Finite differences and Interpolation

Finite differences: Let $f(x)$ be the function and $f_0, f_1, f_2, \ldots, f_n$ be the values of $f(x)$ corresponding to the values of x. Here the independent variable x is called the argument and the corresponding dependent value $f(x)$ is called the entry.

Forward difference: Consider the following table of values:

$$\begin{array}{lcccccccccc} x: & x_0 & x_1 & x_2 & .. & .. & .. & .. & x_{n-1} & x_n \\ f(x): & f_0 & f_1 & f_2 & .. & .. & .. & .. & f_{n-1} & f_n \end{array}$$

If we subtract each value of $f(x)$, (except f_0), we get

$$f_1 - f_0,\quad f_2 - f_1,\quad f_3 - f_2,\quad ..\quad ..\quad ..\quad ..,\quad f_n - f_{n-1}$$

These are called the first differences of $f(x)$, denoted by Δf and defined as

$$\Delta f_r = f_{r+1} - f_r$$

Here the operator Δ (read as delta) is called forward difference operator.

Note: $\Delta^2 f_r = \Delta(\Delta f_r) = f_{r+1} - f_r = \Delta(f_{r+1}) - \Delta(f_r)$

$$= (f_{r+2} - f_{r+1})_r - (f_{r+1} - f_r) = f_{r+2} - 2f_{r+1} + f_r$$

In particular, $\Delta^2 f_0 = f_2 - 2f_1 + f_0$

Here Δ^2 is called second order forward difference operator. In the same way, the higher order differences can be found.

Forward difference table: In many applications displayed, it will be convenient if the successive differences of a function are prominently displayed. This is usually done by constructing a difference table as follows:

$$\begin{array}{cccccc} x_n & f_n & \Delta & \Delta^2 & \Delta^3 & \Delta^4 \\ x_0 & f_0 & & & & \\ & & \Delta f_0 & & & \\ x_1 & f_1 & & \Delta^2 f_0 & & \\ & & \Delta f_1 & & \Delta^3 f_0 & \\ x_2 & f_2 & & \Delta^2 f_1 & & \Delta^4 f_0 \\ & & \Delta f_2 & & \Delta^3 f_1 & \\ x_3 & f_3 & & \Delta^2 f_2 & & \\ & & \Delta f_3 & & & \\ x_4 & f_4 & & & & \end{array}$$

The first term in the table f_0 is called the leading term and the differences $\Delta f_0, \Delta^2 f_0, \ldots\ldots$ are called leading differences.

Note: Consider the following table:

x_n	f_n	Δf_n	$\Delta^2 f_n$	$\Delta^3 f_n$	$\Delta^4 f_n$	$\Delta^5 f_n$	$\Delta^6 f_n$
0	1						
		7					
1	8		12				
		19		6			
2	27		18		0		
		37		6		0	
3	64		24		0		0
		61		6		0	
4	125		30		0		
		91		6			
5	216		36				
		127					
6	343						

Here $\Delta^2 f_0 = f_2 - 2f_1 + f_0 = 27 - 16 + 1 = 12$

It is clear that, if we have seven data points, $\Delta^6 f_0$ will be a constant. In general, if we have $n+1$ data points, we can almost go upto $\Delta^n f_0$ difference.

Backward difference: Consider the following table of values:

$$\begin{array}{lcccccccccc} x: & x_0 & x_1 & x_2 & .. & .. & .. & .. & x_{n-1} & x_n \\ f(x): & f_0 & f_1 & f_2 & .. & .. & .. & .. & f_{n-1} & f_n \end{array}$$

If the differences $f_1 - f_0, \quad f_2 - f_1, \quad f_3 - f_2, \quad .. \quad .. \quad .. \quad .., \quad f_n - f_{n-1}$ represented by $\nabla y_1, \nabla y_2, \nabla y_3, \ldots \nabla y_n$ are known as backward difference. In general, we write

$$\nabla f_r = f_r - f_{r-1}$$

Here the operator ∇ (read as nabla) is called backward difference operator.

Note: $\nabla^2 f_r = \nabla(\nabla f_r) = \nabla(f_r) - \nabla(f_{r-1}) = (f_r - f_{r-1}) - (f_{r-1} - f_{r-2}) = f_r - 2f_{r-1} + f_{r-2}$

Note: The backward difference table as follows:

x_n	f_n	∇	∇^2	∇^3	∇^4
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

Note: Consider the following table:

x_n	f_n	Δf_n	$\Delta^2 f_n$	$\Delta^3 f_n$	$\Delta^4 f_n$	$\Delta^5 f_n$	$\Delta^6 f_n$
0	1						
		7					
1	8		12				
		19		6			
2	27		18		0		
		37		6		0	
3	64		24		0		0
		61		6		0	
4	125		30		0		
		91		6			
5	216		36				
		127					
6	343						

Here $\nabla^2 f_6 = f_6 - 2f_5 + f_4 = 343 - 2(216) + 125 = 36$

Central difference: In some applications, the central difference is found to be more convenient to present the successive differences of a function. The central difference operator δ is defined by

$$\delta f_r = f_{r+\frac{h}{2}} - f_{r-\frac{h}{2}}$$

Central difference table:

x_n	f_n	δ	δ^2	δ^3	δ^4
x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

Shift operator or translation operator (E) : The shift operator E defined as

$$Ef_r = f_{r+1}; \qquad E^2 f_r = f_{r+2}$$

In general, $E^n f_r = f_{r+n}$ or $f_n = E^n f_0$

In the same way, the inverse shift operator E^{-1} defined as,

$$E^{-1} f_r = f_{r-1}$$

In general, $E^{-n} f_r = f_{r-n}$

Some relations between the operator:

1. $\Delta y_n = y_{n+1} - y_n = Ey_n - y_n = (E-1)y_n$

   $\Delta = E - 1$ or $E = \Delta + 1$

Note that, $\Delta^2 y_0 = (E-1)^2 y_0 = E^2 y_0 - 2Ey_0 + y_0 = y_2 - 2y_1 + y_0$

2. $\nabla y_n = y_n - y_{n-1} = y_n - E^{-1} y_n = \left(1 - E^{-1}\right) y_n$

   $\nabla = 1 - E^{-1}$ or $E = (1-\Delta)^{-1}$

3. $\delta f_r = f_{r+\frac{h}{2}} - f_{r-\frac{h}{2}} = E^{1/2} f_r - E^{-1/2} f_r = \left(E^{1/2} - E^{-1/2}\right) f_r$

   $\delta = E^{1/2} - E^{-1/2}$

   $\delta = E^{-1/2}(E-1) = E^{-1/2}\Delta$

   $\delta = E^{1/2}\left(1 - E^{-1}\right) = E^{1/2}\nabla$

4. $y(x+h) = y(x) + hy'(x) + \dfrac{h^2}{2!} y''(x) + ---$ (Taylor series expansion)

   $Ey(x) = \left(1 + hD + \dfrac{h^2}{2!} D^2 + ---\right) y(x)$, where D is the differential operator

   $E = e^{hD}$

Example: Find the 7th term of the sequence 2,7,14,24,37,54 and also find the general term.

Solution: Prepare the forward table as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	2					
		5				
1	7		2			
		7		1		
2	14		3		-1	
		10		0		2
3	24		3		1	
		13		1		
4	37		4			
		17				
5	54					

$$y_6 = (1+\Delta)^6 y_0$$

$$y_6 = y_0 + (6,c,1)\Delta y_0 + (6,c,2)\Delta^2 y_0 + (6,c,3)\Delta^3 y_0 + (6,c,4)\Delta^4 y_0$$
$$+ (6,c,5)\Delta^5 y_0 + (6,c,6)\Delta^6 y_0$$

$$y_6 = 2 + 6(5) + 15(2) + 20(1) + 15(-1) + 6(2) = 79$$

We can find the general term as follows:

$$y_n = y_0 + (n,c,1)\Delta y_0 + (n,c,2)\Delta^2 y_0 + (n,c,3)\Delta^3 y_0 + (n,c,4)\Delta^4 y_0$$
$$+ (n,c,5)\Delta^5 y_0 + (n,c,6)\Delta^6 y_0 + ...$$

$$y_n = 2 + n(5) + \frac{n(n-1)}{2!}(2) + \frac{n(n-1)(n-2)}{3!}(1)$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}(-1) + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}(2) + 0$$

Example: Find y_{-1} if $y_0 = 2, y_1 = 9, y_2 = 28, y_3 = 65, y_4 = 126, y_5 = 217$.

Solution: Prepare the backward table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	2				
		7			
1	9		12		
		19		6	
2	28		18		0
		37		6	
3	65		24		0
		61		6	
4	126		30		
		91			
5	217				

Now, $y_{-1} = y_{5-6} = (1-\nabla)^6 y_5$

$$y_{-1} = y_5 - (6,c,1)\nabla y_5 + (6,c,2)\nabla^2 y_5 - (6,c,3)\nabla^3 y_5 + (6,c,4)\nabla^4 y_5 - (6,c,5)\nabla^5 y_5$$

$$y_{-1} = 217 - 6(91) + 15(30) - 20(6) + 0 = 1$$

Interpolation: Interpolation is the process of finding intermediate values of a function which is not explicitly known from a given set of tabular values of the function. suppose that the following table represents a set a corresponding values of x and y.

$$\begin{array}{llllllll} x: & x_0 & x_1 & x_2 & x_3 & .. & .. & x_n \\ y: & y_0 & y_1 & y_2 & y_3 & .. & .. & y_n \end{array}$$

The process of finding $y = y_i$ corresponding to $x = x_i$ where $x_0 < x_i < x_n$ is called interpolation.

Note: If $x < x_0$ or $x_n < x_i$, then the process of finding $y = y_i$ corresponding to $x = x_i$ is called extrapolation.

Note: We can find interpolation for two different kinds of arguments.

1. Equally spaced arguments
2. Unequally spaced arguments

Newton's forward and backward difference formulas and Stirling formula are comes under equally spaced arguments, whereas Newton's divided difference formula and Lagrange formula are coming under unequally spaced arguments.

Newton's interpolation formula for equal intervals:

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, ..., y_n$ corresponding to the values $x_0, x_1, x_2, ..., x_n$ of the independent variable x. Let the values of x be at equidistant intervals. That is $x_k = x_0 + kh$ be the values of $f(x)$ corresponding to the values of x.

$$\begin{array}{llllllll} x: & x_0 & x_1 & x_2 & x_3 & .. & .. & x_n \\ y: & y_0 & y_1 & y_2 & y_3 & .. & .. & y_n \end{array}$$

$$\begin{aligned} y_k &= E^k y_0 = (1+\Delta)^k y_0 \\ &= y_0 + k\Delta y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-3)}{3!}\Delta^3 y_0 + ... + \Delta^k y_0 \qquad (1) \end{aligned}$$

Since the quantities x_0, x_1, … are equally spread, then

$$x_k = x_0 + kh$$

$$k = \frac{x_k - x_0}{h} = X \text{ (say)}$$

$$x_k = x_0 + hX$$

Equation (1) becomes,

$$y_k = y_0 + \left(\frac{x_k - x_0}{h}\right)\Delta y_0 + \left(\frac{x_k - x_0}{h}\right)\left(\frac{x_k - x_0}{h} - 1\right)\Delta^2 y_0 + ... + \Delta^k y_0$$

Therefore, for a general point $x_k = x$, we have $\frac{x - x_0}{h} = X$ and

$$y_k = y_0 + \left(\frac{x - x_0}{h}\right)\Delta y_0 + \left(\frac{x - x_0}{h}\right)\left(\frac{x - x_0}{h} - 1\right)\Delta^2 y_0 + ... + \Delta^k y_0$$

$$y_k = y_0 + X\Delta y_0 + X\left(X - 1\right)\Delta^2 y_0 + ... + \Delta^k y_0$$

This is known as Newton's forward interpolation formula.
Note: In the similarly way, we write the Newton's backward interpolation formula.

$$y_k = E^k y_0 = \left(1 - \nabla\right)^{-k} y_n$$

$$y_k = y_0 + k\nabla y_n + \frac{k\left(k+1\right)}{2!}\nabla^2 y_k + \frac{k\left(k-1\right)\left(k-3\right)}{3!}\nabla^3 y_k + ... + \nabla^k y_k$$

Problem: Consider the following date

x:	40	50	60	70	80	90
$\theta\left(Temp\right)$:	184	204	226	250	276	304

Find θ when $x = 43$ and $x = 84$.
Solution: From the difference table

x	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$
40	184			
		$20\left(=\Delta\theta_0\right)$		
50	204		2	
		$22\left(=\Delta\theta_1\right)$		0
60	226		2	
		$24\left(=\Delta\theta_2\right)$		0
70	250		2	
		$26\left(=\Delta\theta_3\right)$		0
80	276		2	
		$28\left(=\Delta\theta_4\right)$		
90	304			

Observation: This data fits a second order polynomial. At $x = 43$, use the interpolation formula

$$\theta_{43} = \theta_0 + X\Delta\theta_0 + \frac{X\left(X - 1\right)}{2}\Delta^2\theta_0$$

where, $X = \frac{x - x_0}{h} = \frac{43 - 40}{10} = 0.3$

$$\theta_{43} = 184 + \left(0.3\right)20 + \frac{(0.3)(0.3-1)}{2}2 = 189.79$$

Similarly, at $x = 84$

$$\theta_{84} = \theta_0 + X\Delta\theta_0 + \frac{X\left(X - 1\right)}{2}\Delta^2\theta_0, \text{ where } X = \frac{x - x_0}{h} = \frac{84 - 40}{10} = 4.4$$

$$\theta_{84} = 184 + \left(4.4\right)20 + \frac{(4.4)(4.4-1)}{2}2 = 286.96$$

We can find the polynomial which fits the above data in the following way.

$$\theta = \theta_0 + X\Delta\theta_0 + \frac{X(X-1)}{2}\Delta^2\theta_0, \text{ where } X = \frac{x - x_0}{h} = \frac{x-40}{10}$$

$$\theta = 184 + \left(\frac{x-40}{10}\right)20 + \frac{1}{2}\left(\frac{x-40}{10}\right)\left(\frac{x-40}{10} - 1\right)2$$

$$\theta(x) = 0.01x^2 + 1.1x + 184$$