2.6 Volume by Slicing

Name:______________________

The general volume by slicing formula is

$$\text{Volume} = \int_a^b \text{Area}(x)\,dx.$$

By far the most common case is the volume of revolution, where the cross sectional areas are discs of radius $f(x)$. That is, the volume given by revolving $f(x)$ around the x-axis from a to b is

Volume by Discs:

$$\text{Volume} = \pi \int_a^b (f(x))^2\,dx.$$

For the volume between two functions, $f > g$ we have:

Volume by Washers:

$$\text{Volume} = \pi \int_a^b (f(x))^2 - (g(x))^2\,dx.$$

NOTE: This is NOT the same as $\pi \int_a^b (f(x) - g(x))^2\,dx$. That thing is nothing useful!

1. A solid has cross sectional areas which are squares of side-length $f(x) = 3x - x^2$, for $0 \le x \le 3$. Find the volume. (Hint: $\int_0^3 (3x - x^2)^2\,dx$.)

2. A solid has cross sectional areas which are equilateral triangles of side-length $f(x) = x\sqrt{4-x}$, for $0 \le x \le 4$. Find the volume. (Recall the area of an equilateral triangle of side-length ℓ is $\frac{\sqrt{3}}{2}\ell^2$).

Find the volume given by revolving the function around the x-axis.

1. $f(x) = \sqrt{2x - x^2}$, $0 \le x \le 2$.

2. $f(x) = \sec(x)$, $\frac{-\pi}{3} \le x \le \frac{\pi}{3}$.

Sketch the region between the functions. Then find the volume given by

revolving the region around the x-axis.

1. The region between $f(x) = x^2 + 1$ and $g(x) = 3 - x^2$.

SOLUTION:

$$x^2 + 1 = 3 - x^2 \qquad \Rightarrow \qquad 2x^2 = 2 \qquad \Rightarrow \qquad x = \pm 1.$$

So

$$\begin{aligned} \text{Volume} &= \pi \int_{-1}^{1} (3 - x^2)^2 - (x^2 + 1)^2 \, dx \\ &= \pi \int_{-1}^{1} 8 - 8x^2 \, dx \\ &= \pi \left(8x - \frac{8}{3}x^3 \Big|_{-1}^{1} \right) = \frac{32\pi}{3} \qquad \square. \end{aligned}$$

2. The region between $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x$.

3. The region between $f(x) = x^2 + 1$ and $g(x) = \frac{1}{2}x^2 + 3$.

We can revolve around other lines as well:

Revolve around the horizontal line $y = c$:

$$\text{Volume} = \pi \int_a^b (f(x) - c)^2 - (g(x) - c)^2 \, dx \qquad (c < g(x) < f(x))$$

$$\text{Volume} = \pi \int_a^b (c - g(x))^2 - (c - f(x))^2 \, dx \qquad (g(x) < f(x) < c)$$

1. Find the area given by revolving the region between $y = x^2 + 2$ and $y = 3$ around the line $y = 1$.

2. Find the area given by revolving the region between $y = x^2 + 2$ and $y = 3$ around the line $y = 5$.

Some interesting regions.

1. A sphere of radius 5 has a cylindrical hole of radius 3 drilled through the middle. Find the volume by revolving the region between $f(x) = \sqrt{25 - x^2}$ and $g(x) = 3$ around the x-axis.