6. *Experiments with distance concentration.* High-dimensional spaces are full of strange effects. One of these is *distance concentration*, which we'll explore in this problem.

   Suppose we draw n points $x_1, \ldots, x_n$ uniformly at random from the unit sphere in d dimensions, that is, $S^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ (the superscript is $d-1$ because this is a $(d-1)$-dimensional manifold in $\mathbb{R}^d$). When d is small, say 1 or 2 or 3, the distances between the points will take on a fairly broad range of values in $[0, 2]$. But as d grows, the interpoint distances get more concentrated. We will investigate this phenomenon experimentally.

   First off, how does one generate a random point $X \in S^{d-1}$? Here's an easy way to do it:

   - Let $Z_1, \ldots, Z_d$ each be chosen from a standard normal distribution (a Gaussian with mean zero and variance 1).
   - Define $Z = (Z_1, \ldots, Z_d)$ and $X = Z/\|Z\|$.

   (a) Using the procedure above, write a function that returns n points chosen at random from S^{d-1}, given d and n. (In Python, you can sample from the standard normal using `numpy.random.normal`.) For $d = 2, 5, 10, 20, 100$ and $n = 100$:

   - Generate these samples, compute all $\binom{n}{2}$ interpoint distances, and plot a histogram of these values. There should be a separate histogram for each choice of d. In each case, allot 20 bins for the histogram and have the horizontal axis run from 0 (minimum possible distance) to 2 (maximum possible distance).

   (b) In your histograms from part (a), you should see the distances concentrating around a particular value $v \in [0, 2]$ as d grows. What do you think this value is?

   (c) Now focus on a particular choice of d, say $d = 1000$. In $\mathbb{R}^d$, there can be at most $d + 1$ points that are *exactly* the same distance from each other. But there can be $2^{O(d)}$ points that are *approximately* the same distance from each other. To get a taste of this, try out the following procedure:

   - Pick $x^{(1)}$ at random from S^{d-1}
   - For $i = 2, 3, 4, \ldots, 10000$:
     - Generate $x^{(i)}$ at random from S^{d-1}
     - Compute distances from $x^{(i)}$ to $x^{(1)}, \ldots, x^{(d-1)}$. Let u_i be the largest of these distances and s_i the smallest distance.

   (If your computer is sluggish, you might need to limit i to 5000 rather than 10000.) In a single plot, show both the u_i and s_i values for $i > 1$. Set the vertical axis to stretch from 0 (minimum possible distance) to 2 (maximum possible distance).