Heap Sort

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Heapsort

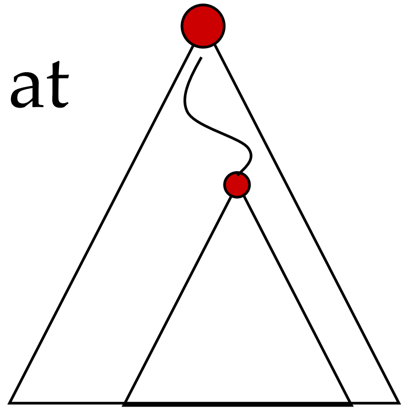
- Combines the better attributes of merge sort and insertion sort.
  - Like merge sort, but unlike insertion sort, running time is $O(n \lg n)$.
  - Like insertion sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
  - Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - Priority Queues

Data Structure Binary Heap

- Array viewed as a nearly complete binary tree.
  - Physically – linear array.
  - Logically – binary tree, filled on all levels (except lowest.)
- Map from array elements to tree nodes and vice versa
  - Root – $A[1]$
  - Left[i] – $A[2i]$
  - Right[i] – $A[2i+1]$
  - Parent[i] – $A[\lfloor i/2 \rfloor]$
- length[A] – number of elements in array A.
- heap-size[A] – number of elements in heap stored in A.
  - heap-size[A] $\leq$ length[A]

Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



- Min-Heap
  - For every node excluding the root, value is at least that of its parent: $A[parent[i]] \leq A[i]$
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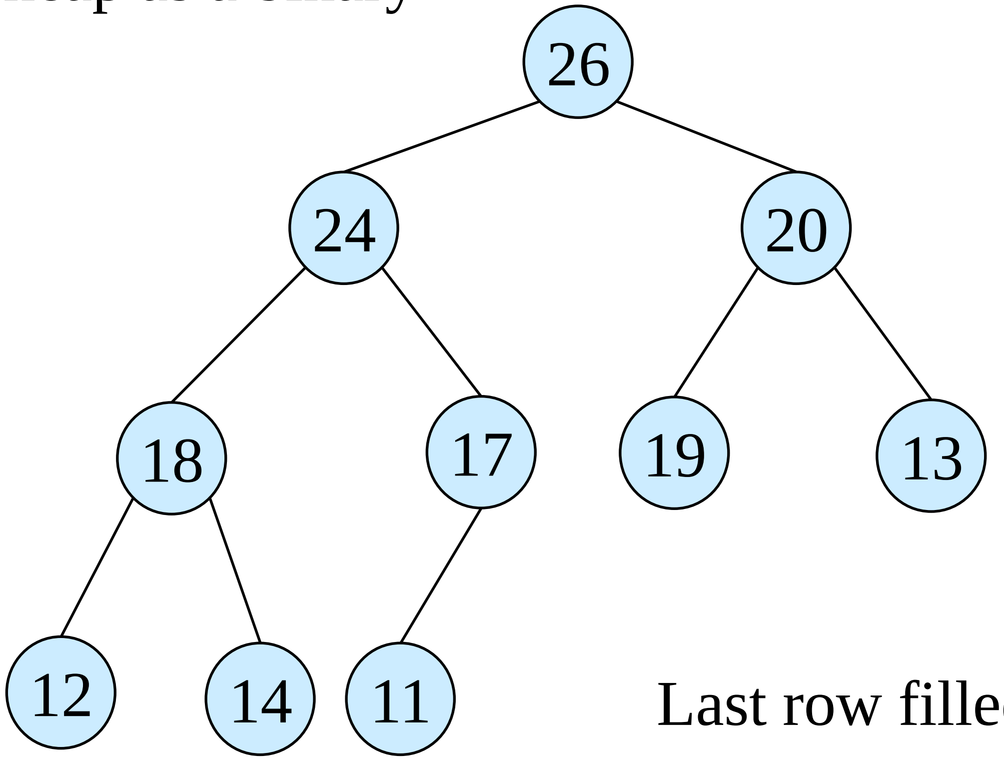
Max-heap as an array.

Max-heap as a binary tree.



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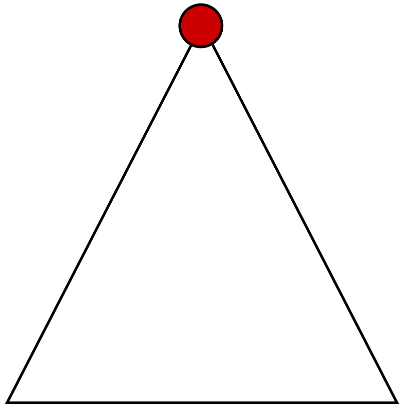
Height

- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.
- *Height of a tree*: the height of the root.
- Height of a heap: $\lfloor \lg n \rfloor$
  - Basic operations on a heap run in $O(\lg n)$ time

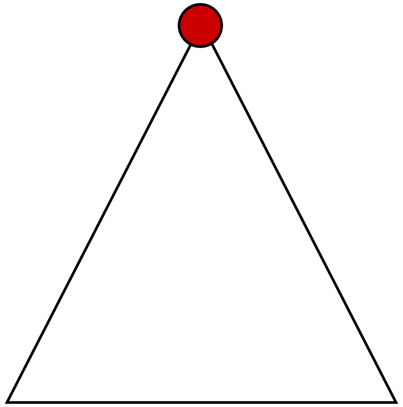


Heaps in Sorting

- Use max-heaps for sorting.
- The array representation of max-heap is not sorted.
- Steps in sorting
  - Convert the given array of size n to a max-heap (*BuildMaxHeap*)
  - Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves $n - 1$ elements to be placed in their appropriate locations.
    - However, the array of first $n - 1$ elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (MaxHeapify)
    - Repeat step 2 until the array is sorted.

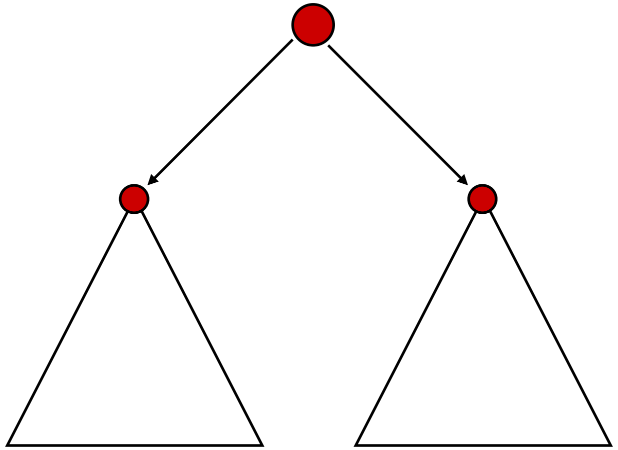
Heap Characteristics

- *Height* $= \lfloor \lg n \rfloor$
- No. of *leaves* $= \lceil n/2 \rceil$
- No. of nodes of height h $\leq \lceil n/2^{h+1} \rceil$



Maintaining the heap property

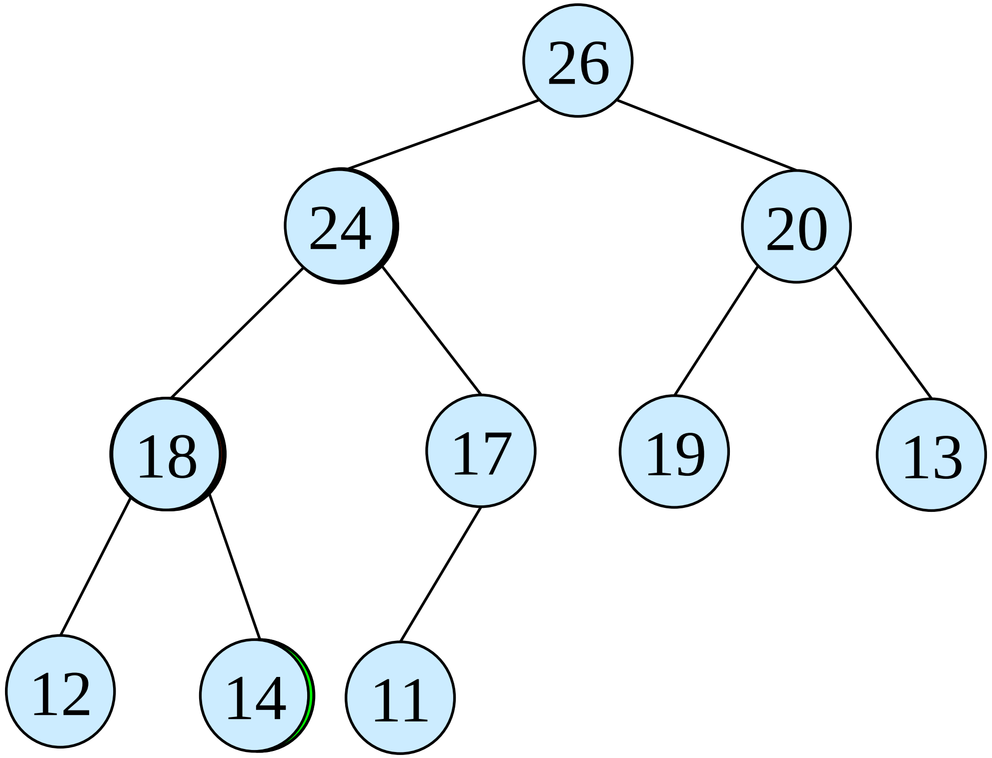
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.



- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - May lead to the subtree at the child not being a heap.
- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example

- MaxHeapify(A, 2)



Procedure MaxHeapify

MaxHeapify(A, i)
1. $l \leftarrow \text{left}(i)$
2. $r \leftarrow \text{right}(i)$
3. **if** $l \leq \textit{heap-size}[A]$ and $A[l] > A[i]$
4. **then** $\textit{largest} \leftarrow l$
5. **else** $\textit{largest} \leftarrow i$
6. **if** $r \leq \textit{heap-size}[A]$ **and** $A[r] > A[\textit{largest}]$
7. **then** $\textit{largest} \leftarrow r$
8. **if** $\textit{largest} \neq i$
9. **then** exchange $A[i] \leftrightarrow A[\textit{largest}]$
10. *MaxHeapify*$(A, \textit{largest})$

Assumption:
Left(i) and Right(i) are max-heaps.

Running Time for MaxHeapify

<u>*MaxHeapify(A, i)*</u>

1. $l \leftarrow \text{left}(i)$
2. $r \leftarrow \text{right}(i)$
3. **if** $l \leq \textit{heap-size}[A]$ and $A[l] > A[i]$
4.   **then** $\textit{largest} \leftarrow l$
5.   **else** $\textit{largest} \leftarrow i$
6. **if** $r \leq \textit{heap-size}[A]$ **and** $A[r] > A[\textit{largest}]$
7.   **then** $\textit{largest} \leftarrow r$
8. **if** $\textit{largest} \neq i$
9.   **then** exchange $A[i] \leftrightarrow A[\textit{largest}]$
10. $\textit{MaxHeapify}(A, \textit{largest})$

Time to fix node *i* and its children = $\mathbf{\Theta(1)}$

PLUS

Time to fix the subtree rooted at one of *i*'s children = *T*(size of subree at *largest*)

Running Time for MaxHeapify(A, n)

- $T(n) = T(largest) + \Theta(1)$
- $largest \le 2n/3$ (worst case occurs when the last row of tree is exactly half full)
- $T(n) \le T(2n/3) + \Theta(1) \Rightarrow T(n) = O(\lg n)$
- Alternately, MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied

Building a heap

- Use *MaxHeapify* to convert an array A into a max-heap.
- How?
- Call MaxHeapify on each element in a bottom-up manner.

BuildMaxHeap(A)

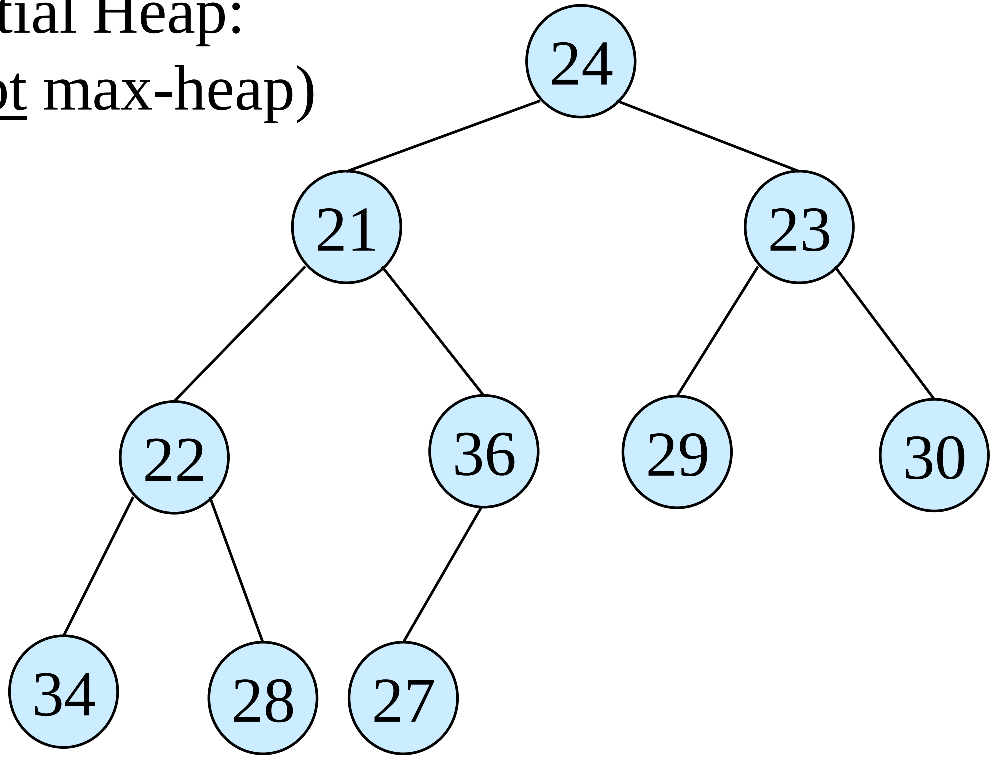
1. $heap\text{-}size[A] \leftarrow length[A]$
2. **for** $i \leftarrow \lfloor length[A]/2 \rfloor$ **downto** 1
3. **do** $MaxHeapify(A, i)$

BuildMaxHeap – Example

- Input Array

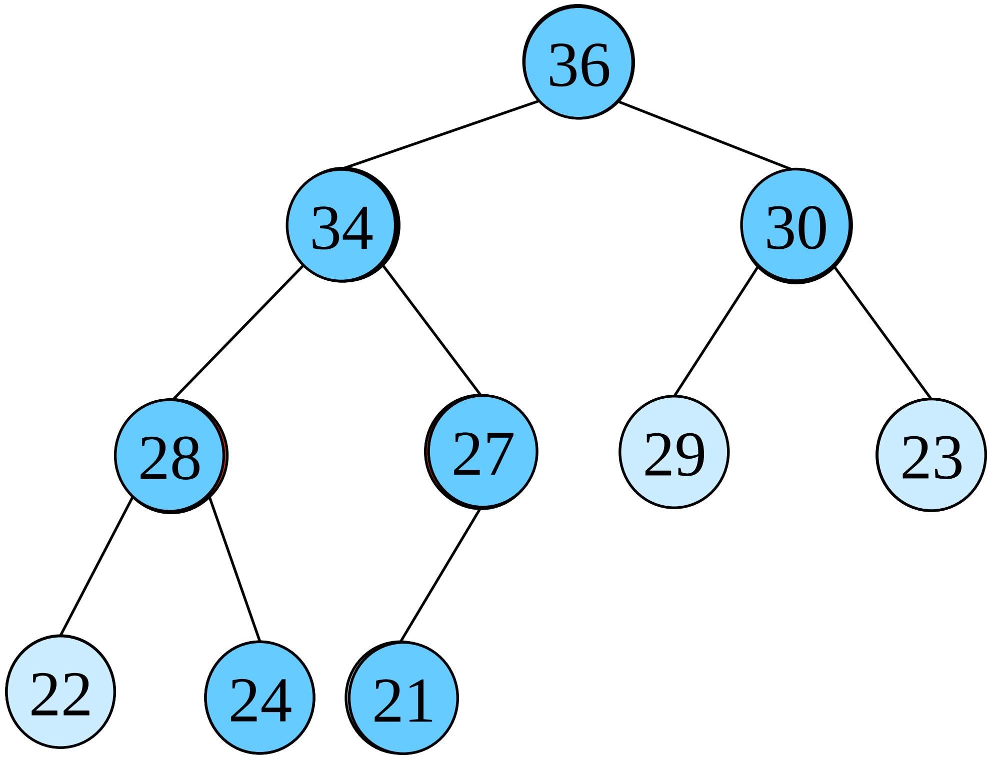
24	21	23	22	36	29	30	34	28	27

Initial Heap:
(<u>not</u> max-heap)



BuildMaxHeap – Example

- MaxHeapify($\lfloor 10/2 \rfloor = 5$)
- MaxHeapify(4)
- MaxHeapify(3)
- MaxHeapify(2)
- MaxHeapify(1)



Correctness of BuildMaxHeap

- <u>Loop Invariant:</u> At the start of each iteration of the **for** loop, each node $i+1$, $i+2$, …, n is the root of a max-heap.
- <u>Initialization:</u>
  - Before first iteration $i = \lfloor n/2 \rfloor$
  - Nodes $\lfloor n/2 \rfloor+1$, $\lfloor n/2 \rfloor+2$, …, n are leaves and hence roots of max-heaps.
- <u>Maintenance:</u>
  - By LI, subtrees at children of node i are max heaps.
  - Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - Decrementing i reestablishes the loop invariant for the next iteration.

Running Time of BuildMaxHeap

- Loose upper bound:
  - Cost of a *MaxHeapify* call × No. of calls to *MaxHeapify*
  - $O(\lg n) \times O(n) = O(n\lg n)$
- Tighter bound:
  - Cost of a call to *MaxHeapify* at a node depends on the height, h, of the node – $O(h)$.
  - Height of most nodes smaller than n.
  - Height of nodes h ranges from 0 to $\lfloor \lg n \rfloor$.
  - No. of nodes of height h is $\lceil n/2^{h+1} \rceil$

Running Time of BuildMaxHeap

- Tighter Bound for $T(BuildMaxHeap)$

  $T(BuildMaxHeap)$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$
$$= O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}$$
$$\leq \sum_{h=0}^{\infty} \frac{h}{2^h} \qquad , x = 1/2 \text{ in (A.8)}$$
$$= \frac{1/2}{(1-1/2)^2}$$
$$= 2$$

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$
$$= O(n)$$

- Can build a heap from an unordered array in linear time

Heapsort

- Sort by maintaining the as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in A.
  - Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
  - Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
  - Decrement heap-size[A].
- Restore the max-heap property on $A[1..n–1]$.
  - Call *MaxHeapify*(A, 1).
- Repeat until heap-size[A] is reduced to 2.

Heapsort(A)

<u>*HeapSort*(A)</u>

1. Build-Max-Heap(A)
2. **for** $i \leftarrow length[A]$ **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
5. $MaxHeapify(A, 1)$

Heapsort – Example

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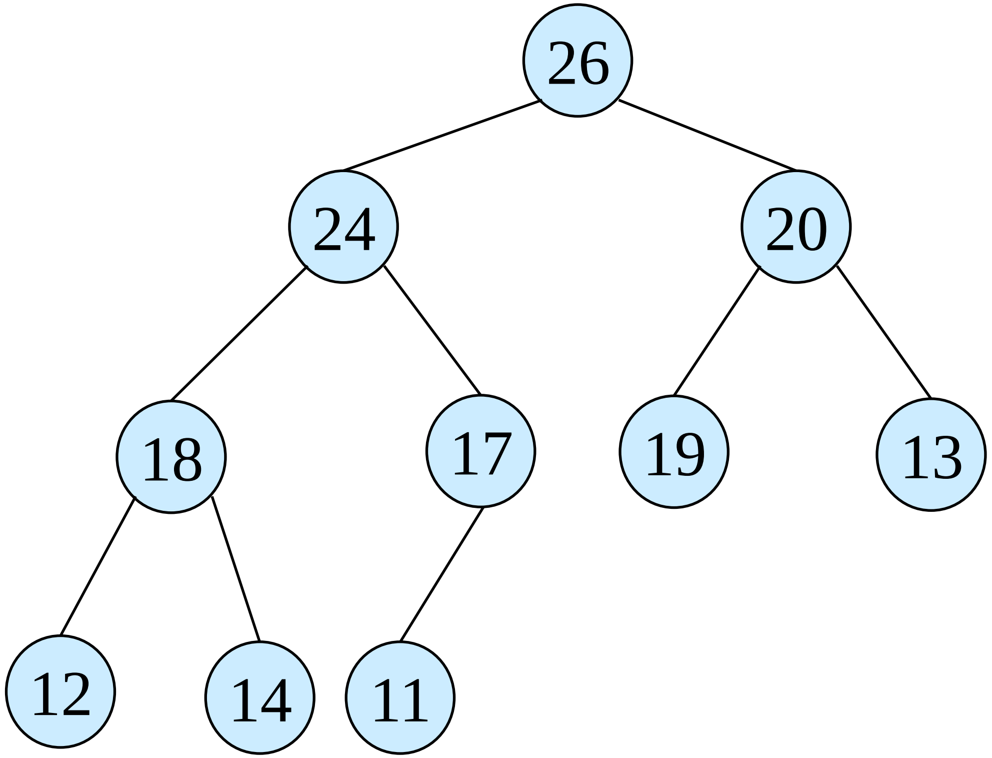


Algorithm Analysis

<u>*HeapSort*(*A*)</u>

1. Build-Max-Heap(A)
2. **for** $i \leftarrow$ *length*[A] **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. *heap-size*[A] $\leftarrow$ *heap-size*[A] – 1
5. *MaxHeapify*(A, 1)

- In-place
- Not Stable
- Build-Max-Heap takes $O(n)$ and each of the n-1 calls to Max-Heapify takes time $O(\lg n)$.
- Therefore, $T(n) = O(n \lg n)$

Heap Procedures for Sorting

- MaxHeapify $O(\lg n)$
- BuildMaxHeap $O(n)$
- HeapSort $O(n \lg n)$

Priority Queue

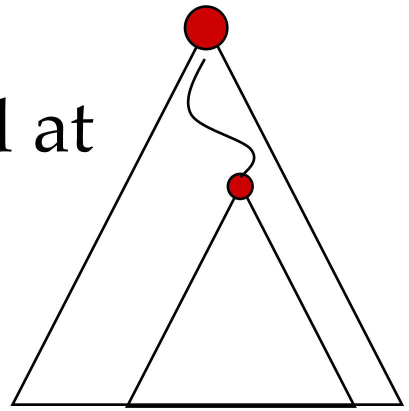
- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
  - Insert(S, x) - inserts the element x into the set S
    - $S \leftarrow S \cup \{x\}$.
  - Maximum(S) - returns the element of S with the largest key.
  - Extract-Max(S) - removes and returns the element of S with the largest key.
  - Increase-Key(S, x, k) – increases the value of element x's key to the new value k.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- Max-Heap
  - For every node excluding the root, value is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



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Heap-Extract-Max(*A*)

Implements the Extract-Max operation.

<u>*Heap-Extract-Max*(*A*)</u>

1. if *heap-size*[*A*] < 1
2. then error "heap underflow"
3. $max \leftarrow A[1]$
4. $A[1] \leftarrow A[heap\text{-}size[A]]$
5. $heap\text{-}size[A] \leftarrow heap\text{-}size[A] - 1$
6. MaxHeapify(*A*, 1)
7. return max

Running time : Dominated by the running time of MaxHeapify
$= O(\lg n)$

Heap-Insert(*A*, *key*)

Heap-Insert(A, key)

1. *heap-size*[*A*] ← *heap-size*[*A*] + 1
2. *i* ← *heap-size*[*A*]
4. **while** *i* > 1 **and** *A*[Parent(*i*)] < *key*
5. **do** *A*[*i*] ← *A*[Parent(*i*)]
6. *i* ← Parent(*i*)
7. *A*[*i*] ← *key*

- Running time is $O(\lg n)$
  - The path traced from the new leaf to the root has length $O(\lg n)$

Heap-Increase-Key(A, i, key)

- *Heap-Increase-Key(A, i, key)*

If $key < A[i]$
then error "new key is smaller than the current key"
$A[i] \leftarrow key$
while $i > 1$ **and** $A[\mathrm{Parent}[i]] < A[i]$
do exchange $A[i] \leftrightarrow A[\mathrm{Parent}[i]]$
$i \leftarrow \mathrm{Parent}[i]$

- *Heap-Insert(A, key)*

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
Heap-Increase-Key$(A, heap\text{-}size[A], key)$

Examples

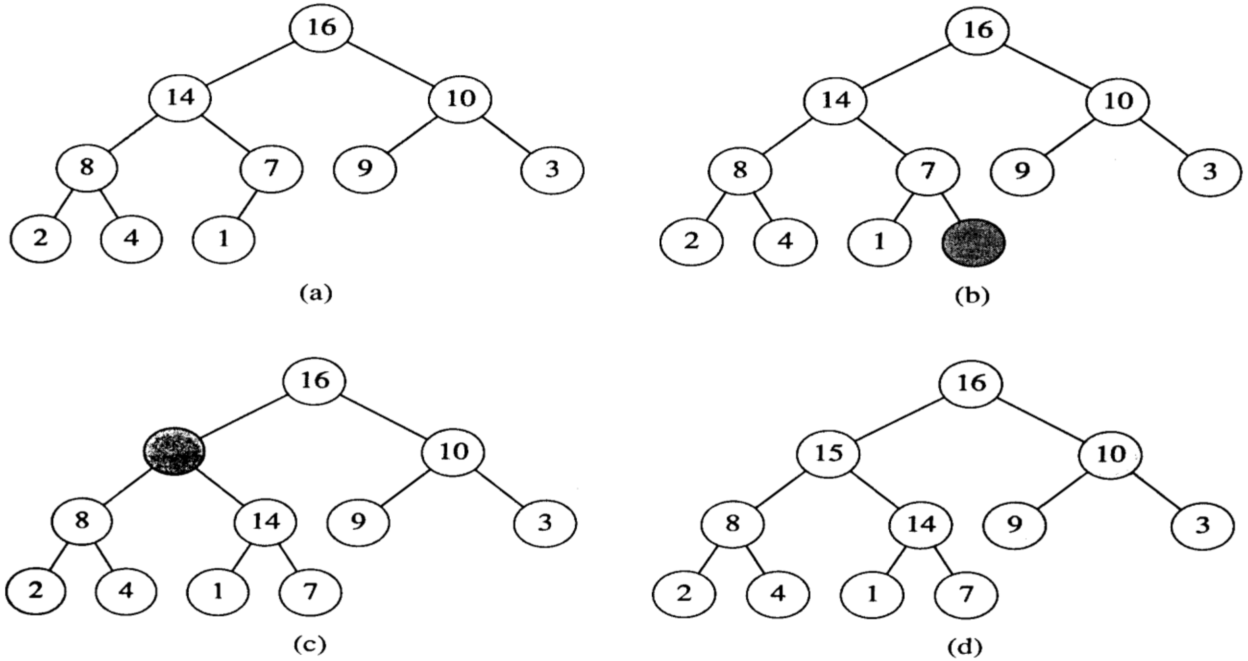

Figure 7.5 The operation of HEAP-INSERT. **(a)** The heap of Figure 7.4(a) before we insert a node with key 15. **(b)** A new leaf is added to the tree. **(c)** Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. **(d)** The key 15 is inserted.

```
void heapify(int A[], int n, int i){
    int parent = i; // root node
    int leftchild = 2*i+1;
    int rightchild = 2*i+2;

    if((leftchild<n) && (A[leftchild] >
A[parent])){
        parent = leftchild;
    }
    if ((rightchild<n) &&
(A[rightchild]>A[parent])){
        parent = rightchild;
    }
    if (parent !=i) {
        swap(A[i], A[parent]);
        heapify(A,n,parent);}
}
```

```
void HeapSort (int A[], int n){
    //Build Heap
    for (int i =n/2-1;i>=0;i--){
        heapify(A,n,i);
    }
    for (int i = n-1;i>=0;i--){
        swap(A[0],A[i]);
        heapify(A,i,0);}
}
```