Mgf alla fördelningar + def.

E[X],V[X] alla fördelningar. Skattning

E[X] OCH V[X] FÖR FÖRDELNINGARNA

	E[X]	V[X]	mgf
Ber(p)	p	p(1-p)	
Bin(n, p)	np	np(1-p)	
Hyp(N, n, p)	np	np(1-p)*(N-n)/(N-1)	
Geo(p)	1/p	$(1 – p)/p^2$	
NegBin(l, p)	l/p	$l(1 – p)/p^2$	
Po(λ)	λ	λ	
Exp(λ)	1/λ	$1/λ^2$	
R(a, b)	(b – a)/2	$(b – a)^2/12$	
Gam(α, λ)	α/λ	$α/λ^2$	
N(μ, σ)	μ	$σ^2$	
$χ^2(n)$	n	2n	
t(student)	0	n/(n – 2)	

	MomentGeneratingFunction	Fördelning
definition	m_x(t) = E(e^(tx))	
Ber(p)		
Bin(n, p)	$(1-p+pe^s)^n$	$\binom{n}{k}\times p^k\times(1-p)^{n-k}$
Hyp(N, n, p)		Se nedan
Geo(p)	$pe^s/(1-(1-p)e^s)$	$P(X=k)=p(1-p)^{k-1}$
NegBin(l, p)	$(pe^s/(1-(1-p)e^s))$^l	
Po(λ)	e^(λ (e^s-1))	$P(X=k)=(e^{-m}m^k/k!)$
Exp(λ)	λ /(λ -s)	P(X=k)= (1/m)*e^(-x/m)
R(a, b)	GLHF	
Gam(α, λ)	$(λ /(λ -s))^r$	
N(μ, σ)	e^(μs+(1/2)σ²s²	(1/(σ* sqrt(2π))*e^(-(x-μ)²/2σ²)
$χ^2(n)$	(1-2s)^(-r/2)	
t(student)	GLHF	

Förhållande mellan densitetf. och fördelningsf.

$$\frac{d}{dx}F(x)=f(x)$$

Definition densitetsfunktion: $F(x)=P(X\leqslant x)$

$$diskret:\sum_x^n f(x)=1 \quad kontinuerlig:\int_x^n f(x)dx=1$$

HYPERGEOMETRISK FÖRDELNING

I en population finns N st individer, varav Np har en viss egenskap. Tag n st ur populationen och sätt X = antalet n som har egenskapen

$$P(X=k)=\frac{\binom{Np}{k}\binom{N(1-p)}{n-k}}{\binom{N}{n}}$$ FDSFD

STORA TALENS LAG

Låt X1, X2, … , Xn vara ober. s.v. $E[X_i] = μ$, $V[X_i] = σ^2$

Sätt $\bar{X}_n=1/n\times\sum X_i$ Då gäller:

$$\forall\varepsilon>0,\; P(\mu-\varepsilon<X_n<\mu+\varepsilon)\to 1\; när\; n\to\infty$$

$$\left(\right)\lim_{n\to\infty}P(|\bar{X}_n-m|>\varepsilon)=0\Leftrightarrow\lim_{n\to\infty}1-P(|\bar{X}_n-m|\leqslant\varepsilon)=0$$

BAYES SATS

$$P(B_i|A)=\frac{P(A|B_i)P(B_i)}{P(A)}=\frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}$$

Bevis:

från definitionen av conditional probability:

$$P(A|B)=P\frac{(A\cap B)}{P(B)}$$

$$P(B|A)=P\frac{(A\cap B)}{P(A)}$$

Får vi att: $P(A|B)P(B)=P(A\cap B)=P(B|A)P(A)$

Dividera båda sidorna med $P(A)$, *där* $P(A)\neq 0$:

$$P(B_i|A)=\frac{P(A|B_i)P(B_i)}{P(A)}=$$

(*Total probability law ger*:
$P(A)=P((A_n B_1)\cup\ldots\cup(A_n B_n))$

$$=\sum_{(i=1)}^n P(B_n A_i)=\sum_{(i=1)}^n P(A|B_i)P(A_i))$$

$$=\frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}$$

MARKOV'S INEQUALITY

$$P(|X|\geq a)\leq\frac{E(|X|)}{a},\; a>0,\;\; X\; är\; en\; rv$$

Bevis:

$$E[X]=\int_0^\infty xf(x)dx=\int_0^a xf(x)dx+\int_a^\infty xf(x)dx$$

$$\leqslant 0+\int_a^\infty xf(x)dx=a\int_a^\infty f(x)dx=aP(x\geq t)$$

$$E\frac{[X]}{a}\geq P(X\geq a)$$

DEFINITION OF INDEPENDENT EVENTS

$P(A\cap B)=P(A)P(B)$

Snitt/Union: $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

BETINGAD SANNOLIKHET

Beroende:

$$P(G|H)=\frac{P(G\cap H)}{P(H)}$$

Oberoende

$$P(G|H)=\frac{P(G\cap H)}{P(H)}=\frac{P(G)\times P(H)}{P(H)}=P(G)$$

Partialbråksuppdelning

$$\frac{f(x)}{(x-1)^n}=\frac{A_1}{(x-1)}+\frac{A_2}{(x-1)^2}+\ldots+\frac{A_n}{(x-1)^n}$$

CHEBYSHOV'S INEQUALITY

$$P[|X-\mu|<k\sigma]\geq 1-\frac{1}{k^{2,}}\quad för\; alla\;\; k>0$$

Bevis:

Anta X är continious med väntevärde μ, sdev σ och densitetsfunktion f. Från definition:

$$\sigma^2=Var(X)=\int_{-\infty}^{\infty}(x-\mu)^{2f}(x)dx$$

låt $k>0$ *och* $c=k^2\sigma^2$

$$\sigma^2=\int_{-\infty}^{\mu-\sqrt{c}}(x-\mu)^2f(x)dx+\int_{\mu-\sqrt{c}}^{\mu+\sqrt{c}}(x-\mu)^2f(x)dx+\int_{\mu+\sqrt{c}}^{\infty}(x-\mu)^2f(x)dx$$

Då $(x-\mu)^2f(x)\geq 0$,

$$\int_{\mu-\sqrt{c}}^{\mu+\sqrt{c}}(x-\mu)^2f(x)\,dx$$

och därför:

$$\sigma^2\geq\int_{-\infty}^{\mu-\sqrt{c}}(x-\mu)^2f(x)dx+\int_{\mu+\sqrt{c}}^{\infty}(x-\mu)^2f(x)\,dx$$

eftersom $(x-\mu)^2\geq c$:

$$\sigma^2\geq\int_{-\infty}^{\mu-\sqrt{c}}cf(x)\,dx+\int_{\mu+\sqrt{c}}^{\infty}c\,f(x)dx$$

$$\sigma^2\geq cP[X\leq\mu-\sqrt{(c)}]+cP[X\leq\mu+\sqrt{(c)}]$$

$$\sigma^2\geq c(P[X-\mu\leq\mu-\sqrt{(c)}]+P[X-\mu\leq\sqrt{(c)}])$$

$$(P[X-\mu\leq\mu-\sqrt{(c)}]+P[X-\mu\leq\sqrt{(c)}])\leq\frac{\sigma^2}{c}$$

$$P[-\sqrt{c}\leq X-\mu\leq\sqrt{c}]\geq 1-\frac{\sigma^2}{c}$$

$$c=k^2\sigma^2\Rightarrow\sqrt{c}=k\sigma$$

$$P[-k\sigma\leq X-\mu\leq k\sigma]\geq 1-\frac{1}{k^2}$$

$$P[|X-\mu|\leq k\sigma]\geq 1-\frac{1}{k^2}$$

Då X är kontinuerlig kan vi dra slutsatsen att

$$P[|X-\mu|<k\sigma]\geq 1-\frac{1}{k^2}$$

Definition E[x]

$E[x^j]=\sum_k k^jf_x(k)$, *diskret*

$\int x^jf_x(x)dx$, *kontinuerlig*

Centrala gränsvärdessatsen:

$$\bar{x}\sim\text{approx }N(\mu,\sqrt{\frac{\sigma^2}{n}})\cdot\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\sim N(0.1)$$

$$P(-Z_{\frac{\alpha}{2}}\leqslant\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\leqslant Z_{\frac{\alpha}{2}})=1-\alpha\Rightarrow\bar{x}\pm Z_{\frac{\alpha}{2}}\cdot\frac{\sigma}{\sqrt{n}}$$

Härledning av μ för binomialfördelning:

$$E(x)=\sum_{x=0}x\frac{n!}{x!(n-x)!}p^xq^{n-x}$$

$$\sum_{x-1}^n\frac{n!}{x!(n-x)!}p^xq^{n-x}$$

$$\sum_{x-1}^n\frac{(n-1)!}{(x-1)!(n-x)!}p^{x-1}q^{n-x}$$

$$np(p+q)^{n-1},(p+q)=1\Rightarrow E(x)=np$$

Härledning:

Var[x] binomialfördelat:

$Var[x]=E(X^2)-[E(X)]^2$

$E(x^2)=E[x(x-1)+x]$

$E[x(x-1)]+E(x)=E[x(x-1)]+np$

$E[X(X-1)]+np$

$$\sum_{X=0}^n x(x-1)f(x)+np$$

$$\sum_{X=0}^n x(x-1)\frac{n!}{x!(n-x)!}p^xq^{n-x}+np$$

$$\sum_{x=2}^n\frac{n!}{(x-2)!(n-x)!}p^xq^{n-x}+np$$

$$n(n-1)p^2\sum_{x=2}^n\frac{n-2!}{(x-2)!(n-x)!}p^{x-2}q^{n-x}+np$$

$n(n-1)p^2(q+p)^{n-2}+np$

$n(n-1)p^2(1)^{n-2}+np=n(n-1)p^2+np$

Härledning för E[x] binomialfördelat:

$$E[x]=\sum_{x=0}^n x\binom{n}{x}p^xq^{n-x}$$

$$\sum_{x=1}^n\frac{n!}{(n-x)!x!}\binom{n}{x}p^xq^{n-x}$$

$$np\sum_{x=1}^n\frac{(n-1)!}{(n-x)!(x-1)!}\binom{n}{x}p^{x-1}q^{n-x}$$

$np(p+q)^{n-1}=np(1)^{n-1}=np$

Väntevärdesriktig skattning:

$E\hat{\theta}=\theta$ (*om unbiased*), $EX=\lambda,\hat{\lambda}=\bar{X}$

$$\bar{X}:=\frac{1}{n}\sum_{i=1}^n EX_i=\mu$$

Matrismultiplikation

$$\begin{pmatrix}A & B & C\\ D & E & F\\ G & H & I\end{pmatrix}^2=$$

$$\begin{pmatrix}A^2+BD+CG & AB+EB+CH & AC+IC+BF\\ AD+ED+FG & E^2+BD+FH & CD+EF+IF\\ AG+IG+DH & BG+EH+IH & CG+FH-1\end{pmatrix}$$

Räkneregler för E[x] och Var[x]

$$Var\,X=\frac{\sigma^2}{n},\sigma^2=Var\,x=E[X^2]-(E[X])^2$$

$$Var(aX+b)=a^2Var(X)$$

$$E[aX+b]=aE[X]+b$$

$$E(Y^2)=m_Y''(0)$$

Räkneregler för E[x] och Var[x]

$Var(aX+bY)=a^2Var(X)+b^2Var(Y)+2\text{abCov}(X,Y)$

$Var(x+y)=E((x+y)^2-(E(x+y))^2)=$

$E(x^2+y^2+2xy)-(Ex+Ey)^2=$

$Ex^2-(Ex)^2+Ey^2-(Ey)^2+2Exy-2ExEy=$

$Var\,x+Var\,y+2\text{Cov}(x,y)$

$Cov(x,y)=E((X-EX)(Y-EY))$,

X,Y *oberoende* $\to cov=0$

$Cov(x,y)=E((X-EX)(Y-EY))$

$=E(XY-XEY-(EX)Y+EXEY)$

$=EXY-EXEY-EXEY+EXEY=EXY-EXEY$

$=EXY-EXEY(X,Y\text{ oberoende})=EXEY-EXEY=0$

$E(X+Y)=EX+EY$

$EXY=EXEY$ *om* X,Y *är oberoende*

$$EXY=\sum_x\sum_y xyf_{xy}(x,y)=\sum_x\sum_y xyf_x(x)f_y(y)$$

$$=\sum_x xf_x(x)\sum_y yf_y(y)=ExEy$$

Correlation coefficient between R.V X.Y

$$Corr(x,y)=\frac{cov(x,y)}{\sqrt{Var\,x\,Var\,y}}$$

Skattning av väntevärde & Varians

<u>Väntevärde:</u> Punktskattning av μ.

Använd stickprovsvärdet

$\hat{\theta}(x_1,x_2,\ldots,x_n)=\bar{X}=1/n\cdot\Sigma X_i$

$\bar{X}$ är en vvr & är en konstant skattn. av μ

<u>Varians:</u> Punktskattning av σ^2

Använd stickprovsvärdet.

$\hat{\theta}(x_1,x_2,\ldots,x_n)=S_2=\{1/(n-1)\}\cdot\Sigma(X_i-\hat{X})^2$

S_2 är en observation på $S_2(x_1,x_2,\ldots,x_n)$

$E[S_2]=\sigma^2\to$ fittlång uträkn.→

$\sigma^2=\{1/(n-1)\}\cdot\Sigma(X_I-\bar{X}_I)^2$

Def. Konfidensintervall

$P(-\lambda_{\alpha/2}\leqslant(\hat{X}-\mu)/(\sigma/\sqrt{n})\leqslant\lambda_{alfa/2})=1-\alpha\Leftrightarrow$

$P(\bar{X}-(\sigma/\sqrt{n})\cdot\lambda_{\alpha/2}\leqslant\mu\leqslant\bar{X}+(\sigma/\sqrt{n})\cdot\lambda_{\alpha/2})=1-\alpha$

Marginal Distribution, Discrete:

$f_{XY}(x,y)=P[X=x\wedge Y=y], f_{XY}(x,y)\geq 0$

$$\sum_{all\,x}\sum_{all\,y}f_{XY}(x,y)=1$$

$$f_X(x)=\sum_{all\,y}f_{XY}(x,y)$$

$$f_Y(y)=\sum_{all\,x}f_{XY}(x,y)$$

Continuous: samma formel med integraler istället för summa. Från a till b och c till d.

Generating function:

$$A(x)=\sum_{n=0}^{\infty}a_nx^n$$

Räkna ut:

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$$

$$\sum_{n=0}^{\infty}(ax)^n=\frac{1}{1-ax}$$

$$\sum_{n=0}^{\infty}(-1)^nx^n=\frac{1}{1+x}$$

$$\sum_{n=0}^{\infty}x^{2n}=\frac{1}{1-x^2}$$

$$\sum_{n=0}^{\infty}\binom{n}{k}x^k=(1+x)^n$$

$$\sum_{n=0}^{\infty}\binom{n+k}{k}x^n=\frac{1}{(1-x)^{k+1}}$$

$$\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}=e^{cx}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}=\frac{e^x+e^{-x}}{2}$$

$$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=\frac{e^x-e^{-x}}{2}$$

Momentgenererande funktioner (MGF):

Defenition:

$m_x(t)=Ee^{tx}$

$\int f(x)\cdot Ee^{tx}$ *kontinuerlig*

$\sum f(x)\cdot Ee^{tx}$ *diskret*

För oberoende r.v:

$m_z(t)=m_x(t)\cdot m_y(t)$

bevis:

$EXY=EX\,EY$

$m_z(t)=Ee^{tz}=Ee^{t(x+y)}=Ee^{tx}e^{ty}$

$Ee^{tx}Ee^{ty}=m_x(t)m_y(t)$

För N:

if x is normally distributed then a+bx is also normally distributed

$x\sim N(\mu,\sigma)$

$$m_y(t)=e^{\mu t+\sigma^2\frac{t^2}{2}},Y=a+bx$$

$$m_y(t)=e^{at},m_x(bt)=e^{at}\cdot e^{\mu t+\sigma^2\frac{t^2}{2}}$$

$$e^{(a+\mu b)t+(\sigma b)^2\frac{t^2}{2}},Y\sim N(a+\mu b,\sigma^2b^2)$$

För exponential distrubition:

$X\sim\exp(\lambda),F(x)=\lambda e^{-\lambda x},t<\lambda$

$$m_x(t)=Ee^{tx}=\int_0^{\infty}e^{tx}\lambda e^{-\lambda x}dx$$

$$\lambda[\frac{e^{(t-\lambda)x}}{t-\lambda}]_0^{\infty}=\lambda(-\frac{1}{t-\lambda})$$

$$\frac{\lambda}{\lambda-t},$$

$$m'(t)=\frac{\lambda}{(\lambda-t)^2}$$

$$m'x(0)=EX=\frac{1}{\lambda}$$

Momentgenerande funktioner (MGF):

För binomal distrubition:

$$X\sim Bin(n,p),f_x(x)=\binom{n}{x}p^xq^{n-x}$$

$$m_x(t)=Ee^{tx}=\sum_{X=0}^n e^{tx}\binom{n}{x}p^xq^{n-x}$$

$$\sum_{X=0}^n\binom{n}{x}(pe^t)^xq^{n-x}=(pe^t+q)^n$$

$(p(e^t-1)+1)^n$

För Geometrisk distrubition:

$$m_x(t)=Ee^{tx}=\sum_{x=1}^{\infty}e^{tx}(1-p)^{x-1}p=$$

$$pe^t\sum_{x=1}^{\infty}e^{t(x-1)}(1-p)^{x-1}p=pe^t=$$

$$pe^t\sum_{x=0}^{\infty}e^{t(x)}(1-p)^xp=$$

$$pe^t\sum_{x=0}^{\infty}e^t(1-p)^x=pe^t\frac{1}{1-e^t(1-p)}=$$

$$\frac{pe^t}{1-e^x(1-p)}$$

$$\sum_{j=0}^{\infty}e^{tj}\left(\frac{e^{-\lambda}\lambda^j}{j!}\right)=e^{-\lambda}\sum_{j=0}^{\infty}(\frac{(\lambda e^t)^j}{j!})=$$

$e^{-\lambda}e^{\lambda e^t}=e^{\lambda(e^t-1)}=e\hat{}(\lambda(e\hat{}t-1))$

Poisson moment generating bevis:

$$\sum_{j=0}^{\infty}e^{tj}\left(\frac{e^{-\lambda}\lambda^j}{j!}\right)=e^{-\lambda}\sum_{j=0}^{\infty}(\frac{(\lambda e^t)^j}{j!})=e^{-\lambda}e^{\lambda e^t}=$$

$e^{\lambda(e^t-1)}=e\hat{}(\lambda(e\hat{}t-1))$

Model assumptions: Simple linear regression:

1. The random variables Y_i are independently and normally distributed.
2. The mean of Y_i is $\beta_0+B_1x_i$
3. The variance of Y_i is σ^2

I allmänhet ser formeln för en regressionslinje ut som följande:

$Y_i=a+b(x_i-\bar{x})+\varepsilon_i$

$$a*(y)=\bar{y}=\frac{1}{n}\times\sum y_i$$

$$b*(y)=\frac{n\sum x_iy_i-\sum x_i\sum y_i}{n\sum x_i^2-(\sum x_i)^2}$$

$Y*(x)=a*(y)+b*(y)\times(x_i-x)$

$$V(b*)=V(\frac{\sum(Y_i-\bar{Y})(x_i-\bar{x})}{\sum(x_i-\bar{x})^2})$$

$$=\frac{V(\sum(Y_i)(x_i-\bar{x}))}{(\sum(x_i-\bar{x})^2)^2}$$

$$=\frac{V(Y_i)\sum(x_i-\bar{x})^2}{(\sum(x_i-\bar{x})^2)^2}=\frac{\sigma}{\sum(x_i-\bar{x})^2}$$

Poissonfördelning för S.V:
Parameter λ X ~ Poisson(λ)

if it has density $f_x(x)=\frac{e^{-\lambda}\lambda^x}{x!}$

x= 0,1,2,3 EX= λ, Bevis:

$$EX=\sum_{X=0}^{\infty}xP(X=x)$$

$$=\sum_{X=0}^{\infty}x\frac{e^{-\lambda}\lambda^x}{x!}=e^{-\lambda}\lambda\sum_{X=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}$$

$$=\lambda e^{-\lambda}\sum_{X=0}^{\infty}\frac{\lambda^x}{x!}=\lambda e^{-\lambda}e^{\lambda}=\lambda$$

För $EX^2=\lambda^2+\lambda$

$$EX^2=\sum_{X=0}^{\infty}x^2P(X=x)=\sum_{X=0}^{\infty}x^2\frac{e^{-\lambda}\lambda^x}{x!}$$

$$=e^{-\lambda}\lambda\sum_{x=1}^{\infty}x\frac{\lambda^{x-1}}{(x-1)!}=e^{-\lambda}\lambda\sum_{X=1}^{\infty}\frac{\lambda^x}{(x-1)!}\frac{d\lambda}{dx}$$

$$=e^{-\lambda}\lambda\sum_{X=0}^{\infty}\frac{\lambda^x}{(x)!}'=\lambda e^{-\lambda}(\lambda e^{\lambda})'$$

$$=\lambda e^{-\lambda}(e^{\lambda}+\lambda e^{\lambda})=\lambda+\lambda^2$$

För Var $X=EX^2-(EX)^2=\lambda+\lambda^2-\lambda^2$
$=\lambda=EX$

Härledning för exponentialfördelning.

$$E[X]=\int_{-\infty}^{\infty}xf(x)dx=\int_{-\infty}^{\infty}x\lambda e^{-\lambda x}=$$

$$=[\frac{-x}{\lambda}]_0^{\infty}-\lambda(\frac{-1}{\lambda})\int_0^{\infty}e^{-\lambda x}dx=$$

$$[\frac{-1}{\lambda}(e^{-\lambda x})]_0^{\infty}=\frac{-1}{\lambda}(-1)=\frac{1}{\lambda}$$

$$E[X^2]=\int_0^{\infty}x^2\lambda e^{-\lambda x}dx=\lambda\int_o^{\infty}x^2e^{-\lambda x}dx=$$

$$[\lambda(\frac{-x}{\lambda})]_0^{\infty}=-(\frac{-2}{\lambda})\int_0^{\infty}x\lambda e^{-\lambda x}dx=\frac{2}{\lambda^2}$$

$$Var[X]=E[X^2]-E[X]^2=\frac{2}{\lambda^2}-(\frac{1}{\lambda})^2=\frac{1}{\lambda^2}$$

Deriveringsregler

$f(x)=Af'(x)=0$

Potensregeln

$f(x)=x^n\to f'(x)=nx^{n-1}$

Produktregeln

$f(x)=g(x)h(x)\to f'(x)=g'(x)h(x)+g(x)h'(x)$

Kvotregeln

$$f(x)=\frac{g(x)}{h(x)}\to f'(x)=\frac{g'(x)h(x)-g(x)h'(x)}{[h(x)]^2}$$

Derivata av exp.funktion

$f(x)=e^{g(x)}\to f'(x)=g'(x)e^{g(x)}$

Logaritmisk derivering

$$f(x)=\ln g(x)\to f'(x)=\frac{g'(x)}{g(x)}$$

Kedjeregeln

$f(x)=g(u(x))\to f'(x)=g'(u(x))*u'(x)$

12 JAN 2011

6a Ge definitionen för den genererande funktionen för talföljden $\{a_n\}_{n=0}^{\infty}$ Ange den genererande funktionen för talföljden $a_n=2^n+3^{(n-1)}$.

$A(x)=\sum_{n=0}^{\infty} a_n x^n, a_n=2^n+3$

$\sum_{n=0}^{\infty} 2^n x^n+3^{n-1}x^n=\sum_{n=0}^{\infty} 2x^n+\sum_{n=0}^{\infty}\frac{1}{3}(3x)^n$

$\sum_{n=0}^{\infty}(2x)^n+\frac{1}{3}\sum_{n=0}^{\infty}(3x)^n=\frac{1}{1-2x}+\frac{1}{3}*\frac{1}{1-3x}$

$\frac{1}{1-2x}+\frac{1}{3(1-3x)}$

AUG 2011

3 Låt X och Y vara två stokastiska variabler med väntervärdena μx och μy, varianserna σ_x^2 och σ_y^2 och kovariansen σ_{xy}.

a Hitta väntervärdet 33X+12Y-11

$E(33x+12y-11)=E33x+E12y-11=$

$33Ex+12Ey-11=33\mu_x+12\mu_y-11$

b Beräkna variansen av 5X+2Y+4

$Var(5x+2y+4)=Var(5x)+Var(2y)+2*2*5cov(x,y$

$=5^2 Var(x)+2^2 Var(y)+20cov(x,y)$

$=25\sigma_x^2+4\sigma_y^2+20\sigma_{xy}$

c Vid oberoende så är covariancen 0

Användningsområden för bevis etc.

Stora talens lag

De stora talens lag kan sägas motsvara uttrycket *Det jämnar ut sig i det långa loppet*, under vissa omständigheter, The LLN is important because it guarantees stable long-term results for random events. For example, while a casino may lose money in a single spin of the roulette wheel, its earnings will tend towards a predictable percentage over a large number of spins. Any winning streak by a player will eventually be overcome by the parameters of the game. It is important to remember that the LLN only applies (as the name indicates) when a large number of observations are considered. There is no principle that a small number of observations will converge to the expected value or that a streak of one value will immediately be *balanced* by the others.

Central Limit Theorem

If you don't know your data's distribution but know that it has a finite mean andvariance and the sample is large enough you can approximate the sample's meandistribution by a normal one and therefore draw inferences about it (e.g. the confidenceintervals)

Kort sammanfattning om Konfidensintervall

För att förstå innebörden av det som konfidensintervalle anger, betrakta en population för vilken man vill skatta någon förbestämd parameter utifrån stickprovsdata. *Den givna populationen kommer*

För att förstå innebörden av det som konfidensintervalle anger, betrakta en population för vilken man vill skatta någon förbestämd parameter utifrån stickprovsdata.

Den givna populationen kommer att samplas upprepade gånger, varpå intervallskattningar för den givna parametern bestäms.

Då är konfidensintervallet det intervall som kommer att innesluta populationsparametern för den andel av samplingarna som bestäms av konfidensgraden.

Exempelvis om konfidensgraden är 95 % så kommer konfidensintervallet innesluta populationsparametern 95 % av samplingarna

Ett ensidigtkonfidensintervall kommer att begränsa populationsparametern från ett håll, antingen från ovanifrån eller underifrån.

Detta erbjuder alltså antingen en övre eller undre begränsning för populationsparameterns magnitud.

Ett tvåsidigt konfidensintervall innesluter populationsparametern både ovanifrån och underifrån.

Markovs Olikhet

Markovs olikhet är, inom sannolikhetsteorin, en uppskattning av en sannolikhet med hjälp av ett väntevärde. att samplas upprepade gånger, varpå intervallskattningar för den givna parametern bestäms.

Då är konfidensintervallet det intervall som kommer att innesluta populationsparametern för den andel av samplingarna som bestäms av konfidensgraden.

Exempelvis om konfidensgraden är 95 % så kommer konfidensintervallet innesluta populationsparametern 95 % av samplingarna

Ett ensidigt konfidensintervall kommer att begränsa populationsparametern från ett håll, antingen från ovanifrån eller underifrån.

Detta erbjuder alltså antingen en övre eller undre begränsning för populationsparameterns magnitud.

Ett tvåsidigt konfidensintervall innesluter populationsparametern både ovanifrån och underifrån.

Chebychefs inequality

In probability theory, Chebyshev's inequality guarantee that inany data sample or *probability distribution*, *nearlyall* values are close to the mean — the precise statement being that no more than 1/k2 of the distribution's values can be more than k standard deviations away from the mean. The inequality has great utility because it can be applied to completely arbitrary distributions (unknown except for mean and variance), for example it can be used to prove the weak law of large numbers.

X har densitet f(x)= $\frac{(x-3)^2}{5}$ $x=3,4,5$

a)Verifiera

$\sum_{3,4,5}\frac{(x-3)^2}{5}=1$

b)Hitta väntevärdet

$E[x]=\sum x\cdot f(x)=\sum_{3,4,5}\frac{(x-3)^2}{5}$

c)Hitta mgf

$mx(t)=e^{tx}\cdot f(x)=\sum_{3,4,5}e^{tx}\frac{(x-3)^2}{5}$

d)Använd mgf för att hitta väntevärdet

$E[X]=\frac{1}{5}e^{t4}+\frac{4}{5}e^{t5}$ $y'=\frac{4}{5}e^{t4}+4e^{t5}$ $y'(0)=\frac{4}{5}+4=4\frac{4}{5}$

f,e)Hitta E[x²] med hälp av mgf

$E[X^2]=y''=\frac{16}{5}e^{t4}+20e^{t5}$ $y''(0)=\frac{16}{5}+20=23\frac{1}{5}$

g)Hitta variansen och standardavvikelsen

$Var[x]=E[X^2]-E[X]^2=23\frac{1}{5}-\frac{24^2}{5^2}=\frac{4}{25}$

$\sigma=\sqrt{\frac{4}{25}}=\frac{2}{5}$

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antag att $Y=\beta_0+\beta_1 X$

där X är s.v med väntevärdet μ_x och variansen σ_X^2 a) Väntevärdet av Y:

$\beta_0+\beta_1\mu_x=E[Y]$

b)Variansen av Y:

$Var[Y]=\beta_1^2\cdot\omega_x^2\rightarrow\beta_1^2\cdot var(x)$

c)Kovariansen mellan X och Y:

$Cov(x,y)=E[XY]-E[X]\cdot E[Y]$

$E[X(\beta_0+\beta_1 x)]-\mu_x\cdot(\beta_0+\beta_1\mu_x)$

$E[x\beta_0]+E[x^2\beta_1 x]-\beta_0\mu_x-\beta_1\mu_x^2$

$\beta_0 E[x]+\beta_1 E[x^2]-\beta_0 E[x]-\beta_1 E[x]^2$

$\beta_1(E[x^2]-E[x]^2)=\beta_1\cdot\sigma_x^2$

Tentauppgifter som kan komma:

Let A and B be two independent events, with $P(A)=0.6$ and $P(B)=0.5$

What is the definition of independence between two events?

$P(A\cap B)=P(A)P(B)$

What is the statistical meaning of independence?

The knowledge about one event does not carry any information about the other.

Calculate $P(A\cup B)$ and $P(A\cap B^c)$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=0.6+0.5-(0.6\cdot 0.5)=0.8$

$P(A\cap B^c)=P(A)-P(A\cap B)$

$=0.6-(0.6\cdot 0.5)=0.6-0.3=0.3$

Give the defintion of conditional probability.

$P(A|B)=\frac{P(A\cap B)}{P(B)}$

What is the conditional probability if the two events are independent?

If A and B is independent, then $P(A|B)=P(A)$

Let X be a random variable with density $f_X(x)=Cx^{-\alpha}$ for $X>0$ and $f_X(x)=0$ for $X\leq 0$, where $\alpha>1$ and C is the normalizing constant

What is the definition of a cumulative distribution function?

$F_X(x)=P(X\leq x)$

What is the relationship between a cumulative distribution function and a density function of a continuous random variable?

$f_X=\frac{d}{dx}F_X(x)$

Calculate the cumulative distribution function of the random variable X

$F_X(x)=\int_{x_0}^{x}Cy^{-\alpha}dy=C\frac{1}{a-\alpha}(x^{1-\alpha}-x_0^{1-\alpha})$

we need to choose some lower bound as there is the problem of integrability at 0.

Studies have shown that the random variable X, the processing time required to do a multiplication on a new 3D computer, is normally distributed with mean μ and standard deviation 2ms. A random sample (in microseconds) of 16 observations is taken. The sample average is $\bar{x}=42.88$.

Give the definition of an unbiased estimator. Is $\bar{X}$ an unbiased estimator of μ and why or why not? What is the distribution of $\bar{X}$ here?

θ is unbiased if

$E[\hat{\theta}]=\theta$, $\bar{X}:=\frac{1}{n}\sum_{i=1}^{n}X_i E[\bar{X}]=\frac{1}{n}\sum_{i=1}^{n}E[X_i]=\mu$

therefore is unbiased, $\bar{X}\sim N(\mu, 0.5)$, X is normal as linear combination of normals then we need to calculate its mean and variance using the properties of the mean and variance of a linear combination of independent random variables

Derive the forumula for $1-\alpha$ level confidence interv for μ (with the variance known). What is the interpretation of a confidence interval?

Motivate $P(-z_{\alpha/2}\leq\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\leq z_{\alpha/2})=1-\alpha$

and transform it to $\mu\in\bar{X}\pm z_{\alpha/2}\cdot\frac{\sigma}{\sqrt{n}}$

If we would have a large number of independent samples of size n then about $(1-\alpha)100\%$ of them would generate confidence intervals which contain the true value of μ.

Assume that $Y=\beta_0+\beta_1 X$, where X is aome random variable with mean μ_X and variance σ_X^2

Find the expected value of Y

$\beta_0+\beta_1\mu_X$

Calculate the variance of Y

$\beta_1^2\sigma_X^2$

Calculate the covariance between X and Y

$\beta_1\sigma_X^2$

The joint density for the pair of discrete random variables (X,Y) is given by $f_{XY}(x,y)=\frac{2}{n(n+1)}$ where $1\leq y\leq x\leq n$ and $n>0$ is a positive integer

Verify that $f_{XY}(x,y)$ satisfies the conditions necessary to be a density.

Use that the sum of the first n integers is $\frac{2}{n(n+1)}$

Find the marginal densities for X and Y.

$P(X=k)=\sum_{y=1}^{k}\frac{2}{n(n+1)}=\frac{2k}{n(n+1)}k=1,\dots,n$

$P(Y=k)=\sum_{x=k}^{n}\frac{2}{n(n+1)}=\frac{2(n-k+1)}{n(n+1)}k=1,\dots,n$

Are X and Y independent?

No, one can calculate this from definition as $P(X=k,Y=r)\neq P(X=k)P(Y=r)$ for some appropriate k and r or just state that if we observe only X then this gives us information about Y, i.e. that it has to be lesser than X (or alternatively observing Y gives us information about X).

What does the knowledge of the independence between X and Y tell us about the value of covariance between them?

Since X and Y are dependent this does not give any information about the covariance between them.

Let X and Y be exponential random variable with parameters $\lambda_X=1$ and $\lambda_Y=1$

Assume that they are independent.

Calculate the variance of 3X − 7Y, knowing that Var[X] = Var[Y] = 1. Clearly state the properties of the variance that you are using.

$9\cdot 1+49\cdot 1=58$, we use that if X and Y are independent then $Var[X+Y]=Var[X]+Var[Y]$ and $Var[aX]=a^2Var[X]$

What is (without doing any calculations) the covariance between X and Y?

$Cov[X,Y]=0$

In general if you know the covariance between two random variables what does this tell you about the dependence between them?

If the covariance is non–zero then they are dependent. A zero covariance does not allow us to draw conclusior except in the case of a normal distribution where zero covariance implies independence.

Central limit theorem

State the CLT

Let $X_1, X_2, \dots$ be a random sample of size n from a distribution with mean μ and varaiance σ^2. Then for large n, $\bar{X}$ is approximately normal with mean μ and variance σ^2/n. Furthermore, for large n, the random variable $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is approximately standard normal.

Discuss what the CLT can be useful for.

If you don't know your data's distribution but know that it has a finite mean and variance and the sample is large enough you can approximate the sample's mean distribution by a normal one and therefore draw inferences about it (e.g. the confidence intervals).

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q=1-p$ that it won't.

Write down the matrix of transition probabilities.

$P=\begin{bmatrix}q & p\\ p & q\end{bmatrix}$

If the digit 1 was put into the machine, write down the formula for the probability distribution after n stages.

$P^n[01]^T$

What is the probability that a 0 remains a 0 after two stages?

$p\cdot p+q\cdot q$

Random questions

State the density function of the binomial distribution with parameters n and p.

How is the binomial distribution related to the Bernoulli distribution with parameter p?

$\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,\dots,n$

the binomial distribution is the sum of n independent trails of a Bernoulli distribution with parameter p.

Derive the mean value and variance of the binomial distribution with parameters n and p.

Mean is np and variance is $np(1-p)$

Let S_n be the number of success in n Bernoulli trials with probability p for success in each trail. Show that for and

$\epsilon>0,\ P(|\frac{S_n}{n}-p|\geq\epsilon)\leq\frac{p(1-p)}{n\epsilon^2}$

Discuss what this inequality could be useful for.

Multiply by n, use Chebyshev with mean and variance of binomial distribution. We use S_n/n as an estimator for p, this gives us a rough bound on the probability that we will err in the estimation by at least σ.

Give the definition of the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$

$A(x)=\sum_{n=0}^{\infty}x^n a_n$

Calculate the generating function of the sequence $a_n=2^n+3^{n-1}$

$A(x)=\frac{1}{1-2x}+\frac{1}{3(1-3x)}$

Express $\frac{x^2}{(x-1)(x+3)(x-5)}$ as a sum of partial fractions.

$\frac{-1/16}{x-1}+\frac{9/32}{x+3}+\frac{25/32}{x-5}$

Provide the definition of the moment generating function of a random variable X.

$M_X(t)=\epsilon[e^{tX}]$

Calculate the mgf of a Poisson random variable with mean value λ.

$e^{\lambda(e^t-1)}$

Calculate the mgf of an exponential random variable with mean $1/\mu$

$\frac{\mu}{\mu-t}$

Let X be Poisson distributed with mean value λ, Y be an exponential random variable with mean $1/\mu$ and assume that X and Y are independent. Calculate the mgf of $X+Y$.

$e^{\lambda(e^t-1)}\cdot\frac{\mu}{\mu-t}$