Criterion for the convergence of Jacobi and Gauss-Seidel method:

Consider the following system of linear equations

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \ldots + b_{1n}x_n &= u_1 \\ b_{21}x_1 + b_{22}x_2 + \ldots + b_{2n}x_n &= u_2 \\ &------------ \\ &------------ \\ b_{n1}x_1 + b_{n2}x_2 + \ldots + b_{nn}x_n &= u_n \end{aligned} \tag{1}$$

We rewrite the equation as

$$\begin{aligned} x_1 &= \frac{\left(u_1 - b_{12}x_2 - b_{13}x_3 \ldots - b_{1n}x_n\right)}{b_{11}} \\ x_2 &= \frac{\left(u_2 - b_{21}x_1 - b_{23}x_3 \ldots - b_{2n}x_n\right)}{b_{22}} \\ &\ldots\ldots\ldots\ldots \\ x_n &= \frac{\left(u_n - b_{n1}x_1 - b_{n2}x_2 \ldots - b_{n,n-1}x_{n-1}\right)}{b_{nn}} \end{aligned} \tag{2}$$

We can write the Eqn. (2) of the form

$$X = AX + V \tag{3}$$

where

$$A = \begin{pmatrix} 0 & b_{12}/b_{11} & b_{13}/b_{11} & \ldots & b_{1n}/b_{11} \\ b_{21}/b_{22} & 0 & b_{23}/b_{22} & \ldots & b_{2n}/b_{22} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n1}/b_{nn} & b_{n2}/b_{nn} & \ldots & \ldots & 0 \end{pmatrix}, \; V = \begin{pmatrix} u_1/b_{11} \\ u_2/b_{22} \\ \ldots \\ \ldots \\ u_n/b_{nn} \end{pmatrix} \tag{4}$$

If the starting vector is X_0, then

$X_1 = AX_0 + V$ (first iteration)

$X_2 = AX_1 + V$ (second iteration)

In general, $X_{k+1} = AX_k + V$

Consider,

$$X_k = AX_{k-1} + V = A\left(AX_{k-2} + V\right) + V = A^2 X_{k-2} + \left(I + A\right)V$$

$$X_k = A^k X_0 + \left(I + A + A^2 + \ldots + A^{k-1}\right)V$$

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Convergence normally requires that, $\underset{k\to\infty}{Lt}\ A^k = 0$. The value of X_k as $k\to\infty$ should not depend on the initial chosen vector X_0. It develops that a necessary and sufficient condition for

$$\underset{k\to\infty}{Lt}\ A^k = 0 \qquad (5)$$

is that all the eigen values of A shall be, in absolute value less than one. If $A = P^{-1}DP$, where D is the diagonal matrix consisting of eigen values of A, then

$$\underset{k\to\infty}{Lt}\ A^k = P^{-1}\begin{pmatrix} \lambda_1^k & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \lambda_n^k \end{pmatrix} P = 0 \text{ implies that } |\lambda| < 1$$

(Proof can also be accomplished if A is not similar to a diagonal matrix). On the strength of the above statement, we can show (it will be shown later) that a necessary and sufficient condition that the matrix series

$$I + A + A^2 + \dots + A^k + \dots$$

converge is that all eigen values of A be less than unity in modulus. In such an event

$$\left(I - A\right)^{-1} = I + A + A^2 + \dots + A^k + \dots \qquad (6)$$

Each element of the matrix on the left is the sum of an infinite series built from the corresponding elements on the right. Thus, if $\underset{k\to\infty}{Lt}\ A^k = 0$, then $X = \underset{k\to\infty}{Lt}\ X_k$ exists and

$$\underset{k\to\infty}{Lt}\ X_k = X = 0 + \left(I - A\right)^{-1} V$$

$$\left(I - A\right) X = V$$

$$X = AX + V$$

Thus convergence hinges on the truth of Eqn. (5). Eqn. (6) is true if and only if all eigen values of A are in modulus less than unity.