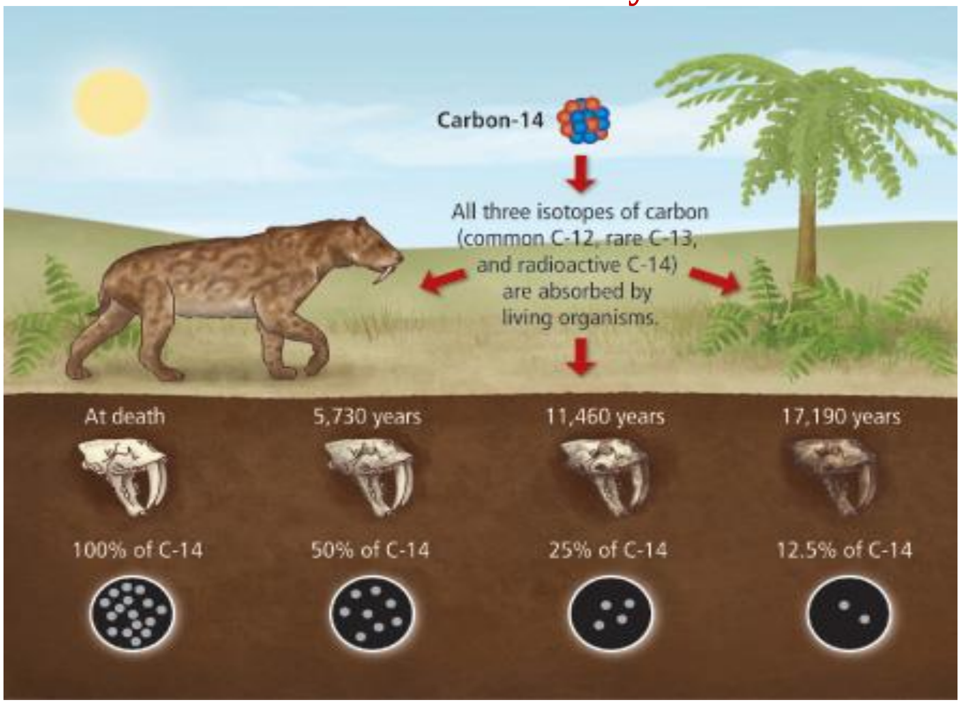
# Formulation of Differential Equation

## Radioactive Decay




**You are presented with a document which purports to contain the recollections of a Mycenaean soldier during the Trojan War. The city of Troy was finally destroyed in about 1250 BC, or about 3250 years ago.**

**Given the amount of carbon-14 contained in a measured sample cut from the document, there would have been about $1.3 \times 10^{-12}$ grams of carbon-14 in the sample when the parchment was new, assuming the proposed age is correct. According to your equipment, there remains $1.0 \times 10^{-12}$ grams.**

**Is there a possibility that this is a genuine document? Or is this instead a recent forgery? Justify your conclusions.**

Let A be the amount of C-14. Since rate of decay is proportional to the amount of substance present at time t,

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA \quad ; A(0) = A_0$$

**Carbon-14 has a half-life of 5730 years.**

## Laws of Cooling/Warming


**Assume you are a police officer at a crime scene with a dead body.**
**The forensics team measures the temperature of the body twice (assuming they do not know the constant of proportionality κ): the 1st time, immediately after their arrival, at 10 pm, the temperature of the body is found to be 33 °C and 1 hour later it is measured at 32.2°C.**
**Furthermore, we know the temperature of a healthy person to be around 37 °C (ignoring variation among people). Lastly, the room, where the body has been found is kept at a constant temperature of 20 °C.**
**Find the hour of the crime with the given information.**
According to Newton's empirical law of cooling/warming, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature. Newton's law of cooling/warming translates into the mathematical statement:

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\boldsymbol{\frac{dT}{dt} = k(T - T_m)}$$

where
$T =$ Temperature of body, $t =$ time
$T_m =$ Temperature of environment
We will see, after learning two simple methods to solve such DE that how its solution takes the form

$$\boldsymbol{T = T_m + ce^{kt}}$$

To know how we can construct a DE, let we have a function $\boldsymbol{y = e^{0.1x^2}}$ differentiable on the interval $(-\infty, \infty)$. Its derivative will be ……. Which lead to form DE

$$\boldsymbol{\frac{dy}{dx} = 0.2xy}$$

where $y$ is dependent variable,
and $x$ is independent variable.

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

## Notations

**Leibniz Notation:**

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \ldots$$

**Prime Notation**: $\quad y', y'', y''', y^{(4)}, \ldots$

**Dot Notation** (independent variable is always $t = time$) $\dot{y}, \ddot{y}$

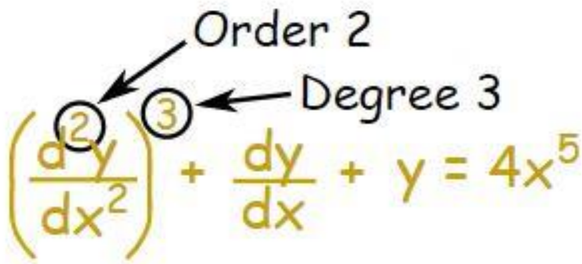
As we know that each model is a differential equation, so we have to learn the method to solve a particular type of differential equation. That's why we will learn types of DEs first and some methods to solve them accordingly.

## Classification by Type

| ODEs | PDEs |
|---|---|
| If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).**<br>For example,<br>$\frac{dy}{dx} = x + y$ | An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE)**.<br>For example,<br>$\frac{\partial y}{\partial x} = -\frac{\partial z}{\partial v}$ |

# Classification by Order & Degree

- The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation.
- **Degree of a differential equation** is the power of the highest derivative.




# Classification by Linearity

**For an ODE to be linear, it should satisfy following properties:**

**1.** The dependent variable and all its derivatives are of the first degree, that is, the power of each term involving dependent variable is 1.

**2.** The coefficients of dependent variable and its derivatives depend at most on the independent variable x.

| Linear ODEs | Non linear ODEs |
|---|---|
| $(y-x)dx + 4xdy = 0$ | $(1-\boldsymbol{y})y' + 2y = x$ |
| $y + y' = x$ | $y'' + \boldsymbol{siny} = 0$ |
| $x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$ | $\frac{dy}{dx} = \boldsymbol{y^2}$ |

## Practice Problems (Ex 1.1)

State the order and degree of the given ordinary differential equation. Determine whether the equation is linear or nonlinear. Also tell which variable is dependent and independent.

1. $(1-x)y'' - 4xy' + 5y = cosx$
2. $x\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^4 + y = 0$
3. $t^5y^{(4)} - t^3y'' + 6y = 0$
4. $(sin\theta)y''' - (cos\theta)y' = 2$
5. $\ddot{x} - \dot{x} + t = 0$

## Existence of Solution

Any function $y$, defined on an interval $I$ and possessing at least $n$ derivatives that are continuous on $I$, which when substituted into an $nth$-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

**Verification of a solution**:

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

a. $\frac{dy}{dx} = xy^{1/2}$ ; $y = \frac{1}{16}x^4$

Let $\frac{dy}{dx} = xy^{1/2}$ be eq. 1.

| L.H.S of eq. 1 | R.H.S of eq. 1 |
|---|---|
| $\frac{dy}{dx} = \frac{d(\frac{1}{16}x^4)}{dx} = \frac{1}{16}(4x^3) = \frac{x^3}{4}$ | $xy^{\frac{1}{2}} = x\left(\frac{x^4}{16}\right)^{\frac{1}{2}} = x.\frac{x^2}{4} = \frac{x^3}{4}$ |

Since L.H.S = R.H.S, $y$ is a solution of the given ODE.

b. $y'' - 2y' + y = 0$ ; $y = xe^x$

*Solve!*

## <u>1.2 Initial Value Problems (IVPs)</u>

An initial value problem is an ODE together with an initial condition which specifies the value of the dependent variable at a given point.

- Modeling a system in Physics or other sciences frequently amounts to solving an IVP.

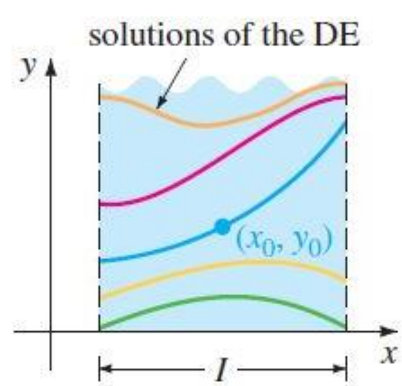
For example:

$$y'' + y = 0 \ ; \ y(0) = 1 \ , \qquad y'(0) = 1$$

| **Initial Conditions** | **Boundary Conditions** |
|---|---|
| These conditions are specified at the same value of the independent variable.<br>e.g.<br>$y'' + y = 0 \ ;$<br>$y(\mathbf{0}) = 1 \ , \qquad y'(\mathbf{0}) = 1$ | These conditions are specified at the extremes of the independent variables in the equation.<br>e.g.<br>$y'' + y = 0 \ ;$<br>$y(\mathbf{0}) = 1 \ , \qquad y(\mathbf{2}) = 1$ |

## <u>Geometric Interpretation of IVPs</u>

If $y' = f(x, y)$ is an ODE subject to the initial condition $y(x_0) = y_0$, we are seeking a solution $y(x)$ of the differential equation $y' = f\ (x, y)$ *on an interval $I$ containing $x_0$ so that its graph passes through the specified point $(x_0, y_0)$. A solution curve is shown in blue in Figure 1.*



If $\frac{d^2y}{dx^2} = f(x, y, y')$ is an ODE subject to the initial conditions

$$y(x_0) = y_0 \text{ and } y'(x_0) = y_1 \ ,$$

*we want to find a solution* $y(x)$ *of the differential equation* $y'' = f(x, y, y')$ *on an interval* $I$ *containing* $x_0$ *so that its graph not only passes through* $(x_0, y_0)$ *but the slope of the curve at this point is the number* $y_1$. *A solution curve is shown in blue in Figure 2.*

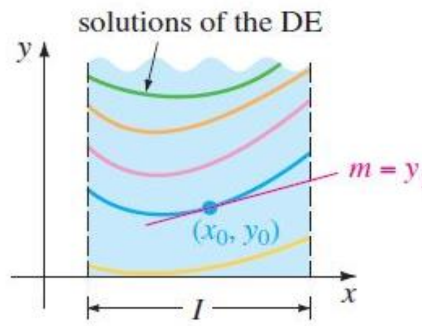



***Figure 1***

## Question:

**$y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.**

- **a. $y(0) = 1 \ , \ y'(0) = 2$**
- **b. $y(1) = 0 \ , \ y'(1) = 3$**

**a) $y(0) = 1 \ , \ y'(0) = 2$**

$$y = c_1e^x + c_2e^{-x}$$

Since $y(0) = 1$,

$$1 = c_1e^0 + c_2e^{-0}$$

$$1 = c_1 + c_2 \qquad (1)$$

Also,

$$y' = c_1e^x - c_2e^{-x} \quad and \quad y'(0) = 2$$

$$2 = c_1e^0 - c_2e^{-0}$$

$$2 = c_1 - c_2 \qquad (2)$$

Solving eq. 1 and eq. 2 gives: $\quad c_1 = 3/2 \quad$ and $\quad c_2 = -1/2$

So, $\quad \boldsymbol{y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}}$

**Note:**

- Here, $y = c_1 e^x + c_2 e^{-x}$ is called the **general solution** and $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$ is called the **particular solution**.

## Practice Problems

Q. Verify that the indicated function is a solution of the given ODEs:

12. $y'' + y = tanx \;\; ; \;\; y = -(cosx)\ln(secx + tanx)$

13. $y'' - 6y' + 13y = 0 \;\; ; \qquad y = e^{3x}cos2x$

Q. Find values of $m$ so that $y = e^{mx}$ is a solution of the given ODEs:

27. $y' + 2y = 0$

30. $2y'' + 7y' - 4y = 0$

## Exercise 1.2

Q7. $x = c_1 cost + c_2 sint$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

a. $x(0) = -1 \;,\; x'(0) = 8$

b. $x(\frac{\pi}{2}) = 0 \;,\; y'\left(\frac{\pi}{2}\right) = 1$

## Work to do

Do questions 1, 3, 6, 9, 35-38

### DIRECTION FIELDS

Let us imagine for the moment that we have in front of us a first-order DE

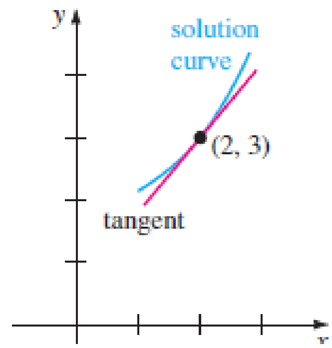
$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

and let us further imagine that we can neither find nor invent a method for solving it analytically. This is not as bad a predicament as one might think, since the DE itself can sometimes "tell" us specifics about how its solutions "behave" or we can say Solutions curves without a solution.

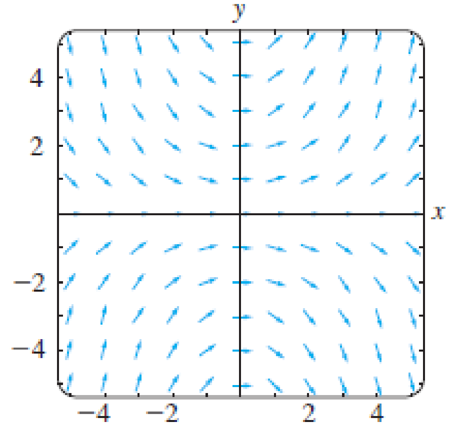
The function *f* in the normal form (1) is called the **slope function** or **rate function.** The slope of the tangent line at $(x, y(x))$ on a solution curve is the value of the first derivative $dy/dx$ at this point, and we know from (1) that this is the value of the slope function $f(x, y(x))$. The value $f(x, y)$ that the function $f$ assigns to the point represents the slope of a line or line segment. For example, consider the equation

$$\frac{dy}{dx} = 0.2xy = f(x, y)$$

At the point (2, 3) the slope of a line is $f(2, 3) = 1.2$. First figure shows a line segment with slope 1.2 passing though (2, 3). As shown in Figure, *if* a solution curve also passes through the point (2, 3), it does so tangent to this line segment; in other words, the lineal element is a miniature tangent line at that point.

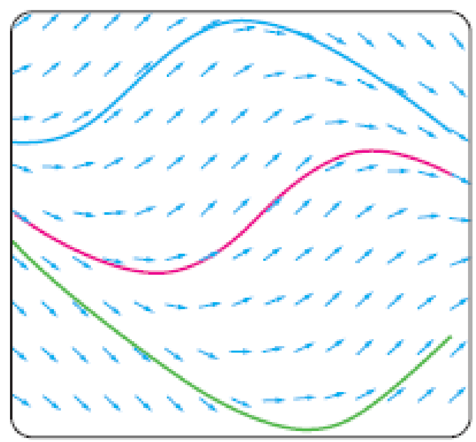



If we systematically evaluate $f$ over a rectangular grid of points in the $xy$-plane and draw a line element at each point $(x, y)$ of the grid with slope $f(x, y)$, then the collection of all these line elements is called a **direction field** or a **slope field** of the DE (1). Visually, the direction field suggests the appearance or shape of a family of solution curves of the DE.
The direction field for the DE $dy/dx = 0.2xy$ shown in Figure (a) was obtained by using computer software.




(a) direction field for

A single solution curve that passes through a direction field must follow the flow pattern of the field; it is tangent to a line element when it intersects a point in the grid. Figure (b) shows a computer-generated direction field of the differential equation $dy/dx = \sin(x + y)$ over a region of the $xy$-plane.



*From the next lecture we will formally move to the first method of solving a simple DE (for that you have to revise some basic integral formulae).