CS70 Extra Problems 0

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January 29, 2019

1 Warm-Up: Liars and Truth-Tellers

You arrive on an island with 2 types of people: liars and truth-tellers. Everyone knows everything about each other, liars always lie, and truth-tellers always tell the truth.

Problem 1.1. You encounter Alice, Bob, and Charlie.
Alice says something but you can't hear her.
Bob says "Alice said she is a liar."
Charlie says "Don't listen to Bob, he is a liar."
What types of people are Bob and Charlie?

Proof. Bob is a liar and Charlie is a truth-teller. If Bob was telling the truth, Alice would be calling herself a liar. This is a contradiction (why?), so Bob is a liar and Charlie is a truth-teller. □

Problem 1.2. You have heard stories of gold on the island, and are allowed to ask a single yes-or-no question to a person from the island. What do you ask?

Proof. We want to determine the value of the proposition $P :=$ "There is gold on the island". One solution is to ask "What would you say if I asked if there was gold?"
If the person is a liar, note that if we directly asked if there was gold, their answer would be $\neg P$. Hence, their answer to our question is "I would say P."
If the person is a truth-teller, if we directly asked about the gold, their answer would be P, and their answer to our final question is also "I would say P."
In both cases, we are able to deduce the true value of P, as desired. □

2 Bite-Sized Questions

Problem 2.1. Prove or disprove: $\sqrt{xy}$ is always irrational for distinct primes x, y.

Proof. True. The proof of irrationality is similar to that of $\sqrt{2}$. First, suppose that $\sqrt{xy}$ were rational, so that $\sqrt{xy} = \frac{p}{q}$ for some integers p and q which share no common factors. By squaring both sides and rearranging, we obtain

$$xy = \frac{p^2}{q^2}$$

$$xyq^2 = p^2$$

which tells us that p^2 is divisible by xy.
As a lemma, we claim that if p^2 is divisible by xy, then p is also divisible by xy. This follows from the fact that if p is not divisible by xy, then squaring the prime factorization of p gives a number indivisible by xy.
The lemma tells us that we can express $p = kxy$ for some integer k. Substituting this into the above equality gives us

$$xyq^2 = k^2x^2y^2$$
$$q^2 = k^2xy$$

Now reapplying our lemma implies that q is also divisible by xy. But this contradicts our earlier assumption that p and q shared no common factors by the definition of a rational number. Hence, $\sqrt{xy}$ cannot be rational for distinct primes x, y. □

Problem 2.2. Prove or disprove: The sum of 2 irrational numbers is always irrational.

Proof. False. Consider $\sqrt{2}, -\sqrt{2}$. □

Problem 2.3. In a group of 400 people, at least k of them share a birthday with another person. What is the minimum value of k?

Proof. $k = 36$. In the minimum case, the first 365 people have distinct birthdays, leaving 35 people to distribute among the remaining dates. Then at least $35 + 1 = 36$ people share a birthday by the pigeonhole principle. □

Problem 2.4. Given 5 people anywhere on earth, prove that there exists a closed hemisphere containing at least 4 of them. (*Hint:* A great circle is defined by _ points)

Proof. Take the great circle defined by any 2 of the points. The hemisphere with at least 2 points in it, when closed, contains 4 points. □

3 Long Division... Forever

Problem 3. In decimal long division, the quotient $\frac{p}{q}$ is given by a recursive algorithm:

$$\text{div}(p, q) = \lfloor \frac{p}{q} \rfloor + 0.1 * \text{div}(p - \lfloor \frac{p}{q} \rfloor,\ 0.1 * q)$$

Using the definition of long division, show that the decimal representation of any rational number will have digits that eventually repeat forever.

Proof. Any rational number can be expressed as $\frac{p}{q}$, where p and q are integers. Using the definition of long division, observe that the remainder $p - \lfloor \frac{p}{q} \rfloor$ is always in $\{0, 1, 2, ..., q - 1\}$. We will eventually get the same remainder, at which point the output of the algorithm and hence the digits of $\frac{p}{q}$ will start to repeat forever. □