Method of bisection:

Let the equation be $f(x)=0$.

Step 1: Suppose $f(x)$ changes sign between $x=x_n$ and $x=x_{n+1}$ (this can be checked by noting that the product of $f(x_{n+1})$ and $f(x_n)$ is negative. That is $f(x_{n+1})\,f(x_n)<0$)

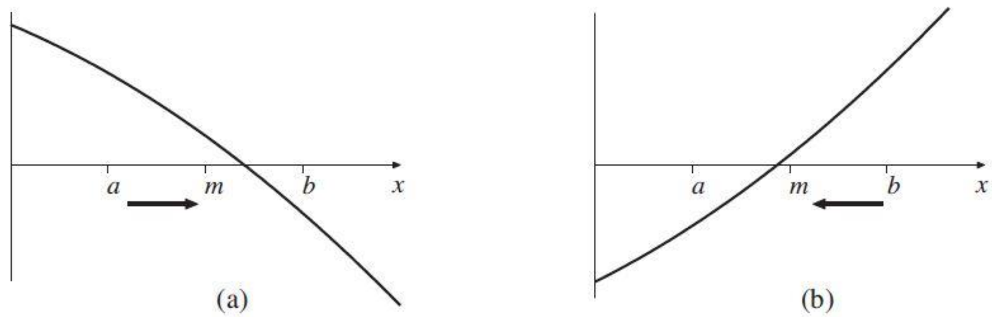
Step 2: Find $f\left(\dfrac{x_n+x_{n+1}}{2}\right)=f(x_{mid})$

If sign of $f(x_{mid})$ is the same as $f(x_n)$, the root is closer to x_{n+1}. Replace x_n by x_{mid} for the next bisection. If sign of $f(x_{mid})$ is the same as $f(x_{n+1})$, replace x_{n+1} by x_{mid} .

Step 3: When successive values of x_{mid} are close enough, i.e., within a certain tolerance, the iteration can be stopped.

Note: The aim with the bisection method is to repeatedly reduce the width of the bracketing interval a < x < b so that it "pinches" the required zero of $f(x)=0$ to some desired accuracy. We begin by describing one iteration of the bisection method in detail.

Let $m=\frac{1}{2}(a+b)$, the mid-point of the interval a < x < b. All we need to do now is to see in which half (the left or the right) of the interval a < x < b the zero is in. We evaluate $f(m)$. There is a (very slight) chance that $f(m)=0$, in which case our job is done and we have found the zero of f. Much more likely is that we will be in one of the two situations shown in Figure below. If f(m)f(b) < 0 then we are in the situation shown in (a) and we replace a < x < b with the smaller bracketing interval m < x < b. If, on the other hand, f(a)f(m) < 0 then we are in the situation shown in (b) and we replace a < x < b with the smaller bracketing interval a < x < m.



(a) (b)

Advantages:

i) The method is easy to understand and remember

ii) It is guaranteed to work if $f(x)$ is continuous in $[a,b]$ and if the values $x=a$ and $x=b$ actually bracket a root.

iii) It is known that the last value x_m differs from the true root by less than $\frac{1}{2}$ the last interval. Error of n iterations less than $\left|\frac{b-a}{2^n}\right|$.

Disadvantages:

i) The major objection is; it is slow to converge

ii) We may choose steps which are too small and involve excessive work or that for more complicate functions we may choose steps which are too big and miss two or more roots which have involved a double change of sign of the function.

Problem 1: Find the root of the equation $f(x)=x^3+x^2-3x-3=0$

Solution: Note that $f(1)<0$ and $f(2)>0$. Root lies between 1 and 2.

Iteration 1: $x_1=1$, $x_2=2$ and $x_3=\frac{1+1.5}{2}=1.75$

$$f(x_3)=f(1.5)=-1.875<0$$

Iteration 2: $x_1=1.5$, $x_2=2$ and $x_3=\frac{1+2}{2}=1.5$

$$f(x_3) = f(1.75) = 0.171875 > 0$$

Iteration 3: $x_1 = 1.5$, $x_2 = 1.75$ and $x_3 = \frac{1.5+1.75}{2} = 1.625$

$$f(x_3) = f(1.75) = -0.943 < 0$$

Iteration 4: $x_1 = 1.625$, $x_2 = 1.75$ and $x_3 = 1.6875$

After 12th iteration, we get

$$x_1 = 1.731934 \ , \ x_2 = 1.732422 \text{ and } x_3 = 1.732178$$

$$f(x_3) = f(1.732178) > 0$$

Iteration 13: $x_1 = 1.731934$, $x_2 = 1.732178$ and $x_3 = 1.732056$

We may stop here and the root is 1.732.

Example 2: find the positive roots of $x - \cos x = 0$ by bisection method correct to three decimal places.

Solution: Let $f(x) = x - \cos x$

We can easily find that,

$$f(0) = -1 < 0 \text{ and } f(1) = 0.4597 > 0$$

As result, the root lies between 0 and 1.

Iteration 1: $x_1 = 0$, $x_2 = 1$ and $x_3 = 0.5$

$$f(x_3) = f(0.5) = -0.37758 < 0$$

After 11 iterations, we get $x = 0.738$

Example 3: The single positive root of the function $f(x) = x\tanh\left(\frac{x}{2}\right) - 1$ models the wave number of water waves at a certain frequency in water depth 0.5 (measured in some units). Find the roots of $f(x)$.

Solution:

At $x = 0$, $\quad f(0) = -1 < 0$

At $x = 0.5$ $\quad f(0.5) = -0.8775 < 0$

At $x=1$ $\quad f(1)=-0.5379<0$

At $x=1.5$ $\quad f(0.5)=-0.0473<0$

At $x=2$ $\quad f(2)=0.5232>0$

The root lies between 1.5 and 2.

Example 4: A function $f(x)$ is known to have a single zero between the points $a=3.2$ and $b=4$. If these values were used as the initial bracketing points in an implementation of the bisection methods, how many iterations would be required to ensure an error less than $\frac{10^{-3}}{2}$?

Solution: We require that,

$$\frac{1}{2^n}\left(\frac{4-3.2}{2}\right)<\frac{10^{-3}}{2}$$

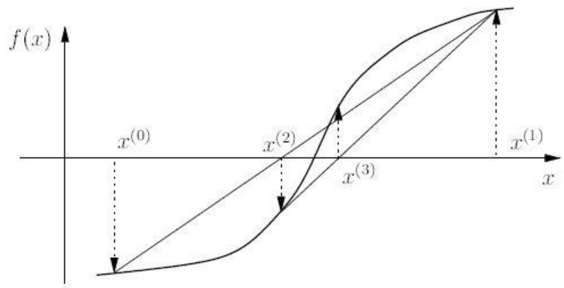
The smallest value of n which satisfies this is $n=10$

The Secant method (False position method):

This method is also based on finding roots of opposite sign and interpolating between them, but by a method which is generally more efficient than bisection.

Step 1: Let $f(x_n)$ and $f(x_{n+1})$ have opposite signs. The false position method interpolates linearly between $[x_n, f(x_n)]$ and $[x_{n+1}, f(x_{n+1})]$, taking the intersection of the interpolating line with the $x-$axis as an improved guess for the root. Using the same procedure as in bisection, the improved guess replaces either the lower bound or upper bound previous guess as the case may be.

Equation of line joining:



$$y - f(x_1) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}(x - x_1)$$

At $y = 0$,

$$x = x_1 - \frac{(x_1 - x_2) f(x_1)}{f(x_1) - f(x_2)}$$

Algorithm:

$$x_{new} = x_n - \frac{(x_{n+1} - x_n) f(x_n)}{f(x_{n+1}) - f(x_n)}$$

Example: Use the false position method to find a root of the equation $f(x) = x^3 - x - 1 = 0$ in the range 1.3 and 1.4.

Solution: Note that $f(1.3) < 0$ and $f(1.4) > 0$.

$$x^{(2)} = 1.3 - f(1.3)\left(\frac{1.4 - 1.3}{f(1.4) - f(1.3)}\right)$$

$$x^{(2)} = 1.3 - f(1.3)\left(\frac{1.4 - 1.3}{f(1.4) - f(1.3)}\right) = 1.32304$$

Note that $f(1.32304) < 0$.

$$x^{(3)} = 1.32304 - f(1.32304)\left(\frac{1.4 - 1.32304}{f(1.4) - f(1.32304)}\right) = 1.3246$$

Note that $f(1.3246) < 0$

$$x^{(4)} = 1.3246 - f(1.3246)\left(\frac{1.4 - 1.3246}{f(1.4) - f(1.3246)}\right) = 1.32471$$

Note that $f(1.32471) \simeq 0.000$

Example 2: Suppose we wish to find a root of the function f(x) = cos(x) + 2 sin(x) + x^2. A closed form solution for x does not exist so we must use a numerical technique. We will use x_0 = 0 and x_1 = -0.1 as our initial approximations. Find the root of the equation using secant method (upto six iterations)

n	x_{n-1}	x_n	x_{n+1}	$\lvert f(x_{n+1})\rvert$	$\lvert x_{n+1} - x_n\rvert$

1	0.0	-0.1	-0.5136	0.1522	0.4136
2	-0.1	-0.5136	-0.6100	0.0457	0.0964
3	-0.5136	-0.6100	-0.6514	0.0065	0.0414
4	-0.6100	-0.6514	-0.6582	0.0013	0.0068
5	-0.6514	-0.6582	-0.6598	0.0006	0.0016
6	-0.6582	-0.6598	-0.6595	0.0002	0.0003

After six iterations, our approximation to the root is -0.6595.

Newton-Rapson method:

We want to find the roots for $f(x)=0$.

$$f(x_{n+1}) = f(x_n+h) = f(x_n) + hf'(x_n) + \frac{h^2}{2!} f''(x_n) + \ldots$$

If x_{n+1} is a root, $f(x_{n+1})=0$. Then

$$f(x_n) + hf'(x_n) = 0 + O(h^2)$$

$$h = -\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n + h = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 1: Find the root of the equation $x^3-3x-5=0$

$$x_{n+1} = x_n - \frac{x_n^3-3x_n-5}{3x_n^2-3}$$

Let $x_0=3$, then we get

$x_1 = 2.46$; $x_2 = 2.295$; $x_3 = 2.279$; $x_4 = 2.279$

Example 2: Find the root of the equation $x^3 - x - 1 = 0$ which is close to 2.

Example 3: Find the root of the equation $x\sin x + \cos x = 0$

Solution: $x_{n+1} = x_n - \dfrac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$. Let $x_0 = \pi = 3.1416$.

x_n	$f(x_n)$	x_{n+1}
3.1416	-1.0	2.8233
2.8233	-0.0662	2.7986
2.7986	-0.0006	2.7984
2.7984	0.0	2.7984

Newton's method convergence condition: If the inequality $\left| \dfrac{f(x) f''(x)}{\left(f'(x)\right)^2} \right| < 1$ holds for all values of x in an interval around a root of f, then the method will converge to the root for any starting value x_0 in the interval.