Linear Equations

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (1)$$

is said to be a **linear equation** in the dependent variable y.

Standard Form: By dividing both sides of (1) by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form,** of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x) \qquad (2)$$

SOLVING A LINEAR FIRST-ORDER EQUATION

(*i*) Put a linear equation of form (1) into the standard form (2).

(*ii*) From the standard form identify $P(x)$ and then find the integrating factor $e^{\int P(x)dx}$.

(*iii*) Multiply the standard form of the equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

(*iv*) Integrate both sides of this last equation.

Example 1 Solve $\dfrac{dy}{dx} - 3y = 0$.

SOLUTION This linear equation can be solved by separation of variables. Alternatively, since the equation is already in the standard form (2), we see that $P(x) = -3$, and so the integrating factor is $e^{\int(-3)dx} = e^{-3x}$. We multiply the equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0 \qquad \text{is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 0.$$

Integrating both sides of the last equation gives $e^{-3x}y = c$. Solving for y gives us the explicit solution $y = ce^{3x}$, $-\infty < x < \infty$. ■

Example 2

Solve $\dfrac{dy}{dx} - 3y = 6.$

SOLUTION The associated homogeneous equation for this DE was solved in Example 1. Again the equation is already in the standard form (2), and the integrating factor is still $e^{\int(-3)dx} = e^{-3x}$. This time multiplying the given equation by this factor gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}, \qquad \text{which is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

Integrating both sides of the last equation gives $e^{-3x}y = -2e^{-3x} + c$ or $y = -2 + ce^{3x}$, $-\infty < x < \infty$. ■

Example 3

Solve $x\dfrac{dy}{dx} - 4y = x^6e^x.$

SOLUTION Dividing by x, we get the standard form

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x.$$

From this form we identify $P(x) = -4/x$ and $f(x) = x^5e^x$ and further observe that P and f are continuous on $(0, \infty)$. Hence the integrating factor is

we can use $\ln x$ instead of $\ln |x|$ since $x > 0$

$$e^{-4\int dx/x} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}.$$

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \qquad \text{as} \qquad \frac{d}{dx}[x^{-4}y] = xe^x.$$

It follows from integration by parts that the general solution defined on the interval $(0, \infty)$ is $x^{-4}y = xe^x - e^x + c$ or $y = x^5e^x - x^4e^x + cx^4$. ■

Example 4

Find the general solution of $(x^2 - 9)\dfrac{dy}{dx} + xy = 0$.

SOLUTION We write the differential equation in standard form

$$\frac{dy}{dx} + \frac{x}{x^2 - 9}y = 0$$

and identify $P(x) = x/(x^2 - 9)$. Although P is continuous on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$, we shall solve the equation on the first and third intervals. On these intervals the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2-9)} = e^{\frac{1}{2}\ln|x^2-9|} = \sqrt{x^2 - 9}.$$

$$\frac{d}{dx}\left[\sqrt{x^2 - 9}\,y\right] = 0.$$

Integrating both sides of the last equation gives $\sqrt{x^2 - 9}\,y = c$. Thus for either $x > 3$ or $x < -3$ the general solution of the equation is $y = \dfrac{c}{\sqrt{x^2 - 9}}$. ■

Practice Questions: [Exercise 2.3 of Book: Differential Equations by D.G. Zill]

1. $\dfrac{dy}{dx} = 5y$
2. $\dfrac{dy}{dx} + 2y = 0$
3. $\dfrac{dy}{dx} + y = e^{3x}$
4. $3\dfrac{dy}{dx} + 12y = 4$
5. $y' + 3x^2y = x^2$
6. $y' + 2xy = x^3$
7. $x^2y' + xy = 1$
8. $y' = 2y + x^2 + 5$
9. $x\dfrac{dy}{dx} - y = x^2 \sin x$
10. $x\dfrac{dy}{dx} + 2y = 3$
11. $x\dfrac{dy}{dx} + 4y = x^3 - x$
12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$
13. $x^2y' + x(x + 2)y = e^x$

14. $xy' + (1 + x)y = e^{-x} \sin 2x$

15. $y\,dx - 4(x + y^6)\,dy = 0$

16. $y\,dx = (ye^y - 2x)\,dy$

17. $\cos x \dfrac{dy}{dx} + (\sin x)y = 1$

18. $\cos^2 x \sin x \dfrac{dy}{dx} + (\cos^3 x)y = 1$

19. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$

20. $(x + 2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy$

21. $\dfrac{dr}{d\theta} + r \sec \theta = \cos \theta$

22. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$