Thomas Algorithm for Tridiagonal systems:

Consider the tridiagonal system of the form

$$\begin{aligned}
d_1x_1 + a_1x_2 \qquad\qquad\qquad &= c_1 \\
b_2x_1 + d_2x_2 + a_2x_3 \qquad\qquad &= c_3 \\
----------- \\
b_ix_{i-1} + d_ix_i + a_ix_{i+1} \qquad &= c_i \\
------------ \\
b_nx_{n-1} + d_nx_n \qquad &= c_n
\end{aligned} \tag{1}$$

In the matrix form we write,

$$\begin{bmatrix} d_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & .. & .. & .. & 0 & 0 & 0 \\ & & & b_i & d_i & a_i & 0 & 0 \\ 0 & 0 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_i \\ .. \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ .. \\ .. \\ c_i \\ .. \\ c_n \end{bmatrix}$$

We first demonstrate the validity of a recursion relation of the form (backward substitution relation)

$$x_i = w_i - E_i\, x_{i+1} \tag{2}$$

in which the constants w_i and E_i are to be determined. Substituting (2) inti the i[th] equation gives,

$$b_i\left(w_{i-1} - E_{i-1}x_i\right) + d_ix_i + a_ix_{i+1} = c_i$$

$$x_i\left(d_i - b_iE_{i-1}\right) = \left(c_i - b_iw_{i-1}\right) - a_ix_{i+1}$$

$$x_i = \frac{\left(c_i - b_iw_{i-1}\right)}{\left(d_i - b_iE_{i-1}\right)} - \frac{a_i}{\left(d_i - b_iE_{i-1}\right)}x_{i+1}$$

which satisfies the assumed relation (2), subject to recursion relation

$$w_i = \frac{\left(c_i - b_iw_{i-1}\right)}{\left(d_i - b_iE_{i-1}\right)}; \quad E_i = \frac{a_i}{\left(d_i - b_iE_{i-1}\right)} \tag{3}$$

(Note that E_n cannot exist, since a_n does not exist.)

Also, form (1), we get,

$$x_1 = \frac{c_1}{d_1} - \frac{a_1}{d_1}x_2$$

Whence,

$$w_1 = \frac{c_1}{d_1}; \quad E_1 = \frac{a_1}{d_1} \tag{4}$$

Since, w_1 and E_1 are known, the equations (3) are valid for $i = 2,3,4...,n$

Note: From (2), $x_n = w_n - E_n\, x_{n+1}$, E_n does not exist. As a result, we expect $x_n = w_n$. Let us now show $x_n = w_n$.

Substitute the recursion relation (2) in the last equation of (1)

$$b_n\left(w_{n-1} - E_{n-1}x_n\right) + d_n x_n = c_n$$

$$x_n\left(d_n - b_n E_{n-1}\right) = c_n - b_n w_{n-1}$$

$$x_n = \frac{c_n - b_n w_{n-1}}{d_n - b_n E_{n-1}} = w_n$$

Thomas Algorithm is:

$$x_i = w_i - E_i\, x_{i+1}\text{, where } i = (n-1),(n-2),\ldots,3,2,1$$

$$x_n = w_n$$

$$w_i = \frac{\left(c_i - b_i w_{i-1}\right)}{\left(d_i - b_i E_{i-1}\right)}\text{, where } i = 2,3,4\ldots,n$$

$$w_1 = \frac{c_1}{d_1}$$

$$\beta_i = d_i - b_i E_{i-1}\text{, where } i = 2,3,4\ldots,n$$

$$E_i = \frac{a_i}{\beta_i}\text{, where } i = 2,3,4\ldots,n-1$$

$$E_1 = \frac{a_1}{d_1}$$

Example: Find the solution of the following tridiagonal system using Thomas algorithm.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution: The general form is

$$\begin{bmatrix} d_1 & a_1 & 0 & 0 & 0 \\ b_2 & d_2 & a_2 & 0 & 0 \\ 0 & b_3 & d_3 & a_3 & 0 \\ 0 & 0 & b_4 & d_4 & a_4 \\ 0 & 0 & 0 & b_5 & d_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

$$w_1 = \frac{c_1}{d_1} = \frac{-1}{2} \text{ and } E_1 = \frac{a_1}{d_1} = \frac{-1}{2}$$

To find E_i and w_i:

$\beta_i = d_i - b_i E_{i-1}$, where $i = 2,3,4,5$

$\beta_i = d_i - b_i E_{i-1}$ where $i = 2,3,4,5$	$E_i = \frac{a_i}{\beta_i}$, where $i = 2,3,4$	$w_i = \frac{(c_i - b_i w_{i-1})}{(d_i - b_i E_{i-1})}$, where $i = 2,3,4,5$
$\beta_2 = (-2) + \left(\frac{1}{2}\right) = \frac{-3}{2}$	$E_2 = \frac{-2}{3}$	$w_2 = \frac{-1}{3}$
$\beta_3 = (-2) - \left(\frac{-2}{3}\right) = \frac{-4}{3}$	$E_3 = \frac{-3}{4}$	$w_3 = \frac{-1}{4}$
$\beta_4 = (-2) - \left(\frac{-3}{4}\right) = \frac{-5}{4}$	$E_4 = \frac{-4}{5}$	$w_4 = \frac{-1}{5}$
$\beta_5 = (-2) - \left(\frac{-4}{5}\right) = \frac{-6}{5}$		$w_5 = \frac{-1}{6}$

Backward setup: $x_i = w_i - E_i\, x_{i+1}$ for $i = (n-1),(n-2),...,3,2,1$

$$x_5 = w_5 = \frac{-1}{6}$$

$$x_4 = w_4 - E_4\, x_{4+1} = \left(\frac{-1}{5}\right) - \left(\frac{-4}{5}\right)\left(\frac{-1}{6}\right) = \frac{-1}{3}$$

$$x_3 = w_3 - E_3\, x_{3+1} = \left(\frac{-1}{4}\right) - \left(\frac{-1}{3}\right)\left(\frac{-3}{4}\right) = \frac{-1}{2}$$

$$x_2 = w_2 - E_2\, x_{2+1} = \left(\frac{-1}{3}\right) - \left(\frac{-1}{2}\right)\left(\frac{-2}{3}\right) = \frac{-2}{3}$$

$$x_1 = w_1 - E_1\, x_{1+1} = \left(\frac{-1}{2}\right) - \left(\frac{-2}{3}\right)\left(\frac{-1}{2}\right) = \frac{-5}{6}$$