DIRECTION FIELDS

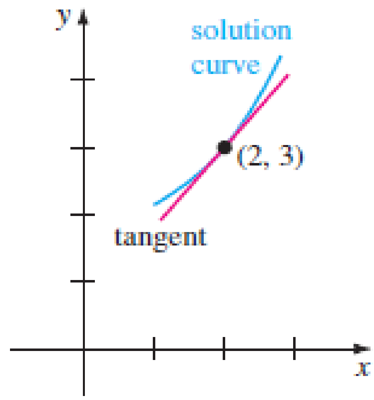
Let us imagine for the moment that we have in front of us a first-order DE

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

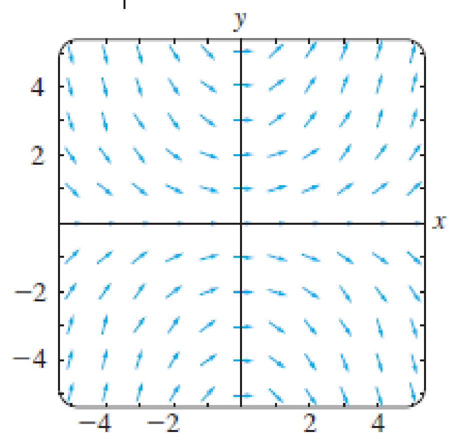
and let us further imagine that we can neither find nor invent a method for solving it analytically. The function *f* in the normal form (1) is called the **slope function** or **rate function.** The slope of the tangent line at (x, $y(x)$) on a solution curve is the value of the first derivative dy/dx at this point, and we know from (1) that this is the value of the slope function $f(x, y(x))$. The value $f(x, y)$ that the function *f* assigns to the point represents the slope of a line or line segment. For example, consider the equation

$$\frac{dy}{dx} = 0.2xy = f(x, y)$$

At the point (2, 3) the slope of a line is $f(2, 3) = 1.2$. First figure shows a line segment with slope 1.2 passing though (2, 3). As shown in Figure, *if* a solution curve also passes through the point (2, 3), it does so tangent to this line segment; in other words, the lineal element is a miniature tangent line at that point.

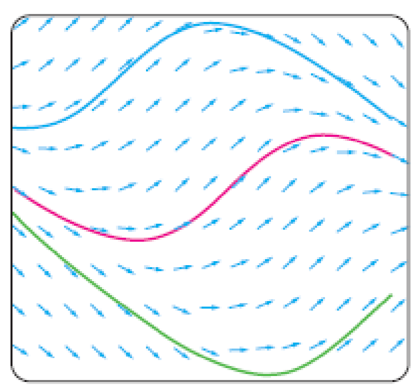


If we systematically evaluate *f* over a rectangular grid of points in the *xy*-plane and draw a line element at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all these line elements is called a **direction field** or a **slope field** of the DE (1). Visually, the direction field suggests the appearance or shape of a family of solution curves of the DE.
The direction field for the DE $dy/dx = 0.2xy$ shown in Figure (a) was obtained by using computer software.



(a) direction field for $dy/dx = 0.2xy$

A single solution curve that passes through a direction field must follow the flow pattern of the field; it is tangent to a line element when it intersects a point in the grid. Figure (b) shows a computer-generated direction field of the differential equation $dy/dx = \sin(x + y)$ over a region of the *xy*-plane.



*From the next lecture we will formally move to the first method of solving a simple DE (for that you have to revise some basic integral formulae).

2.2 SEPARABLE VARIABLES

We begin our study of how to solve differential equations with the simplest of all differential equations: first-order equations with separable variables. Because the method in this section and many techniques for solving differential equations involve integration, you are urged to refresh your memory on important formulas.

DEFINITION
A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**
For example, the equations

$$\frac{dy}{dx} = y^2xe^{3x+4y} \quad \text{and} \quad \frac{dy}{dx} = y + \sin x$$

are separable and nonseparable, respectively. In the first equation we can factor

$$\frac{dy}{dx} = (xe^{3x})(y^2e^{4y})$$

but in the second equation there is no way of expressing $y + \sin x$ as a product of a function of *x* times a function of *y*.

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \quad \leftarrow \text{laws of exponents}$$

$$= |1 + x|\,e^{c_1} \qquad \leftarrow \begin{cases} |1 + x| = 1 + x, & x \geq -1 \\ |1 + x| = -(1 + x), & x < -1 \end{cases}$$

$$= \pm e^{c_1}(1 + x).$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = -3$.

SOLUTION Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. This solution of the differential equation represents a family of concentric circles centered at the origin.

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$. Thus the initial-value problem determines the circle $x^2 + y^2 = 25$ with radius 5. Because of its simplicity we can solve this implicit solution for an explicit solution that satisfies the initial condition.

Q1. Solve the DE by separation of variables

$$e^x y\,\frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

Solution:

$$e^x y\,\frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

Implies

$$e^x y\,\frac{dy}{dx} = e^{-y}(1 + e^{-2x})$$

$$ye^y\,\frac{dy}{dx} = (1 + e^{-2x})e^{-x}$$

$$ye^y\,dy = (e^{-x} + e^{-3x})dx$$

Integrating both sides gives

$$\int ye^y\,dy = \int (e^{-x} + e^{-3x})\,dx$$

So, the general solution is

$$e^y(y-1) = -e^{-x} - \frac{1}{3}e^{-3x} + c$$

EXERCISES 2.2

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1–22 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}dy = 0$

4. $dy - (y - 1)^2 dx = 0$

5. $x\dfrac{dy}{dx} = 4y$

6. $\dfrac{dy}{dx} + 2xy^2 = 0$

7. $\dfrac{dy}{dx} = e^{3x+2y}$

8. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

9. $y \ln x \dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$

10. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$

11. $\csc y\, dx + \sec^2 x\, dy = 0$

12. $\sin 3x\, dx + 2y \cos^3 3x\, dy = 0$

13. $(e^y + 1)^2 e^{-y}\, dx + (e^x + 1)^3 e^{-x}\, dy = 0$

14. $x(1 + y^2)^{1/2}\, dx = y(1 + x^2)^{1/2}\, dy$

15. $\dfrac{dS}{dr} = kS$

16. $\dfrac{dQ}{dt} = k(Q - 70)$

17. $\dfrac{dP}{dt} = P - P^2$

18. $\dfrac{dN}{dt} + N = Nte^{t+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

20. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$

21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$

22. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28 find an explicit solution of the given initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$

24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$

25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$

26. $\dfrac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$

27. $\sqrt{1 - y^2}\, dx - \sqrt{1 - x^2}\, dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

28. $(1 + x^4)\, dy + x(1 + 4y^2)\, dx = 0, \quad y(1) = 0$

In Problems 29 and 30 proceed as in Example 5 and find an explicit solution of the given initial-value problem.

29. $\dfrac{dy}{dx} = ye^{-x^2}, \quad y(4) = 1$

30. $\dfrac{dy}{dx} = y^2 \sin x^2, \quad y(-2) = \frac{1}{3}$

31. **(a)** Find a solution of the initial-value problem consisting of the differential equation in Example 3 and the initial conditions $y(0) = 2$, $y(0) = -2$, and $y(\frac{1}{4}) = 1$.