CS 228: Logic for Computer Science
Quiz 1 Solutions

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August 2023

Question 1

Marking Scheme:

1 mark for writing down the correct CNF for $\bigoplus_{i=1}^{n} p_i$ in part (a) - no marks only for writing the CNF for specific values of n.

$\frac{1}{2}$ mark in each part for getting the key idea of the correct proof - in part (a), this is noticing all of the following:

- If a clause C in the CNF does not contain the variable p, and some αC, then $\alpha\, [p \to 1 - b]\, C$, where $b = \alpha(p)$

Question 2

Marking Scheme:
3 marks for correct proof

-0.25 mark for incorrect rule/reference; max drop to 1 mark

-0.5 mark for not mentioning sequent at end

Proof:

1.	$p \lor q$	premise
2.	$r \lor s$	premise
3.	p	$\lor\ e$ (1)
4.	r	$\lor\ e$ (1.1)
5.	$p \land r$	$\land\ i_1$ $3, 4$
6.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\land\ i_1$ 5
7.	s	$\lor\ e$ (1.2)
8.	$p \land s$	$\land\ i_1$ $3, 7$
9.	$(p \land r) \lor (p \land s)$	$\land\ i_2$ 8
10.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\land\ i_1$ 9
11.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\lor\ e$ $2, 4-6, 7-10$
12.	q	$\lor\ e$ (2)
13.	r	$\lor\ e$ (2.1)
14.	$q \land r$	$\land\ i_1$ $12, 13$
15.	$(p \land r) \lor (p \land s) \lor (q \land r)$	$\land\ i_2$ 14
16.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\land\ i_1$ 15
17.	s	$\lor\ e$ (2.2)
18.	$q \land s$	$\land\ i_1$ $12, 17$
19.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\land\ i_2$ 18
20.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\lor\ e$ $2, 13-16, 17-19$
21.	$(p \land r) \lor (p \land s) \lor (q \land r) \lor (q \land s)$	$\lor\ e$ $1, 3-11, 12-20$

Question 3

Marking Scheme:
1 mark for Complete + 1 mark for justifying

1 mark for Not Sound + 1 mark for justifying

Justifying completeness: Adding proof rules does not affect completeness of our proof engine. Since our old proof was engine was sound and complete, any statement that is provably true, can also to be shown to be semantically true using only the old proof rules. Hence, the new proof engine is also complete.

Justifying not soundness: This can be justified by showing that the new proof rules allow us to conclude contradicting results or $\perp$ - this can be done in a variety of ways by taking a suitable assignment or by a truth table. All valid methods have been awarded marks.

Question 4

Marking scheme:

0.5 marks if only a satisfying assignment is given

1 mark if the formula has been converted into CNF and a satisfying assignment is given

3 marks for a completely correct solution which uses Horn's algorithm

Solution: Given the formula $\neg p_1 \vee (p_3 \wedge \neg p_2 \wedge p_1)$, we observe that the formula is not in CNF. To apply Horn's algorithm, we need our starting formula to be in CNF. Using distributivity of union and intersection, we get that the equivalent CNF formula for the given formula is:

$$(\neg p_1 \vee p_3) \wedge (\neg p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_1)$$

Notice that we can drop the last clause as it is equivalent to $\top$. Expressing the formula as a conjunction of impications, we have,

$$(p_1 \to p_3) \wedge ((p_1 \wedge p_2) \to \bot)$$

Applying Horn's algorithm to the above formula, notice that at the first step, since there's no clause of the form $\top \to p$, none of the variables get marked. Consequently, for the second step too, it follows that none of the variables are marked. Since none of the variables of the formula are marked, after the third step, the algorithm yields Sat. A satisfying assignment can be obtained by setting p_1 to false.

Question 5

Part a (3 marks)

We know that any formula can be written in terms of $\neg, \wedge, \vee, \implies$, so if we can construct these we will be able to construct any formula. (actually only $\neg, \wedge$ are enough)

$$\neg p \equiv op(p, \bot, \top)$$

$$p \vee q \equiv op(p, q, p \wedge q)$$

$$p \implies q \equiv op(p, p, q)$$

Hence $C = \{\top, \bot, \wedge, op\}$ is adequate.

Part b (3 marks)

We will show that any 3 element subset of C is not adequate. There are four separate cases:

1. $\{\top, \bot, \wedge\}$ - If the formula contains even one $\bot$ then we see that it always evaluates to false. If the formula does not contain $\bot$ then when we set all variables to 1 the formula must evaluate to true. Hence in either case it cannot represent $\neg p$.

2. $\{\top, \wedge, op\}$ - If all variables are assigned 1 then any formula we make using these will be true. Hence it cannot represent $\neg p$.

3. $\{\bot, \wedge, op\}$ - If all variables are assigned 0 then any formula we make using these will be false. Hence it cannot represent $\neg p$.

4. $\{\top, \bot, op\}$ - We can show that the number of solutions to any formula constructed using these is even. Hence it cannot represent $p \wedge q$.