Interpolating polynomials (with unequal intervals).

Consider the following data to fit a 3rd degree polynomial.

x :	3.2	2.7	1.0	4.8	5.6
$f(x)$:	22	17.8	14.2	38.3	51.7

First, we need to select the points that determine our polynomial. We know that the maximum degree of polynomial is always one less than the number of points. Suppose, we choose the first four points.

Let ax^3+bx^2+cx+d be the general form of cubic polynomial. Then we write four equations involving the unknown coefficients a, b, c and d:

when $x=3.2$; $a(3.2)^3+b(3.2)^2+c(3.2)+d=22$

$x=2.7$: $a(2.7)^3+b(2.7)^2+c(2.7)+d=17.8$

$x=1.0$: $a(1.0)^3+b(1.0)^2+c(1.0)+d=14.2$

$x=4.8$: $a(4.8)^3+b(4.8)^2+c(4.8)+d=38.3$

On solving these equations, we get

$a=-0.5275$; $b=6.4952$; $c=-16.1177$

and $d=24.3499$

$\therefore$ Therefore, our required polynomial is

$-0.5275x^3+6.4952x^2-16.1177x+24.3499.$

At $x=3$, the estimated value is 20.212.

Comment: Finding such interpolating polynomials, in this way is awkward. Furthermore, this method leads to an ill-conditioned system of equations.

Lagrange interpolation:

The Lagrangian polynomial is perhaps the simplest way to exhibit the existance of a polynomial for interpolation with unevenly spaced data.

Suppose that we know a function f exactly at a few points and that we want to approximate how the function behaves between those points. Let us suppose that the data are in the form

$$(x_1, f_1), (x_2, f_2), (x_3, f_3), \ldots$$

Our aim is to find a polynomial which passes exactly through the given data points. That is we want to find $p(x)$ such that

$$p(x_1) = f_1; \quad p(x_2) = f_2; \quad p(x_3) = f_3, \ldots$$

We now define Lagrange polynomials, $L_1, L_2, L_3 \ldots$ which have the following properties:

$L_1(x) = 1$ at $x = x_1$ and $L_1(x) = 0$ at $x = x_2, x_3, x_4 \ldots$

$L_2(x) = 1$ at $x = x_2$ and $L_2(x) = 0$ at $x = x_1, x_3, x_4 \ldots$

$L_3(x) = 1$ at $x = x_3$ and $L_3(x) = 0$ at $x = x_1, x_2, x_4 \ldots$

$\vdots \qquad \vdots \qquad \vdots$

Here, these polynomials are constructed in such way that ②

i) if you evaluate it at a point other than it's own value, the value we get is 0

ii) if you evaluate, any of these polynomials at its own value, the value we get is 1.

These two properties are enough to be able to write down what $p(x)$ must be:

$$p(x) = t_0 p_0(x) + t_1 p_1(x) + t_2 p_2(x) + \cdots$$

For example at $x = x_1$,

$$p(x_1) = t_0 \underbrace{L_0(x_1)}_{0} + t_1 \underbrace{L_1(x_1)}_{1} + t_2 \underbrace{L_2(x_2)}_{0} + \cdots$$

And we write Lagrange polynomials in the following way,

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}$$ and so on.

Note that the numerator of $L_i(x)$ does not contain $(x-x_i)$ and denominator of $L_i(x)$ does not contain $(x_i - x_i)$.

$$\hookrightarrow P(x_1) = t_0 \underbrace{L_0(x_1)}_{0} + t_1 \underbrace{L_1(x_1)}_{1} + t_2 \underbrace{L_2(x_1)}_{0} + \cdots$$

Ex: Find a polynomial of degree 3, which passes through the first four points of the previous data and use it to find the interpolation at $x = 3.0$

x : $x_0 = 3.2$ $\quad x_1 = 2.7$ $\quad x_2 = 1.0$ $\quad x_3 = 4.8$

f : $f_0 = 22$ $\quad f_1 = 17.8$ $\quad f_2 = 14.2$ $\quad f_3 = 38.3$

$$P_3(x) = f_0 L_0(x) + f_1 L_1(x) + f_2 L_2(x) + f_3 L_3(x)$$

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot f_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_1)} \cdot f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot f_3$$

$$P_3(3.0) = \frac{(3.0-2.7)(3.0-1.0)(3.0-4.8)}{(3.2-2.7)(3.2-1)(3.2-4.8)}(22)$$

$$+ \frac{(3.0-3.2)(3.0-1.0)(3.0-4.8)}{(2.7-3.2)(2.7-1.0)(2.7-4.8)}(17.8)$$

$$+ \frac{(3.0-3.2)(3.0-2.7)(3.0-4.8)}{(1.0-3.2)(1.0-2.7)(1.0-4.8)}(14.2)$$

$$+ \frac{(3.0-3.2)(3.0-2.7)(3.0-1.0)}{(4.8-3.2)(4.8-2.7)(4.8-1.0)} \cdot (38.3)$$

$$P_3(3.0) = 20.21.$$

Ex!. A designer wants a curve on a diagram he is preparing to pass through the points

x:	0.25	0.5	0.75	1
$f(x) = y$:	0.32	0.65	0.43	0.1

He decides to do this by using an interpolating polynomial $p(x)$. what is the y-value corresponding to $x = 0.8$?

Solution: $p(0.8) = f_0 L_0(0.8) + f_1 L_1(0.8) + f_2 L_2(0.8) + f_3 L_3(0.8)$

$p(0.8) = 0.32(0.032) + (0.65)(-0.176) + (0.43)(1.056) + (0.088)$

$p(0.8) = 0.358720$

Note!. There are two disadvanges with Lagrangian polynomial for interpolation.

i) It involves more arithmetic operations

ii) If we desire to add or subtract a point, then the interpolation coefficients are required to be recalculated.

This labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called "divided differences".

Ex: Let $p(x)$ be the polynomial of degree 3 which interpolates the data:

x:	0.8	1	1.4	1.6
$f(x)$:	-1.82	-1.73	-1.4	-1.11

Evaluate $p(1.1)$.

Solution: $p_3(1.1) = f_0 L_0(1.1) + f_1 L_1(1.1) + f_2 L_2(1.1) + f_3 L_3(1.1)$.

$$L_0(1.1) = \frac{(1.1-x_1)(1.1-x_2)(1.1-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(1.1-1)(1.1-1.4)(1.1-1.6)}{(0.8-1)(0.8-1.4)(0.8-1.6)}$$

$$L_0(1.1) = -0.15625.$$

Similar calculations for the other Lagrange polynomials:

$L_1(1.1) = 0.93750$; $L_2(1.1) = 0.31250$; $L_3(1.1) = -0.09375$

$$\therefore p_3(1.1) = (-1.82)(-0.15625) + (-1.73)(0.93750)$$
$$(-1.4)(0.31250) + (-1.11)(-0.09375)$$

$$p_3(1.1) = -1.670938$$

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Divided differences :

If $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ .. be the given points, then the first divided difference for the arguments x_0, x_1 is defined by the relation.

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

In general, $[x_s, x_t] = \dfrac{y_t - y_s}{x_t - x_s}$ is the first difference between x_s and x_t. Observe that the order of the points is immaterial

i.e $[x_s, x_t] = [x_t, x_s]$.

The 2nd order divided difference for x_0, x_1, x_2 is defined

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The 3rd order divided difference for x_0, x_1, x_2, x_3 is defined

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$ and so on.

Divided difference table:

x_i	y_i	$[x_i, x_{i+1}]$	$[x_i, x_{i+1}, x_{i+2}]$	$[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	y_0			
		$[x_0, x_1]$		
x_1	y_1		$[x_0, x_1, x_2]$	
		$[x_1, x_2]$		$[x_0, x_1, x_2, x_3]$
x_2	y_2		$[x_1, x_2, x_3]$	
		$[x_2, x_3]$		$[x_1, x_2, x_3, x_4]$
x_3	y_3		$[x_2, x_3, x_4]$	
		$[x_3, x_4]$		
x_4	y_4			

Newton's divided difference formula:

Let $y_0, y_1, \ldots y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots x_n$. Then we write Newton's divided difference formula in the following way:

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \cdots$$
$$+ (x - x_0)(x - x_1)(x - x_2)\cdots(x - x_n)[x_0, x_1, x_2 \ldots x_n].$$

Ex). Consider the same following data to find the interpolation at $x = 3.0$.

x:	3.2	2.7	1.0	4.8	5.6
f:	22	17.8	14.2	38.3	51.7

Solution.
First construct the divide difference in the following way:

x_i	f_i	$[x_i, x_{i+1}]$	$[x_i, x_{i+1}, x_{i+2}]$	$[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$[x_i, \dots x_{i+4}]$
$x_0 = 3.2$	22				
		8.400			
			2.856		
$x_1 = 2.7$	17.8			-0.528	
		2.118			0.256
$x_2 = 1.0$	14.2		2.012		
				0.0865	
		6.342			
$x_3 = 4.8$	38.3		2.263		
		16.75			
$x_4 = 5.6$	51.7				

We wish to construct the interpolating polynomial of degree 3 that fits the data points from $x_0 = 3.2$ to $x_3 = 4.8$.

From the Newton divide difference interpolation,

$$P_3(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3].$$

$$P_3(x) = 22 + (x - 3.2)\, 8.4 + (x - 3.2)(x - 2.7)\, 2.856$$
$$+ (x - 3.2)(x - 2.7)(x - 1)(-0.528)$$

$$P_3(3) = 22 + 8.4(3-3.2) + 2.856(3-3.2)(3-2.7) - 0.528(3-3.2)(3-2.7)(3-1)$$

$$P_3(3) = 20.212$$

<u>Comment</u>: Since we have 5 data point, we can construct a interpolating polynomial of degree 4. Using Newton's divide difference formula, we can easily construct this interpolating polynomial of degree 4 by adding one more term to $P_3(x)$.

$$P_4(x) = P_3(x) + 0.256(x-3.2)(x-2.7)(x-1.0)(x-4.8).$$