2.3 Fundamental Theorem of Calculus

Name: _____

Fundamental Theorem of Calculus I: If

$$F(x) = \int_a^x f(t)\,dt,$$

Then

$$F'(x) = f(x).$$

There are a few generalizations of this that are also very useful to know. For instance, if $g(x)$ is another function, and

$$G(x) = \int_a^{g(x)} f(t)\,dt$$

Then we have $G(x) = F(g(x))$, so that the chain rule gives

$$G'(x) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Also, the endpoints can be switched. For instance,

$$\frac{d}{dx}\int_x^a f(t)\,dt = \frac{d}{dx}\left(-\int_a^x f(t)\,dt\right) = -f(x).$$

Most generally, we have

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))h'(x) - f(g(x))g'(x).$$

71-84 Use FTC I to find the derivative of the given function.

1.

$$F(x) = \int_{\pi x}^{x^3} \frac{\sin(t)}{t}\,dt.$$

SOLUTION:

$$\begin{aligned}
\frac{d}{dx}\int_{\pi x}^{x^3} \frac{\sin(t)}{t}\,dt &= \frac{d}{dx}\int_0^{x^3} \frac{\sin(t)}{t}\,dt + \frac{d}{dx}\int_{\pi x}^{0} \frac{\sin(t)}{t}\,dt \\
&= \frac{d}{dx}\int_0^{x^3} \frac{\sin(t)}{t}\,dt - \frac{d}{dx}\int_0^{\pi x} \frac{\sin(t)}{t}\,dt \\
&= \left(\frac{\sin(x^3)}{x^3}\right)(3x^2) - \left(\frac{\sin(\pi x)}{\pi x}\right)(\pi) \\
&= \frac{3\sin(x^3) - \sin(\pi x)}{x} \quad \square.
\end{aligned}$$

2.

$$F(x) = \int_{-2}^{\cos(x)} \sqrt{1+t^3}\, dt.$$

3.

$$F(x) = \int_{5x+1}^{3x-2} \cos(t^2)\, dt.$$

4.

$$F(x) = \int_{-x^2}^{x^2} \frac{\sin(t)}{t}\, dt.$$

The second fundamental theorem of calculus tells us that area can be evaluated by antiderivatives.

> Fundamental Theorem of Calculus II: If f is continuous on $[a,b]$ and F is an antiderivative of f, then
>
> $$\int_a^b f(t)\, dt = F(b) - F(a).$$

Since we will see $F(b) - F(a)$ so much in the future, we often abbreviate it as $F(x)|_a^b$.

29-54 Evaluate the integrals by the Fundamental Theorem of Calculus.

1.
$$\int_{-1}^{2} x^2\,dx.$$

SOLUTION:

$$\begin{aligned}\int_{-1}^{2} x^2\,dx &= \frac{1}{3}x^3\Big|_{-1}^{2} \\ &= \left(\frac{1}{3}(2^3)\right) - \left(\frac{1}{3}(-1)^3\right) \\ &= \frac{8}{3} + \frac{1}{3} = 3 \quad \square.\end{aligned}$$

2.
$$\int_{-1}^{2} x^2 - 2x + 2\,dx$$

3.
$$\int_{1}^{2} \frac{2}{x^2}\,dx.$$

4.
$$\int_{0}^{\pi} \sin(x)\,dx$$

5.
$$\int_{0}^{\frac{\pi}{3}} \sec^2(x)\,dx$$