In Lecture 1 we saw how a first-order DE could be used as a mathematical model in the study of population growth, radioactive decay, cooling of bodies. Using the methods of Chapter 2, we are now able to solve some of the linear DEs that commonly appear in applications. Let's go back to the models presented in Lecture 1 and solve those models to answer the questions asked there.

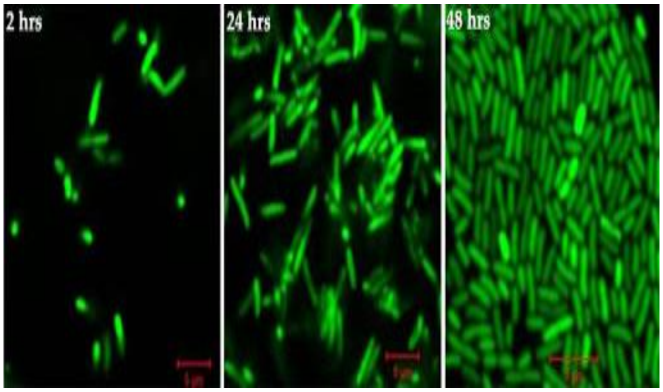
Population Model

We are studying a population of bacteria undergoing binary fission. In particular, the population doubles every three hours.

1. How many bacteria are present after 51 hours if a culture is inoculated with 1 bacterium?

2. With how many bacteria should a culture be inoculated if there are to be 81,920 bacteria present on hour 42?

3. How long would it take for an initial population of 6 to reach a size of 12,288 bacteria?



How to Model it?

Let P represents the population of bacteria at any time t. Since,

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP \quad ; \qquad P(0) = P_0$$

Notice that the above DE is both separable and linear, so solving this IVP using Separation of Variables or First Order Linear equation, we get

$$P = P_0 e^{kt}$$

Using the information that the population doubles every three hours.

$$P(3) = 2P_0$$

$$2P_0 = P_0 e^{3k}$$

$$k = \frac{ln2}{3} = 0.2310490602$$

$$\boldsymbol{P = P_0 e^{0.2310490602t}}$$

1. How many bacteria are present after 51 hours if a culture is inoculated with 1 bacterium?

$$Since\ P_0 = 1\ \ and\ t = 51$$

$$P = e^{0.2310490602(51)}$$

$$\boldsymbol{P = 131,072\ bacteria}$$

2. With how many bacteria should a culture be inoculated if there are to be 81,920 bacteria present on hour 42?

$$Since\ P = 81{,}920\ \ and\ t = 42$$

$$81{,}920 = P_0 e^{0.2310490602(42)}$$

$$\boldsymbol{P_0 \cong 5\ bacteria}$$

3. How long would it take for an initial population of 6 to reach a size of 12,288 bacteria?

$$Since\ P_0 = 6\ and\ P = 12{,}288$$

$$12{,}288 = 6e^{0.2310490602t}$$

$$\boldsymbol{t = 32.9}$$

EXAMPLE 1. A culture initially has P_0 number of bacteria. At t =1h the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

SOLUTION. We first solve the differential equation

$$\frac{dP}{dt} = kP\ \ ; \qquad P(0) = P_0$$

and get the solution

$$P = P_0 e^{kt}$$

We then use the empirical observation $P(1) = \frac{3}{2}P_0$ to determine the constant of proportionality k in the solution obtained, that is

$$\frac{3}{2}P_0\ = P_0 e^{k(1)}$$

Therefore $k = \ln\frac{3}{2} = 0.4055$, gives updated solution as

$$P = P_0 e^{0.4055t}$$

To find the time at which the number of bacteria has tripled, we solve

$$3P_0 = P_0 e^{0.4055t}$$

for t and found that $t = \frac{\ln 3}{0.4055} \approx 2.71$ hours.

Also, we can have model of human population of a town or country to estimate population after few years.

Exercise 3.1

GROWTH AND DECAY

1. The population of a community is known to increase at a rate proportional to the number of people present at time *t*. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?

2. Suppose it is known that the population of the community in Problem 1 is 10,000 after 3 years. What was the initial population P_0? What will be the population in 10 years? How fast is the population growing at $t = 10$?

3. The population of a town grows at a rate proportional to the population present at time *t*. The initial population of 500 increases by 15% in 10 years. What will be the population in 30 years? How fast is the population growing at $t = 30$?

4. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time *t*. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?