Problem Set 7

1. For each of the following, draw an NFA and/or FO formula whenever possible:
   (a) The set of strings over $x \in \{0,1\}^*$ such that $\#_0(x).\#_1(x)$ is even. The automaton can have no more than 4 states.
   (b) the set of strings over $\{a, b\}$ containing atleast 3 occurrences of two consecutive a's, overlapping allowed (for example, $aaaa$ is accepted). While drawing the automaton, draw one which has 7 states.
   (c) $(0 + 1(01^*0)^*1)^*$

2. Let L_1, L_2 be FO-definable. Is $L_1.L_2$ FO-definable?

3. Let L_1, L_2 be two languages. Define the shuffle operation on two words $w_1 = a_1a_2 \dots a_n$ and $w_2 = b_1b_2 \dots b_m$ as follows. shuffle(w_1, w_2) contains words $x_1y_1 \dots x_ky_k$ such that $x_1 \dots x_k = w_1, y_1y_2 \dots y_k = w_2$, and x_i, y_i are strings (not necessarily symbols). Extend the definition of shuffle from words to languages in the obvious way by shuffling every pair of words in the two languages.
   (a) If L_1 is finite and L_2 is FO definable, is shuffle(L_1, L_2) FO-definable?
   (b) If L_1, L_2 are FO-definable, is shuffle(L_1, L_2) regular?

4. Let $h : \Sigma^* \to \Gamma^*$ be a mapping defined as follows. $h(\epsilon) = \epsilon$, h is defined on all symbols of Σ and $h(a)$ is a unique word for each $a \in \Sigma$. Moreover, $h(w_1w_2) = h(w_1)h(w_2)$ for all words w_1w_2. If $L \subseteq \Sigma^*$, define $h(L) = \{h(w) \mid w \in L\}$.
   (a) If L is regular, is $h(L)$ regular?
   (b) If L is FO-definable, is $h(L)$ FO-definable?

5. Which of the following formulae are valid? Which are satisfiable? In each case, answer your question by constructing equivalent automata.
   - $\exists y[\neg first(y) \wedge Q_b(y) \wedge \forall x[x < y \to Q_a(x)]]$
   - $\forall x[Q_a(x) \to \exists y[(x < y) \wedge Q_b(y) \wedge \exists z_1 \exists z_2(z_1 < z_2) \wedge x < z_1 \wedge \neg(Q_b(z_1) \wedge Q_b(z_2)))]]$
6. Write second order logic formulae to capture the following:
   (a) There is a path from node s to node t in the graph. The signature is $\tau = \{E\}$.
   (b) Every bounded non empty set has a least upper bound. The signature is $\tau = \{\leq\}$
7. Let Σ be a finite alphabet. The atomic formulae in MSO defined over Σ^* are $x = y, x < y, S(x, y), X(x)$ and $Q_a(x)$, $a \in \Sigma$. Consider the following logic called MSO_0 having atomic formulae of the following forms:
   $$Sing(X), X \subseteq Y, X < Y, S(X, Y), Q_a(X)$$
   where

- $Sing(X)$ means that X is a SO variable of cardinality 1;
- $X \subseteq Y$ means that every element of the SO variable X is contained in the SO variable Y;
- $X < Y$ means that SO variables X, Y have cardinality 1, and that the element in Y is greater than the element in X;
- $S(X, Y)$ means that SO variables X, Y have cardinality 1, and Y contains the successor of the element in X; and,
- $Q_a(X)$ means that all positions in X are decorated by $a \in \Sigma$.

If φ is an atomic formula in MSO, then $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$, $\forall x\ \varphi$ and $\forall X\ \varphi$ are formulae in MSO. Similarly, if φ is an atomic formula in MSO_0, then, $\varphi \wedge \varphi, \neg\varphi, \varphi \vee \varphi$ and $\forall X\ \varphi$ are formulae in MSO_0.
Compare the expressiveness of MSO and MSO_0.

8. For the formula $\exists x \forall y(x < y \rightarrow Q_a(y))$ give an equivalent MSO_0 formula.
9. Write an MSO formula for $x < y$ without using $<$. You can use S in your MSO formula.
10. Consider the following NFA $N = (\{0,1,2,3\}, \{a,b\}, \Delta, \{0\}, \{1\})$ with $\Delta(0,b) = \{1\}$, $\Delta(1,a) = \{2\}$, $\Delta(2,a) = \{2\}$, $\Delta(2,b) = \{3\}$ and $\Delta(3,b) = \{0\}$. Write an MSO formula with two SO variables that characterizes $L(N)$.