Central difference interpolation:

In the previous section, we have discussed Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end respectively of tabulated values. Suppose we need the value of a function near the middle of a table. In that case, the Newton's forward and backward interpolation formulae are not suitable. To obtain more accuracy near the middle of a table, we have to use another difference and also develop the formulae which use differences close to the central of the table. The difference is called central difference and the formulae are known as central difference interpolation formulae.

Central differences and the central difference table:

The relation between central difference and shift operators is

$$\delta = E^{1/2} - E^{-1/2} \text{ or } \delta = E^{-1/2}\left(E-1\right) = \Delta E^{-1/2}$$

Let $y = f\left(x\right)$ be a function which takes the values $y_0, y_1, y_2, ..., y_n$ corresponding to the x values $x_0, x_0+h, x_0+2h, ..., x_0+nh$. When $x = x_0-h, x_0-2h, ...$ the corresponding values of y are $y_{-1}, y_{-2}, ...$.

Let us consider the following table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0-3h	y_{-3}						
		Δy_{-3}					
x_0-2h	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_0-h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_0+h	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_0+2h	y_2		$\Delta^2 y_1$				
		Δy_2					
x_0+3h	y_3						

This forward difference table can also be written in terms of central difference operator in the following way:

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0-3h	y_{-3}						
		$\delta y_{-5/2}$					
x_0-2h	y_{-2}		$\delta^2 y_{-2}$				
		$\delta y_{-3/2}$		$\delta^3 y_{-3/2}$			
x_0-h	y_{-1}		$\delta^2 y_{-1}$		$\delta^4 y_{-1}$		
		$\delta y_{-1/2}$		$\delta^3 y_{-1/2}$		$\delta^5 y_{-1/2}$	
x_0	y_0		$\delta^2 y_0$		$\delta^4 y_0$		$\delta^6 y_0$
		$\delta y_{1/2}$		$\delta^3 y_{1/2}$		$\delta^5 y_{1/2}$	
x_0+h	y_1		$\delta^2 y_1$		$\delta^4 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_0+2h	y_2		$\delta^2 y_2$				
		$\delta y_{5/2}$					
x_0+3h	y_3						

Gauss forward difference interpolation formulae:

The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-3)}{3!}\Delta^3 y_0 + \ldots + \Delta^p y_0$$

where $p = \dfrac{x-x_0}{h}$.

Note that from the above difference table, we have

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}, \text{ that is } \Delta^2 y_0 = \Delta^3 y_{-1} + \Delta^2 y_{-1}$$

Similarly, we get

$$\Delta^3 y_0 = \Delta^4 y_{-1} + \Delta^3 y_{-1} \text{ and } \Delta^4 y_0 = \Delta^5 y_{-1} + \Delta^4 y_{-1}$$

And also, $\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$ that is $\Delta^3 y_{-1} = \Delta^4 y_{-2} + \Delta^3 y_{-2}$

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$

Substituting for $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0 \ldots$ in Newton's forward interpolation formulae, we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots$$

This is known as gauss's forward interpolation formula.

Note1: In the central difference notation, we can write Gauss's forward interpolation formula in the following way

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{p(p-1)(p+1)}{3!}\delta^3 y_{1/2} + \frac{p(p-1)(p+1)(p-2)}{4!}\delta^4 y_0 + \ldots$$

Note 2: It employs odd differences just below the central line and even difference on the central line.

Note 3: This formula is used to interpolate the values of y for $0 < p < 1$ measured forwardly from the origin.

Gauss backward difference interpolation formulae:

The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-3)}{3!}\Delta^3 y_0 + ... + \Delta^p y_0$$

where $p = \dfrac{x - x_0}{h}$.

Note that from the above difference table, we have

$\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$, that is $\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$

Similarly, we get $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ and $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$

And also, $\Delta^4 y_{-2} = \Delta^3 y_{-1} - \Delta^3 y_{-2}$ that is $\Delta^3 y_{-1} = \Delta^4 y_{-2} + \Delta^3 y_{-2}$

Similarly, $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$

Substituting for Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0 \ldots$ in Newton's forward interpolation formula, we get

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-2} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

This is known as Gauss's backward interpolation formula.

Note1: In the central difference notation, we can write Gauss's backward interpolation formula in the following way:

$$y_p = y_0 + p\delta y_{-1/2} + \frac{p(p+1)}{2!}\delta^2 y_0 + \frac{p(p-1)(p+1)}{3!}\delta^3 y_{-1/2} + \frac{p(p-1)(p+1)(p-2)}{4!}\delta^4 y_0 + ...$$

Note 2: It employs odd differences above the central line and even difference on the central line.

Note 3: This formula is used to interpolate the values of y for $-1 < p < 0$ measured forwardly from the origin.

Stirling's formula: Consider Gauss's forward and backward interpolation formulas

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + ... \tag{1}$$

$$y_p = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-2} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + ... \tag{2}$$

Taking the mean of Eqn. (1) and Eqn. (2), we get

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + ...$$

This is known as Stirling's interpolation formula. Stirling's interpolation is very effective when the value p lies between $-0.25 < p < 0.25$

Note: The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data. The following rules will be found useful:

i) To find a tabulated value near the beginning of the table, use Newton's forward formula
ii) To find a value near the end of the table, use Newton's backward formula
iii) To find an interpolated value near the centre of the table, use Stirling's formula

Example 1: Apply the Gauss forward interpolation formula to find $y(25)$ for the following data:

x: 20	*24*	*28*	*32*
y: 2845	*3162*	*3544*	*3992*

Solution: Gauss forward interpolation formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + ...$$

where $p = \dfrac{x - x_0}{h}$.

Let us take $x = 24$ as the origin. Then $p = \frac{25-24}{4} = \frac{1}{4} = 0.25$

The difference table now becomes,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-1} = 20$	2854			
		308		
$x_0 = 24$	3162		74	
		382		-8
$x_1 = 28$	3544		66	
		448		
$x_2 = 32$	3992			

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1}$$

$$y(25) = 3162 + (0.25)(382) + \frac{(0.25)(0.25-1)}{2!}(74) + \frac{(0.25)(0.25-1)(0.25+1)}{3!}(-8)$$

$$y(25) = 3162 + 95.5 - 6.938 + 0.313 = 3250.875$$

Example 2: Use Gauss forward interpolation to evaluate y_{30}, given that $y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $y_{37} = 15.5154$

Solution: Let us take, $x = 29$ as the origin. Then, $p = \frac{30-29}{4} = \frac{1}{4} = 0.25$

The difference table now becomes,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 21$	18.4708				
		-0.6564			
$x_{-1} = 25$	17.8144		-0.0510		
		-0.7074		-0.0054	
$x_0 = 29$	17.1070		-0.0564		-0.0022
		-0.7638		-0.0076	
$x_1 = 33$	16.3432		-0.0640		
		-0.8278			
$x_2 = 37$	15.5154				

Gauss forward interpolation formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots$$

$$y_{30} = 17.1070 + (0.25)(-0.7632) + \frac{(0.25)(0.25-1)}{2!}(-0.0564) +$$
$$\frac{(0.25)(0.25-1)(0.25+1)}{3!}(-0.0076) + \frac{(0.25)(0.25-1)(0.25+1)(0.25-2)}{4!}(0.0022)$$

$$y_{30} = 17.1070 - 0.1908 + 0.0053 + 0.0003 - 0.0009 = 16.9209$$

Example 3: Apply the Gauss backward interpolation formula to find $y(32)$ for the following data:

x: 25	*30*	*35*	*40*
y: 0.2707	*0.3027*	*0.3386*	*0.3794*

Solution: Let us take, $x=35$ as origin. Then, $p=\dfrac{32-35}{5}=\dfrac{-3}{5}=-0.6$

Now, we prepare the following central difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-2}=25$	0.2707			
		0.032		
$x_{-1}=30$	0.3027		0.0039	
		0.0359		0.0010
$x_0=35$	0.3386		0.0049	
		0.0408		
$x_1=40$	0.3794			

Gauss backward interpolation formula,

$$y_p=y_0+p\Delta y_{-1}+\frac{p(p+1)}{2!}\Delta^2 y_{-1}+\frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-2}+....$$

$$y(32)=0.3386+(-0.6)(0.0359)+\frac{(-0.6)(-0.6+1)}{2!}(0.0049)+$$

$$\frac{(-0.6)(-0.6+1)}{2!}(0.0049)+\frac{(-0.6)(-0.6+1)(-0.6-1)}{3!}(0.0010)$$

$$y(32)=0.3386-0.0215-0.00069+0.00006=0.31657$$

Example 4: If $\cos 50^0=0.6428$, $\cos 51^0=0.6293$, $\cos 52^0=0.6157$, $\cos 53^0=0.6018$ and $\cos 54^0=0.5878$, find $\cos 50^0 42'$ by using Gauss backward interpolation formula.

Solution: Let us take, $x=52^0$ as origin. Then, $p=\dfrac{51^0 42'-52^0}{1^0}=\dfrac{-18'}{60'}=-0.3$

Now, we prepare the following central difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2}=50^0$	0.6428				
		-0.0135			
$x_{-1}=51^0$	0.6293		-0.0001		
		-0.0136		-0.0002	
$x_0=52^0$	0.6157		-0.0003		0.0004
		-0.0139		0.0002	
$x_1=53^0$	0.6018		-0.0001		
		-0.0140			
$x_2=54^0$	0.5878				

Gauss backward interpolation

formula,

$$y\left(51^0 42'\right)=0.6157+(-0.3)(-0.0136)+\frac{(-0.3)(-0.3+1)}{2!}(-0.0003)$$
$$+\frac{(-0.3)(-0.3+1)(-0.3-1)}{3!}(-0.0002)+\frac{(-0.3)(-0.3+1)(-0.3-1)(-0.3+2)}{5!}(-0.0004)$$

$$y\left(51^0 42'\right)=0.6198$$

Example 5: The following table gives the values of x and $y=\sqrt{x}$:

x	*1.0*	*1.05*	*1.10*	*1.15*	*1.2*	*1.25*	*1.30*
y	*1.000*	*1.0242*	*1.0480*	*1.0714*	*1.0944*	*1.1170*	*1.1392*

Using Stirling's interpolation formula find $\sqrt{1.12}$.

Solution: Since the point 1.12 is very close to 1.20, we take $x_0=1.1$ as origin.

Then, $p=\dfrac{1.12-1.1}{0.05}=0.4$

Now, we prepare the following difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	1.0000			
		0.0242		
1.05	1.0242		-0.0004	
		0.0238		0
1.1	1.0480		-0.0004	
		0.0234		0
1.15	1.0714		-0.0004	
		0.0230		0
1.2	1.0944		-0.0004	
		0.0226		0
1.25	1.1170		-0.0004	
		0.0222		
1.3	1.1392			

The Stirling's interpolation formula is,

$$y_p=y_0+p\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right)+\frac{p^2}{2!}\Delta^2 y_{-1}+\frac{p\left(p^2-1\right)}{3!}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right)+\frac{p^2\left(p^2-1\right)}{4!}\Delta^4 y_{-2}+\ldots.$$

$$y(1.12)=1.0480+(0.4)\left(\frac{0.0238+0.0234}{2}\right)+\frac{(0.4)^2}{2!}(-0.0004)$$

$$y(1.12)=1.0574$$