3. (LINEAR) REPRESENTATION

13/5/15

Def: A (linear) representation of a group G is a group homomorphism:

$\rho: G \longrightarrow \boxed{GL(V) = \{\text{invertible, linear transf. of } V\}}$ ↗ isomorphism from an object to itself ~ "symmetry group of the object" loosely speak.

V (↑ repr. space) — general linear group on V
↳ group of all automorphisms of V, i.e. set of all bijective linear transformations $V \to V$ together with functional composition as group operation.

V: vector space

Remark: $\boxed{\dim(V) = n < \infty}$ ← dimension of the representation

then by choosing a basis, one can think of a representation as a collection of invertible $n \times n$-matrices $\rho(g)$, $g \in G_T$ G ?

s.t. $\rho(e) = \mathbb{1}_n$

$\rho(g)\rho(g') = \rho(gg') \quad \forall g, g' \in G$

Examples

(i) $G = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ representations on $V = \mathbb{C}$

$GL(V) = \mathbb{C}^+ = \mathbb{C} \setminus [0]$ → $\dim(V) = n = 1$
↳ complex without zero

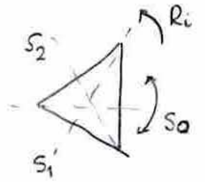
$\rho([a]) = e^{\frac{2\pi i a}{n}}$

$\rho([a])\rho([b]) = e^{\frac{2\pi i}{n}(a+b)} = \rho([a+b])$

(ii) $V = \mathbb{R}^2$

$$\rho([a]) = \begin{pmatrix} \cos(\frac{2\pi}{n}a) & \sin(\frac{2\pi}{n}a) \\ -\sin(\frac{2\pi}{n}a) & \cos(\frac{2\pi}{n}a) \end{pmatrix}$$

(iii) $G = D_n$, $V = \mathbb{R}^2$ $\dim(V) = n = 2$



$R_i R_j = R_{i+j}$
$R_i S_j = S_{i+j}$
$S_i R_j = S_{i-j}$
$S_i S_j = R_{i-j}$

$$\rho(R_j) = \begin{pmatrix} \cos(\frac{2\pi j}{n}) & \sin(\frac{2\pi j}{n}) \\ -\sin(\frac{2\pi j}{n}) & \cos(\frac{2\pi j}{n}) \end{pmatrix}$$

$$\rho(S_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\rho(S_j) = \begin{pmatrix} \cos(\frac{2\pi j}{n}) & -\sin(\frac{2\pi j}{n}) \\ -\sin(\frac{2\pi j}{n}) & -\cos(\frac{2\pi j}{n}) \end{pmatrix}$$

$$\rho(R_a)\rho(R_b) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}\begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} = \begin{pmatrix} \cos a\cos b - \sin a \sin b & \cos a \sin b + \sin a \cos b \\ -\sin a \cos b - \cos a \sin b & -\sin a \sin b + \cos a \cos b \end{pmatrix} = \begin{pmatrix} \cos(a+b) & \sin(a+b) \\ -\sin(a+b) & \cos(a+b) \end{pmatrix} = \rho(R_{a+b})$$

$$\rho(R_a)\rho(S_b) = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}\begin{pmatrix} \cos b & -\sin b \\ -\sin b & -\cos b \end{pmatrix} = \begin{pmatrix} \cos a\cos b - \sin a \sin b & -\cos a \sin b - \sin a \cos b \\ -\sin a \cos b - \cos a \sin b & +\sin a \sin b - \cos a \cos b \end{pmatrix} = \begin{pmatrix} \cos(a+b) & -\sin(a+b) \\ -\sin(a+b) & -\cos(a+b) \end{pmatrix} = \rho(S_{a+b})$$

- Every group has a trivial representation:

  $\dim(V) = 1 \quad \rho(g) = 1$ ← maps everything to one

  $\rho(g)\rho(g') = \rho(gg')$

this
⇝ ~~a~~ representation does not contain any information about the group.

- Representation containing all information about G are called faithful:

  $\rho: G \hookrightarrow GL(V)$, which is injective — no information has been lost

  This means: $\rho(g) = \rho(g') \to g = g'$

  if it is not true $\exists g \neq e \;\; \rho(g) = id_V$ (identity in V) ← degenerate representation

  Recall: $\ker(\rho) \triangleleft G$

  ⇝ ρ descends to a faithful representation of $\boxed{G/\ker(\rho)}$

  e.g. $G = S_n$

  (i) has a trivial one-dimensional representation, $\rho(g) = 1$

  (ii) has a non-trivial one-dim repr.: sign-representation

  $\rho(g) = (-1)^{\text{length}(g)}$ $\quad$ [remember: $(-1)^{\text{length}(g)} : S_n \to \mathbb{Z}_n$]

  $\ker(\rho) = A_n$ alternating group

  ρ descends to a faithful representation of $S_n/A_n = \mathbb{Z}_n$

- Building representations:

  Given a representation $\rho_1: G \to GL(V_1)$
  $\rho_2: G \to GL(V_2)$

  * All the representations that we will consider are representations of a GL of a vector space.

  (i) Direct sum:

  $$\rho_1 \oplus \rho_2(g) = \rho_1(g) \oplus \rho_2(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right)$$

  $V_{\rho_1 \oplus \rho_2} = V_1 \oplus V_2$

  $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$

  dim. of the representation-vector space

(ii) Tensor product: $V_{\rho_1 \otimes \rho_2} = V_{\rho_1} \otimes V_{\rho_2}$

$$\rho_1 \otimes \rho_2 (g) = \rho_1(g) \otimes \rho_2(g)$$

$$\dim(\rho_1 \otimes \rho_2) = \dim(\rho_1) \cdot \dim(\rho_2)$$

(iii) Dual representation

$$\rho : G \longrightarrow GL(V)$$

dual vs. $V^* := \mathrm{Hom}(V, \mathbb{K})$

$$\bar{\rho} : G \longrightarrow GL(V^*)$$

$$\bar{\rho}(g) = (\rho(g^{-1}))^*$$

↑ dual map

About dual maps

$A : V \longrightarrow W$

$A^* : V^* \rightarrow W^*$

$W^* = \mathrm{Hom}(W, \mathbb{K}) \ni w^*$

$A^*(w^*) = w^* \circ A : V \longrightarrow \mathbb{K}$

$\in \mathrm{Hom}(V, \mathbb{K}) = V^*$

Prop. of dual maps:

$A : V \rightarrow W$

$B : W \rightarrow U$

$(BA)^* = A^* B^*$

$\bar{\rho}$ is representation

$$\bar{\rho}(g)\bar{\rho}(h) = (\rho(g^{-1}))^*(\rho(h^{-1}))^* =$$
$$= (\rho(h^{-1})\rho(g^{-1}))^* =$$
$$= (\rho(h^{-1}g^{-1}))^* =$$
$$= (\rho((gh)^{-1}))^* = \bar{\rho}(gh)$$

- Subrepresentations:

Def: An invariant subspace W of a representation $\rho : G \rightarrow GL(V)$ is a subspace $W \subset V$, s.t.

$$\rho(g)(W) \subset W$$

$\Rightarrow \rho(g)|_W$ is a subrepresentation.

$\rho|_W : G \longrightarrow GL(W)$ is a representation on its own.

Def: ρ is called irreducible if there are no invariant subspaces, except $\{0\}$ and V.

Otherwise (if it's not irreducible) it is called reducible.

In matrix form this means:

$$\rho(g) = \left(\begin{array}{c|c} \rho|_W(g) & * \\ \hline 0 & * \end{array} \right)$$

↙ reducible representation

(in a particular basis)

∘ A repr. ρ is called <u>decomposable</u> if it is a direct sum $\rho = \rho_1 \oplus \rho_2$

In matrix form : (in a particular basis)

$$\left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right) = \rho(g)$$

If it is not decomposable, it is called <u>indecomposable</u>.

∘ A repr. is called <u>fully decomposable</u> if it can be decomposable in a direct sum of irreducible repr. :

$$\rho = \rho_1 \oplus \dots \oplus \rho_n$$

irreducible representation

<u>Ex</u>: i) $\rho: \mathbb{Z} \to GL(2)$

$$\rho(n) \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad , \ \rho(n)\rho(m) = \rho(n+m)$$

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+n \\ 0 & 1 \end{pmatrix}$$

There is an invariant subspace $\Rightarrow$ reducible

$\mathbb{R}^2 = \mathrm{span}(e_1, e_2)$, $W = \mathrm{span}(e_1)$ is invariant

But it is <u>not decomposable</u>: because $\rho(n)$ cannot be diagonalised

$\rho|_\omega$ is the trivial representation

(ii) $G = S_3$, $V = \mathrm{span}\{e_1, e_2, e_3\}$

$$\rho(\pi)(e_i) = e_{\pi(i)}$$

In basis: $\rho((12)) = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$

$$\rho((23)) = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

Invariant subspace: $W = \mathrm{span}\{e_1+e_2+e_3\}$, because

$$\rho(\pi)(e_1+e_2+e_3) = e_{\pi(1)} + e_{\pi(2)} + e_{\pi(3)} = e_1+e_2+e_3$$

$$W^\perp = \mathrm{span}\{V_1 = e_1 - e_2, \ V_2 = e_2 - e_3\}$$

$\rho((12))|_{W^\perp}$: $V_1 \longmapsto -V_1$, $V_2 \longmapsto V_1 + V_2$ In matrix form $\rho((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$\rho\big|_{W^\perp}((23))$: $v_1 \longmapsto v_1 + v_2$ In matrix form: $\rho((23)) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

$v_2 \longmapsto -v_2$

$\rightarrow \rho|_{W^\perp}$ is irreducible

$\rho = \rho|_W \oplus \rho|_{W^\perp}$ $\rightsquigarrow$ Here we had a repr. $\rightarrow$ irreducible

$\rightarrow$ fully decomposable

$\rho|_W$: trivial, $\rho|_{W^\perp}$: 2dim irred. repr.

Examples: for building of representations

$\rho_r : \mathbb{Z}_n \longrightarrow GL(\mathbb{C})$

$\rho_r([a]) = e^{\frac{2\pi i r a}{n}}$

(i) $\rho_r \oplus \rho_s([a]) = \begin{pmatrix} e^{\frac{2\pi i r a}{n}} & 0 \\ 0 & e^{\frac{2\pi i s a}{n}} \end{pmatrix}$

direct sum

(ii) tensor product

$\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$ $\rightsquigarrow$ we'll get again a 1-dim representation

$\rho_r \otimes \rho_s([a]) = \rho_r([a]) \otimes \rho_s([a]) = e^{\frac{2\pi i (r+s)a}{n}} = \rho_{r+s}([a])$

(iii) dual representation

$\mathbb{C}^* = \mathbb{C}$

$\bar{\rho}_r([a]) = \rho_r([a]^{-1}) = \rho_r([-a]) = e^{-\frac{2\pi i r a}{n}} = \rho_{-r}([a])$

$\bar{\rho}_r = \rho_{-r}$

$\rho_r \otimes \rho_s = \rho_{r+s}$

Special repr. compatible with some structures:

- <u>Unitary representation</u>:

<u>Def</u>: A <u>unitary representation</u> ρ is a repr. on a Hermitian vector space $(V, \langle , \rangle)$

s.t. $\rho: G \longrightarrow \mathcal{U}(V) = \{ T \in \mathcal{U}(V),$ $\qquad (\langle a,b \rangle = \overline{\langle b,a \rangle})$

$$\langle Ta, Tb \rangle = \langle a, b \rangle \quad \forall a, b \in V \ \} \subset GL(V)$$

$$(T^*)^t \cancel{T} = T^+ T = id_V$$

Unitary representations are nice:

<u>Fact</u>: Unitary repr. are fully decomposable.

$\ulcorner$ Assume $\rho: G \longrightarrow U(V)$, $W \subset V$ is invariant subspace

$\Rightarrow W^\perp$ is invariant subspace aswell.

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \quad \forall w \in W \}$$

$\forall v \in W^\perp$, $w \in W$: $0 = \langle v, w \rangle = \langle v, \rho(g) w \rangle = \langle \rho(g^{-1}) v, w \rangle$

$\Rightarrow W^\perp$ is invariant ~~under~~ subspace as well

$\Rightarrow \rho = \rho|_W \oplus |_{W^\perp}$

Iteratively $\rightsquigarrow \rho = \bigoplus_i \rho_i$
irreducible representation

$\lrcorner$

$\rightarrow$ building blocks of unitarity repres. are the irreducible repr.

- Maps between representations:

Def: An intertwiner between two representations:

$$\rho_1 : G \longrightarrow GL(V_1)$$
$$\rho_2 : G \longrightarrow GL(V_2)$$

is a linear map $f: V_1 \to V_2$

s.t. $\rho_2(g) \circ f = f \circ \rho_1(g) \quad \forall g \in G$

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & V_1 \arrow[d, "f"] \\ V_2 \arrow[r, "\rho_2(g)"'] & V_2 \end{tikzcd}$$

only one f for every g

If f is invertible $\Rightarrow$ equivalence of representations

$$\rho_1(g) = f^{-1} \rho_2(g) f$$

change of basic transf.

(requires $\dim(V_1) = \dim(V_2)$)

Remark: $\mathrm{Im}(f)$ and $\mathrm{Ker}(f)$ are invariant subspaces.

image of f

- $\rho_2(g) \underbrace{f(v)}_{\in \mathrm{Im}(f)} \overset{\text{def of intertwiner}}{=} \underbrace{f \circ \rho_1(g)(v)}_{\in \mathrm{Im}(f)} \Rightarrow \mathrm{Im}(f)$ is invariant subspace.

- $f(v) = 0$, i.e. $v \in \mathrm{Ker}(f)$

$$\underset{\overset{\shortparallel}{0}}{\rho_2(g) f(v)} = f \circ \rho_1(g) v \Rightarrow \rho_1(g) v \in \mathrm{Ker}(f)$$

$\to \mathrm{Ker}(f)$ is invariant subspace. $\perp$

(*)(*)(*)

- Schur's Lemma:

If ρ_1 and ρ_2 are irreducible repr. of a group G and $f: V_1 \to V_2$ is an intertwiner, then:

(i) either $f \equiv 0$ or f is an isomorphism.

(ii) if $V_1 \cong V_2 \Rightarrow f = \lambda \, id$, $\lambda \in \mathbb{C}$, (over $\mathbb{C}$)

$\ulcorner$ (i) $f: V_1 \longrightarrow V_2$ is intertwiner

$\ker(f)$ is invariant subspace $\xrightarrow{\text{irreducibility of } \rho_2}$ $\ker(f) = V_1$ or $\ker(f) = \{0\}$

$f \equiv 0$ $\quad$ f injective

$\mathrm{Im}(f)$ is invariant subspace $\underset{\text{irreducibility of } \rho_2}{\Rightarrow}$ $\mathrm{Im}(f) = \{0\}$ or $\mathrm{Im}(f) = V_2$

$f \equiv 0$ $\quad$ surjective

This shows (i).

(ii) Over $\mathbb{C}$: since $\mathbb{C}$ is algebraically closed, f has at least one e-value λ.

We have already set $V = V_1 = V_2$

$f' = (f - \lambda \, id_V)$ is still an intertwiner.

but now: $\ker(f') \neq \{0\}$ by irreducibility of $\ker(f') = V$

$\Rightarrow f' = (f - \lambda id) \equiv 0$

this shows (ii). $\lrcorner$

<u>Remark</u>: If two irreducible representations over $\mathbb{C}$ are equivalent as in (ii), then all the intertwiners are multiples of each other.

$$f_1, f_2 : V_1 \longrightarrow V_2 \quad , f_2 \neq 0$$

$$\Rightarrow \underbrace{f_2^{-1} f_1}_{\lambda id} : V_1 \longrightarrow V_1 \quad \Rightarrow \boxed{f_1 = \lambda f_2} \; , \; \lambda \in \mathbb{C}$$

Note: this works over $\mathbb{C}$!

<u>Aside</u>: $\mathbb{R}$ is not algebraically closed

$\rightsquigarrow$ not every intertwiner has eigenvalue.

$\rightsquigarrow$ arguments above does not work.

$\rightsquigarrow$ Not all intertwiners over $\mathbb{R}$ are multiplets of each other.

↳ This is the bad news. However $\mathbb{R}$ still have enough structure:

The intertwiners form an <u>associative division algebra</u>.

- can add them
- can multiply them
- if they are not zero, one can divide by them.
- since repr. by matrices, associative

- Division algebra over $\mathbb{C}$; $\mathbb{C}$

↳ associative

- Division algebras over $\mathbb{R}$: $\begin{cases} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H}\text{, quaternions} \end{cases}$