bivariate regression

Adam Okulicz-Kozaryn
adam.okulicz.kozaryn@gmail.com

this version: Tuesday 23rd January, 2024 19:39

outline

bivariate regression

stat significance (hypothesis testing)

basic measurement

discussions

- everyone saw discussion i posted on canvas?

math

- today we start some math
- important you understand it
- memorizing formulas is not enough to pass this class
- again, start working on this and ask questions early!
- good idea to go over slides again after the class
- note hats: $\hat{\beta}$ v β
- instead of $\sum_{i=1}^{n}$ i may just use $\sum$

outline

bivariate regression

stat significance (hypothesis testing)

basic measurement

the idea

- $Y \leftarrow \quad X$, there is a directional relationship
- like in correlation, but here there is a direction
  - (almost causality, but to argue causality you need also research design!)
- so we have outcome, or dependent variable predicted or affected by:
  - independent variable (does not depend on the dependent variable),

why regression?

- ols is the most fundamental technique for soc sci
  - anova, t-test, z-test, chi-sq test, etc are obsolete!
- just run regression! indeed, no studies use these anymore
  - from qm1 only use des sta esp graphs
- if you want to figure out what predicts something, run regression
  - eg what will make you live longer, or which year wine is good
  - eg lexp=weighted avg(diet, exercise, smoking, etc)
  - eg lexp=50+2*(veggie serv/day)+3*(hrs at gym)-10*(packs of cigarettes per day)

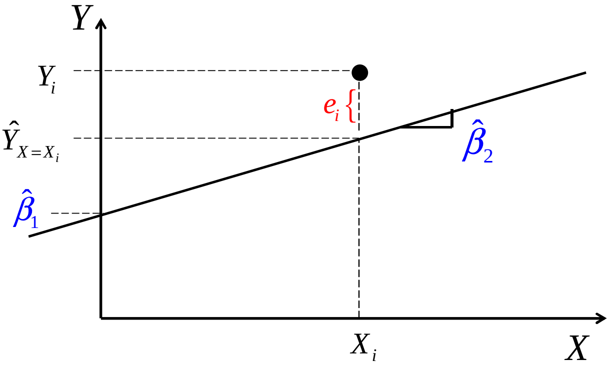
http://ianayres.yale.edu/prediction-tools

“regression” sounds scary

- regression is easy (yes, we will do all the tedious calculations), but all that regression does it fits a line that ...
  minimizes the sum of the squared vertical distances in a scatter plot; hence “OLS” !
  - sounds complicated but it’s easy, too
- that’s it ! we will be just showing some math that can fit this line

regression function

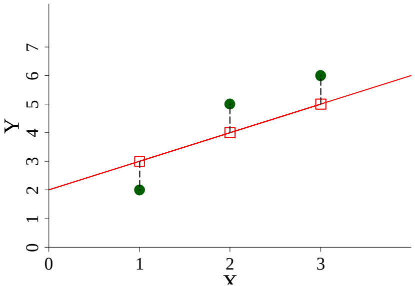
- $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i \quad Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + e_i$



- (e_i) are errors of prediction

first guess

Y_i	X_i
2	1
5	2
6	3

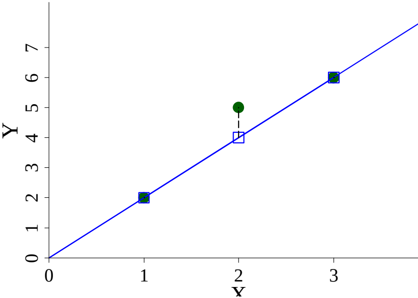


- (1) $Y_i = 2 + X_i \rightarrow \sum e_i^2 = 3$

second guess

-

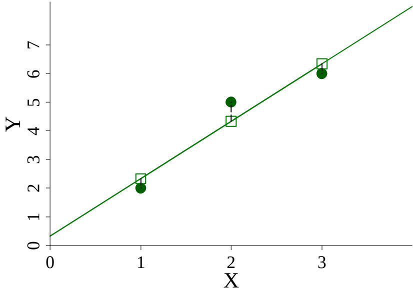
Y_i	X_i
2	1
5	2
6	3



- (1) $Y_i = 2 + X_i \rightarrow \sum e_i^2 = 3$
- (2) $Y_i = 0 + 2X_i \rightarrow \sum e_i^2 = 1$

example – you cannot beat ols!

Y_i	X_i
2	1
5	2
6	3



- (1) $Y_i = 2 + X_i \rightarrow \sum e_i^2 = 3$
- (2) $Y_i = 0 + 2X_i \rightarrow \sum e_i^2 = 1$
- (3) $Y_i = 0.33 + 2X_i \rightarrow \sum e_i^2 = 0.67$
- dofile: guessing can use these est to predict like lexp eg

ols

- $Y_i = \hat{\beta}_1 - \hat{\beta}_2 X_i + e_i \rightarrow e_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i$
- chose estimators to minimize $\sum e_i^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2$
- [*] for elaboration and derivations see gujarati

intercept and slope

- $$\boxed{\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2\bar{X}}$$
- $$\boxed{\hat{\beta}_2 = \frac{\sum_{i=1}^{n} Y_i X_i - n\bar{X}\bar{Y}}{(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2)}}$$
- $\hat{\beta}_2 = \frac{\sum Y_i X_i - n\bar{Y}\bar{X}}{\sum X_i^2 - n\bar{X}^2}$
- $\hat{\beta}_2 = \frac{\sum (Y_i - \bar{Y})(X_i - \bar{X})}{\sum (X_i - \bar{X})^2}$
-
- $\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2}$ $y_i = Y_i - \bar{Y}$ $x_i = X_i - \bar{X}$
- slope is the covariance of Y and X divided by the variance of X; variance is always positive, so the numerator (the covariance) determines the sign of the slope

solving the problem [blackboard from scratch]

	Y_i	X_i	$(Y_i - \bar{Y}) = y_i$	$(X_i - \bar{X}) = x_i$	y_i^2	x_i^2	$y_i x_i$
	2	1	–2.33	–1	5.53	1	2.33
	5	2	0.67	0	0.45	0	0
	6	3	1.67	1	2.79	1	1.67
Σ	13	6	0	0	8.67	2	4
mean	4.33	2					

-
- $\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} = \frac{4}{2} = 2$
- $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 4.33 - (2)(2) = 0.33$

example: age(18-80) and fear(0-15) [blackboard]

The Data

obs	X_i	Y_i
1	22	2
2	35	7
3	47	6
4	56	14
5	72	13
$\sum$	232	42

$$\bar{X} = \frac{232}{5} = 46.4$$

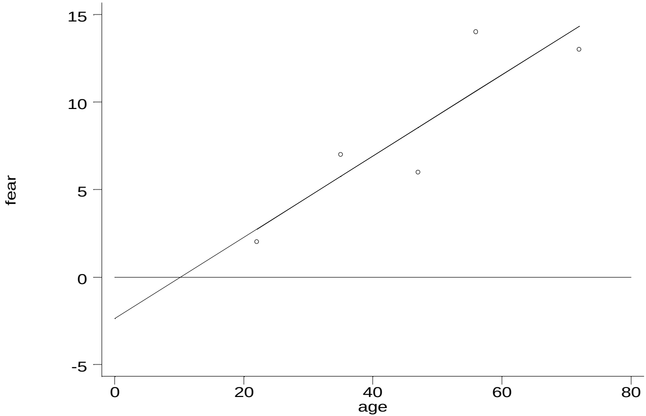
$$\bar{Y} = \frac{42}{5} = 8.4$$

Deviations from the means

Obs	x_i	x_i^2	y_i	y_i^2	$x_i y_i$
1	–24.4	595.36	–6.4	40.96	156.16
2	–11.4	129.96	–1.4	1.96	15.96
3	0.6	0.36	–2.4	5.76	–1.44
4	9.6	92.16	5.6	31.36	53.76
5	25.6	655.36	4.6	21.16	117.76
$\sum$	0	1473.2	0	101.2	342.2

-
- $\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} = \frac{342}{1473} = .232$
- $\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X} = 8.4 - (.232)(46.4) = -2.365$
- $\hat{Y}_i = \hat{\beta}_1 + \beta_2 X_i = -2.365 + .232 X_i$
- interpretation?

the estimated regression line



variance and std error of regression

- ok, we know how to calculate betas and fit the line (that min the sum of the squared resid)
- but some lines fit better and some worse
- need a measure of uncertatinty, ie how line fits the
- the fit is measured with residuals...
- ... so our measure of uncertainty has to do with residuals!
- $s^2 = \frac{\sum_{i=1}^{n}(e_i - \bar{e})^2}{n-2} = \frac{\sum_{i=1}^{n} e_i^2}{n-2}$
- $s = \sqrt{\frac{\sum_{i=1}^{n} e_i^2}{n-2}}$

  the mean of the residuals is 0 so $\bar{e}$ drops out
- s measures spread of the points around the regression line

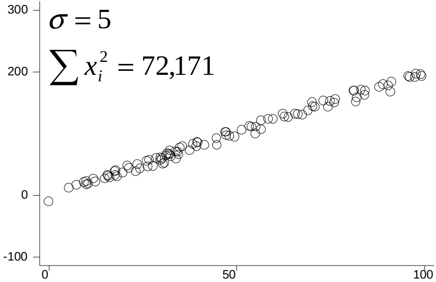
from $\hat{Y}$ to s (se of reg) to $s_{\hat{\beta}_2}$ (se of slope)

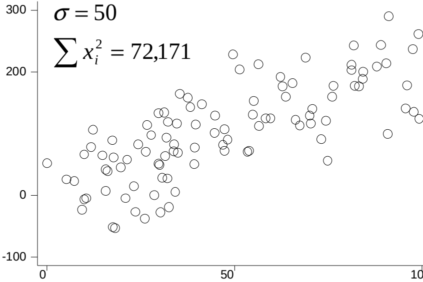
i	$\hat{Y}_i$	e_i	e_i^2
1	2.739	–0.739	0.546
2	5.755	1.245	1.556
3	8.539	–2.539	6.447
4	10.627	3.373	11.377
5	14.339	–1.339	1.793
Σ		0	21.713

- $s = \sqrt{\frac{\sum_{i=1}^{5} e_i^2}{n-2}} = \sqrt{\frac{21.7}{3}} =$ 2.7
- $s_{\hat{\beta}_2} = \frac{s}{\sqrt{\sum_{i=1}^{5} x_i^2}} = \frac{2.7}{\sqrt{1473}} =$ .07
- calc yhats and se of beta
- yahts important! like our lexp we predicted earlier
- $t = \frac{\hat{\beta}}{s_{\hat{\beta}}}$

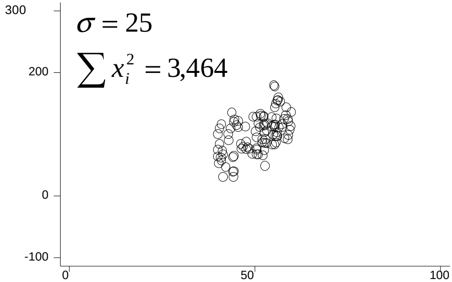
Standard Error of the Slope Coefficient

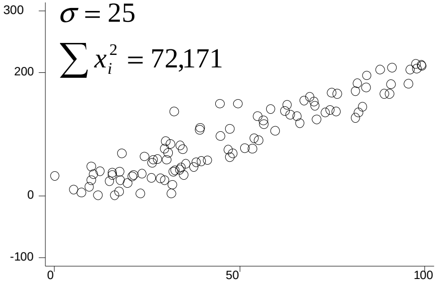
Numerator -- variance of disturbance term





Denominator -- variation in X





ucla: hands-on dofile

- https://stats.idre.ucla.edu/stata/webbooks/reg
- let's just see a first reg output (you'll do it for ps2)
  - what is bivariate regression command?
  - where is β_1 and β_2
- excellent for self study!!
- do it at home; and do ask me questions about it if any
- this is especially an excellent resource for final paper

finish first class here

- finish first class here

outline

bivariate regression

stat significance (hypothesis testing)

basic measurement

basic calculations blackboard; dofile

Y	X	y	y2	x	x2	xy
1	17					
3	13					
5	8					
7	10					
9	2					

Sum:

25	50

$$\bar{Y}=5 \quad \bar{X}=10$$

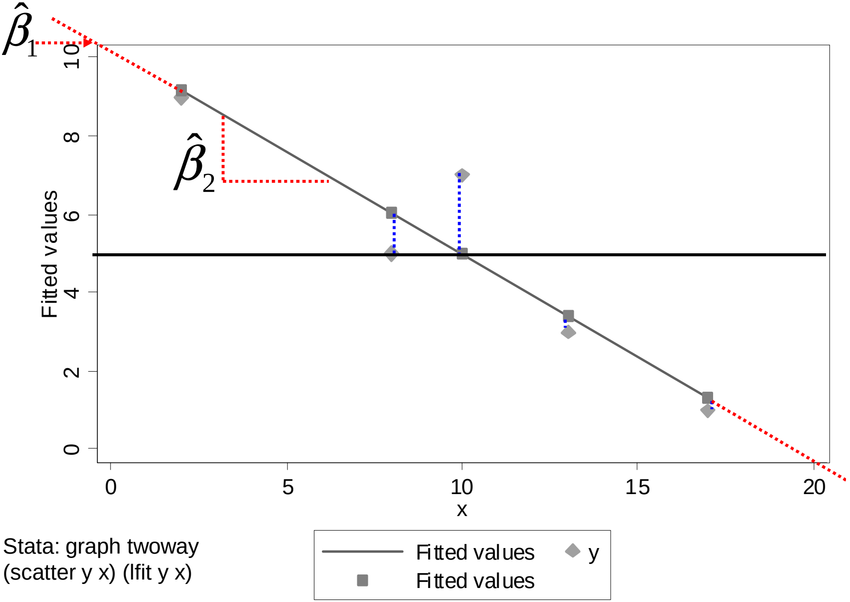
the coefficients–interpretation

- Beta hat two is the slope coefficient. Thus, a one unit change in X leads to a 0.524 decrease in Y. Beta hat one is the intercept term. It is the predicted value for Y when X is equal to zero.

predicted val and resid **blackboard; dofile**

Y	X	Y_hat	e	e^2
1	17			
3	13			
5	8			
7	10			
9	2			
- $\hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$
- for obs 1:
- $\hat{Y}_1 = 10.24 + (-0.524)(17) = 1.332$
- $e_1 = 1 - 1.33 = -0.33$

regression plot again



Stata: graph twoway
(scatter y x) (lfit y x)

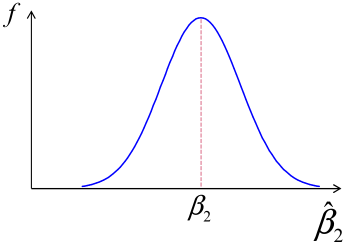
se of the slope **blackboard; dofile**

- $\sum e_i^2 = 5.42$
- $s = \sqrt{\frac{\sum e_i^2}{n-2}} =$
- $s_{\hat{\beta}_2} = \frac{s}{\sqrt{\sum x_i^2}}$

◦ it gives us info about reliability (like sd or se) of slope

sampling distribution of the slope

probability distribution of $\hat{\beta}_2$ is centered on the true value of the parameter (i.e. unbiased) and is normally distributed with variance:



- $s_{\hat{\beta}_2} = \frac{s}{\sqrt{\sum x_i^2}}$

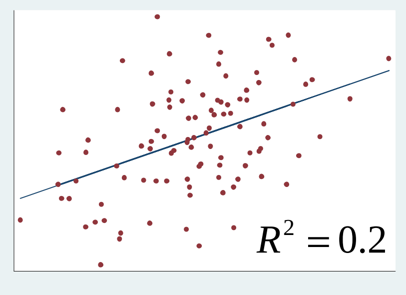
hypothesis test dofile

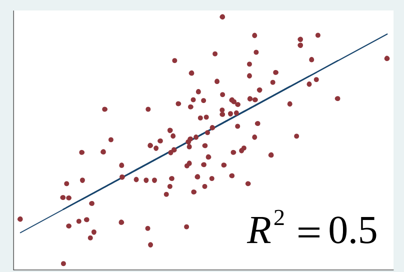
- the null is that slope ("the unobserved true parameter")
  - is zero (ie no effect)
- $H_0 : \beta_2 = 0$
- $H_A : \beta_2 \neq 0$
- $t = \frac{\hat{\beta}_2 - \beta_2}{s_{\hat{\beta}_2}} = \frac{\hat{\beta}_2}{s_{\hat{\beta}_2}}$
- CI: $\hat{\beta}_2 \pm (t_{n-2,\frac{\alpha}{2}})(s_{\hat{\beta}_2})$
- lets do it and calculate all by hand! incl crit val

accounting for variation in Y **blackboard in 3 colors**

- before regression $E[Y] = \bar{Y}$
  - TSS total sum of squares [like RSS before reg; we were off by this much!]
    $TSS = \sum_{i=1}^{n} (Y_i - \bar{Y})^2$
- after regression
  $E[Y|X_i] = \hat{Y}_i = \hat{\beta}_1 + \hat{\beta}_2 X_i$
  - ESS explained sum of squares
    $ESS = \sum_{i=1}^{n} (\hat{Y}_i - \bar{Y})^2$
  - RSS residual sum of squares
    $RSS = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} e_i^2$
- $TSS = ESS + RSS$

R^2 variation explained



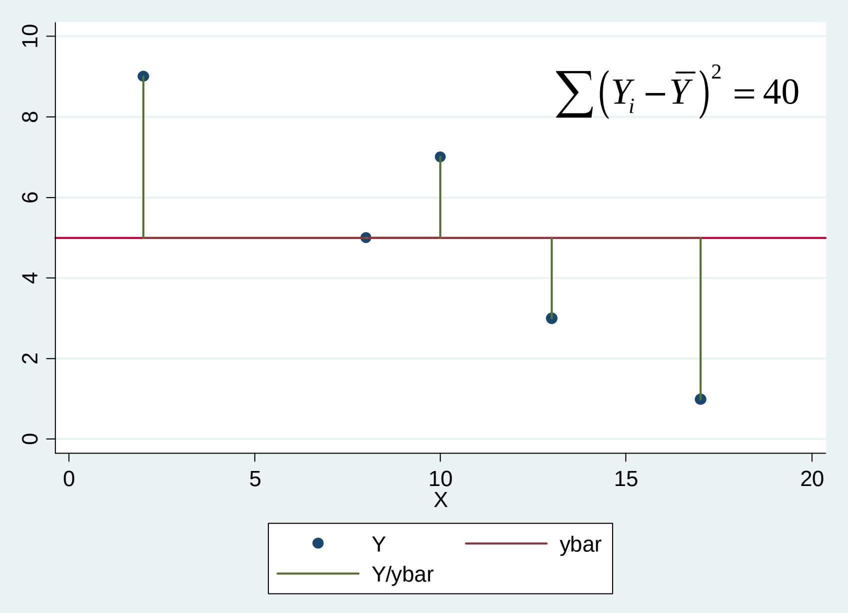


- $TSS = ESS + RSS$
- $1 = \frac{ESS}{TSS} + \frac{RSS}{TSS}$
- $R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum e_i^2}{y_i^2}$
- R^2 : the percent of the variance in the dependent variable explained by the model

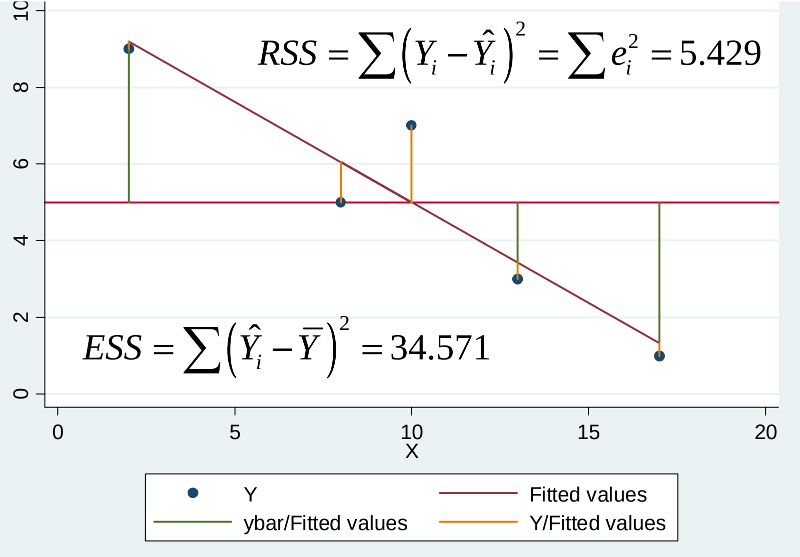
partitioning variance in Y dofile

- before regression $E[Y_i] = \bar{Y}$
  - $TSS = \sum(Y_i - \bar{Y})^2 = \sum y_i^2 = 40$
- after regression $E[Y_i|X_i] = \hat{Y}_i$
  - $RSS = \sum(Y_i - \hat{Y}_i)^2 = \sum e_i^2 = 5.43$
  - $ESS = TSS - RSS = 40 - 5.4 = 34.57$
-
- $R^2 = 1 - \frac{\sum e_i^2}{\sum y_i^2}$
- proportion of the total variance in the Y explained by Xs
- $0 \leq R^2 \leq 1$

TSS



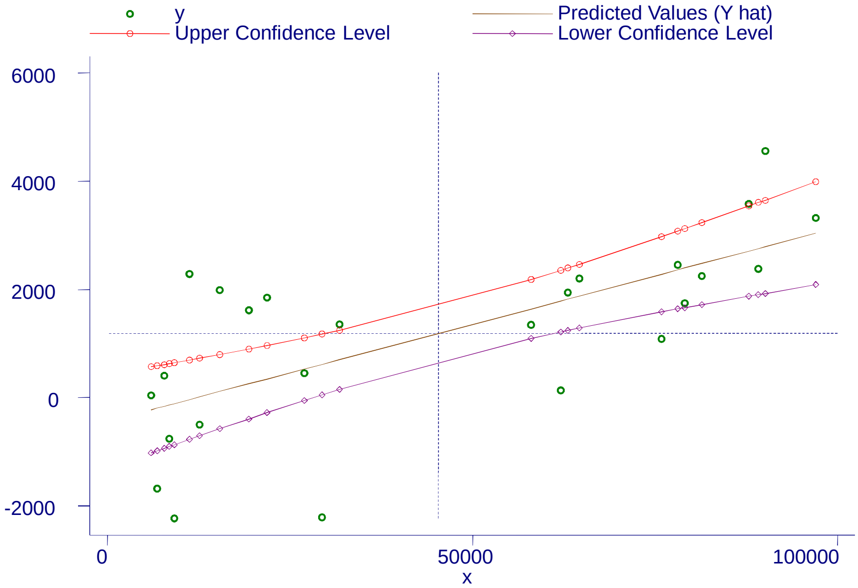
RSS


$RSS = \sum \left(Y_i - \hat{Y}_i\right)^2 = \sum e_i^2 = 5.429$
$ESS = \sum \left(\hat{Y}_i - \bar{Y}\right)^2 = 34.571$
0
2
4
6
8
10
0
5
10
15
20
X
Y
Fitted values
ybar/Fitted values
Y/Fitted values

reliability of predict val (se of $E(Y|X)$)

- We have discussed the fact that parameter estimates are random variables, and so they have standard errors. Predicted values are also random variables because they are linear combinations of the coefficients.
- The further from the mean of X, the wider the confidence interval around the predicted value.
- leave it to software, no need to know the formula

se of $E(Y|X$ illustration dofile



anatomy of stata output [biv] dofile: outlier

```
. regress DV IV
```

Source	SS	df	MS
Model	$ESS = \sum\left(\hat{Y}_i - \bar{Y}\right)^2$	1	....
Residual	$RSS = \sum e_i^2$	$n-2$	$s^2 = \dfrac{RSS}{n-2}$
Total	$TSS = \sum\left(Y_i - \bar{Y}\right)^2$	$n-1$	$s_Y^2 = \dfrac{TSS}{n-1}$

Number of obs	=	n
$\mathrm{F}(1, n-2)$	=	....
Prob > F	=	....
R-squared	=	r^2
Adj R-Squared	=	....
Root MSE	=	s

DV	Coef.	Std.Err.	t	P>|t|	[95% Conf.	Interval]
IV	$\hat{\beta}_2$	$s_{\hat{\beta}_2}$	$\left(\dfrac{\hat{\beta}_2}{s_{\hat{\beta}_2}}\right)$	p val. for H_0 that $\beta_2 = 0$	$\hat{\beta}_2 - t_{0.025} s_{\hat{\beta}_2}$	$\hat{\beta}_2 + t_{0.025} s_{\hat{\beta}_2}$
Intercept	$\hat{\beta}_1$	$s_{\hat{\beta}_1}$	$\left(\dfrac{\hat{\beta}_1}{s_{\hat{\beta}_1}}\right)$	p val. for H_0 that $\beta_1 = 0$	$\hat{\beta}_1 - t_{0.025} s_{\hat{\beta}_1}$	$\hat{\beta}_1 + t_{0.025} s_{\hat{\beta}_1}$

outline

bivariate regression

stat significance (hypothesis testing)

basic measurement

intuition

- what happens to betas if we change variables' measurement?
  - millions of dollars as opposed to dollars
  - curved grades (each person gets extra 10 points)
  - proportion of people in poverty v percent in poverty
- income per capita v income per 100k people

add constant c to X or Y (say curved grades)

- if you add c to each obs, mean of var would change by that much
- but demeaned var doesn't change:
- $x_i^{'} = (X_i^{'} - \bar{X}^{'}) = [(X_i + c) - (\bar{X} + c)] = x_i$ same for Y
- $\hat{\beta}_2 = \frac{\sum y_i x_i^{'}}{\sum x_i^{'2}} = \frac{\sum y_i x_i}{\sum x_i^2}$ only demeaned vars so no change
- and nobody cares about intercept anyway, so let's spare our brain

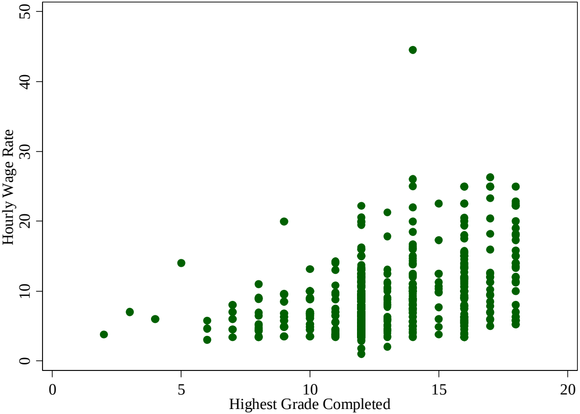
multiply X or Y by constant (say months, not years)

- think about it, assume some example
  - say year of educ produces \$2 increase in wage
- how about a month of educ? should be $1/12$ of \$2 !
- to convert yr to mo, multiply years by 12, right?
  - if a person has 2yr of educ, that's 24mo
- so if i multiply X by c, say 12, I need to divide $\hat{\beta}_2$ by 12
- what if multiply Y?
  - again, say year of educ produces \$2 increase in wage
  - ...or 200 cent increase in wage
- to get cents from dollars, I multiply dollars by 100
  - so if I multiply Y by 100, i get β_2 100x bigger

fun fact1: correlation v bivariate regression

- $r = \frac{\sum y_i x_i}{\sqrt{(\sum x_i^2)(\sum y_i^2)}} \quad \hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2}$
- bivariate slope equals corr coef scaled by std dev of Y and X:
  $\hat{\beta}_2 = \frac{\sum y_i x_i}{\sum x_i^2} = r\left(\frac{s_Y}{s_X}\right)$

education and wages dofile




```
. corr wage educ
(obs=534)

             |     wage     educ
-------------+------------------
        wage |   1.0000
        educ |   0.3819   1.0000
```

```
. sum wage educ
    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
        wage |       534    9.023939    5.138876          1       44.5
        educ |       534    13.01873    2.615373          2         18
```

education and wages dofile

```
. regress wage educ

  Source |       SS       df       MS              Number of obs =     534
---------+------------------------------           F(  1,   532) =   90.86
   Model |  2053.22494     1  2053.22494           Prob > F      =  0.0000
Residual |  12022.2635   532  22.5982396           R-squared     =  0.1459
---------+------------------------------           Adj R-squared =  0.1443
   Total |  14075.4884   533  26.4080458           Root MSE      =  4.7538

------------------------------------------------------------------------------
    wage |      Coef.   Std. Err.       t     P>|t|       [95% Conf. Interval]
---------+--------------------------------------------------------------------
    educ |   .7504488     .07873      9.532   0.000       .5957891    .9051086
   _cons |   -.745949   1.045404     -0.714   0.476      -2.799576    1.307678
------------------------------------------------------------------------------
```

The estimated regression line:

$$\widehat{wage}_i = \hat{\beta}_1 + \hat{\beta}_2 educ_i = -0.75 + 0.75 educ_i$$

Interpret the coefficients.

fun fact2: Z scores bivariate regression=correlation

- $z_{Yi} = \beta_1 + \beta_2 z_{Xi} + u_i$
  $z_{Xi} = \frac{X_i - \bar{X}}{s_X} = \frac{x_i}{s_x}$
  $z_{Yi} = \frac{Y_i - \bar{Y}}{s_U} = \frac{y_i}{s_Y}$
- z scores always have a mean of 0 and a variance (and standard deviation) of 1:
  $\hat{\beta}_2 = r_{Z_Y Z_X} \frac{s_{Z_Y}}{s_{Z_X}} = r_{YX}$
  $\hat{\beta}_1 = \bar{z}_Y - \hat{\beta}_2 \bar{z}_X = 0 - r(0) = 0$
- Thus, a regression of the z scores of Y on the z scores of X produces a slope equal to the correlation coefficient of X and Y and a zero intercept.

exercise 2: if no time do at home: see dofile

- confirm the above in stata using our simple data we started today's lecture with
- run regression of Y on X
- modify X or Y and check what happened