Module2-System of Linear Equations and Eigen Value Problems

Dr. K.Mahipal Reddy

Assistant Professor

VIT-AP University, Amaravati

February 27, 2021

Syllabus

- Gauss Elimination and Back Substitution
- LU Decomposition Method
- Tridiagonal matrix algorithm (or) Thomas algorithm
- Eigenvalue-Power Method, Jacobi Method
- Thomas Algorithm

System of linear equations

- A linear system of m equation in n unknown $x_1, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- The constants $a_{11}, \ldots, a_{mn}$ are known as the coefficients.
- If the right hand side constants i.e. $b_1, \ldots, b_m$ are all zero then system is known as the homogeneous system of equations otherwise it is known as non-homogeneous system of equations.

- The linear system of equation in matrix form written as:

$$Ax = b$$

where the $A = [a_{ij}]$ is known as the coefficient matrix of order $m \times n$.

i.e., $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

The method of the linear algebraic system of equations are classified in two types

1. **Direct Methods:** These methods produce the exact solution after a finite number of steps.
    - Gauss Elimination Method
    - LU decomposition Method
    - Tridiagonal System
2. **Iterative methods:** These methods give a sequence of approximate solutions, which converges when the number of steps tend to infinity.
    - Jacobi iteration method
    - Gauss-seidel iteration method

- Step-1 Convert the system of linear equations into triangular form by using elementary row operations.
- Step-2 Solve the triangular system by back substitution.

Example 1: Solve the system of equations

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30 \end{aligned}$$

Answer Augmented matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \tilde{\mathbf{A}} \simeq \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

Gauss Elimination and Back Substitution

- Step-1 Convert the system of linear equations into triangular form by using elementary row operations.
- Step-2 Solve the triangular system by back substitution.

Example 1: Solve the system of equations

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30 \end{aligned}$$

Answer Augmented matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \tilde{\mathbf{A}} \simeq \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

$2x_1 + 5x_2 = 2$

$13x_2 = -26$

$\implies x_2 = -2$ and $x_1 = 6$.

Solve the system of equations

$$\begin{aligned} 2x_1 + 6x_2 + 8x_3 &= 16 \\ 4x_1 + 15x_2 + 19x_3 &= 38 \\ 2x_1 + 3x_3 &= 6 \end{aligned}$$

Answer : Augmented matrix $\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 2 & 6 & 8 & 16 \\ 4 & 15 & 19 & 38 \\ 2 & 0 & 3 & 6 \end{array}\right]$

$$\xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 2 & 6 & 8 & 16 \\ 0 & 3 & 3 & 6 \\ 0 & -6 & -5 & -10 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 2 & 6 & 8 & 16 \\ 0 & 3 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]; \qquad x_3 = 2, x_2 = 0, x_1 = 0.$$

3. Example of Gauss Elimination

Solve the system of equations

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6 \end{aligned}$$

Answer : Augmented matrix $\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \begin{array}{l} \mathsf{R}_2 - \frac{2}{3}\mathsf{R}_1 \to \mathsf{R}_2 \\ \mathsf{R}_3 - 2\mathsf{R}_1 \to \mathsf{R}_3 \end{array}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \mathsf{R}_3 - 6\mathsf{R}_2 \to \mathsf{R}_3 \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Solve the system of equations

$$\begin{aligned}3x_1 + 2x_2 + x_3 &= 3\\ 2x_1 + x_2 + x_3 &= 0\\ 6x_1 + 2x_2 + 4x_3 &= 6\end{aligned}$$

Answer : Augmented matrix $\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3\\ 2 & 1 & 1 & 0\\ 6 & 2 & 4 & 6\end{array}\right]$ $\begin{array}{l} \mathsf{R}_2 - \frac{2}{3}\mathsf{R}_1 \to \mathsf{R}_2\\ \mathsf{R}_3 - 2\mathsf{R}_1 \to \mathsf{R}_3\end{array}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3\\ 0 & -\frac{1}{3} & \frac{1}{3} & -2\\ 0 & -2 & 2 & 0\end{array}\right] \mathsf{R}_3 - 6\mathsf{R}_2 \to \mathsf{R}_3 \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3\\ 0 & -\frac{1}{3} & \frac{1}{3} & -2\\ 0 & 0 & 0 & 12\end{array}\right]$$

rank$(\mathbf{A}) = 2 \neq 3 =$rank$(\tilde{\mathbf{A}})$.

3. Example of Gauss Elimination

Solve the system of equations

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 4x_3 &= 6 \end{aligned}$$

Answer : Augmented matrix $\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$ $\begin{array}{l} \mathsf{R}_2 - \frac{2}{3}\mathsf{R}_1 \rightarrow \mathsf{R}_2 \\ \mathsf{R}_3 - 2\mathsf{R}_1 \rightarrow \mathsf{R}_3 \end{array}$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \mathsf{R}_3 - 6\mathsf{R}_2 \rightarrow \mathsf{R}_3 \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

rank$(\mathbf{A}) = 2 \neq 3 =$rank$(\tilde{\mathbf{A}})$.No solution.

A factorization of a square matrix A as

$$A = LU$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_{(n-1)1} & l_{(n-1)2} & l_{(n-1)3} & \cdots & l_{(n-1)(n-1)} & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{(n-1)(n-1)} & l_{nn} \end{bmatrix},$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1(n-1)} & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2(n-1)} & u_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & u_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix},$$

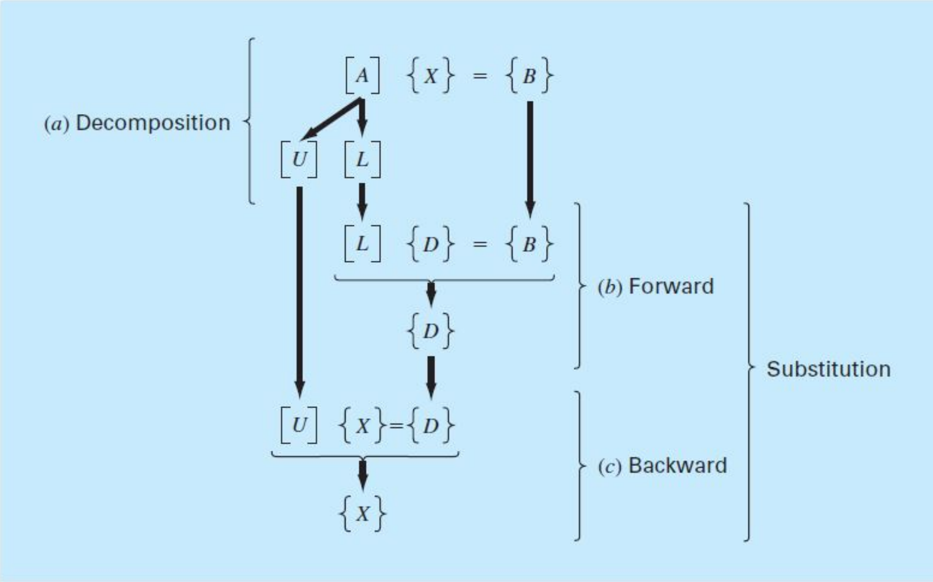
- $A = LU$
- L and U are lower and upper triangular matrix, is called an LU-decomposition(or LU-factorization) of A. when we choose
  1. $l_{ii} = 1$, the method is known as Doolttle's Method
  2. $u_{ii} = 1$, the method is known as Crout's method.

Produce of LU decomposition Method

The Method of LU-Decomposition to solve the System of Equations

- **Step 1** Rewrite the system $Ax = b$ as $LUx = b$
- **Step 2** Define a new $n \times 1$ matrix y by $Ux = y$
- **Step 3** Use Step(2) to rewrite Step(1) as $Ly = b$ and solve this system for y.
- **Step 4** Substitute y in step(2) and solve for x.
- The inverse of A can be $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$.
- Note: If the matrix is invertible, then a LU decomposition exists only if the leading principal minors are non-zero.

Produce of LU decomposition Method



Solve the Linear system

$$x + 2y = 5$$

$$4x + 9y = 21$$

by using the LU factorization.

Solution: The matrix of linear system $Ax = b$ where $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$.

$\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$, Here $l_{11} = l_{22} = 1$

We can be written as $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} \end{bmatrix}$.

After comparing the above matrices, we obtain

$u_{11} = 1, u_{12} = 2, l_{21} = 4, u_{22} = 1.$

Triangular matrices are $L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

$$Ax = b \implies (LU)x = b \implies L(Ux) = b$$

$$Ly = b, \quad \text{where} \quad y = Ux$$

$$Ly = b \implies \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \implies y_1 = 5, \; y_2 = 1.$$

And, $Ux = y \implies \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \implies x_1 = 3, \; x_2 = 1.$

1. Solve the system of equations

$$7x_1 - 2x_2 + x_3 = 12$$
$$14x_1 - 7x_2 - 3x_3 = 17$$
$$-7x_1 + 11x_2 + 18x_3 = 5$$

by using the LU factorization. **Solution:x=(3,4,-1)**

2. Solve the system of equations

$$x_1 + 3x_2 - 2x_3 = -4$$
$$3x_1 + 7x_2 + x_3 = 4$$
$$-2x_1 + x_2 + 7x_3 = 7$$

Band Matrix

$$
\begin{array}{cc}
 & \leftarrow \quad a \quad \rightarrow \\
\begin{array}{c} \uparrow \\ \\ b \\ \downarrow \\ \\ \\ \\ \end{array} &
\begin{bmatrix}
\times & \times & \times & 0 & 0 & 0\ldots & 0 \\
\times & \times & \times & \times & 0 & 0\ldots & 0 \\
\times & \times & \times & \times & \times & 0\ldots & 0 \\
\times & \times & \times & \times & \times & \times\ldots & 0 \\
0 & \times & \times & \times & \times & \times\ldots & 0 \\
0 & 0 & \times & \times & \times & \times\ldots & 0 \\
. & . & . & . & . & . & . \\
\end{bmatrix}
\end{array}
$$

Band width is $a + b - 1$ More details:

Tridiagonal system

In numerical linear algebra, the tridiagonal matrix algorithm, also known as the Thomas algorithm, is a simplified form of Gaussian elimination that can be used to solve tridiagonal systems of equations. A tridiagonal system for n unknowns may be written as

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i,\ \ i = 1, 2, \ldots, n \text{ where } a_1 = 0 \ \text{ and } \ b_n = 0.$$

A tridiagonal system has a bandwidth of 3 and can be expressed generally as:

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & \vdots & 0 & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}$$

- The matrix has three diagonal one main diagonal second upper diagonal and third lower diagonal.
- To solve this system we use the Thomas algorithm which is again based on the LU decomposition method. We convert the given coefficient matrix in to lower and upper triangular matrix.

Example of Tridiagonal sytem

Solve the following Tridigonal system

$$\begin{aligned} 3x_1 - x_2 &= -1 \\ -x_1 + 3x_2 - x_3 &= 7 \\ -x_2 + 3x_3 &= 7 \end{aligned}$$

Solution: As with Gauss elimination, the first step involves transforming the matrix to upper triangular form.

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}$$

Applying row operation on above matrix, we obtain

$(R_1 \to \frac{R_1}{2},\ R_2 \to 3R_2 + R_1)$

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 \\ 0 & 8 & -3 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} \\ 20 \\ 7 \end{bmatrix}$$

$R_2 \to \frac{R_2}{8},\ R_3 \to 8R_3 + R_2$

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{-3}{8} \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} \\ \frac{20}{8} \\ 76 \end{bmatrix}$$

$R_2 \to \frac{R_2}{8},\ R_3 \to 8R_3 + R_2$

$$\begin{bmatrix} 1 & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{-3}{8} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} \\ \frac{20}{8} \\ \frac{76}{21} \end{bmatrix}$$

Now back substitution can be applied to generate the final solution:

- $x_3 = \frac{76}{21} = 3.619$
- $x_2 - \frac{3}{8}x_3 = \frac{20}{8} \Rightarrow x_2 = \frac{20}{8} + \frac{3}{8}\frac{76}{21} = \frac{27}{7} = 3.857$
- $x_1 + \frac{-1}{3}x_2 = \frac{-1}{3} \Rightarrow x_1 = \frac{-1}{3} + \frac{1}{3}\frac{27}{7} = \frac{76}{21} = 0.952$
- **Solution** is $x_1 = 0.952,\ x_2 = 3.857,\ x_3 = 3.619.$

Thomas Algorithm

- The tridiagonal system $Ax = B$, where A is tridiagonal matrix.
- **Step1: Triangularization: Forward sweep with normalization**

$$\alpha_1 = b_1, \quad \alpha_k = b_k - \frac{a_k}{\alpha_{k-1}} * c_{k-1}, \text{ for } \quad k = 2, 3, \ldots, n-1.$$

$$\beta_1 = d_1, \quad \beta_k = d_k - \frac{a_k}{\alpha_{k-1}} * \beta_{k-1} \text{ for } \quad k = 2, 3, \ldots, n.$$

- This sequence of operations finally results in the following system of equation

$$\begin{bmatrix} \alpha_1 & c_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & c_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & 0 & \alpha_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

- **Step 2:Backward sweep leads to solution vector**
  $x_n = \beta_n, \quad x_k = \frac{\beta_k - c_k x_{k+1}}{\alpha_k}, \quad$ for $\; k = n-1, n-2, \ldots, 1.$

Example2 of Tridiagonal system

Solve the following tridiagonal system with the Thomas algorithm

$$\begin{aligned} 2.04T_1 - T_2 &= 4.08 \\ -T_1 + 2.04T_2 - T_3 &= 0.8 \\ -T_2 + 2.04T_3 - T_4 &= 0.8 \\ -T_3 + 2.04T_4 &= 2.08 \end{aligned}$$

Solution Given tridiagonal system is

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 4.08 \\ 0.8 \\ 0.8 \\ 2.08 \end{bmatrix},$$

Example2 of Tridiagonal system (Contin...)

Tridigonal system is

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$

Above system, we can be written as

$$\begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 \\ 0 & 0 & \alpha_3 & c_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

- $\alpha_1 = b_1 = 2.04, \;\; \alpha_2 = b_2 - \frac{a_2}{\alpha_1}c_1 = 2.04 - \frac{-1}{2.04}(-1) = 2.04 - 0.4902 = 1.5498$
- $\alpha_3 = b_3 - \frac{a_3}{\alpha_2}c_2 = 2.04 - \frac{-1}{1.5498}(-1) = 2.04 - 0.6452 = 1.3948$
- $\alpha_4 = b_4 - \frac{a_4}{\alpha_3}c_3 = 2.04 - \frac{-1}{1.3948}(-1) = 1.3230$
- $\beta_1 = d_1 = 4.08, \;\; \beta_2 = d_2 - \frac{a_2}{\alpha_1}\beta_1 = 0.8 - \frac{-1}{2.04}(4.08) = 0.8 + 2 = 2.8$
- $\beta_3 = d_3 - \frac{a_3}{\alpha_2}\beta_2 = 0.8 - \frac{-1}{1.5498}(2.8) = 0.8 + 1.8067 = 2.6067$
- $\beta_4 = d_4 - \frac{a_4}{\alpha_3}\beta_3 = 2.08 - \frac{-1}{1.3948}(2.6067) = 2.08 + 1.8689 = 3.9489$
- $\alpha_i, \beta_i, c_i, i = 1, 2, 3, 4$ values substitute in above system, we obtain matrix form is
- $$\begin{bmatrix} 2.0400 & -1 & 0 & 0 \\ 0 & 1.5498 & -1 & 0 \\ 0 & 0 & 1.3948 & -1 \\ 0 & 0 & 0 & 1.3230 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 4.0800 \\ 2.8000 \\ 2.6067 \\ 3.948 \end{bmatrix}$$

- $T_4 = \frac{\beta_4}{\alpha_4} = \frac{3.948}{1.323} = 2.9848$
- $T_3 = \frac{\beta_3 - c_3 x_4}{\alpha_3} = \frac{2.6067-(-1)2.9848}{1.3948} = \frac{4.7466}{1.3948} = 4.0089$
- $T_2 = \frac{\beta_2 - c_2 x_3}{\alpha_2} = \frac{2.8-(-1)4.0089}{1.5498} = \frac{6.8089}{1.3948} = 4.3934$
- $T_1 = \frac{\beta_1 - c_1 x_2}{\alpha_1} = \frac{4.08-(-1)4.3934}{2.04} = \frac{8.4734}{1.3948} = 4.1536$
- Solution is $T_1 = 4.1536,\ T_2 = 4.3934,\ T_3 = 4.0089$ and $T_4 = 2.9848$.

Example3 of Tridiagonal system

Solve the following tridiagonal system by using Thomas algorithm

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 1 \\ -x_3 + x_4 &= 0 \end{aligned}$$

Solution Given tridiagonal system is

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

Example3 of Tridiagonal system (Contin...)

Tridigonal system is

$$\begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 \\ 0 & 0 & \alpha_3 & c_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

- $\alpha_1 = b_1 = 2,\ \alpha_2 = b_2 - \frac{a_2}{\alpha_1}c_1 = 2 - \frac{-1}{2}(-1) = 2 - 0.5 = \frac{3}{2}$
- $\alpha_3 = b_3 - \frac{a_3}{\alpha_2}c_2 = 2 - \frac{-1}{\frac{3}{2}}(-1) = 2 - \frac{2}{3} = \frac{4}{3}$
- $\alpha_4 = b_4 - \frac{a_4}{\alpha_3}c_3 = 2 - \frac{-1}{\frac{4}{3}}(-1) = 1 - \frac{3}{4} = \frac{1}{4}$
- $\beta_1 = d_1 = 0, \beta_2 = d_2 - \frac{a_2}{\alpha_1}\beta_1 = 0 - \frac{-1}{2}(0) = 0$
- $\beta_3 = d_3 - \frac{a_3}{\alpha_2}\beta_2 = 1 - \frac{-1}{\frac{3}{2}}(0) = 1$
- $\beta_4 = d_4 - \frac{a_4}{\alpha_3}\beta_3 = 0 - \frac{-1}{\frac{4}{3}}(1) = \frac{3}{4}$
- $\alpha_i, \beta_i, c_i, i = 1, 2, 3, 4$ values substitute in above system, we obtain matrix form is
- $$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{3}{4} \end{bmatrix}$$

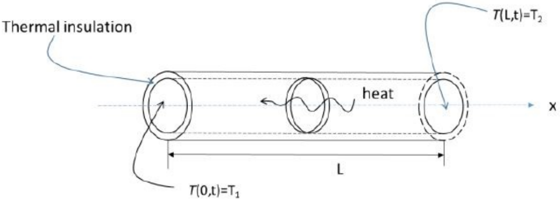
Example3 of Tridiagonal system (Contin...)

- $x_4 = \frac{\beta_4}{\alpha_4} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$
- $x_3 = \frac{\beta_3 - c_3 x_4}{\alpha_3} = \frac{1-(-1)(3)}{\frac{4}{3}} = \frac{3(1+3)}{4} = 3$
- $x_2 = \frac{\beta_2 - c_2 x_3}{\alpha_2} = \frac{0-(-1)3}{\frac{3}{2}} = \frac{2}{3}(3) = 2$
- $x_1 = \frac{\beta_1 - c_1 x_2}{\alpha_1} = \frac{0-(-1)2}{2} = 1$
- Solution is $x_1 = 1,\ x_2 = 2,\ x_3 = 3$ and $x_4 = 3$.

1.Appication of Thomas algorithm

A steady-state heat balance for a copper wire can be represented as

$$\frac{d^2T}{dx^2} - 0.15T = 0$$
$$T(0) = T_1 = 240$$
$$T(10) = T_6 = 150$$



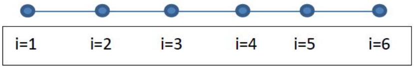
1.Appication of Thomas algorithm(Contin..)

Obtain a tridiagonal matrix (suitable for numerical solution) for a wire of L=10-m length.

Use N=6 knots (ix=6) and the running index from 1 to 6. There are 6 temperatures [T_1, T_2, T_3, T_4, T_5, T_6] and the x-vector [x_1, x_2, x_3, x_4, x_5, x_6]. This problem is developed with indices starting in one to match the indexes in the development of Thomas Algorithm.

The step size and discretization:

$$\Delta x = \frac{(x_6 - x_1)}{N - 1} = \frac{(10 - 0)}{5} = 2$$



Discretization of the BCs yields: T_1=240 and T_6=150

The Relation between Differences and Derivatives.

$$\frac{df}{dx} \approx \frac{f_{i+1} - f_i}{\Delta x}$$
$$\frac{d^2 f}{dx^2} \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2}$$

Discretization of the PDE by central difference approximation:

$$\frac{(T_{i-1} - 2T_i + T_{i+1})}{(\Delta x)^2} - 0.15T_i = 0$$

Plugging in $\Delta x = 2$, yields the recurrence formula:

$$T_{i-1} - 2.6T_i + T_{i+1} = 0$$

There are four interior knots, therefore we need four equations:

$$T_1 - 2.6T_2 + T_3 = 0$$

$$T_2 - 2.6T_3 + T_4 = 0$$

$$T_3 - 2.6T_4 + T_5 = 0$$

$$T_4 - 2.6T_5 + T_6 = 0$$

After plugging in the BC's you obtain:

$$\begin{pmatrix} -2.6 & 1 & 0 & 0 \\ 1 & -2.6 & 1 & 0 \\ 0 & 1 & -2.6 & 1 \\ 0 & 0 & 1 & -2.6 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} -240 \\ 0 \\ 0 \\ -150 \end{pmatrix}$$

Identify each number above with Thomas Algorithm notation:

$$\begin{pmatrix} q_1 & r_1 & 0 & 0 \\ p_2 & q_2 & r_2 & 0 \\ 0 & p_3 & q_3 & r_3 \\ 0 & 0 & p_4 & q_4 \end{pmatrix} \begin{pmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}$$

Thomas algorithm involves the following formulas:

2.Appication of Thomas algorithm(Contin..)

The following differential equation results from a steady-state mass balance for a chemical in a one-dimensional canal

$$D\frac{d^2c}{dx^2} - U\frac{dc}{dx} - kc = 0$$

where c=concentration, t=time, x=distance, D=diffusion coefficient, fluid velocity, and $k =$ a first-order decay rate. Convert this differential equation to an equivalent system of simultaneous algebraic equations. Given $D = 2,\ U = 1,\ k = 0.2, c(0) = 80$ and $c(10) = 20$, solve these equations from x = 0 to 10 and develop a plot of concentration versus distance.

Iterative Method-Jacobi iteration formula

A linear system of m equation in n unknown $x_1, \ldots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_m \end{aligned}$$

Jacobi iteration formula

$$\begin{aligned} {x_1}^{k+1} &= -\frac{1}{a_{11}}\left[a_{12}{x_2}^k + a_{13}{x_3}^k \cdots + a_{1n}{x_n}^k - b_1\right] \\ {x_2}^{k+1} &= -\frac{1}{a_{22}}\left[a_{21}{x_1}^k + a_{23}{x_3}^k \cdots + a_{2n}{x_n}^k - b_2\right] \\ \ldots \\ {x_n}^{k+1} &= -\frac{1}{a_{nn}}\left[a_{n1}{x_1}^k + a_{n2}{x_2}^k \cdots + a_{n(n-1)}{x_n}^k - b_n\right] \end{aligned}$$

Example of Jacobi iteration formula

Use the Jacobi method to approximate the solution of the following sustem of linear equations

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximation are identical when rounded to three significant digits.

- **Solution:** Choose initial approximaation $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0.$

Jacobi iteration formula

$$
\begin{aligned}
{x_1}^{k+1} &= -\frac{1}{a_{11}}\left[a_{12}{x_2}^k + a_{13}{x_3}^k \cdots + a_{1n}{x_n}^k - b_1\right] \\
{x_2}^{k+1} &= -\frac{1}{a_{22}}\left[a_{21}{x_1}^k + a_{23}{x_3}^k \cdots + a_{2n}{x_n}^k - b_2\right] \\
\cdots \\
{x_n}^{k+1} &= -\frac{1}{a_{nn}}\left[a_{n1}{x_1}^k + a_{n2}{x_2}^k \cdots + a_{n(n-1)}{x_n}^k - b_n\right]
\end{aligned}
$$

For $k = 0$,

$$
\begin{aligned}
{x_1}^{1} &= -\frac{1}{a_{11}}\left[a_{12}{x_2}^0 + a_{13}{x_3}^0 \cdots + a_{1n}{x_n}^0 - b_1\right] \\
{x_2}^{1} &= -\frac{1}{a_{22}}\left[a_{21}{x_1}^0 + a_{23}{x_3}^0 \cdots + a_{2n}{x_n}^0 - b_2\right] \\
\cdots \\
{x_n}^{1} &= -\frac{1}{a_{nn}}\left[a_{n1}{x_1}^0 + a_{n2}{x_2}^0 \cdots + a_{n(n-1)}{x_n}^0 - b_n\right]
\end{aligned}
$$

- To begin, write the system in the form

$$
\begin{aligned}
x_1^1 &= \frac{-1 + 2x_2^0 - 3x_3^0}{5} \\
x_2^1 &= \frac{2 + 3x_1^0 - x_3^0}{9} \\
x_3^1 &= \frac{-3 + 2x_1^0 - x_2^0}{7}
\end{aligned}
$$

- Substitute the initial approximaation value $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0.$ in the above system, we get

$$
\begin{aligned}
x_1^1 &= \frac{-1 + 2(0) - 3(0)}{5} = -\frac{1}{5} = -0.2 \\
x_2^1 &= \frac{2 + 3(0) - (0)}{9} = \frac{2}{9} = 0.222 \\
x_3^1 &= \frac{-3 + 2(0) - (0)}{7} = \frac{-3}{7} = -0.429
\end{aligned}
$$

- First approximaation value $x_1^1 = -0.2,\ x_2^1 = 0.222,\ x_3^1 = -0.429.$

- $$\begin{aligned} x_1^2 &= \frac{-1 + 2x_2^1 - 3x_3^1}{5} \\ x_2^2 &= \frac{2 + 3x_1^1 - x_3^1}{9} \\ x_3^2 &= \frac{-3 + 2x_1^1 - x_2^1}{7} \end{aligned}$$

- Take the first approximaation value
  $x_1^1 = -0.2,\ x_2^1 = 0.222,\ x_3^1 = -0.429.$, we get

$$\begin{aligned} x_1^2 &= \frac{-1 + 2(0.222) - 3(-0.429)}{5} = -\frac{0.7310}{5} = 0.1462 \\ x_2^2 &= \frac{2 + 3(-0.2) - (-0.4290)}{9} = \frac{1.8290}{9} = 0.2032 \\ x_3^2 &= \frac{-3 + 2(-0.2) - (0.222)}{7} = \frac{-1.2073}{7} = -0.5174 \end{aligned}$$

- Second approximaation value
  $x_1^2 = 0.1462,\ x_2^2 = 0.2032,\ x_3^2 = -0.5174.$

- Continuing this procedure, you obtain the sequence of approximation shown bellow table

n	0	1	2	3	4	5	6	7
x_1	0	-0.2	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

- Because thelast two columnssin table are identical,you can conclude that to three singnificant digits the solution is

$$x_1^6 = 0.186,\ x_2^6 = 0.331,\ x_3^6 = -0.423.$$

- The Jacobimethod is said to converge. That is repeated iterations sucseed in producing an approximation that is correct to three significant digits.

Note: It is convenient to split the matrix A into three parts. We write

$$A = L + D + U$$

where L consists of the elements of A strictly below the diagonal and zeros elsewhere; D is a diagonal matrix consisting of the diagonal entries of A and U consists of the elements of A strictly above the diagonal. Note that L and U here are not the same matrices as appeared in the LU decomposition. The current L and U are much easier to find.

For example:

$$\underbrace{\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}}_{U}$$

and $$\underbrace{\begin{bmatrix} 2 & -6 & 1 \\ 3 & -2 & 0 \\ 4 & -1 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & -1 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & -6 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

more generally for 3×3 matrix,

Jacobi Method(Contin...)

$$\underbrace{\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} \bullet & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \bullet \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & \bullet & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

Note 2: Jacobi iteration: The basic idea is to use the $A = L + D + U$ partitioning of A to write the system $AX = B$ in the form

$$(L + D + U)X = B$$

$$DX = -(L + U)X + B$$

We use this equation as the motivation to define the iterative process

$$DX^{(n+1)} = -(L + U)X^{(n)} + B$$

Example2 of Jacobi Method

Example 2: use the Jacobi iteration to approximate the solution of the system

$$\begin{pmatrix} 8 & 2 & 4 \\ 3 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

. Use the initial guess $X^{(0)} = [0 \quad 0 \quad 0]^T$

Solution: Here $D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $L+U = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

First iteration: $DX^{(1)} = -(L+U)X^{(0)} + B$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

Taking this information row by row we see that

$8x_1^{(1)} = -16$; $5x_2^{(1)} = 4$ and $4x_3^{(1)} = -12$

Example2 of Jacobi Method(Contin...)

Thus the first Jacobi iteration gives us $X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} -2 \\ 0.8 \\ -3 \end{pmatrix}$

Continuing this procedure, we obtain the sequence of approximations shown below.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=20$	$n=40$
$x_1^{(n)}$	0	-2	-0.7	-0.155	-0.765	-1.277	-0.839	-0.9959	-1
$x_2^{(n)}$	0	0.8	2.6	1.666	2.39	1.787	2.209	2.0043	2
$x_3^{(n)}$	0	-3	-2.2	-3.3	-2.64	-3.215	-2.808	-2.9959	-3

Theorem

If A is a strictly diagonally dominant matrix, then the Jacobi iteration scheme converges for any initial starting vector.

Gauss-seidel iteration method

$$
\begin{aligned}
{x_1}^{k+1} &= -\frac{1}{a_{11}}\left[a_{12}{x_2}^k + a_{13}{x_3}^k + \cdots + a_{1n}{x_n}^k b_1\right] \\
{x_2}^{k+1} &= -\frac{1}{a_{22}}\left[a_{21}{x_1}^{k+1} + a_{23}{x_3}^k \cdots + a_{2n}{x_n}^k - b_2\right] \\
\cdots \\
{x_n}^{k+1} &= -\frac{1}{a_{nn}}\left[a_{n1}{x_1}^{k+1} + a_{n2}{x_2}^{k+1} \cdots + a_{n(n-1)}{x_n}^{k+1} - b_n\right]
\end{aligned}
$$

$k = 0, 1, , \ldots$

Example of Gauss-seidel iteration method

- Use the Gauss-seidel iteration method to approximate the solution of the following sustem of linear equations

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

  Continue the iterations until two successive approximation are identical when rounded to three significant digits.

- Choose initial approximation $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0.$

- To begin, write the system in the form

$$\begin{aligned}
x_1^1 &= \frac{-1 + 2x_2^0 - 3x_3^0}{5} \\
x_2^1 &= \frac{2 + 3x_1^1 - x_3^0}{9} \\
x_3^1 &= \frac{-3 + 2x_1^1 - x_2^1}{7}
\end{aligned}$$

-

$$x_1^1 = \frac{-1 + 2(0) - 3(0)}{5} = \frac{-1}{5} = -0.2$$

Now that you have a new value for x_1, however use it tocompute a new value for x_2. That is

$$x_2^1 = \frac{2 + 3(-0.2) - 0}{9} = \frac{1.4}{9} = 0.1556 \equiv 0.156$$

Similarly,use $x_1 = -0.2,\ x_2 = 0.156$ tocompute a new value for x_3. That is

$$x_3^2 \quad = \quad \frac{-3 + 2x_1^1 - x_2^1}{7} = \frac{-3 + 2(-0.2) - 0.156}{7} = \frac{-3.5560}{7} = -0.5080$$

So the first approximation is $x_1 = -0.2,\ x_2 = 0.156$ and $x_3 = -0.508$. Continued iterations produce the sequence of approximations in bellow

n	0	1	2	3	4	5
x_1	0	-0.2	0.167	0.191	0.186	0.186
x_2	0	0.156	0.334	0.333	0.331	0.331
x_3	0	-0.508	-0.429	-0.422	-0.423	-0.423

Note that after only five iterations of the Gauss-Seidel method, you achieved the same accuracy as was obtained with seven iterations of the Jacobi method in example.

Remark: A sufficient condition for convergence of the Jacobi method is that the system of equations is diagonally dominant,that is, the coefficient matrix A is diagonally dominant. We can verify that

$$|a_{ii}| \geq \sum_{j=1,\ i \neq j}^{n} |a_{ij}|.$$

This implies that convergence may be obtained even if the system is not diagonally dominant. If the system is not diagonally dominant, we may exchange the equations, if possible, such that the new system is diagonally dominant and convergence is guaranteed.

Note1: Jacobi method or method of simultaneous displacement

In matrix form $Ax = b$, the method can be written as

$$A = L + D + U$$

and

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b,$$

where D is the Diagonal matrix, L is the Lower triangular matrix, U is Upper triangular matrix.

Note2: Gauss-seidel iteration method or method of succesive displacement

In matrix form $Ax = b$, the method can be written as

$$A = L + D + U$$

and

$$x^{(k+1)} = -(L+D)^{-1}Ux^{(k)} + (L+D)^{-1}b,$$

where D is the Diagonal matrix, L is the Lower triangular matrix, U is Upper triangular matrix.

- A general linear iterative method for the solution of the system may be defined in the form

$$x^{k+1} = Hx^{K} + c, k = 0, 1, 2, \ldots,$$

where x^{k+1} and x^{k} are the approximations for $x$at the $(k+1)$ iterations, respectively , H is called the iteration matrix depending on A and c is a column vector.

- The necessary and sufficient condition for convergence of the Gauss-Jacobi and Gauss-Seidel iteration methods is that the spectral radius of the iteration matrix H is less than one unit, that is, $\rho(H) < 1$, where $\rho(H)$ is the largest eigen value in magnitude of H

The Power Method

There are many applications in which some vector $\mathbf{x}_0$ in R^n is multiplied repeatedly by an $n \times n$ matrix A to produce a sequence

$$\mathbf{x}_0, \quad A\mathbf{x}_0, \quad A^2\mathbf{x}_0, \ldots, \quad A^k\mathbf{x}_0, \ldots$$

We call a sequence of this form a ***power sequence generated by A***. In this section we will be concerned with the convergence of power sequences and how such sequences can be used to approximate eigenvalues and eigenvectors. For this purpose, we make the following definition.

DEFINITION 1 If the *distinct* eigenvalues of a matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_k$, and if $|\lambda_1|$ is larger than $|\lambda_2|, \ldots, |\lambda_k|$, then λ_1 is called a ***dominant eigenvalue*** of A. Any eigenvector corresponding to a dominant eigenvalue is called a ***dominant eigenvector*** of A.

EXAMPLE 1 Dominant Eigenvalues

Some matrices have dominant eigenvalues and some do not. For example, if the distinct eigenvalues of a matrix are

$$\lambda_1 = -4, \quad \lambda_2 = -2, \quad \lambda_3 = 1, \quad \lambda_4 = 3$$

then $\lambda_1 = -4$ is dominant since $|\lambda_1| = 4$ is greater than the absolute values of all the other eigenvalues; but if the distinct eigenvalues of a matrix are

$$\lambda_1 = 7, \quad \lambda_2 = -7, \quad \lambda_3 = -2, \quad \lambda_4 = 5$$

then $|\lambda_1| = |\lambda_2| = 7$, so there is no eigenvalue whose absolute value is greater than the absolute value of all the other eigenvalues.

THEOREM 9.2.1 *Let A be a symmetric $n \times n$ matrix that has a positive** *dominant eigenvalue λ. If $\mathbf{x}_0$ is a unit vector in R^n that is not orthogonal to the eigenspace corresponding to λ, then the normalized power sequence*

$$\mathbf{x}_0, \quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|}, \quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|}, \ldots, \quad \mathbf{x}_k = \frac{A\mathbf{x}_{k-1}}{\|A\mathbf{x}_{k-1}\|}, \ldots \tag{1}$$

converges to a unit dominant eigenvector, and the sequence

$$A\mathbf{x}_1 \cdot \mathbf{x}_1, \quad A\mathbf{x}_2 \cdot \mathbf{x}_2, \quad A\mathbf{x}_3 \cdot \mathbf{x}_3, \ldots, \quad A\mathbf{x}_k \cdot \mathbf{x}_k, \ldots \tag{2}$$

converges to the dominant eigenvalue λ.

Remark In the exercises we will ask you to show that (1) can also be expressed as

$$\mathbf{x}_0, \quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|}, \quad \mathbf{x}_2 = \frac{A^2\mathbf{x}_0}{\|A^2\mathbf{x}_0\|}, \ldots, \quad \mathbf{x}_k = \frac{A^k\mathbf{x}_0}{\|A^k\mathbf{x}_0\|}, \ldots \tag{3}$$

This form of the power sequence expresses each iterate in terms of the starting vector $\mathbf{x}_0$, rather than in terms of its predecessor.

The Power Method with Euclidean Scaling

Step 0. Choose an arbitrary nonzero vector and normalize it, if need be, to obtain a unit vector $\mathbf{x}_0$.

Step 1. Compute $A\mathbf{x}_0$ and normalize it to obtain the first approximation $\mathbf{x}_1$ to a dominant unit eigenvector. Compute $A\mathbf{x}_1 \cdot \mathbf{x}_1$ to obtain the first approximation to the dominant eigenvalue.

Step 2. Compute $A\mathbf{x}_1$ and normalize it to obtain the second approximation $\mathbf{x}_2$ to a dominant unit eigenvector. Compute $A\mathbf{x}_2 \cdot \mathbf{x}_2$ to obtain the second approximation to the dominant eigenvalue.

Step 3. Compute $A\mathbf{x}_2$ and normalize it to obtain the third approximation $\mathbf{x}_3$ to a dominant unit eigenvector. Compute $A\mathbf{x}_3 \cdot \mathbf{x}_3$ to obtain the third approximation to the dominant eigenvalue.

Continuing in this way will usually generate a sequence of better and better approximations to the dominant eigenvalue and a corresponding unit eigenvector.*

EXAMPLE 2 The Power Method with Euclidean Scaling

Apply the power method with Euclidean scaling to

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{with} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Stop at $\mathbf{x}_5$ and compare the resulting approximations to the exact values of the dominant eigenvalue and eigenvector.

Solution We will leave it for you to show that the eigenvalues of A are $\lambda = 1$ and $\lambda = 5$ and that the eigenspace corresponding to the dominant eigenvalue $\lambda = 5$ is the line represented by the parametric equations $x_1 = t$, $x_2 = t$, which we can write in vector form as

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

Setting $t = 1/\sqrt{2}$ yields the normalized dominant eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \approx \begin{bmatrix} 0.707106781187\ldots \\ 0.707106781187\ldots \end{bmatrix} \tag{7}$$

Now let us see what happens when we use the power method, starting with the unit vector $\mathbf{x}_0$.

$$A\mathbf{x}_0 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \frac{1}{3.60555} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \approx \begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix}$$

$$A\mathbf{x}_1 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix} \approx \begin{bmatrix} 3.60555 \\ 3.32820 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \frac{1}{4.90682} \begin{bmatrix} 3.60555 \\ 3.32820 \end{bmatrix} \approx \begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix}$$

$$A\mathbf{x}_2 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix} \approx \begin{bmatrix} 3.56097 \\ 3.50445 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \frac{1}{4.99616} \begin{bmatrix} 3.56097 \\ 3.50445 \end{bmatrix} \approx \begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix}$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix} \approx \begin{bmatrix} 3.54108 \\ 3.52976 \end{bmatrix} \qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \frac{1}{4.99985} \begin{bmatrix} 3.54108 \\ 3.52976 \end{bmatrix} \approx \begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix}$$

$$A\mathbf{x}_4 \approx \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix} \approx \begin{bmatrix} 3.53666 \\ 3.53440 \end{bmatrix} \qquad \mathbf{x}_5 = \frac{A\mathbf{x}_4}{\|A\mathbf{x}_4\|} \approx \frac{1}{4.99999} \begin{bmatrix} 3.53666 \\ 3.53440 \end{bmatrix} \approx \begin{bmatrix} 0.70733 \\ 0.70688 \end{bmatrix}$$

$$\lambda^{(1)} = (A\mathbf{x}_1) \cdot \mathbf{x}_1 = (A\mathbf{x}_1)^T \mathbf{x}_1 \approx \begin{bmatrix} 3.60555 & 3.32820 \end{bmatrix} \begin{bmatrix} 0.83205 \\ 0.55470 \end{bmatrix} \approx 4.84615$$

$$\lambda^{(2)} = (A\mathbf{x}_2) \cdot \mathbf{x}_2 = (A\mathbf{x}_2)^T \mathbf{x}_2 \approx \begin{bmatrix} 3.56097 & 3.50445 \end{bmatrix} \begin{bmatrix} 0.73480 \\ 0.67828 \end{bmatrix} \approx 4.99361$$

$$\lambda^{(3)} = (A\mathbf{x}_3) \cdot \mathbf{x}_3 = (A\mathbf{x}_3)^T \mathbf{x}_3 \approx \begin{bmatrix} 3.54108 & 3.52976 \end{bmatrix} \begin{bmatrix} 0.71274 \\ 0.70143 \end{bmatrix} \approx 4.99974$$

$$\lambda^{(4)} = (A\mathbf{x}_4) \cdot \mathbf{x}_4 = (A\mathbf{x}_4)^T \mathbf{x}_4 \approx \begin{bmatrix} 3.53666 & 3.53440 \end{bmatrix} \begin{bmatrix} 0.70824 \\ 0.70597 \end{bmatrix} \approx 4.99999$$

$$\lambda^{(5)} = (A\mathbf{x}_5) \cdot \mathbf{x}_5 = (A\mathbf{x}_5)^T \mathbf{x}_5 \approx \begin{bmatrix} 3.53576 & 3.53531 \end{bmatrix} \begin{bmatrix} 0.70733 \\ 0.70688 \end{bmatrix} \approx 5.00000$$

Thus, $\lambda^{(5)}$ approximates the dominant eigenvalue to five decimal place accuracy and $\mathbf{x}_5$ approximates the dominant eigenvector in (7) to three decimal place accuracy. ◀

THEOREM 9.2.2 *Let A be a symmetric $n \times n$ matrix that has a positive dominant eigenvalue λ. If $\mathbf{x}_0$ is a nonzero vector in R^n that is not orthogonal to the eigenspace corresponding to λ, then the sequence*

$$\mathbf{x}_0, \quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)}, \quad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)}, \ldots, \quad \mathbf{x}_k = \frac{A\mathbf{x}_{k-1}}{\max(A\mathbf{x}_{k-1})}, \ldots \tag{8}$$

converges to an eigenvector corresponding to λ, and the sequence

$$\frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1}, \quad \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2}, \quad \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3}, \ldots, \quad \frac{A\mathbf{x}_k \cdot \mathbf{x}_k}{\mathbf{x}_k \cdot \mathbf{x}_k}, \ldots \tag{9}$$

converges to λ.

Remark In the exercises we will ask you to show that (8) can be written in the alternative form

$$\mathbf{x}_0, \quad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)}, \quad \mathbf{x}_2 = \frac{A^2\mathbf{x}_0}{\max(A^2\mathbf{x}_0)}, \ldots, \quad \mathbf{x}_k = \frac{A^k\mathbf{x}_0}{\max(A^k\mathbf{x}_0)}, \ldots \tag{10}$$

which expresses the iterates in terms of the initial vector $\mathbf{x}_0$.

The Power Method with Maximum Entry Scaling

Step 0. Choose an arbitrary nonzero vector $\mathbf{x}_0$.

Step 1. Compute $A\mathbf{x}_0$ and multiply it by the factor $1/\max(A\mathbf{x}_0)$ to obtain the first approximation $\mathbf{x}_1$ to a dominant eigenvector. Compute the Rayleigh quotient of $\mathbf{x}_1$ to obtain the first approximation to the dominant eigenvalue.

Step 2. Compute $A\mathbf{x}_1$ and scale it by the factor $1/\max(A\mathbf{x}_1)$ to obtain the second approximation $\mathbf{x}_2$ to a dominant eigenvector. Compute the Rayleigh quotient of $\mathbf{x}_2$ to obtain the second approximation to the dominant eigenvalue.

Step 3. Compute $A\mathbf{x}_2$ and scale it by the factor $1/\max(A\mathbf{x}_2)$ to obtain the third approximation $\mathbf{x}_3$ to a dominant eigenvector. Compute the Rayleigh quotient of $\mathbf{x}_3$ to obtain the third approximation to the dominant eigenvalue.

Continuing in this way will generate a sequence of better and better approximations to the dominant eigenvalue and a corresponding eigenvector.

- The method for finding the largest eigen value in magnitude and the corresponding eigen vector of the eigen value problem

$$Ax = \lambda x,$$

is called the power method.

- Let $\lambda_1,\ \lambda_2, \lambda_3, \ldots, \lambda_n$ are distinct eigen values such that

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \cdots > |\lambda_n|$$

- Let $v_1, v_2, \ldots, v_n$ be the eigen vectors corresponding to the eigen values $\lambda_1,\ \lambda_2, \lambda_3, \ldots, \lambda_n$, respectively. The method is applicable if a complete system of n linearly independent eigen vectors exist, even though some of the eigen values $\lambda_2, \lambda_3, \ldots, \lambda_n$, may not be distinct.

- The eigen value λ_1 is obtained as the ratio of the corresponding components of $A^{k+1}v$ and $A^k v$. That is,

$$\lambda_1 = \lim_{k\to\infty} \frac{(A^{k+1}v)_r}{(A^k v)_r},\ r = 1, 2, \ldots, n$$

  where the suffix r denotes the r^{th} component of the vector.
- When do we stop the iteration: The iterations are stopped when all the magnitudes of the differences of the ratios are less than the given error tolerance.

For any given matrix we write the characteristic equation (1.50), expand it and find the roots $\lambda_1, \lambda_2, ..., \lambda_n$, which are the *eigen values*. The roots may be real, repeated or complex. Let $\mathbf{x}_i$ be the solution of the system of the homogeneous equations (1.49), corresponding to the eigen value λ_i. These vectors $\mathbf{x}_i$, $i = 1, 2, \ldots, n$ are called the *eigen vectors* of the system.

There are several methods for finding the eigen values of a general matrix or a symmetric matrix. In the syllabus, only the power method for finding the largest eigen value in magnitude of a matrix and the corresponding eigen vector, is included.

1.3.2 Power Method

The method for finding the largest eigen value in magnitude and the corresponding eigen vector of the eigen value problem $\mathbf{Ax} = \lambda\, \mathbf{x}$, is called the power method.

What is the importance of this method? Let us re-look at the Remarks 20 and 23. The necessary and sufficient condition for convergence of the Gauss-Jacobi and Gauss-Seidel iteration methods is that the spectral radius of the iteration matrix $\mathbf{H}$ is less than one unit, that is, $\rho(\mathbf{H}) < 1$, where $\rho(\mathbf{H})$ is the largest eigen value in magnitude of $\mathbf{H}$. If we write the matrix formulations of the methods, then we know $\mathbf{H}$. We can now find the largest eigen value in magnitude of $\mathbf{H}$, which determines whether the methods converge or not.

We assume that $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct eigen values such that

$$| \lambda_1 | > | \lambda_2 | > \ldots > | \lambda_n |. \tag{1.52}$$

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, respectively. The method is applicable if a complete system of n linearly independent eigen vectors exist, even though some of the eigen values $\lambda_2, \lambda_3, ..., \lambda_n$, may not be distinct. The n linearly independent eigen vectors form an n-dimensional vector space. Any vector $\mathbf{v}$ in this space of eigen vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ can be written as a linear combination of these vectors. That is,

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\,\mathbf{v}_n. \tag{1.53}$$

Premultiplying by $\mathbf{A}$ and substituting $\mathbf{Av}_1 = \lambda_1\mathbf{v}_1$, $\mathbf{Av}_2 = \lambda_2\mathbf{v}_2, ..., \mathbf{Av}_n = \lambda_n\mathbf{v}_n$, we get

$$\mathbf{Av} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots + c_n\lambda_n\mathbf{v}_n$$

$$= \lambda_1 \left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)\mathbf{v}_n \right].$$

Premultiplying repeatedly by $\mathbf{A}$ and simplifying, we get

$$\mathbf{A}^2\mathbf{v} = \lambda_1^2 \left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^2\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^2\mathbf{v}_n \right]$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\mathbf{A}^k\mathbf{v} = \lambda_1^k \left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n \right]. \tag{1.54}$$

$$\mathbf{A}^{k+1}\mathbf{v} = \lambda_1^{k+1}\left[c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1}\mathbf{v}_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1}\mathbf{v}_n\right]. \tag{1.55}$$

As $k \to \infty$, the right hand sides of (1.54) and (1.55) tend to $\lambda_1^k c_1\mathbf{v}_1$ and $\lambda_1^{k+1} c_1\mathbf{v}_1$, since $|\lambda_i/\lambda_1| < 1$, $i = 2, 3, \ldots, n$. Both the right hand side vectors in (1.54), (1.55)

$$[c_1\mathbf{v}_1 + c_2(\lambda_2/\lambda_1)^k\,\mathbf{v}_2 + \ldots + c_n\,(\lambda_n/\lambda_1)^k\,\mathbf{v}_n],$$

and

$$[c_1\mathbf{v}_1 + c_2(\lambda_2/\lambda_1)^{k+1}\,\mathbf{v}_2 + \ldots + c_n\,(\lambda_n/\lambda_1)^{k+1}\,\mathbf{v}_n]$$

tend to $c_1\mathbf{v}_1$, which is the eigen vector corresponding to λ_1. The eigen value λ_1 is obtained as the ratio of the corresponding components of $\mathbf{A}^{k+1}\mathbf{v}$ and $\mathbf{A}^k$v. That is,

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{A}^{k+1}\mathbf{v})_r}{(\mathbf{A}^k\mathbf{v})_r}, \quad r = 1, 2, 3, \ldots, n \tag{1.56}$$

where the suffix r denotes the rth component of the vector. Therefore, we obtain n ratios, all of them tending to the same value, which is the largest eigen value in magnitude, $|\lambda_1|$.

When do we stop the iteration The iterations are stopped when all the magnitudes of the differences of the ratios are less than the given error tolerance.

Remark 24 The choice of the initial approximation vector $\mathbf{v}_0$ is important. If no suitable approximation is available, we can choose $\mathbf{v}_0$ with all its components as one unit, that is, $\mathbf{v}_0 = [1, 1, 1, \ldots, 1]^T$. However, this initial approximation to the vector should be non-orthogonal to $\mathbf{v}_1$.

Remark 25 Faster convergence is obtained when $|\lambda_2| << |\lambda_1|$.

As $k \to \infty$, premultiplication each time by **A,** may introduce round-off errors. In order to keep the round-off errors under control, we normalize the vector before premultiplying by **A.** The normalization that we use is to make the largest element in magnitude as unity. If we use this normalization, a simple algorithm for the power method can be written as follows.

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k, \tag{1.57}$$

$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1} \tag{1.58}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. Now, the largest element in magnitude of $\mathbf{v}_{k+1}$ is one unit. Then (1.56) can be written as

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3, \ldots., n \tag{1.59}$$

and $\mathbf{v}_{k+1}$ is the required eigen vector.

Remark 26 It may be noted that as $k \to \infty$, m_{k+1} also gives $|\lambda_1|$.

Remark 27 Power method gives the largest eigen value in magnitude. If the sign of the eigen value is required, then we substitute this value in the determinant $|\mathbf{A} - \lambda_1\mathbf{I}|$ and find its value. If this value is approximately zero, then the eigen value is of positive sign. Otherwise, it is of negative sign.

Example 1.24 *Determine the dominant eigen value of* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *by power method.*

(*A.U. Nov/Dec 2004*)

Solution Let the initial approximation to the eigen vector be $\mathbf{v}_0$. Then, the power method is given by

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k,$$
$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. The dominant eigen value in magnitude is given by

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3, ..., n$$

and $\mathbf{v}_{k+1}$ is the required eigen vector.

Let $\mathbf{v}_0 = [1 \quad 1]^T$. We have the following results.

$$\mathbf{y}_1 = \mathbf{A}\mathbf{v}_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \quad m_1 = 7, \quad \mathbf{v}_1 = \frac{\mathbf{y}_1}{m_1} = \frac{1}{7}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 0.42857 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_2 = \mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.42857 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.42857 \\ 5.28571 \end{bmatrix}, \; m_2 = 5.28571,$$

$$\mathbf{v}_2 = \frac{\mathbf{y}_2}{m_2} = \frac{1}{5.28571}\begin{bmatrix} 2.42857 \\ 5.28571 \end{bmatrix} = \begin{bmatrix} 0.45946 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_3 = \mathbf{A}\mathbf{v}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45946 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45946 \\ 5.37838 \end{bmatrix}, \; m_3 = 5.37838,$$

$$\mathbf{v}_3 = \frac{\mathbf{y}_3}{m_3} = \frac{1}{5.37838}\begin{bmatrix} 2.45946 \\ 5.37838 \end{bmatrix} = \begin{bmatrix} 0.45729 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_4 = \mathbf{A}\mathbf{v}_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45729 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45729 \\ 5.37187 \end{bmatrix}, \; m_4 = 5.37187,$$

$$\mathbf{v}_4 = \frac{\mathbf{y}_4}{m_4} = \frac{1}{5.37187}\begin{bmatrix} 2.45729 \\ 5.37187 \end{bmatrix} = \begin{bmatrix} 0.45744 \\ 1 \end{bmatrix}$$

$$\mathbf{y}_5 = \mathbf{A}\mathbf{v}_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45744 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45744 \\ 5.37232 \end{bmatrix}, \; m_4 = 5.37232,$$

$$\mathbf{v}_5 = \frac{\mathbf{y}_5}{m_5} = \frac{1}{5.37232}\begin{bmatrix} 2.45744 \\ 5.37232 \end{bmatrix} = \begin{bmatrix} 0.45743 \\ 1 \end{bmatrix}.$$

$$\mathbf{y}_6 = \mathbf{A}\mathbf{v}_5 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45743 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45743 \\ 5.37229 \end{bmatrix}.$$

Now, we find the ratios

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r} \qquad r = 1, 2.$$

We obtain the ratios as

$$\frac{2.45743}{0.45743} = 5.37225, \quad 5.37229.$$

The magnitude of the error between the ratios is $| 5.37225 - 5.37229 | = 0.00004 < 0.00005$. Hence, the dominant eigen value, correct to four decimal places is 5.3722.

Example 1.25 *Determine the numerically largest eigen value and the corresponding eigen vector of the following matrix, using the power method.*

$$\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \qquad \text{(A.U. May/June 2006)}$$

Solution Let the initial approximation to the eigen vector be $\mathbf{v_0}$. Then, the power method is given by

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k,$$
$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. The dominant eigen value in magnitude is given by

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \qquad r = 1, 2, 3, \ldots, n$$

and $\mathbf{v}_{k+1}$ is the required eigen vector.

Let the initial approximation to the eigen vector be $\mathbf{v_0} = [1, 1, 1]^T$. We have the following results.

$$\mathbf{y_1} = \mathbf{A}\mathbf{v_0} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 28 \\ 4 \\ -2 \end{bmatrix}, \; m_1 = 28,$$

$$\mathbf{v}_1 = \frac{1}{m_1} \mathbf{y}_1 = \frac{1}{28} \begin{bmatrix} 28 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.14286 \\ -0.07143 \end{bmatrix}.$$

$$\mathbf{y}_2 = \mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.14286 \\ -0.07143 \end{bmatrix} = \begin{bmatrix} 25.0000 \\ 1.42858 \\ 2.28572 \end{bmatrix}, \; m_2 = 25.0,$$

$$\mathbf{v}_2 = \frac{1}{m_2} \mathbf{y_2} = \frac{1}{25.0} \begin{bmatrix} 25.00000 \\ 1.14286 \\ 2.28572 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.05714 \\ 0.09143 \end{bmatrix},$$

$$\mathbf{y}_3 = \mathbf{A}\mathbf{v_2} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.05714 \\ 0.09143 \end{bmatrix} = \begin{bmatrix} 25.24000 \\ 1.17142 \\ 1.63428 \end{bmatrix}, \; m_3 = 25.24,$$

$$\mathbf{v}_3 = \frac{1}{m_3}\,\mathbf{y}_3 = \frac{1}{25.24}\begin{bmatrix}25.24000\\1.17142\\1.63428\end{bmatrix} = \begin{bmatrix}1\\0.04641\\0.06475\end{bmatrix},$$

$$\mathbf{y}_4 = \mathbf{A}\mathbf{v}_3 = \begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}\begin{bmatrix}1\\0.04641\\0.06475\end{bmatrix} = \begin{bmatrix}25.17591\\1.13923\\1.74100\end{bmatrix},\ m_4 = 25.17591,$$

$$\mathbf{v}_4 = \frac{1}{m_4}\,\mathbf{y}_4 = \frac{1}{25.17591}\begin{bmatrix}25.17591\\1.13923\\1.74100\end{bmatrix} = \begin{bmatrix}1\\0.04525\\0.06915\end{bmatrix},$$

$$\mathbf{y}_5 = \mathbf{A}\mathbf{v}_4 = \begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}\begin{bmatrix}1\\0.04525\\0.06915\end{bmatrix} = \begin{bmatrix}25.18355\\1.13575\\1.72340\end{bmatrix},\ m_5 = 25.18355,$$

$$\mathbf{v}_5 = \frac{1}{m_5}\,\mathbf{y}_5 = \frac{1}{25.18355}\begin{bmatrix}25.18355\\1.13575\\1.72340\end{bmatrix} = \begin{bmatrix}1\\0.04510\\0.06843\end{bmatrix},$$

$$\mathbf{y}_6 = \mathbf{A}\mathbf{v}_5 = \begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}\begin{bmatrix}1\\0.04510\\0.06843\end{bmatrix} = \begin{bmatrix}25.18196\\1.13530\\1.72628\end{bmatrix},\ m_6 = 25.18196,$$

$$\mathbf{v}_6 = \frac{1}{m_6}\,\mathbf{y}_6 = \frac{1}{25.18196}\begin{bmatrix}25.18196\\1.13530\\1.72628\end{bmatrix} = \begin{bmatrix}1\\0.04508\\0.06855\end{bmatrix},$$

$$\mathbf{y}_7 = \mathbf{A}\mathbf{v}_6 = \begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}\begin{bmatrix}1\\0.04508\\0.06855\end{bmatrix} = \begin{bmatrix}25.18218\\1.13524\\1.72580\end{bmatrix},\ m_7 = 25.18218,$$

$$\mathbf{v}_7 = \frac{1}{m_7}\,\mathbf{y}_7 = \frac{1}{25.18218}\begin{bmatrix}25.18218\\1.13524\\1.72580\end{bmatrix} = \begin{bmatrix}1\\0.04508\\0.06853\end{bmatrix},$$

$$\mathbf{y}_8 = \mathbf{A}\mathbf{v}_7 = \begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}\begin{bmatrix}1\\0.04508\\0.06853\end{bmatrix} = \begin{bmatrix}25.18214\\1.13524\\1.72588\end{bmatrix},\ m_8 = 25.18214.$$

Now, we find the ratios

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3.$$

We obtain the ratios as

$$25.18214,\ \frac{1.13524}{0.04508} = 25.18279,\ \frac{1.72588}{0.06853} = 25.18430.$$

The magnitudes of the errors of the differences of these ratios are 0.00065, 0.00216, 0.00151, which are less than 0.005. Hence, the results are correct to two decimal places. Therefore, the largest eigen value in magnitude is $|\ \lambda_1\ | = 25.18$.

The corresponding eigen vector is $\mathbf{v}_8$,

$$\mathbf{v}_8 = \frac{1}{m_8}\ \mathbf{y}_8 = \frac{1}{25.18214}\begin{bmatrix} 25.18214 \\ 1.13524 \\ 1.72588 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.04508 \\ 0.06854 \end{bmatrix}.$$

In Remark 26, we have noted that as $k \to \infty$, m_{k+1} also gives $|\ \lambda_1\ |$. We find that this statement is true since $|\ m_8 - m_7\ | = |\ 25.18214 - 25.18220\ | = 0.00006$.

If we require the sign of the eigen value, we substitute λ_1 in the characteristic equation. In the present problem, we find that $|\ \mathbf{A} - 25.18\ \mathbf{I}\ | = 1.4018$, while $|\ \mathbf{A} + 25.18\ \mathbf{I}\ |$ is very large. Therefore, the required eigen value is 25.18.

REVIEW QUESTIONS

1. When do we use the power method?

Solution We use the power method to find the largest eigen value in magnitude and the corresponding eigen vector of a matrix **A**.

2. Describe the power method.

Solution Power method can be written as follows.

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k,$$
$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. Now, the largest element in magnitude of $\mathbf{v}_{k+1}$ is one unit. The largest eigen value in magnitude is given by

$$\lambda_1 = \lim_{k \to \infty} \frac{(\mathbf{y}_{k+1})_r}{(\mathbf{v}_k)_r}, \quad r = 1, 2, 3, \ldots, n$$

and $\mathbf{v}_{k+1}$ is the required eigen vector. All the ratios in the above equation tend to the same number.

3. When do we stop the iterations in power method?

Solution Power method can be written as follows.

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{v}_k,$$
$$\mathbf{v}_{k+1} = \mathbf{y}_{k+1}/m_{k+1}$$

where m_{k+1} is the largest element in magnitude of $\mathbf{y}_{k+1}$. Now, the largest element in magnitude of $\mathbf{v}_{k+1}$ is one unit. The largest eigen value is given by