Newton-Rapson method:

We want to find the roots for $f(x)=0$.

$$f(x_{n+1}) = f(x_n+h) = f(x_n) + hf'(x_n) + \frac{h^2}{2!} f''(x_n) + \ldots$$

If x_{n+1} is a root, $f(x_{n+1})=0$. Then

$$f(x_n) + hf'(x_n) = 0 + O(h^2)$$

$$h = -\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n + h = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example 1: Find the root of the equation $x^3-3x-5=0$

$$x_{n+1} = x_n - \frac{x_n^3-3x_n-5}{3x_n^2-3}$$

Let $x_0=3$, then we get

$$x_1 = 2.46;\ x_2 = 2.295;\ x_3 = 2.279;\ x_4 = 2.279$$

Example 2: Find the root of the equation $x^3-x-1=0$ which is close to 2.

Example 3: Find the root of the equation $x\sin x+\cos x=0$

Solution: $x_{n+1} = x_n - \dfrac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$. Let $x_0 = \pi = 3.1416$.

x_n	$f(x_n)$	x_{n+1}
3.1416	-1.0	2.8233
2.8233	-0.0662	2.7986
2.7986	-0.0006	2.7984
2.7984	0.0	2.7984

Newton's method convergence condition: If the inequality $\left|\dfrac{f(x)\,f''(x)}{\left(f'(x)\right)^2}\right|<1$ holds for all values of x in an interval around a root of f, then the method will converge to the root for any starting value x_0 in the interval.

Comment: Newton's method does not always converge.

Newton-Raphson method for multiple roots:

Let α be a multiple root of $f(x)=0$ with multiplicity m, then the iteration formula will be

$$x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)}$$

Example 1: Find the multiple root of $x^4-6x^3+12x^2-10x+3=0$ with the multiplicity 3, by the Newton-Raphson method with an initial guess $x_0=0$.

Solution: Here $f(x)=x^4-6x^3+12x^2-10x+3$ and $f'(x)=4x^3-18x^2+24x-10$

The Newton-Raphson method for multiple roots is, $x_{n+1}=x_n-m\dfrac{f(x_n)}{f'(x_n)}$

Iteration 1: $x_1=x_0-m\dfrac{f(x_0)}{f'(x_0)}=3-3\left(\dfrac{3}{-10}\right)=0.9$

Iteration 2: $x_2=x_1-m\dfrac{f(x_1)}{f'(x_1)}=0.9-3\dfrac{f(0.9)}{f'(0.9)}$

$$x_2=0.9-3\left(\frac{(0.9)^4-6(0.9)^3+12(0.9)^2-10(0.9)+3}{4(0.9)^3-18(0.9)^2+24(0.9)-10}\right)=0.998$$

Iteration 3: $x_3=x_2-m\dfrac{f(x_2)}{f'(x_2)}=0.998-3\dfrac{f(0.998)}{f'(0.998)}$

$$x_3=0.998-3\left(\frac{(0.998)^4-6(0.998)^3+12(0.998)^2-10(0.998)+3}{4(0.998)^3-18(0.998)^2+24(0.998)-10}\right)=1.096$$

Iteration 4: $x_4 = x_3 - m\dfrac{f(x_3)}{f'(x_3)} = 1.096 - 3\dfrac{f(1.096)}{f'(1.096)}$

$$x_4 = 1.096 - 3\left(\frac{(1.096)^4 - 6(1.096)^3 + 12(1.096)^2 - 10(1.096) + 3}{4(1.096)^3 - 18(1.096)^2 + 24(1.096) - 10}\right) = 1.0965$$

The required root $x = 1.096$ (which is correct to 3 decimals).

Note that the exact roots are $x = 1,1,1,3$

Newton – Rapson Method for system of equations (linear & nonlinear)

Consider the equations

$$\begin{aligned} f(x_1, x_2) &= 0 \\ g(x_1, x_2) &= 0 \end{aligned} \qquad (1)$$

Suppose we make a guess (x_1^k, x_2^k) which is close to a root (x_1^{k+1}, x_2^{k+1}). Then, by Taylor expansion

$$\begin{aligned} f\left(x_1^{k+1}, x_2^{k+1}\right) &= f\left(x_1^k + h_1, x_2^k + h_2\right) \\ &= f\left(x_1^k, x_2^k\right) + h_1 \frac{\partial}{\partial x_1} f\left(x_1^k, x_2^k\right) + h_2 \frac{\partial}{\partial x_2} f\left(x_1^k, x_2^k\right) + \ldots \end{aligned} \qquad (2)$$

Similarly,

$$\begin{aligned} g\left(x_1^{k+1}, x_2^{k+1}\right) &= g\left(x_1^k + h_1, x_2^k + h_2\right) \\ &= g\left(x_1^k, x_2^k\right) + h_1 \frac{\partial}{\partial x_1} g\left(x_1^k, x_2^k\right) + h_2 \frac{\partial}{\partial x_2} g\left(x_1^k, x_2^k\right) + \ldots \end{aligned} \qquad (3)$$

where $h_1 = x_1^{k+1} - x_1^k = \Delta x_1$ and $h_2 = x_2^{k+1} - x_2^k = \Delta x_2$.

Let us neglect higher derivatives than the first; we still get an improved estimate than the previous one. From Eqn. (2 - 3), we get

$$-f = \Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2}$$

$$-g = \Delta x_1 \frac{\partial g}{\partial x_1} + \Delta x_2 \frac{\partial g}{\partial x_2}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \begin{pmatrix} -f \\ -g \end{pmatrix} \tag{4}$$

Thus for a system of N simultaneous nonlinear equations, we must solve N simultaneous linear equations to find changes Δx_1 and Δx_2 in every iteration. The matrix on the left side of Eqn. (4) is called the Jacobin matrix J and its determinant simply "the Jacobian". Clearly, if $|J| = 0$ the method fails, and if it is close to zero slow convergence is anticipated. Note,

$$\begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \end{pmatrix} = \begin{pmatrix} x_1^{k} \\ x_2^{k} \end{pmatrix} + \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}$$

Problem 1: Use Newton-Rapson method to find a root of $xy = 1$, $x^2 + y^2 = 4$ which are close to $x = 1.8, y = 0.5$.

Here $f(x, y) = xy - 1 = 0$; $g(x, y) = x^2 + y^2 - 4 = 0$ and $J = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$

$$J \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -f \\ -g \end{pmatrix}$$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = J^{-1} \begin{pmatrix} -f \\ -g \end{pmatrix}, \text{ where } J^{-1} = \frac{1}{2y^2 - 2x^2} \begin{pmatrix} 2y & -x \\ -2x & y \end{pmatrix}$$

1st iteration: Starting values are $x = 1.8, y = 0.5$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -0.1672 & 0.3010 \\ 0.6020 & -.0836 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.51 \end{pmatrix} = \begin{pmatrix} 0.1368 \\ 0.0176 \end{pmatrix}$$

2nd iteration: Starting values are $x = 1.8 + 0.1368 = 1.9368, y = 0.5 + 0.0176 = 0.5176$

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -0.1486 & 0.2780 \\ 0.5560 & -0.0743 \end{pmatrix} \begin{pmatrix} -0.0025 \\ -0.0191 \end{pmatrix} = \begin{pmatrix} -0.0049 \\ 0.0000 \end{pmatrix}$$

Roots are: $x = 1.9368 - 0.0049 = 1.9319$ and $y = 0.5176 - 0.0000 = 0.5176$

Note: We may use a slightly different algorithm which looks like generalized Newton's method.

In Eqn. (2), suppose $h_2 = 0$ and in Eqn. (3) $h_1 = 0$, we get

$$x_1^{k+1} = x_1^k - \frac{f\left(x_1^k, x_2^k\right)}{\frac{\partial}{\partial x_1} f\left(x_1^k, x_2^k\right)} \text{ and } x_2^{k+1} = x_2^k - \frac{g\left(x_1^k, x_2^k\right)}{\frac{\partial}{\partial x_2} g\left(x_1^k, x_2^k\right)}$$

If we put the convergence factor also, as in generalized Newton's method, the final algorithm will be

$$x_1^{k+1} = x_1^k - \frac{w\, f\left(x_1^k, x_2^k\right)}{\frac{\partial}{\partial x_1} f\left(x_1^k, x_2^k\right)} \text{ and } x_2^{k+1} = x_2^k - \frac{w\, g\left(x_1^k, x_2^k\right)}{\frac{\partial}{\partial x_2} g\left(x_1^k, x_2^k\right)}$$

The choice of w different from unity can increase the convergence rate appreciably. Normally $0 < w < 2$. The algorithm is similar for any number of variables.

Problem 2: Consider the transcendental system

$$f_1\left(x_1, x_2\right) = -e^{x_1} - x_1 + 3x_2 + 3 = 0$$

$$f_2\left(x_1, x_2\right) = e^{x_2} + x_2 - 2x_1 + 1 = 0 .$$

Solution:

$$x_1^{k+1} = x_1^k - \frac{w\left(e^{x_1^k} + x_1^k - 3x_2^k - 3\right)}{\left(e^{x_1^k} + 1\right)} \tag{1}$$

$$x_2^{k+1} = x_2^k - \frac{w\left(e^{x_2^k} + x_2^k - 2x_1^k + 1\right)}{\left(e^{x_2^k} + 1\right)} \tag{2}$$

Let $x_1^0 = x_2^0 = 0$. Choose $w = 1.5$

$$x_1^1 = 0 - \frac{(1.5)\left(e^0 + 0 - 3(0) - 3\right)}{\left(e^0 + 1\right)} = 1.5$$

$$x_2^1 = 0 - \frac{(1.5)\left(e^0 + 0 - 2(1.5) + 1\right)}{\left(e^0 + 1\right)} = 0.75$$

These results would then be interested into Eqns. (1-2) to produce x_1^2, x_2^2 and iteration would continue. The advantage with this algorithm is that it can be easily extended to n system of equations.

Comment: There are several other methods. It is rarely possible to determine, a priori. When an iteration will converge and yield a solution. It is better to verify the solution by direct substitution.

Problem 3: Find the solution of the system of nonlinear equations

$$2x_1^2 + x_2^2 = 4.32 \text{ and } x_1^2 - x_2^2 = 0$$

Newton-Rapson Method:

$$f(x_1, x_2) = 2x_1^2 + x_2^2 - 4.32 = 0 \text{ and } g(x_1, x_2) = x_1^2 - x_2^2 = 0$$

Starting values: $x_1 = 1$ and $x_2 = 1$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 4x_1 ; \ \frac{\partial f}{\partial x_2} = f_{x_2} = 2x_2 ; \ \frac{\partial g}{\partial x_1} = g_{x_1} = 2x_1 \text{ and } \frac{\partial g}{\partial x_2} = g_{x_2} = -2x_2$$

$$\begin{pmatrix} 4x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{pmatrix}_{(x_{1i}, x_{2i})} \begin{pmatrix} \Delta x_{1i} \\ \Delta x_{2i} \end{pmatrix} = \begin{pmatrix} -f \\ -g \end{pmatrix}_{(x_{1i}, x_{2i})}$$

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} = \begin{pmatrix} 1.32 \\ 0 \end{pmatrix}$$

On solving we get, $\Delta x_1 = \Delta x_2 = 0.22$

After first iteration the roots are: $x_1 = x_2 = 1.22$