

# bivariate regression

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## outline

bivariate regression

hands-on: dofile

interesting properties

stat significance (hypothesis testing)

basic measurement

## math

- ◇ today we start some math
- ◇ important you understand it
- ◇ memorizing formulas is not enough to pass this class
- ◇ again, ask questions early!
- ◇ good idea to go over slides again after the class
- ◇ note hats:  $\hat{\beta}$  v  $\beta$
- ◇ instead of  $\sum_{i=1}^n$  i may just use  $\sum$

## looking ahead

- ◇ repeat today's material next week
  - and extend a bit: esp stat significance
  - and talk about measurement time
  - over time, esp after midterm, class will get more applied
  - and we will have more examples

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## the idea

- ◇  $Y \leftarrow X$ , there is a directional relationship
- ◇ like in correlation, but here there is a direction
  - (almost causality, but to argue causality you need also research design!)
- ◇ so we have outcome, or dependent variable predicted or affected by:
  - independent variable (does not depend on the dependent variable),

## why regression?

- ◇ ols is the most fundamental technique for soc sci
  - and quite powerful
  - things like anova, t-test, z-test, chi-sq test, etc are obsolete!
- ◇ just run regression! indeed, no studies use these anymore
  - the only thing to remember from qm1 is descriptive stats, esp graphs
- ◇ if you want to figure out what predicts something, run regression
  - eg what will make you live longer, or which year wine is

good

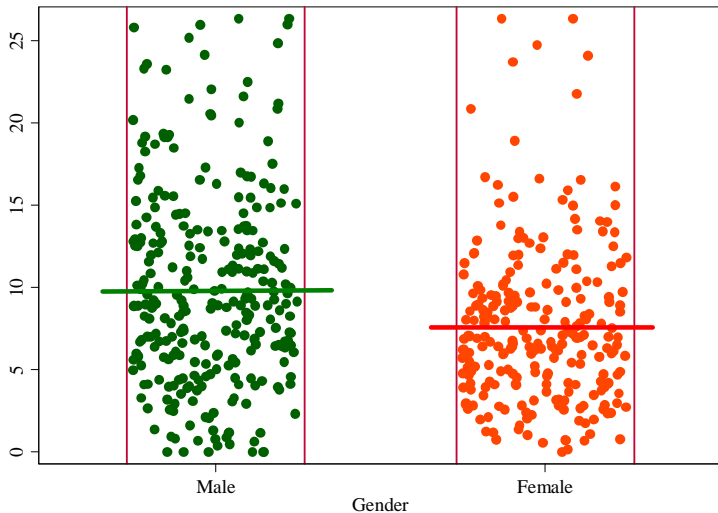
## examples

- ◇ see some of the useful things you can predict
    - eg  $\text{lexp} = \text{weighted avg}(\text{diet}, \text{exercise}, \text{smoking}, \text{etc})$
    - eg  $\text{lexp} = 50 + 2 * (\text{veggie serv/day}) + 3 * (\text{hrs at gym}) - 10 * (\text{packs of cigarettes per day})$
- life expectancy <http://www.northwesternmutual.com/learning-center/the-longevity-game.aspx>
- [\*]<http://ianayres.yale.edu/prediction-tools>



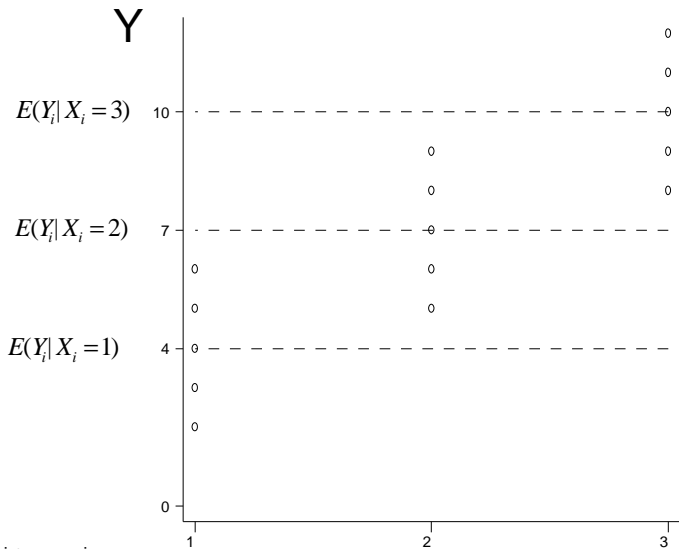
## conditional mean

◇ (useful way to start thinking about regression)



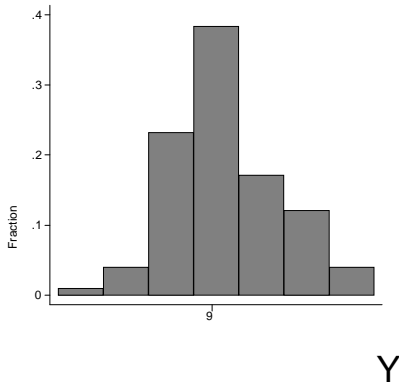
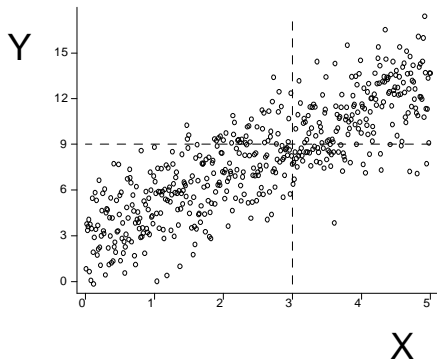
## conditional mean of y depends on x

◇ for each value of  $x(1,2,3)$   $E(y)$  is different



## distribution of Y around the Expected Value

◇ e.g:  $E(Y|X = 3) = 9$ ; Values of Y cluster around 9

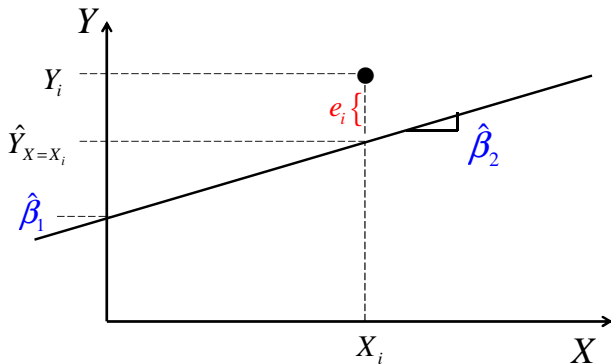


## “regression” sounds scary

- ◇ regression is easy (yes, we will do all the tedious calculations), but all that regression does it fits a line that ...  
minimizes the sum of the squared vertical distances in a scatter plot; hence “OLS” !
- sounds complicated but it's easy, too
- ◇ that's it ! we will be just showing some math that can fit this line

## regression function

◇  $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$     $Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + e_i$



◇  $(e_i)$  are errors of prediction

## what are the disturbance terms?

- ◇  $Y_i = \beta_1 + \beta_2 X_i + u_i$
- ◇  $u_i = Y_i - \beta_1 - \beta_2 X_i = Y_i - E(Y|X_i)$
- ◇ the combined effect of all other variables not in the model
- ◇ random events that affect the outcome
- ◇ errors of measurement in  $Y$  and  $X$

## parameters v estimators

parameters (PRF)	estimators (SRF)
------------------	------------------

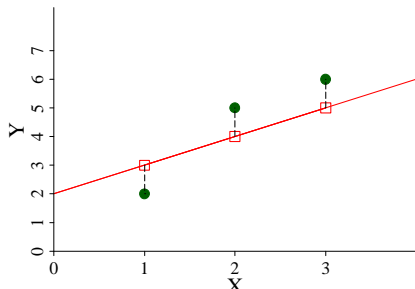
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$\beta_1$	$\hat{\beta}_1$
$\beta_2$	$\hat{\beta}_2$
$\mu$	$\bar{X}$
$\rho$	$\hat{\rho}$
$\sigma$	$s$
$\mu_i$	$e_i$

- ▶ estimators are based on samples
- ▶ parameters are fixed (and usually unknown)
- ▶ estimators have sampling distributions

## first guess

$Y_i$	$X_i$
2	1
5	2
6	3



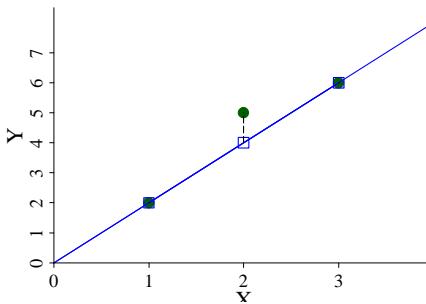
◇ (1)  $Y_i = 2 + X_i \rightarrow \sum e_i^2 = 3$



## second guess

◇

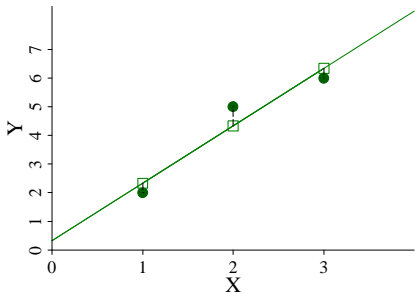
$Y_i$	$X_i$
2	1
5	2
6	3



- ◇ (1)  $Y_i = 2 + X_i \rightarrow \sum e_i^2 = 3$
- ◇ (2)  $Y_i = 0 + 2X_i \rightarrow \sum e_i^2 = 1$

## example – you cannot beat ols!

$Y_i$	$X_i$
2	1
5	2
6	3



- ◇ (1)  $Y_i = 2 + X_i \rightarrow \sum e_i^2 = 3$
- ◇ (2)  $Y_i = 0 + 2X_i \rightarrow \sum e_i^2 = 1$
- ◇ (3)  $Y_i = 0.33 + 2X_i \rightarrow \sum e_i^2 = 0.67$
- ◇ **dofile: guessing**; then can use these est to predict like in

lexp eg

## ols

- ◇  $Y_i = \hat{\beta}_1 - \hat{\beta}_2 X_i + e_i \rightarrow e_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i$
- ◇ chose estimators to minimize
$$\sum e_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$$
- \* for elaboration and derivations see gujarati...

## intercept

◇ Intercept:  $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$

Note: sum of the residuals is zero:  $\sum_{i=1}^n (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)$

## slope

$$\diamond \hat{\beta}_2 = \frac{\sum_{i=1}^n Y_i X_i - n \bar{X} \bar{Y}}{(\sum_{i=1}^n X_i^2 - n \bar{X}^2)}$$

$$\diamond \hat{\beta}_2 = \frac{\sum Y_i X_i - n \bar{Y} \bar{X}}{\sum X_i^2 - n \bar{X}^2}$$

$$\diamond \hat{\beta}_2 = \frac{\sum (Y_i - \bar{Y})(X_i - \bar{X})}{\sum (X_i - \bar{X})^2}$$

$$\diamond \hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} \quad y_i = Y_i - \bar{Y} \quad x_i = X_i - \bar{X}$$

- Another way to look at the slope coefficient is the covariance of Y and X divided by the variance of X. Since the variance is always positive, the numerator (the covariance) will determine the sign of the slope.

## solving the problem

	$Y_i$	$X_i$	$(Y_i - \bar{Y})$ $= y_i$	$(X_i - \bar{X})$ $= x_i$	$y_i^2$	$x_i^2$	$y_i x_i$
	2	1	-2.33	-1	5.53	1	2.33
	5	2	0.67	0	0.45	0	0
	6	3	1.67	1	2.79	1	1.67
$\Sigma$	13	6	0	0	8.67	2	4
<i>mean</i>	4.33	2					

◇

$$\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} = \frac{4}{2} = 2$$

◇

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 4.33 - (2)(2) = 0.33$$

## example: age and fear

- ◇ In this example, imagine that we have some sort of survey that measures people's fear of crime, and that our hypothesis is that fear of crime increases with age. Assume the fear measure is an index ranging from 0 to 15.
- ◇ First, we calculate the means. Second, we calculate the deviations from the means and the their squares for each observation, as well as the co-product of the X and Y deviations. Finally, we sum these up.
- ◇ **blackboard! all steps!**

## example: age and fear

The Data

obs	$X_i$	$Y_i$
1	22	2
2	35	7
3	47	6
4	56	14
5	72	13
$\Sigma$	232	42

$$\bar{X} = \frac{232}{5} = 46.4$$

$$\bar{Y} = \frac{42}{5} = 8.4$$

Deviations from the means

Obs	$x_i$	$x_i^2$	$y_i$	$y_i^2$	$x_i y_i$
1	-24.4	595.36	-6.4	40.96	156.16
2	-11.4	129.96	-1.4	1.96	15.96
3	0.6	0.36	-2.4	5.76	-1.44
4	9.6	92.16	5.6	31.36	53.76
5	25.6	655.36	4.6	21.16	117.76
$\Sigma$	0	1473.2	0	101.2	342.2

◇

$$\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} = \frac{342}{1473} = .232$$

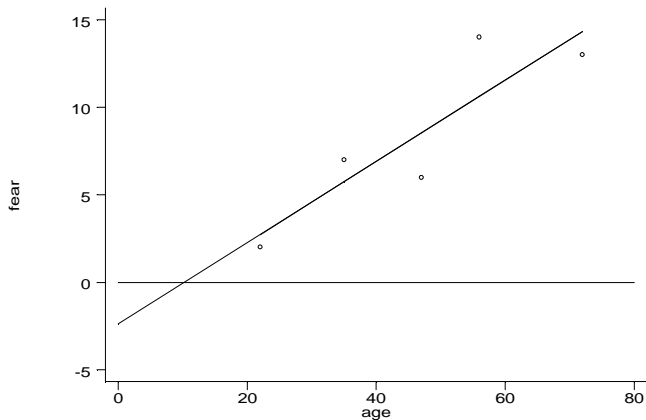
$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 8.4 - (.232)(46.4) = -2.365$$

$$\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i = -2.365 + .232 X_i$$

◇ how would you interpret this?



## the estimated regression line



## variance and std error of regression

- ◇ ok, we know how to calculate betas and fit the line (that min the sum of the squared resid)
  - ◇ but there are lines that fit better and lines that fit worse in different samples
- draw good and bad fits with same betas
- ◇ we need a measure of uncertainty, i.e. how well our line fit the data...
  - ◇ and the fit is measured by residuals...
  - ◇ ... so our measure of uncertainty has to do with residuals !

## variance and std error of regression

$$\diamond s^2 = \frac{\sum_{i=1}^n (e_i - \bar{e})^2}{n-2} = \frac{\sum_{i=1}^n e_i^2}{n-2}$$

$$\diamond s = \sqrt{\frac{\sum_{i=1}^n e_i^2}{n-2}}$$

again, the mean of the residuals is zero (hence,  $\bar{e}$  drops out)

◇ why divide by  $n-2$ ?

◇  $s^2$  and  $s$  are measures of the spread of the points around the estimated regression line.

◇ they are estimators of the variance and standard deviation of the disturbance terms:  $\sigma^2$  and  $\sigma$

from  $\hat{Y}$  to  $s$  (se of reg) to  $s_{\hat{\beta}_2}$  (se of slope)

$i$	$\hat{Y}_i$	$e_i$	$e_i^2$
1	2.739	-0.739	0.546
2	5.755	1.245	1.556
3	8.539	-2.539	6.447
4	10.627	3.373	11.377
5	14.339	-1.339	1.793
$\Sigma$		0	21.713

$$\diamond s = \sqrt{\frac{\sum_{i=1}^5 e_i^2}{n-2}} = \sqrt{\frac{21.7}{3}} = 2.7$$

$\diamond$  what is it measuring?

$$\diamond s_{\hat{\beta}_2} = \frac{s}{\sum_{i=1}^5 x_i^2} = \frac{2.7}{\sqrt{1473}} = .07$$

$\diamond$  how does it differ from  $s$ ?

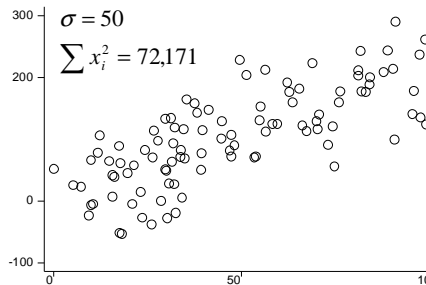
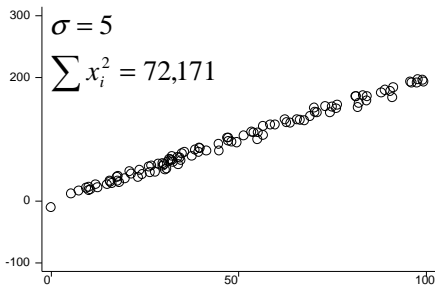
$\diamond$

calc yhats and se of beta!!

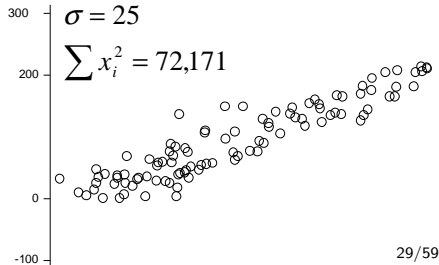
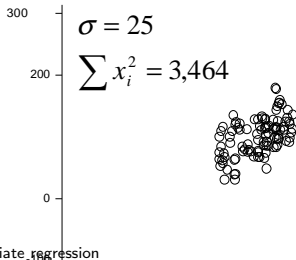
$\diamond$  yahts are important! like our lexp we tried to predict earlier

# Standard Error of the Slope Coefficient

Numerator -- variance of disturbance term



Denominator -- variation in X



## ucla: hands-on dofile

- ◇ <https://stats.idre.ucla.edu/stata/webbooks/reg>
- ◇ let's just see a first reg output (you'll do it for ps2)
  - what is bivariate regression command?
  - where is  $\beta_1$  and  $\beta_2$
- ◇ excellent for self study!!
- ◇ do it at home; and do ask me questions about it if any
- ◇ this is especially an excellent resource for final paper

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## basic calculations **blackboard; dofile**

Y	X	y	y2	x	x2	xy
1	17					
3	13					
5	8					
7	10					
9	2					

Sum:

25      50

$$\bar{Y}=5 \quad \bar{X}=10$$



## the coefficients—interpretation

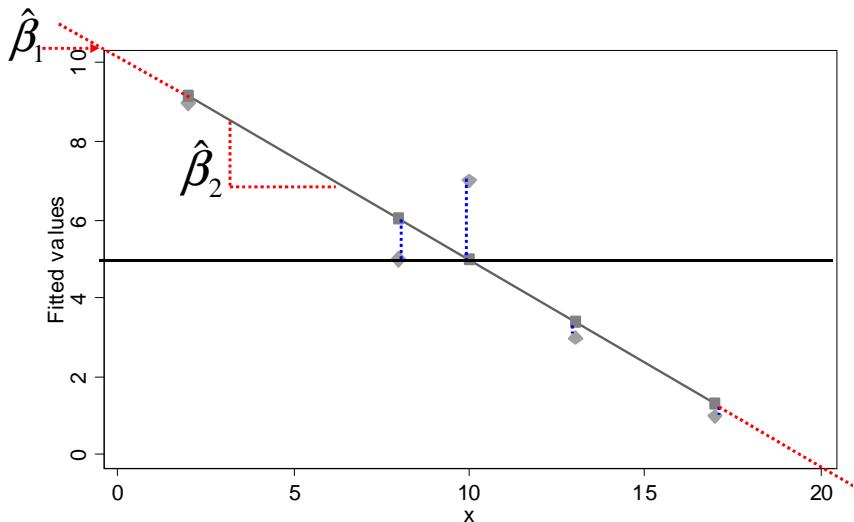
- ◇ Beta hat two is the slope coefficient. Thus, a one unit change in  $X$  leads to a 0.524 decrease in  $Y$ . Beta hat one is the intercept term. It is the predicted value for  $Y$  when  $X$  is equal to zero.

## predicted val and resid **blackboard; dofile**

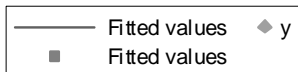
Y	X	Y hat	e	e <sup>2</sup>
1	17			
3	13			
5	8			
7	10			
9	2			

- ◇  $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$
- ◇ for obs 1:
- ◇  $\hat{Y}_1 = 10.24 + (-0.524)(17) = 1.332$
- ◇  $e_1 = 1 - 1.33 = -0.33$

## regression plot again



Stata: graph twoway  
(scatter y x) (lfit y x)  
stat significance (hypothesis testing)



## se of the slope **blackboard; dofile**

$$\diamond \sum e_i^2 = 5.42$$

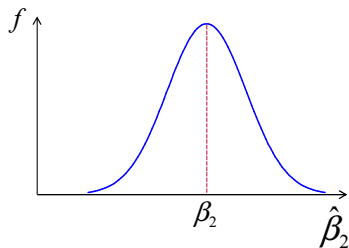
$$\diamond s = \sqrt{\frac{\sum e_i^2}{n-2}} =$$

$$\diamond s_{\hat{\beta}_2} = \frac{s}{\sqrt{\sum x_i^2}}$$

- it gives us info about reliability (like sd or se) of slope

## sampling distribution of the slope

probability distribution of  $\hat{\beta}_2$  is centered on the true value of the parameter (i.e. unbiased) and is normally distributed with variance:



$$\diamond s_{\hat{\beta}_2}^2 = \frac{s^2}{\sum x_i^2}$$

$$\diamond s_{\hat{\beta}_2} = \sqrt{\frac{s^2}{\sum x_i^2}} = \frac{s}{\sqrt{\sum x_i^2}}$$

## hypothesis test **dofile**

- ◇ the null is that slope (“the unobserved true parameter”)
  - is zero (ie no effect)
- ◇  $H_0 : \beta_2 = 0$
- ◇  $H_A : \beta_2 \neq 0$
- ◇  $t = \frac{\hat{\beta}_2 - \beta_2}{s_{\hat{\beta}_2}} = \frac{\hat{\beta}_2}{s_{\hat{\beta}_2}}$
- ◇ CI:  $\hat{\beta}_2 \pm (t_{n-2, \frac{\alpha}{2}})(s_{\hat{\beta}_2})$

## accounting for variation in $Y$ blackboard in 3 colors

◇ before regression  $E[Y] = \bar{Y}$

- TSS total sum of squares

$$TSS = \sum_{i=1}^n (Y_i - \bar{Y})^2$$

◇ after regression

$$E[Y|X_i] = \hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$$

- ESS explained sum of squares

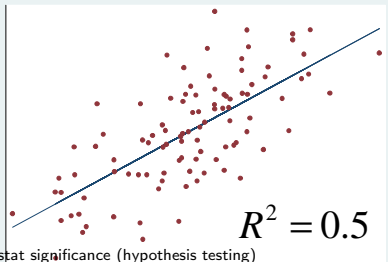
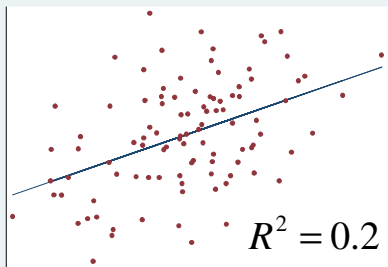
$$ESS = \sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2$$

- RSS residual sum of squares

$$RSS = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^n e_i^2$$

◇  $TSS = ESS + RSS$

## $R^2$ variation explained



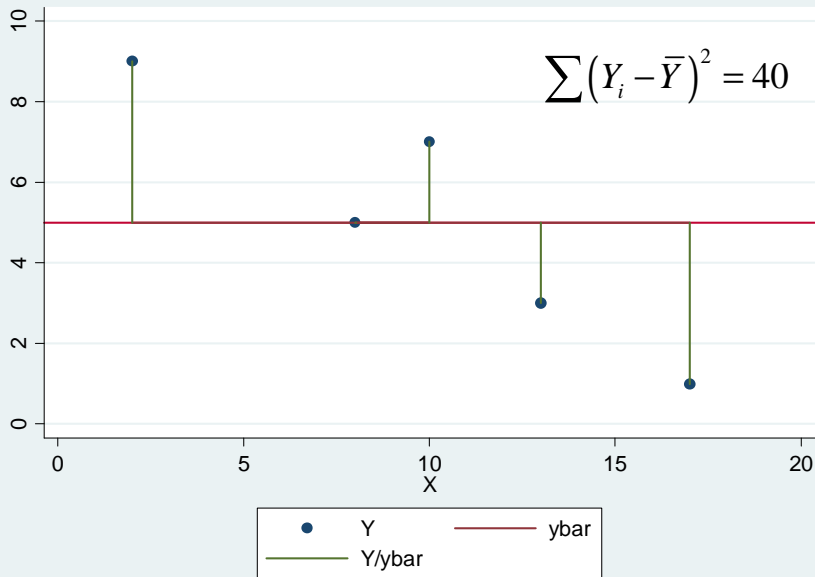
- ◇  $TSS = ESS + RSS$
- ◇  $1 = \frac{ESS}{TSS} + \frac{RSS}{TSS}$
- ◇  $R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum e_i^2}{y_i^2}$
- ◇  $R^2$  : the percent of the variance in the dependent variable explained by the model



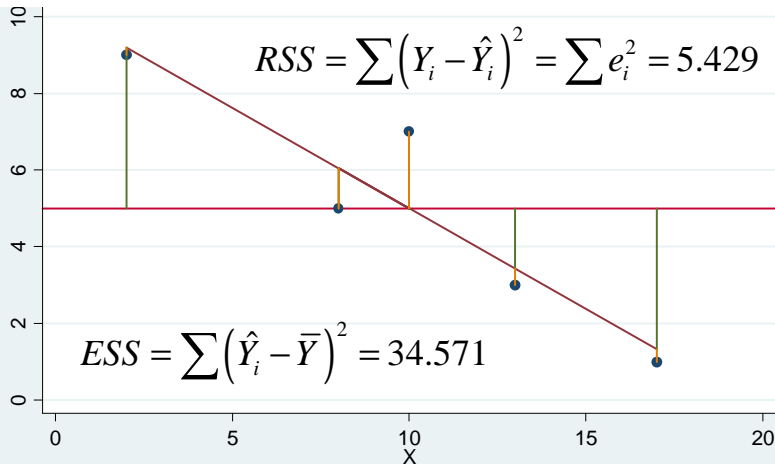
## partitioning variance in Y **dofile**

- ◇ before regression  $E[Y_i] = \bar{Y}$ 
  - $TSS = \sum (Y_i - \bar{Y})^2 = \sum y_i^2 = 40$
- ◇ after regression  $E[Y_i|X_i] = \hat{Y}_i$ 
  - $RSS = \sum (Y_i - \hat{Y}_i)^2 = \sum e_i^2 = 5.43$
  - $ESS = TSS - RSS = 40 - 5.4 = 34.57$
- ◇  $R^2 = 1 - \frac{\sum e_i^2}{\sum y_i^2}$
- ◇ proportion of the total variance in the Y explained by Xs
- ◇  $0 \leq R^2 \leq 1$

# TSS



# RSS



## exercise 1 **dofile**

- ◇ you regressed car's price on its weight

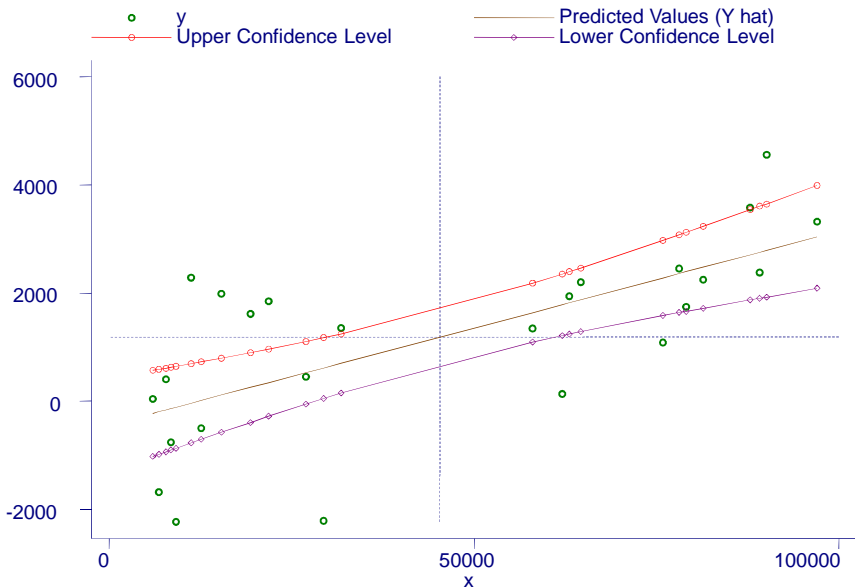
```
-----  
           price |           Coef.   Std. Err.  
-----+-----  
           weight |    2.044063    .3768341
```

- ◇ interpret the coefficient
- ◇ is it significant ?
- ◇ calculate 95% CI

## reliability of predict val (se of $E(Y|X)$ )

- ◇ We have discussed the fact that parameter estimates are random variables, and so they have standard errors. Predicted values are also random variables because they are linear combinations of the coefficients.
- ◇ The further from the mean of  $X$ , the wider the confidence interval around the predicted value.
- ◇ leave it to software, no need to know the formula

# se of $E(Y|X)$ illustration **dofile**



# anatomy of stata output **dofile: outlier**

. regress DV IV

Source	SS	df	MS	Number of obs	=	$n$
Model	$ESS = \sum (\hat{Y}_i - \bar{Y})^2$	1	....	$F(1, n-2)$	=	....
Residual	$RSS = \sum e_i^2$	$n-2$	$s^2 = \frac{RSS}{n-2}$	Prob > F	=	....
Total	$TSS = \sum (Y_i - \bar{Y})^2$	$n-1$	$s_Y^2 = \frac{TSS}{n-1}$	R-squared	=	$r^2$
				Adj R-Squared	=	....
				Root MSE	=	$s$

DV	Coef.	Std.Err.	t	P> t	[95% Conf.	Interval]
IV	$\hat{\beta}_2$	$s_{\hat{\beta}_2}$	$\left( \frac{\hat{\beta}_2}{s_{\hat{\beta}_2}} \right)$	p val. for $H_0$ that $\beta_2 = 0$	$\hat{\beta}_2 - t_{0.025} s_{\hat{\beta}_2}$	$\hat{\beta}_2 + t_{0.025} s_{\hat{\beta}_2}$
Intercept	$\hat{\beta}_1$	$s_{\hat{\beta}_1}$	$\left( \frac{\hat{\beta}_1}{s_{\hat{\beta}_1}} \right)$	p val. for $H_0$ that $\beta_1 = 0$	$\hat{\beta}_1 - t_{0.025} s_{\hat{\beta}_1}$	$\hat{\beta}_1 + t_{0.025} s_{\hat{\beta}_1}$

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## intuition

- ◇ what happens to betas if we change variables' measurement?
  - millions of dollars as opposed to dollars
  - curved grades (each person gets extra 10 points)
  - proportion of people in poverty v percent in poverty
- ◇ income per capita v income per 100k people

## add constant $c$ to $X$ or $Y$ (say curved grades)

- ◇ if you add  $c$  to each obs, mean of var would change by that much
- ◇ but demeaned var doesn't change:
- ◇  $x'_i = (X'_i - \bar{X}') = [(X_i + c) - (\bar{X} + c)] = x_i$  same for  $Y$
- ◇  $\hat{\beta}_2 = \frac{\sum y_i x'_i}{\sum x_i'^2} = \frac{\sum y_i x_i}{\sum x_i^2}$  only demeaned vars so no change
- ◇ and nobody cares about intercept anyway, so let's spare our brain

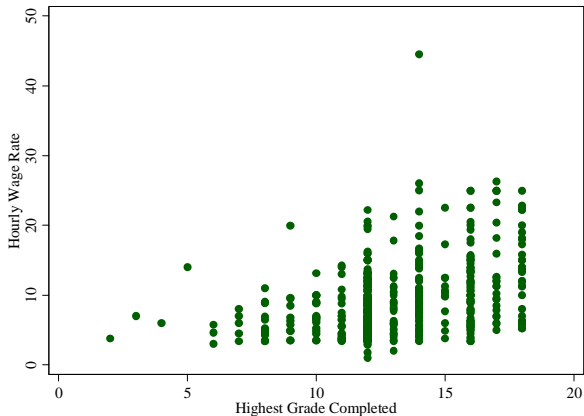
## multiply X or Y by constant (say months, not years)

- ◇ think about it, assume some example
  - say year of educ produces \$2 increase in wage
- ◇ how about a month of educ? should be 1/12 of \$2 !
- ◇ to convert yr to mo, multiply years by 12, right?
  - if a person has 2yr of educ, that's 24mo
- ◇ so if i multiply X by c, say 12, I need to divide  $\hat{\beta}_2$  by 12
- ◇ what if multiply Y?
  - again, say year of educ produces \$2 increase in wage
  - ...or 200 cent increase in wage
- ◇ to get cents from dollars, I multiply dollars by 100
  - so if I multiply Y by 100, i get  $\beta_2$  100x bigger

## fun fact1: correlation v bivariate regression

- ◇  $r = \frac{\sum y_i x_i}{\sqrt{(\sum x_i^2)(\sum y_i^2)}} \quad \hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2}$
- ◇ bivariate slope equals corr coef scaled by std dev of Y and X:  
$$\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} = r \left( \frac{s_Y}{s_X} \right)$$

# education and wages **dofile**



```
. corr wage educ
(obs=534)
```

	wage	educ
wage	1.0000	
educ	0.3819	1.0000

```
. sum wage educ
```

Variable	Obs	Mean	Std. Dev.	Min	Max
wage	534	9.023939	5.138876	1	44.5
educ	534	13.01873	2.615373	2	18

## education and wages **dofile**

```
. regress wage educ
```

Source	SS	df	MS	Number of obs =	534
-----+-----				F( 1, 532) =	90.86
Model	2053.22494	1	2053.22494	Prob > F	= 0.0000
Residual	12022.2635	532	22.5982396	R-squared	= 0.1459
-----+-----				Adj R-squared	= 0.1443
Total	14075.4884	533	26.4080458	Root MSE	= 4.7538

wage	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
-----+-----						
educ	.7504488	.07873	9.532	0.000	.5957891	.9051086
_cons	-.745949	1.045404	-0.714	0.476	-2.799576	1.307678
-----+-----						

The estimated regression line:

$$\widehat{wage}_i = \hat{\beta}_1 + \hat{\beta}_2 educ_i = -0.75 + 0.75 educ_i$$

Interpret the coefficients.

## fun fact2: Z scores bivariate regression=correlation

- ◇  $z_{Yi} = \beta_1 + \beta_2 z_{Xi} + u_i$   
 $z_{Xi} = \frac{X_i - \bar{X}}{s_X} = \frac{x_i}{s_x}$   
 $z_{Yi} = \frac{Y_i - \bar{Y}}{s_Y} = \frac{y_i}{s_y}$
- ◇ z scores always have a mean of 0 and a variance (and standard deviation) of 1
- ◇  $\hat{\beta}_2 = r_{ZY} \frac{s_{ZY}}{s_{ZX}} = r_{YX}$   
 $\hat{\beta}_1 = \bar{z}_Y - \hat{\beta}_2 \bar{z}_X = 0 - r(0) = 0$
- ◇ Thus, a regression of the z scores of Y on the z scores of X produces a slope equal to the correlation coefficient of X and Y and a zero intercept.

exercise 2: if no time do at home: see **dofile**

- ◇ confirm the above in stata using our simple data we started today's lecture with
- ◇ run regression of  $Y$  on  $X$
- ◇ modify  $X$  or  $Y$  and check what happened