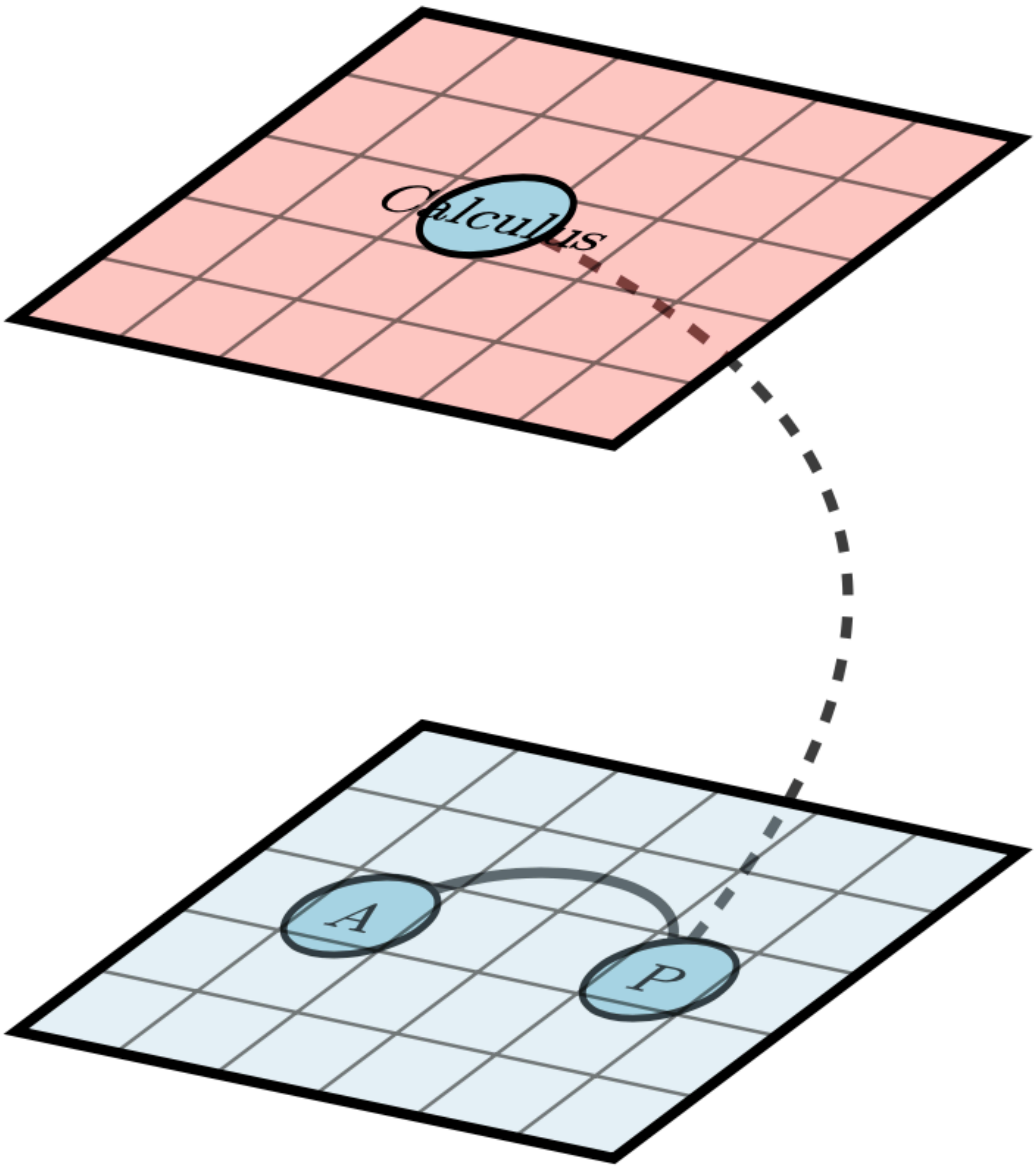
AP Calculus

by: ____________________



Contents

Chapter 0

Essentials for Calculus

§1 Partial Fraction Decomposition

Consider the problem of adding two rational expressions:

$$\frac{3}{x+4} \text{ and } \frac{2}{x-3}$$

The result is:

The reverse procedure, starting with $\dfrac{5x-1}{x^2+x-12}$ and ending with two simpler fractions is called **partial fraction decomposition**.

Recall that a rational expression is the ratio of two polynomials, say P and $Q \neq 0$. Assume that P and Q have no common factors. The rational expression $\frac{P}{Q}$ is **proper** if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Otherwise, the rational expression is **improper**.

There are 4 distinct cases that we must know how to apply to decompose $\frac{P}{Q}$. We begin with the Case 1.

Decompose $\frac{P}{Q}$ when:

Case 1: Q has only non repeated linear factors

Under the assumption Q has only non repeated linear factors, the polynomial Q has the form:

$$Q(x) = (x-a_1)(x-a_2)\cdot \ldots \cdot (x-a_n)$$

Where no two of the numbers $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$, are equal. In this case, the partial fraction decomposition of $\frac{P}{Q}$ is of the form:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \ldots + \frac{A_n}{(x-a_n)}$$

Where the numbers $A_1, A_2, A_3, \ldots, A_n$ are to be determined using manual algebra (or matrices).

Example 1. Find the partial fraction decomposition of $\dfrac{x}{x^2 - 5x + 6}$.

DIY 1. Find the partial fraction decomposition of $\dfrac{3x}{(x+2)(x-4)}$.

Case 2: Q has repeated linear factors
If the polynomial Q has a repeated linear factor, say

$$(x-a)^n, n \geq 2, n \text{ an integer},$$

then the partial fraction decomposition of $\frac{P}{Q}$, allow for the terms:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}.$$

Where the numbers $A_1, A_2, \ldots A_n$ are to be determined.

Example 2. Find the partial fraction decomposition of $\dfrac{x+2}{x^3-2x^2+x}$.

DIY 2. Find the partial fraction decomposition of $\dfrac{x^2}{(x-1)^2(x+1)}$

Case 3: Q contains a non-repeated irreducible quadratic factor

If Q contains a non repeated irreducible quadratic factor of the form

$$ax^2 + bx + c,$$

then in the partial fraction decomposition of $\frac{P}{Q}$, allow for the term

$$\frac{Ax + B}{ax^2 + bx + c}$$

Where the numbers A and B are to be determined.

Example 3. Find the partial fraction decomposition of $\dfrac{3x - 5}{x^3 - 1}$.

Case 4: Q contains a repeated irreducible quadratic factor

If the polynomial Q contains a repeated irreducible quadratic factor

$$(ax^2 + bx + c)^n, n \geq 2, n \text{ an integer},$$

then in the partial fraction decomposition of $\frac{P}{Q}$ allows for the terms

$$\frac{P(x)}{Q(x)} = \frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

Where $A_1, B_1, A_2, B_2, \ldots A_n, B_n$ are to be determined.

Example 4. Find the partial fraction decomposition of $\dfrac{x^3 + x^2}{(x^2 + 4)^2}$.

DIY 3. Find the partial fraction decomposition of $\dfrac{x^2+2x+3}{(x^2+4)^2}$.

§2 Sequences

Definition 0.1: Sequences Definition

A sequence is a list of things generated by a rule.

More formally, a sequence is a function whose domain is the set of positive integers, or natural numbers, n such that $n \in \mathbb{N} = \{1, 2, 3, ...\}$. The range of the function are called the terms in the sequence,

$$a_1, a_2, a_3, ...a_n$$

.

Where a_n is called the nth term (or rule of the sequence), and we denote the sequence by $\{an\}$. The sequence can be expressed by either

1. An ample number of terms in the sequence, separated by commas
2. A recursive function including a first term and a rule using the previous term of the sequence.
3. A rule for the sequence given as an explicit function set off in curly braces.

Arithmetic Sequences

Definition 0.2: Arithmetic Sequences Recursive Definition

Given a_1 , then

$$a_n = a_{n-1} + d$$

where d is called the common difference and can be positive or negative.

Example 5. Consider $a_n = a_{n-1} + 3$ and $a_1 = 5$.

a) Determine the value of a_2, a_3, a_4

b) How many times was 3 added to 5 to produce a_4?

c) Use the ideas in part b to find a_{50}.

d) Find an **explicit** relation for this sequence.

Geometric Sequences

Definition 0.3: Geometric Sequences Recursive Definition

Given a_1, then

$$a_n = a_{n-1} \cdot r$$

where r is called the common ratio and can be positive or negative and is often fractional.

DIY 4. Consider $a_n = a_{n-1} \cdot 3$ and $a_1 = 2$.

a) Determine the value of a_2, a_3, a_4

b) How many times was 2 multiplied by 3 to produce a_4?

c) Use the ideas in part b to find a_{10}.

d) Find an **explicit** relation for this sequence.

Series

A series is the sum of the terms in a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. Informally, a series is the result of adding any number of terms from a sequence together:

$$a_1 + a_2 + a_3 + \ldots$$

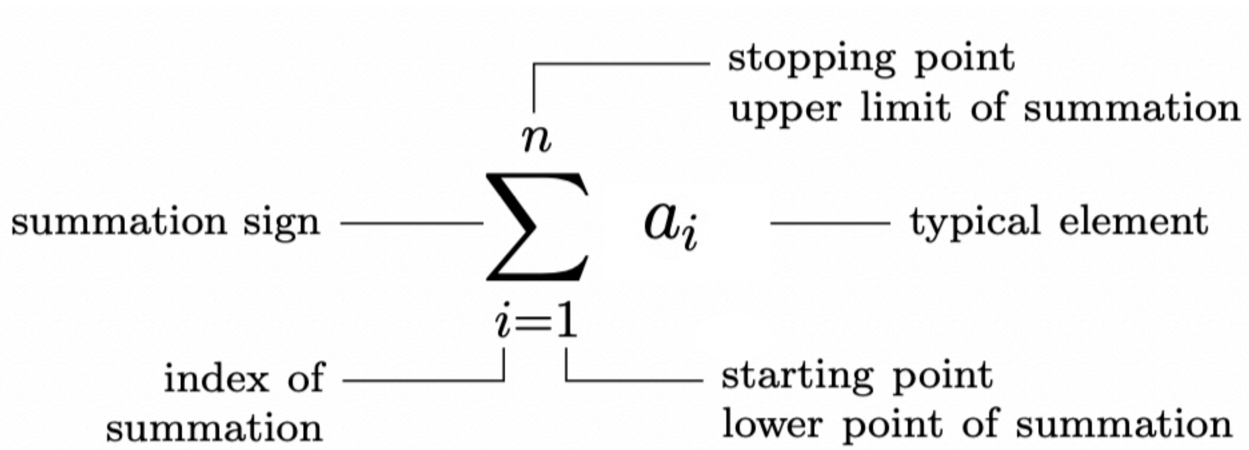
A series can be written more succinctly using the summation notation.

Definition 0.4: Series

If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an infinite collection of numbers, the expression

$$\sum_{i=1}^{n} a_n = a_1 + a_2 + a_3 + \ldots + a_n$$

is called a finite series.



For infinite series, we can look at the sequence of partial sums to get an idea of the behavior of the sequence. In general, the nth partial sum is denoted by S_n.

DIY 5. Find the first 5 terms of the sequence of partial sums, and list them below.

a) $\sum_{i=1}^{n} \left(\frac{1}{i!}\right)$

b) $\sum_{i=1}^{n} \left(\frac{3}{2}\right)^{i}$

§3 Trig Essentials

The following will be useful, so we will explore their derivations:
Trig Definitions:

Given a circle of radius r[1].

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y}, \qquad \sec\theta = \frac{r}{x}, \qquad \cot\theta = \frac{x}{y}$$

Quotient Identities:

$$\sin\theta = \frac{1}{\csc\theta}, \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities:[2]

Derive and prove the following identities.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

[1] Recall that the equation of a circle is $x^2 + y^2 = r^2$
[2] -same as above- but now we let $r = 1$

.

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

Sum and Difference Formulas[3]

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double-Angle Formulas

Derive and prove the following identities.

$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2 - 1 = 1 - 2\sin^2\theta$$

[3]These are not obvious at all so these must be learned!!

Chapter 1

Limits

1.1 Limits of Functions Using Numerical Approaches

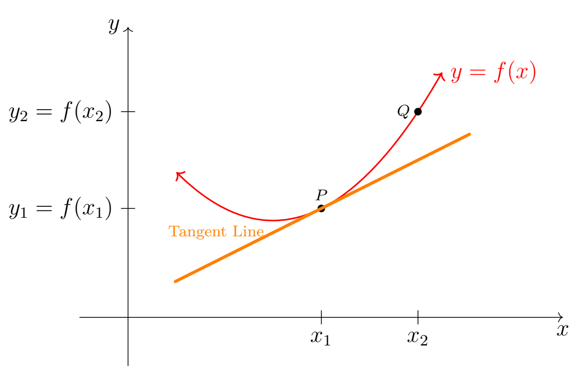
- Consider a function f and a point P on its graph. What is the slope of the tangent line to the graph of f at the point P?
- Consider a non-negative function f whose domain is the closed interval $[a, b]$. What is the area of the region enclosed by the graph of f, the $x-$axis, the vertical lines $x = a$ and $x = b$.

The limit is fundamental to the study of calculus. The concept is the defining distinction between other courses you have taken in the past, It is important to attain a strong working knowledge of the limit before moving forward. So, lets get started!

Geometric Motivation: Tangent and Velocity Problem

Differential Calculus was partly motivated by the idea of finding the instantaneous velocity of an object. This idea, at first, was difficult to approach and answer, but two great minds found independent solutions that were one in the same. [1]. We will motivate this discussion with an example.

NO FRUIT WAS HARMED IN THE MAKING OF THIS PROBLEM.



[1]Issac Newton and Gottfried Wilhelm (von) Leibniz are the men responsible for this great discovery

Example 6. Suppose a large watermelon is thrown straight up in the air at $t = 0$ seconds. The watermelon leaves the thrower's hand at highs speed, slows down until it reaches its maximum height, then speeds up in the downward direction and finally, **"SPLAT"!**

a) How far does the watermelon travel during the 1st second? During the 2nd second? What does this tell us about the motion of the watermelon?

b) What is the watermelon's *average rate of change* over the interval $1 \leq t \leq 3$? $4 \leq t \leq 5$? What is the significance of the sign?

t sec.	0	1	2	3	4	5	6
$H(t) ft.$	4	26	38	40	32	14	-14

DIY 6. The function $C(t)$ below gives the average cost, in dollars, of gallons of milk t years after 2000. Find the average rate of change between 2003-2006 and 2003-2008. Interpret these values.

t years	2	3	4	5	6	7	8
$C(t)$ dollars	1.47	1.69	1.94	2.30	5.90	2.64	3.01

As we have seen, the **AROC** of a function/scenario has a certain amount of meaning and importance. Although this is true, a more relevant and more useful calculation is the **instantenous rate of change** or in other words, the rate of change at an **exact instant**.

If you look at the speedometer of your car when you travel to school, you may notice that the needle never stays still for more than an instant (i.e. the velocity is not constant), but without just our bare eye, it is relatively impossible to state the exact velocity of a moving object at a specific moment.

Graphically, the above examples are solved using the principle of a secant line between two points of a graph.

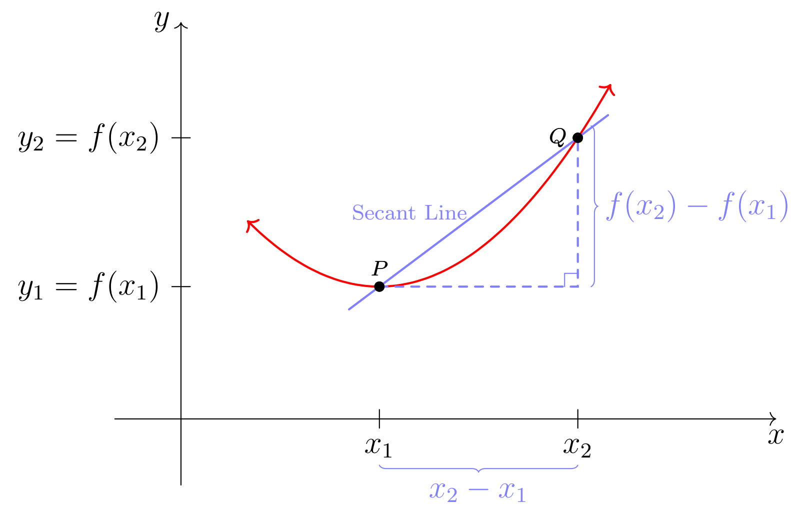
Definition 1.1: Average Rate of Change (AROC)

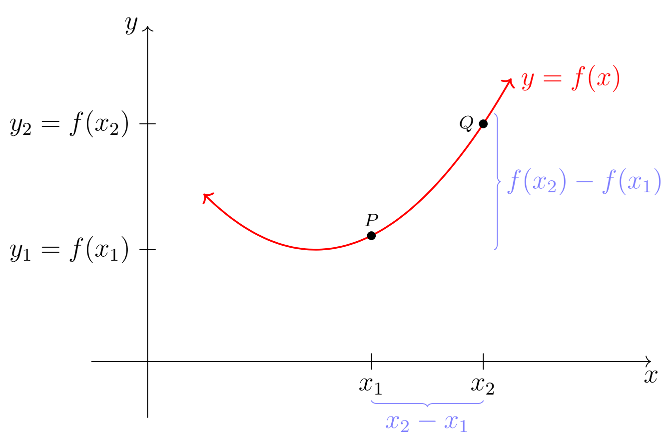
If x_1 and x_2, $x_1 \neq x_2$, are in the domain of a function $y = f(x)$, the **average rate of change of** f from x_1 to x_2 is defined as

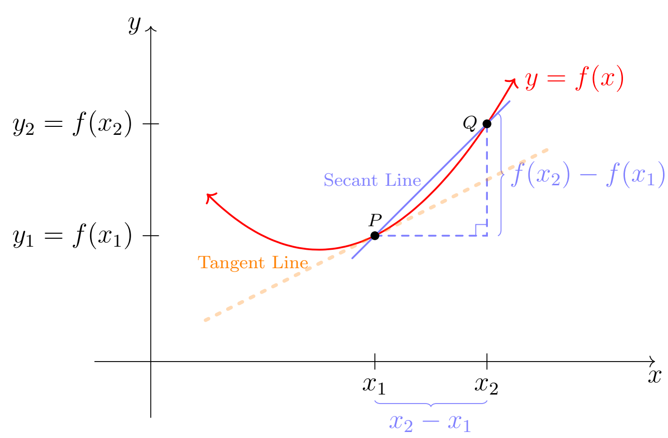
$$\text{Average Rate of Change} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

The units on a rate of change are "output units per input units"

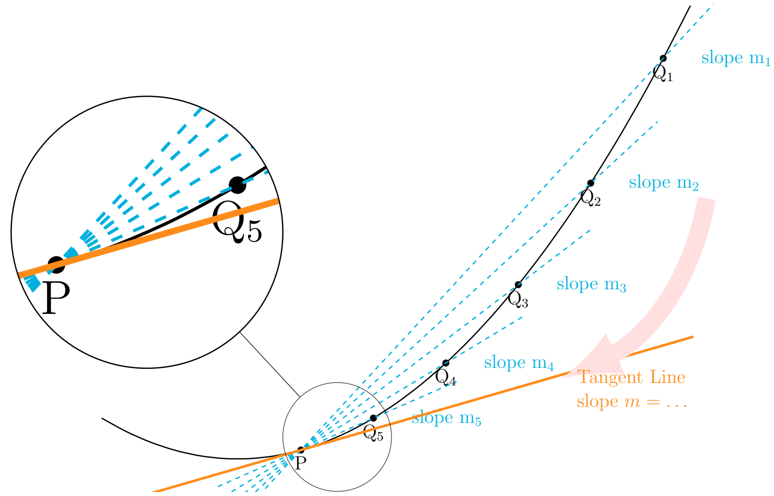
Geometrically: The average rate of change is the slope of the **Secant Line** containing the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.







Notice how we begin with our traditional approach from Algebra I; we are simply finding the slope between two points. We then keep picking points Q_i such that they keep approaching P from the *right*. Eventually, a point that is sufficiently close to P will eventually lead to a secant line "almost identical to our tangent line. If the limiting position of the secant line is the tangent line, then the limit of the slopes of the secant lines should equal the slope of the tangent line.



As $Q_i \to P$, we see that the secant limit will be the tanget line.

The Concept of a Limit

This process of *approaching* a point from a certain *direction* is the fundamental idea behind the limit. So this begs the question:

What does it mean for a function f to have a limit L as x approaches some fixed number c?

Definition 1.2: Naive Definition of Limits

To have a limit at $x = c$, the function f must be defined in a local-open interval containing c, except possibly at c, and L must be a number[a] In math notation,

$$\lim_{x \to c} f(x) = L$$

which is read as

"the limit as x approaches c of $f(x)$ is equal to the number L."

Fact: The value $f(x)$ can be made as close as we please to L for x sufficiently close to c, but not equal to c. [b]

This implicitly is stating that if the above is true, then the function should behave"nicely" and "predictable" around c.

In other words, limits are like fortune tellers, they tell us where we are headed, regardless of if we get there.

[a] or infinity. We will discuss this special case later.

[b] The more formal treatment of limits and its definition is called the $(\epsilon - \delta)$ definition of the limit and will be discussed in an Upper Division Real Analysis course...or in BC.

Example 7. Interpret the following limits in words:

a) In words, $\lim_{x \to -1} \dfrac{x^2 - 1}{x + 1} = -2, x \neq -1$, is read as :

b) The limit may be interpreted as: "The value of ________________ can be made as ________ as we please to -2 by choosing ________ sufficiently close to, but not equal to ________.

c) What is the importance of $x \neq -1$ being mentioned?

DIY 7. The limit as x approaches c of a function f is written symbolically as:

Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish between these behaviors.

Using Tables and Graphs to Evaluate Limits

Example 8. Evaluate the following limit using a table. Confirm your solution using a graph.

$$\lim_{x \to 2}(2x + 5) =$$

.

x	1.99	1.999	1.9999	1.99999	$\to 2 \leftarrow$	2.00001	2.0001	2.001	2.01
$f(x) = 2x + 5$	8.98	8.998	8.9998	8.99998	$f(x)$approaches ___	9.00002	9.0002	9.002	9.02

DIY 8. Evaluate the following limit using a table. Confirm your solution using a graph.

$$\lim_{x \to 5}(3x - 8) =$$

.

x	4.99	4.999	4.9999	4.99999	$\to 5 \leftarrow$	5.00001	5.0001	5.001	5.01
$f(x) = 3x - 8$					$f(x)$approaches ___				

When picking values to use for our tables we notice that we used values that were less than our desired c value and greater than our desired c value. We will now formally address these limits as the **left handed and right handed limits of** f, respectively and give them their own notation.

Definition 1.3: Left and Right Handed Limits

The limit as x approaches c from the left of $f(x)$ (i.e. $x > c$) is known as the **left handed limit** of f and is written as :

$$\lim_{x\to c^-} f(x) = L_{left}$$

The limit as x approaches c from the right of $f(x)$ (i.e. $x < c$) is known as the **right handed limit** of f and is written as:

$$\lim_{x\to c^+} f(x) = L_{right}$$

Example 9. Evaluate the following limit using a table. Confirm your solution using a graph.

$$\lim_{x\to 0} \frac{\sin x}{x} =$$

x	-0.01	-0.001	-0.0001	-0.00001	$\to 0 \leftarrow$	0.00001	0.0001	0.001	0.01
$f(x) = \dfrac{\sin x}{x}$					$f(x)$approaches ___				

Piece-Wise Defined Functions

Example 10. Use a graph to evaluate $\lim\limits_{x\to 2} f(x)$ if

$$f(x) = \begin{cases} 3x+1 & \text{if } x \neq 2 \\ 10 & \text{if } x = 2 \end{cases}$$

DIY 9. Use a graph to evaluate $\lim_{x\to 0} f(x)$ if

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x+1 & \text{if } x < 0 \end{cases}$$

The prior two examples reveals to us that we can now define what it means for a limit to exist at a point with a bit more accuracy.

Theorem 1: Limit Existence Theorem

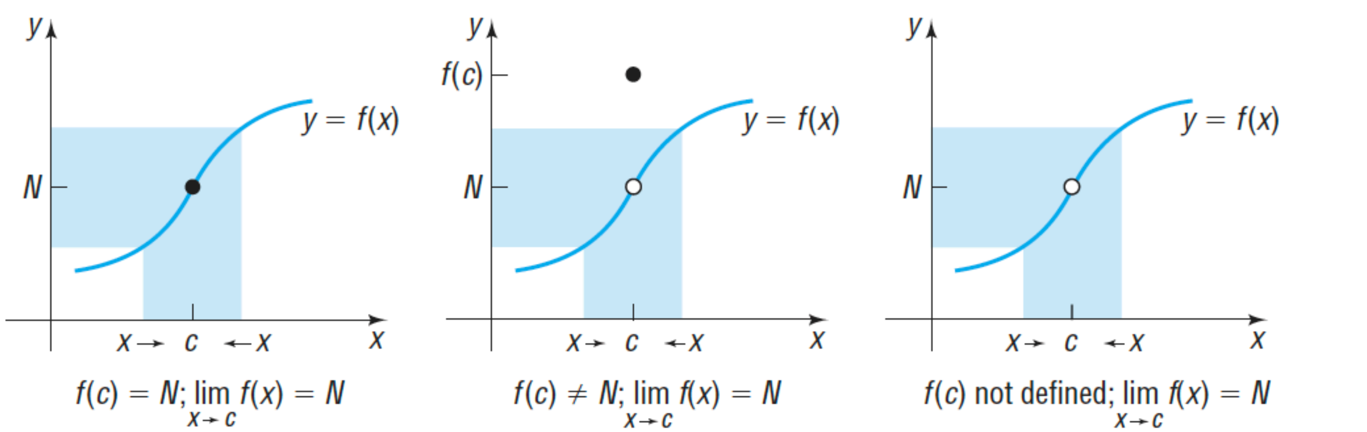
The limit of a function $y = f(x)$ as x approaches a number c exists if and only if

- both one-sided limits exist at c
- both one-sided limits are equal

That is,

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$$

The Limit Existence Theorem allows us now evaluate functions that behave like the following:



Example 11. Use the graph to evaluate the following limits:

A) $\lim_{x \to 0} f(x)$

B) $\lim_{x \to 6^-} f(x)$

C) $\lim_{x \to 2^-} f(x)$

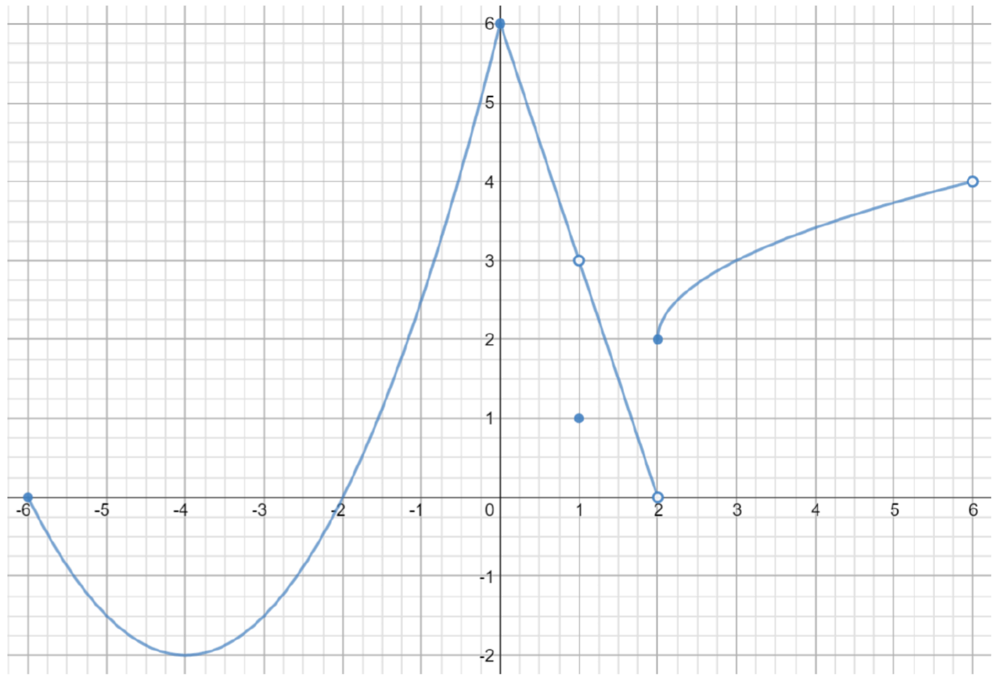
D) $\lim_{x \to -6} f(x)$

E) $\lim_{x \to 1} f(x)$

F) $f(1)$

G) $f(2)$

H) $\lim_{x \to 2} f(x)$



Using tables and graphs gives us an idea as to how a function might behave around a point c but caution must be taken since these methods only ***suggest*** a limit(-ing value). The following is a classic counterexample to this common misconception.

DIY 10. Evaluate the following limit using a table. Confirm your solution using a graph.

$$\lim_{x \to 0} \sin \frac{\pi}{x^2} =$$

x	-0.01	-0.001	-0.0001	-0.00001	$\to 0 \leftarrow$	0.00001	0.0001	0.001	0.01
$f(x) = \sin \frac{\pi}{x^2}$					$f(x)$approaches ___				

x	$-\frac{2}{3}$	$-\frac{2}{5}$	$-\frac{2}{7}$	$-\frac{2}{9}$	$\to 0 \leftarrow$	$\frac{2}{9}$	$\frac{2}{7}$	$\frac{2}{5}$	$\frac{2}{3}$
$f(x) = \sin \frac{\pi}{x^2}$					$f(x)$approaches ___				

$\varepsilon - \delta$ Definition of Limits

The above example showed us that using tables and graphs fail at finding the limit with certainty. So what do we do? We turn to our $\varepsilon - \delta$ definition of the limit. Before presenting the definition, let us explore why we were able to evaluate previous limits with certainty.

Example 12. We claimed that $\lim_{x \to 2}(2x + 5) = 9$

a) How close must x be to 2 so that $f(x) = 2x + 5$ differs from 9 by less than 0.1?

b) How close must x be to 2 so that $f(x) = 2x + 5$ differs from 9 by less than 0.05?

Definition 1.4: $\varepsilon - \delta$ Definition of the Limit

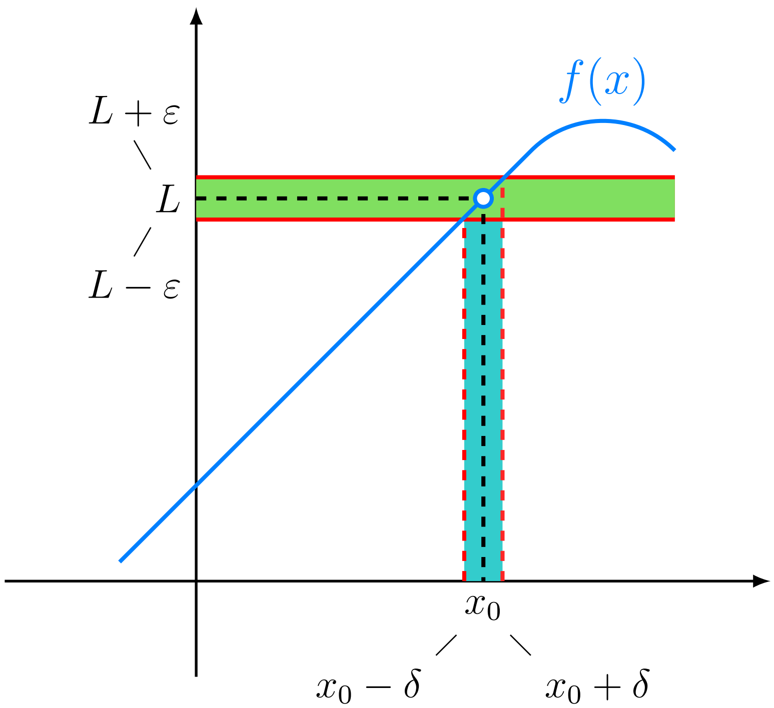
Let f be a function defined everywhere in an open interval containing c, except possibly at c and L is a real number. We say that

$$\lim_{x \to c} f(x) = L$$

If for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all $x \in I$,

$$\text{whenever } 0 < |x - c| < \delta \text{ then } |f(x) - L| < \varepsilon.$$

The image you should have in your mind is the following:



1.2 Limits of Functions using Algebraic Methods

We have just evaluated limits from a numerical/visual standpoint. Although useful, we do seek a more efficient method. This is where those years of algebra come in!

Theorem 2: Limit Properties

Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exists. Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, where $k, L, M \in \mathbb{R}$

$\lim_{x\to c} A = A$	$\lim_{x\to c} k \cdot f(x)$
$\lim_{x\to c} x = c$	$\lim_{x\to c} [f(x)]^{\frac{r}{s}} = L^{\frac{r}{s}}$
$\lim_{x\to c} [f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm M$	$\lim_{x\to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$
$\lim_{x\to c} [f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot M$	$\lim_{x\to c} [f(g(x))] = f\left(\lim_{x\to c} g(x)\right) = f(M)$

Example 13. Suppose $\lim_{x\to c} f(x) = -4$ and $\lim_{x\to c} g(x) = 3$. Evaluate the following, if the limit does not exist, explain.

a) $\lim_{x\to c} 3g(x)$

b) $\lim_{x\to c} [f(x) + g(x)]$

c) $\lim_{x\to c} [f(x)g(x)]$

d) $\lim_{x\to c} \frac{f(x)}{g(x)}$

DIY 11. Suppose $\lim_{x\to c} f(x) = 3, \lim_{x\to c} g(x) = -1, \lim_{x\to c^-} h(x) = \infty$ and $\lim_{x\to c^+} h(x) = 6$. Evaluate the following, if the limit does not exist, explain.

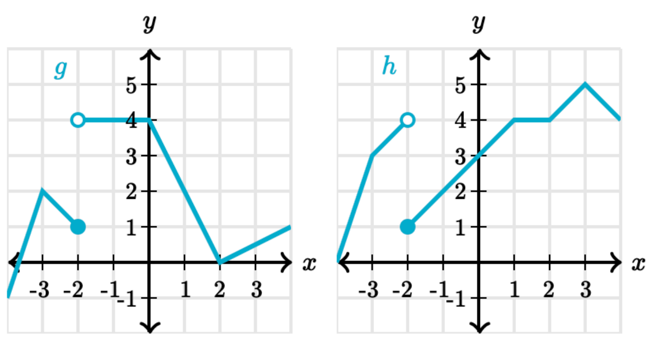
a) $\lim_{x\to c} f(x) - 3g(x)$

b) $\lim_{x\to c^+} 2f(x)g(x)h(x)$

c) $\lim_{x\to c^-} f(x)g(x)h(x)$

d) $\lim_{x\to c^-} \dfrac{f(x)}{g(x)h(x)}$

DIY 12. Evaluate each limit using the graphs provided. If the limit does not exist, explain.



a) $\lim_{x\to -2} [g(x) + h(x)]$

b) $\lim_{x\to 2} h(g(x))$

c) $\lim_{x\to -2} g(x)$

d) $\lim_{x\to 1} \dfrac{g(x)}{h(x)}$

From our limit properties we have a very useful corollary for evaluating limits of polynomials.
Corollary 1. *If* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0$ *then* $\lim_{x \to c} p(x) = p(c)$. [2]

Method 1: Substitution

Direct substitution means that you plug-in the value that the variable approaches and determine what the expression evaluates to. As long as the expression does not evaluate to an undefined value, direct substitution will work. **This should always be your first course of action!!!**
Example 14. Evaluate the following limits:

a) $\lim_{x \to 3}(2x + 1)$

b) $\lim_{x \to 4}(5x - 9)$

c) $\lim_{x \to 1} \dfrac{x^2 - 1}{x + 1}$

d) $\lim_{x \to 2} x^2 - 4x + 9$

e) $\lim_{x \to \frac{\pi}{2}} \dfrac{x}{2} \cot(3x)$

f) $\lim_{x \to -1} \dfrac{x^2 - 9}{x^2 + 1}$

[2] In other words, plug in the value! Although we have yet to discuss continuity, the main reason this theorem is true is due to the fact that all polynomials are continuous.

If direct substitution yields $\frac{0}{0}$, otherwise known as indeterminate form, then you cannot determine the limit in its current form. Encountering this form means you should try another technique. We will now discuss 3 techniques to solve these limits.

Method 2: Factoring

Example 15. Evaluate the following limits:

a) $\displaystyle\lim_{x\to -1} \frac{x^2-3x-4}{x+1}$

b) $\displaystyle\lim_{x\to -1} \frac{x^2-1}{x+1}$

c) $\displaystyle\lim_{x\to -1} \frac{2x^2-x-3}{x+1}$

Method 3: Conjugation/Rationalize the Denominator

Example 16. Evaluate the following limits:

a) $\displaystyle\lim_{x\to -1} \frac{x+1}{\sqrt{x+5}-2}$

b) $\displaystyle\lim_{x\to 7} \frac{\sqrt{x-3}-2}{x-7}$

Method 4: Simplifying Complex Fractions

Example 17. Evaluate the following limits:

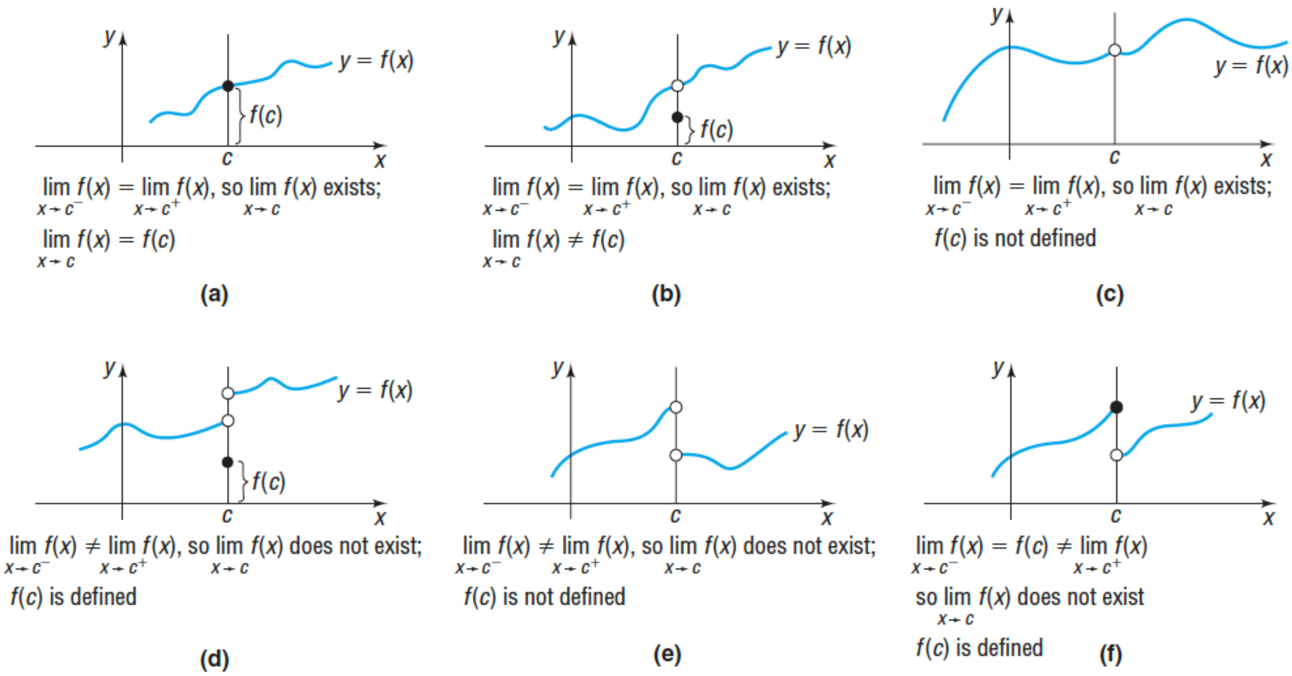
a) $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x}$

b) $\lim\limits_{x \to 2} \dfrac{\dfrac{1}{x+1} - \dfrac{1}{3}}{x-2}$

1.3 Continuity

When we hear the word "continuity" there are numerous examples we can think about. In Pre-Calculus we were content with the following statement:

Naive Definition 1. A function f is continuous if you can draw its graph without picking up your pencil.



From the six graphs below, the only one that is continuous would be (a).

Definition 1.5: Continuity at a Point

A function f is continuous at a number c if the following conditions are true:

- $f(c)$ exists
- $\lim_{x\to c} f(x)$ exists
- $\lim_{x\to c} f(x) = f(c)$

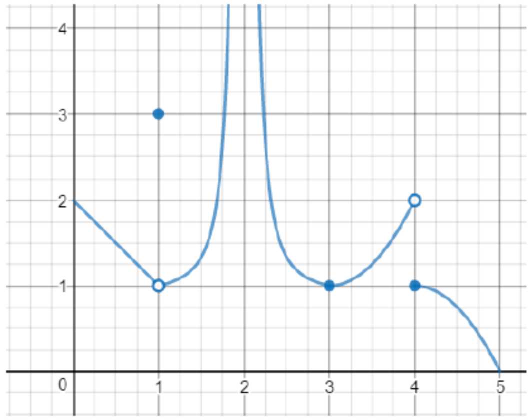
If we can show that one of the conditions for continuity fail (at a point) then we claim that the function is discontinuous.

Definition 1.6: Discontinuity

A function that is not continuous is said to be **discontinuous**.

There are two types of discontinuities, removable and non-removable. A hole in the graph is an example of a removable discontinuity. It is considered removable because you can easily make the graph continuous again by filling the hole. Vertical asymptotes and jumps are examples of non-removable discontinuities. They cannot be made continuous without drastically changing the function itself.

Example 18. Using the graph of f, determine whether f is continuous at $c = 1, 2,$ and 4. Take notice in the importance of each part of the definition of continuity.



More often than not, you will not be provided with the graph. In these cases you can decide to graph or use some algebra to evaluate the limit.

Theorem 3: Continuity of a Sum, Difference, Product, and Quotient

If f and g are continuous, then the functions $f+g$, $f-g$, $f \cdot g$, and $\dfrac{f}{g}$ (when $g(c) \neq 0$) are also continuous.

"Combinations of continuous objects remain continuous."

Example 19. Determine whether each function is continuous or not. If it is not continuous, use the definition of continuity to explain why.

a) $f(x) = 3x^2 - 5x + 4$ when $c = 1$

b) $g(x) = \dfrac{\sqrt{x^2+2}}{x^2-4}$ when $c = 2, 0$

c) $h(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ -1 & \text{if } x = 2 \\ x^2 - 2 & \text{if } x > 2 \end{cases}$ when $x = 2$

Example 20. Determine whether each function is continuous:

a) $F(x) = x^2 + 5 - \dfrac{x}{x^2+4}$

b) $G(x) = x^5 + x + \dfrac{x^3}{x^2-1}$

c) $H(x) = \dfrac{x^2-1}{x^2-4} + \sqrt{x-1}$

Example 21. If g is to be continuous, find the value of k.

$$g(x) = \begin{cases} \dfrac{x^2+5x+6}{x+3} & \text{if } x \neq -3 \\ k & \text{if } x = -3 \end{cases}$$

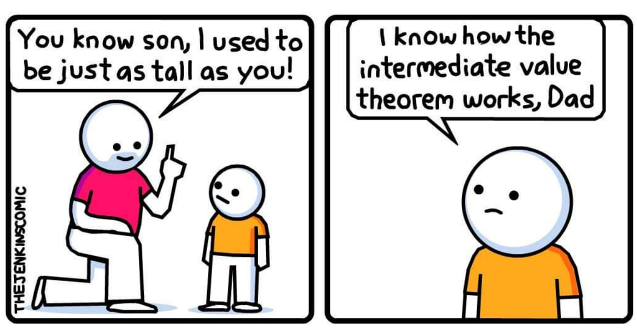
DIY 13. If β is to be continuous for $x \geq -5$, find the value of k.

$$\beta(x) = \begin{cases} \dfrac{x+1}{\sqrt{x+5}-2} & \text{if } x \geq -5, x \neq -1 \\ k & \text{if } x = -1 \end{cases}$$

DIY 14.

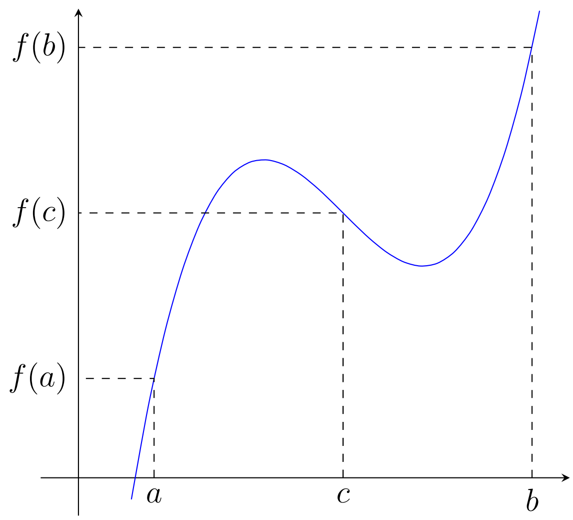
$$h(x) = \begin{cases} x^2 - 1 & \text{if } x < 3 \\ 4bx & \text{if } 3 \leq x \leq 6 \\ a(x-1)^2 - 4 & \text{if } x > 6 \end{cases}$$

Functions that are continuous on a closed interval have many important properties. One of them is stated in the **Intermediate Value Theorem**. This is one of the BIG THEOREMS for AP CALCULUS.



Theorem 4: Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$ then f takes every value between $f(a)$ and $f(b)$. Suppose k is a value between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = d$.



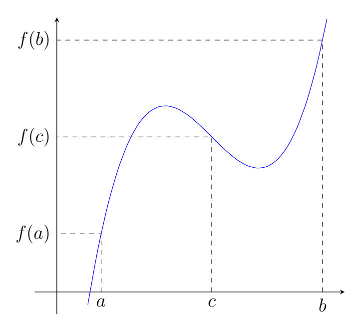
Examples to keep in mind:

- Temperature
- Speed
- Height
- Fluid Flow

The Intermediate Value Theorem tells you that at least one c exists, but it does not give you a method of finding c. This theorem is an example of an **existence theorem.**

Example 22. In the Intermediate Value Theorem,

a) What are the requirements in order to apply this theorem?



b) d is on which axis?

c) c is on which axis?

Example 23. Verify that the Intermediate Value Theorem applies to the following function $f(x)$ over the interval $\left[\frac{5}{2}, 4\right]$, explain why the IVT guarantees an $x-$value of c where $f(c) = 6$, and find c.

$$f(x) = \frac{x^2 + x}{x - 1}$$

There is a special corollary that allows us to use the IVT to locate the zeros of a function.

Theorem 5: Bolzano's Theorem

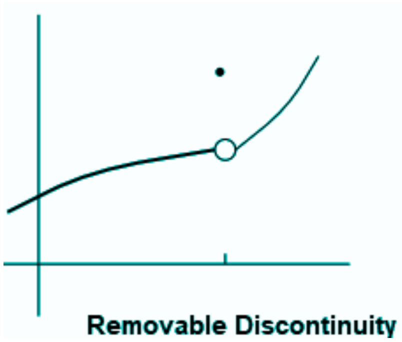
If a continuous function defined on an interval is sometimes positive and sometimes negative, it must be 0 at some point.

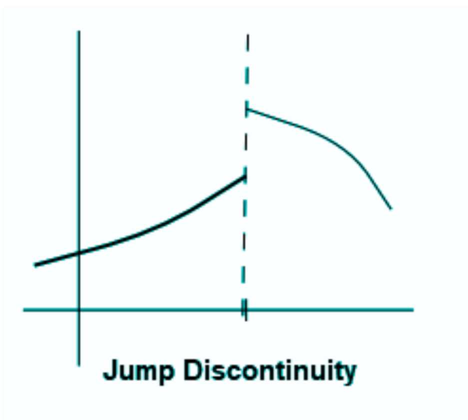
DIY 15. Verify that the Intermediate Value Theorem applies to the following function $g(x)$ over the interval $[1, 2]$, explain why the IVT guarantees a zero.

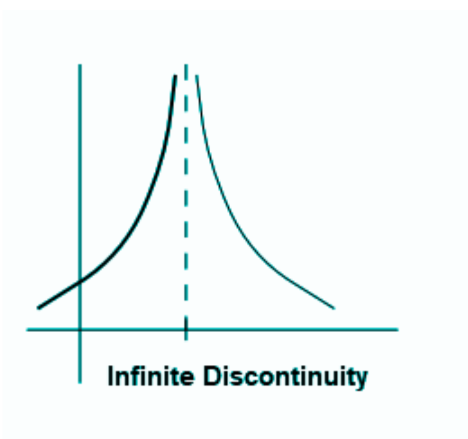
$$g(x) = x^3 + x^2 - x - 2$$

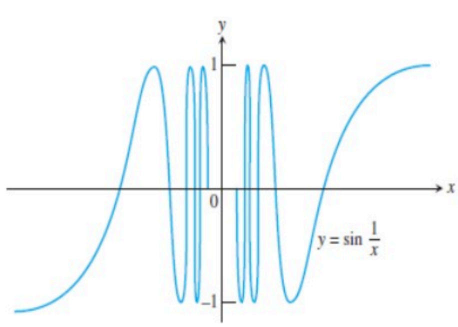
Types of Discontinuity

Although we will be mostly interested in functions that are continuous on its domain, we will now explore the different types of discontinuities we might encounter and also discuss their properties.









1.4 Squeeze's Theorem and Transcendental Functions

In section 1.2 we discussed the limit properties and found that limits could be evaluated by applying those techniques. Unfortunately, not all limits are able to be evaluated this way. For those remaining cases we may use **Squeeze Theorem**.

Theorem 6: Squeeze Theorem

If $f(x) \leq g(x) \leq h(x), \forall x \neq c$, in some interval about c, and

$$\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x),$$

then

$$\lim_{x \to c} g(x) = L.^{a}$$

[a] Also known as, Sandwich Theorem and Pinching Theorem

Example 24. Use the Squeeze Theorem to evaluate the following limits.

a) $\displaystyle\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right)$

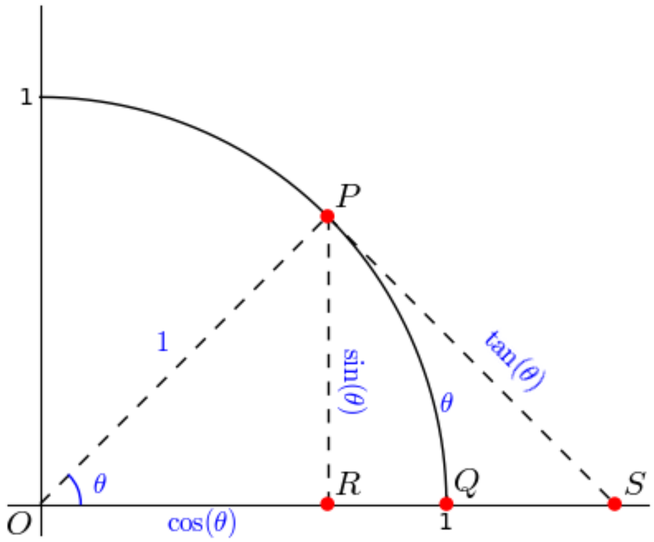
b) $\displaystyle\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

DIY 16. Use the Squeeze Theorem to evaluate the following limits.

a) If $5 - 3x - x^2 \leq g(x) \leq x + 9$, find $\displaystyle\lim_{x \to -2} g(x)$

b) $\displaystyle\lim_{x \to 0} x^4 \sin\left(\frac{7}{x}\right)$

Special Trig Limits



Example 25. Evaluate and prove your result.

a) [3] $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

b) $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x}$

The 2 function in Example 19 are the building blocks for more complex limits involving trigonometric functions.

[3] To ensure we are talking about an angle in **radians** it is traditionally written as $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta}$.

What about Exponentials and Logarithms?

Exponentials and logarithms are related to each other since they are both inverse functions of each other. From the graphs, we can see that these two functions contain no discontinuities, therefore we state that the exponential and logarithmic functions are both **continuous**.

Theorem 7: Continuity of Inverse Functions

If f is a one to one function on its domain, then its inverse function f^{-1} is also continuous on its domain.

Due to the above theorem we can now investigate the continuity of the inverse trigonometric functions.

1.5 Infinite Limits and Limits at Infinity

Through our exploration of limits we have seen that all of our theorems and results have assisted us in being able to describe the "behaviour" of a function around a *particular point* c. We will now extend these notions so that L and c can be replaced with ∞.

The first case allows us describe functions have grown without bounds at a point c.

The second case allows us to predict a functions end behaviour.

Infinite Limits(Vertical Asymptotes)

Example 26. Suppose $f(x) = \frac{1}{x}$. Using a graph, investigate $\lim_{x\to 0} \frac{1}{x}$.

DIY 17. Suppose $f(x) = \frac{1}{x^2}$. Using a graph, investigate $\lim_{x\to 0} \frac{1}{x^2}$.

The following theorem gives us a way to be able to accurately identify if a limit indeed goes to ∞.

Theorem 8: Infinite Limits

Let f be defined in an open interval about c except at c. If:

1. $\forall x \neq c, f(x)$ is positive
2. $\lim_{x\to c} \left[\frac{1}{f(x)}\right] = 0$

Then $\lim_{x\to c} f(x) = \infty$.

Similarly, if $f(x)$ is negative, and $\lim_{x\to c} \left[\frac{1}{f(x)}\right] = 0$, then $\lim_{x\to c} f(x) = -\infty$

Example 27. Show that $\lim_{x\to 1} \dfrac{1}{(x-1)^2} = \infty$

DIY 18. Show that $\lim_{x\to 3^+} \dfrac{2x+1}{x-3} = \infty$

DIY 19. Show that $\lim_{x\to 1^-} \dfrac{x^2}{\ln x} = \infty$

Limits at Infinity(Horizontal Asymptotes)

We are now interested in finding out what happens as x becomes really large (unbounded). Keep in mind that being unbounded does not just mean ∞ but rathe it also includes unusually large growth in the negative direction $-\infty$.

Before we continue, let's review some skills from Algebra 2.

PS : Finding Limits of Rational Functions - Algebra 2 Methods

Suppose

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$$

1. If $n = m$, then there is a Horizontal Asymptote at $y = \frac{a_n}{b_n}$.
2. If $n < m$, then there is a Horizontal Asymptote at $y = 0$.
3.
4. If $n = m + 1$, then we have an oblique asymptote.
5. If $n \geq m + 2$ then there are no H.A.

Now, using limits, we can re-write the above:

PS : Finding Limits of Rational Functions - Calculus Methods

Suppose

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0}$$

1. $\lim\limits_{x \to \infty} R(x) = \frac{a_n}{b_n}$ when $n = m$
2. $\lim\limits_{x \to \infty} R(x) = 0$ when $n < m$
3. $\lim\limits_{x \to \infty} R(x) = \pm\infty$ when $n > m$ [a]

(Polynomials) If $q(x) = 1$ and $p(x) > 0$ then $R(x) = p(x)$ is a polynomial, then

$$\lim_{x \to \infty} R(x) = \begin{cases} \infty & \text{if } n \text{ is even} \\ \infty & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \lim_{x \to -\infty} R(x) = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$$

[a] we will not be investigating oblique asymptotes in this course

Example 28. a) Suppose $f(x) = \frac{1}{x}$. Using a graph, investigate $\lim\limits_{x \to \infty} \frac{1}{x}$ and $\lim\limits_{x \to -\infty} \frac{1}{x}$.

DIY 20. a) Suppose $f(x) = \frac{1}{x^2}$. Using a graph, investigate $\lim_{x\to\infty} \frac{1}{x^2}$ and $\lim_{x\to-\infty} \frac{1}{x^2}$.

Corollary 2. *If $p > 0$ is a rational number, then*

$$\lim_{x\to\infty} \frac{1}{x^p} = 0 \textit{ and } \lim_{x\to-\infty} \frac{1}{x^p} = 0$$

when $x^p \neq 0$.

Example 29. Evaluate the following limits.

a) $\lim_{x\to\infty} (3x^3 - x^2 + 5x + 1)$

b) $\lim_{x\to\infty} (2x^4 - x^2 + 2x + 8)$

c) $\lim_{x\to\infty} \frac{3x^2 - 2x + 8}{x^2 + 1}$

d) $\lim_{x\to\infty} \frac{4x^5 - 3x^2 + 3}{6x^5 + 100x^2 - 10}$

e) $\lim_{x\to-\infty} \frac{3x^3 - 2x^2 + 7}{6x^4 + x^3 - 2x + 100}$

f) $\lim_{x\to\infty} \frac{4x^4 - 3x^3}{250x^2 + 1000}$

Up to this point we have used Algebra 2 techniques to reach these conclusions. At times we will be asked to use some algebra to find evaluate these limits.
Example 30. Evaluate the following limits.

a) $\lim\limits_{x\to\infty} \dfrac{x^2}{1-x^2}$

b) $\lim\limits_{x\to-\infty} \dfrac{3x^3-2x^2+7}{6x^4+x^3-2x+100}$

DIY 21. Evaluate the following limits.

1. $\lim\limits_{x\to\infty} \dfrac{8x+2}{3x-1}$

2. $\lim\limits_{x\to\infty} \dfrac{4x^4-3x^3}{250x^2+1000}$

Example 31. Evaluate the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{16x^4-3}}{2x^2+3}$

b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2-3x}}{4x+5}$

DIY 22. Evaluate the following limits.

a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{5x^2+2x}}{x}$

b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{5x^2+2x}}{x}$

c) $\displaystyle\lim_{x\to-\infty} (x+\sqrt{x^2+ax})$

Example 32. Evaluate the following limits:

a) $\lim_{x \to -\infty} \log_{10} \left(\dfrac{x^6 - 500}{x^6 + 500} \right)$

b) $\lim_{x \to -\infty} \cos \left(\dfrac{x}{x^2 + 10} + \dfrac{\pi}{3} \right)$

DIY 23. Evaluate the following limits:

a) $\lim_{x \to -\infty} \dfrac{e^x}{4 + 5e^{3x}}$

b) $\lim_{x \to \infty} \dfrac{5^x}{3^x + 2^x}$

Chapter 2

The Derivative from First Principles

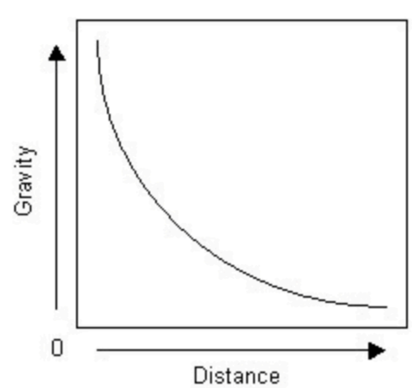
2.1 Rate of Change and the Derivative

In Chapter 1 we discussed how the underpinning of the tangent problem required us to introduce the concept of the limit. Here we will illustrate the discovery of the derivative and general formula for the derivative from **First Principles**. In order to do so, we will allow the story to be told from the perspective of Isaac Newton.

Newton one day, while sitting under an apple tree, noticed that an apple descended and before he was able to flinch, he was struck right on the head. After a few moments of laughter and frustration, Newton then had a strange thought: "Gravity, gravity is responsible for making the apple fall, BUT WHY DOES IT NOT AFFECT THE MOON WITH ITS IMMENSE SIZE?!"

As Newton began to take empirical data, he began to notice that the effect of gravity began to behave with a behavior with respect to distance (in this case square of the distance).

The graph below illustrates this phenomena:



DIY 24. Explain the relationship between distance and gravity in this graph. What happens if the distance increases? What happens if the distance decreases? How does this explain the difference between the apple and the moon question Newton had?

After a close inspection, Newton noticed that the graph was steeper at first and not so steep the farther right we go. Newton started to ask "how does gravity *change* over distance?" The way this question is posed leads us to use our concept of the Average Rate of Change **(AROC)**.

As we know, finding the AROC of a linear relationship leads us to simply finding the slope of the graph. We will use the concept of AROC to lead us to Newton's approach to this empirical problem.

Definition 2.1: Average Rate of Change (AROC)

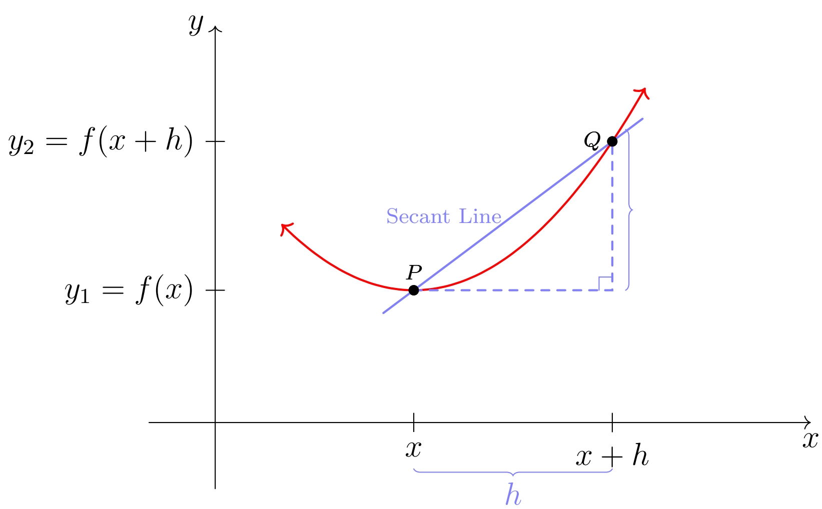
If x_1 and x_2, $x_1 \neq x_2$, are in the domain of a function $y = f(x)$, the **average rate of change of** f from x_1 to x_2 is defined as

$$\text{Average Rate of Change} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

The units on a rate of change are "output units per input units"

Geometrically: The average rate of change is the slope of the **Secant Line** containing the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

From the above definition, we can extract the following graphic:



Now, if we were to take points Q_i that get closer and closer to P we notice that the distance h becomes smaller and smaller, but in turn, our secant line becomes more and more similar to its tangent line. With the language of calculus, we are now lead to the following result:

Definition 2.2: Instantaneous Rate of Change (IROC)

The **instantaneous rate of change** of a function f at the input value $x = c$,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided that the limit exists.

The units on a rate of change are "output units per input units"

Geometrically: The instantaneous rate of change is the slope of the **Tangent Line** containing the point $(c, f(c))$.

Although the above definition is useful, we will now attempt to generalize this concept for all points x on the graph of f. We will return to the geometric discussion, but for now we will venture into the algebraic approach. This leads us to the following definition:

The Derivative from an Algebraic Perspective

Definition 2.3: The Limit Definition of the Derivative

The **derivative** of a function f with respect to the variable x is the function f', read as "$f - prime$", whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

The units on a rate of change are "output units per input units"

If the derivative exists along an interval (a, b) we say that the function is ***differentiable*** on (a, b).

The following observations can also be made:

- The derivative f' is a function that gives us the slope of a function at **ANY** point.
- The expression $\dfrac{f(x+h) - f(x)}{h}$ is called the difference quotient.
- Our goal will be to simplify the difference quotient until the h in the denominator has been factored out and then we will evaluate the limit as $h \to 0$.
- The derivative is defined on an open interval.
- We have NOT imposed any conditions on f.

We'll usually find the derivative as a function of x and then plug in $x = c$. This allows us to quickly find the value of the derivative of multiple values of f without having to evaluate a limit for each.

Example 33. Use the definition of the derivative to find $f'(x)$.

a) $f(x) = 3x + 2$

b) $f(x) = x^3 + x^2$

c) $f(x) = \dfrac{1}{x+1}$

d) $f(x) = \sqrt{4x+1}$

DIY 25. Use the definition of the derivative to find $f'(x)$.

a) $f(x) = x^2 - 4x$

b) $f(x) = \dfrac{1}{x^2}$

The above problems have shown us how to evaluate a few derivatives using the standard definition to the derivative, but if we change our persepective, we are able to attain an **Alternate Definition of the Derivative**.

Definition 2.4: The Alternate Definition of the Derivative

An alternate definition of the derivative of f at a given point c is:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

provided that the limit exists.

The following observations can also be made:

- The derivative f' is a function that gives us the slope of a function at **ANY** point c.
- The expression $\dfrac{f(x) - f(c)}{x - c}$ is called the AROC.
- Our goal will be to simplify the expression until the $x - c$ in the denominator has been factored out and then we will evaluate the limit as $x \to c$.
- We have NOT imposed any conditions on f.

Example 34. Use the alternate definition of the derivative to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$ for any $c \in \mathbb{R}$.

Example 35. Find the rate of change of the function $f(x) = x^2$ at:

a) $c = 2$

b) Any $c \in \mathbb{R}$

DIY 26. Find the rate of change of the function $f(x) = x^2 - 5x$ at:

a) $c = 2$

b) Any $c \in \mathbb{R}$

DIY 27. In a metabolic experiment, the mass M of glucose decreases according to the function

$$M(t) = 4.5 - 0.03t^2$$

where M is measured in grams (g) and t is measured in hours (h).

a) Find the reaction rate $M'(t)$, measure in g/h units, at $t = 1$.

b) Interpret your solution in part (a).

The Derivative from an Geometric Perspective

Alright, now that we have a few examples down, we are ready to find the equation of the tangent line. We begin with the definition of the average rate of change and show how similar construction lead to the results we desire.

Definition 2.5: Equation of Secant Line

The equation of a secant line between $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by :

$$y - f(x_1) = \mathbf{m}(x - x_1) \text{ OR } y - f(x_2) = \mathbf{m}(x - x_2)$$

where m = AROC

From this, we know we are able to approximate the slope of the tangent line by taking the limit of secant lines, thus we have the following theorem.

Theorem 9: Equation of a Tangent Line

If $f'(c)$ exists, then the equation of the tangent line to the graph at the point $(c, f(c))$ is

$$y - f(c) = f'(c)(x - c)$$

If $f'(c) = 0$, the tangent line is horizontal and the equation of the tangent line is $y = f(c)$.

The line perpendicular to the tangent at a point P on the graph of f is called the **normal line** to the graph of f at the point P. From Algebra 1 we know a relationship between the equation of a line and its perpendicular, so we will now take advantage of it keeping in mind that we now call it the **"normal"**.

Theorem 10: Equation of the Normal Line

If $f'(c) \neq 0$ exists, then the equation of the normal line to the graph at the point $(c, f(c))$ is

$$y - f(c) = \frac{1}{f'(c)}(x - c).$$

If $f'(c) = 0$, the tangent line is horizontal and the normal line is vertical, and the equation of the normal is $x = c$.

Example 36. Let $f(x) = \dfrac{1}{x+1}$.

a) Find the $f'(x)$.

b) Find the slope of the curve at $x = 2$.

c) Write the equation of the tangent line to the curve at $x = 2$.

d) Write the equation of the normal line to the curve at $x = 2$.

DIY 28. Let $f(x) = x^2$.

a) Find the $f'(x)$.

b) Find the slope of the curve at $x = -2$.

c) Write the equation of the tangent line to the curve at $x = -2$.

d) Write the equation of the normal line to the curve at $x = -2$.

Example 37. Let $f(x) = \sqrt{4 - x^2}$.

a) Find the $f'(x)$.

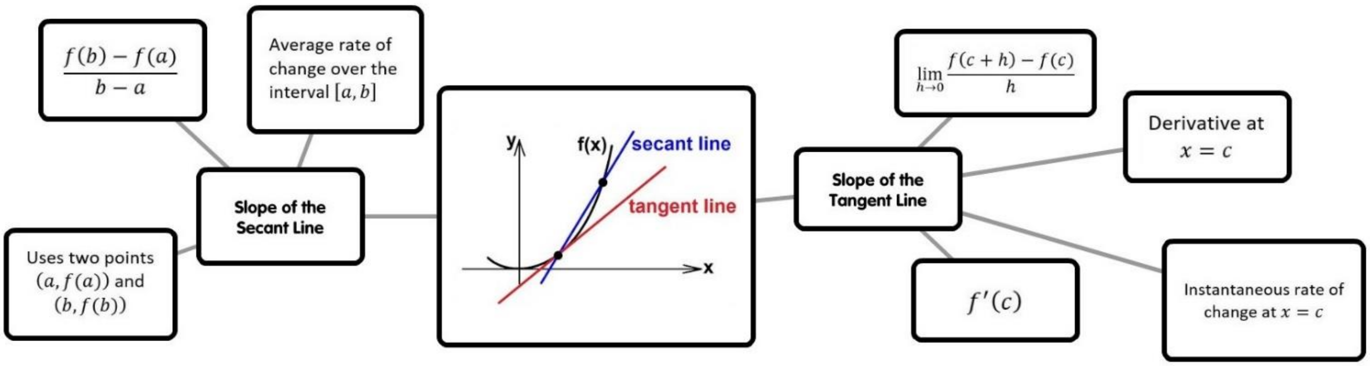
b) Find the slope of the curve at $x = -2$.

c) Write the equation of the tangent line to the curve at $x = -2$.

d) Write the equation of the normal line to the curve at $x = -2$.

With this we summarize the two interpretations we have discussed:

- *Geometric Interpretation:* If $y = f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph at the point $(c, f(c))$.
- *Rate of change of a function Interpretation:* If $y = f(x)$, the derivative $f'(c)$ is the rate of change of f at c.



The above graphic formally lays out the differences between secant lines and tangent lines.

We conclude this section with a discussion on approximating the derivative if we are only given a data set.

Approximating the Derivative using a Table

Example 38. The table below lists several values of a function $y = f(x)$ that is continuous on the interval $[-1, 5]$ and has a derivative at each number in the interval $(-1, 5)$. Approximate the derivative of f at 2.

x	0	1	2	3	5
$f(x)$	0	3	12	33	72

DIY 29. Using the above example, approximate the derivative of f at 4.

DIY 30. For $-2 \leq x \leq 10$, the function f and f' are continuous. Selected values of $f'(x)$ are given in the table below.

x	-2	0	2	3	4
$f'(x)$	-8	-2	2	6	8

a) Approximate $f''(-1)$.

b) Is there a time c, $-2 \leq c \leq 4$ at which $f'(c)$ equals this approximation?

2.2 Differentiability

The focus of the previous section was on the evaluation and meaning of the derivative. Now we will aim to determine when a function fails to have a derivative. We begin by first being a bit more explicit in our definition of the derivative and then by exploring 3 particular examples.

Theorem 11: Differentiability at a point x=c

A function f is differentiable at c if and only if:

$$\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = L = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c},$$

provided that $L \in \mathbb{R}$.

The above theorem explicitly tells us that the derivative from the left must equal the derivative from the right, but implicitly it also states that differentiability implies continuity.

Theorem 12: Differentiability implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Example 39. Prove that Differentiability implies Continuity.

DIY 31. If you are given that f is differentiable at $x = 2$, then explain why each statement below is true.

a) $\lim_{x\to 2} f(x)$ exists.

b) $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$ exists.

c) $f(2)$ exists.

d) $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$ exists.

Although the above might seem clear and straight forward, we must consider why the opposite case is not true (i.e., why does continuity NOT imply differentiability?).

Example 40. Let $f(x) = |x-3|$ and answer the following.

a) Graph f.

b) What is $f'(x)$ as $x \to 3^-$?

c) What is $f'(x)$ as $x \to 3^+$?

d) Is f continuous at $x = 3$?

e) Is f differentiable at $x = 3$?

DIY 32. Let $f(x) = x^{\frac{2}{3}}$ and answer the following.

a) Graph f.

b) Describe the derivative of f as x approaches 0 from the left and the right.

c) Suppose you found $f'(x) = \dfrac{2}{3\sqrt[3]{x}}$. Using this equation, what is the value of the derivative when $x = 0$.

DIY 33. Let $f(x) = \sqrt[3]{x}$ and answer the following.

a) Graph f.

b) Describe the derivative of f as x approaches 0 from the left and the right.

c) Suppose you found $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$. Using this equation, what is the value of the derivative when $x = 0$.

The last three examples show us three cases where the function f is **continuous**, but the function is not differentiable everywhere.

- The first graph had a **corner**, or a sharp turn, and the derivative from the left and right hand side failed to agree.
- The second graph had a **cusp** where the slope approached positive infinity from one side and negative infinity from the other.
- The third graph had a **vertical tangent line**, where the slopes approach positive (or negative) infinity from both sides.

This then leads us to conclude the following:
Corollary 3. *Functions are not differentiable at:*

- *corners,*
- *cusps,*
- *points where the function is discontinuous,*
- *or at points where there is a vertical tangent line.*

DIY 34. If f is a function such that $\lim_{x\to -3} \dfrac{f(x)-f(-3)}{x+3} = 2$, which of the following must be true?

A) The limit of $f(x)$ as x approaches -3 does not exist.

B) f is not defined at $x = -3$.

C) The derivative of f at $x = -3$ is 2.

D) f is continuous at $x = 2$

E) $f(-3) = 2$

We conclude this section with a clever piece-wise problem. Let us recall that in Chapter 1, we were interested in a piecewise function was continuous.

Example 41. Determine whether f is differentiable at $x = 1$.

a) $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

b) $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$

DIY 35. Determine whether f is differentiable.

a) $f(x) = \begin{cases} 2x+2 & \text{if } x < 3 \\ 5 & \text{if } x = 3 \\ x^2-1 & \text{if } x > 3 \end{cases}$

b) $f(x) = \begin{cases} -2x^2 + 4 & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \geq 1 \end{cases}$ at $x = 1$.

Notation, Equivalent Symbols, and the WHY behind it all

If we recall our AROC discussion, you might recall that we wrote

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(c)}{x - c}$$

and used this expression to develop the derivative. Now, notice the LHS expression. If we let

$$\Delta x = x - c, \text{ then as } x \to c, \Delta x \to 0.$$

This then tells us that we can re-write our limit definition of the derivative as follows:

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}$$

where df and dx are meant to symbolize VERY VERY VERY VERY SMALL differences. Now, this makes sense as we should expect that a smaller Δx would lead to a smaller Δf.

Definition 2.6: Differentials

The differentials are interpreted as infinitesimals. There are several methods of defining infinitesimals rigorously, but it is sufficient to say that an infinitesimal number is smaller in absolute value than any positive real number, just as an infinitely large number is larger than any real number.

This leads us to the following common notation. We will be using most of these in different contexts so learn them:

2.3 Derivative of Transcendentals

Derivative of Exponentials

None of the differentiation rules developed so far allow us to find the derivative of an exponential function $f(x) = a^x$. Let us use the limit definition of the derivative and see what we find:

Definition 2.7: Derivative of Exponentials: Limit Definition

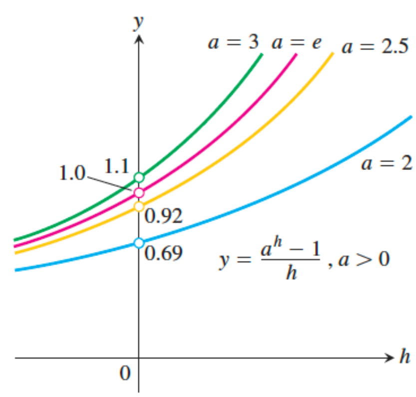
Suppose $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The derivative is given by:

$$f'(x) = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}.$$

provided that $\lim_{h \to 0} \frac{a^h - 1}{h}$ exists.

Notice:

- $f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$.
- $f'(x)$ is a multiple of $f(x)$. (i.e. ROC is proportional to itself)
- $f'(x) = a^x \cdot f'(0)$.



Notice that in the above definition, the value of

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} = L$$

depends on the base a.

Now, inspecting the graphs we have on the right, we can see that when $2.5 < a < 3$, the value of L becomes closer and closer to 1. Specifically, L is going to indeed equal 1 when $a = e \approx 2.718281828...$. That is:

Theorem 13: Derivative of the Exponential Function $y = e^x$

If $f(x) = e^x$, then $f'(0) = 1$ so that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

and moreover,

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(e^x) = e^x.$$

Example 42. Determine the derivative of the following:

a) $y = \dfrac{5e^x}{3e^x + 1}$

b) $y = (x^2 - 6x + 3)e^x$

DIY 36. Determine the derivative of the following:

a) $y = \dfrac{1 - e^x}{3e^x + 1}$

b) $y = x^3 e^x - 2e^x$

Derivative of Trigonometric Functions

In the previous section we were presented with a few useful rules for the sum, difference, product, and quotient of two functions (given that they satisfy the conditions to exist). From our discussions from Precalculus we know that there is still one function operation which we have not discussed, the composition of functions. We will defer that for the next section, for now we will explore the derivative of our Trigonometric Functions (a.k.a Circular Functions).

Using the derivatives of $\sin x$ and $\cos x$, you can find the derivatives of the other four trig functions. Let's start by finding the derivative of the sine function using the limit definition of derivative. We will need the following two results established in Chapter 1 for this section:

$$\lim_{x\to 0}\frac{\sin x}{x} = \underline{\hspace{3cm}} \qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = \underline{\hspace{3cm}}$$

Example 43. If $f(x) = \sin x$, find $f'(x)$.

DIY 37. If $f(x) = \cos x$, find $f'(x)$.

Example 44. Prove: If $f(x) = \tan x$, then $f'(x) = \sec^2 x$.

DIY 38. Prove: If $f(x) = \cot x$, then $f'(x) = -\csc^2 x$.

Example 45. Prove: If $f(x) = \sec x$, then $f'(x) = \sec x \cdot \tan x$.

DIY 39. Prove: If $f(x) = \csc x$, then $f'(x) = -\csc x \cdot \cot x$.

Example 46. Find the derivative of each function:

a) $f(x) = x^2 \sin x$

b) $g(x) = \dfrac{\cos x}{x}$

c) $h(x) = \dfrac{e^x}{\sin x}$

d) $p(x) = \sqrt{x} + \sec x$

DIY 40. Find the derivative of each function:

a) $f(x) = x^2 \cos x$

b) $g(x) = \dfrac{\sin x}{x^2}$

c) $h(x) = \dfrac{e^x}{\cos x}$

d) Find all the tangent lines that are parallel to the x-axis of $p(x) = x + \sin x$.

e) Find $f''\left(\dfrac{\pi}{4}\right)$ when $f(x) = \sec x$.

Chapter 3

Differentiation

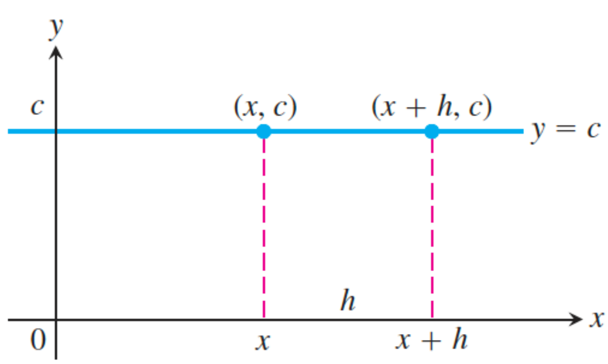
3.1 Rules of Differentiation

In chapter 2 we discussed the concept of the derivative and its intrinsic properties(ROC and Tangent line). Although useful, the limit definitions (either one) of the derivative are rather cumbersome to deal with. In this chapter we will explore all the different rules and techniques mathematicians have devised and formalized over the years. Keep in mind that underneath all of these rules, properties, and principles, the concepts of the limit and continuity are being used. We start with a familiar rule: the rule of derivative of every constant function is zero.

Theorem 14: Derivative of a Constant

If f has the constant value $f(x) = A$, then

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(A) = 0.$$



Example 47. Differentiate the following:

a) $f(x) = 5$

b) $f(x) = \pi$

c) $f(x) = \sqrt{3}$

d) $f(x) = \ln 3 + e^2$

The following result is useful when dealing with algebraic functions.

Theorem 15: Power Rule

If n is any real number, and $f(x) = x^n$, then

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}(x^n) = n \cdot x^{n-1}.$$

for all x where x^n and x^{n-1} are defined.

Example 48. Differentiate the following:

a) $f(x) = x^5$

b) $g(x) = x^{10}$

c) $h(x) = x^{2/3}$

d) $m(x) = x^{\sqrt{2}}$

e) $d(x) = \dfrac{1}{x^4}$

DIY 41. Differentiate the following:

a) $f(x) = x^4$

b) $g(x) = 10x$

c) $h(x) = \sqrt[3]{x^2}$

d) $m(x) = \dfrac{1}{x^{-4/3}}$

The following two rules are helpful when dealing with a function with various terms.

Theorem 16: Derivative of Constant Multiple

If f is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cf(x)) = c \cdot \frac{df}{dx}.$$

DIY 42. Differentiate the following:

a) $f(x) = 5x^2$

b) $h(x) = -\dfrac{1}{2}x^2$

c) $m(x) = \pi^4 x^3$

Theorem 17: Sum Rule of the Derivative

If f and g are differentiable function of x, then their sum $(f + g)$ is differentiable at every point where f and g are both differentiable. At such points,

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}.$$

These two rules encapsulate the **linearity property of the derivative**.

Theorem 18: Linearity of the Derivative

For any functions f and g, if

$$h(x) = c \cdot f(x) + g(x),$$

then

$$h'(x) = c \cdot f'(x) + g'(x),$$

where $c \in \mathbb{R}$.

DIY 43. Differentiate the following:

a) $f(x) = 3x^2 + 8$

b) $g(x) = x^3 + 4x^2 - 2x + 7$

c) $h(x) = \dfrac{3}{(-2x)^4} - \dfrac{x}{2} + \dfrac{1}{4}$

d) $T(x) = \sqrt{x} + 9\sqrt[3]{x^7} - \dfrac{2}{\sqrt[5]{x^2}}$

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives and the derivative of the quotient of two functions is not the quotient of their derivatives.

Theorem 19: Product Rule

If f and g are differentiable functions and if $F(x) = f(x)g(x)$, then F is differentiable, and the derivative is

$$F'(x) = \frac{dF}{dx} = [f(x)g(x)]' = \underbrace{f(x)g'(x)}_{\text{first dee second}} + \underbrace{f'(x)g(x)}_{\text{second dee first}}$$

Example 49. Differentiate the following:

a) $y = (3 + 2\sqrt{x})(5x^3 - 7)$

b) $h(x) = (3x - 2x^2)(4 + 5x)$

DIY 44. Find the derivative of $y = (x^2 + 1)(x^3 + 3)$ by:

a) using the product rule.

b) multiplying out the factors and use the power rule.

Theorem 20: Quotient Rule

If f and g are differentiable functions and if $F(x) = \dfrac{f(x)}{g(x)}$, then F is differentiable, and the derivative is

$$F'(x) = \frac{dF}{dx} = \left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Note: *low dee high minus high dee low all over low squared*

Example 50. Differentiate the following:

a) $y = \dfrac{x^2+1}{2x-3}$

b) $m(x) = \dfrac{5x^2}{x^2+1}$

DIY 45. Differentiate the following:

a) $y = \dfrac{x}{x^2+1}$

b) $h(t) = \dfrac{t^2-1}{t^3+1}$

The Reasoning behind the Product and Quotient Rule

These two rules were given above without proof. The proofs of these concepts are a bit long and do not really illuminate what is really going on **geometrically**. For this, I will resort to **3Blue1Brown** to explain this using some nice visuals:

- Explanation of Derivative of $y = x^2$.:

  Derivative formulas through geometry — Essence of calculus, chapter 3 (up to 8:00)

- Explanation of Sum Rule and Product Rule

  Visualizing the chain rule and product rule — Essence of calculus, chapter 4 (up to 9:05)

Second and Higher Order Derivatives

The first derivative of y with respect to x is denoted $y', f'(x)$, or $\frac{dy}{dx}$. The second derivative with respect to x is denoted $y'', f''(x)$, or $\frac{d^2y}{dx^2}$. The second derivative is an example of a higher order derivative. We can continue to take derivatives (as long as they exist) using the following notation:

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
Nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

DIY 46. Find the indicated derivative of each of the following.

a) $\frac{d^4}{dx^4}[-5x^6 + 2x^5 - 9x^3 + 32x - 1]$

b) $\frac{d^2}{dx^2}\left[\frac{x}{x-1}\right]$

Derivative Rules in Various Forms

Example 51. If $f(x) = 4x^3 - 12x^2 + 2$,

a) Find the points on the graph of f, where f has a horizontal tangent line.

b) Where is $f'(x) > 0$? Where is $f'(x) < 0$?

DIY 47. If $f(x) = \dfrac{e^x}{x^2+1}$,

a) Find the points on the graph of f, where f has a horizontal tangent line.

b) Where is $f'(x) > 0$? Where is $f'(x) < 0$?

DIY 48. For each of the following, write an expression for $f'(x)$ and find $f'(2)$ given the information below:

$$g(2) = 3 \qquad g'(2) = -2,$$
$$h(2) = -1 \qquad h'(2) = 4,$$

a) $f(x) = 2g(x) + h(x)$

b) $f(x) = g(x)h(x)$

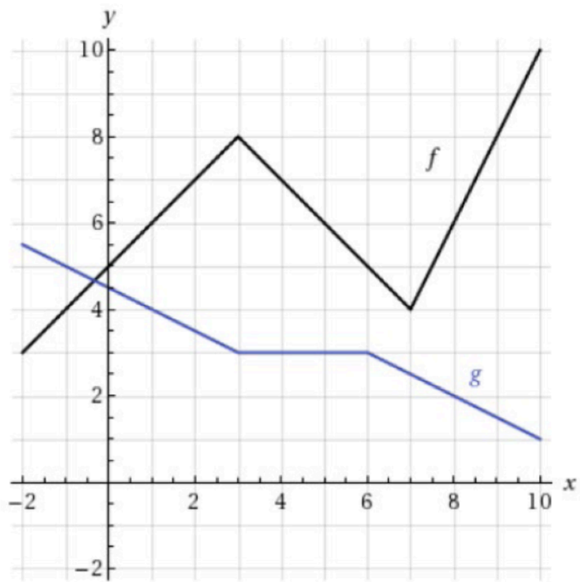
c) $f(x) = 4 - h(x)$

d) $f(x) = \dfrac{g(x)}{h(x)}$

DIY 49. Find the values of a and b so that $g(x)$ is both continuous and differentiable at $x = 1$.

$$g(x) = \begin{cases} x^2 + 2 & \text{if } x \leq 1 \\ a\left(x - \dfrac{1}{x}\right) + b & \text{if } x > 1 \end{cases}$$

Example 52. If $p(x) = f(x)g(x)$ and $q(x) = \dfrac{f(x)}{g(x)}$, use the graphs of f and g below to answer the following.



We conclude this section with a discussion of limit expressions that are actually derivatives in disguise.

DIY 50. Evaluate the following limits.

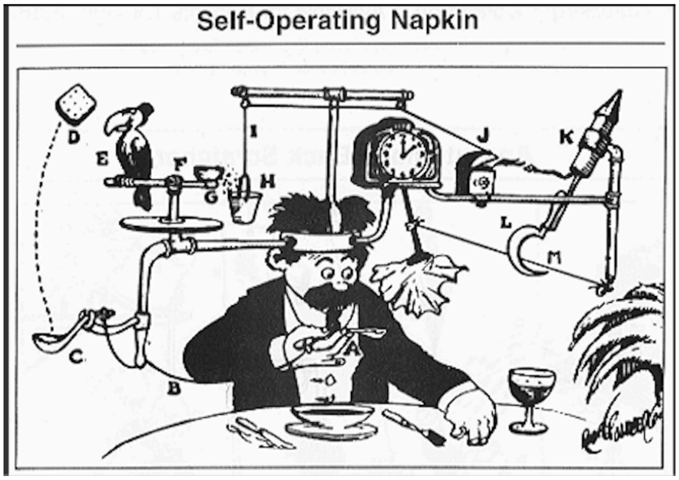
a) $\displaystyle\lim_{h\to 0} \frac{8(3+h)^3 - 8(3)^3}{h}$

b) $\displaystyle\lim_{h\to 0} \frac{\frac{5}{(2+h)^2} - \frac{5}{(2)^2}}{h}$

c) $\displaystyle\lim_{h\to 0} \frac{7e^4 - 7e^{4+h}}{h}$

3.2 The Chain Rule: Derivatives of Composition

The Chain Rule actually gets its name because it is a similar chain reaction whereby one action triggers another, which triggers another, which triggers another. For the sake of an analogy, a Rube Goldberg is the perfect example to keep in mind.



In the image above, the motion of the spoon at point A travels, eventually moving the napkin at N. If we wanted to know the change of N with respect to A, we would need to start by finding the rate of change of B, the next stage in the chain after A, and work our way all the way through to N.

For example purposes we are going to assume our napkin will gently move at a point E.

The Chain Rule is our weapon for deriving composite functions, or functions within other functions. Here are some examples of the types of functions the chain rule will allow us to differentiate.

Do Not Need Chain Rule	**Need Chain Rule**
$y = x^2 - 1$	$y = \sqrt{x^2 - 1}$
$y = \sin x$	$y = \sin 5x$
$y = 3x - 2$	$y = (x^2 - 1)^7$
$y = x - \tan x$	$y = x - \tan(x^2)$

Theorem 21: The Chain Rule

If a function g is differentiable at x and a function f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

For a differentiable function $y = f(u)$ and $u = g(x)$, the Chain rule takes the form

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

DO NOT TREAT THESE AS FRACTIONS ALTHOUGH THE INTUITION IS TRUE.

Before evaluate, lets identify the inside function and the outside functions.

Example 53. Identify the inside function and the outside function in each of the following composite functions.

a) $f(x) = (x^3 - 4x + 1)^{100}$

b) $y = \cos(5x^2)$

c) $f(x) = \sin^2(x)$

d) $y = \dfrac{1}{5x+1}$

e) $f(x) = \cos\left(3x - \dfrac{\pi}{4}\right)$

Example 54. Find the derivative of each of the following functions.

a) $y = \cos(5x^2)$

b) $f(x) = \sin^2(x)$

c) $y = \dfrac{1}{5x+1}$

d) $f(x) = e^{x^2-4}$

DIY 51. Find the derivative of each of the following functions.

a) $f(x) = (x^3 - 4x + 1)^{100}$

b) $f(x) = \cos\left(3x - \frac{\pi}{4}\right)$

c) $y = \sqrt{4x + 1}$

d) $f(x) = \cos(4e^x)$

Example 55. Find the derivative of the following function.

a) $y = \sin^3 \sqrt{4x+1}$

DIY 52. Find the derivative of the following function.

a) $y = \dfrac{3}{\sqrt{\tan(4x^2+1)}}$

Chain Rule with Multiple Functions

Example 56. Find the derivative of each of the following functions.

a) $y = (x^2+1)\sqrt{2x-3}$

b) $f(x) = \left(\dfrac{3x+2}{4x^2-5}\right)^5$

Example 57. For each of the following, use the following to find $f'(5)$ that

$$g(5) = -3, g'(5) = 6, h(5) = 3, \text{and } h'(5) = -2$$

, if possible. If it is not possible, state what additional information is needed to find the value.

1. $f(x) = g(x)h(x)$

2. $f(x) = g(h(x))$

3. $f(x) = \dfrac{g(x)}{h(x)}$

4. $f(x) = [g(x)]^3$

DIY 53. Suppose $h = f(g(x))$. Find $h'(1)$ provided that,

$$f(1) = 2, f'(-1) = 1, f'(2) = -4, g'(1) = -3, \text{and } g'(2) = 5.$$

Now that we are equipped with the chain rule we can find the general form of the derivative of an exponential:

Definition 3.1: Derivative of $y = a^x$

Suppose $f(x) = a^x$, where $a > 0$ and $a \neq 1$. The derivative is given by:

$$f'(x) = a^x \ln a.$$

DIY 54. Find the derivative of the following functions.

a) $y = 2^x$

b) $f(x) = 3^{-x}$

AB Calculus: Implicit Differentiation **Name:** ______________________________

Equations that are solved for y are called **explicit** functions, whereas equations that are not solved for y are called **implicit** functions. For instance, the equation $x + 2y - 3 = 0$ implies that y is a function of x, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using explicit functions of y expressed in the form $y = f(x)$ such as

$$y = \frac{x+1}{x+2} \qquad \text{or} \qquad y = \sin x$$

If we have an equation that involves both x and y in which y has not been solved for x, then we say the equation defines y as an implicit function of x. In this case, we may (or may not) be able to solve for y in terms of x to obtain an explicit function (or possibly several functions).

Example 1 Find the derivative of the following expressions

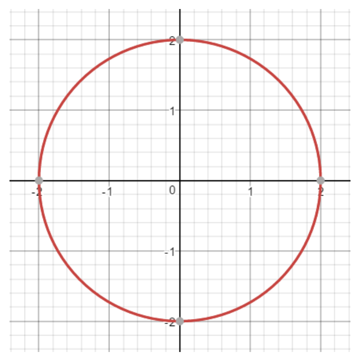
a) Find $\frac{dy}{dx}$ of x^3

b) Find $\frac{dy}{dx}$ of y^3

c) Find $\frac{dy}{dx}$ of xy

d) Find $\frac{dy}{dx}$ of $x^3 + y^2 - 3xy$

Example 2 The graph to the right is a circle with the equation $x^2 + y^2 = 4$.



a) Solve the equation for y and find the derivative of the resulting function(s)

b) Find the derivative by differentiating implicitly

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

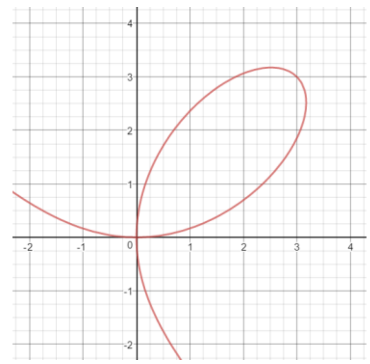
If we have y written as an explicit function of x, $y = f(x)$, then we can find the derivative using the rules we have previously learned. For an equation which defines y as an implicit function of x, we can compute the derivative without solving for y in terms of x with the following procedure. The key to this entire procedure is to remember that even though you did not (or cannot) write y as a function of x, y is implicitly defined as a function of x.

Steps for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x (y is a function of x, so use chain rule).
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation if there is more than one $\frac{dy}{dx}$ term.
4. Solve for $\frac{dy}{dx}$ (It is okay to have both x's and y's in your answer).

Note: To find $\frac{dy}{dx}$ at a given point, take the derivative of both sides then immediately plug in the point and simplify.

Example 4 Given the curve $x^3 + y^3 = 6xy$,



a) Find $\frac{dy}{dx}$

b) Find the equation of the tangent line and normal line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Example 5 Find $\frac{dy}{dx}$ at the point $(0, 0)$ of the function $\tan(x + y) = x$.

Example 6 Find $\frac{d^2y}{dx^2}$ of $2x^3 - 3y^2 = 8$

AB Calculus: Derivatives of Inverse Functions **Name: ______________________________**

Example 1: Suppose $y = \sin^{-1} x$. Find $\frac{dy}{dx}$ using implicit differentiation.

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

Notes:

- Domains are restricted to make them functions so do not worry about $\sin^{-1} x$ versus $\mathrm{Sin}^{-1} x$.
- $\sin^{-1} x$ and $\arcsin x$ are the same thing. Both refer to the inverse sine function.

Example 2: Find $\frac{d}{dt}[\sin^{-1}(t^2)]$

Example 3: Find $\frac{d}{dx}\left[\tan^{-1}(\sqrt{x-1})\right]$

Example 4: Find $\frac{d}{dx}[x\sec^{-1}(3x)]$

Derivatives of Other Inverse Functions

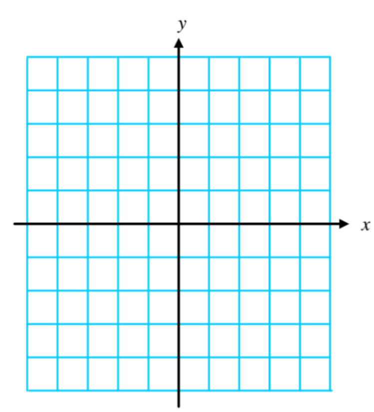
Example 5 Graph the line $y = 2x + 1$

a) What is the slope of the line?

b) Find the inverse of the function and graph it.

c) What is the slope of the inverse function

d) If $(2, 5)$ is on the original line, what point does it correspond to on the inverse function?



The slope of the line at $(2, 5)$ on the original function is the ____________ of the slope of the inverse. The difference is that the inverse is calculated using the point ____________ instead of $(2, 5)$.

There are two methods you can use to find the derivative of an inverse function. Method Number 1:

Steps for finding derivatives of Inverse Functions Using Implicit Differentiation

1. Find the point if you are only given one coordinate by substituting into the original function. Remember, an x-value on an inverse is the y-value on the original
2. Find the point on the inverse function
3. Interchange x and y in the equation and derive implicitly
4. Substitute the inverse point into the equation and solve for $\frac{dy}{dx}$.

The other method is to use the property that the derivative of an inverse function at (p, q) is going to be the reciprocal of the derivative of the original function at (q, p).

Derivative of the Inverse Function at (p, q)

$$(f^{-1})'(p) = \frac{1}{f'(q)}$$

The derivative of $f^{-1}(x)$ at the point (p, q) is the reciprocal of the derivative of $f(x)$ at the point (q, p).

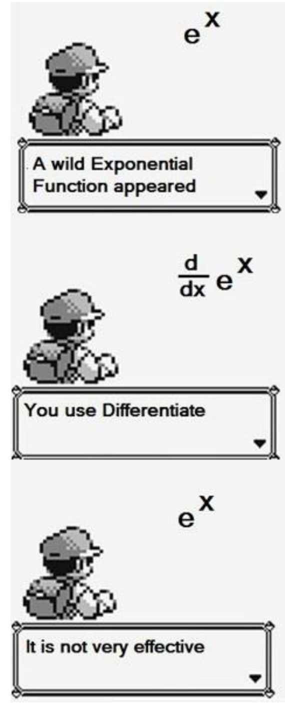
Example 6 Let $f(x) = x^5 + 2x - 1$. Find of $(f^{-1})'(-1)$ at the point $(0, -1)$ using both methods.

Example 7 Let $f(x) = x^3 + 2x - 1$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=2}$ using both methods. You can use your calculator to help you find the missing coordinate.

AB Calculus: Derivatives of Exp. and Log Functions **Name:** ____________________

Derivative of e^x

$$\frac{d}{dx}[e^x] = e^x$$



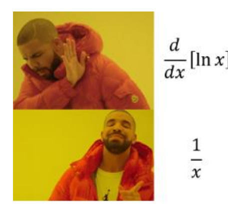
Example 1 $\frac{d}{dx}[e^{2x-1}]$

Example 2 $\frac{d}{dx}\left[e^{\frac{3}{x}}\right]$

Example 3 Find y' if $y^2 + e^y = 2x^2$

Derivative of $\ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$



Example 4 prove $\frac{d}{dx}[\ln x] = \frac{1}{x}$ using implicit differentiation

Example 5 Find y' if $y = \ln(2x + 2)$

Example 6 Let $f(x) = \ln(\tan x)$. Find $f'(x)$.

Logarithmic Differentiation

The properties of logarithms can be used to simplify some problems. Here is a review of the properties

Name	**Mathematical Property**	**Example**
Definition of Logarithm	If $b^c = a$, then $log_b a = c$	If $2^4 = 16$, then $log_2 16 = 4$
Addition Rule	$log_b(MN) = log_b(M) + log_b(N)$	$log_2(5x) = log_2(5) + log_2(x)$
Subtraction Rule	$log_b\left(\frac{M}{N}\right) = log_b(M) - log_b(N)$	$log_2\left(\frac{5}{x}\right) = log_2(5) - log_2(x)$
Exponent Rule	$log_b(M^k) = k \cdot log_b(M)$	$log_2(5^3) = 3 \cdot log_2(5)$
Change of Base	$log_b a = \frac{\ln a}{\ln b}$	$log_2 3 = \frac{\ln 3}{\ln 2}$

Example 7 Rewrite $f(x)$ using properties of logs and find $f'(x)$

$f(x) = log_5\sqrt{x}$

Example 8 Use the properties of logarithms to rewrite $f(x)$ and find $f'(x)$ in terms of x.

$$f(x) = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$$

By utilizing the rules of logarithms and implicit differentiation, you can turn an exponential equation into an equation involving logarithms that is usually easier to deal with.

Example 9 $\frac{d}{dx}[2^x]$

Example 10 $\frac{d}{dx}[3^x]$

Derivative of a^x where a is a constant

$$\frac{d}{dx}[a^x] = \ln a \cdot a^x$$

Example 11 Find the derivative of $f(x) = e^{5x} + 7^{2x} + \ln(x^2 + 4)$

Example 12 Find the derivative of $f(x) = e^{\tan 3x} + 6^{x^2} + \ln(\sec x)$

Chapter 4

Applications of the Derivative pt. 1

4.1 Extreme Value Theorem

Often problems in engineering or economics seeks to find an optimal solution to a problem. If a problem can be modeled by a function, then finding the max/min values of the function solves the problem.

We must first establish some precise vocabulary:

Definition 4.1: Absolute Extrema

1. $f(c)$ is the **absolute (global) minimum** of f on an interval I, if

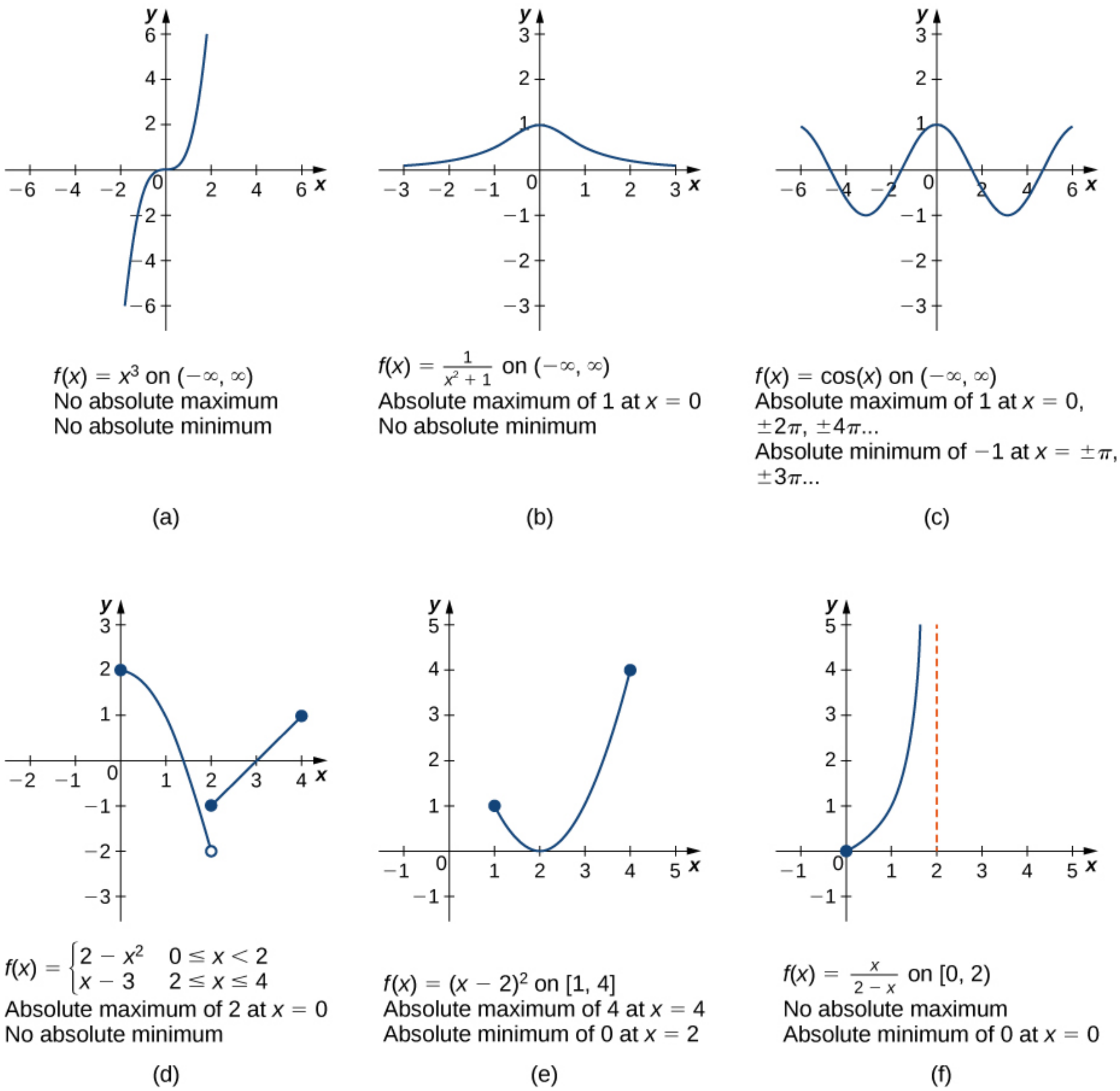
$$f(c) \leq f(x) \text{ for all } x \in I$$

2. $f(c)$ is the **absolute (global) maximum** of f on an interval I, if

$$f(c) \geq f(x) \text{ for all } x \in I$$

The absolute maximum and the absolute minimum value, if they exist, are the largest and smallest values, respectively, of a function f on the interval I. In this case the interval may be **open,closed, or neither**.

Examine the following:



Based on our investigation, we can now call upon a useful **Existence Theorem**.

Definition 4.2: The Extreme Value Theorem

If a function f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

That is, there are numbers $x_1, x_2 \in [a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$, and

$$m \le f(x) \le M$$

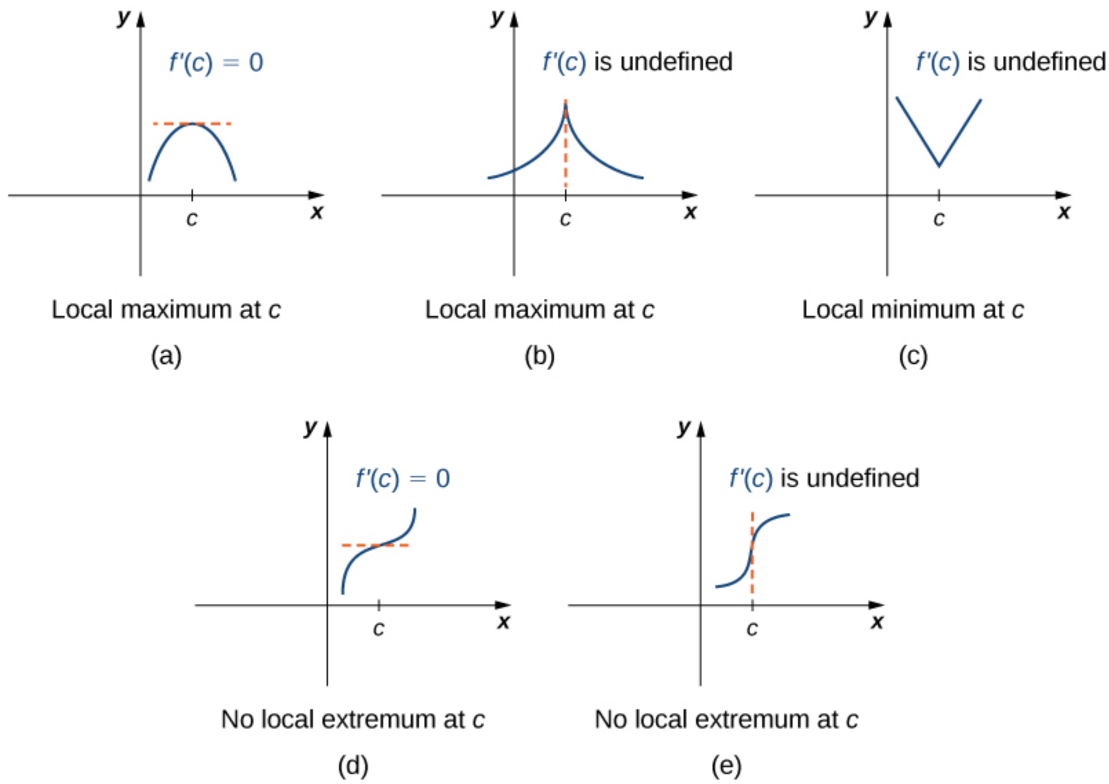
for every other $x \in [a, b]$.

Now, with our investigation we noticed that we were able to identify absolute extrema but now we turn our attention to **local maxima and minima**.

Definition 4.3: Local Extrema

1. A function f has a local minimum value at a point c within an interval I if $f(x) \geq f(c)$ for all x in I lying in some **OPEN INTERVAL** (maybe smaller) containing c.
2. A function f has a local maximum value at a point c within an interval I if $f(x) \leq f(c)$ for all x in I lying in some **OPEN INTERVAL** (maybe smaller) containing c.

Now, we have investigated local and absolute extremas but only using graphs. This thus leads us to ask the questions, How do we find these extrema when only the function f is provided?



Local maximum at c (a) Local maximum at c (b) Local minimum at c (c)

No local extremum at c (d) No local extremum at c (e)

The above graphic allows us to *informally* state the following theorem without proof.

Theorem 22: Conditions for a Local Maxima/Minima

If a function f has a local maximum or minimum at the number c, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition 4.4: Critical Number

A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ is undefined.

Example 58. Find any critical number in the following functions:

a) $f(x) = x^3 - 6x^2 + 9x + 2$

b) $R(x) = \dfrac{1}{x-2}$

DIY 55. Find any critical number in the following functions:

a) $f(x) = \sin(x)$

b) $R(x) = \dfrac{(x-2)^{2/3}}{x}$

Example 59. Determine the absolute maximum and absolute minimum value over the stated interval by applying the Extreme Value Theorem.

a) $f(x) = x^2 + 4x + 4$ on $[-4, 0]$

b) $R(x) = x^3 - 3x + 1$ on $(\frac{-3}{2}, 3)$

DIY 56. Determine the absolute maximum and absolute minimum value over the stated interval by applying the Extreme Value Theorem.

a) $f(x) = x^3 - 6x^2 + 9x + 2$ on $[0, 2]$

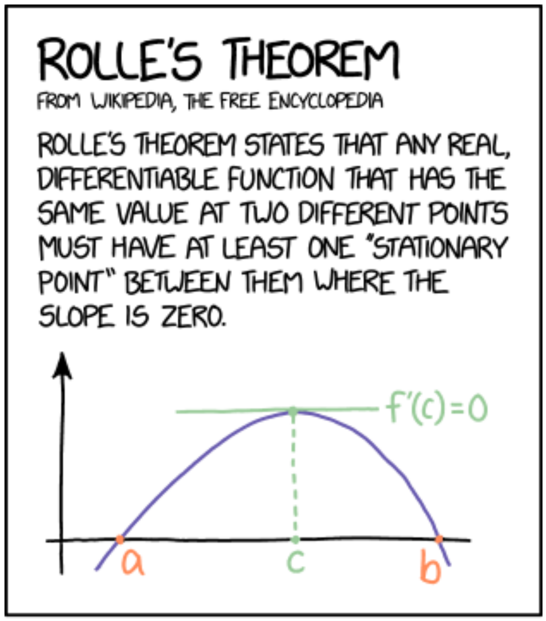
b) $R(x) = \dfrac{(x-2)^{2/3}}{x}$ on $[1, 10]$

Theorem 23: Rolle's Theorem

Let f be a function defined on a closed interval $[a, b]$. If

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b),
3. $f(a) = f(b)$

Then there is at least one number c in the open interval (a, b) for which $f'(c) = 0$.



EVERY NOW AND THEN, I FEEL LIKE THE MATH EQUIVALENT OF THE CLUELESS ART MUSEUM VISITOR SQUINTING AT A PAINTING AND SAYING "C'MON, MY KID COULD MAKE THAT."

4.2 Mean Value Theorem

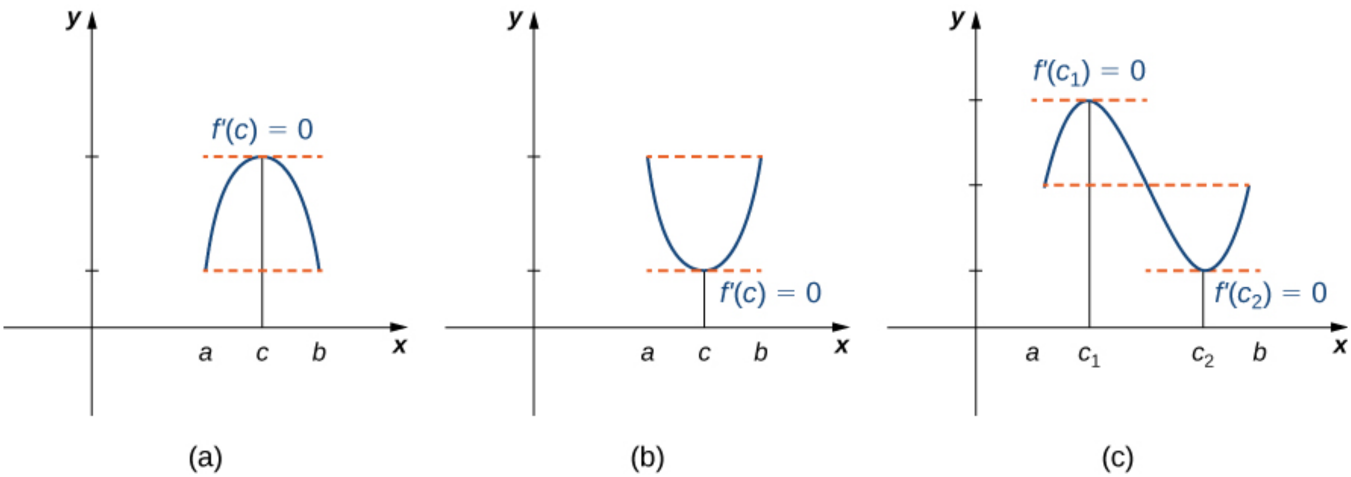
Theorem 24: Rolle's Theorem

Let f be a function defined on a closed interval $[a, b]$. If

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b),
3. $f(a) = f(b)$

Then there is at least one number c in the open interval (a, b) for which $f'(c) = 0$.[a]

[a] An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold.



Proof. Because f is continuous on a closed interval $[a, b]$, the Extreme Value Theorem guarantees that f has an absolute maximum and minimum on $[a, b]$. If we let $k = f(a) = f(b)$,there are two possibilities:

1. Case 1: $f(x) = k$ for all $x \in (a, b)$.
   (i.e., $f(x)$ is constant)

2. Case 2: There exists an $x_1 \in (a, b)$ such that $f(x_1) > k$ or $f(x_1) < k$.
   (i.e., $f(x)$ is **NOT** constant)

□

Example 60. For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a) $f(x) = x^3 - 4x$ on $[-2, 2]$

b) $g(x) = |x| - 1$ on $[-1, 1]$

DIY 57. Find the $x-$intercepts of $f(x) = x^2 - 5x + 6$, and show that $f'(c) = 0$ for some number c in the open interval formed by the two $x-$intercepts. Find c.

Rolle's Theorem was proved in 1691 by Michel Rolle. It took over a 100 years for another mathematician, Augustin Cauchy, to find more generalized result. This result is what is now known as the Mean Value Theorem, can be obtained from Rolle's Theorem, and as we will see, this extension will now allow us to obtain many result, many of which have a wide array of applications.

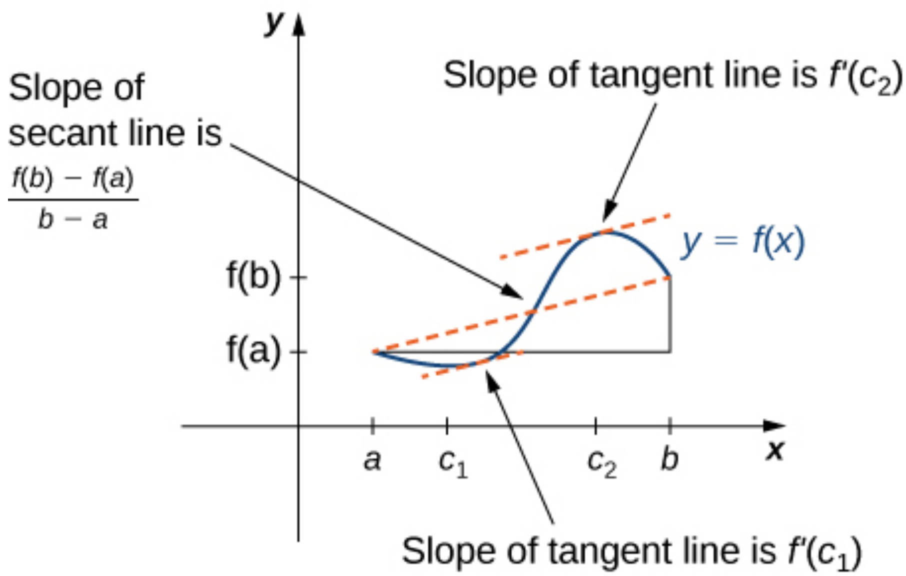
Theorem 25: Mean Value Theorem

Let f be a function defined on a closed interval $[a, b]$. If

1. f is continuous on $[a, b]$
2. f is differentiable on (a, b),

Then there is at least one number c in the open interval (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof. The most efficient proof for the MVT requires us to use Rolle's Theorem. In order to do so, we need a function that is continuous on the closed interval $[a, b]$ and that has the same endpoint values. We begin by noticing:

$$m_{sec} = \frac{f(b) - f(a)}{b - a}.$$

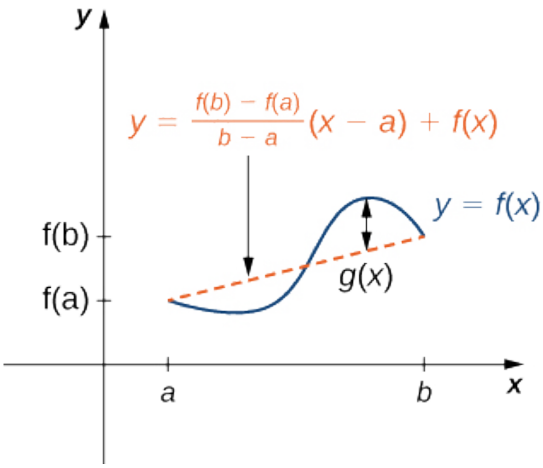
. Therefore, the equation of the secant line using $(a, f(a))$ is defined as

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \text{ or } y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

We now DEFINE, the following function

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$$

It remains to check that g is continuous on $[a, b]$ and differentiable on (a, b), as well as the value for $g(a)$ and $g(b)$. I will leave it to you to verify these result.



In conclusion we have found a function that satisfies Rolle's Theorem, or in other words

$$g'(c) = f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right] = 0 \text{ or } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\square$

Unfortunately, the Mean Value Theorem is another **Existence Theorem**, which means that we know that a c value exists but we do not have a systematic way of finding the value...but in AP Calculus we can often find it.

Example 61. Answer the following for the given functions:

- Verify that each function satisfies the condition of the Mean Value Theorem.
- Find the numbers c guaranteed by the Mean Value Theorem.
- Interpret the number(s) geometrically.

a) $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1, 3]$

b) $f(x) = \ln \sqrt{x}$ on $[1, e]$

DIY 58. Answer the following for the given functions:

- Verify that each function satisfies the condition of the Mean Value Theorem.
- Find the numbers c guaranteed by the Mean Value Theorem.
- Interpret the number(s) geometrically.

a) $f(x) = \dfrac{x^2}{x+1}$ on $[0, 1]$

b) $g(x) = x^{-1/3} - x$ on $[-1, 1]$

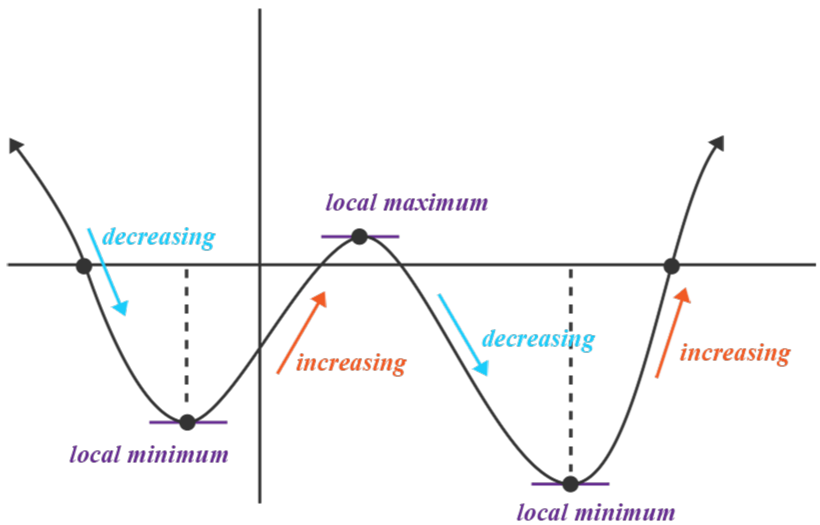
3 Important Corollaries of the MVT

The following 3 results are consequences of the MVT:

Theorem 26: Corrollary #1; Increasing/Decreasing Test

Let f be a differentiable function on the open interval (a, b) and:

- If $f'(x) > 0$, then f is increasing on (a, b).
- If $f'(x) < 0$, then f is decreasing on (a, b)



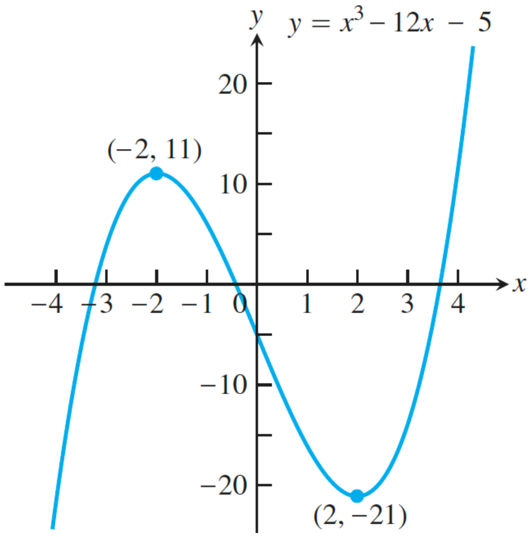
With this we can now call upon the theorem that allows us to verify if a critical point is indeed a local extrema or not.

Theorem 27: First Derivative Test

Let f be a function that is continuous on an interval I. Suppose that c is a critical number of f and (a, b) is an open interval I containing c:

- If $f'(x) > 0$ (is increasing) to the left of c and $f'(x) < 0$ (is decreasing) to the right of c, then $f(c)$ is a local maximum.
- If $f'(x) < 0$ (is decreasing) to the left of c and $f'(x) < 0$ (is increasing) to the right of c, then $f(c)$ is a local minimum.
- If $f'(x)$ stays the same on both sides, then $f(c)$ is neither type of local extrema.

Example 62. Using the graph of f determine the critical numbers, where $f'(x) > 0$, and where $f'(x) < 0$. Confirm you answer by using the First Derivative Test on $f(x) = x^3 - 12x - 5$.



Example 63. Find the local extrema of $f(x) = x^4 - 4x^3$.

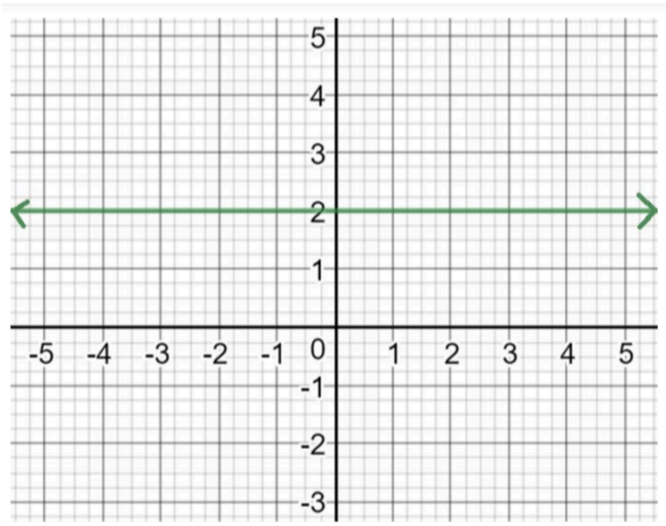
DIY 59. Find the local extrema of $f(x) = x^{2/3}(x - 5)$.

Anti-Derivatives

Example 64. Suppose that a friend came up to you and said $f'(x) = 3$. What could $f(x)$ be? Is there more than one answer?

DIY 60. A friend gives you the following graph of $f'(x)$.

a) Draw 3 possibilities for the graph of $f(x)$.



The three functions you drew should only differ by a constant (differ by a vertical transformation). If you let C represent this constant, then you can represent the **family of all antiderivatives of** $f'(x)$ to be:

$$f(x) = 2x + C$$

where $f(x)$ is the anti-derivative.

b) Suppose your friend also said that $f(3) = -2$, what would C be?

Theorem 28: Corrollary #2; Zero Derivative from Constant Function(Curve Sketching)

If a function f is continuous on the closed interval $[a, b]$ and is differentiable on the open interval (a, b), and if $f'(c) = 0$ for all numbers x in (a, b), then f is constant on (a, b).

Theorem 29: Corrollary #3;Anti-Derivatives

If a functions f and g are differentiable on the open interval (a, b), and if $f'(x) = g'(x)$ for all numbers x in (a, b), then there is a constant C such that

$$f(x) = g(x) + C \text{ on } (a, b).$$

4.3 Concavity, POI, and Second Derivative Test

In the prior section we discussed the utility of the first derivative. In an analogous manner, the second derivative also gives us a lot of useful information. Before we continue, let us unpack what the second derivative really is using the limit definition.

Definition 4.5: Second Derivative Definition

Suppose the f is a twice differentiable function such that f' exists, then the second derivative, f'', is defined as

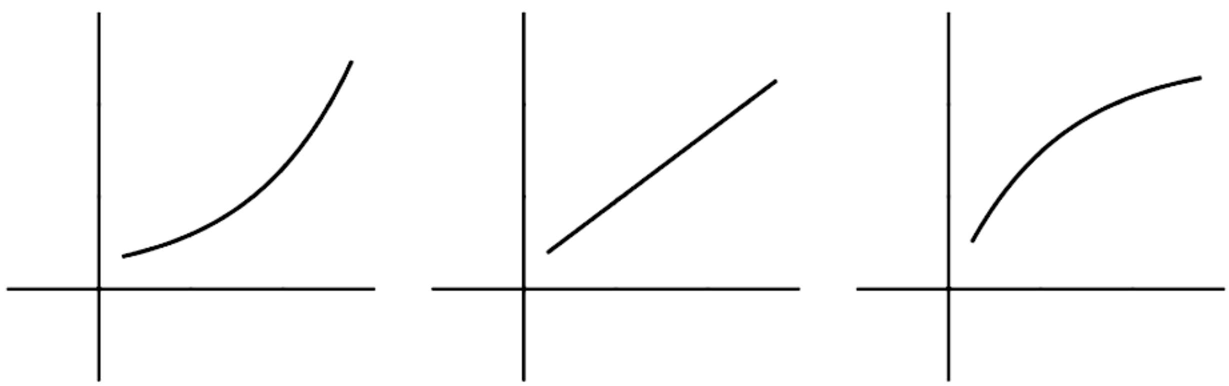
$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

The units on the second derivative are “units of output per unit of input per unit of input”, or more commonly known as ”units of output per units of input **squared**”.

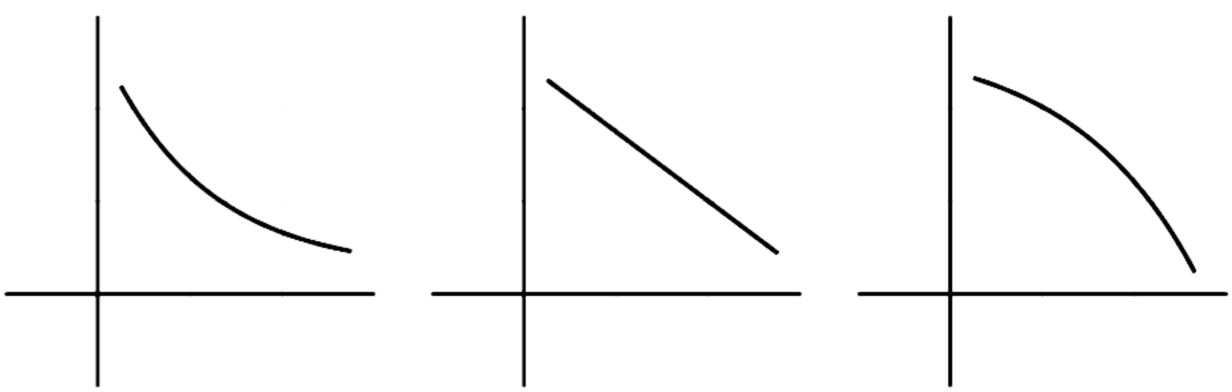
In other words, the second derivative measures the rate of change of the first derivative (which in itself is a rate of change)!

Given this definition of the second derivative, we can now clearly see that it is “the rate of the rate”, so aside from knowing where a function is increasing and decreasing, we are now also going to be interested in finding out ***HOW*** a function increases or decreases.

As a bit of motivation we can look at the following three graphs.



We can see that in these three cases our function is “increasing” but increasing in different manners. A similar case can be made about the following three decreasing graphs.



**If we were to draw a sequence of tangent lines from left to right, it would reveal the information we seek in regards to the second derivative.

Concavity

The above discussion can leads us to the notion of concavity, which provides an easier language to communicate what is going on with our original function.

Let us now explore the majority of the cases using the functions $f(x) = x^2$ and $g(x) = -x^2$. [1].

- **Concave Up**
- **Concave Down**

- **Properties of Concave ↑**
- **Properties of Concave ↓**

[1]We could have also used e^x and $-e^x$(why?)

Based on our exploration above we are now ready to formally define concavity:

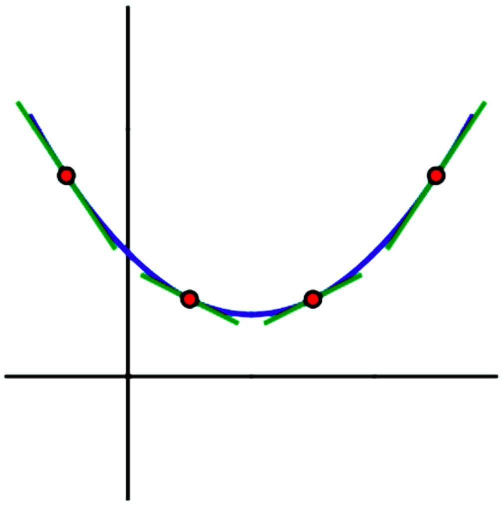
Definition 4.6: Concavity

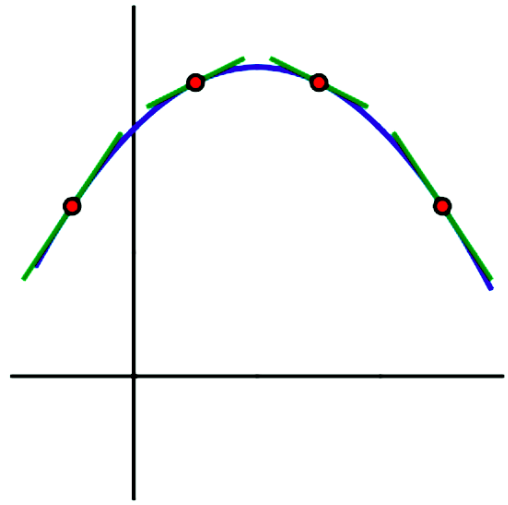
Let f be a differentiable function on an interval (a, b).

Then f is said to be **concave up** if and only if f' is increasing on (a, b).

And, f is said to be **concave down** iff f' is decreasing on (a, b).

As a helpful reminder, I note that for concave up, UP, starts with a U, (shape of a parabola) and then I dissect it using the breakdown from the previous page. Similarly, for concave down, the thought of a *person who is down* reminds me of a frown....





The following image with (exaggerated) drawn out tangent lines encompasses the above definition. We will come back to this image when we discuss Linearization and Quadratic Approximations.

Anyways, with the above definition and image we can now see some meaning coming from the second derivative.

Using the MVT, we noticed that there was a clear connection between the first derivative, f' and f. Our hope is to apply the same techniques to f' and see what the connection will be between f'' and f'.

- f **and** f'

- f' **and** f''

So, we know that when $f'' > 0$, we have a concave up portion of f and when $f'' < 0$ we have a concave down portion of f. The final question that remains unanswered is, what happens when $f'' = 0$ or when f'' is undefined? And moreover, as a natural extension, what happens when f'' changes sign?!

Definition 4.7: Point of Inflection

A point of inflection is a point, (x, y), on the graph of f such that the concavity of f changes.

Theorem 30: Points of Inflection

If $(c, f(c))$ is a point of inflection on the graph of f, then $f'' = 0$ or f'' is undefined at c.

Example 65. Let f, g, h be the functions defined. Determine on which open intervals the functions are concave up and concave down. Identify any points of inflection. Justify all responses.

a) $f(x) = 4x^3 + 21x^2 + 36x - 20$

b) $h(x) = x^{5/3}$

c) $g(x) = \dfrac{x^2 + 1}{x^2 - 4}$

DIY 61. Let f, g, h be the functions defined. Determine on which open intervals the functions are concave up and concave down. Identify any points of inflection. Justify all responses.

a) $f(x) = x^4 + 2x^2$

b) $h(x) = x^{1/3}$

c) $g(x) = x^4 - 8x^3$

Example 66. Let f, g, h be the functions defined. Determine on which open intervals the functions are concave up and concave down. Identify any points of inflection. Justify all responses.

a) $f(x) = e^x$

b) $h(x) = e^{2/x}$

c) $g(x) = x - 2\cos(x), 0 \leq x \leq 2\pi$

Second Derivative Test

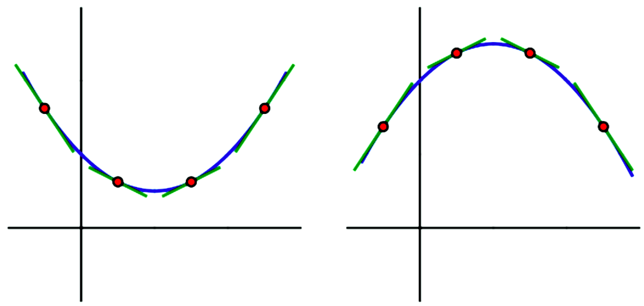
The following theorem comes to us (almost for free) intuitively if we look at it from the right perspective.

Theorem 31: Second Derivative Test

Let f be a **twice differentiable** function on an open interval I. Suppose c is a critical number of f on the interval I, then:

- If $f''(c) < 0$, then $f(c)$ is a local maximum value.
- If $f''(c) > 0$, then $f(c)$ is a local minimum value.

The test is inconclusive if $f'' = 0$ or if f'' is undefined.



At first, the utility of this theorem does not seem apparent, but if we were to encounter a function where the First Derivative Test is rather "difficult" to apply, then we can resort to the Second Derivative Test and hope that the computation will be a bit more straightforward.[2]

Example 67. Use the Second Derivative Test to identify any local extrema. Justify all responses.

a) $g(x) = x - 2\cos(x), 0 \le x \le 2\pi$

b) $f(x) = x^4$

[2]This will usually be the case for non algebraic functions.

DIY 62. Find the local extrema of each function using:

- The First Derivative Test
- The Second Derivative Test

and discuss which was easier to compute.

a) $g(x) = 3x^5 + 5x^4 + 1$

b) $f(x) = (x-3)^2 e^x$

4.4 f, f', and f'' Relationships

DIY 63. If $f(x)$ is continuous at $x = 3$ and

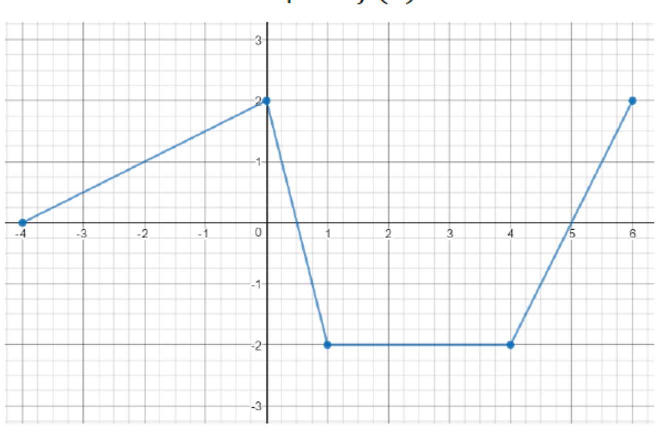
$$f'(3) = 0, \quad f''(3) = -5$$

Prove that f has a relative max at $x = 3$.

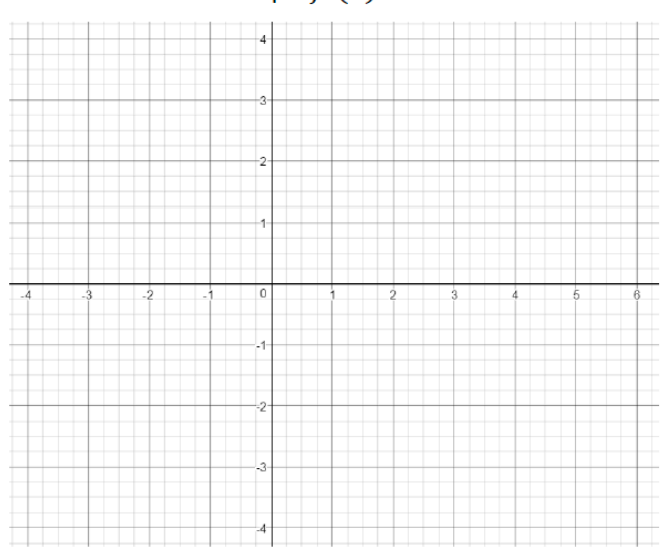
DIY 64. Sketch a graph whose derivative is always positive and another whose derivative is always negative.

DIY 65. Sketch the graph of the derivative of f below.

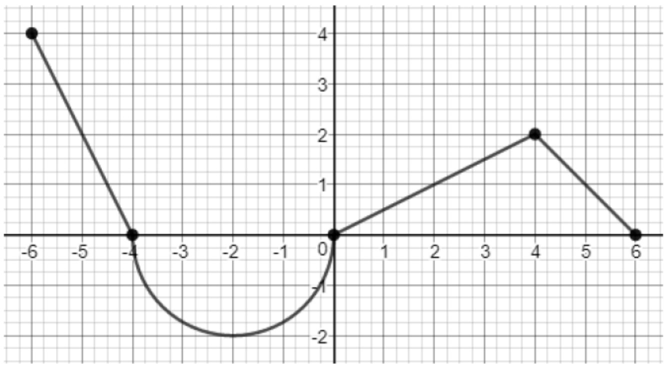
Graph of $f(x)$



Graph $f'(x)$ here



Example 68. The function f is differentiable on the closed interval $[-6, 6]$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown.



a) Find the $x-$coordinate of each critical point of $y = f(x)$ on the interval $-6 < x < 6$. Justify your response.

b) Find the $x-$coordinate of each relative extrema of $y = f(x)$ on the interval $-6 < x < 6$. Label each as a minimum or a maximum. Justify your response.

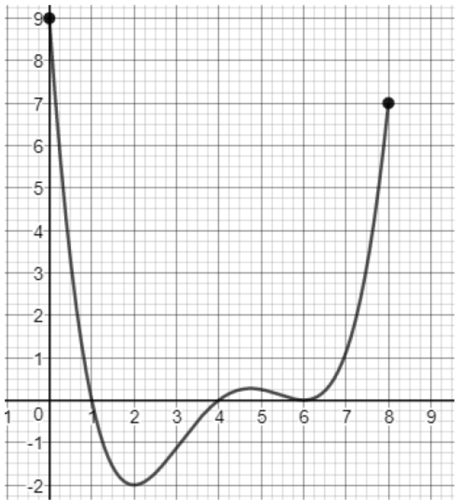
c) Find the open intervals over which the function $y = f(x)$ is increasing and decreasing on the interval $-6 < x < 6$. Justify your response.

d) Find the $x-$coordinate of each point of inflection for $y = f(x)$ on the interval $-6 < x < 6$. Justify your response.

e) Find the interval over which the function $f(x)$ is concave up and concave down on the interval $-6 < x < 6$. Justify your response.

f) If $f(-2) = -4$, Find the equation of the tangent line to $y = f(x)$ at $x = -2$

DIY 66. The function f is defined and differentiable on the closed interval $[0, 8]$. The graph $y = f(x)$, the derivative of f, is shown in the figure below and has horizontal tangents at $x = 2, x = 4.5$, and $x = 6$.



a) Find each $x-$coordinate of the critical points of $y = f(x)$ on the interval $0 < x < 8$. Justify your answer.

b) Find where f is increasing and decreasing on the interval $0 < x < 8$. Justify your answer.

c) Find the $x-$coordinate of each relative extrema of f on the interval $0 < x < 8$. Determine if these locations are minima or maxima and justify your response.

d) Find the $x-$coordinate of each point of inflection for f on the interval $0 < x < 8$. Justify your answer.

e) Find the intervals over which the function f is concave up and concave down on the interval $0 < x < 8$. Justify your answer.

f) Let $g(x) = \frac{1}{6}x^3 - f(x)$. Find $g'(7)$.

Curve Sketching

Example 69. Analyze and sketch the graph of $f(x) = \dfrac{(x^2 - 9)}{x^2 - 4}$.

a) Determine the x and $y-$intercepts.

b) Determine the equations of any horizontal and vertical asymptotes.

c) Determine the first and second derivative of the function.

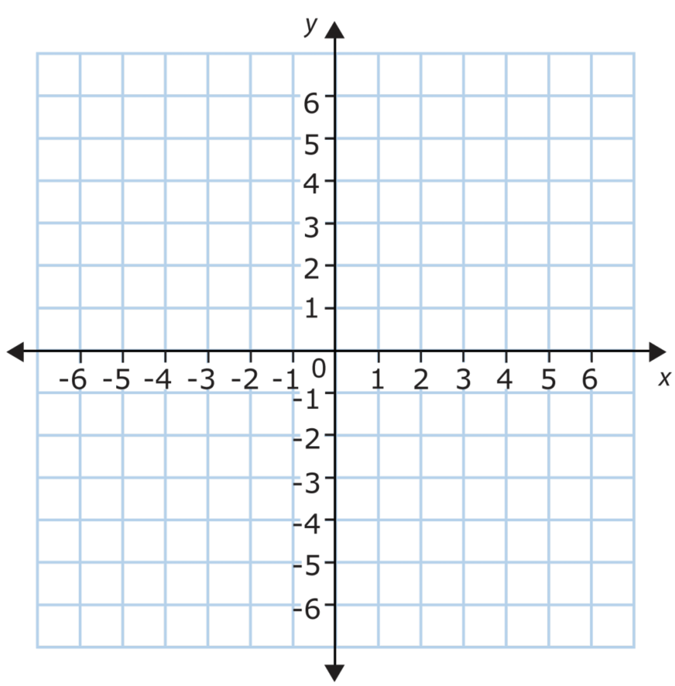
d) Determine where f is increasing and decreasing.

e) Determine any relative extrema.

f) Determine where f is concave up and/or concave down.

g) Identify any points of inflection.

h) Sketch:

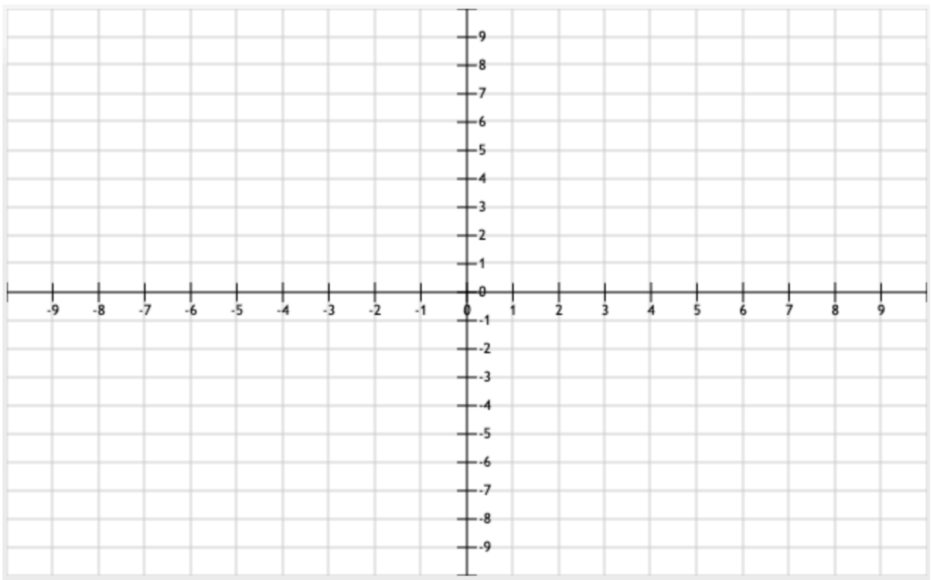


DIY 67. Sketch a curve using the following information:

$$f(-2) = 8 \qquad f(0) = 4 \qquad f(2) = 0$$

$$f'(x) < 0 \text{ for } |x| < 2 \qquad f'(2) = f'(-2) = 0 \qquad f'(x) > 0 \text{ for } |x| > 2$$

$$f''(x) > 0 \text{ for } x > 0 \qquad f''(x) < 0 \text{ for } x < 0$$

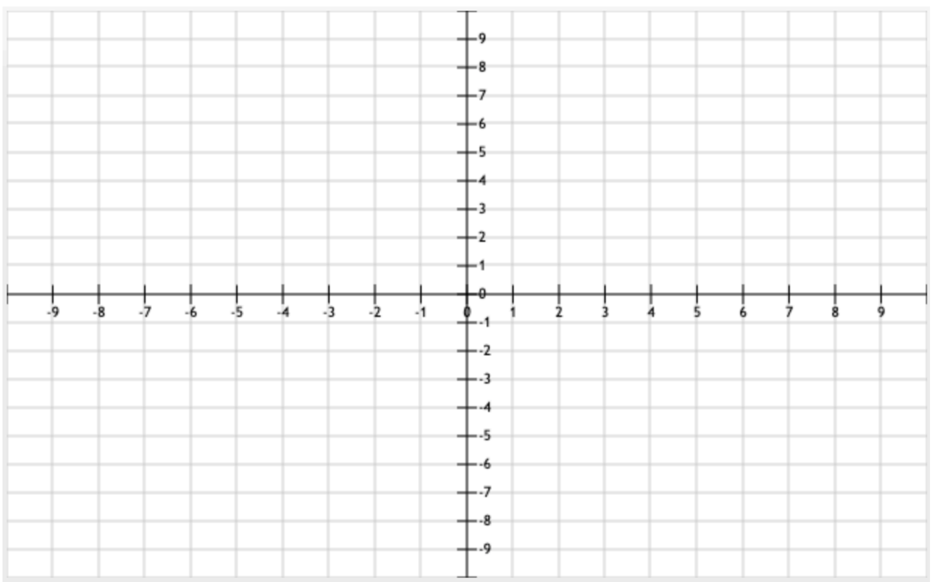


DIY 68. Sketch a curve using the following information:

$$f(0) = 2 \qquad \lim_{x\to\infty} f(x) = -2 \qquad \lim_{x\to 0^-} f(x) = \infty$$

$$f'(x) < 0 \text{ for } -4 < x < -1 \qquad f'(x) > 0 \text{ for } -1 < x < 0$$
$$f'(x) > 0 \text{ for } 0 < x < 2 \qquad f'(x) < 0 \text{ for } 2 < x < \infty$$

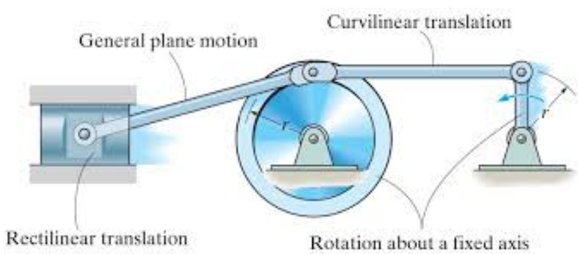
$$f''(x) < 0 \text{ for } -4 < x < -1 \qquad f''(x) > 0 \text{ for } -1 < x < 0$$
$$f''(x) < 0 \text{ for } 0 < x < 4 \qquad f''(x) > 0 \text{ for } 4 < x < \infty$$

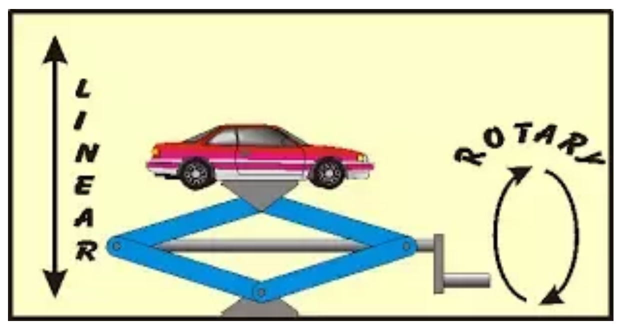


Chapter 5

Applications of Derivatives pt.2

This chapter will deal with more"physical" and "tangible" real world examples. We will continue our previous discussion between f, f', f'' but we will now focus on the utility of these functions when making approximations. We will then delve into the applications of the Chain Rule and EVT.





5.1 Particle Motion: PVA

We concluded the previous chapter with a discussion about the relationship between f, f', f''. Newton noted that physicist already had a similar relationship already defined that they could now define using the language of calculus. We will focus our attention to the topics of objects moving along a straight-line, which in physics is called **Rectilinear Motion**.

Definition 5.1: Position, Velocity, Acceleration(Physics Version)

- **Position/Displacement:** The location of a particle/object as a function of time and with respect to a reference frame.
- **Velocity:** The rate of change of displacement over time.
- **Acceleration:** The rate of change of velocity over time.
- **Speed**: A **scalar** quantity with no sense of direction, only magnitude.

Definition 5.2: Position, Velocity, Acceleration (Mathematician Version)

- **Position Function:** A parametrized function relating the position of a moving object with respect to time. Usually denoted as follows:

$$x(t) \text{ or } s(t)$$

- **Velocity Function:** The derivative of the position function. Usually denoted as follows:

$$\dot{x} = x'(t) = \frac{dx}{dt} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{ds}{dt} = \dot{s} = s'(t)$$

- **Acceleration Function:** The derivative of the velocity function OR the second derivative of the position function. Usually denoted as follows:

$$v'(t) = a(t)$$

&

$$\ddot{x} = x''(t) = \frac{d^2x}{dt^2} = \frac{\text{change in velocity}}{\text{change in time}} = \frac{d^2s}{dt^2} = s''(t) = \ddot{s}$$

- **Speed**: The absolute value of velocity.

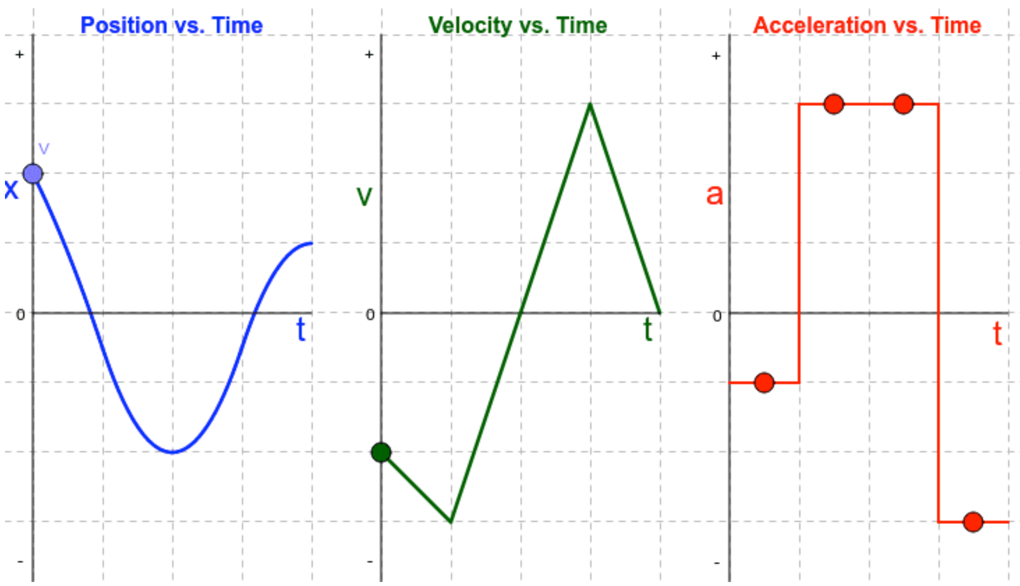
$$\text{Speed } = |v(t)|$$

Reflecting the parts of the velocity graph that lie below the $t-$axis will give you the graph of the speed.

Definition 5.3: Properties of 1-D PVA

Suppose that a particle is moving along the $x-$axis.

- If $v(t) = \dfrac{dx}{dt} < 0$, then the particle is moving left.
- If $v(t) = \dfrac{dx}{dt} > 0$, then the particle is moving right.
- If $v(t) = \dfrac{dx}{dt} = 0$, then the particle is not moving.
- If the velocity and acceleration have the same sign, the speed is increasing.
- If the velocity and acceleration have different signs, the speed is decreasing.



Example 70. The path of a particle can be traced using $x(t) = t^3 - 9t^2 + 24t + 5$, where t is in seconds.

a) Find the velocity and acceleration function $v(t)$ and $a(t)$.

b) In which direction is the particle moving initially? Provide reasoning.

c) At what time t does the particle change direction?

d) Find the time t when the acceleration is 0.

e) Find the intervals of time when the particle is slowing down and speeding up.

f) What is the total distance covered between $t = 0$ and $t = 3$?

5.2 Linearization and Differentials

Recall from Section 4.2 the following discussion:

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$

where df and dx are meant to symbolize VERY VERY VERY VERY SMALL differences. Now, this makes sense as we should expect that a smaller Δx would lead to a smaller Δf.

Definition 5.4: Differentials

Let $y = f(x)$ be a differentiable function and let Δx denote a change in x.

The **differential of** x, denoted by dx, is defined as $dx = \Delta x \neq 0$.

The **differential of** y, denoted by dy, is defined as

$$dy = f'(x)dx$$

Example 71. For the function $f(x) = xe^x$.

a) Find an equation of the tangent line to the graph to the graph of f at $(0, 0)$.

b) Find the differential dy.

c) Compare dy to Δy when $x = 0$ and $\Delta x = 0.5$

d) Compare dy to Δy when $x = 0$ and $\Delta x = 0.1$

Based on our work we begin to notice that as dx becomes closer and closer to actual 0, the difference between Δy and dy is almost negligible, or in other words:

$$\Delta y \approx dy$$

$$f(x) - f(x_0) \approx f'(x_0)(x - x_0)$$
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

In other words, the equation of a tangent line at a point $(x_0, f(x_0))$ closely approximates the function f for a certain window around the point $x = x_0$. This approximation is called the **Linear Approximation** of a function.

Definition 5.5: The Linear Approximation

The linear approximation $L(x)$ to a differentiable function f at $x = x_0$ is given by

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

- If $f''(c) < 0$, $f(x)$ is concave down at $x = c$ and $L(x)$ is an overestimate.
- If $f''(c) > 0$, $f(x)$ is concave down at $x = c$ and $L(x)$ is an underestimate.

The closer x is to x_0 the better the approximation.

Example 72. Approximate $\sqrt[3]{70}$.

Example 73. Find the linearization $y = \sqrt{1+x}$ at $x = 0$ and use it to approximate $\sqrt{1.02}$ without a calculator.

DIY 69. Approximate 3.01^5. Determine if the linearization is a over estimate or an under estimate.

DIY 70. Approximate $\sin^{-1}(0.45)$. Determine if the linearization is a over estimate or an under estimate.

5.3 Related Rates

We have already seen how to differentiate functions that are defined with respect to t, time. In the natural sciences and related social sciences there are quantities that are related to each other but this relationship overall varies over time. Problems involving the rates of change of related variables are called **related rate problems**.

Example 74. A golfer hits a ball into a pond and causes a circular ripple. If the radius of the circle increases at the constant rate of $0.5m/s$, how fast is the area of the circle increasing when the radius of the ripple is 3m?

Example 75. A spherical balloon is inflated at the rate of 10 m^3/min. Find the rate at which the surface area of the balloon is increasing when the radius of the sphere is 3m.

Example 76. A person standing at the end of a pier is docking a boat by pulling a rope at the rate of $2m/s$. The end of the rope is $3m$ above water level. How fast is the boat approaching the base of the pier when $5m$ of rope are left to pull in?

Example 77. A 50ft. ladder is placed against a large building. The base of the ladder is resting on an oil spill, and it slips at the rate of 4ft. per minute. Find the rate of change of the height of the top of the ladder above the ground at the instant when the base of the ladder is 30 ft. from the base of the building.

Example 78. A person who is 6 feet tall is walking away from a lamp post at the rate of 40 feet per minute. When the person is 10 feet from the lamp post, his shadow is 20 feet long. Find the rate at which the length of the shadow is increasing when he is 30 feet from the lamp post.

Example 79. Consider a conical tank whose radius at the top is 4 feet and whose depth is 10 feet. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is at depth 5 feet?

5.4 L'Hopitals Rule

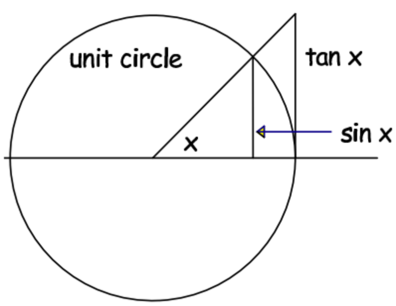
Early on in the course we tackled a particularly awkward limit:

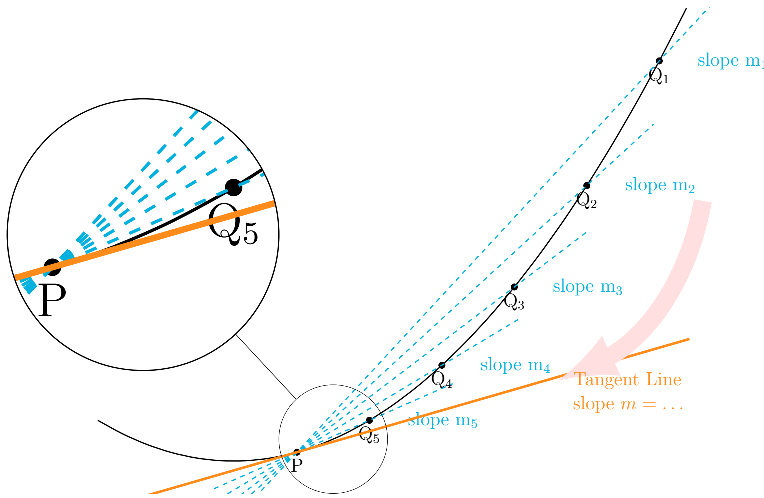
$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

When we tried to directly evaluate via direct substitution, this led to our first encounter with a "$\frac{0}{0}$" expression. We also encounter this situation when we define the derivative:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The way that we have tackled these two limit questions have been through the point of view of geometry.





Before we delve deeper, let's define these an "indeterminate form".

Definition 5.6: Indeterminate Form

The following outputs/expressions are defined as **indeterminate forms**:

$$\frac{0}{0}\ ,\ \frac{\infty}{\infty}\ , 0 \times \infty\ \ , \infty - \infty\ \ , 0^0\ \ , 1^\infty\ \ , \infty^0$$

If after directly evaluating a limit expression, the result reads an indeterminate form, then the evaluation is inconclusive.

Theorem 32: L'Hpitals Rule

If (f) and (g) are continuous functions such that:

- $(\lim_{x\to a} f(x) = 0)$ and $(\lim_{x\to a} g(x) = 0)$

- $(\lim_{x\to a} f(x) = \infty)$ and $(\lim_{x\to a} g(x) = \infty)$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)},$$

provided this limit exists.

If this is still of the indeterminate form $(\frac{0}{0})$ OR $(\frac{\infty}{\infty})$, then derivatives may be taken again, and so on. **NOTE: YOU MUST VERIFY THAT THE NUMERATOR AND DENOMINATOR EVALUATE TO 0 OR ∞ BEFORE USING L'HOPITALS**

Proof. We begin with

$$\lim_{x\to a} \frac{f(x)}{g(x)} =$$

□

Example 80. Evaluate the following:

a) $\lim_{x\to 0} \dfrac{e^{3x}-1}{x}$

b) $\lim_{x\to\infty} \dfrac{\ln x}{2\sqrt{x}}$

c) $\lim\limits_{x \to -\infty} \dfrac{x^2}{e^{-x}}$

d) $\lim\limits_{x \to \infty} e^{-x}\sqrt{x}$

e) $\lim\limits_{x \to 0^+} x^x$

DIY 71. Find the values for which the following piece-wise function will be continuous at $x = \dfrac{\pi}{2}$

$$f(x) = \begin{cases} \dfrac{\sin x - 1}{x - \dfrac{\pi}{2}} & \text{if } x \neq \dfrac{\pi}{2} \\ k & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

Chapter 6

Differential Equations

In our discussion of Related Rates and Optimization we ran into derivatives that expressed the relationship between physical quantities. In the studies of physical, biological, and social sciences we tend to run into phenomena where we can only attempt to find mathematical laws from models created from observed behaviour. This observed behaviour usually comes in the form of an equation that involves one or more derivatives and constants. Here are a few examples:

$$\frac{dy}{dt} = ky \qquad \frac{dy}{dt} = -ky \qquad \frac{dT}{dt} = -k(T - T_S) \qquad \frac{d^2u}{dt^2} = c^2\frac{d^2u}{dx^2} \qquad \frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The ideal goal is to be able to find general solutions to these equations as these would produce the FUNDAMENTAL LAWS OF PHYSICS.

Definition 6.1: Differential Equations

A differential equation is any equation that contains derivatives and constants. A **solution** to a differential equation is any function that can satisfy the differential equation.

$$\text{If } \frac{dy}{dx} + 4y = 0, \text{ then } y = Ce^{-4x} \text{ is a solution.}$$

A **DE** with an initial condition is called an **initial value problem (IVP)**. It has a unique solution, called the **particular solution** to the differential equation.

The **order** of a differential equation is the highest derivative involved in the equation.

Two general (common) types of differential equations:

- **Ordinary Differential Equations**(ODE's): equations that contain differentials with respect to one variable.
- **Partial Differential Equations**(PDE's): equations that contain differentials with respect to multiple independent variables.

As we will see some DE's are solvable and their methods are not too complex, but unfortunately not all DE's are solvable, so we will also go over a few common **numerical** and **graphical** solution methods.

6.1 Separation of Variables and U-Sub

We have already discussed how for each **differentiable** function f, there exists a function f', which we called the derivative. From this process we were able to establish a lot of distinct properties and theorems that later assisted us in solving "real" world problems.

Now we will begin to tackle the inverse question:

For a given function f, is there a function F whose derivative is f?

$$F'(x) = \frac{dF}{dx} = f(x)?$$

If such a function F can be found, it is called an **anti-derivative** of f.

We previously discussed this as a Corollary from the MVT:

Theorem 33: MVT Corrollary #3;Anti-Derivatives

If a functions f and g are differentiable on the open interval (a, b), and if $f'(x) = g'(x)$ for all numbers x in (a, b), then there is a constant C such that

$$f(x) = g(x) + C \text{ on } (a, b).$$

We now take this as a definition with a few changes in notation:

Definition 6.2: Anti-Derivative/Inverse Derivative

A function F is called an anti-derivative of the function f if $F'(x) = f(x)$ for all x in the domain of f.

Using a **long 's'** we can re-write this explicitly in the solution form as

$$F(x) = \int f(x)dx = G(x) + C$$

Where C is the **constant of integration**. Since C can be any value, we call $G(x) + C$ the **general solution**.

In other words, we will find F by undoing the derivative on f. This inverse derivative process is called **integration**.[a] Therefore, the above expression, is called **the integral of** f.

[a] We will discuss the integral from the POV of limits in the next chapter, for now, this definition will suffice.

The Linearity property of the derivative carries over to the anti-derivative (integral).

Theorem 34: Linearity of the Integral

If $k \in \mathbb{R}$ is any constant and the functions $k \cdot F$ and G are antiderivatives of the functions $k \cdot f(x)$ and $g(x)$ respectively, then $k \cdot F \pm G$ is an anti-derivative of $f \pm g$.

$$k \cdot F(x) \pm G(x) = \int k \cdot f(x)\, dx \pm \int g(x)\, dx = \int \Big(k \cdot f(x) \pm g(x)\Big)\, dx$$

Separation of Variables

What the above definition does, is allow us to find solutions to differential equations. In particular let us suppose we have any of the following

$$\frac{dy}{dx} = f(x) \qquad \frac{dy}{dx} = \frac{x^2}{2y} \qquad \frac{dy}{dx} = \sin x$$

These three functions are examples of **Separable Differential Equations** since it is possible to separate all the x and y variables and their derivatives.

Definition 6.3: Separation of Variables

A separable differential equation is a DE such that

$$\text{If } \frac{dy}{dx} = f(x), \text{ then } dy = f(x)dx.$$

We can find the solution $y(x)$ by taking the integral of both sides such that

$$\int dy = \int f(x)dx \Longrightarrow y(x) = F(x) + C$$

WARNING: ON THE AP EXAM YOU MUST SHOW THE SEPARATION TO RECEIVE POINTS FOR THIS SOLUTION METHOD. AS TRIVIAL OR AS EASY IT MAY FEEL, WRITE OUT THE DIFFERENTIAL FORM.

Example 81. Solve $\dfrac{dy}{dx} = \sin x$ given $y(0) = 2$.

Example 82. Find the general solution for $\dfrac{dy}{dx} = e^x + 20(1+x^2)^{-1}$. Find the particular solution given $y(0) = -2$.

Example 83. The acceleration of a particle can be modeled by $a(t) = cos(t)$. Find the velocity and position functions if $v(0) = -1$ and $s(0) = 1$.

Example 84. Find the general solution for $\dfrac{dy}{dx} = ky$. Find the particular solution given $y(0) = -2$.

DIY 72. Find the general solution for $\dfrac{dy}{dx} = 3y$. Find the particular solution given $y(0) = 4$.

DIY 73. Solve $y' - y \sin x = 0$ given $y(0) = 2$.

U-Sub

This method of dealing with integrals/DE's is used when an expression consists of multiple rules/functions. As we will see, this method is the **reverse of the chain rule**.

Example 85. Evaluate the following:

$$\int \cos(7x + 5)dx$$

Example 86. Evaluate the following:

$$\int \sqrt{4x - 1}\, dx$$

Example 87. Evaluate the following:

$$\int \cos(x^2 + 2x - 3)(x + 1)\, dx$$

DIY 74. Evaluate the following:

$$\int (x^2 + 2x - 3)(x + 1)\, dx$$

Example 88. Evaluate the following:

$$\int \frac{\sin x}{1 - \sin^2 x}\, dx$$

DIY 75. Evaluate the following:

$$\int \tan x\, dx$$

Detour through Newtons Law's

According to Galileo, free falling objects obey

$$F = -mg$$

where g is the acceleration due to gravity. From Newton's Second Law of Motion,

$$F = ma$$

Therefore, we must have that

$$-mg = ma \qquad \text{or } a = -g$$

We seek the formulas for the velocity, v and distance, s, from Earth of a free falling object.

6.2 Slope Fields and Euler's Method

In the previous section we discussed how to solve differential equations by using the method of **separation of variables**. Not all differential equations are solvable using this method and in fact, some differential equations are not even solvable.

When we work with differential equations, we are dealing with expressions in which the derivative appears as an expression involving one (or more) variables. For example:

$$\frac{dy}{dx} = x^2 \qquad \frac{dy}{dx} = 2x \qquad \frac{dy}{dx} = 2y \qquad \frac{dy}{dx} = x + y$$

Although different, these 4 expressions describe **how** a function is changing. Let's remind ourselves of our usual goal when we are given a differential equation:

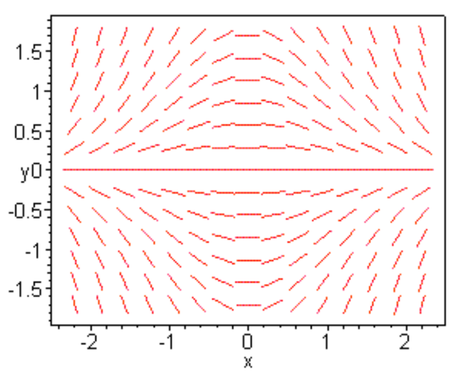
Find the original function whose derivative would be the DE given.

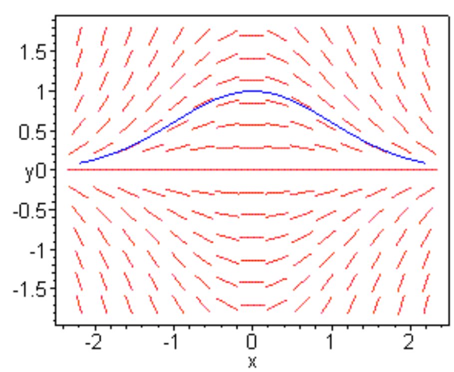
In our above examples this means that our goal is to find y for the first two, find x for the third, and find $f(x, y)$ for the fourth. Let us pretend like we do not know how to integrate the above functions. It is possible to learn a great deal about a differential equation, even when we don't know how to solve the equation, by looking at a picture produced by the DE and relating it to the trajectories of the unknown function.

Definition 6.4: Slope Field

A slope field, sometimes called, a direction field, is a graphical representation for a Differential Equation, usually ODE's. By finding the **tangents of the function** at particular points, we can produce a "field" of unit-tangent lines.

In other words, to every point (x, y) in the domain of f, assign a unit vector with slope $\left.\frac{dy}{dx}\right|_{(x,y)}$.



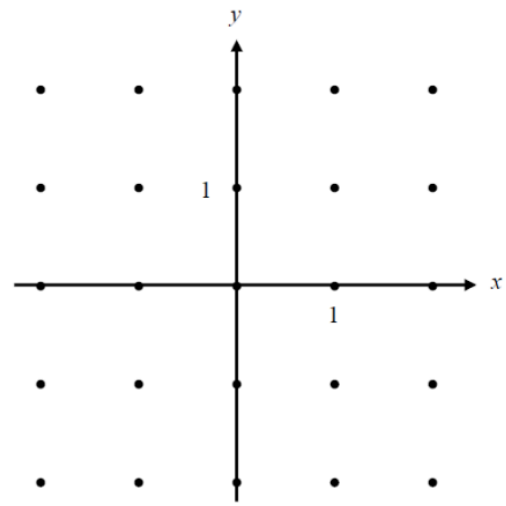


An initial condition determines a particular solution by requiring that a solution curve pass through a given point. If the curve is continuous, this pins down the solution on the entire domain.

If the curve is discontinuous, the initial condition only pins down the continuous piece of the curve that passes through the given point. In this case, the domain of the solution must be specified.

Example 89. Draw a slope field for the differential equation at the indicated points.

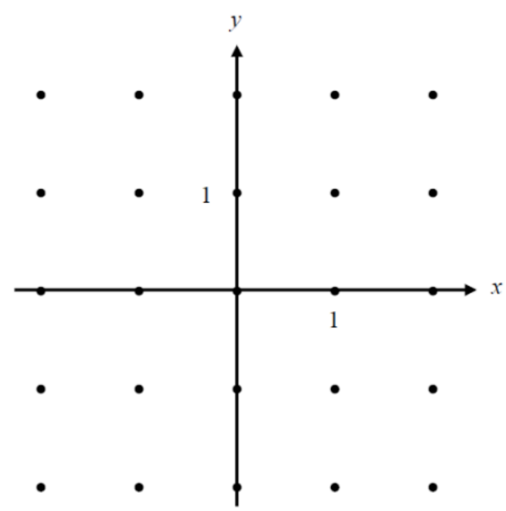
$$\frac{dy}{dx} = x^2$$



Example 90. Draw a slope field for the differential equation at the indicated points.

$$\frac{dy}{dx} = x + y$$

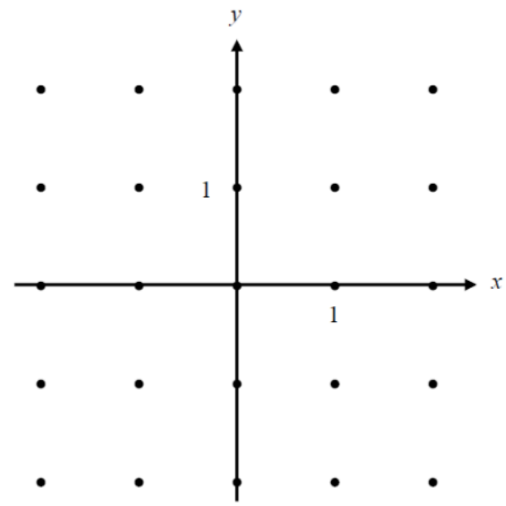
Sketch the solution curve that satisfies the initial value $y(0) = 1$.



DIY 76. Draw a slope field for the differential equation at the indicated points.

$$\frac{dy}{dx} = 2y$$

Sketch the solution curve that passes through the origin. Then, solve the DE using separation of variables.



DIY 77. The slope field for the differential equation $\dfrac{dy}{dx} = \dfrac{x^2y + y^2}{4x + 2y}$ will have vertical segments when

A) $y = 2x$

B) $y = -2x$

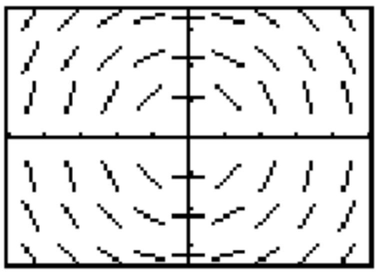
C) $y = -x^2$

D) $y = 0$

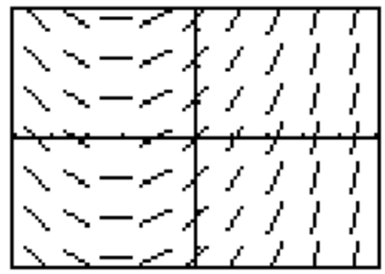
E) C and D

Example 91. Match the following slope field with their differential equation.

(A)

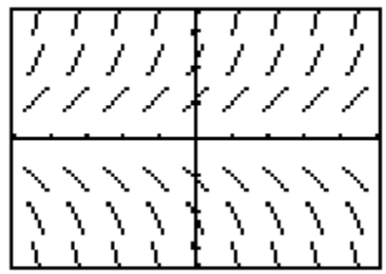


(B)



(C)

(D)



$$\frac{dy}{dx} = \frac{1}{2}x + 1 \qquad \frac{dy}{dx} = x - y \qquad \frac{dy}{dx} = y \qquad \frac{dy}{dx} = \frac{-x}{y}$$

Example 92. The differential equation $\dfrac{dy}{dx} = x + y$ is shown below.

a) Sketch the solution curve through $(0, 1)$.

b) Sketch the solution curve through $(-3, 0)$.

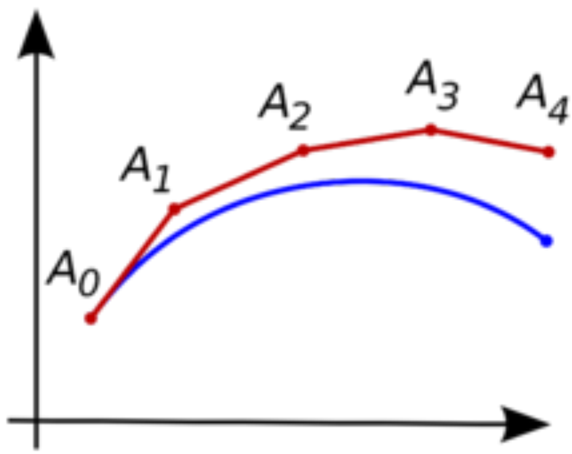
c) Approximate $y(-3.1)$ by using the equation of the tangent line to $y = f(x)$ at the point $(-3, 0)$.

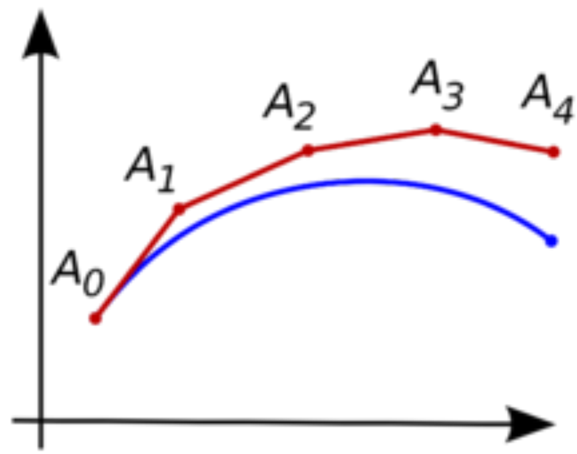
Euler's Method

Okay, but what if you really really need a numerical solution (model)?! Meanwhile there are many approximation methods and algorithms, such as the Linear Approximation method we discussed in our Linearization section we now hope to find a better approximation method. This method stems from the Linear Approximation method and it comes from the 18th century mathematician that we know and love, EULER.

Definition 6.5: Euler's Method

Euler's Method is a numerical procedure for solving ODE's with a given initial value. This method is a type of "predictor-corrector" method due to the process we take in evaluating the solution.





Instead of walking along the same line the whole time as in a tangent line approximation, we change tangent lines with each step (of length dx). This involves recalculating the point and slope after each step. This will produce a much more accurate approximation than simply using the original tangent line.

Use the following table to as a tool for these problems:

x	y	$m = \left.\frac{dy}{dx}\right\|_{(x,y)}$	$dy = m(dx)$	y_{new}
a	$y(a)$			

1. The problem will usually ask us to take a certain number of steps; define this value to be n.
2. To find the dx, the length of each step, we divide the length the interval by the number of steps:

$$dx = \frac{b-a}{n}$$

Example 93. Given the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$.

a) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps.

b) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

Example 94. If $\frac{dy}{dx} = 2x - y$ and $y = 3$ when $x = 2$, use Euler's method with 3 equal steps to approximate y when $x = 1.7$.

6.3 Exponential Models

DIY 78. Solve the following DE:

$$\frac{1}{x}\frac{dy}{dx} = 4y$$

DIY 79. Solve the following DE:

$$\frac{dy}{dx} + 5y = 20; y(0) = 2$$

Example 95. Solve the following DE:

$$\cos^2 x \cdot \frac{dy}{dx} = \tan x$$

Uninhibited Growth

Exponential growth and decay are most commonly used to model the change of a population, as the rate of population growth or decay is usually proportional to the population itself. For example, as mold(population) growth increases, there is more mold(individuals) capable of reproduction so the rate of increase also increases. Exponential growth and decay can also be used to model an investment with interest compounded continuously, the processes of a solution being diluted by fresh water, a capacitor being discharged, the time of death of an individual, and even the temperature of an object that is cooling(heating).

Definition 6.6: Diff. Eq of Exponential Growth and Decay

Suppose that y is varies at a rate directly proportional to y

$$\therefore \frac{dy}{dt} = ky = k \cdot y(t)$$

where k represents the **rate constant** of the equation.

If $k > 0$, then we have exponential growth. If $k < 0$, then we have exponential decay.

Definition 6.7: Solution of Diff. Eq of Exponential Growth and Decay

Suppose that y is varies at a rate **directly proportional** to y

$$\therefore \frac{dy}{dt} = ky = k \cdot y(t) \text{ ,where } y(t) = y_0 e^{kt}$$

where k represents the **rate constant** of the equation.

If $k > 0$, then we have exponential growth. If $k < 0$, then we have exponential decay.

Example 96. Bacteria in a culture increased from 400 to 1600 in three hours. Assuming that the rate of increase is directly proportional to the population,

a) Find an appropriate equation to model the population.

b) Find the number of bacteria at the end of six hours.

The decay of a radioactive substance also follows this law. As the amount of substance decreases, the rate of decay (the change of substance capable of producing radiation) also decreases.

Usually problems might tell us the doubling time of an investment or the half life of a radioactive substance, like Carbon-14, so as a helpful tip:

Theorem 35: Doubling/Halving Relationship

If a problem asks for the doubling rate/time or half life, then use

$$\ln 2 = k \cdot t$$

where $t > 0$ and the sign of k is determined by the circumstance/problem given.

Example 97. Carbon-14 has a half life of approximately 5730 years (every 5730 years, the amount of radioactive substance will be halved). It is often used in carbon dating to find the age of artifacts and fossils since the amount of carbon-14 in the atmosphere is known. Assume that a certain fossil has 30% as much carbon-14 as its present-day equivalent should have. Approximate the age of the fossil.

DIY 80. Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon-14. If the half life of carbon-14 is 5730, approximately when was the tree cut and burned?

Inhibited Growth

Now, in the real world, the quantity is not unlimited, so in many cases, the exponential model is unrealistic as we let $t \to \infty$.

Definition 6.8: Inhibited Growth

If the rate of growth is limited by a constant K, called the carrying capacity, then we can model the rate of growth as

$$\frac{dy}{dt} = r(K - y), r > 0.$$

The Inhibited Growth model can model the sales of a newly advertised product, in which case there exists a maximum limit of the product sales. It can also model the processes of an object cooling down to a certain temperature. Other cases include the processes of a solution being diluted by another of different concentration, a capacitor being charged, and certain learning patterns.

Definition 6.9: Solution of Diff. Eq of Inhibited Growth

If the rate of growth is limited by a constant K, called the carrying capacity, then we can model the rate of growth as

$$\therefore \frac{dy}{dt} = r(K - y), r > 0, \text{ where } y(t) = K + (y_0 - K)e^{-rt}$$

Example 98. According to Newton's law of cooling, the rate at which an object cools (or warms) is directly proportional to the temperature difference between the environment and the object itself. If a pot of boiling water (100^0C) is left at room temperature (22^0C) and after five minutes the water is only 70^0C, find its temperature after another 5 minutes.

DIY 81. Police arrive at the scene of a murder at 12 am. They immediately take and record the body's temperature, which is 90^0, and thoroughly inspect the area. By the time they finish the inspection, it is 1:30 am. They again take the temperature of the body, which has dropped to 87^0, and have it sent to the morgue. The temperature at the crime scene has remained steady at 82^0. When was the person murdered?

6.4 Logistic Models

We have discussed cases in which the rate of change of quantity P is either directly proportional to itself (P), or to its remaining room for growth ($L - P$). Logistic growth deals with growth rates that are directly proportional to both of these quantities.

Definition 6.10: Diff Eq. of Logistic Model

If the rate of growth is directly proportional itself and its remaining room of growth, then we can model the rate of growth as

$$\frac{dy}{dt} = r'y(L - y)$$

or alternatively as

$$\frac{dy}{dt} = ky(1 - \frac{y}{L}), \text{ where } k = r'L$$

Here L is our carrying capacity and k is the intrinsic growth rate.

As y nears L, the rate will shrink toward 0 , resulting in an S-shaped curve and the population will become stable. If y were to exceed L, then the rate would become negative and the population would decrease towards the carrying capacity.

Definition 6.11: Solution to Diff Eq. of Logistic Model

If

$$\frac{dy}{dt} = r'y(L-y)$$

then

$$y(t) = \frac{L}{1+Ce^{-(Lr')t}}$$

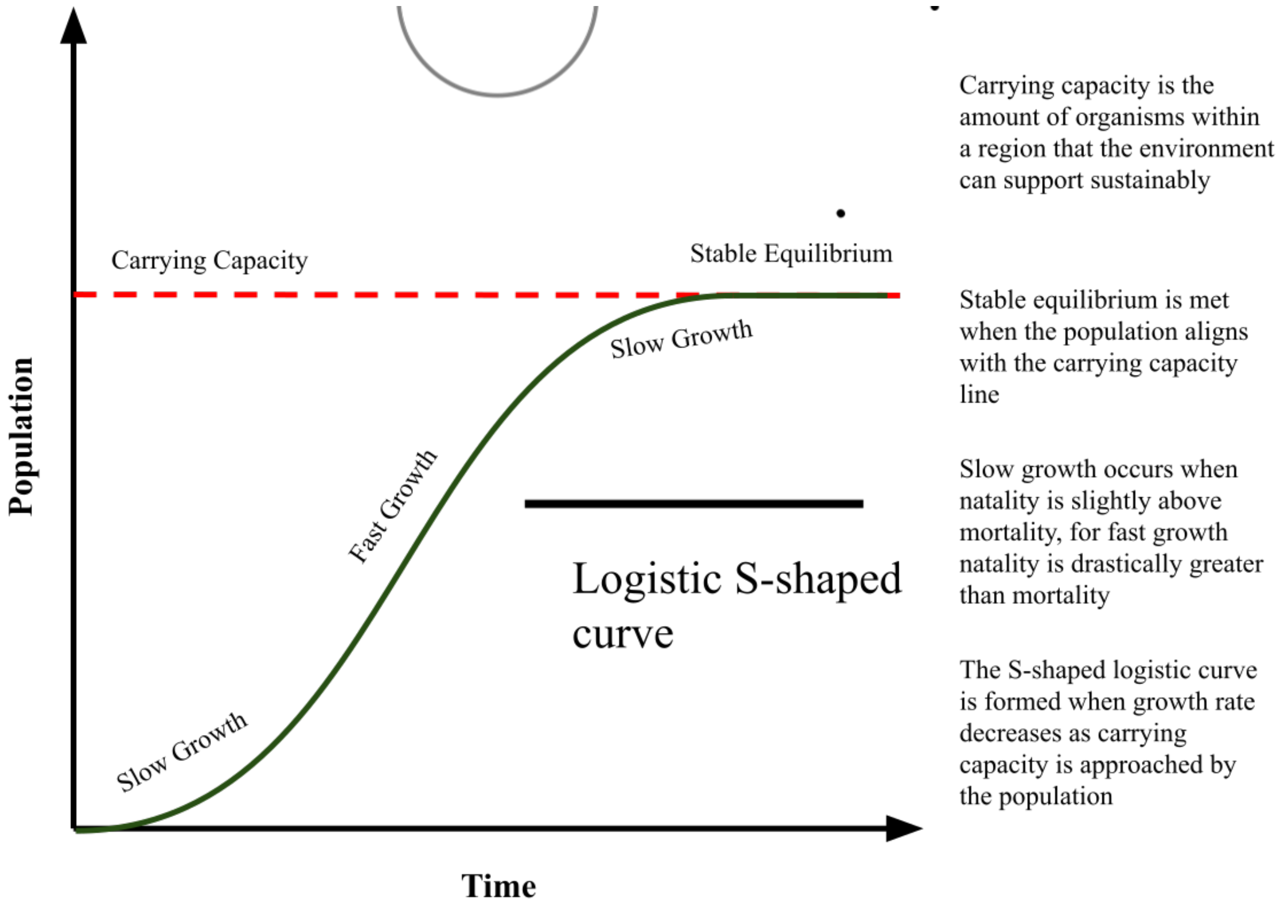
or alternatively as

$$\frac{dy}{dt} = ky\left(1-\frac{y}{L}\right), \text{ where } k = r'L$$

then

$$y(t) = \frac{L}{1+Ce^{-kt}}$$

Here L is our carrying capacity and $k = r'L$ is the intrinsic growth rate.



Chapter 7

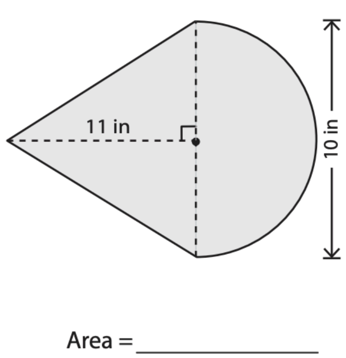
Integration: From Riemann Sums to the Fundamental Theorem of Calculus

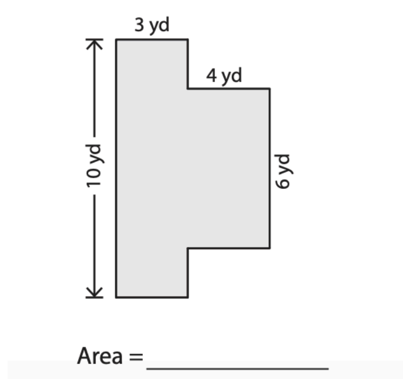
7.1 Riemann Sums → Riemann Integrals

Finding Areas pt.1

We have already discussed the connection between the derivative and the integral. In this final chapter, we will be discussing two more representations for the integral. Through these two concepts, we will gain much more insight into the utility of the integral in real world applications. Integrals can be used to calculate cumulative totals, averages, and areas.

To begin though, let's begin by tackling the following question:
Example 99. Find the area of the following figures

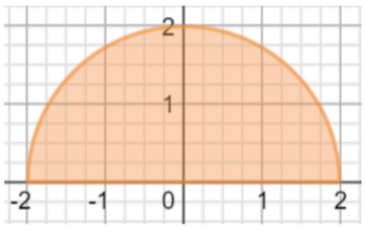


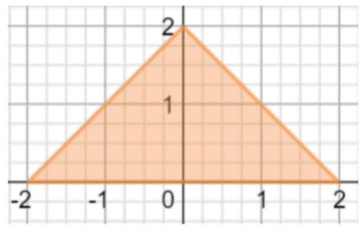


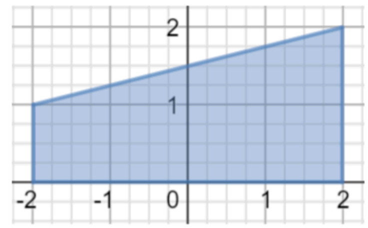
We can make these questions a bit more interesting if we include a coordinate system. Recall, that by definition, one unit of area is the area of a square whose sides are one unit long, so if we were to sum up all of these little squares and use our creativity to make squares from the "smaller" pieces, then we can find the area of any general shaded region.

DIY 82. Use any previous geometry knowledge go answer the following:

Find the area of the following shapes

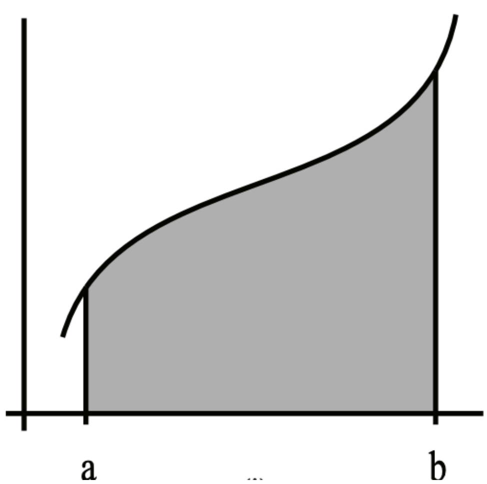


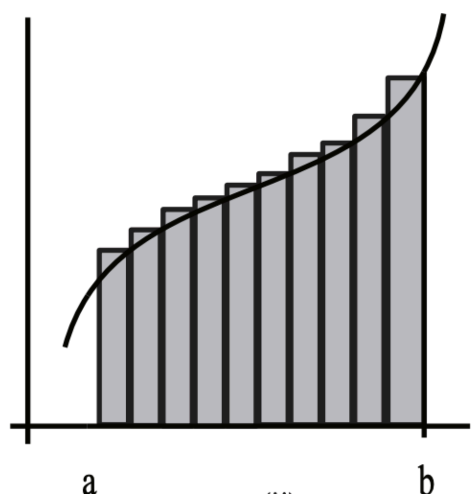




Area: ____________________ Area: ____________________ Area: ____________________

So this process of cutting our region into little squares and counting out all the whole squares that we can create works...but its really inefficient. Mathematicians and scientist, for a long long long time used similar methods to approximate things that were still not fully understood. Luckily for us, Leibniz and Newton did all the heavy lifting for us and other great mathematicians, such as Riemann and Cauchy, made this process a lot more rigorous that it now stands the test of time!





Let us look at what they discovered through an example:
Example 100. What is the area bounded between $y = f(x) = x^2$ and the $x-$axis on the interval $[a, b]$, where $a = 0$ and b is arbitrary.

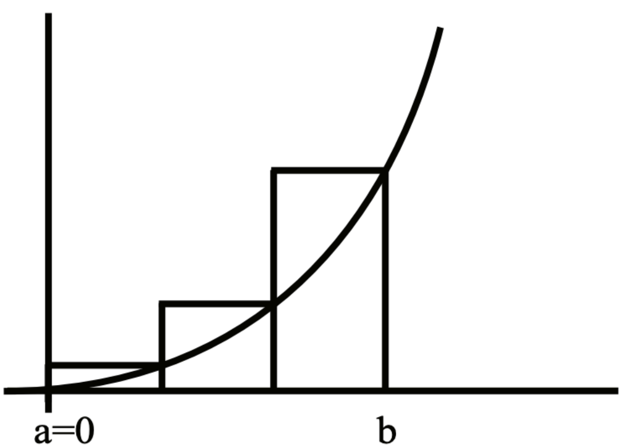
1. Divide into n intervals. (Length b/n = base of rectangle)
2. Heights:
   (a) 1st: $x_0 = \dfrac{b}{n}$, height$= \left(\dfrac{b}{n}\right)^2$
   (b) 2nd: $x_1 = \dfrac{2b}{n}$, height$= \left(\dfrac{2b}{n}\right)^2$
   (c) 3rd: $x_2 = \dfrac{3b}{n}$, height$= \left(\dfrac{2b}{n}\right)^2$
   (find and establish the patterns) ...
3. Sum of areas of rectangles:

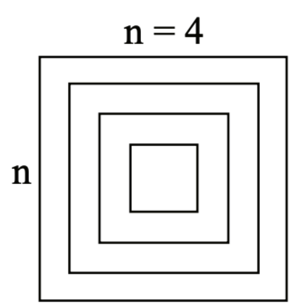
$$\left(\frac{b}{n}\right)\cdot\left(\frac{b}{n}\right)^2+\left(\frac{b}{n}\right)\cdot\left(\frac{2b}{n}\right)^2+\left(\frac{b}{n}\right)\cdot\left(\frac{3b}{n}\right)^2+\ldots+\left(\frac{b}{n}\right)\cdot\left(\frac{(n-1)b}{n}\right)^2+\left(\frac{b}{n}\right)\cdot\left(\frac{n\cdot b}{n}\right)^2=$$

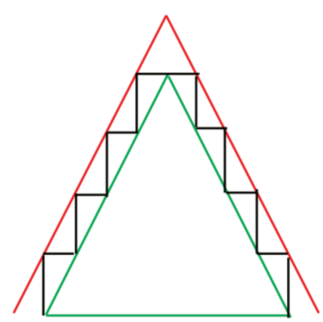
$$\left(\frac{b^3}{n^3}\right)\cdot\left(1^2+2^2+3^2+\cdots+n^2\right)$$

4. Find the limit as $n \to \infty$.

We will need estimate the sum using some 3-Dimensional thinking.







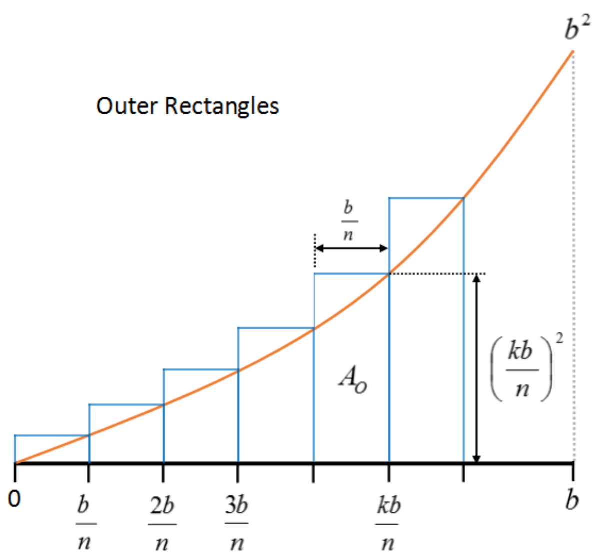
With this we now have

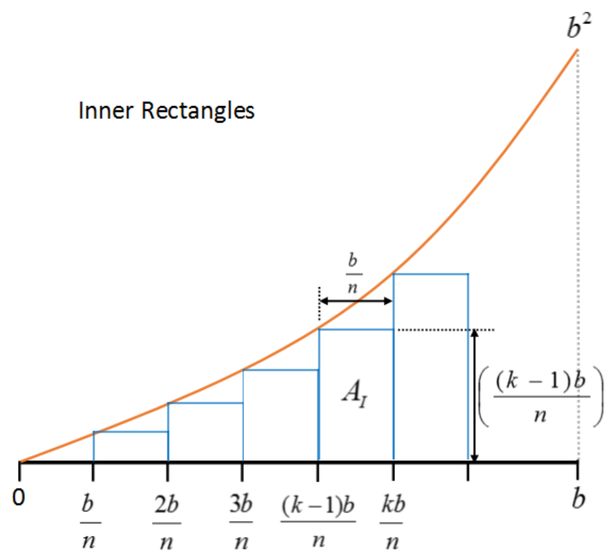
$$\frac{\frac{1}{3} \cdot n^3}{n^3} < \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3} < \frac{\frac{1}{3} \cdot (n+1)^3}{n^3}$$

Therefore,

$$\lim_{n \to \infty} \frac{b^3}{n^3} \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) =$$

Underneath step 4, we cleverly disguised the following two approximations and then let $n \to \infty$:





In this case, k serves as a **COUNTER VARIABLE** because it is keeping count of how many times we have added our evenly divided width.

DIY 83. So, what does it really mean to let $n \to \infty$? and how does our solution to the above example fit with this explanation?

Definition 7.1: Rectangular Approximation Method (R.A.M)

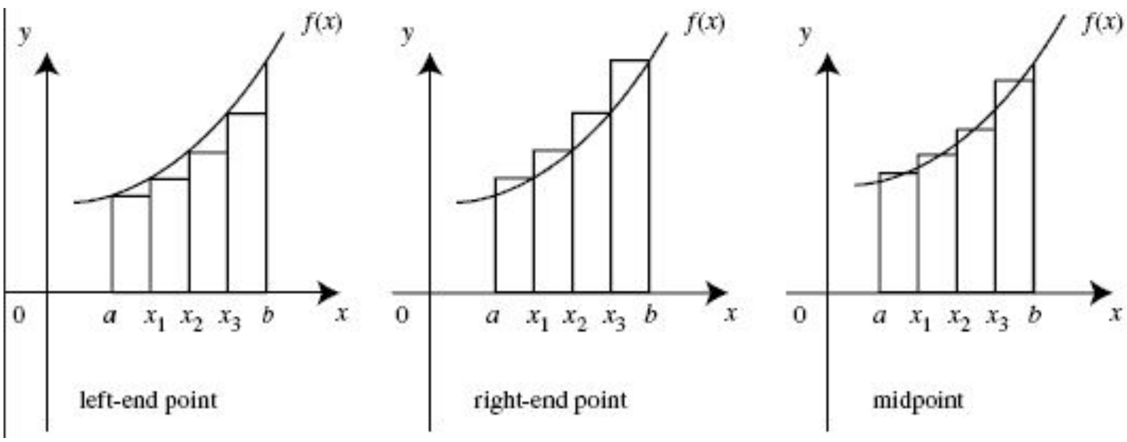
Suppose we wanted to know the area of the region bounded by a curve, the $x-$axis, and the lines $x = a$ and $x = b$.

The first step is to divide the interval from a to b into subintervals, we can then draw rectangles using the width of each subinterval as the base.

$$\Delta x = \frac{(b-a)}{n} = \text{width of individual interval}$$

The height of each rectangle is determined by the function value at a point in the specific subinterval, and can be determined using three different methods.

We could use the left endpoint of each subinterval (called **LRAM**), the right endpoint of each subinterval (called **RRAM**), or the midpoint of each subinterval (called **MRAM**).



Nonetheless, we plug in the following sequence based on the selected method:

$$a = x_0, x_1, x_2, \ldots, x_n = b$$

Such that

$$f(x_k) \cdot \Delta x =$$

$$\text{(height at } k^{th} \text{ location)} \cdot (\text{ width of individual interval})$$

And the sum of areas becomes

$$f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \ldots + f(x_n) \cdot \Delta x$$

Notice that the sum runs through the n values.

EACH METHOD WILL COME WITH A CERTAIN AMOUNT OF ERROR, THE GOAL IS TO FIND THE LEAST AMOUNT OF ERROR.

With this new definition, let us use this approach to find an approximation.
Example 101. What is the area bounded between $y = f(x) = x^2$ and the $x-$axis on the interval $[a, b]$, where $a = 0$ and $b = 2$. Estimate the area with $n = 4$ subintervals by finding:

a) a left Riemman Sum (LRAM)

b) a right Riemman Sum (RRAM)

c) a midpoint Riemman Sum (MRAM)

d) Average Approximation (TRAP)

Additonally, determine the order of approximations.

DIY 84. What is the area bounded between $y = f(x) = x^2$ and the $x-$axis on the interval $[a, b]$, where $a = 1$ and $b = 5$. Estimate the area with $n = 4$ subintervals by finding:

a) a left Riemman Sum (LRAM)

b) a right Riemman Sum (RRAM)

c) a midpoint Riemman Sum (MRAM)

d) Average Approximation (TRAP)

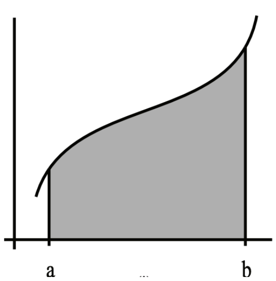
Additionally, determine the order of approximations.

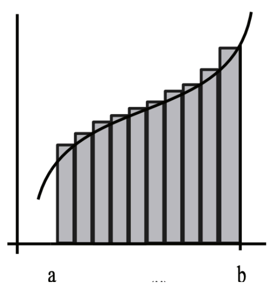
Finding Areas pt.2

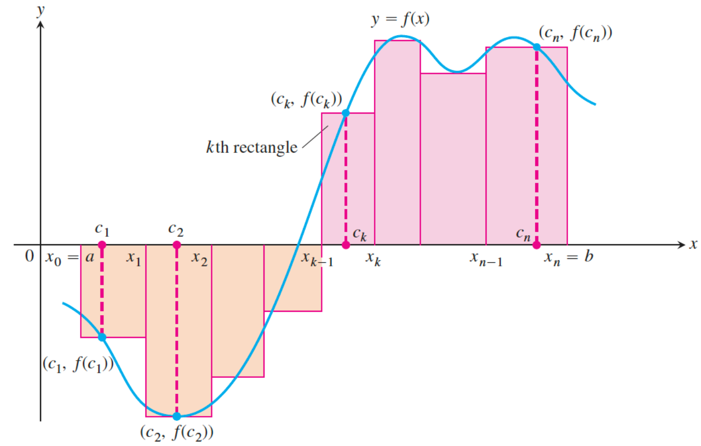
We now have a understanding of how the "3-Dimensional" part of our initial example was just the consideration of the different types of approximation method. (**LRAM,RRAM,MRAM**) For all of these methods, we have assumed either uniform widths as well as fixed locations for the endpoints of our approximations. With this under our belts, let us now dive a bit deeper into the story:

What is the area bounded between a positive function $y = f(x)$ and the $x-$axis on the interval $[a, b]$, where a and b are arbitrary?

We will now tackle the most GENERAL idea and hope to find the most GENERAL solution.



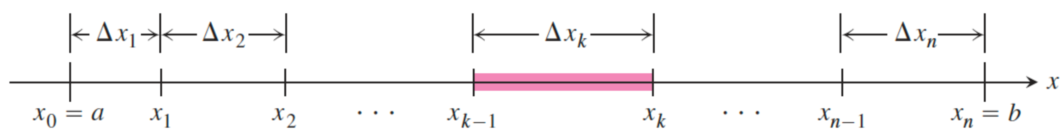




Bernhard Riemann, was the first mathematician to attempt creating a rigorous definition behind these ideas because he noticed that there was a nice generalization once we took $n \to \infty$.

Here is a summary of Riemann's conclusions:

1. **(PARTITIONS)** Suppose we have have an interval $[a, b]$. We can choose to divide it nicely so that the width of our approximation is uniform OR we can take individual widths, such that we have individual locations for each c_k and for each individual width, call them Δx_k.



   If this is the case, k corresponds to the individual **location** and the individual **width**.

2. **(INDIVIDUAL AREAS)** We can thus form the product $f(x_k) \cdot \Delta x_k$ for each sub-interval in our partition. This product can be postive, negative, or zero, depending on the value of $f(c_k)$.

3. **(SUMMING UP THE PIECES)** Define the sum to equal S_n.

$$f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 + f(c_3) \cdot \Delta x_3 + \ldots + f(c_{n-1}) \cdot \Delta x_{n-1} + f(c_n) \cdot \Delta x_n = S_n$$

$$S_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

4. **(TAKE IT TO THE LIMIT)** Find the limit of S_n:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k = \int\limits_a^b f(x)\,dx$$

Definition 7.2: Definite Integral

Let f be a function defined on a closed interval $[a,b]$. For **any** partition on $[a,b]$, and **any** arbitrary c_k chosen within each sub-interval, we define the **Definite Integral** as the limit of taking sums:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k = \int\limits_a^b f(x)\,dx = I$$

Where I is a real number.

Assuming equal width, $\Delta x = \dfrac{(b-a)}{n}$, and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x = \int\limits_a^b f(x)\,dx = I$$

Where $\int\limits_a^b f(x)\,dx$ is read as "the integral from a to b of f of x dee x" and it is always equal to some value I as long as the interval widths go to zero.

Notice how Greek $\to$ Roman.

Greek $\sum$ becomes Roman $\int$ (S).

Greek Δ becomes Roman d.

Theorem 36: Existence of Definite Integrals

All continuous functions f are integrable.

In other words, if a function is continuous on an interval $[a,b]$, then its definite integral over $[a,b]$ exists and is well defined.

Now, this is marvelous news, but let us compare this to what we know about the derivative:

Continuity and The Derivative **Continuity and Integrals**

Sums $\rightleftarrows$ Integrals

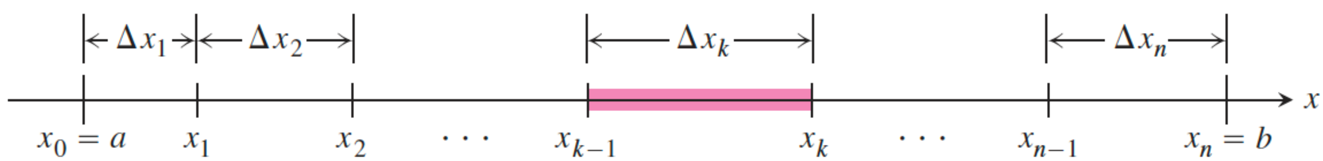
AP L.O.V.E.S using the notation we just went over and enjoys asking you to convert Sums to Integrals, and vice-versa. Here is a collection of examples. I recommend you review these questions before the exam. That being said, take good notes that future you will appreciate and understand.

There are two common ways to write a definite integral as sums:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k \qquad \textbf{OR} \qquad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} f(a+i\Delta x)\cdot \Delta x$$

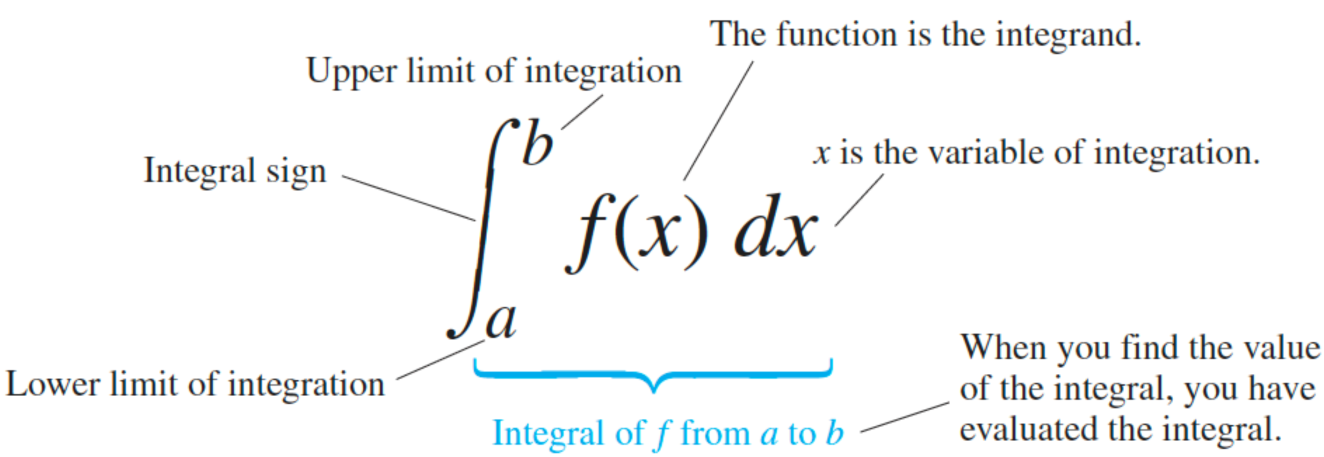
In the latter we assume uniform width $\Delta x = \dfrac{(b-a)}{n}$.

$f(a+i\Delta x)$ comes about from the sub-intervals.



More explicitly, we hope to re-write all sums to the following form:

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right) = \int_a^b f(t)\, dt.$$



This being said, let's do some examples.
Example 102. Write the following sums as definite integrals.

1. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(\frac{k}{n}\right) \cdot \frac{1}{n}$

2. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left[\left(2+\frac{3k}{n}\right)^2 - 2\left(2+\frac{3k}{n}\right)\right]\frac{3}{n}$

3. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left(\frac{6n}{9n^2+4k^2}\right)$

4. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n\left(2+\frac{i}{n}\right)\ln\left(2+\frac{i}{n}\right)}$

5. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left[4\left(-3+\frac{5k}{n}\right)^3 + 3\left(3+\frac{5k}{n}\right) + 6\right]\frac{5}{n}$

6. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{\sin(b/n) + \sin(2b/n) + \sin(3b/n) + \sin(4b/n) \ldots + \sin((n-1)b/n) + \sin(nb/n)}{n}$

7. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \dfrac{2^{(b/n)} + 2^{(2b/n)} + 2^{(3b/n)} + 2^{(4b/n)} \ldots + 2^{((n-1)b/n)} + 2^{(nb/n)}}{n}$

8. $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \sqrt{\left(4 + \dfrac{5i}{n}\right)} \cdot \dfrac{5}{n}$

Example 103. Express the following definite integrals as limits of Riemann Sums.

1. $\int\limits_2^5 \left(2x^2 - \dfrac{5}{x}\right)\, dx$

2. $\int\limits_6^{10} 4x \ln x \; dx$

3. $\int\limits_2^5 \dfrac{1}{5}x^2 \; dx$

4. $\int\limits_0^\pi \cos(x)\, dx$

5. $\int\limits_0^\pi e^x\, dx$

6. $\int\limits_1^e \ln x\, dx$

7.2 FTC 1 & Definite Integrals

In our previous section, we noted how the definite integral is defined through the process of creating an approximation with n rectangles and then taking the limit as $n \to \infty$.

In this section we will take this as a definition and discuss a few important properties that come about about the Integral.

To begin, we noted that **ALL** continuous functions are integrable, so let us see what happens when we attempt to integrate different types of continuous functions.

Definite Integrals from Common Geometric Equations

Example 104. Sketch the region corresponding to each definite integral. Then calculate each integral using a geometric formula.

a) $\int\limits_1^3 4\, dx$

b) $\int\limits_{-1}^1 |x|\, dx$

c) $\int\limits_0^3 (x+2)\, dx$

d) $\int\limits_{-2}^2 \sqrt{(4-x^2)}\, dx$

e) $\int\limits_{0}^{2} (2x + 5)\ dx$

f) $\int\limits_{-1}^{1} (1 - |x|)\ dx$

DIY 85. Sketch the region corresponding to each definite integral. Then calculate each integral using a geometric formula.

a) $\int\limits_{-2}^{2} -\sqrt{(4 - x^2)}\ dx$

b) $\int\limits_{-1}^{1} (1 - |x|)\ dx$

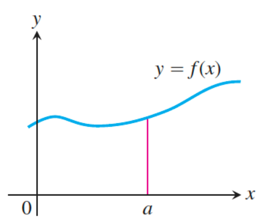
c) $\int\limits_{2}^{6} -\sqrt{(4 - (x-4)^2}\ dx$

d) $\int\limits_{4}^{6} \sqrt{(4 - (x-4)^2}\ dx$

General Properties

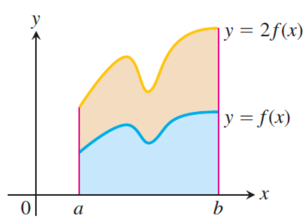
The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums. The graphical representation of these properties has also been included.

Rules for Definite Integrals			
#	**Rule**	**Notation**	**Statement**
1.	Order of Integration	$\int_a^b f(x)dx = -\int_b^a f(x)dx$	If you reverse the order of integration, you get the opposite answer.
2.	Zero	$\int_a^a f(x)dx = 0$	This should make sense if you think about the area of a rectangle with no width.
3.	Constant Multiple	$\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$	Taking the constant out of the integral many times makes it simpler to integrate.
4.	Sum and Difference	$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$	This allows you to integrate functions that are added or subtracted separately.
5.	Additivity	$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$	Pay close attention to the limits of integration ... this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.



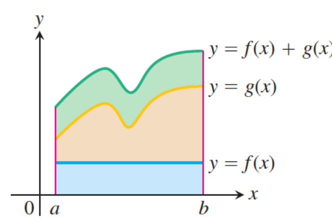
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



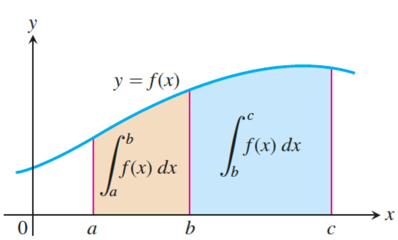
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



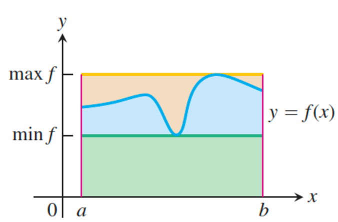
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



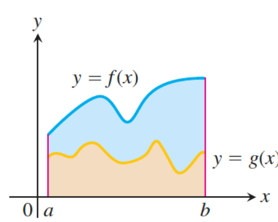
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx$$
$$\le \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Example 105. Given $\int\limits_2^6 f(x)\,dx = 10$ and $\int\limits_2^6 g(x)\,dx = -2$. Find the following:

a) $\int\limits_2^6 [f(x) + g(x)]\,dx$

b) $\int\limits_2^6 [g(x) - f(x)]\,dx$

c) $\int\limits_2^6 3f(x)\,dx$

d) $\int\limits_6^2 [3f(x) + 2g(x)]\,dx$

DIY 86. Given $\int\limits_0^5 f(x)\,dx = 10$ and $\int\limits_5^7 f(x)\,dx = 3$. Find the following:

a) $\int\limits_0^7 f(x)\,dx$

b) $\int\limits_5^0 f(x)\,dx$

c) $\int_{5}^{5} -4f(x)\,dx$

d) $\int_{5}^{7} 2f(x)\,dx$

Fundamental Theorem of Calculus pt.1 (Evaluation)

We present the Fundamental Theorem of Calculus, which is the central theorem of integral calculus. It connects integration and differentiation, enabling us to compute integrals using an antiderivative of the integrand function rather than taking limits of Riemann sums or having to use a calculator. The theory of calculus is fully encapsulated in the Fundamental Theorem and it traditionally comes in two parts. The first, we will call the evaluation rule of the FTC.

Definition 7.3: FTC pt.1

If f is continuous at every point in $[a,b]$ and F is any antiderivative of f on $[a,b]$, then

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a)$$

The Evaluation Theorem is important because it says that to calculate the definite integral of f over an interval $[a,b]$ we need do only two things:

1. Find an antiderivative F of f
2. Calculate the number $F(b) - F(a)$, which is equal to $\int_{a}^{b} f(x)\,dx$

Example 106. Evaluate the following integrals:

a) $\int\limits_0^{\pi} \cos(x)\, dx$

b) $\int\limits_0^{\pi} \sin(x)\, dx$

c) $\int\limits_0^{1} \frac{dx}{x+1}$

d) $\int\limits_0^{1} \frac{dx}{x^2+1}$

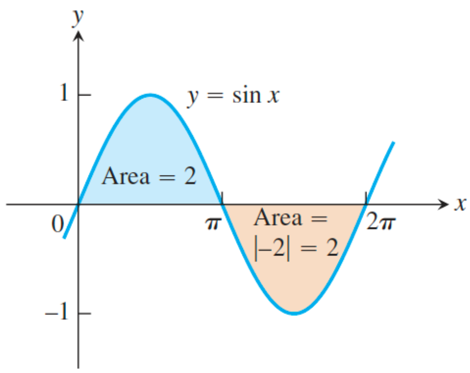
DIY 87. .

1. Evaluate the following indefinite integral:

$$\int \sin(x)\, dx$$

2. Evaluate the definite integral and compare your solution to the following image and identify the difference and justify why this is the case:

$$\int_0^{2\pi} \sin(x)\, dx$$



The previous example tells us that the definite integral is NOT the **TOTAL AREA**, instead, it is the **NET AREA**: the difference between the area **ABOVE** the $x-$axis and the area **BELOW** the $x-$axis.

Definition 7.4: Total Area

To find the area between the graph of $f(x)$ and the $x-$axis over the interval $[a, b]$:

1. Find all the zeros of f on the interval $[a, b]$.
2. Integrate f over each subinterval.
3. Add the absolute value of each integral together.

DIY 88. Find the area of the region between the $x-$axis and the graph of

a) $f(x) = x^3 - x^2 - 2x, -1 \leq x \leq 2$

b) $f(x) = |x| - 2, -2 \leq x \leq 2$

c) $f(x) = \cos(x^2), -\pi \leq x \leq \pi$

These 3 problems also show us that we can use our Pre-Calculus skills in a clever way. We can use factoring and even/odd properties to simply a complex looking integrals.

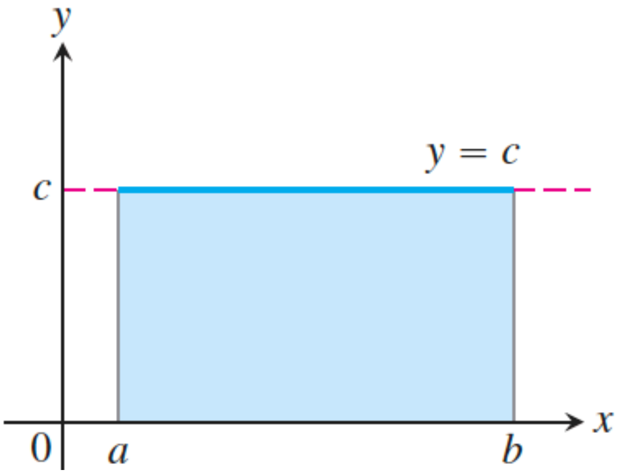
Average Value of a Function

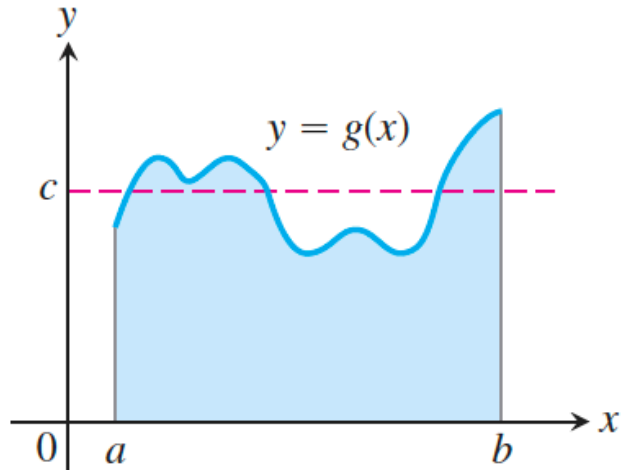
The temperature at a certain location in a town is a continuous function that goes up and down each day. What does it mean to say that the average temperature in the town over the course of a day is 73 degrees?

The average value of a collection of n numbers is obtained by adding them together and dividing by n.

But what is the average value of a continuous function f on an interval $[a, b]$? Such a function can assume infinitely many values.

We can use the following reasoning. We start with the idea from arithmetic that the average of n numbers is their sum divided by n. A continuous function f on $[a, b]$ may have infinitely many values, but we can still sample them in an orderly way. When a function is constant, this question is easy to answer, but if it is non-constant, then the story changes a bit.





A function with constant value c on an interval $[a, b]$ has average value c. When c is positive, its graph over $[a, b]$ gives a rectangle of height c. The average value of the function can then be interpreted geometrically as the area of this rectangle divided by its width $b - a$.

Alternatively, we can use the following reasoning for the non-constant case. We can think of this graph as a snapshot of the height of some water that is sloshing around in a tank between enclosing walls at $x = a$ and $x = b$. As the water moves, its height over each point changes, but its average height remains the same. To get the average height of the water, we let it settle down until it is level and its height is constant.

Since the average value of a collection of n numbers is obtained by adding them together and dividing by n. For functions, we take a Riemann Sum and recall that $n = b - a$

Definition 7.5: Average Value of a Function (Mean)

If f is integrable on $[a, b]$, then its average value on $[a, b]$, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{F(b) - F(a)}{b - a}$$

Do you think that there is any point during the day that the temperature reading on the thermometer is the exact value of the average temperature? Is there a point in the above graphs that equal the average value c?

This line of reasoning leads us to the Mean Value Theorem for Integrals:

Definition 7.6: MVT for Integrals

If f is integrable on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

This is an existence theorem. It tells us that c exists, but does not help you find it.

AROC $\neq$ Average value of f

DIY 89. (2014 AP Calculus AB Free Response Question #1)

Grass clippings are placed in a bin, where they decompose. For $0 \leq t \leq 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

a) Find the average rate of change of $A(t)$ over the interval $0 \leq t \leq 30$. Indicate units of measure.

b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

c) Find the time t for which the amount grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \leq t \leq 30$.

d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pounds of grass clippings remaining in the bin. Show the work that leads to your answer.

7.3 FTC 2 & Accumulation Functions

Recall the following:

Theorem 37: FTC pt.1

If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The Evaluation Theorem is important because it allows us to find definite integrals by finding the values of the antiderivative at each endpoint.

Theorem 38: Existence of Definite Integrals

All continuous functions f are integrable.

In other words, if a function is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists and is well defined.

Now, it is true that ALL continuous functions have antiderivatives which we can use to find the **NET AREA under the curve**.

However, the following functions are examples of functions that we CANNOT find the antiderivative to directly:

$$\int e^{-x^2}\,dx, \qquad \int \frac{\sin x}{x}\,dx, \qquad \int \sin(x^2)\,dx, \quad \text{and} \int \cos(x^2)\,dx$$

Because of the previous theorem we know that there must exist some solution to these 4 integrals, but we do **NOT** have **COMMON FUNCTIONS** that will equal the antiderivative of each.

(Go ahead and look at your repertoire of functions and integration methods; none of them apply.)

YES, these antiderivatives do exist. NO, it is not a function that we have met before: IT IS A NEW FUNCTION.

This new function is defined as

$$F(x) = \int_0^x e^{-t^2}\,dt$$

Such that the following property holds[1] :

$$F'(x) = e^{-x^2}$$

Where t is a dummy variable (place holder...because we need it there to define a integral.)

[1]Notice the roles of the x and t variables.

The other three integrals are also examples of how we can define the integrals as functions by stating that the derivative equals the function being integrated. This process is so useful because it not only gives us solutions to unsolvable integrals, but it also shows us how the derivative and integral are indeed related and how this relationship can be used. For this reason, we call this conclusion and overall process the **Fundamental Theorem of Calculus pt. 2**

Theorem 39: FTC 2

Let f be a function that is continuous on a closed interval $[a, b]$. The function F defined as

$$F(x) = \int_a^x f(t)\, dt$$

then

$$F'(x) = f(x)$$

more explicitly:

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$$

In other words, the **derivative** of ANY integral defined function is equal to the function inside of the integral $f(x)$.

Example 107. Use the Fundamental Theorem of Calculus to find dy/dx.

a) $\displaystyle\int_1^x \frac{1}{t^2}\, dt$

b) $\displaystyle\int_a^x \sqrt{t+1}\, dt$

c) $\displaystyle\int_2^x \frac{s^3-1}{2s^2+s+1}\, ds$

d) $\displaystyle\int_4^{3x^2+1} \sqrt{e^t+t}\, dt$

DIY 90. Use the Fundamental Theorem of Calculus to find dy/dx.

a) $\displaystyle\int_a^x (t^3+1)\,dt$

b) $\displaystyle\int_x^5 3t\sin t\,dt$

c) $\displaystyle\int_1^{x^2} \cos(\theta)\,d\theta$

d) $\displaystyle\int_{3x^2+1}^5 \frac{1}{e^t+3}\,dt$

Example 108. Solve the following:

a) $\displaystyle\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f'(t)\,dt\right]$

b) $\displaystyle\frac{d}{dx}\left[\int_{x^2}^{2x} f(t)\,dt\right]$

In conclusion, this means that the FTC2 can be extended as follows:

Theorem 40: FTC2 w/ Chain Rule

If f is continuous then

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right] = f(g(x))g'(x) - f(h(x))h'(x)$$

DIY 91. Evaluate the following:

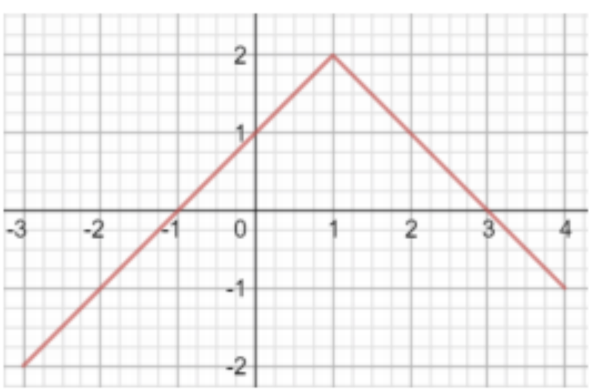
a) $\dfrac{d}{dx}\left[\displaystyle\int_{6x}^{4x^2} \sqrt{1+t^4}\,dt\right]$

b) $\dfrac{d}{dx}\left[\displaystyle\int_{\cos x}^{x^4} \sqrt{2-u}\,du\right]$

The following example shows us what it means to define an integral function.

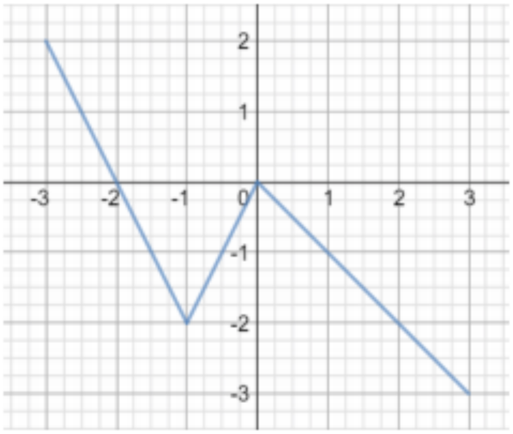
Example 109. .

Let $g(x) = \displaystyle\int_{-2}^{x} w(t)\,dt$ where the graph of $w(t)$ is given below.



Find $g(0)$, $g(2)$, and $g(-3)$.

DIY 92. The function $f(t)$ is graphed below and $g(x) = \displaystyle\int_{-1}^{x} f(t)\, dt$.



a) Complete the table.

x	-3	-2	-1	0	1	2	3
g(x)							

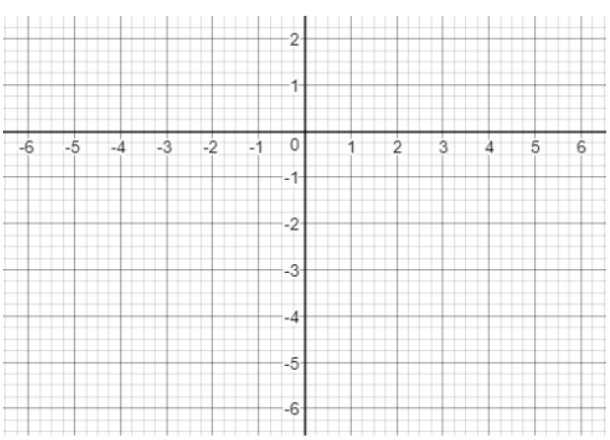
b) What are the intervals on which g is increasing or decreasing? Justify your response.

c) What are the intervals on which g is concave up or concave down? Justify your response.

d) For what values of x does g have a relative minimum or maximum? Justify your response.

e) For what values of x does g have an inflection point? Justify your response.

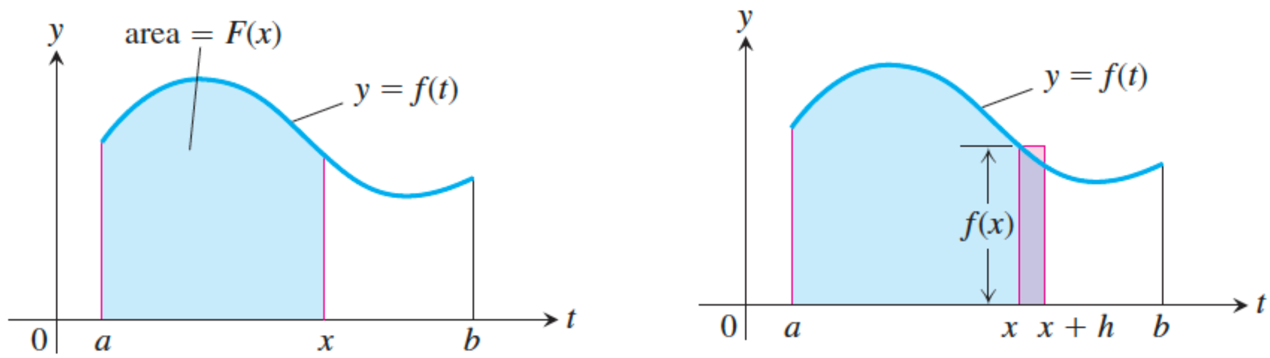
f) Graph $g(x)$.



We will briefly discuss a geometrical proof of FTC2:

Since $F(x)$= AREA under the curve between a and x we have the following graphic. The second image considering what happens when we increase x by a small increment h.

The question is: What is change in $F, (\Delta F)$, after we move x by h units?



Therefore

$$\Delta F = F(x+h) - F(x)$$

$$\Delta F \approx (\text{ base })(\text{ height }) \approx (h) \cdot f(x)$$

After re-writing:

$$\frac{\Delta F}{h} \approx f(x)$$

Thus,

$$\lim_{h\to 0} \frac{\Delta F}{h} = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

But, by the definition of the derivative,

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = F'(x) \text{ and } \therefore F'(x) = f(x)$$

Because of this we can view an integral defined function as an **Accumulation Function** if f continuous on a closed interval $[a, b]$ since its accumulating all the area up to the value of x. In this case, the farther x is from a, the more we have accumulated, and the closer x is from a, implies we have accumulated a little bit. The fact that our problems revolve around a closed interval $[a, b]$ means that a is our starting accumulation point and b is our stopping point.

Theorem 41: Accumulation Function

Let f be a function that is continuous on the closed interval $[a, b]$. The accumulation function F associated with f over the interval $[a, b]$ is defined as

$$F(x) = \int_a^x f(t)dt, \qquad a \leq x \leq b$$

From FTC2, we were able to establish that $F'(x) = f(x)$ as long as F is an antiderivative of f. If this is true, this means we can re-write FTC1 (Evaluation Theorem) as follows:

By FTC2

$$F'(x) = f(x)$$

therefore,

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$$

In other words:

Theorem 42: Net Change Theorem

The net change in a function $F(x)$ over a closed interval $[a, b]$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

This Net Change Theorem in conjunction of our Accumulation Function Theorem tells us the following:

$$\int_a^x f'(t)\,dt = f(x) - f(a) \qquad \text{Accumulation = Final - Initial}$$

$$f(a) + \int_a^x f'(t)\,dt = f(x) \qquad \text{Initial + Accumulation = Final}$$

Example 110. The population of the United States is growing at a rate of $P'(t) = 2.867(1.009)^t$ and is measured as million people per year, where t is the number of years since 2015.

a) Find and interpret: $\displaystyle\int_0^6 P'(t)dt$

b) If the population of the United States was 320 million people in 2015, what is the projected population in 2021?

DIY 93. An investment in a hedge fund is growing at a continuous rate of $A'(t) = 1105.17(1.105)^t$ dollars per year.

a) Find and interpret in the context of the problem: $\displaystyle\int_0^{10} A'(t)dt$

b) If initially \$1000 is invested, what is the value of the investment after 10 years?

PVA Round 2

Recall that in our discussion of PVA in the past we realized the following:

Definition 7.7: Position, Velocity, Acceleration (Mathematician Version)

- **Position Function:** A parametrized function relating the position of a moving object with respect to time. Usually denoted as follows:

$$x(t) \text{ or } s(t)$$

- **Velocity Function:** The derivative of the position function. Usually denoted as follows:

$$x'(t) = v(t)$$

- **Acceleration Function:** The derivative of the velocity function OR the second derivative of the position function. Usually denoted as follows:

$$x''(t) = v'(t) = a(t)$$

- **Speed**: The absolute value of velocity.

$$\text{Speed } = |v(t)|$$

Reflecting the parts of the velocity graph that lie below the $t-$axis will give you the graph of the speed.

Based on the conclusion and relationships we have made thus far, we can now update these definitions as follows:

To Find	Verbally	Mathematically
Displacement (Change in Position)	Integrate the rate of change over the interval	$\int_a^b v(t)dt$
Distance Traveled	Integrate the speed over the interval *Recall that speed is the absolute value of velocity	$\int_a^b speed\ dt = \int_a^b \lvert v(t)\rvert dt$
New Position	Old positon + change in position	$s(b) = s(a) + \int_a^b v(t)dt$

Displacement v. Distance: The difference between these two concepts is vital to understand as they are distinct. To aid with this distinction either think of the following phrase or think of the following graphic.

"Two steps forward and one step back"

Example 111. Suppose the velocity of a particle moving along the $x-$axis is given by

$$v(t) = 6t^2 - 18t + 12, \qquad 0 \leq t \leq 2$$

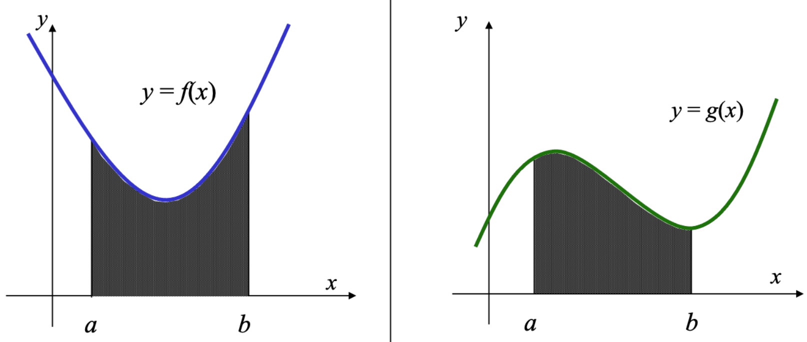
a) When is the particle moving to the right? When is the particle moving to the left? When is it stopped?

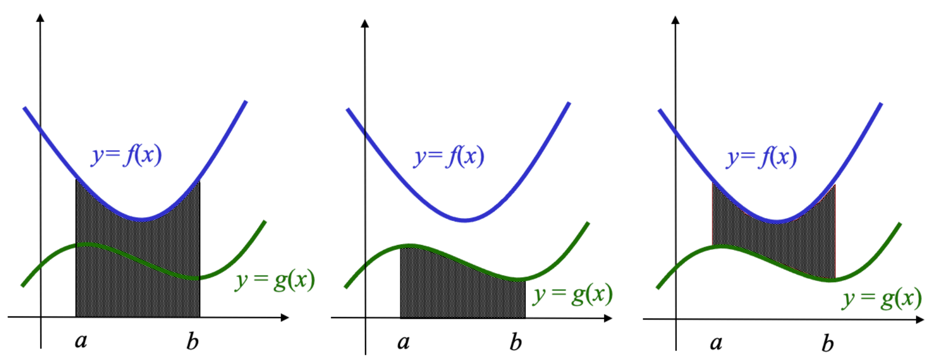
b) Find the particle's displacement over the time interval.

c) Setup, but do not solve, an integral to find the particle's total distance without using absolute value. Find the total distance using a calculator.

7.4 Area Between Curves

General Idea





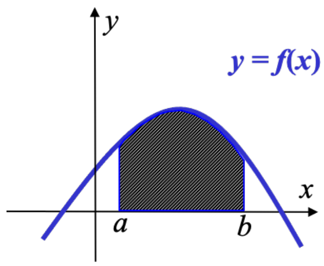
Area under $f(x)$ = $\int_a^b f(x)dx$ Area under $g(x)$ = $\int_a^b g(x)dx$ Area between $f(x)$ and $g(x)$

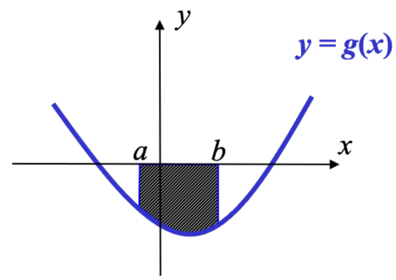
$$= \int_a^b [f(x) - g(x)]dx$$

$$\boxed{\text{Area Between Two Curves in } [a\,,\,b] = \int_a^b [\textit{Top Function} - \textit{Bottom Function}]dx}$$

Example 112. Find the area A bounded between the graph $y = e^x$ and $y = \sqrt{x}$ and the lines $x = 0$ and $x = 1$.

What happens when $g(x)$ is below $x-$axis?





Example 113. Find the area A bounded between the graph $y = x^2 - 4$ and $y = 2x - 1$.

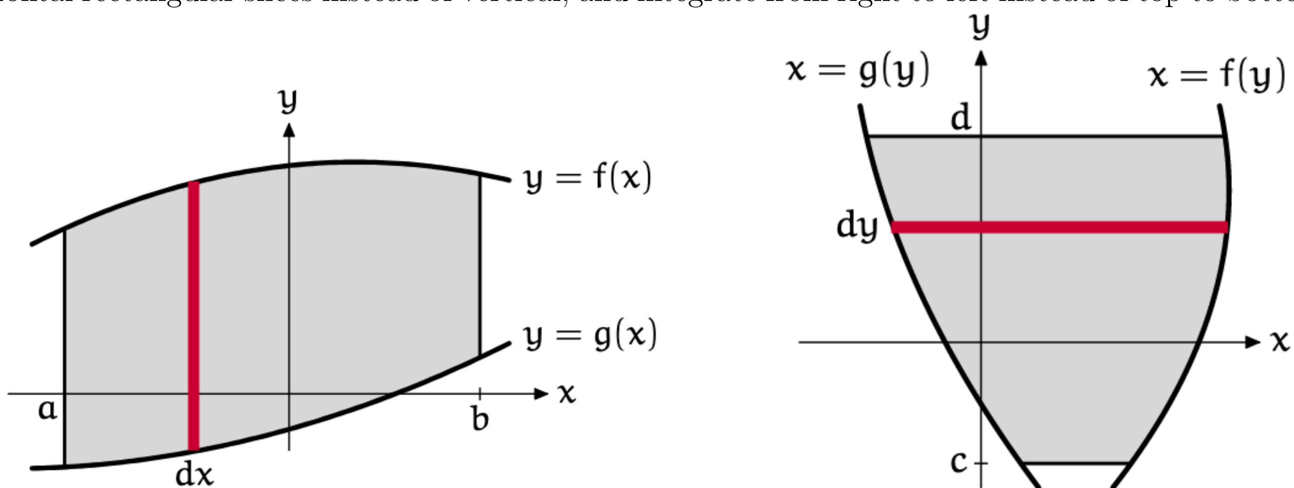
DIY 94. Find the area A bounded between the graph $y = 10x - x^2$ and $y = 3x - 8$.

Example 114. Find the area A bounded between the $x-$axis, the graph $y = x^2 - 4$ and the lines $x = -1$ and $x = 3$.

DIY 95. Find the area A bounded between the graph $y = x^3$ and $y = x$.

Integration w/respect to y; Horizontal Rectangles

Sometimes, we will be given equations that are in terms of y or it will be easier to calculate the area if the functions are in terms of y. In either case, the steps are similar, except that we use horizontal rectangular slices instead of vertical, and integrate from right to left instead of top to bottom.



V.

$$\textbf{Area between two curves on } [c, d] = \int_c^d (RIGHT - LEFT)\, dy$$

Example 115. Find the area A bounded between the graph $y = x^2 + 2$, $y = 4$ and the $y-$axis.

DIY 96. Find the area A bounded between the graph $y = \sqrt{4-4x}$, $y = \sqrt{4-x}$ and the $x-$axis.

DIY 97. Find the area A bounded between the graph $y = 1$, $y = \ln x$ and the $x-$ and $y-$axes.

7.5 Volumes and Arcs

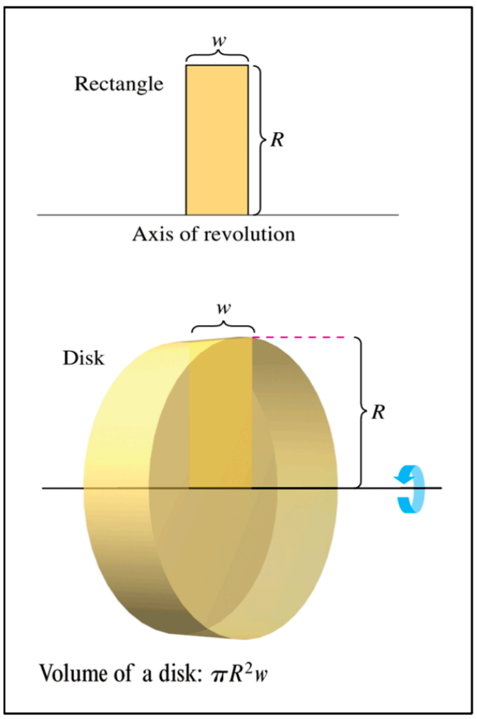
We will now use the powers of the definite integral to find the volume of 3-D solids that have common cross-sections. In particular, we will start with circular cross sections and then generalize by working out square, triangular, and semi-circular cross sections.

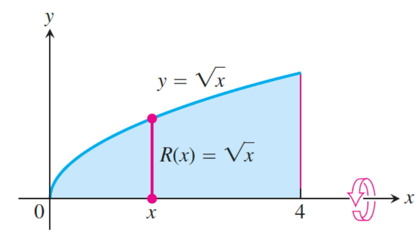
The Disk Method

Definition 7.8: Solid of Revolution

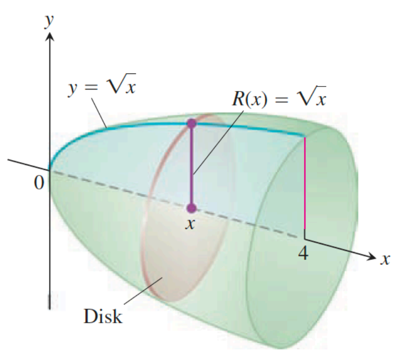
If a bounded region in a plane is a revolved about a line, the resulting 3-D solid is called a **solid of revolution**, and that line is called the **axis of revolution**.

The simplest such solid is a right circular cylinder...or a DISK, which is found by observing that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. Thus we have a revolving rectangle about an axis adjacent to one side of the rectangle.





(a)



(b)

$$\text{Volume of one disk} = (\text{area of disk}) \cdot (\text{width of disk}) \;= \pi R^2 \cdot w = \pi R^2 \cdot \Delta x$$

The approximation appears to get better and better as $\Delta x \to 0$, which only happens if we increase the number of cross sections (i.e. $n \to \infty$).

$$\text{Volume of Solid} = \lim_{n\to\infty} \pi \sum_{i=1}^{n} \Big[R(x)\Big]^2 \Delta x = \pi \int_a^b \Big[R(x)\Big]^2 \, dx$$

A similar formula can be derived if the axis of rotation is vertical.

Theorem 43: The Disk Method

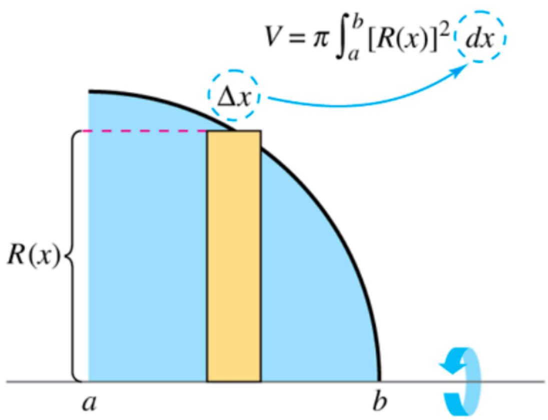
To find the volume of a solid of revolutions, using the disk method, the integral will have the following form:

$$Volume = V = \pi \int_a^b \left[R(x)\right]^2 dx$$

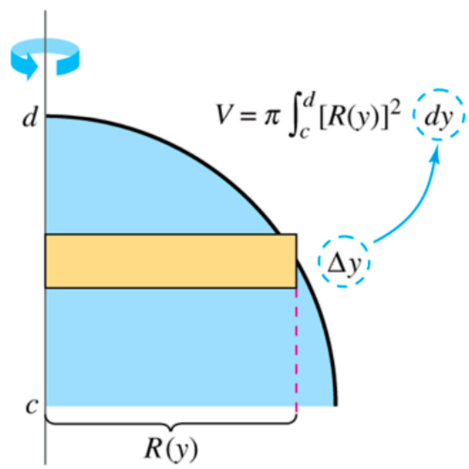
Horizontal Axis of Revolution

$$Volume = V = \pi \int_c^d \left[R(y)\right]^2 dy$$

Vertical Axis of Revolution



Horizontal axis of revolution



Vertical axis of revolution

To determine the variable of integration place a representative rectangle "perpendicular" to the axis of revolution. If the width of the rectangle is Δx, integrate with respect to x, and if the width of the rectangle is Δy, integrate with respect to y.

Example 116. Find the volume of the solid formed by revolving about the $x-$axis the region under the curve

$$y = \sqrt{x} \text{ from } x = 0 \text{ to } x = 1$$

.

Example 117. Find the volume of the solid formed by revolving the region bounded by the curve

$$y = 2x^2, y = 0 \text{ and } x = 2$$

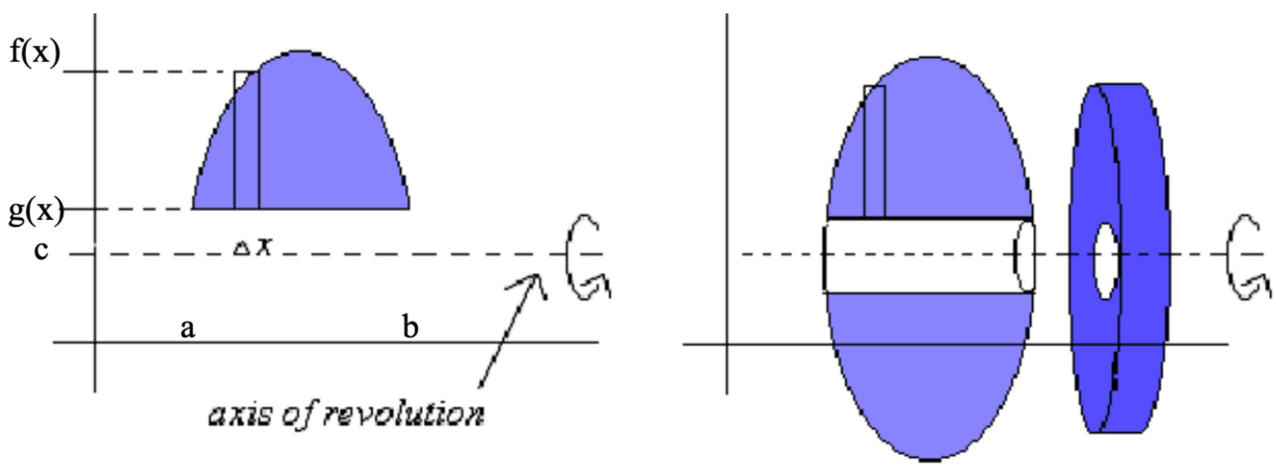
about the line $x = 2$.

The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it, therefore our representative disk is now a representative WASHER with dimensions

$$\text{Outer radius: } R(x) = f(x) - c$$

$$\text{Inner radius: } r(x) = g(x) - c$$



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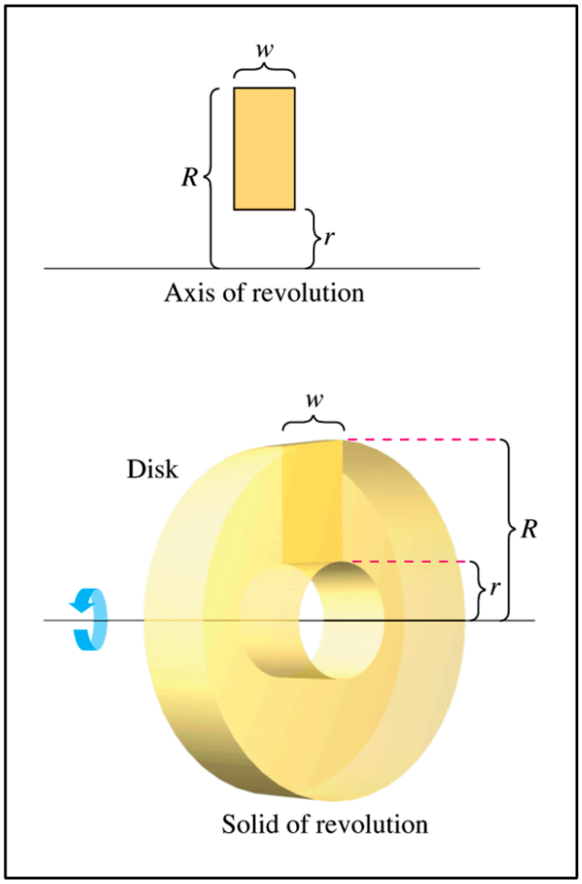
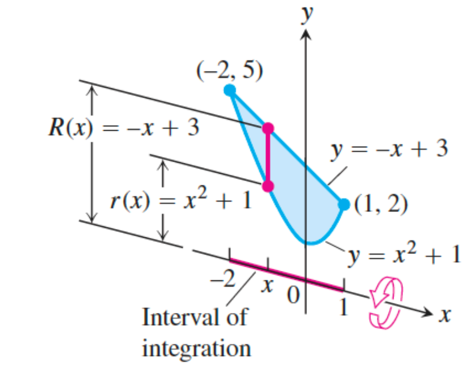
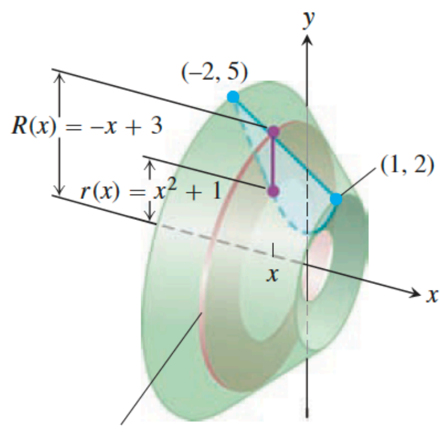




(a)



Washer cross-section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

(b)

$$\text{Volume of One Representative Washer} =$$
$$(\text{area of washer})(\text{width of washer}) =$$
$$(\pi R^2 - \pi r^2)w =$$
$$\pi(R^2 - r^2)\Delta x$$

Theorem 44: The Washer Method

To find the volume of a region bounded by an outer radius $R(x)$ and an inner radius $r(x)$. The volume of the solid of revolution generated is given by:

$$\underbrace{Volume = V = \pi \int_a^b \left[R(x)\right]^2 - \left[r(x)\right]^2 \, dx}_{\text{Horizontal Axis of Revolution}} \qquad \underbrace{Volume = V = \pi \int_c^d \left[R(y)\right]^2 - \left[r(y)\right]^2 \, dy}_{\text{Vertical Axis of Revolution}}$$

Example 118. Find the volume of the solid formed by revolving the region bounded by

$$y = \sec x, \quad y = 0, \quad 0 \leq x \leq \frac{\pi}{3}$$

about the line $y = 4$.

Example 119. Find the volume of the solid formed by revolving the region bounded by

$$y = 2 - \frac{x^2}{4} \text{ and } y = 1$$

about the $x-$axis.

Generalized: Any Cross Section

With the Disk method, we found the volume of the solid having a **circular** cross section whose area is $A = \pi R^2$. This method can be generalized as long as we have a way to compute the area of one representative cross section.

Theorem 45: Volume of Solids with Known Cross Section

1. If the cross section is **perpendicular to the $x-$axis** and it's area is a function of x, say $A(x)$, then the volume, V, of the solid on $[a, b]$ is given by
$$V = \int_a^b A(x)\, dx$$

2. If the cross section is **perpendicular to the $y-$axis** and it's area is a function of y, say $A(y)$, then the volume, V, of the solid on $[a, b]$ is given by
$$V = \int_c^d A(y)\, dy$$

Before we continue, it will prove helpful to know the area of the following situations:

1. A square with sides of length x
$$A = x^2$$

2. A square with diagonals of length x
$$A = \left(\frac{\sqrt{2}}{2}x\right)^2 = \frac{1}{2}x^2$$

3. A semicircle with radius x
$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi x^2$$

4. A semicircle with diameter x
$$A = \frac{1}{2}\pi r^2 = \frac{1}{8}\pi x^2$$

5. An equilateral triangle with sides of length x
$$A = \frac{1}{2}b \cdot h = \frac{1}{2}\Big(x\Big) \cdot \left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$$

6. An isosceles right triangle with legs of length x
$$A = \frac{1}{2}b \cdot h = \frac{1}{2}x^2.$$

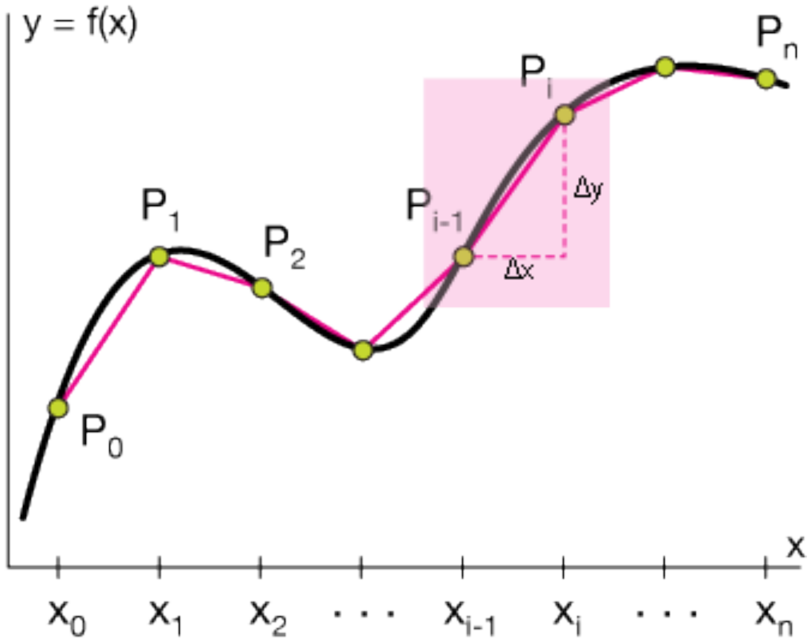
Example 120. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$ and has the following cross sections that are perpendicular to the $x-$axis.

- Squares
- Equilateral Triangles
- Right Isosceles Triangles
- Semi-Circles

DIY 98. Find the volume of the solid whose base is bounded by the line $4x - 3y = 9$ and has semi-circles for cross sections that are perpendicular to the $y-$axis.

DIY 99. (Calculator Active) A paperweight is to be made so that its base is the shape of the region between the $x-$axis and once arch of the curve $y = 2\sin x$. Each cross section cut perpendicular to the $x-$axis is a semicircle whose diameter runs from the $x-$axis to the curve. Find the volume of the paperweight.

Arc Length

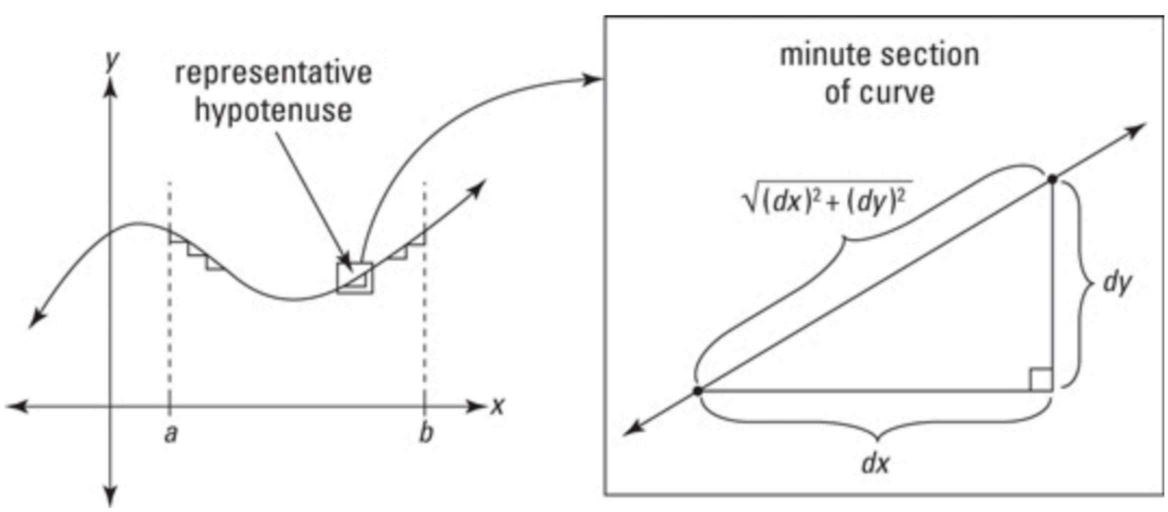


Theorem 46: Arc Length of a Smooth Curve

If a smooth curve begins at the point (a, c) and ends at the point (b, d), then the length of the curve, called the **arc length**, is given by:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y \text{ is a smooth function of } x \text{ on } [a, b]$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx \qquad \text{if } x \text{ is a smooth function of } y \text{ on } [c, d]$$



Example 121. Find the exact length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad \text{for } 0 \leq x \leq 1.$$

DIY 100. (Calculator Active)

Find the exact length of the curve

$$y = x^{1/3} \qquad \text{from } (-8, 2) \text{ to } (8, 2).$$

Why must we integrate with respect to y?

7.6 Advanced Methods of Integration

TRIGONOMETRIC IDENTITIES

RECIPROCAL IDENTITIES

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

PYTHAGOREAN IDENTITIES

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

SUM AND DIFFERENCE IDENTITIES

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

DOUBLE-ANGLE IDENTITIES

$$\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1 + \tan^2 x}$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x} \end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

HALF-ANGLE IDENTITIES

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

U-Sub w/ Trig

Evaluate the following:

a) $\displaystyle\int 3^{\cos x} \sin x \, dx$

b) $\displaystyle\int \tan(x) \, dx$

c) $\displaystyle\int \csc(x) \, dx$

d) $\displaystyle\int \sin^2(x) \, dx$

e) $\displaystyle\int \tan^2(x) \, dx$

Inverse Trig

Definition 7.9: Inverse Trig with Stretch

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

a) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\, dx$

b) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\, dx$

c) $\displaystyle\int \frac{1}{1+x^2}\, dx$

d) $\displaystyle\int \frac{1}{9+25x^2}\, dx$

Integration by Parts(Reverse Product Rule)

We begin with the product of two functions $u(x)$ and $v(x)$ and find the derivative via the product rule.

$$(uv)' = u\,dv + v\,du$$

How do we go about identifying $u(x)$? FOLLOW

$$L.I.P.E.T$$

a) $\displaystyle\int x\cos(x)\,dx$

b) $\displaystyle\int 2xe^{4x}\,dx$

c) $\displaystyle\int x\ln x\,dx$

Tricky Problems

a) $\displaystyle\int \ln x\,dx$

b) $\displaystyle\int \arctan(x)\,dx$

c) $\displaystyle\int x^2\sin(x)\,dx$

Definition 7.10: Tabular Method for IBP

Suppose the product of $f(x)$ and $g(x)$. If

1. If f can be differentiated multiple times until it equals 0.
2. If g can be "easily" integrated multiple times.

a) $\displaystyle\int x^2\sin(x)\,dx$

b) $\int x^3 \cos(2x)\, dx$

Partial Fractions (Linear)

Evaluate the following:

a) $\displaystyle\int \frac{3x+11}{x^2-x-6}\,dx$

b) $\displaystyle\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$

7.7 Improper Integrals

Infinite Limits of Integration

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 122. Evaluate the following integrals.

a) $\int_1^{\infty} \frac{1}{x^{3/2}} dx$

b) $\int_1^{\infty} \frac{1}{x^2} dx$

c) $\int_1^\infty \frac{1}{x^3} dx$

d) $\int_1^\infty \frac{1}{x} dx$

P-Series Integrals

If $a > 0$, then $\int_a^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$. These are called **p-series integrals**.

If $a = 1$ and $p > 1$, then $\int_a^\infty \frac{1}{x^p} dx$ converges to $\frac{1}{p-1}$.

DIY 101. Evaluate the following integrals.

a) $\int_1^\infty \frac{1}{x^{3/2}} dx$

b) $\int_1^\infty \frac{1}{x^{1.1}} dx$

c) $\int_1^\infty \frac{1}{x^{2/3}} dx$

Example 123. Evaluate the following integrals.

a) $\int_{-\infty}^{1} e^x dx$

DIY 102. Evaluate the following integrals.

a) $\int_1^\infty \left(\frac{1}{x} + \frac{5}{1+5x}\right) dx$

b) $\int_1^\infty \frac{dx}{2x-1}$

c) $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$

Integrands with Vertical Asymptotes

DEFINITION Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

Example 124. Evaluate the following integrals.

a) $\int_0^1 \frac{1}{x^{1/3}} dx$

b) $\int_0^1 \frac{1}{x^3} dx$

c) $\int_0^{27} \frac{dx}{\sqrt[3]{27-x}}$

DIY 103. Evaluate the following integral:

$$\int_0^{\infty} \frac{1}{x^2} dx$$

.

Chapter 8

Parametric and Vector Valued Functions

8.1 Parametric Equations

A path of a moving object in $x - y$ plane (or in space) need NOT pass the vertical line test, so it CANNOT be described as the graph of a function. However, we can describe the path using TWO (or more) functions.

$$x = f(t) \qquad\qquad y = g(t)$$

where f and g are continuous.

Definition 8.1: Parametric Equations

If x and y are given functions,

$$x = f(t) \qquad\qquad y = g(t)$$

over an interval I of $t-$ values, then the set of points (x, y)

$$(x, y) = \Big(f(t), g(t)\Big)$$

define a **parametric curve.**

The equations are parametric equations where t is the **parameter.** The equation and interal, together, constitute a **parametrization**.

Example 125. Suppose

$$x = t^2 \qquad y = t + 1, \qquad -\infty < t < \infty$$

TABLE 11.1 Values of $x = t^2$ and $y = t + 1$ for selected values of t.

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

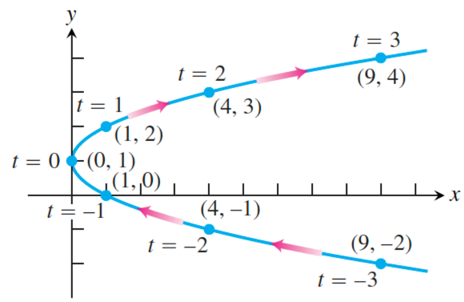


FIGURE 11.2 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 1).

DIY 104. Graph the following sets of parametrization using Desmos and identify characteristics of the graph and the orientation.

- $x = \cos(t) \qquad y = \sin(t), \qquad 0 \leq t \leq 2\pi$

- $x = \sin(t) \qquad y = \cos(t), \qquad 0 \leq t \leq 2\pi$

- $x = \cos(2t) \qquad y = \sin(2t), \qquad 0 \leq t \leq 2\pi$

- $x = 2\cos(t) \qquad y = 2\sin(t), \qquad 0 \leq t \leq 2\pi$

- $x = 3\cos(t) \qquad y = 2\sin(t), \qquad 0 \leq t \leq 2\pi$

Example 126. Suppose $x = \sqrt{t} \qquad y = t, \qquad t \geq 0$. Graph and characterize this parametrization.

DIY 105. Suppose $x = t \qquad y = t^2, \qquad -\infty < t < \infty$. Graph and characterize this parametrization.

Parametric Differentiation

By the chain rule we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

therefore we can divide by $\dfrac{dx}{dt}$ and we now have a direct relation ship to find the derivative

$$\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt}$$

By letting $y' = \dfrac{dy}{dx}$ we can define how to find the second PARAMETRIC derivative:

$$\frac{d^2y}{dx^2} = \frac{dy'/\,dt}{dx/\,dt}$$

In other words, take your first derivative and derive it with respect to t and then divide by $\dfrac{dx}{dt}$.

Example 127. Find the tangent to the curve

$$x = \sec(t) \qquad y = \tan(t), \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$ where $t = \dfrac{\pi}{4}$.

Example 128. Find $\dfrac{d^2y}{dx^2}$ for

$$x = t - t^2 \qquad y = t - t^3$$

DIY 106. Given

$$x = t^2 - 5 \qquad y = 2\sin(t)$$

- Find the highest point.
- Find the point of inflection.

Arc Length of Parametrized Functions

For a smooth curve, $y = f(x)$ we define the arc length as

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

based on the fact that the Pythagorean Theorem states $\sqrt{(\Delta x)^2 + (\Delta y)^2}$.

For parametric curve, $x = f(t)$ and $y = g(t)$ we have the analogous are length definition

Theorem 47: Arc Length for Parametrized Curves

The arc length of a parametrized curve that is traversed ONCE as t increases from t_1 to t_2 is measured as

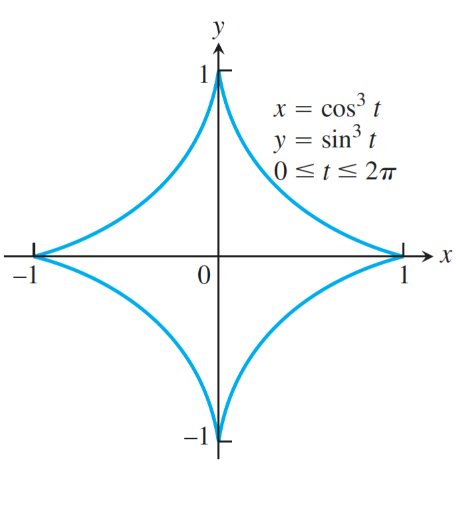
$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{t_1}^{t_2} \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Example 129. Find the length of

$$x = r\cos(t) \qquad y = r\sin(t) \qquad 0 \le t \le 2\pi$$

DIY 107. Find the length of the astroid

$$x = \cos^3(t) \qquad y = \sin^3(t) \qquad 0 \le t \le 2\pi$$



8.2 Polar Coordinates

Polar Coordinate System

Throughout your education in math you have plotted points and graphed all functions on the rectangular coordinate system. A point is represented using (x, y) as our coordinate pair. We are now going to define a new coordinate system called the polar coordinates system. The way we represent a point in the polar coordinate system is by using a distance and angle from a reference point In this way, it is a more intuitive way to describe a location since locations in the real world are usually described by a distance and direction. (Ex: Mr.Lopez's house is 5 miles northeast from Crespi.)

In a polar coordinate system, we select a point, called the polar axis. Comparing the rectangular and polar coordinate systems we notice that the origin in rectangular coordinates coincides with the pole in the polar coordinate, and the positive $x-$axis in rectangular coordinates coincides with the polar axis in polar.

Plot points using polar coordinates

A point P in a polar coordinate system is represented by an ordered pair of numbers (r, θ). If $r > 0$, then r is the distance of the point from the pole, θ is an angle (in degrees or radians) formed by the polar axis and a ray from the pole through the point.

If $r < 0$, draw the angle, but instead of the point being on the terminal side of the angle, it is on the ray in the opposite direction of the terminal side.
Example 130. Graph the polar coordinate:

a) $(2, \frac{\pi}{4})$ b) $(-3, \frac{2\pi}{3})$

c) $(-2\sqrt{2}, \frac{-3\pi}{4})$ d) $(2\sqrt{2}, \frac{\pi}{4})$

DIY 108. Plot the points with the following polar coordinates:

a) $(3, \dfrac{5\pi}{3})$ b) $(2, -\dfrac{\pi}{4})$

c) $(3, 0)$ d) $(-2, \dfrac{\pi}{4})$

Finding Several Polar Coordinates of a single point

One aspect of polar coordinates that makes this system very versatile is that one point can be defined using distinct coordinates.

Example 131. The point P has polar coordinates $(2, \frac{\pi}{4})$. What are some other coordinates that will give this same point?

Example 132. Plot the point P with the polar coordinates $(3, \frac{\pi}{6})$, and find the other polar coordinates (r, θ) of this same point for which:

a) $r > 0, 2\pi \leq \theta < 4\pi$ b) $r < 0, 0 \leq \theta < 2\pi$ c) $r > 0, -2\pi \leq \theta < 0$

Summary: A point with polar coordinates (r, θ), θ radians, can also be represented by either of the following:

$$(r, \theta + 2\pi k) \text{ or } (-r, \theta + \pi + 2\pi k) \text{ where } k \text{ is any integer.}$$

Example 133. Plot each point given in polar coordinates, and find other polar coordinates (r, θ) of the point for which:
a) $r > 0, 2\pi \leq \theta < 4\pi$ b) $r < 0, 0 \leq \theta < 2\pi$ c) $r > 0, -2\pi \leq \theta < 0$

1) $(4, \frac{3\pi}{4})$

2) $(-3, 4\pi)$

Conversion from Polar to Rectangular Coordinates

Theorem 48: Polar to Cartesian

If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by:

$$x = r\cos\theta \qquad y = r\sin\theta$$

where

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}$$

Example 134. Graph the polar coordinates, then convert from polar coordinates to rectangular.

a) $(6, \frac{\pi}{6})$

b) $(-4, -\frac{\pi}{4})$

DIY 109. Graph the polar coordinates, then convert from polar coordinates to rectangular coordinates:

a) $(4, \frac{3\pi}{2})$

Transform Equations from Polar to Rectangular Form

There are two techniques to transforms equations from polar to rectangular:

1. Multiply both sides of the equation by r.
2. Squaring both sides of the equation.

Example 135. Transform the following equations from polar to rectangular.
a) $r = 6\cos\theta$ b) $r = \cos\theta + 1$

DIY 110. Transform $r = \sin\theta + 1$ into rectangular form.

DIY 111. Transform $r = \dfrac{4}{2\cos\theta - \sin\theta}$ into rectangular form.

Converting from Rectangular Coordinates to Polar Coordinates

1. Determine if the point lies on an axis or in the quadrant.
2. Find r. If the point lies on an axis, r is the distance on the axis. If the point lies in the quadrant then we find the magnitude of r with $r = \sqrt{x^2 + y^2}$
3. Find θ. If the point lies on the axis, θ is a quadrantal angle $(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2})$. If it lies in the quadrant then:
   (a) Quadrant I or IV: $\theta = \tan^{-1} \frac{y}{x}$
   (b) Quadrant II or III: $\theta = \pi + \tan^{-1} \frac{y}{x}$

Example 136. Find polar coordinates of a point whose rectangular coordinates are:
a)$(-3, 0)$ b)$(0, 3)$

DIY 112. Find the polar coordinate of a point whose rectangular coordinates are:
a)$(5, 0)$ b)$(0, -4)$

Example 137. Find the polar coordinates of a point whose rectangular coordinates are:
a) $(2, -2)$ b)$(-1, -\sqrt{3})$

DIY 113. Find the polar coordinates of a point whose rectangular coordinates are $(-3, 3)$.

Transform an Equation From Rectangular to Polar

Example 138. Convert the equation $4xy = 9$ into polar form.

DIY 114. Convert the equation $x^2 + (y-3)^2 = 9$ into polar form.

Parametric Equations of Polar Curves

It is possible to use our previous definitions for parametric equations since the graphs produced using polar coordinates can also be the path of a moving object. Since the radius r varies dependently on θ we can view r as being a function of θ. Therefore, we have the following

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\cos\theta = f(\theta)\sin\theta$$

Since this is a parametrized curve of with parameter θ our previous parametric derivative formula becomes the following:

$$\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt} \Longrightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

The tricky part of this concept comes when we try to find the second derivative of this relationship. Given that $y' = \dfrac{dy}{dx}$, we define the second derivative as

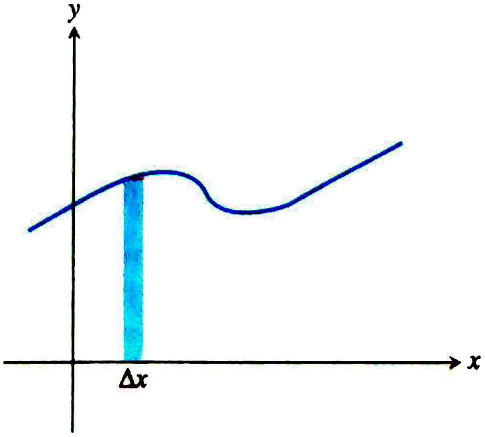
$$\frac{d^2y}{dx^2} = \frac{dy'/d\theta}{dx/d\theta}$$

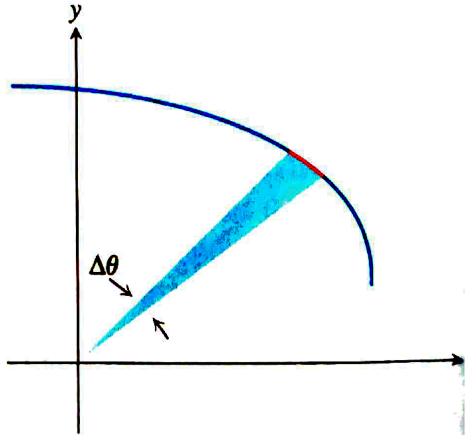
Example 139. Find the equation of the tangent line of the rose curve $r = 2\sin(3\theta)$ at the point where $\theta = \pi/6$.

DIY 115. Find the equation of the tangent line of the rose curve $r = -1 + \cos(\theta)$ at the point where $\theta = \pm\pi/2$.

Area and Arc Length in Polar Coordinates

While a small change in x produces a thin rectangle with a strip of area, a small change in θ produces a thin ***circular*** sector of area.





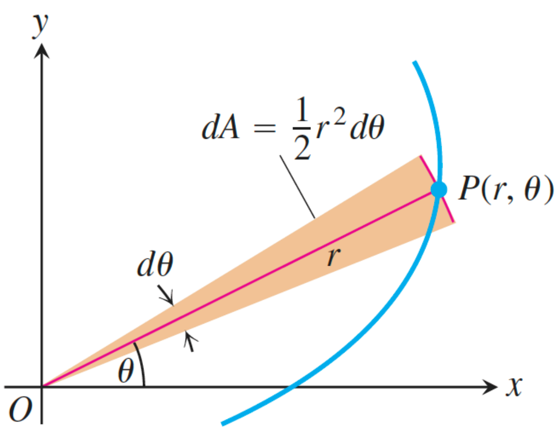
From trigonometry we have that the area of a sector is

$$A = \frac{1}{2}r^2\theta$$

where θ is measured in radians.

If we replace θ with the differential $d\theta$, we get the ***area differential***

$$dA = \frac{1}{2}r^2\,d\theta$$



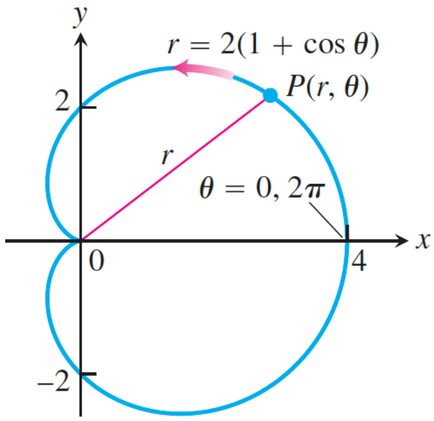
Theorem 49: Area in Polar Coordinates

The area of the region between the origin and the curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 d\theta$$

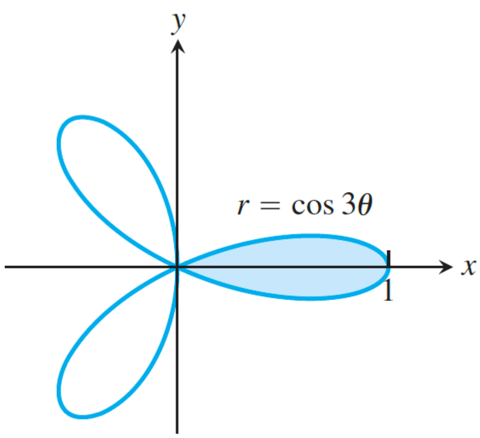
Example 140. Find the area of the region in the plane enclosed by the cardioid

$$r = 2(1 + \cos\theta)$$



Example 141. Find the area of the region inside one leaf of the three leafed rose

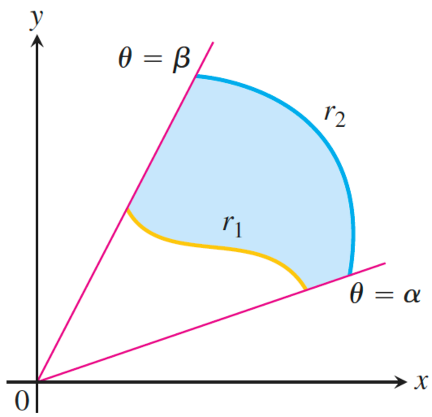
$$r = \cos(3\theta)$$



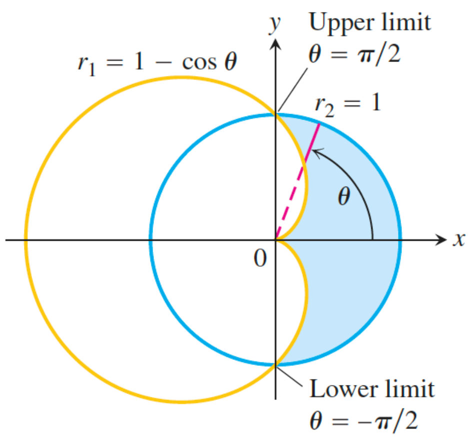
Theorem 50: Area Enclosed by Polar Curves

If $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$ and $\alpha \leq \theta \leq \beta$ the area enclosed is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2}(r_2^2 - r_1^2) d\theta$$



Example 142. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.



Chapter 9

Sequences and Series

9.1 Sequences

The theory of infinite sums of numbers has many important applications in various areas of study. The most notable use of these infinite sums is their ability to approximate irrational values to a certain degree of precision. Before we can explore this we must develop our intuition for a special class of functions called a **sequence**.

Definition 9.1: Sequence

A **sequence** is a function whose domain is the set of positive integers and whose range is a subset of the real numbers.

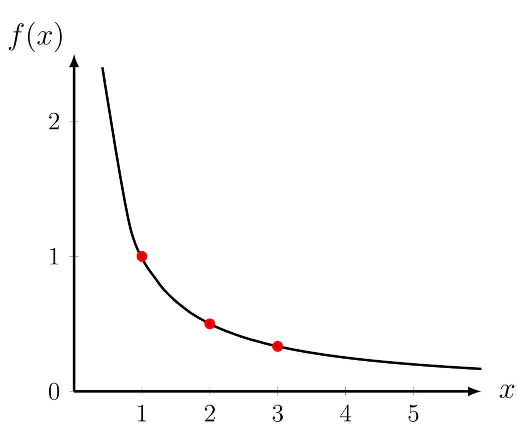
The range elements of the function are called the **terms** of the sequence

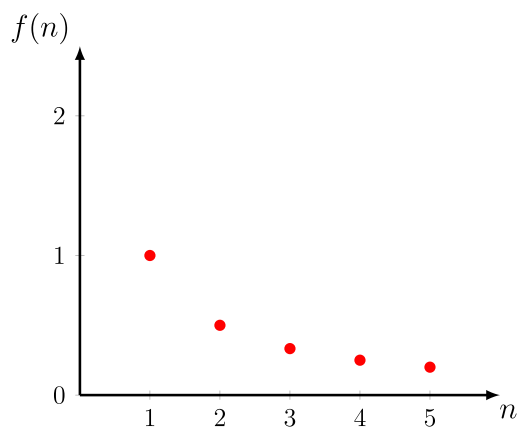
$$a_1, a_2, a_3, \ldots, a_{n_1}, a_n$$

Where a_n is called the n^{th}-term and we denote the sequence by:

$$\{a_n\} \text{ or } \{a_n\}_{n=0}^{\infty}$$

To give an intuitive ideas as to what the above definition is describing let's explore the following function.





Example 143. Write the first five terms of each sequence:

a) $\{a_n\}_{n=0}^{\infty} = \left\{ \dfrac{1}{3n-2} \right\}_{n=0}^{\infty}$

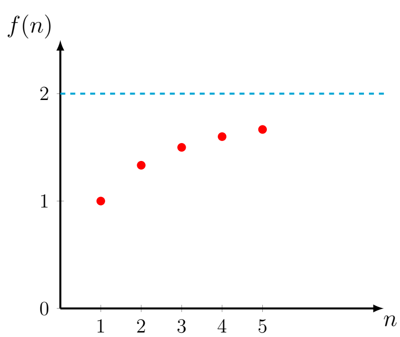
b) $\{b_n\}_{n=0}^{\infty} = \left\{ \dfrac{2n-1}{n^3} \right\}$

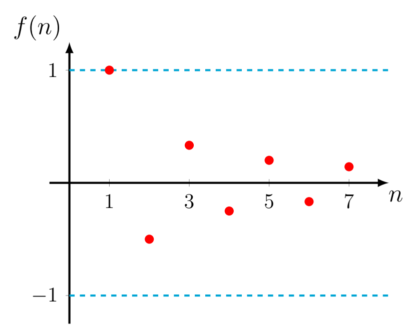
DIY 116. Write the first five terms of each sequence:

a) $\{t_n\} = \left\{ (-1)^n \left(\dfrac{1}{2} \right)^n \right\}$

Example 144. Find the n^{th} term of each of the following sequences. Assume that the indicated pattern continues.

Sequences	n^{th}**term**
a) $e, \frac{e^2}{2}, \frac{e^3}{3}, \frac{e^4}{4}, \ldots$	
b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$	
c) $1, 4, 9, 16, 25, \ldots$	
d) $\frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5} \ldots$	
e) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5} \ldots$	
f) $1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{6}, \ldots$	





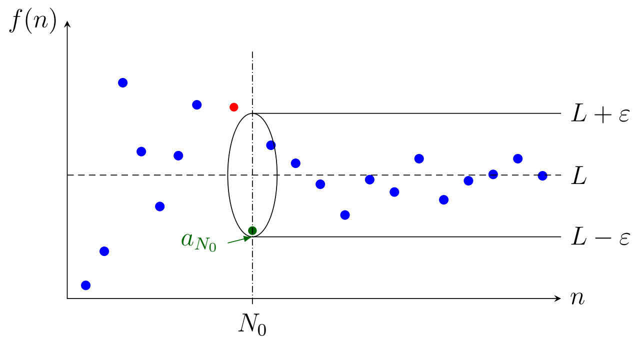
Now, if we look closely at the sequence graphs of part ***d*** and ***e*** we can see that both sequences begin to *settle* at a particular value. Let us distinguish sequences whose elements approach a single point as n increases from those sequences whose elements do not.

Definition 9.2: Convergent Sequence

A sequence $\{a_n\}$ **converges** to the real number L, $\lim_{n\to\infty} a_n = L$, if for every $\varepsilon > 0$ there is an integer N such that

$$|a_n - L| < \varepsilon \text{ whenever } n > N.$$

In other words, the value of the a_n's approach L as we go farther down the list of terms. If a sequence does not converge, then we say it is **divergent**.



PS : Possibilities for Sequences

Let $\{a_n\}$ be a sequence of real numbers,

- If $\lim_{n\to\infty} a_n = L$, then $\{a_n\}$ converges to L.
- If $\lim_{n\to\infty} a_n = \pm\infty$, then $\{a_n\}$ diverges to positive/negative infinity.
- If $\lim_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation.

Example 145. Determine whether the following converge or diverge.

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{n}{n+1}, \cdots$

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots, \frac{1}{2^n}, \cdots$

c) $a_n = \{3 + (-1)^n\}$

d) $a_n = \left\{\frac{n}{1-2n}\right\}$

e) $a_n = \left\{\frac{\ln n}{n}\right\}$

f) $a_n = \left\{\frac{n!}{(n+2)!}\right\}$

g) $a_n = \left\{\frac{2n!}{(n-1)!}\right\}$

h) $a_n = \left\{\frac{n + (-1)^n}{n}\right\}$

i) $a_n = \left\{\frac{(-1)^n(n-1)}{n}\right\}$

j) $a_n = \left\{\frac{2^n}{(n+1)!}\right\}$

k) $a_n = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$

l) $a_n = \left\{\frac{(2n)!}{(2n-2)!}\right\}$

Theorem 51: Properties of Convergent Sequences

Suppose $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exists. Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, where $k, L, M \in \mathbb{R}$

$\lim_{n\to\infty} A = A$	$\lim_{x\to c} k \cdot f(x)$
$\lim_{n\to\infty} [a_n \pm b_n] = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L \pm M$	$\lim_{n\to\infty} [a_n]^{\frac{r}{s}} = L^{\frac{r}{s}}$
$\lim_{n\to\infty} [a_n \cdot b_n] = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = L \cdot M$	$\lim_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$

DIY 117. Determine whether the following sequences converge or diverge.

a) $\{s_n\} = \left\{\frac{2}{n} + 3\right\}$

b) $\{t_n\} = \left\{\frac{4}{n^2}\right\}$

c) $\{z_n\} = \left\{\sqrt[3]{\frac{12n^2 + 3n}{4n^2}}\right\}$

Theorem 52

Let a_n be a sequence of real numbers. If $\lim_{n\to\infty} a_n = L$ and if f is a function that is continuous at L and is defined for all numbers s_n, then $\lim_{n\to\infty} f(a_n) = f(L)$.

Example 146. Show that $\left\{ \ln\left(\frac{2}{n}+3\right)\right\}$ converges and find its limit.

DIY 118. Show that $\left\{ \ln\left(\frac{n^2+2}{2n^2+3}\right)\right\}$ converges and find its limit.

Theorem 53: Related Function of a Sequence

A related function f of the sequence $\{a_n\}$ has the following two properties:

- f is defined on the open interval $(0, \infty)$;
- $f(n) = a_n$ for all integers $n \geq 1$.

Suppose L is a real number,

$$\text{If } \lim_{x\to\infty} f(x) = L, \text{ then } \lim_{n\to\infty} a_n = L.$$

If $\lim_{x\to\infty} f(x)$ is infinite, then $\{a_n\}$ diverges.

If $\lim_{x\to\infty} f(x)$ does not exists, then this is all inconclusive.

Example 147. Show that the following sequences converge and find its limit.

a) $\left\{\dfrac{4n^2+3n-1}{8n^2-7n+2}\right\}$

b) $\left\{\dfrac{n}{e^n}\right\}$

DIY 119. Show that the following sequences converge and find its limit.

a) $\left\{\dfrac{n^2-4}{n^2-7n+2}\right\}$

b) $\left\{\dfrac{n^2}{3^n}\right\}$

Theorem 54: Convergence/Divergence of a Geometric Sequence

The geometric sequence $a_n = r^n$, where r is a real number,

- converges to 0, if $-1 < r < 1$[a]
- converges to 1, if $r = 1$
- diverges for all other values of r

[a]Usually written as: $|r| < 1$

Example 148. Determine whether the following geometric sequences converge or diverge:

a) $s_n = \left(\frac{3}{4}\right)^n$

b) $t_n = \left(\frac{4}{3}\right)^n$

Definition 9.3: Bounded Sequence

A sequence a_n is **bounded from above** if every term of the sequence is less than or equal to some number M.

$$a_n \leq M \text{ for all } n$$

Similarly, a sequence is **bounded from below** if every term of the sequence is greater than or equal to some number m.

$$a_n \geq m \text{ for all } n$$

A sequence is said to be **bounded** if it is bounded from both above and below. For a bounded sequence, there exists a value K such that

$$|a_n| \leq K \text{ for all } n \geq 1$$

Theorem 55: Boundedness Theorem

1. A convergent sequence is bounded.[a]
2. If a sequence is not bounded from above and below, then it is divergent.

[a]But a bounded sequence need not be convergent. Ex: $\{(-1)^n\}$

Example 149. Determine whether each sequence is bounded.

a) $a_n = \dfrac{3n}{n+2}$

b) $b_n = \dfrac{4n}{3}$

c) $c_n = \ln n$

Divergent Sequences to Infinity

To be a bit more precise, a sequence is divergent if it falls between one of two categories.

Definition 9.4: Divergence of a Sequence

Divergence occurs if:

- The sequence does not settle to a particular value L
- given any value M, there is a positive integer N so that whenever $n > N$, $a_n > M$. That is,
$$\lim_{n\to\infty} a_n = \infty.$$

9.2 Infinite Series: Geometric and Harmonic Series

A series is the sum of the terms in a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. Informally, a series is the result of adding any number of terms from a sequence together:

$$a_1 + a_2 + a_3 + \dots$$

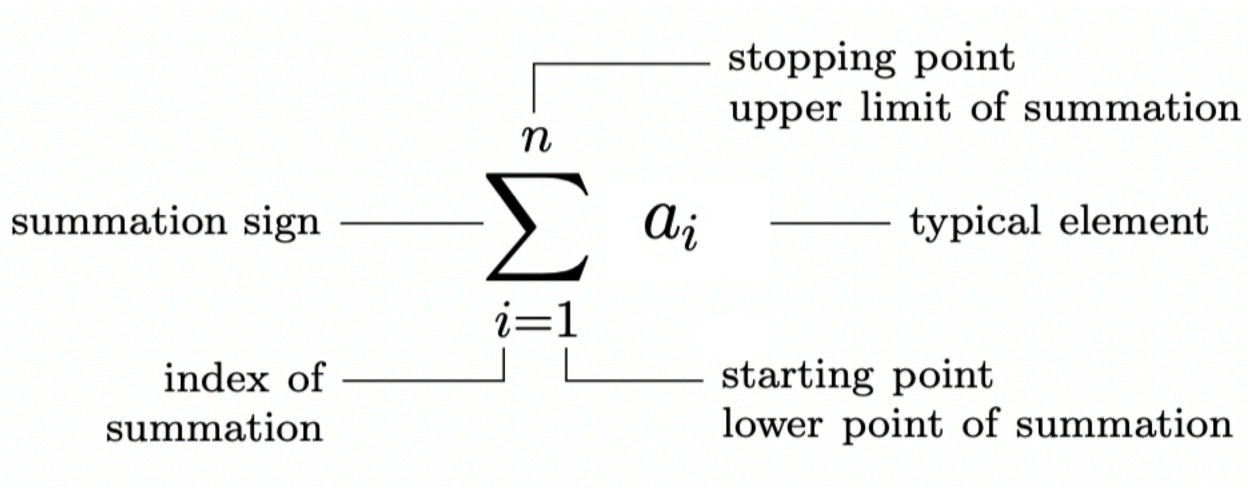
A series can be written more succinctly using the summation notation.

Definition 9.5: Infinite Series

If $a_1, a_2, a_3, \dots, a_n, \dots$ is an infinite collection of numbers, the expression

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is called an infinite series.



For infinite series, we can look at the sequence of partial sums to get an idea of the behavior of the sequence. In general, the nth partial sum is denoted by S_n. This can be explored on a calculator by adding sequential terms to the ongoing sum.

Example 150. For both $a_n = \dfrac{1}{n}$ and $b_n = \dfrac{1}{n^2}$, generate the sequence of partial sums $S_1, S_2, \dots, S_n$, for each. Determine whether the series converges or diverges.

DIY 120. Find the first 5 terms of the sequence of partial sums, and list them below. Based on this sequence, determine whether the series converges or diverges?

a) $\sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)$

b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

Definition 9.6: Convergence/Divergence of an Infinite Series

If the sequence $\{S_n\}$ of partial sums of a series $\sum_{n=1}^{\infty} a_n$ has a limit S, then the series converges. That is:

$$\text{If } \lim_{n\to\infty} S_n = S, \text{ then } \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots = S$$

The series is said to diverge if S is not a real number.

Telescoping Series

Example 151. Show that:

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

b) $\sum_{n=1}^{\infty} (-1)^n$ diverges.

DIY 121. Determine whether the following series converge or diverge. If it converges, find its sum.

a) $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$

b) $\sum\limits_{n=1}^{\infty} n$

We will now discuss a special class of series that arise in the study of waves and harmonics as well as investment theory[1]. The geometric series is also a good first example of a principle we will see throughout future courses: plug in some values, see if a pattern or special case occurs, and keep note of all special properties that can be identified.

Definition 9.7: Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n + \ldots$$

where $a \neq 0$, is called a **geometric series** with a ratio r.

Example 152. Prove the convergence of the Geometric Series by letting $r = -1, 1, 0$ and values larger than 1.

[1]Calculating Present Value

Theorem 56: Convergence of a Geometric Series

- If __________, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ **converges**, and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

- If __________, the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

Example 153. Determine whether each geometric series converges of diverges. If it converges, find its sum.

a) $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 8 \cdot \left(\frac{2}{5}\right)^{n-1}$

b) $\sum_{n=1}^{\infty} \left(-\frac{5}{9}\right)^{n-1}$

c) $\sum_{n=1}^{\infty} 3 \cdot \left(\frac{3}{2}\right)^{n-1}$

d) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

e) $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n-1}$

f) $\sum_{n=1}^{\infty} \frac{10^{n+1}}{9^{n-2}}$

Applications of the Geometric Series

Example 154. Express the repeating decimal $0.0909090\ldots$ as a quotient of two integers.

DIY 122. Prove $0.99999\ldots = 1$.

Example 155. A ball is dropped from a height of $16m$. Each time it strikes the ground, it bounces back to height three-fourths from which it fell. Find the total distance travelled by the ball.

Definition 9.8: The Harmonic Series

The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is called the harmonic series. Its name derives from harmonics in musics where the wavelengths of the overtones of a vibrating string are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc.[a]

Interesting occurences:

- music
- electrical systems (A.C.)
- Leaning Tower of Lire
- Architecture of the Baroque Era
- Physics of Sound

[a]https://www.youtube.com/watch?v=0Rfushlee0U

Example 156. Show that the Harmonic Series Diverges.

In economics, geometric series are used to represent the present value of an annuity (a sum of money to be paid in regular intervals). The present value is usually less than the future value because money has interest-earning potential, a characteristic referred to as the **time value of money**. Time value can be described with the simplified phrase,"A dollar today is worth more than a dollar tomorrow". The present value formula is the core formula for the time value of money. The standard equation is

$$PV = \frac{FV}{(1+i)^n}$$

.

DIY 123. If you are to receive \$1000 in five years, and the effective annual interest rate during this period is 10% (or 0.10), then the present value of this amount is ...?

These calculations are used to estimate the present value of expected stock dividends, the terminal value of a security, or to compute the APR of a loan.

9.3 nth Term Test, Integral Test, & p-Series

We have recently explored the concept of convergence and divergence of a infinite series. Our discussion led us to use the sequence of partial sums and then examine $\lim_{n\to\infty} S_n$. Like all things, it is rare when we can encounter a straightforward case in reality, so we are going to have to resort to alternate techniques to determine if a series is convergent or divergent. Our discussion will now defer the idea of finding an exact sum and only focus on determining convergence or divergence. We begin with a clean result.

Theorem 57: Convergent Series; Convergent Sequence

If the infinite series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Note that the above theorem does not guarantee anything since we can find sequence which converges to 0 but a series that does not. (Can you think of one?). This leads us to the following theorem which will prove to be our **"go-to"**.

Theorem 58: nth Term Test for Divergence

If the infinite series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$.

If $\lim_{n\to\infty} a_n = 0$, then test is inconclusive.

Example 157. Use the *nth* term test to determine whether the following series diverge.

a) $\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$

b) $\sum_{n=1}^{\infty} \frac{2n!}{3n!+3}$

DIY 124. Use the *nth* term test to determine whether the following series diverge.

a) $\sum\limits_{n=1}^{\infty} \dfrac{4n^2 + n^3}{10n - 2n^3}$

b) $\sum\limits_{n=1}^{\infty} \dfrac{e^{2n}}{n}$

Theorem 59: Sum and Difference of Convergent Series

Suppose $\sum\limits_{n=1}^{\infty} a_n = S$ and $\sum\limits_{n=1}^{\infty} b_n = T$ are two convergent series, then the series sum and difference evaluate as follows:

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = S \pm T$$

Theorem 60: Constant Multiple of a Series

Suppose $\sum\limits_{n=1}^{\infty} a_n = S$ converges and $c \in \mathbb{R}$, then

$$\sum_{n=1}^{\infty}(c \cdot a_n) = c \sum_{n=1}^{\infty} a_n = c \cdot S.$$

Example 158. Determine whether each series converges of diverges. If it converges, find its sum.

a) $\sum\limits_{n=5}^{\infty} \dfrac{1}{n}$

b) $\sum\limits_{n=1}^{\infty} \frac{2}{n}$

c) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} + \frac{1}{3^{n-1}} \right)$

DIY 125. Determine whether each series converges of diverges. If it converges, find its sum.

a) $\sum\limits_{n=1}^{\infty} \frac{10}{n}$

b) $\sum\limits_{n=1}^{\infty} \sin\left(\frac{\pi}{2} k \right)$

c) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{3n} + \frac{1}{4n} \right)$

Now let's consider a general series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ where each term $a_n > 0$.

This leads us to the following truth which we will be referring to as we progress.

Theorem 61: General Convergence Test

An infinite series of positive terms converges if and only if its sequence of partial sums is bounded. The sum of such an infinite series will not exceed an upper bound.

The Integral Test

Theorem 62: The Integral Test

Let f be a function that is continuous, positive, and non-increasing (decreasing or equal to) on the interval $[1, \infty)$. Let $a_n = f(n)$ for all postive n integers. Then

- if the improper integral $\int_1^\infty f(x)dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- if the improper integral $\int_1^\infty f(x)dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 159. Determine whether the following converge or diverge.

a) $\sum_{n=1}^{\infty} \dfrac{4}{n^2+1}$

b) $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$

c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

p-Series

Definition 9.9: p-Series

A **p-series** is an infinite series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \frac{1}{k^p} + \cdots$$

where p is a positive real number.

If $p = 1$, then we have the **harmonic** series.

Fortunately for us, we have a nice and simple theorem which will tell us if the p-Series is convergent or divergent.

Theorem 63: p-Series Test

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \frac{1}{k^p} + \cdots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 160. Use the p-Series Test to determine whether the following series diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$

DIY 126. Prove the p-Series Test. (Hint: Use the Integral Test)

9.4 Comparison Tests

We have seen how to determine the convergence of geometric series, p-series, and a few others. We can test the convergence of many more series by comparing their terms to those of a series whose convergence is known.

Theorem 64: The Comparison Test

Let $\sum a_n, \sum c_n$, and $\sum d_n$ be series with non-negative terms. Suppose that for some integer value N,

$$d_n \leq a_n \leq c_n. \text{ for all } n > N.$$

a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

It is important to note that the early terms in the series have no effect on the overall convergence or divergence of a series.

Example 161. Apply Theorem 37 to show whether the following converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

b) $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

c) $\sum_{n=1}^{\infty} \frac{1}{n!}$

d) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

e) $\sum_{n=1}^{\infty} \frac{1}{2^n + \sqrt{n}}$

f) $\sum_{n=1}^{\infty} \frac{n+3}{n(n+2)}$

Although useful, the Comparison Test require some clever algebraic thinking. **The Limit Comparison Test** on the other hand is a bit more straightforward since it sets certain condition on the limit of the ratio.

Theorem 65: The Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are both series of positive terms. [a]

- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L, 0 < L < \infty$, then both series converge or both series diverge.
- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and if $\lim\limits_{n\to\infty} b_n$ converges, then $\lim\limits_{n\to\infty} a_n$ converges.
- If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and if $\lim\limits_{n\to\infty} b_n$ diverges, then $\lim\limits_{n\to\infty} a_n$ diverges.

[a] The Limits Comparison Test is particularly useful for series in which is a_n is a rational function of n.

Example 162. Apply The Limit Comparison Test to determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2k^{3/2}+5}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

d) $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n - 1}$

e) $\sum\limits_{n=1}^{\infty} \dfrac{1 + n \ln n}{n^2 + 5}$

Limit Comparison Test with a p-Series

Example 163. Determine whether $\sum\limits_{n=1}^{\infty} \dfrac{3\sqrt{n} + 2}{\sqrt{n^3 + 3n^2 + 1}}$ converges or diverges.

DIY 127. Determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{3n+1}}{\sqrt{4n^2-2n+1}}$ converges or diverges.

DIY 128. Determine whether $\sum_{n=1}^{\infty} \frac{2n^2+5}{3n^3-5n^2+2}$ converges or diverges.

9.5 The Ratio and Root Test

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio $\frac{a_{n+1}}{a_n}$. For a geometric series $\sum ar^n$, this rate is a constant r and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result.

Theorem 66: Ratio Test

[a]Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

Then,

- the series converges if $\rho < 1$
- the series diverges if $\rho > 1$ or ρ is infinite
- the test is inconclusive if $\rho = 1$.

[a]The Ratio Test is often effective when the terms of a series contain factorials of expressions involving n or expressions raised to a power involving n.

Example 164. Apply The Ratio Test to determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

Theorem 67: Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho^{a}$$

Then,

- the series converges if $\rho < 1$
- the series diverges if $\rho > 1$ or ρ is infinite
- the test is inconclusive if $\rho = 1$.

[a] The Root Test works well for series of nonzero terms whose nth term involves an nth power.

Example 165. Apply The Root Test to determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

DIY 129. Apply The Root Test to determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

b) $\sum_{n=1}^{\infty} \left(\frac{8n+3}{5n-2} \right)^n$

9.6 Alternating Series & Absolute Convergence

Alternating Series and Error Approximations

A series in which the terms are alternately positive and negative is an alternating series.

Definition 9.10: Alternating Series

An alternating series is a series of the following form:

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

or of the form:

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \dots$$

The following test allows us to determine whether an alternating series is convergent; WARNING: THIS CANNOT BE USED TO SHOW DIVERGENCE.

Theorem 68: Alternating Series Test

The series

$$\sum_{n=0}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

converges if all three of the following conditions are satisfied:

1. The a_n's are all positive.
2. The terms of the sequence are nonincreasing.
3. $\lim\limits_{n\to\infty} a_n = 0$

Example 166. Show that the alternate harmonic series converges.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

DIY 130. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty}(-1)^n \frac{n}{n-1} = 2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \ldots$$

DIY 131. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty}(-1)^n \frac{2n^2+9}{n^2+1}$$

When tasked with proving that an alternating series converges, we need to find a systematic way to figure out if our sequence of terms a_n are indeed non-increasing. We can do so using the following three methods:

PS : Determing whether a sequence is nonincreasing

1. Use the Algebraic Difference: Show that $a_{n+1} - a_n \leq 0$ for all $n \geq 1$.
2. Use the Algebraic Ratio: Show that $\dfrac{a_{n+1}}{a_n} \leq 1$ for all $n \geq 1$.
3. Use the derivative of the related function f such that $a_n = f(n)$:

   Show that $f'(x) \leq 0$ for all $x > 0$.

Example 167. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty}(-1)^n \frac{10n}{n^2+16}$$

DIY 132. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}$$

A graphical interpretation of the partial sums shows how an alternating series converges to its limit L when the three conditions of The Alternating Series Test are satisfied.

Theorem 69: The Alternating Series Estimation Theorem

For a convergent alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

that satisfies the conditions for the Alternating Series Test, the error E_n between the partial sum S_n and the actual sum sum value S is numerically less than or equal to the $(n+1)$st term in the series. That is

$$|E_n| = |S - S_n| \leq a_{n+1}.$$

Example 168. Approximate the sum S of the alternating series so that the error is less than or equal to 0.0001.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}$$

DIY 133. Approximate the sum S of the alternating series so that the error is less than or equal to 0.004.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$

Absolute and Conditional Convergence

In the event that we encounter a series that is not necessarily alternating, but does contain positive and negative terms, then we will have to determine whether the sequence is *Absolutely Convergent* or *Conditional Convergent*. To do so, we will apply the tests for convergence studied before to the series of absolute values of a series with both positive and negative terms.

Definition 9.11: Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \dots$$

is convergent.

Definition 9.12: Conditional Convergence

A series that converges but does not converge absolutely converges conditionally.

This then leads to a nice theorem:

Theorem 70: Absolute Convergence Test

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Example 169. Determine whether the series converges.

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \dots$$

DIY 134. Determine whether the series converges.

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

DIY 135. Determine whether the series converges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$$

DIY 136. Determine for what values of p the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

9.7 Power Series

Now that we can test many infinite series of numbers for convergence, we can study sums that look like "infinite polynomials." We call these sums **power series** because they are defined as infinite series of powers of some variable, in our case x. Just like a polynomial consists of a *finite* number of monomials, a power series is the sum of an *infinite* number of monomials. Power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

Definition 9.13: Power Series

If x is a variable, then a series of the form:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 \cdots + a_n x^n + \cdots$$

where the coefficients $a_0, a_1, a_2, \ldots$ are constants, is called a **power series centered at** $x = 0$. A series of the form:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 \cdots + a_n(x-c)^n + \cdots$$

where c is a constant, is called a **power series centered at** $x = c$.

Let us now discuss a couple familiar series that arise from this definition:

- Let $x = 0$ or equivalently let $x = c$.

- Let $a_n = 1$ for all n.

Determining Convergence of Power Series

The second discussion from above allowed us to generalize the results of the geometric series test. We will return to that example and its usefulness but for now we will detour into determining whether or not a power series will converge using the **Ratio Test** and on occasion the **Root Test**.

Example 170. Find all the numbers x for which each power series in x converges.

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{4^n}$

c) $\displaystyle\sum_{n=0}^{\infty} n!x^n$

DIY 137. Find all the numbers x for which each power series in x converges.

a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

b) $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

The previous example illustrated how a power series might converge. The next result shows that if a power series converges at more than one value, then it converges over an entire interval of values. The interval might be finite or infinite and contain one, both, or none of its endpoints.

Theorem 71: Convergence/Divergence Theorem for Power Series

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$.

If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

DIY 138. If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 6$, then what can we say about the series when:

a) $x = 3$

b) $x = 8$

c) $x = -7$

As a consequence of the above theorem we can now categorize every power series into of of 3:

Theorem 72: Types of Power Series Convergence

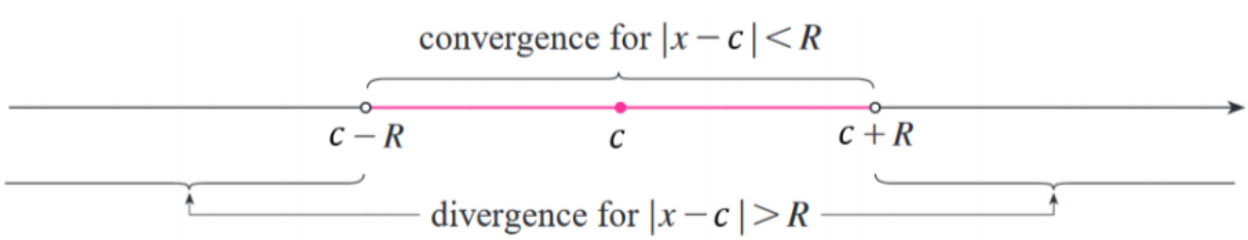
For a power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

exactly one of the following is true:

- The series converges only for $x = c$.
- The series converges absolutely for all x.
- There is a positive number R for which the series converges absolutely for all x, $|x-c| < R$, and diverges for all x, $|x - c| > R$.

The behavior of the series at $|x - c| = R$ must be determined separately. The number R mentioned above is called the **radius of convergence**. We call the set of all numbers x for which the power series converges the **interval of convergence**. The interval of convergence may be open, closed, or half-open, depending on the particular series.



DIY 139. Determine the radius of convergence and interval of convergence for the following:

a) $\sum\limits_{n=0}^{\infty} \dfrac{nx^n}{4^n}$

b) $\sum\limits_{n=1}^{\infty} \dfrac{x^{2n}}{n}$

c) $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{(n+2)^{2n}}$

d) $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x-2)^n}{n+1}$

Functions as Power Series

A power series $\sum_{n=0}^{\infty} a_n x^n$, defines a function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots,$$

where the domain of f is the interval of convergence of the power series.

At the beginning of this discussion we noticed that if we let the sequence of coefficients equal 1 our power series resulted in an expression that resembled the geometric series.

Example 171. Suppose that f is defined by the power series $f(x) = \sum_{n=0}^{\infty} x^n$.

a) Find the domain of f.

b) Evaluate $f\left(\frac{1}{2}\right)$ and $f\left(-\frac{1}{3}\right)$.

c) Find f by summing the series.

Definition 9.14: Power Series Result

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n, \text{for } -1 < x < 1.$$

Example 172. Represent the following function as a power series and determine the interval of convergence.

$$h(x) = \frac{1}{3+x}.$$

DIY 140. Represent the following function as a power series and determine the interval of convergence.

a) $g(x) = \dfrac{1}{1-2x^2}$

b) $H(x) = \dfrac{x^2}{1-x}$

Properties of Power Series

We conclude this section with a quick overview of the properties of Power Series. As always, the usefulness of a new object in mathematics is found in its inherent properties. Unfortunately, the proofs for these are out of the scope of this course, so we will only list them and emphasize on those that AP requires.

Theorem 73: Properties of Power Series

If $\sum\limits_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Define

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 + a_2 + \dots, \text{ for } -R < x < R.$$

Then the following are true:

- Multiplication: The product of two power series can be taken as term by term products.
- Continuity: $\lim\limits_{x \to c} \left(\sum\limits_{n=0}^{\infty} a_n x^n \right) = \sum\limits_{n=0}^{\infty} \left(\lim\limits_{x \to c} a_n x^n \right) = \sum\limits_{n=0}^{\infty} a_n c^n$, for $-R < c < R$.
- Differentiation: $\dfrac{d}{dx}\left(\sum\limits_{n=0}^{\infty} a_n x^n \right) = \sum\limits_{n=0}^{\infty} \left(\dfrac{d}{dx} a_n x^n \right) = \sum\limits_{n=0}^{\infty} n \cdot a_n x^{n-1}$.

  In other words, term by term differentiation.
- Integration: $\int_0^x \left(\sum\limits_{n=0}^{\infty} a_n t^n \right) = \sum\limits_{n=0}^{\infty} \left(\int_0^x a_n t^n \right) = \sum\limits_{n=0}^{\infty} \dfrac{a_n x^{n+1}}{n+1}$.

  In other words, term by term integration.

It can be shown that the power series obtained by differentiation or integration leads to the same radius of convergence as the original power series, but the **interval of convergence** is may be different at the endpoints.

Example 173. Find the power series representation for $f(x) = \dfrac{1}{(1-x)^2}$.

DIY 141. Find the power series representation for $f(x) = \ln\left(\frac{1}{1-x}\right)$. Finds its interval of convergence, as well as the value of $\ln 2$.

Example 174. Find the power series representation of $\tan^{-1} x$. Find the radius of convergence as well as the interval of convergence.

9.8 Taylor and Maclaurin Series

We have investigated a few examples of power series and noted that we could create a series expansion representation of functions. In addition, we also created new expansion and representation by taking advantage of the differentiation and integration property of power series.

We are now interested in the following, what happens if we do not have an initial series? If a power series representation exists, what should it look like? Is there a general form? How confident can we be that our functions is closely approximated with the power series expansion?

To begin, consider the power series about $x = c$.

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 \cdots + a_n(x-c)^n + \cdots$$

and suppose that the interval of convergence is the open interval $(c-R, c+R), R > 0$.

Theorem 74: Taylor Series; Maclaurin Series

Let f be a function with derivatives of all orders throughout some open interval containing c, $(c-R, c+R), R > 0$. Then the power series generated by f at $x = c$ can be expressed as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f^{(1)}(c)}{1!}(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

for all numberx x in the open interval. A power series of this form is called a **Taylor Series** of the function f.

The Taylor series centered at $x = 0$ is called a **Maclaurin Series**.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(c) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Example 175. Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $c = 2$. Where, if anywhere, does the series converge to $\frac{1}{x}$?

If we were to only take the first two terms of our Taylor series expansion, we would have the **Linearization** of f.

$$f(x) = f(c) + f'(c)(x - c)$$

Simliarly, if we took the first three terms, then we would have the **Quadratic Approximation** of f.

We have used these methods to approximate $f(x)$ at values of x near c. It just so happens that if f has derivatives of higher order, then it is possible to also attain a higher order polynomial approximation.

Definition 9.15: Taylor Polynomials

Let f be a function with derivatives of order n for $n = 1, 2, 3, \ldots, N$ in some interval containing c. Then for any integer n from 0 to N, the **Taylor polynomial of order** n generated by f at $x = c$is the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

Example 176. Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$. (i.e. Find the Maclaurin Series of $f(x) = e^x$.)

DIY 142. Find the Taylor series and the Taylor polynomials generated by $f(x) = \cos(x)$ at $x = 0$. (i.e. Find the Maclaurin Series of $f(x) = \cos(x)$.)

DIY 143. If

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

although difficult, it can be shown that the function has derivatives of all orders at $x = 0$. Find the Maclaurin Series of f.

Convergence of a Taylor Series

If we consider the Taylor polynomial of order n then we will be assuming a certain amount of round-off. Having this perspective means that we should be able to re-write our function f as

$$f(x) = P_n(x) + R_n(x), \text{ for } x \in I$$

where R_n is the **remainder**.

With this our question boils down to, well, how can we be sure R_n will be small enough such that $f(x) = P_n(x)$ and moreover, is it possible for $f(x)$ to be represented **infinite Taylor polynomial**? Well, as we have seen, adding more and more terms to our Taylor polynomial will make our approximation more and more accurate. With this, we can now say something about $R_n(x)$.

But before we do, we must equip ourselves with a theorem that Generalizes the Mean Value Theorem, Taylor's Theorem.

Theorem 75: Taylor's Theorem

If f and its first n derivatives are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

So, let's now tie this all together by holding changing a to c and keeping it fixed and changing b to x and allowing this to be the independent variable:

Theorem 76: Taylor's Formula

Let f be a function whose first $n+1$ derivatives are continuous on an open interval I containing the number c. Then for every x in I, there is a number c between x and a for which

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(u)}{(n+1)!}(x-c)^{n+1}, \text{for some } u \text{ between } x \text{ and } c$$

is the remainder after the $n+1$ terms. R_n is known as the **Lagrange Form** of the remainder.

When possible we will also approximate the error:

Theorem 77: Lagrange Error Bound

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and c inclusive, then the remainder R_n in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-c|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then (infinite) series converges to $f(x)$.

Example 177. Show that the Taylor series for $f(x) = e^x$ centered at $x = 0$ converges for all x.

DIY 144. Show that the Taylor series for $f(x) = \cos(x)$ centered at $x = 0$ converges for all x.

DIY 145. Find the Maclaurin expansion for $f(x) = \sin(x)$ and show that it converges to $\sin(x)$ for all x.

Example 178. Using known series, find the first few terms of the Taylor series for the given function using power series operations.

a) $\frac{1}{3}(2x + x\cos(x))$

b) $e^x \cos(x)$

c) $\cos(2x)$

d) $\sin(2x)$