Problem 1

The problem is to find a general solution to:

$$u_{xx} + u = 6y$$

in terms of arbitrary functions; where $u = u(x,t)$. Here, the general solution is seen to be:

$$u = A(y)\sin(x) + B(y)\cos(x) + 6y$$

where $A(y)$ and $B(y)$ are arbitrary. This general solution can be verified directly:

$$u_{xx} + u = [-A(y)\sin(x) - B(y)\cos(x)] + [A(y)\sin(x) + B(y)\cos(x) + 6y] = 6y$$

as required.

Problem 2

The problem is to find a general solution to:

$$u_{tx} + u_x = 1$$

in terms of arbitrary functions; where $u = u(x,t)$. To begin, one may integrate both sides with respect to x:

$$\int u_{tx}dx + \int u_x dx = \int 1dx \Rightarrow u_t + u = x + f(t)$$

for some arbitrary function $f(x)$. At this point, we may solve for u using the method of integration factors:

$$u = \frac{\int (x + f(t))\, e^{\int 1dt} + g(x)}{e^{\int 1dt}} = \left[\int (x + f(t))\, e^t dt + g(x)\right] e^{-t}$$

where $g(x)$ is an arbitrary function. This general solution may be verified directly:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\left(\left[\int (x + f(t))\, e^t dt + g(x)\right] e^{-t}\right)\right) + \frac{\partial}{\partial x}\left(\left[\int (x + f(t))\, e^t dt + g(x)\right] e^{-t}\right)$$

and since:

$$\frac{\partial}{\partial x}\left(\left[\int (x + f(t))\, e^t dt + g(x)\right] e^{-t}\right) = \frac{\partial}{\partial x} e^{-t} \int x e^t dt + \frac{\partial}{\partial x} e^{-t} \int f(t)\, e^t dt + e^{-t} g'(x)$$

$$= 1 + g'(x)\, e^{-t}$$

we get:

$$\Rightarrow u_{tx} + u_x = \frac{\partial}{\partial t}\left(1 + g'(x)\, e^{-t}\right) + 1 + g'(x)\, e^{-t} = -g'(x)\, e^{-t} + 1 + g'(x)\, e^{-t} = 1$$

$$\Rightarrow u_{tx} + u_x = 1$$

as required.

Problem 3

The problem is to show that the general solution $u = u(x,y)$ of $yu_x - xu_y = 0$ is given by $u = \psi\left(x^2 + y^2\right)$, where ψ is an arbitrary function. First, consider the parameterization:

$$x = r\cos(\theta),\, y = r\sin(\theta)$$

From this, we see that:

$$\left.\begin{array}{c} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}\right\} \Rightarrow \left.\begin{array}{c} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{array}\right\}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r\cos(\theta)}{r} = \cos(\theta)$$

$$\Rightarrow \frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2}} \cdot 2y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r\sin(\theta)}{r} = \sin(\theta)$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = -\frac{y}{\left(1 + \left(\frac{y}{x}\right)^2\right) x^2} = -\frac{y}{x^2 + y^2} = -\frac{\sin(\theta)}{r}$$

$$\Rightarrow \frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos(\theta)}{r}$$

Next, consider an auxiliary function defined by:

$$\upsilon(r,\theta) = u(r\cos(\theta), r\sin(\theta))$$

Writing the given PDE in terms of υ gives:

$$yu_x - xu_y = 0 \Rightarrow r\sin(\theta)\left(\frac{\partial \upsilon}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \upsilon}{\partial \theta}\frac{\partial \theta}{\partial x}\right) - r\cos(\theta)\left(\frac{\partial \upsilon}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \upsilon}{\partial \theta}\frac{\partial \theta}{\partial y}\right) = 0$$

$$\Rightarrow r\sin(\theta)\cos(\theta)\,\upsilon_r - \sin^2(\theta)\,\upsilon_\theta - r\sin(\theta)\cos(\theta)\,\upsilon_r - \cos^2(\theta)\,\upsilon_\theta = 0$$

$$\Rightarrow -\left[\sin^2(\theta) + \cos^2(\theta)\right]\upsilon_\theta = 0 \Rightarrow \upsilon_\theta = 0 \Rightarrow \upsilon(r,\theta) = f(r)$$

where $f(r)$ is an arbitrary function. Since $r = \sqrt{x^2 + y^2}$, we may conclude the u is an arbitrary function of $\sqrt{x^2 + y^2}$ (or, equivelantly, an arbitrary function of $x^2 + y^2$):

$$u(x,y) = \psi\left(x^2 + y^2\right)$$

as required.

Problem 4

The problem is to determine the regions in the plane where the equation $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ is parabolic, elliptic, or hyperbolic. The discriminant is computed to be $(-2)^2 - 4xy = 4(1 - xy)$. Each case is now considered in turn:

(a) (Parabolic): For this case, we must have $4(1 - xy) = 0 \Rightarrow 1 - xy = 0 \Rightarrow y = \frac{1}{x}$. So, $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ is parabolic along the points on $y = \frac{1}{x}$

(b) (Elliptic): For this case, we must have $4(1 - xy) < 0 \Rightarrow xy > 1 \Rightarrow \begin{cases} y < {}^1\!/_x & \text{for } x < 0 \\ y > {}^1\!/_x & \text{for } x > 0 \end{cases}$. So, $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ is elliptic in the portion of the plane complimentry to the area between the two curves which comprise $y = \frac{1}{x}$.

(c) (Hyperbolic): For this case, we must have $4(1-xy) > 0 \Rightarrow xy < 1 \Rightarrow \begin{cases} y > 1/x & \text{for } x < 0 \\ y < 1/x & \text{for } x > 0 \end{cases}$. So, $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ is hyperbolic in the area of the plane between the two curves which comprise $y = \frac{1}{x}$.

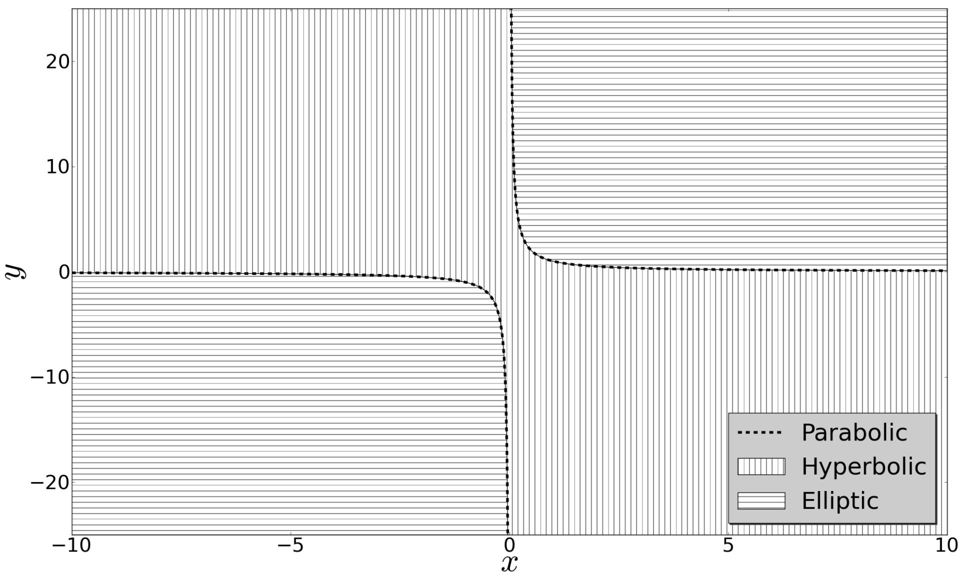

Fig. 1: (Problem 22) A plot showing where the equation $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ is parabolic, elliptic, and hyberbolic.

Problem 5

The problem is to show that the growth-diffusion equation $u_t = D\Delta u + ru$ can be transformed into a pure diffusion equation via the transformation $v = ue^{-rt}$. The definition of the transformation gives:

$$v = ue^{-rt} \Rightarrow u = ve^{rt}$$

and thus:

$$u_t = v_t e^{rt} + vre^{rt}$$

Since the Laplacian operator treats t as a constant, we have:

$$\Delta u = \Delta\left[ve^{rt}\right] = e^{rt}\Delta v$$

So the original PDE can be written as:

$$v_t e^{rt} + vre^{rt} = \Delta ve^{rt} + vre^{rt} \Rightarrow v_t e^{rt} = e^{rt}\Delta v \Rightarrow v_t = \Delta v$$

which is in the form of a pure diffusion equation.

Problem 6

The problem is to prove the non-optimal Poincaré inequality:

$$\|u\|_2^2 \leqslant \frac{L^2}{4}\|u_x\|_2$$

for all sufficiently smooth functions u satisfying $u(0) = u(L) = 0$. To begin, we know from the fundamental theorem of calculus and the given conditions $u(0) = u(L) = 0$ that:

$$u = \int_0^x u_x = -\int_0^L u_x$$

Using the first part of the above equation, note that:

$$|u| = \left|\int_0^x u_x\right| \leqslant \int_0^x |u_x|$$

From the Cauchy-Schwartz inequality $\int_0^L |fg| \leqslant \|f\|_2\|g\|_2$, we have:

$$\int_0^x |u_x| \leqslant \left[\int_0^x |u_x|^2\right]^{\frac{1}{2}}\left[\int_0^x |1|^2\right]^{\frac{1}{2}} = \left[\int_0^x |u_x|^2\right]^{\frac{1}{2}} x^{\frac{1}{2}} \leqslant \left[\int_0^L |u_x|^2\right]^{\frac{1}{2}} x^{\frac{1}{2}}$$

and thus:

$$|u|^2 \leqslant x\int_0^L |u_x|^2 \Rightarrow |u|^2 \leqslant x\|u_x\|_2^2$$

From $u = -\int_0^L u_x$, we get:

$$|u| = \left|-\int_0^L u_x\right| \leqslant \int_0^L |u_x|$$

and, again, from the Cauchy-Schwartz inequality:

$$\int_0^x |u_x| \leqslant \left[\int_0^L |u_x|^2\right]^{\frac{1}{2}}\left[\int_0^L |1|^2\right]^{\frac{1}{2}} = \left[\int_0^L |u_x|^2\right]^{\frac{1}{2}} x^{\frac{1}{2}} \leqslant \left[\int_0^L |u_x|^2\right]^{\frac{1}{2}} (L-x)^{\frac{1}{2}}$$

thus:

$$|u|^2 \leqslant (L-x)\int_0^L |u_x|^2 \Rightarrow |u|^2 \leqslant (L-x)\|u_x\|_2^2$$

Now, since $\|u(x)\|_2^2 = \int_0^L |u(x)|^2 dx$, we may combine our two derived bounds on $|u(x)|$ by seperating the integration from 0 to $L/2$ and $L/2$ to L:

$$\|u(x)\|_2^2 = \int_0^L |u(x)|^2 dx \leqslant \int_0^{L/2} t\,\|u_x\|_2^2\,dt + \int_{L/2}^{L} (L-t)\,\|u_x\|_2^2\,dt$$

$$\Rightarrow \|u(x)\|_2^2 \leqslant \|u_x\|_2^2 \left(\left[\frac{t^2}{2}\right]\Bigg|_0^{L/2} + \left[Lt - \frac{t^2}{2}\right]\Bigg|_{L/2}^{L} \right) = \|u_x\|_2^2 \left(\frac{L^2}{8} + L^2 - \frac{L^2}{2} - \frac{L^2}{2} + \frac{L^2}{8} \right)$$

$$\Rightarrow \|u(x)\|_2^2 \leqslant \frac{L^2}{4}\,\|u_x\|_2^2$$

as required.

Problem 7

The problem is to find the equations which determine the solution to the isometric problem:

$$\min_{y\in a} : \int_a^b \left(p(x)\,y'^2 + q(x)\,y^2 \right) dx$$

$$\text{subject to: } \int_a^b r(x)\,y^2 dx = 1$$

where p, q, and r are given functions, $y(a) = y(b) = 0$, with the admissible class of functions $a = \left\{ y \in C^2(a,b) \cap C[a,b] \middle|\, y(a) = y(b) = 0 \right\}$. In a formulation analogous to the method of Lagrange multipliers, consider:

$$\int_a^b \left(p(x)\,y'^2 + q(x)\,y^2 \right) dx + \lambda \int_a^b r(x)\,y^2 dx$$

$$= \int_a^b \left(p(x)\,y'^2 + q(x)\,y^2 + \lambda r(x)\,y^2 \right) dx = \int_a^b \left(p(x)\,y'^2 + y^2\left(q(x) + \lambda r(x)\right) \right) dx$$

Define $\mathcal{L}(y, y', \lambda)$ to be the integrand of the above expression:

$$\mathcal{L}\left(y, y', \lambda\right) = p(x)\,y'^2 + y^2\left(q(x) + \lambda r(x)\right)$$

Now, we look for extremals of $\int_a^b \mathcal{L}(y, y', \lambda)\,dx$ by solving the Euler-Lagrange equation:

$$-\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'} + \frac{\partial \mathcal{L}}{\partial y} = 0, \text{ subject to: } \int_a^b r(x)\,y^2 dx = 1 \text{ and } y(a) = y(b) = 0$$

Computing $\frac{\partial \mathcal{L}}{\partial y}$ gives:

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial y}\left(p(x)\,y'^2 + y^2\left(q(x) + \lambda r(x)\right) \right) = 2y\left(q(x) + \lambda r(x)\right)$$

Computing $\frac{\partial \mathcal{L}}{\partial y'}$ gives:

$$\frac{\partial \mathcal{L}}{\partial y'} = \frac{\partial}{\partial y'}\left(p(x)\,y'^2 + y^2\left(q(x) + \lambda r(x)\right) \right) = 2p(x)\,y'$$

so, $\frac{d}{dx}\frac{\partial \mathcal{L}}{\partial y'}$ is computed to be:

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{d}{dx}\left[2p(x)\,y'\right] = 2p'(x)\,y' + 2p(x)\,y''$$

Therefore, the Euler-Lagrange equation for this problem is computed to be:

$$-2p'(x)\,y' - 2p(x)\,y'' + 2y\left(q(x) + \lambda r(x)\right) = 0$$

$$\Rightarrow -p(x)\,y'' - p'(x)\,y' + y\left(q(x) + \lambda r(x)\right) = 0$$

So an extremal solution to the original minimization problem is determined by the equation:

$$p(x)\,y'' + p'(x)\,y' - y\left(q(x) + \lambda r(x)\right) = 0$$

subject to the conditions:

$$\int_a^b r(x)\,y^2 dx = 1 \text{ and } y(a) = y(b) = 0$$

Problem 8

For this problem, the energy method must be used to show uniqueness of the solution to the initial BVP:

$$u_t = \Delta u, \mathbf{x} \in \Omega, t > 0$$
$$u(\mathbf{x}, 0) = f(\mathbf{x}), \mathbf{x} \in \Omega$$
$$u(\mathbf{x}, t) = g(\mathbf{x}), \mathbf{x} \in \partial\Omega, t > 0$$

Let u_1, u_2 both be solutions to the above problem, and define $w(x,t) = u_1(x,t) - u_2(x,t)$. So, w must be the solution to the problem:

$$w_t = \Delta w, x \in \Omega, t > 0$$

$$w(x, 0) = 0, x \in \Omega$$
$$w(x, t) = 0, x \in \partial\Omega, t > 0$$

To prove that a solution to the original initial BVP must be unique, it is sufficient to show that $w(x,t) = 0 \forall x \in \Omega, t > 0$, since this would imply $u_1(x,t) = u_2(x,t)\,\forall x \in \Omega, t > 0$. To show this, define:

$$E(t) = \int_\Omega w^2(x,t)\,dx$$

Problem 8 continued on next page...

From this, we see that:

$$\frac{d}{dt}E\left(t\right)=\frac{d}{dt}\int\limits_{\Omega}w^{2}\left(x,t\right)dx=\int\limits_{\Omega}\frac{d}{dt}w^{2}\left(x,t\right)dx=2\int\limits_{\Omega}w\left(x,t\right)w_{t}\left(x,t\right)dx$$

$$=2\int\limits_{\Omega}w\Delta w dx$$

From Green's 1st identity $\int\limits_{\Omega}\left(u\Delta w+\nabla u\cdot\nabla w\right)dx=\int\limits_{d\Omega}u\frac{dw}{dn}dA$, we have:

$$E'\left(t\right)=2\left(\int\limits_{d\Omega}u\frac{du}{dn}dA-\int\limits_{\Omega}\left(\nabla u\cdot\nabla u\right)dx\right)$$

From derived conditions: $w\left(x,t\right)=0, x\in\partial\Omega\Rightarrow\int\limits_{d\Omega}u\frac{du}{dn}dA=0$. So, this gives:

$$\Rightarrow E'\left(t\right)=\underbrace{-2\int\limits_{\Omega}\underbrace{\left(\nabla u\cdot\nabla u\right)}_{\geqslant 0}dx}_{\leqslant 0}$$

Therefore, $E(t)$ is stricly non-increasing $\forall x\in\Omega, t>0$. So, $E\left(t\right)\leqslant E\left(0\right)=0\forall t>0$. Hence:

$$w\left(x,t\right)=0\forall x\in\Omega, t>0$$

which implies:

$$u_{1}\left(x,t\right)=u_{2}\left(x,t\right)\forall x\in\Omega, t>0$$

as required.