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Numerical Integration – Gaussian Quadrature

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1 Gaussian-quadrature two-point formula

Consider the numerical evaluation of the integral

$$\int_{-1}^{1} f(x)\ dx.$$

The numerical integration techniques described so far involve equally spaced values of the interval. However, for a fixed number of points the accuracy may be increased, if we do not insist that the points are equidistant. Gauss derived a formula which uses the same number of function values but with different spacing and gives better accuracy.

Guass' formula is expressed in the form

$$\int_{-1}^{1} F(x)\ dx = \sum_{i=1}^{n} s_i F(x_i).$$

Thus, assuming that we can compute a specified number of values of the integrand (at arbitrary points), we shall construct a formula by selecting arguments (or abscissas) within the range of integration in order to arrive at a most accurate integration rule. An obvious advantage of this formula is that the 'abscissae and weights' are symmetrical with respect to the middle point of the interval.

2 Gauss-Legendre Two-Point Formula

Let

$$y = f(x), \quad -1 \le x \le 1$$

and let

$$\int_{-1}^{1} f(x)\ dx \approx w_1 f(x_1) + w_2 f(x_2).$$

We need to find the weights w_1, w_2 and abscissas x_1, x_2. We do this by method of undetermined coefficients such that $\int_{-1}^{1} f(x)\ dx$ is exact for cubic polynomials i.e., $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Since four coefficients w_1, w_2, x_1, x_2 are need to be determined we require four conditions. Now,

$$f(x) = 1 \Rightarrow \int_{-1}^{1} 1 \, dx = 2 = w_1 + w_2$$

$$f(x) = x \Rightarrow \int_{-1}^{1} x \, dx = 0 = w_1 x_1 + w_2 x_2$$

$$f(x) = x^2 \Rightarrow \int_{-1}^{1} x^2 \, dx = 2/3 = w_1 x_1^2 + w_2 x_2^2$$

$$f(x) = x^3 \Rightarrow \int_{-1}^{1} x^3 \, dx = 0 = w_1 x_1^3 + w_2 x_2^3$$

Solving the above four equations, we obtain

$$w_1 = w_2 = 1; \quad x_1 = -\frac{1}{\sqrt{3}} = -0.5773502692, \quad x_2 = \frac{1}{\sqrt{3}} = 0.5773502692$$

So,

$$\int_{-1}^{1} f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \approx f(-0.57735) + f(0.57735) = G_2(f)$$

Note: Above formula has degree of precession $n = 3$.

If $f \in C^4[-1, 1]$ then

$$\int_{-1}^{1} f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + E_2(f),$$

where $E_2(f) = \frac{f^{(4)}(c)}{135}$, here $f^{(4)}(c) = \max_{-1 \leq x \leq 1} |f^{(4)}(x)|$

3 Gauss-Legendre Three-Point Formula

Let

$$y = f(x), \quad -1 \leq x \leq 1$$

and let

$$\int_{-1}^{1} f(x) \, dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

Following the same procedure as above, we obtain

$$\begin{aligned} \int_{-1}^{1} f(x) \, dx &\approx \frac{1}{9} \left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right) \right] \\ &\approx \frac{1}{9} \left[5f(-0.774597) + 8f(0) + 5f(0.774597) \right] = G_3(f). \end{aligned}$$

$G_3(f)$ has a degree of precession $n = 5$. For $f \in C^6[-1, 1]$ error is $E_3(f) = \frac{f^{(6)}(c)}{15750}$ where $f^{(6)}(c) = \max\limits_{-1 \le x \le 1} |f^{(6)}(x)|$.

4 Gauss-Legendre Translation

Let

$$\int\limits_a^b f(t)\, dt$$

be the integral to be evaluated. We can apply Gauss-Legendre formula by transforming the interval $[a, b]$ to $[-1, 1]$. This can be done by the change of variable

$$t = \frac{b+a}{2} + \frac{b-a}{2}x \quad \text{and} \quad dt = \frac{b-a}{2}\, dx.$$

So,

$$\int\limits_a^b f(t)\, dt = \int\limits_{-1}^1 f\left(\frac{b+a}{2} + \frac{b-a}{2}x\right) \frac{b-a}{2}\, dx = \frac{b-a}{2} \int\limits_{-1}^1 g(x)\, dx$$

where $g(x) = f\left(\frac{b+a}{2} + \frac{b-a}{2}x\right)$.

Problem. *Evaluate* $\int\limits_{-1}^1 \frac{1}{x+2}\, dx$ *by Gauss-Legendre two-point formula*

Solution. Given integral value by Gauss-Legendre two-point formula is:

$$G_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.70291 + 0.38800 = 1.09091$$

Now, evaluate the same integral by Trapezoidal as well as Simpons's $1/3-$rule, we obtain

$$T_2(f) = \frac{2}{2}\left[f(-1) + f(1)\right] = 1 + 0.33333 = 1.33333$$
$$S_2(f) = \frac{1}{3}\left[f(-1) + 4f(0) + f(1)\right] = \frac{1}{3}\left[1 + 2 + \frac{1}{3}\right] = 1.11111$$

Exact answer is: $ln(3) - l(2) \approx 1.09861$. Observe that the Gauss-Legendre two-point formula is very close the exact answer compared to the values obtained by Trapezoidal and Simpons's rules.

Problem. *Evaluate* $\int\limits_{-1}^1 5x^4\, dx$ *using Gauss-Legendre three-point formula*

Solution. The given integral value using Gauss-Legendre three-point formula is:

$$G_3(f) = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] = \frac{1}{9}\left[5(5)(0.6)^2 + 0 + 5(5)(0.6)^2\right] = \frac{18}{9} = 2$$

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Problem. *Evaluate* $\int_1^5 \frac{1}{t}\, dt$ *by Gauss-Legendre three-point formula.*

Solution. The first step is to change the limits. Using the change of variable $t = \frac{1}{2}[6 + 4x] = 3 + 2x$, we have $dt = 2dx$ and that,

$$\int_1^5 \frac{1}{t} dt = \int_{-1}^{1} \frac{1}{3+2x}\ (2dx) = 2\int_{-1}^{1} f(x)\ dx$$

Now, applying Gauss-Legendre three-point formula, we have

$$\begin{aligned} 2\int_{-1}^{1} f(x)\ dx =& \frac{2}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] \\ =& \frac{2}{9}[3.4464 + 2.66671 + 1.0991] = 1.6027 \end{aligned}$$

Problem. *Evaluate* $\int_0^{\pi/2} \sin t dt$ *by Gaussian quadrature of two-point formula.*

Solution. Using the change of variables, we have $t = \frac{1}{2}[\frac{\pi}{2} + \frac{\pi}{2}x] = \frac{\pi}{4}(x+1)$; $dt = \frac{\pi}{4}\ dx$.
So,

$$\begin{aligned} \int_0^{\pi/2} \sin t\, dt =& \frac{\pi}{4}\int_{-1}^{1} \sin\left(\frac{\pi}{x}(x+1)\right)\ dx \\ =& \frac{\pi}{4}\int_{-1}^{1} g(x)\ dx \ \ \text{where } g(x) = \sin\left(\frac{\pi}{x}(x+1)\right) \\ =& \frac{\pi}{4}\left[g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)\right] \approx 0.998473 \end{aligned}$$

Problem. *Evaluate* $I = \int_0^1 x\, dx$ *by two-point and three-point*

Solution. Let $x = \frac{1}{2}(1+t)$ and $dx = \frac{dt}{2}$.

Two-point formula:

$$\begin{aligned} I = \int_0^1 x\, dx =& \frac{1}{2}\int_{-1}^{1} \frac{1}{2}(1+t)\, dt = \frac{1}{4}\int_{-1}^{1} f(t)\, dt, \quad \text{where } f(t) = 1+t \\ =& \frac{1}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{1}{4}\left[0.4226 + 1.5774\right] = 0.5 \end{aligned}$$

Three-point formula:

$$\begin{aligned} I = \int_0^1 x\, dx =& \frac{1}{4}\int_{-1}^{1} (1+t)\, dt \\ =& \frac{1}{4}\cdot\frac{1}{9}\left[5(1 - 0.774597) + 8(1+0) + 5(1 + 0.774597)\right] \\ =& \frac{1}{4}\cdot\frac{1}{9}[5(0.2254) + 8(1) + 5(1.7746)] = 0.5. \end{aligned}$$