2.1 Definite Integrals.

Name: ________________________________

We can evaluate the area under a function $f(x)$ using limits of the formulas from the previous section. This is called the definite integral. The right endpoint formula gives

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x.$$

$$\Delta x = \frac{b-a}{n}.$$

$$x_k = a + \left(\frac{b-a}{n}\right)k.$$

If $f(x) > 0$, this is the area between $f(x)$ and the x-axis, from a to b. If $f(x)$ changes sign, we have the 'net area', where the areas under the x-axis are subtracted.

Use the limit formula above to find the definite integral.

1.

$$f(x) = \int_1^3 6 + 2x\,dx.$$

SOLUTION:

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}.$$

$$x_k = 1 + \left(\frac{2}{n}\right)k = 1 + \frac{2k}{n}.$$

So

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(6+2\left(1+\frac{2k}{n}\right)\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(8+4\frac{k}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{16}{n}\right)+\left(\sum_{k=1}^{n}\frac{8k}{n^2}\right) \\
&= \lim_{n\to\infty}\frac{16}{n}\sum_{k=1}^{n}1+\frac{8}{n^2}\sum_{k=1}^{n}k \\
&= \lim_{n\to\infty}\frac{16}{n}(n)+\frac{8}{n^2}\frac{n(n+1)}{2} \\
&= \lim_{n\to\infty}16+4\left(\frac{n^2+n}{n^2}\right) \\
&= 16+4(1) = 20 \quad \square.
\end{aligned}$$

2.

$$\int_0^3 2x - 3\,dx.$$

(Hint: You should get 0. Draw a picture to see why.)

3.

$$\int_0^2 x^3\,dx$$

4.

$$\int_1^2 3x^2 + 1\,dx$$

5.

$$\int_{-1}^{1} 3x^2\,dx.$$

Evaluate the definite integral by sketching the region and using an elementary school geometry formula.

1.

$$\int_{-5}^{0} \sqrt{25 - x^2}\,dx.$$

SOLUTION: $y = \sqrt{25 - x^2}$ is the top half of the circle $x^2 + y^2 = 25$. The area is that of a quarter-circle of radius 5. So

$$\text{Area} = \frac{1}{4}(\pi(5^2)) = \frac{25\pi}{4} \quad \square.$$

2.

$$\int_{-3}^{3} \sqrt{9 - x^2}\,dx.$$

3.
$$\int_0^3 |x-1|\,dx.$$

4.
$$\int_0^3 2-x\,dx.$$

(Hint: Net area!)

5.
$$\int_{-1}^2 2-|x|\,dx.$$

6. $\int_{-1}^1 f(x)\,dx$, where

$$f(x) = \begin{cases} x+1 & -1 \le x < 0 \\ \sqrt{1-x^2} & 0 \le x \le 1 \end{cases}$$