Assignment 4

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Roll# L1S22BSCS0413
Section:- E7

Q1a) i) t^2

solution:-

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2\,dt$$

Integration using DI method.

	D	I
+	t^2	e^{-st}
-	$2t$	$-e^{-st}/s$
+	2	e^{-st}/s^2
-	0	$-e^{-st}/s^3$

$$= \lim_{b\to\infty} -\frac{t^2e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\Bigg|_0^b$$

$$= \left(-\frac{b^2e^{-sb}}{s} - \frac{2be^{-sb}}{s^2} - \frac{2e^{-sb}}{s^3}\right) - \left(-0-0-\frac{2}{s^3}\right)$$

as $b\to\infty$, $e^{-sb}\to 0$

$$= 0 + \frac{2}{s^3}$$

$$= \frac{2}{s^3}$$

ii) e^{-5t}

Solution

$$\mathcal{L}\{e^{-5t}\} = \int_0^\infty e^{-st} e^{-5t} \, dt$$

$$= \int_0^\infty e^{t(-s-5)} \, dt$$

$$= \lim_{b \to \infty} \left. \frac{e^{t(-s-5)}}{-s-5} \right|_0^b$$

$$= \frac{e^{b(-s-5)}}{-s-5} - \frac{1}{-s-5}$$

$$= 0 - \frac{1}{-s-5} = \frac{1}{s+5}$$

iii) $\sinh 3t$

Solution

$$\mathcal{L}\{\sinh 3t\} = \int_0^\infty e^{-st} \sinh 3t \, dt$$

Integration by parts using DI method.

	D	I
+	$\sinh 3t$	e^{-st}
−	$3\cosh 3t$	$-e^{-st}/s$
+	$9\sinh 3t$	e^{-st}/s^2

$$= \lim_{b \to \infty} \left. -\frac{e^{-st}}{s} \sinh 3t - \frac{e^{-st}}{s^2} 3\cosh 3t \right|_0^b + \frac{9}{s^2} \int_0^\infty e^{-st} \sinh 3t \, dt$$

$$= 0 - \left(\frac{-\sinh 0}{s} - \frac{3\cosh 0}{s^2} \right) + \frac{9}{s^2} \mathcal{L}\{\sinh 3t\}$$

$$= 0 + \frac{3}{s^2} + \frac{9}{s^2} \mathcal{L}\{\sinh 3t\}$$

$$\mathcal{L}\{\sinh 3t\} = \frac{3}{s^2} + \frac{9}{s^2} \mathcal{L}\{\sinh 3t\}$$

$$\mathcal{L}\{\sinh 3t\} - \frac{9}{s^2}\{\sinh 3t\} = \frac{3}{s^2}$$

$$s^2 \mathcal{L}\{\sinh 3t\} - 9\{\sinh 3t\} = 3$$

$$\mathcal{L}\{\sinh 3t\}(s^2 - 9) = 3$$

$$\mathcal{L}\{\sinh 3t\} = \frac{3}{s^2 - 9}$$

Q1) b)

i) $\sin(5t+2)$

solution

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

$$\sin(5t+2) = \sin 5t \cos 2 + \cos 5t \sin 2$$

$$\mathcal{L}\{\sin(5t+2)\} = \mathcal{L}\{\sin 5t \cos 2\} + \mathcal{L}\{\cos 5t \sin 2\}$$

$$= \cos 2\, \mathcal{L}\{\sin 5t\} + \sin 2\, \mathcal{L}\{\cos 5t\}$$

$$= \cos 2 \left(\frac{5}{s^2+25} \right) + \sin 2 \left(\frac{s}{s^2+25} \right)$$

$$= \frac{s \sin 2 + 5\cos 2}{s^2+25}$$

ii) e^{-5t}

solution

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{-5t}\} = \frac{1}{s+5}$$

iii) $t \sin t$

solution

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right)$$

$$\mathcal{L}\{t \sin(t)\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= -\left(\frac{0 - 2s}{(s^2+1)^2}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

ii) $e^{-t}\sin 2t$

Solution:- $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$\mathcal{L}\{e^{-t}\sin 2t\} = F(s+1)$

$F(s) = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$\cancel{F(s+1) = \frac{2}{s^2+4} + 1}$

$\cancel{= \frac{2+s^2+4}{s^2+4} = \frac{s^2+6}{s^2+4}}$

$F(s+1) = \frac{2}{(s+1)^2+4}$

Q1) c)

i) $\frac{s-2}{s^2+3}$

solution

$\frac{s-2}{s^2+3} = \frac{s}{s^2+3} - \frac{2}{s^2+3}$

$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+3} - \frac{2}{s^2+3}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{3})^2}\right\} - \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\}$

$$= \cos\sqrt{3}t - \frac{2}{\sqrt{3}}\sin\sqrt{3}t$$

ii) $\dfrac{3}{s(s^2+4)}$

Solution.

$$\mathcal{L}^{-1}\left\{\frac{3}{s(s^2+4)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1/4}{s} - \frac{s}{4(s^2+2^2)}\right\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1/4}{s} - \left(\frac{1}{4}\right)\left(\frac{s}{s^2+2^2}\right)\right\}$$

$$= 3\left(\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}\right)$$

$$= 3\left(\frac{1}{4}(1) - \frac{1}{4}\cos 2t\right)$$

$$= \frac{3}{4} - \frac{3\cos 2t}{4}$$

Q1) d)

i) $y'' + 2y' + 5y = 0$

$y(0) = 1$

$y'(0) = 1$

solution:-

$y'' + 2y' + 5y = 0$

Taking Laplace Transform on b.s

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = 0$$

$$s^2 Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 5Y(s) = 0$$

$$s^2 Y(s) - s - 1 + 2sY(s) - 2 + 5Y(s) = 0$$

$$Y(s)(s^2 + 2s + 5) = s + 3$$

$$Y(s) = \frac{s+3}{s^2+2s+5}$$

$$Y(s) = \frac{s+1+2}{s^2+2s+1+4}$$

$$Y(s) = \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}$$

Taking inverse Laplace

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + \mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$y(t) = \mathcal{L}^{-1}\{F_1(s+1)\} + \mathcal{L}^{-1}\{F_2(s+1)\}$$

$$F_1(s) = \mathcal{L}\{\cos 2t\}$$

$$F_2(s) = \mathcal{L}\{\sin 2t\}$$

we Know

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$, where $F(s)=\mathcal{L}\{f(t)\}$

So,

$y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t.$

ii) $y''+9y = 3\sin 3t$

$y(0)=1$

$y'(0)=0$

solution

$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\sin 3t\}$

$s^2Y(s) - sy(0) - y'(0) + 9(Y(s)) = \frac{3}{s^2+9}$

$Y(s)(s^2+9) - s - 0 = \frac{3}{s^2+9}$

$Y(s) = \frac{3}{(s^2+9)^2} + \frac{s}{s^2+9}$

Taking inverse Laplace Transform.

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{(s^2+9)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{18}\cdot\frac{2.3^3}{(s^2+3^2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$

$= \frac{1}{18}\mathcal{L}^{-1}\left\{\frac{2.3^3}{(s^2+3^2)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$

We Know,

$$\mathcal{L}^{-1}\left\{\frac{2a^3}{(s^2+a^2)^2}\right\} = \sin at - at\cos at$$

$$y(t) = \frac{1}{18}(\sin 3t - 3t\cos 3t) + \cancel{\sin}\ \cos 3t$$

Q2) $2y''' + 3y'' - 3y' - 2y = e^{-t}$

$y(0) = 0$

$y'(0) = 0$

$y''(1) = 2$

solution

The problem is $y''(1) = 2$, it should be $y''(0) = 2$

so,

$$2\mathcal{L}\{y'''\} + 3\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$2[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 3[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] - 2[Y(s)] = \frac{1}{s+1}$$

$$Y(s)(2s^3+3s^2-3s-2) - 4 = \frac{1}{s+1}$$

$$Y(s)(2s^3+3s^2-3s-2) = \frac{4s+5}{s+1}$$

$$Y(s) = \frac{4s+5}{(s+1)(2s^3+3s^2-3s-2)}$$

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -3 & -2 \\ & & 2 & 5 & 2 \\ \hline & 2 & 5 & 2 & \vert\, 0 \end{array}$$

$$Y(s) = \frac{4s+5}{(s+1)(s-1)(2s^2+5s+2)}$$

$$Y(s) = \frac{4s+5}{(s+1)(s-1)(s+2)(2s+1)}$$

$$\frac{4s+5}{(s+1)(s-1)(s+2)(2s+1)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+2} + \frac{D}{2s+1}$$

$$4s+5 = A(s-1)(s+2)(2s+1) + B(s+1)(s+2)(2s+1) + C(s-1)(s+1)(2s+1) + D(s-1)(s+1)(s+2)$$

$s = 1$

$9 = B(2)(3)(3)$

$B = \frac{9}{18} = \frac{1}{2}$

$s = -1$

$1 = A(-2)(1)(-1)$

$A = \frac{1}{2}$

$s = -2$

$-3 = C(-3)(-1)(-3)$

$C = \frac{1}{3}$

$s = -\frac{1}{2}$

$3 = D\left(\frac{1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-3}{2}\right)$

$\cdot D = \frac{8}{3}$

$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$
$+ \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{8}{3}\mathcal{L}^{-1}\left\{\frac{1}{2s+1}\right\}$

$y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} + \frac{1}{3}e^{-2t} + \frac{4}{3}e^{-t/2}$

Q3)

i) $y'' - xy = \sin x$

Solution

$y'' - xy = \sin x$

$y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$

$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$

$m\cancel{a} = n-2$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m$$

$K = n+1$

$$\sum_{K=1}^{\infty} a_{K-1} x^K$$

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} 2x^m - \sum_{K=1}^{\infty} x^K = \sin(x)$$

$$(0+2)(0+1)a_2 + \sum_{m=1}^{\infty} (m+2)(m+1)a_{m+2}\, a_{m-1}\, x_m = \sin x$$

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$x^0 = 2a_2 = 0$

$x^1 = 6a_3 - a_0 = 1$

$x^2 = 12a_4 - a_2 = \frac{1}{6}$

$x^3 = 20a_5 - a_2 = -\frac{1}{6}$

$a_2 = 6$

$a_3 = \frac{1}{2} a_0$

$a_4 = \frac{1}{12} a_1$

$a_5 = -\frac{1}{120}$

$$y(x) = 1 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots$$

ii) $y'' - xy = x^2$

$y(0) = 1$

$y'(0) = 0$

Solution

$$= \sum_{n=2}^{\infty} n(n-1) a_n a^{n-2} - \sum_{n=0}^{\infty} a_n a^{n+1} = x^2$$

$$m = n - 2$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_m x^m$$

$$\sum_{k=1}^{\infty} a_{k-1} k$$

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{k=1}^{\infty} a_{k-1} x^k = x^2$$

$$x^0 = 2a_2 = 0$$

$$x^1 = 6x_3 - a_0 = 0$$

$$x_2 = 12a_x - a_1 = 1$$

$$x_3 = 2a_x - a_2 = 0$$

$$a_3 = \frac{1}{6}$$

$$a_4 = \frac{1}{16}$$

$$a_5 = 0$$

$$y(x) = 1 + \frac{1}{6} x^3 + \frac{1}{12} x^4$$