Iterative methods for solution of linear systems:

There are several occasions where direct methods (like Gauss elimination or LU decomposition etc) are not the best way to solve a system of equations. an alternative method is to use an iterative method. The most important iterative methods are Jacobi and Gauss-Seidel methods.

Diagonally dominant: A system is said to be diagonally dominant if each diagonal entry of the coefficient matrix of the system is larger than in magnitude than the sum of the magnitudes of the other coefficients in that row. Formally, we say that an $n \times n$ matrix A is diagonally dominant if and only if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for each } i = 1, 2, 3, ..., n$$

The Jacobi method: Suppose we have a system equations of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\
&-------------- \\
&-------------- \\
a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n
\end{aligned}$$

To begin the Jacobi method, solve the first equation for x_1, the second equation for x_2 and so on, as follows,

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3 - ... - a_{1n}x_n\right) \\
x_2 &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1 - a_{23}x_3 - ... - a_{2n}x_n\right) \\
&::::: \\
x_n &= \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1 - a_{n2}x_2 - ... - a_{n,n-1}x_{n-1}\right)
\end{aligned}$$

In general,

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)} - \ldots - a_{1n}x_n^{(n)}\right)$$
$$x_2^{(n+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(n)} - a_{23}x_3^{(n)} - \ldots - a_{2n}x_n^{(n)}\right)$$
$$:::::$$
$$x_n^{(n+1)} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(n)} - a_{n2}x_2^{(n)} - \ldots - a_{n,n-1}x_{n-1}^{(n)}\right)$$

Then make an initial approximation of the solution and substitute these initial values on the right side of the above equations, to obtain the first approximation. In the same way, the second approximation is formed by substituting the first approximation values into the right hand side of the rewritten equations. by repeated iterations, you will form a sequence of approximations that often converges to the actual solution.

Note: In Jacobi method, we assume that the system has unique solution and diagonally dominant.

Example 1: Use the Jacobi method to approximate the solution of the following system of linear equations (chose the initial values $x_1 = x_2 = x_3 = 0$)

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue the iterations until two successive approximations are identical when rounded to three significant digits.

Solution: To begin, we write the system of the form

$$\begin{aligned} x_1 &= \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3 \\ x_2 &= \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3 \\ x_3 &= -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2 \end{aligned}$$

The first approximation is

$$x_1 = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.2$$

$$x_2 = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

The second approximation is

$$x_1 = \frac{-1}{5} + \frac{2}{5}(0.222) - \frac{3}{5}(-0.429) = 0.146$$

$$x_2 = \frac{2}{9} + \frac{3}{9}(-0.2) - \frac{1}{9}(-0.429) = 0.203$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}(-0.2) - \frac{1}{7}(0.222) = -0.517$$

Continuing this procedure, we obtain the sequence of approximations shown below.

	0th	1st	2nd	3rd	4th	5th	6th	7th
x_1	0	-0.2	0.416	0.192	0.181	0.185	0.186	0.816
x_2	0	0.222	0.203	0.328	0.332	0.329	0.331	0.331
x_3	0	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

Note that the values in the 6th and 7th iterations are same. Therefore, the solution is

$$x_1 = 0.186; \quad x_2 = 0.332 \quad x_1 = -0.423$$

Note: It is convenient to split the matrix A into three parts. We write

$$A = L + D + U$$

where L consists of the elements of A strictly below the diagonal and zeros elsewhere; D is a diagonal matrix consisting of the diagonal entries of A and U consists of the elements of A strictly above the diagonal. Note that L and U here are not the same matrices as appeared in the LU decomposition. The current L and U are much easier to find.

For example:

$$\underbrace{\begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}}_{U}$$

and $$\underbrace{\begin{bmatrix} 2 & -6 & 1 \\ 3 & -2 & 0 \\ 4 & -1 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & -1 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & -6 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

more generally for 3×3 matrix,

$$\underbrace{\begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & \bullet & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} \bullet & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \bullet \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & \bullet & \bullet \\ 0 & 0 & \bullet \\ 0 & 0 & 0 \end{bmatrix}}_{U}$$

Note 2: Jacobi iteration: The basic idea is to use the $A = L + D + U$ partitioning of A to write the system $AX = B$ in the form

$$(L + D + U)X = B$$

$$DX = -(L + U)X + B$$

We use this equation as the motivation to define the iterative process

$$DX^{(n+1)} = -(L + U)X^{(n)} + B$$

Example 2: use the Jacobi iteration to approximate the solution of the system

$\begin{pmatrix} 8 & 2 & 4 \\ 3 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$. Use the initial guess $X^{(0)} = [0 \quad 0 \quad 0]^T$

Solution: Here $D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $L + U = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

First iteration: $DX^{(1)} = -(L + U)X^{(0)} + B$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = -\begin{pmatrix} 0 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} -16 \\ 4 \\ -12 \end{pmatrix}$$

Taking this information row by row we see that

$8x_1^{(1)} = -16$; $5x_2^{(1)} = 4$ and $4x_3^{(1)} = -12$

Thus the first Jacobi iteration gives us $X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix} = \begin{pmatrix} -2 \\ 0.8 \\ -3 \end{pmatrix}$

Continuing this procedure, we obtain the sequence of approximations shown below.

	n=0	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=20*	*n=40*
$x_1^{(n)}$	0	-2	-0.7	-0.155	-0.765	-1.277	-0.839	-0.9959	-1
$x_2^{(n)}$	0	0.8	2.6	1.666	2.39	1.787	2.209	2.0043	2
$x_3^{(n)}$	0	-3	-2.2	-3.3	-2.64	-3.215	-2.808	-2.9959	-3