CS 228 (M) - Logic in CS
Tutorial IV - Solutions

Ashwin Abraham

IIT Bombay

29th August, 2023

Table of Contents

Question 1

The signature of $\varphi = \forall x \exists y R(x, y) \wedge \exists y \forall x \neg R(x, y)$ can be any $(\mathcal{F}, \mathcal{R})$ where $R^2 \in \mathcal{R}$. Note that the set of variables, *Vars* must contain x and y. A structure $\mathcal{A}$ with a countably infinite universe over which φ is satisfiable, is one where $u(\mathcal{A}) = \mathbb{N}$ and R^2 models the successor relation over the naturals, ie $R^{\mathcal{A}} = \{(a, a+1) : a \in \mathbb{N}\}$. It is easy to see that for each x, we can choose $y = x + 1$ to satisfy $R(x, y)$, but for $y = 0$, we have $\forall x \neg R(x, y)$ (no element exists whose successor is 0).

Question 2

One can verify that

- Assuming a brother is a distinct non-female who shares a parent in common with you, we have
  $\varphi_B(x, y) = \neg(x = y) \wedge \neg F(x) \wedge \exists z\, [P(z, x) \wedge P(z, y)]$
- Assuming an aunt is a female who shares a parent in common with a parent of yours, we have
  $\varphi_A(x, y) = F(x) \wedge \exists z\, [P(z, y) \wedge \exists w(P(w, z) \wedge P(w, x))]$
- Assuming cousins are distinct people who have distinct parents who have a parent in common, we can write $\varphi_C(x, y)$ as
  $\exists a \exists b \exists c\, [\neg(a = b) \wedge P(a, x) \wedge P(b, y) \wedge P(c, a) \wedge P(c, b)] \wedge \neg(x = y)$
- Assuming an only child is a person whose parents have no other children, we have $\varphi_O(x) = \forall y \forall z\, [P(z, x) \wedge P(z, y) \implies x = y]$
- Since the only relationships modelled here are parent-child relationships and whether a person is female, a relationship such as marriage, ie $\varphi_M(x, y)$ that says x is married to y cannot be defined

Question 3

It can be seen that (note that pq is short for $p \times q$ everywhere)

- $Zero(a) = \forall p(p + a = p)$
- $One(a) = \forall p(ap = p)$
- $Even(a) = \exists p(a = 2p)$
- $Odd(a) = \forall p \neg(a = 2p)$
- $\forall p \forall q \left[a = pq \implies ((p = 1 \wedge q = a) \vee (p = a \wedge q = 1))\right] \wedge \neg(a = 1)$ is the formula for $Prime(a)$
- $Goldbach = \forall x \exists y \exists z \left[4 + 2x = y + z \wedge Prime(y) \wedge Prime(z)\right]$

Question 4

One can verify that the axioms can be rewritten in an FOL like so:

1. $\forall x \forall y \forall z \, [op(x, op(y, z)) = op(op(x, y), z)]$
2. $\forall x (op(x, c) = x)$
3. $\forall x \exists y (op(x, y) = c)$
4. $\forall x \forall y \forall z \, [op(x, z) = op(y, z) \implies x = y]$

Question 5

- A structure $\mathcal{A}$ that satisfies ψ can be one with $u(\mathcal{A}) = \mathbb{Z}$, $0^{\mathcal{A}} = 0$ and $+^{\mathcal{A}} = +_{\mathbb{Z}}$
- A structure that does not satisfy $\mathcal{A}$ can be one where $u(\mathcal{A}) = \mathbb{Z}$, $0^{\mathcal{A}} = 0$ and $+^{\mathcal{A}}(x, y) = 2x + 3y$. You can in fact verify that none of φ_1, φ_2 and φ_3 are satisfied
- It is not. An example of a structure $\mathcal{A}$ that does not satisfy $\psi \implies \alpha$ is one where $u(\mathcal{A}) = \{M \in \mathbb{R}^{n\times n} : |M| \neq 0\}$ (the set of invertible real valued $n \times n$ matrices), $0^{\mathcal{A}} = I_n$, and $+^{\mathcal{A}}(A, B) = AB$. It can be verified that this structure satisfies ψ but not α (ie it is not commutative)
-
  1. A structure $\mathcal{A}$ satisfying $\varphi_1 \wedge \varphi_2$ but not ψ is one where $u(\mathcal{A}) = \mathbb{Z}$, $0^{\mathcal{A}} = 1$ and $+^{\mathcal{A}}(x, y) = xy$
  2. A structure $\mathcal{A}$ satisfying $\varphi_2 \wedge \varphi_3$ but not ψ is one $u(\mathcal{A}) = \mathbb{N}$, $0^{\mathcal{A}} = 0$ and $+^{\mathcal{A}}(x, y) = |x - y|$
  3. A structure $\mathcal{A}$ satisfying $\varphi_1 \wedge \varphi_3$ but not ψ is one where $u(\mathcal{A}) = \mathbb{Z}$, $0^{\mathcal{A}} = 1$, and $+^{\mathcal{A}}(x, y) = x + y$

Question 6

The first one states that there exists some x in the universe such that for every y in the universe there is some z in the universe (which may depend on y) such that the statement holds.
The second one states that for every x in the universe, there is some y in the universe (which may depend on x) such that for every z in the universe the statement holds.
The difference between the two is illustrated with an example in the following question.

Question 7

The first sentence is actually satisfied only by the complete graph K_n, where $E^{\mathcal{A}} = u(\mathcal{A}) \times u(\mathcal{A})$. To see this, assume there is some $(a, b) \notin E^{\mathcal{A}}$. If $\forall x \exists y \forall z\, [E(x, y) \wedge E(x, z) \wedge E(y, z)]$, then we can choose $x = a$, and then, for any y that we choose, choosing $z = b$ will cause $E(x, y) \wedge E(x, z) \wedge E(y, z)$ to not be satisfied, since $E(a, b)$ is not satisfied. This structure will also clearly satisfy the second sentence.

For an example of a structure that satisfies the second sentence but not the first, consider the following graph:

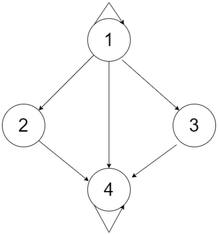

Figure: $u(\mathcal{A}) = \{1, 2, 3, 4\}, E^{\mathcal{A}} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 4), (3, 4), (4, 4)\}$

Question 8

- $\exists^{\geq k}x(x = x) \wedge \neg\exists^{\geq k+1}x(x = x)$ is an FOL sentence with counting quantifers true iff $|u(\mathcal{A})| = k$
- $\exists x_1 \cdots \exists x_n \bigwedge_{1\leq i<j\leq n} \neg(x_i = x_j)$ is an FOL sentence equivalent to $\exists^{\geq n}x(x = x)$

Question 9

Using the counting quantifiers we discussed earlier, such a sentence would be $\exists^{\geq n} x(x = x) \wedge \neg \exists^{\geq m+1} x(x = x)$. Removing the counting quantifiers, we get the sentence

$$\left[\exists x_1 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)\right] \wedge \neg \left[\exists x_1 \cdots \exists x_{m+1} \bigwedge_{1 \leq i < j \leq m+1} \neg(x_i = x_j)\right].$$

One can show that this sentence is equivalent to

$$\left[\exists x_1 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)\right] \wedge \left[\forall x_1 \cdots \forall x_{m+1} \bigvee_{1 \leq i < j \leq m+1} (x_i = x_j)\right]$$