Proof

Mathematical Analysis: Why Newton Step Fails for Bounded Variables

Interior Point Solver Analysis

December 27, 2025

1 Problem Setup

Consider the simple 2D LP:

$$\begin{aligned}
\min_{x,y} \quad & x + y \\
\text{s.t.} \quad & x + y = 1 \\
& 0 \leq x, y \leq 1
\end{aligned}$$

2 Interior Point Method with Barrier

The barrier problem is:

$$\min_{x,y} \quad f(x, y) = x + y - \mu(\log x + \log y + \log(1 - x) + \log(1 - y))$$

subject to $x + y = 1$.

2.1 KKT Conditions

The correct KKT system for box-constrained LP is:

1. **Primal feasibility**: $Ax = b \Rightarrow x + y = 1$
2. **Dual feasibility**: $c - A^T y - s + z = 0 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} y - s + z = 0$
3. **Lower bound complementarity**: $XSe = \mu e \Rightarrow x_i s_i = \mu$
4. **Upper bound complementarity**: $(U - X)Ze = \mu e \Rightarrow (1 - x_i)z_i = \mu$

where:

- $s \in \mathbb{R}^n$: dual variables for $x \geq 0$
- $z \in \mathbb{R}^n$: dual variables for $x \leq u$
- $X = \text{diag}(x)$, $S = \text{diag}(s)$, $Z = \text{diag}(z)$

3 Current Implementation Bug

The current code uses only ONE dual variable s and tries to encode both lower and upper bounds into it.

From `interior_point.cpp`:

```
Vector r_dual = -prob.c + A_sparse.transpose() * y + s;
r_dual(i) -= mu_ / x(i);            // Lower: -/x
r_dual(i) += mu_ / gap_upper;        // Upper: +/(u-x)
```

This attempts to solve:

$$-c + A^T y + s - \frac{\mu}{x} + \frac{\mu}{u - x} = 0$$

But this is mathematically incorrect! The correct formulation needs TWO separate dual variables:

$$\begin{aligned} -c + A^T y + s - z &= 0 \\ s_i &= \mu / x_i \\ z_i &= \mu / (u_i - x_i) \end{aligned}$$

4 Why Current Newton Step Fails

4.1 Hessian Computation

Current code (INCORRECT):

```
H_diag(i) = s(i) / x(i) + mu_ / (gap_upper^2);
```

This mixes the dual variable s (which should only correspond to lower bounds) with the upper bound barrier term.

Correct Hessian should be:

$$H = SX^{-1} + Z(U - X)^{-1}$$

$$H_{ii} = \frac{\mu}{x_i^2} + \frac{\mu}{(u_i - x_i)^2}$$

4.2 Reduced System

For our 2D problem with $A = [1, 1]$:

Current (wrong):

$$(AH^{-1}A^T)dy = r_{prim} - AH^{-1}r_{dual}$$

where H incorrectly mixes s with the barrier term, leading to an inconsistent system.

5 Proof of Failure

Theorem 1. *The current implementation cannot converge for problems with finite upper bounds.*

Proof.

1. The dual variable s is initialized from $s = c - A^T y$
2. The Newton step updates ds via:

$$ds(i) = (\mu - s(i) \cdot dx(i))/x(i) - s(i)$$

3. This formula assumes s corresponds ONLY to lower bounds: $x \cdot s = \mu$
4. But the gradient residual includes BOTH $-\mu/x$ and $+\mu/(u - x)$ terms
5. These two requirements are contradictory: s cannot simultaneously satisfy:
   - $x_i s_i = \mu$ (lower bound complementarity)
   - AND encode information about $(u_i - x_i)$ (upper bound complementarity)
6. From lower complementarity: $s_i = \mu/x_i$
7. From incorrectly using same s for upper: $s_i = \mu/(u_i - x_i)$
8. These are equal only if: $x_i = u_i - x_i \Rightarrow x_i = u_i/2$
9. This only holds at the midpoint! For general x, the system is overdetermined.

Therefore, the KKT system has no solution. □ □

6 Solution

Add explicit dual variables z for upper bounds:

```
struct LPSolution {
    Vector x;    // Primal
    Vector y;    // Dual for Ax = b
    Vector s;    // Dual for x  0
    Vector z;    // Dual for x  u  [NEW]
    ...
};
```

Update Newton step to solve the correct 4-block system: (dx, dy, ds, dz) instead of (dx, dy, ds).