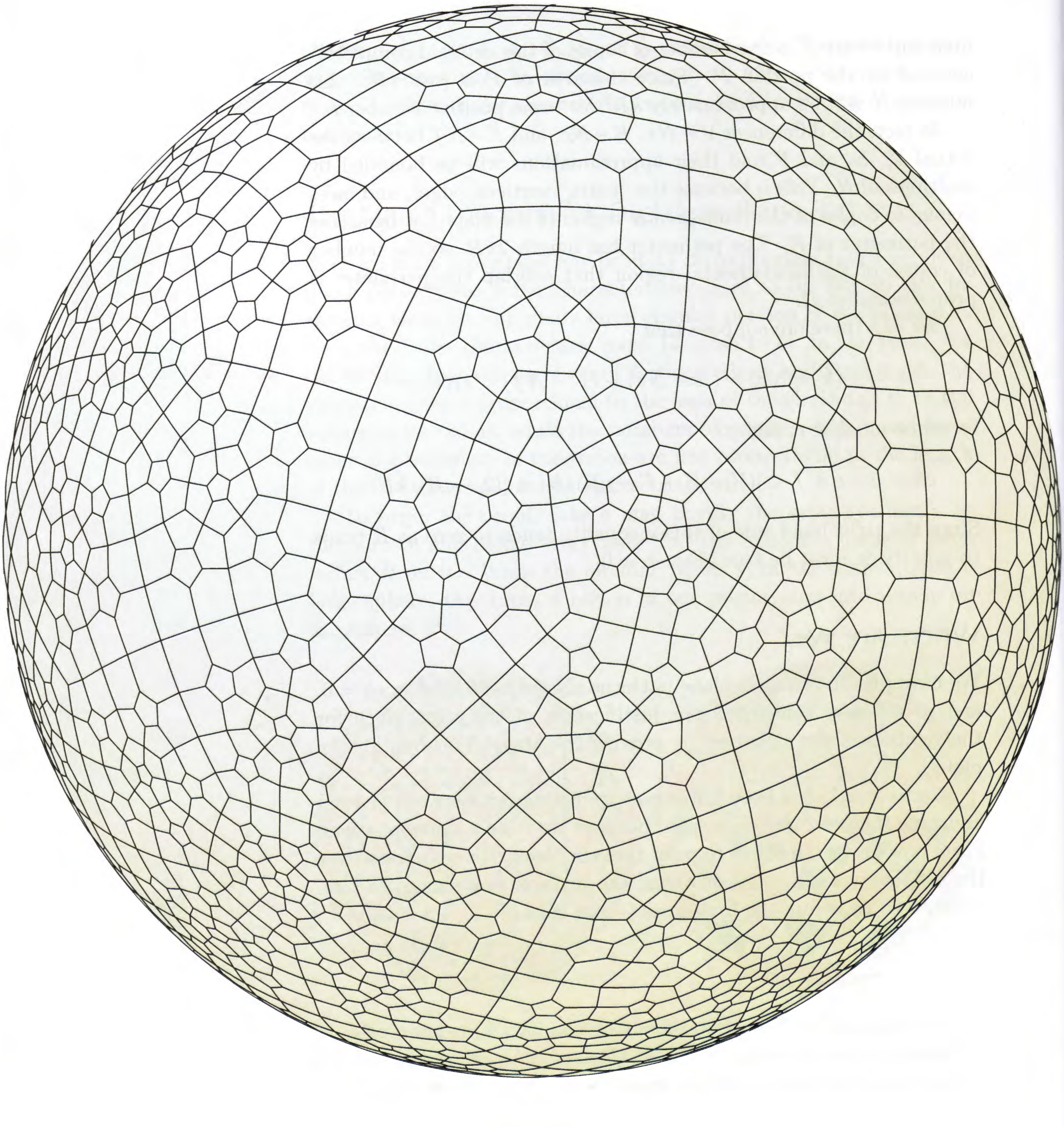


– 7 –

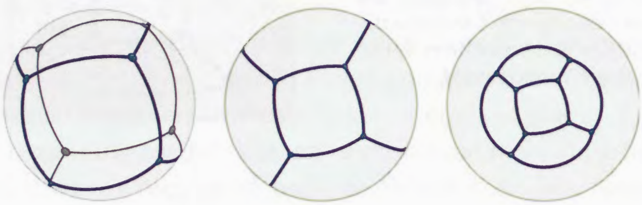
Euler's Map Theorem

We've made some powerful deductions from Euler's Theorem that $V - E + F = 2$ for maps on the sphere. Now we'll prove it!

This is really quite familiar—we've all seen maps of the Earth in the plane.

Proof of Euler's Theorem

We can copy any map on the sphere into the plane by making one of the faces very big, so that it covers most of the sphere.



We'll think of this big face as the ocean, the vertices as towns (the largest being Rome), the edges as dykes or roads, and ourselves as barbarian sea-raiders! (See Figure 7.1.)

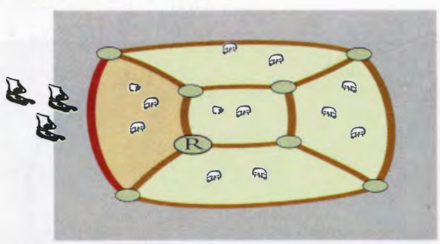
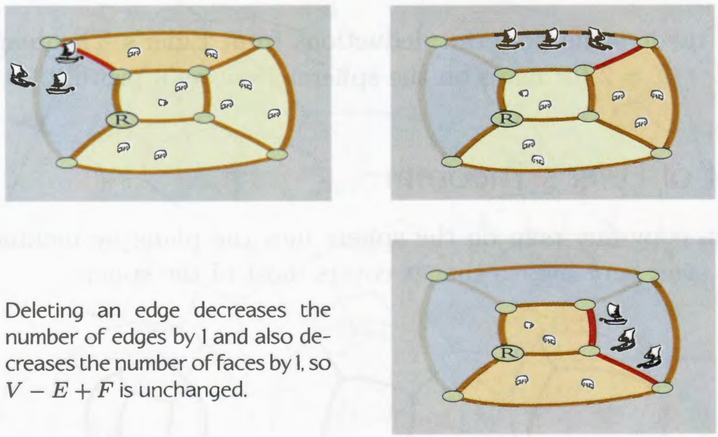


Figure 7.1. Our prey.

(opposite page) Like all maps on the sphere, this beautiful map (signature $*532$) has $V + F - E = 2$.

In this new-found role, our first aim is to flood all the faces as efficiently as possible. To do this, we repeatedly break dykes that separate currently dry faces from the water and flood those faces. This removes just $F-1$ edges, one for each face other than the ocean, by breaking $F-1$ dykes.



Deleting an edge decreases the number of edges by 1 and also decreases the number of faces by 1, so $V-E+F$ is unchanged.

We next repeatedly seek out towns other than Rome that are connected to the rest by just one road, sack those towns, and destroy those roads. (See Figure 7.2.)

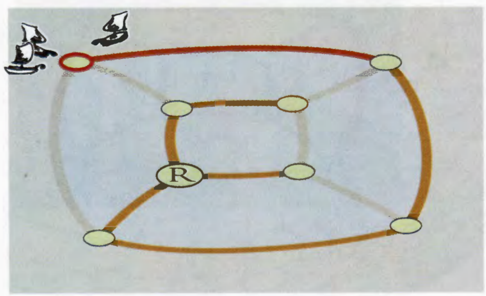
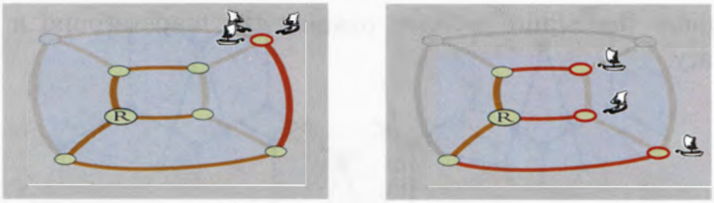
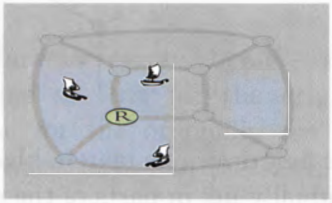


Figure 7.2. Our raid continues!





Deleting a vertex and the edge joining it to the rest of the tree does not change $V - E + F$ either, so $V - E + F$ must always have been 2.

We have sacked $V - 1$ towns by destroying $V - 1$ roads, one for each town other than Rome. The number of edges in the original map must therefore have been $(F - 1) + (V - 1) = V + F - 2 = E$. Therefore, $V + F - E = 2$, proving Euler's Theorem.

(Did we sack every town other than Rome? Yes; an unsacked town furthest from Rome would have two paths back to Rome, which however must enclose some dry fields, a contradiction. Did we destroy all remaining roads? Yes; an undestroyed road must be between unsacked towns, which must both be Rome; but then again it must enclose some dry fields.)

We have tacitly assumed that each face is a topological disk, and we will continue to suppose this. We have also taken for granted some intuitively obvious facts about the topology of the sphere whose formal proofs are surprisingly difficult.

The number 2 is Euler's characteristic number for the sphere. Every surface has such a number.

The Euler Characteristic of a Surface

Theorem 7.1 *Any two maps on the same surface have the same value of* $V-E+F$*, which is called the* Euler characteristic *for that surface.*

We prove that any two maps on the same surface have the same Euler characteristic $V - E + F$ by considering a larger map obtained by drawing them both together. We shall suppose that no two edges