CS 228 (M) - Logic in CS
Tutorial V - Solutions

Ashwin Abraham

IIT Bombay

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Question 1

(a) Since our alphabet is $\{a, b\}$, this is equivalent to the set of languages that end and start with the same alphabet. The FO formula for this would be:

$$\forall x\,[\mathit{first}(x) \implies Q_a(x) \wedge \mathit{last}(x) \implies Q_a(x)]$$
$$\vee$$
$$\forall x\,[\mathit{first}(x) \implies Q_b(x) \wedge \mathit{last}(x) \implies Q_b(x)]$$

where $\mathit{first}(x) \equiv \neg\exists y\,[y < x]$ and $\mathit{left}(x) \equiv \neg\exists y\,[x < y]$

(b)

$$\exists x[Q_{\#}(x) \wedge \forall y(Q_{\#}(y) \implies y = x) \wedge \forall y(y < x \implies Q_a(x)) \wedge \forall y(x < y \implies Q_b(x))]$$

(c) $\neg\exists x\exists y\,[S(x, y) \wedge Q_b(x) \wedge Q_a(y)]$

(d) $\exists x, y, z, w\,[\mathit{first}(x) \wedge S(x, y) \wedge Q_0(y) \wedge \mathit{last}(z) \wedge S(w, z) \wedge Q_0(w)]$

Question 1

Recall that $S(x, y)$ is redundant in FOL over words, and can be defined in terms of $<$ as $x < y \wedge \neg\exists z\,[x < z \wedge z < y]$

(e) A necessary and sufficient condition for the top row to be larger than the bottom row is that at the first point of difference (which should exist), the top row should have bit 1 and the bottom row should have bit 0. In FOL, this can be expressed as:

$$\exists x \left[Q_{\binom{1}{0}}(x) \wedge \forall y \left(y < x \implies \left[Q_{\binom{0}{0}}(y) \vee Q_{\binom{1}{1}}(y) \right] \right) \right]$$

Note that since all of these languages are FO definable, they are all also regular.

Question 2

Firstly, note that the answers to (c) and (d) would both always be yes, as all FO definable languages are regular, and FO definable languages are closed under complementation. By inspection, we observe that the answers to (a) and (b) for the languages defined by the FO formulas are:

(1) (a) ε
    (b) $\Sigma\Sigma^* = \Sigma^+$
(2) (a) $\Sigma^* ba \Sigma^*$
    (b) $[(\Sigma - b)^* [b(\Sigma - a)]^*]^* + [(\Sigma - b)^* [b(\Sigma - a)]^*]^* b$
(3) (a) $\Sigma^* a \Sigma$
    (b) $\varepsilon + \Sigma + \Sigma^*(\Sigma - a)\Sigma$
(4) (a) $(ab)^+ = a(ba)^* b$
    (b) $\Sigma^*(\Sigma - a - b)\Sigma^* + \varepsilon + b(a+b)^* + (a+b)^* a + (a+b)^*(aa+bb)(a+b)^*$

Question 3

By observation, $L = b(a^+b^3)^*$ and we also have $L = L(\varphi)$ where φ is

$$\exists x\,[\mathit{first}(x) \wedge Q_b(x)] \wedge \forall x[\neg \mathit{first}(x) \implies ([Q_a(x) \wedge \exists y(x < y \wedge Q_b(y))] \\ \vee [C_1(x) \vee C_2(x) \vee C_3(x)])]$$

where $C_i(x)$ expresses that x is the i^{th} b in the a^+b^3 element, $i = 1, 2, 3$.

$$C_1(x) \equiv \exists p, q, r[Q_a(p) \wedge S(p, x) \wedge Q_b(x) \wedge S(x, q) \wedge Q_b(q) \\ \wedge S(q, r) \wedge Q_b(r) \wedge \forall s(S(r, s) \implies Q_a(s))]$$

$$C_2(x) \equiv \exists p, q, r[Q_a(p) \wedge S(p, q) \wedge Q_b(q) \wedge S(q, x) \wedge Q_b(x) \\ \wedge S(x, r) \wedge Q_b(r) \wedge \forall s(S(r, s) \implies Q_a(s))]$$

$$C_3(x) \equiv \exists p, q, r[Q_a(p) \wedge S(p, q) \wedge Q_b(q) \wedge S(q, r) \wedge Q_b(r) \\ \wedge S(r, x) \wedge Q_b(x) \wedge \forall s(S(x, s) \implies Q_a(s))]$$