Numerical solution for system of equations

Suppose we have a system equations of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&-----------\\
&-----------\\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}$$

Equivalently, we write the above system in the matrix form, $AX = B$

We can solve the system $AX = B$ in two ways namely, direct method and iterative method. Gaussian elimination, **LU factorization,** Cholesky's method etc. are examples of direct methods. On the other hand, Jacobi and Gauss-Seidel methods are examples of iterative methods.

Note 1: Consider the following system of equations

$$\begin{aligned}
3x_1 + x_2 - x_3 &= 0 \\
2x_2 + x_3 &= 12 \\
x_3 &= 6
\end{aligned}$$

The last equation can be solved immediately to give $x_3 = 2$

Substituting this value in second equation, we get $x_2 = 5$

Substituting the values of x_2 and x_3 in the first equation, we get $x_1 = -1$

The solution is $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} -1 & 5 & 2 \end{bmatrix}^T$

Finding the solution of the given system in this process is called back-substitution.

In the matrix form the system of equations can be written as

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix}$$

Note 2: The coefficient matrix is said to be upper triangular if all elements below the leading diagonal are zero. Any system of equations in which the coefficient matrix is triangular (whether upper or lower) will be particularly easy to solve.

Gauss elimination method:

We have seen earlier, solution of the system of equations using back substitution. Now we wish to find the solution of the systems by converting into triangular form using row operations. Find the solution in this way is known as Gauss elimination method. This method has three stages.

i) In the first stage the equations are written in matrix form

ii) In the second stage the matrix equations are replaced by a system of equations having the same solution but which are in triangular form

iii) In the final stage the new system is solved by back-substitution.

Note 1: The possible row operations are:

1. Interchange any two rows
2. Multiply or divide a row by a non-zero constant factor
3. Add to or subtract from, one row a multiple of another row.

Note 2: these row operations neither create nor destroy solutions so that at every step the system of equations has the same solution as the original system.

Example 1: Find the solution of the following system using Gauss elimination method.

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 2x_1 - x_2 - 3x_3 &= 3 \\ 4x_1 + 5x_2 - x_3 &= 7 \end{aligned}$$

Solution: Write the given system in matrix form as

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 7 \end{bmatrix}$$

Now we combine the coefficient matrix with the column matrix to produce the augmented matrix:

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]$$

In general for the system $AX = B$, the augmented matrix is written in the form $[A \mid B]$.

The second stage proceeds by first eliminating x_1 from the second and third equations using row operations.

$$\begin{array}{c} \\ R_2-2R_1 \\ R_3-4R_1 \end{array}\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \quad \text{implies that} \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

Finally, we eliminate x_2 from the third row.

$$\begin{array}{c} \\ -R_2 \\ R_2-R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right] \quad \text{implies that} \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

Now the system is in triangular form. The system of equations now become,

$$\begin{aligned} x_1+3x_2+5x_3 &= 14 \\ 7x_2+13x_3 &= 25 \\ 8x_3 &= 24 \end{aligned}$$

From the last equation, we get $x_3=3$

Substituting the value of x_3 in second equations, we get $x_2=-2$

Finally, using these two values we get from the first equation, $x_1=5$.

The solution is therefore, $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 & 3 \end{bmatrix}^T$

Solution using LU factorization:

Consider the system $AX=b$. During the conversion of A to upper triangular form, (as in Gaussian elimination method) it is necessary to operate on b. Therefore, if we have to solve for different b the conversion of A into triangular form would be necessary for every b using Gaussian method. We therefore seek a method so that multiple right hand side b vectors can be processed after only a single decomposition of A into triangular form. It can be shown that matrix can always be written as the product $A=LU$, where L is a lower triangular matrix and U is a upper triangular matrix in the forms

$$L=\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \text{ and } U=\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

It is conventional to assume that either l_{kk} or u_{kk} are unity. Thus,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Comment: Consider the system $AX=b$ and let $A=LU$ then we have the system becomes

$$LUX=b$$

Let $UX = Y$ then $LY = b$

The forward substitution gives Y, then $UX = Y$ yields by backward substitution vector X.

Example: Use LU factorisation to solve

$$2x_1 - 3x_2 + x_3 = 7;\ x_1 - x_2 - 2x_3 = -2;\ 3x_1 + x_2 - x_3 = 0$$

Solution: $\begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & -2 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & -2 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix}$$

Here there are 9 unknowns.

$$u_{11} = 2; \quad u_{12} = -3; \quad u_{13} = 1$$

$$l_{21} = 0.5; \quad u_{22} = 0.5; \quad u_{23} = -2.5$$

$$l_{31} = 1.5; \quad l_{32} = 11; \quad u_{33} = 25$$

Thus,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & 11 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 0.5 & -2.5 \\ 0 & 0 & 25 \end{pmatrix}$$

Now the system is $AX = b$, where $A = LU$

$LUX = b$, let $UX = Y$, then the system becomes $LY = b$

Forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & 11 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 0 \end{pmatrix}$$

$$y_1 = 7; \quad y_2 = -5.5; \quad y_3 = 50$$

Backward substitution: Now $UX = Y$

$$\begin{pmatrix} 2 & -3 & 1 \\ 0 & 0.5 & -2.5 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5.5 \\ 50 \end{pmatrix}$$

$$x_3 = 2; \quad x_2 = -1; \quad x_1 = 1$$

Cholesky's method:

When the coefficient matrix A is symmetric and positive definite, a slightly different factorisation can be obtained by forcing U to be transpose of L. Thus,

$$A = LL^T$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

Example: Use Cholesky's factorisation to solve

$$16x_1 + 4x_2 + 8x_3 = 16;\ 4x_1 + 5x_2 - 4x_3 = 18;\ 8x_1 - 4x_2 + 22x_3 = -22$$

Solution: The system is $AX = b$, where $A = \begin{pmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{pmatrix}$ is a symmetric matrix.

Let us check, whether A is positive definite.

The leading principle minors of A are:

$$16; \quad \begin{vmatrix} 16 & 4 \\ 4 & 5 \end{vmatrix} = 64; \quad \det(A) = 576$$

Thus, by Sylvester's condition, the matrix A is positive definite.

Then $A = LL^T$, where L can be found as

$$L = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -3 & 3 \end{pmatrix}$$

Therefore, the system can be written as, $LL^T X = b$. Now, let $L^T X = Y$ then $LY = b$

Forward substitution:

$$\begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 18 \\ -22 \end{pmatrix}$$

$$y_1 = 4; \quad y_2 = 7; \quad y_3 = -3$$

Backward substitution: Now $L^T X = Y$

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix}$$

$$x_3 = -1; \quad x_2 = 2; \quad x_1 = 1$$

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