The power method:

The power method is an iterative method to find the largest eigen value and the corresponding eigen vector of the matrix. we begin this by choosing a starting vector (usually the column vector $[1,1,1]^T$ is a good starting vector). In this method, we follow the procedure given below:

i) Multiply $A*X$
ii) Normalize the resulting vector by dividing each component by the largest in magnitude
iii) Repeat steps 1 and 2 until the change in the normalizing factor is negligible. At that time, the normalization factor is an eigen value and the final vector is an eigen vector.

Example: Find the largest eigen value and corresponding eigen vector of the matrix

$$A=\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \text{ (choose the staring vector } [1,1,1]^T\text{)}$$

Solution:

1st iteration: $AX=\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Normalization gives, $2*\begin{bmatrix} 1 & 0.5 & 0 \end{bmatrix}^T$. The 1st iteration gives the eigen vector $\begin{bmatrix} 1 & 0.5 & 0 \end{bmatrix}^T$

2nd iteration: $AX=\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}=\begin{bmatrix} 3.5 \\ -4 \\ 0.5 \end{bmatrix}$

Normalization gives, $(-4)*\begin{bmatrix} -0.875 & 1 & -0.125 \end{bmatrix}^T$.

The eigen vector now becomes, $\begin{bmatrix} -0.875 & 1 & -0.125 \end{bmatrix}^T$

3rd iteration: $AX=\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -0.875 \\ 1 \\ -0.125 \end{bmatrix}=\begin{bmatrix} -3.625 \\ 6.125 \\ -1.125 \end{bmatrix}$

Normalization gives, $(6.125)*\begin{bmatrix} -0.5918 & 1 & -0.1837 \end{bmatrix}^T$.

The eigen vector now becomes, $\begin{bmatrix} -0.5918 & 1 & -0.1837 \end{bmatrix}^T$

4th iteration: $AX=\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -0.5918 \\ 1 \\ -0.1837 \end{bmatrix}=\begin{bmatrix} -2.7755 \\ 5.7347 \\ -1.1837 \end{bmatrix}$

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Normalization gives, $(5.7347)*[-0.4840 \quad 1 \quad -0.2064]^T$.

The eigen vector now becomes, $[-0.4840 \quad 1 \quad -0.2064]^T$

After 14 iterations, we get, $AX = \begin{bmatrix} -2.21113 \\ 5.47743 \\ -1.22333 \end{bmatrix}$

Normalization gives, $(5.47743)*[-0.40368 \quad 1 \quad -0.22334]^T$

That is, the largest eigen value is 5.37743 and the eigen vector $[-0.40368 \quad 1 \quad -0.22334]^T$

The fourth iteration shows a negligible change in the normalizing factor. We have approximated the largest eigen value and the corresponding eigen vector. (Note that Twenty iterations will give even better values.) The exact eigen values and the corresponding eigen vectors of the given matrix are:

Eigen values: 5.47735, 2.44807, 0.074577

Corresponding eigen vectors: $[-0.40365 \quad 1 \quad -0.22335]^T$

$[1 \quad 0.55193 \quad -0.38115]^T$

$[0.31633 \quad 0.92542 \quad 1]^T$

Observations:

i) If the initial column vector X^0 is an eigen vector of the matrix A other than that corresponding to the dominant eigen value, then the method will fail (since the iteration will converge to the wrong eigen value)

ii) It is possible to show that the speed of convergence of the power method depends on the ration given below:

$$\frac{magnitude\ of\ dominant\ eigen\ value}{magnitude\ of\ next\ largest\ eigen\ value}$$

If this ratio is small the method is slow to converge.

In particular, if the dominant eigen value is complex the power method will fail completely to converge.

iii) The power method only gives one eigen value, the dominant one (although this is often the most important in applications)

iv) This method is simple and easy to implement

v) It gives the eigen vector corresponding to λ as well λ itself.

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Note 1: The matrix $A=\begin{bmatrix}4 & 5\\ 6 & 5\end{bmatrix}$ has eigen values $\lambda_1=10$ and $\lambda_2=-1$. The ratio $\frac{|10|}{|-1|}=10$. For this matrix, only FOUR iterations are required to obtain successive approximations (that agree when rounded to three significant digits)

Note 2: The matrix $A=\begin{bmatrix}-4 & 10\\ 7 & 5\end{bmatrix}$ has eigen values $\lambda_1=10$ and $\lambda_2=-9$. The ratio $\frac{|10|}{|-9|}=1.1$. For this matrix, the power method does not produce successive approximations that agree to three significant digits until sixty-eight iterations have been performed.

Finding smallest eigen value:

If A has dominant eigen value λ_1 then its inverse A^{-1} has an eigen value $\frac{1}{\lambda_1}$. Clearly $\frac{1}{\lambda_1}$ will be the smallest magnitude of A^{-1}. Conversely if we obtain the largest magnitude eigen value, say λ, of $A^{-1}$by the power method then the smallest eigen value of A is the reciprocal $\frac{1}{\lambda}$.

Example: Find the smallest eigen value of the matrix $A=\begin{bmatrix}3 & -1 & 0\\ -2 & 4 & -3\\ 0 & -1 & 1\end{bmatrix}$

(choose the staring vector $[1,1,1]^T$)

Solution: The reciprocal of largest eigen value of A^{-1} is the smallest eigen value of A.

$$A^{-1}=\begin{bmatrix}1 & 1 & 3\\ 2 & 3 & 9\\ 2 & 3 & 10\end{bmatrix}$$

Using $X^0=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ in the power method applied to A^{-1} gives the largest eigen value $\lambda=13.4090$. Hence, the smallest magnitude eigen value of the matrix A is $\frac{1}{13.4090}=0.0746$ and the corresponding eigen vector is $[0.3163, 0.9254, 1]^T$

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