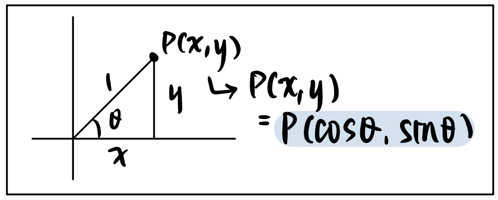
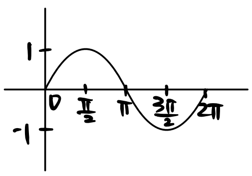
Periodic functions: repeating pattern
↳ Period = n units, where n is the length from one point to the next corresponding point

Base functions: $y=\sin x$, $y=\cos x$, $y=\tan x$ — trigonometric/circle functions
↳ **Sinusoidal** "wave like" → any sine/cosine functions

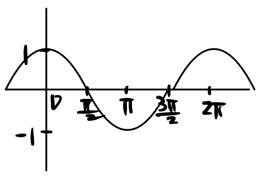


1. $y=\sin x$



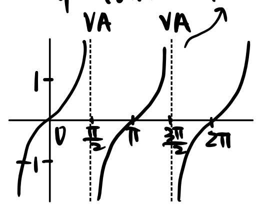
Period $=2\pi$
Range: $-1 \leq y \leq 1$

2. $y=\cos x$



Period $=2\pi$
Range: $-1 \leq y \leq 1$

3. $y=\tan x$ periodic discontinuous function



Period $=\pi$
No restricted range

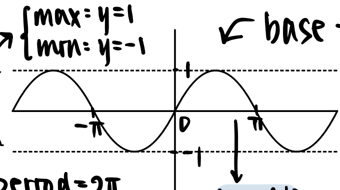
$\sin(\frac{\pi}{2}-\theta)=\cos\theta$
$\cos(\frac{\pi}{2}-\theta)=\sin\theta$
Also $=\cos(\theta-\frac{\pi}{2})$

$\sin x$ and $\cos x$ are horizontal translations of each other

Functions' transformations: $f(x) \rightarrow f(x)=af[b(x-c)]+d$ — horizontal, vertical

General sine function:
$f(x)=a\sin[b(x-c)]+d$

{max: y=1
{min: y=−1



← base function $f(x)=\sin x$
} amplitude (a) = 1
↓
maximum displacement from axis

period $=2\pi$

Axis (d): middle line between top and bottom → $y=0$

a: vertical stretch by a factor of a
↳ $a<0$: reflection in x-axis
b: horizontal stretch by a factor of $\frac{1}{b}$
↳ $b<0$: reflection in y-axis
c&d: translated by a vector of $\binom{c}{d}$

Sinusoidal function transformations:
- Ex. $\boxed{f(x)=-2\sin(\frac{1}{3}(x-\frac{\pi}{4}))-2}$

$a=-2$,
Amplitude $=|a|=2$,
reflected over x-axis

$b=\frac{1}{3}$,
period $=\frac{2\pi}{b}=6\pi$
↳ $6\pi \times \frac{1}{2}=3\pi$ ↗ per ∧ or ∨

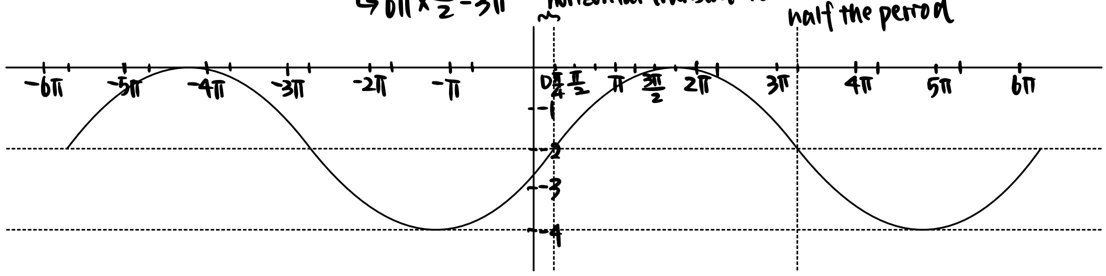
phase shift by $\frac{\pi}{4}$

$c=\frac{\pi}{4}$, $d=-2$
↳ axis $=d=-2$

Graphing:
1. Graph axis & top/bottom lines
2. Note visible increments on the x-axis that can demonstrate precise phase shifts & period

horizontal translation

half the period



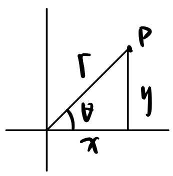
Trig equations & identities:

Equation: "solve for", only true for some values
Identity: "prove" "show that", true for all values

$\underbrace{\text{"prove" "show that"}}$ Solve LS & RS separately
↳ solve the more complicated side first

Axiom: unprovable rule accepted as true because it is self-evident

Basic essential trig identities (no need to prove):



$\rightarrow \boxed{\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x}}$

↳ $\frac{\sin\theta}{\cos\theta} = \frac{y}{r} / \frac{x}{r} = \frac{y}{x} = \tan\theta$

$\therefore \frac{\sin\theta}{\cos\theta} = \tan\theta$ → quotient identity

Pythagorean identities:

$\boxed{x^2+y^2=r^2} \xrightarrow{\text{divide both sides by } r^2} \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$

$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$, $\therefore \cos^2\theta + \sin^2\theta = 1$ → pythagorean identity

divide both sides by x^2:

$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$

$1 + \tan^2\theta = \sec^2\theta$

divide both sides by y^2:

$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}$

$\cot^2\theta + 1 = \csc^2\theta$

$\csc\theta = \frac{1}{\sin\theta}$ → cosecant

$\sec\theta = \frac{1}{\cos\theta}$ → secant

$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$ → cotangent

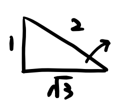
Solving trig equations:

pay attention to domain! more than 1 solution

• Ex. 1) $2\cos\theta - \sqrt{3} = 0,\ -\pi \le \theta \le 2\pi$

$2\cos\theta = \sqrt{3}$

$\cos\theta = \frac{\sqrt{3}}{2}$



reference angle $= \frac{\pi}{6}$

$\theta = \frac{\pi}{6}, \underbrace{\frac{11\pi}{6}, -\frac{\pi}{6}}_{\text{the same place on the unit circle}}$

cos →

$^-$S	A$^+$
$_-$T	C$_+$

3) $2\underbrace{\sin\theta}_{\text{cannot divide across by } \sin\theta\text{!}}\cos\theta = \sin\theta,\ \theta \in [-\pi, 2\pi]$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0,\ \theta = 0, \pi, 2\pi, -\pi$

$2\cos\theta = 1,\ \cos\theta = \frac{1}{2},\ \theta = \frac{\pi}{3}, \frac{5\pi}{3}, -\frac{\pi}{3}$

2) $2\sin x - \cos x = 4 - x,\ x \in [0, 2\pi]$

alternative way to express the domain

★ paper 2 question, not possible to isolate sine or cosine

↓

Use calculator to graph LS & RS, change window to domain

→ take x values of points of intersection

Other identities (in data booklet):

Double angle identities:

1. $\sin 2x = 2\sin x \cos x$
2. $\cos 2x = \cos^2 x - \sin^2 x$
$\quad = 2\cos^2 x - 1$
$\quad = 1 - 2\sin^2 x$

3 possibilities

derived from

Compound angle identities:

1. $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$
2. $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

★ Special angles → $\frac{6\pi}{12}, \frac{4\pi}{12}, \frac{3\pi}{12}, \frac{2\pi}{12}$

compound angles most often have denominator = 12

· Ex. Show that $\frac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1} = \tan\theta$

LS

$$= \frac{2\sin\theta\cos\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1}$$

$$= \frac{\sin\theta(2\cos\theta - 1)}{\cos 2\theta - \cos\theta + 1}$$

$$= \frac{\sin\theta(2\cos\theta - 1)}{2\cos^2\theta \cancel{-1} - \cos\theta \cancel{+1}}$$

$$= \frac{\sin\theta(2\cancel{\cos\theta} - 1)}{\cos\theta(2\cancel{\cos\theta} - 1)}$$

$$= \tan\theta = RS$$

· Ex. Evaluate $2\sin(-\frac{\pi}{12}) + 3\cos(\frac{5\pi}{12})$

$$= 2\sin(\tfrac{3\pi}{12} - \tfrac{4\pi}{12}) + 3\cos(\tfrac{2\pi}{12} + \tfrac{3\pi}{12})$$

$$= 2(\sin\tfrac{\pi}{4}\cos\tfrac{\pi}{3} - \cos\tfrac{\pi}{4}\sin\tfrac{\pi}{3})$$
$$+ 3(\cos\tfrac{\pi}{6}\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{6}\sin\tfrac{\pi}{4})$$

$$= 2\left(\tfrac{\sqrt{2}}{2} \times \tfrac{1}{2} - \tfrac{\sqrt{2}}{2} \times \tfrac{\sqrt{3}}{2}\right) + 3\left(\tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{2}}{2} - \tfrac{1}{2} \times \tfrac{\sqrt{2}}{2}\right)$$

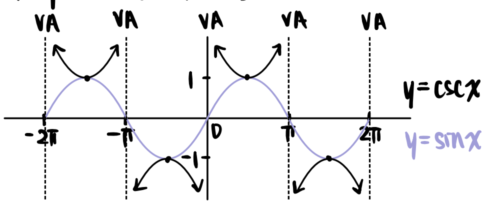
$$= 2\left(\tfrac{\sqrt{2}}{4} - \tfrac{\sqrt{6}}{4}\right) + 3\left(\tfrac{\sqrt{6}}{4} - \tfrac{\sqrt{2}}{4}\right)$$

$$= \frac{2\sqrt{2}}{4} - \frac{2\sqrt{6}}{4} + \frac{3\sqrt{6}}{4} - \frac{3\sqrt{2}}{4}$$

$$= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

Reciprocal trig functions: NOT INVERSE

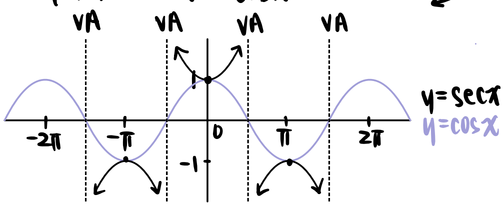
1. $f(x) = \csc x = \frac{1}{\sin x}$



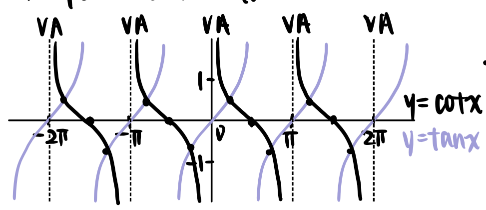
- Points at peaks/troughs of $\sin x$ remain ($\frac{1}{1} = 1$, $\frac{1}{-1} = -1$)
- When $\sin x = 0$, $\csc x = \frac{1}{0} =$ undef. → VA
- As $\sin x$ approaches 0, csc tends to ∞
- No values from $-1 \le x \le 1$

Same idea for $f(x) = \sec x$

2. $f(x) = \sec x = \frac{1}{\cos x}$



3. $f(x) = \cot x = \frac{1}{\tan x}$



- Multiples of $\frac{\pi}{4}$ stay
  ↳ $\tan(\frac{\pi}{4}) = 1$, $\frac{1}{1} = 1$
- $\frac{1}{\tan(\frac{\pi}{2})} = \frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} = 0$
  ↳ Multiples of $\frac{\pi}{2}$ become the zeros

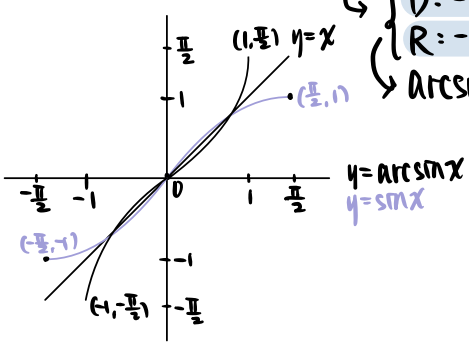
Inverse trig functions: reflected over $y = x$ → arc $\square \neq \square^{-1}$

- In order for a function to have an inverse function, it must be one-to-one ↑ pass vertical & horizontal line test

1. $f(x) = \arcsin x$ → defined function only when restricting domain & range

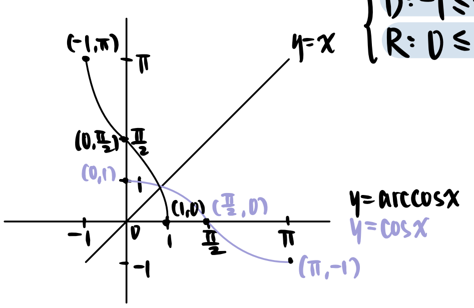
↳ $\begin{cases} D: -1 \le x \le 1 \to D \text{ of } \arcsin x = R \text{ of } \sin x \\ R: -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}$

↳ arcsin x can ONLY BE FOUND with these angles



2. $f(x) = \arccos x$

$\begin{cases} D: -1 \le x \le 1 \\ R: 0 \le y \le \pi \end{cases}$



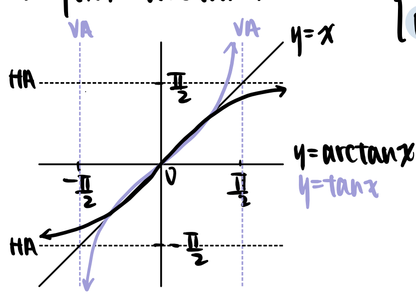
3. $f(x) = \arctan x$

For tan x, $R: y \in \mathbb{R}$

$\begin{cases} D: x \in \mathbb{R} \\ R: -\frac{\pi}{2} \le y \le \frac{\pi}{2} \end{cases}$

↳ 1 section of tan x inverted

VAs become HAs



Solving inverse trig functions:

PAY ATTENTION TO DOMAIN & RANGE

- Ex. 1) Find arccos 1 → inside range
  ↳ $\cos x = 1$, $x = 0$ → inside domain
- 2) Solve: $\arccos x = -\frac{\pi}{4}$ → not inside range
  ↳ $\cancel{\cos(-\frac{\pi}{4}) = x,\ x = \frac{1}{\sqrt{2}}}$ NO SOLUTION

Powers & roots of complex numbers:

conjecture: something true for all cases

De Moivre's Theorem: $(r\,\mathrm{cis}\,\theta)^n = r^n\,\mathrm{cis}(n\theta)$

· Ex. Write $\frac{1+i}{(1-i)^5}$ in cartesian form — general range for arg: $-\pi \leq \theta \leq \pi$

Let $z = 1-i$, $|z| = \sqrt{2}$, $\arg(z) = -\frac{\pi}{4}$, $z = \sqrt{2}\,\mathrm{cis}(-\frac{\pi}{4})$, $z^5 = \sqrt{2}^5\,\mathrm{cis}(-\frac{5\pi}{4})$

Let $w = 1+i$, $|w| = \sqrt{2}$, $\arg(w) = \frac{\pi}{4}$, $w = \sqrt{2}\,\mathrm{cis}(\frac{\pi}{4})$

final angle for polar form

$$\frac{w}{z} = \frac{\sqrt{2}\,\mathrm{cis}(\frac{\pi}{4})}{\sqrt{2}^5\,\mathrm{cis}(-\frac{5\pi}{4})} = \sqrt{2}^{-4}\,\mathrm{cis}\left(\frac{\pi}{4}+\frac{5\pi}{4}\right) = \frac{1}{4}\mathrm{cis}\left(\frac{3\pi}{2}\right) = \frac{1}{4}\mathrm{cis}\left(-\frac{\pi}{2}\right)$$

$$= \frac{1}{4}\cos\left(-\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(-\frac{\pi}{2}\right) = -\frac{1}{4}i$$

If $r_1\,\mathrm{cis}\,\theta = r_2\,\mathrm{cis}\,\theta$, then $r_1 = r_2$ & $\theta_1 = \theta_2 + 2k\pi$, $k \in \mathbb{Z}$

Fundamental theorem of algebra: n solutions to an equation $z^n = w$ — roots of z^n all have the same modulus $|z|$

· Ex. find the 4^{th} roots of unity (4 distinct roots) → use n^{th} root method

$1 = \mathrm{cis}\,0$, let $z = r\,\mathrm{cis}\,\theta$ → $r^4\,\mathrm{cis}\,4\theta = \mathrm{cis}\,0$

↳ $r = 1$ (r is modulus, always positive)

↳ $4\theta = 0 + 2\pi k$, $\theta = \frac{2k\pi}{4} = \frac{k\pi}{2}$

$z = \mathrm{cis}\frac{k\pi}{2}$, $k = 0, 1, 2, 3$

↳ always go in ascending order from 0

OR: $(r\,\mathrm{cis}\,\theta)^4 = \mathrm{cis}\,0$

$r\,\mathrm{cis}\,\theta = \mathrm{cis}(2\pi k)^{\frac{1}{4}}$

$= 1^{\frac{1}{4}}\,\mathrm{cis}(\frac{1}{4} \times 2\pi k)$

$= \mathrm{cis}(\frac{\pi k}{2})$

$k_1 = 0$, $z_1 = \mathrm{cis}\,0 = 1$

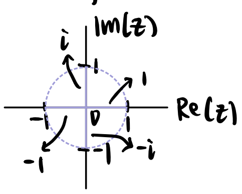
$k_2 = 1$, $z_2 = \mathrm{cis}\frac{\pi}{2} = i$

$k_3 = 2$, $z_3 = \mathrm{cis}\,\pi = -1$

$k_4 = 3$, $z_4 = \mathrm{cis}\frac{3\pi}{2} = -i$

↓

Plotting on an Argand plane:



All roots will be evenly spaced around a circle → same radius

↳ radius = modulus

In real numbers, complex roots occur in conjugate pairs (NOT FOR COMPLEX)

Real function: has all real coefficients

Conjugate root theorem: a zero of an equation is a value z such that $P(z) = 0$

↳ & complex roots of a real function occur in conjugate pairs

· Ex. Given that $1-i$ is a root of $z^3 + 2z^2 - 6z + k = 0$, $k \in \mathbb{R}$, find k

$\because k \in \mathbb{R}$ $\therefore$ Real function & another root is $1+i$

Let the unknown factor $= az+b$

↳ $(z-(1-i))(z-(1+i))(az+b) = (z-1+i)(z-1-i)(az+b)$

$= (z^2 - z - iz - z + 1 + i + iz - i + 1)(az+b) = (z^2 - 2z + 2)(az+b)$

$= (az^3 + bz^2 - 2z^2 - 2bz + 2az + 2b) = z^3 + 2z^2 - 6z + k$

↳ $b - 2 = 2$, $b = 4$, $2b = k$, $k = 8$