The Wreath Product

Motivation:

- Abstract Algebra as a class itself is usually represented, to the naive spectator, as an archetypical embodiment of, a Rubik's cube.
  - Symmetries of a Rubik's Cube
  - Rubik's Cube Group
- Rooted Trees
  - Sylow p-subgroup of S_{p^n} for $n \geq 1$ (Generalized Symmetric Group)
  - Young's Lattice (Bratteli's Diagrams)

Consider the following problem:

Classify the groups of order 18*.*

$\rightarrow$ By the Sylow Theorems we have that $|G| = 2 \cdot 3^2$.

So it follows by the Third Sylow Theorem that $n_2 \in 1, 3, 9$ and $n_3 \in 1, 2$

Now, it must follow that $n_3 = 1$ as $2 \not\equiv 1(\text{mod}3)$. Thus, we must have a subgroup of G such that the subgroup is of order 9.

Let H=(subgroup of order 9); therefore up to isomorphism we have

$$H \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \qquad \text{or} \qquad H \cong \mathbb{Z}/9\mathbb{Z}$$

. We could proceed and find the groups of order 18 up to isomorphism by considering cases, but this was done in [2].

So, we instead turn to the (internal) semi-direct product we have

$$G \cong H \rtimes \mathbb{Z}/2\mathbb{Z}$$

. Now we have the following classifications based on whether $\mathbb{Z}/2\mathbb{Z}$ acts on H via $+1$ or -1.

$$H \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

Under $+1$**:**

$$G \cong \left(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}\right) \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \qquad \text{or} \qquad G \cong \mathbb{Z}/3\mathbb{Z} \times S_3 \ (W\text{reath Product})$$

Under -1**:**

$$G \cong \left(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}\right) \rtimes \mathbb{Z}/2\mathbb{Z} \ (F\text{robenius Group})$$

$$H \cong \mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

Under $+1$**:**

$$G \cong \left(\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\right) \cong \mathbb{Z}/18\mathbb{Z} \qquad \text{or} \qquad G \cong D_{18}$$

Now we look closer at $\mathbb{Z}/3\mathbb{Z} \times S_3$ since it is our first encounter with a Wreath Product, but first a definition:

Definition 0.1: Wreath Product

Consider G and H as groups, then

$$\underbrace{G \times G \times \ldots \times G}_{\text{H - times}} \rtimes H$$

Where the (internal) semi-direct product has $H \curvearrowright G^H$ and $H \to Aut(G^H)$.

Example 1. Sylow Subgroups of Symmetric Groups:

Consider finding the Sylow 3-subgroups of S_{27}.

Example 2. Sylow 2-subgroups of S_{13}.

Proposition: Every $p-$subgroup is within some S_n. In particular, it must be in one of the wreaths.

This proposition might help us try and classify all p-subgroups, but this wouldn't be too efficient, so we instead focus on another aspect of the Wreath Product for now. We will now define the Wreath Product from a more generalized perspective.

Definition 0.2: Wreath Product (D&F)

Let G and L be groups. Let $n \in \mathbb{Z}^+$, and $\rho : G \to S_n$ be a homomorphism. Define

$$H := \underbrace{L \times L \times L \ldots \times L}_{n-times}$$

Recall that $\psi : S_n \to Aut(H)$ is an injective homomorphism by

$$\psi_\pi(l_1, l_2, \ldots, l_n) = (l_{\pi^{-1}(1)}, l_{\pi^{-1}(2)}, \ldots, l_{\pi^{-1}(n)}) \text{ and } \psi_{\pi_1} \circ \psi_{\pi_2} = \psi_{\pi_1\pi_2}$$

where $\pi \in S_n$ is a fixed.

Then $\psi \circ \rho$ is a homomorphism from $G \to Aut(H)$. We define the wreath product of L by G as the internal semi-direct product $H \rtimes G$ with respect to this homomorphism. We denote the wreath by:

$$L \wr G$$

If ρ is the left regular representation of G, then

$$\rho_g : h \to gh \qquad \forall h \in G$$

and

$$(\rho_g f)(x) = f(\rho_{g^{-1}}(x)) = f(g^{-1}(x))$$

Now considering to arbitrary elements under this new group, we define the operation $(\star)$ for this group as :

$$(f, b) \star (f', b') = (fbf', bb')$$

where we have $(fbf')(\gamma) = f(\gamma)f'(b^{-1}\gamma)$.

Now, as coordinates, we have the following:

$$(f, b) \star (f', b') = \Big((f(1), f(2), f(3), \ldots f(n)), b\Big) \star \Big((f'(1), f'(2), f'(3), \ldots f'(n)), b\Big)$$

Equipped with this we can now understand the symmetries of a Rubik's cube with a bit more depth, we can find a representation of the Generalized Symmetric Group, and we can show an analogous theorem to Fermat's Little theorem with Sylow $p-$subgroups. For this we consider groups A, B such that they are of finite order.

Definition 0.3: Wreath Product as Exponentiation of Sets

The wreath product of A by B is defined by the semi-direct product group $A^B \rtimes B$, where B acts on the index set of A^B via left multiplication. We write any element of $A \wr B$ in the canonical form where for $f \in A^B$ such that $f(b) = a$ if $b = b_0 \in B$ which we can write as $\sigma_a(b_0)$ and more explicitly as

$$\sigma_{a_1}(b_1) \ldots \sigma_{a_n}(b_n)\tau(b)$$

Theorem 1

Consider two finite groups, A and B. Let $|B| = p$, and $gcd(|A|, p) = 1$ Then, the number of p-Sylow subgroups in the wreath product $A \wr B$ is $|A|^{p-1}$. Thus we have $|A|^{p-1} \equiv 1 (\text{mod } p)$.

Theorem 2

Let A, B be finite groups, then the normalizer of B in $A \wr B$ is equal to $C \times B$, where C is the subgroup defined as:

$$C := \{\sigma_a(b_1) \ldots \sigma_a(b_n) : a \in A\}$$

and $B = \{b_1, \ldots, b_n\}$.

Theorem 3

Let A, B be finite solvable groups of order a, b with $gcd(a, b) = 1$. Then, the subgroups of order b in $A \wr B$ are conjugate and, are a^{b-1} in number.

References

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