Gauss-Seidel method:

In this method we begin exactly as with the Jacobi method by rearranging the equations, solving each equation for the variable whose coefficient is dominant in terms of the others. We proceed to improve each x value in turn, using always the most recent approximations of the other variables. the rate of convergence is more rapid for the Jacobi method.

Suppose we have a system equations of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&-----------\\
&-----------\\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}$$

Rearranging the equations,

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n\right)$$
$$x_2 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n\right)$$
$$:::::$$
$$x_n = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n,n-1}x_{n-1}\right)$$

From the first equation, we will find the value of x_1 and this recent value of x_1 will be used for the second equation to find x_2. In general,

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)} - \dots - a_{1n}x_n^{(n)}\right)$$
$$x_2^{(n+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(n+1)} - a_{23}x_3^{(n)} - \dots - a_{2n}x_n^{(n)}\right)$$
$$:::::$$
$$x_n^{(n+1)} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(n+1)} - a_{n2}x_2^{(n+1)} - \dots - a_{n,n-1}x_{n-1}^{(n)}\right)$$

Example 1: Use the Jacobi method to approximate the solution of the following system of linear equations (chose the initial values $x_1 = x_2 = x_3 = 0$)

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ x_1 + 2x_2 - 5x_3 &= -1 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \end{aligned}$$

Solution: Before we begin our iteration method, we must first reorder the equations so that the coefficient matrix is diagonally dominant.

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \\ -2x_1 + 7x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 - 5x_3 &= -1 \end{aligned}$$

The iterative methods depend on the rearrangement of the equations in the following manner.

$$\begin{aligned} x_1 &= \frac{11}{6} + \frac{2}{6}x_2 - \frac{1}{6}x_3 = 1.8333 + 0.3333x_2 - 0.1667x_3 \\ x_2 &= \frac{5}{7} + \frac{2}{7}x_1 + \frac{2}{7}x_3 = 0.7143 + 0.2857x_1 - 0.2857x_3 \\ x_3 &= \frac{1}{5} + \frac{1}{5}x_1 + \frac{2}{5}x_2 = 0.2000 + 0.2000x_1 + 0.4000x_2 \end{aligned}$$

The Gauss-Seidel iteration may become,

$$\begin{aligned} x_1^{(n+1)} &= 1.8333 + 0.3333x_2^{(n)} - 0.1667x_3^{(n)} \\ x_2^{(n+1)} &= 0.7143 + 0.2857x_1^{(n+1)} - 0.2857x_3^{(n)} \\ x_3^{(n+1)} &= 0.2000 + 0.2000x_1^{(n+1)} + 0.4000x_2^{(n+1)} \end{aligned}$$

First iteration:

$$\begin{aligned} x_1^{(1)} &= 1.8333 + 0.3333x_2^{(0)} - 0.1667x_3^{(0)} = 1.8333 + 0.3333(0) - 0.1667(0) = 1.8333 \\ x_2^{(1)} &= 0.7143 + 0.2857x_1^{(1)} - 0.2857x_3^{(0)} = 0.7143 + 0.2857(1.8333) - 0.2857(0) = 1.238 \\ x_3^{(1)} &= 0.2000 + 0.2000x_1^{(1)} + 0.4000x_2^{(1)} = 0.2000 + 0.2000(1.8333) + 0.4000(1.238) = 1.062 \end{aligned}$$

Continuing this procedure, we obtain the sequence of approximations shown below.

	0^{th}	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
x_1	0	1.833	2.069	1.998	1.9999	2.0000
x_2	0	1.238	1.002	0.995	1.0000	1.0000
x_3	0	1.062	1.015	0.998	1.0000	1.0000

Note that for the same system the Jacobi methods takes EIGHT iteration to get these values.

Note : As we discussed earlier, the matrix is expressed of the form $A = L + D + U$. The approach is very similar to that used in Jacobi method. The only difference is that we use new approximations to the entries of X as soon as available. And the arranging $(L + D + U)X = B$ slightly differently from what we did for Jacobi.

$$(L + D + U)X = B$$

$$(D + L)X = -UX + B$$

We use this equation as the motivation to define the iterative process

$$(D + L)X^{(n+1)} = -UX^{(n)} + B$$

Example 2: Use the Gauss-Seidel iteration to approximate the solution of the system

$$\begin{pmatrix} 8 & 2 & 4 \\ 3 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$. Use the initial guess $X^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$

Solution: Here $D + L = \begin{pmatrix} 8 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 1 & 4 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

First iteration: $(D + L)X^{(1)} = -UX^{(0)} + B$

$$\begin{pmatrix} 8 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 0 & 0 \\ 3 & 5 & 0 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

Taking this information row by row we see that

$$8x_1^{(1)} = -16;\ 3x_1^{(1)} + 5x_2^{(1)} = 4 \text{ and } 2x_1^{(1)} + x_2^{(1)} + 4x_3^{(1)} = -12$$

Thus the first Gauss-Seidel iteration gives us $X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2.5 \end{pmatrix}$

Note that, the new approximations to x_1 and x_2 were used immediately after they were found.

Continuing this procedure, we obtain the sequence of approximations shown below.

	n=0	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=7*
$x_1^{(n)}$	0	-2	-1.25	-1.0687	-1.0195	-1.0056	-1.0016	-1.0005
$x_2^{(n)}$	0	2	2.05	2.0187	2.0058	2.0017	2.0005	2.0001
$x_3^{(n)}$	0	-2.5	-2.8875	-2.9703	-2.9917	-2.9976	-2.9993	-2.9998

	n=8	*n=9*
$x_1^{(n)}$	-1.0001	-1.0000
$x_2^{(n)}$	2.0000	2.0000
$x_3^{(n)}$	-2.9999	-3.0000

The Gauss-Seidel iteration method has converged to FOUR decimal points in NINE iterations. It took the Jacobi method almost 40 iterations to achieve this.