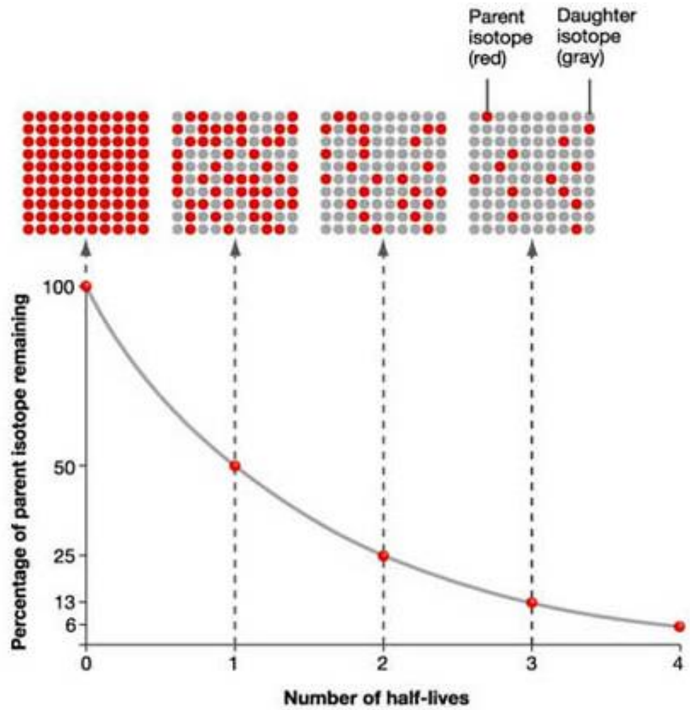
RADIOACTIVE DECAY & HALF-LIFE



In physics the **half-life** is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another element.
The longer the half-life of a substance, the more stable it is. For example radium, Ra-226, over time the highly radioactive radium, Ra-226, transmutes into the radioactive gas radon, Rn-222. To model the phenomenon of **radioactive decay,** it is assumed that the rate dA/dt at which the nuclei of a substance decay is proportional to the amount (more precisely, the number of nuclei) $A(t)$ of the substance remaining at time t.

$$\frac{dA}{dt} \propto A$$

$$\boldsymbol{\frac{dA}{dt} = kA, \qquad A(0) = A_0}$$

We can see that decay model and population models are represented by almost same initial value problem. For growth, $k < 0$, and for decay, $k > 0$.

EXAMPLE 2. A breeder reactor converts relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

SOLUTION.

Let $A(t)$ denote the amount of plutonium remaining at time t. As the solution of the initial-value problem is

$$A(t) = A_0 e^{kt}$$

If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. To find the decay constant k, we use $0.99957A_0 = A(15)$ —that is,

$$0.99957A_0 = A_0e^{15k}$$

Solving for k then gives $k = \frac{1}{15}\ln 0.99957 = \ 0.00002867.$
Hence

$$A(t) = A_0e^{-0.00002867t}$$

Now the half-life is the value of time at which $A(t) = \frac{1}{2}A_0$. Solving for t

$$\frac{1}{2}A_0 = A_0e^{-0.00002867t}$$

gives

$$t = \frac{-\ln 2}{-0.00002867} \approx 24{,}180 \text{ years.}$$

CARBON DATING

Let's now have a particular case of radioactive decay, the carbon dating. Here in this model we have an additional calculated information that Carbon-14 has a half-life of 5730 years. That already known information will reduce our calculations, making problem solution very direct.

EXAMPLE 3. A fossilized bone is found to contain one-thousandth of the C-14 level found in living matter. Estimate the age of the fossil.

SOLUTION. The starting point is again

$$A(t) = A_0e^{kt}$$

To determine the value of the decay constant k, we use the fact $\frac{1}{2}A_0 = A(5730)$ or

$$\frac{1}{2}A_0 = A_0e^{5730k}$$

From $5730k = \ln 1/2$, we get

$$k = \frac{-\ln 2}{5730} = \ -0.000120968.$$

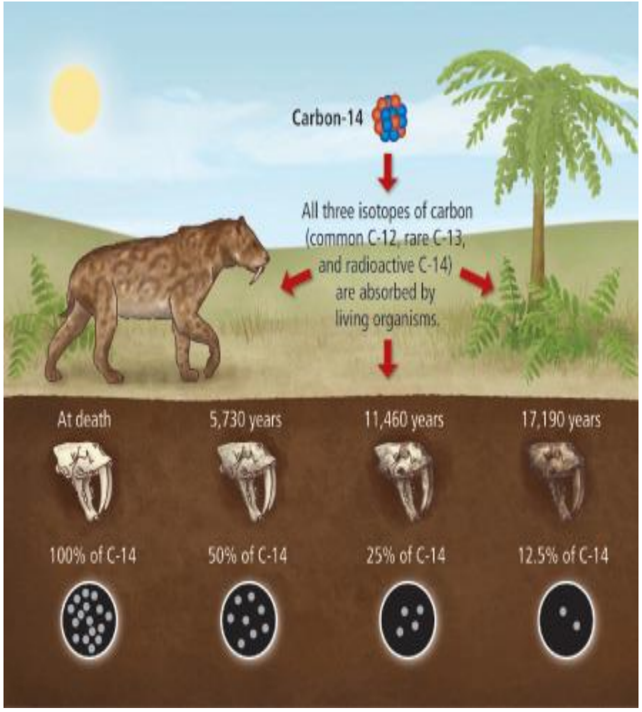
Therefore, we have

$$A(t) = A_0e^{-0.000120968t}$$

Thus the age of the fossil for one-thousandth of the C-14 level found is about

$$t = \frac{-\ln 1000}{-0.000120968} \approx 57{,}104 \text{ years.}$$

You are presented with a document which purports to contain the recollections of a Mycenaean soldier during the Trojan War. The city of Troy was finally destroyed in about 1250 BC, or about 3250 years ago.



Given the amount of carbon-14 contained in a measured sample cut from the document, there would have been about 1.3×10^{-12} grams of carbon-14 in the sample when the parchment was new, assuming the proposed age is correct. According to your equipment, there remains 1.0×10^{-12} grams.

Is there a possibility that this is a genuine document? Or is this instead a recent forgery? Justify your conclusions.

Model. Let A be the amount of C-14 remaining at any time t. So, model is

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

To find that document is fake or original, we solve the model and reach (as above)

$$A(t) = A_0 e^{-0.000120968t}$$

$$1.0 \times 10^{-12} = 1.3 \times 10^{-12} \, e^{-0.000120968t}$$

$$\frac{1}{1.3} = e^{-0.000120968t}$$

$$t = \frac{\ln 1.3}{0.000120968} \approx 2{,}168.87 \text{ years.}$$

The parchment is about 2170 years old, much less than the necessary 3250 years ago that the Trojan War took place. But the parchment is indeed old, so this isn't a total fake!

EXERCISE 3.1

5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at time *t* and has a half-life of 3.3 hours. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay?

6. Initially 100 milligrams of a radioactive substance was present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time *t*, find the amount remaining after 24 hours.

7. Determine the half-life of the radioactive substance described in Problem 6.

12. The shroud of Turin, which shows the negative image of the body of a man who appears to have been crucified, is believed by many to be the burial shroud of Jesus of Nazareth. See Figure. In 1988 the Vatican granted permission to have the shroud carbon-dated. Three independent scientific laboratories analyzed the cloth and concluded that the shroud was approximately 660 years old,* an age consistent with its historical appearance.

Using this age, determine what percentage of the original amount of C-14 remained in the cloth as of 1988.