CS 228 (M): Logic in CS
2022 Quiz 1 Solutions

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21st August, 2023

1. It is satisfiable, the assignment $\{p \to 0, q \to 1, r \to 1\}$ is a satisfying assignment.
2. It is valid, as can be verified by truth table.
3. (a) $\{p \vee q, r\} \vdash p \vee q$ (Assumption)
   (b) $\{p \vee q, r, p\} \vdash p$ (Assumption)
   (c) $\{p \vee q, r, p\} \vdash r$ (Assumption)
   (d) $\{p \vee q, r, p\} \vdash p \wedge r$ ($\wedge$ introduction on (b) and (c))
   (e) $\{p \vee q, r, p\} \vdash (p \wedge r) \vee (q \wedge r)$ ($\vee$ introduction on (d))
   (f) $\{p \vee q, r, q\} \vdash q$ (Assumption)
   (g) $\{p \vee q, r, q\} \vdash r$ (Assumption)
   (h) $\{p \vee q, r, q\} \vdash q \wedge r$ ($\wedge$ introduction on (f) and (g))
   (i) $\{p \vee q, r, q\} \vdash (p \wedge r) \vee (q \wedge r)$ ($\vee$ introduction on (h))
   (j) $\{p \vee q, r\} \vdash (p \wedge r) \vee (q \wedge r)$ ($\vee$ elimination on (a), (e), (i))
4. Drawing the parse tree, and using size to denote the depth (number of nodes from root to node, both inclusive) of the deepest node, we get the size as 8.
5. Completeness would not be affected by adding new rules, so this would be complete. It would not be sound, however. To see this, note that by our normal proof rules we can derive $\mathcal{H} \vdash (p \vee \neg p) \vee \bot$. Now, using proof rule 2, we get $\mathcal{H} \vdash \bot$, which must be true for all $\mathcal{H}$, even satisfiable ones, that do not have $\mathcal{H} \vDash \bot$. So clearly our new proof system is no longer sound.
6. By truth table, you can conclude that the formula is valid, and therefore $\mathcal{H} \vDash \varphi$ for all $\mathcal{H}$. By the completeness of formal proof system, we have $\mathcal{H} \vdash \varphi$ for all $\mathcal{H}$. Therefore, we have (a), (b), (d), (e) as the answer.
7. Removing proof rules clearly cannot affect soundness. In this case, since LEM is a derived rule, completeness is not affected either (any proof step using LEM can be replaced by the sequence of steps that derives LEM).
8. Clearly, (c) and (d)
9. None, by the definition of Horn Formulae.
10. Not in syllabus
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12. The size of a DNF is the number of clauses it contains. The DNF equivalent to the given formula is $(x \wedge \neg y) \vee (y \wedge \neg x)$, which has size 2.
13. None of the above.

    Counterexamples:

    (a) p satisfies (a) but isn't valid
    (b) No conjunction of literals can ever be valid, so any valid DNF like $p \vee \neg p$ is a counterexample
    (c) $(p \wedge \neg p) \vee p \vee \neg p$ is valid but doesn't satisfy (c)
    (d) $p \vee \neg p$ is valid but doesn't satisfy (d)