



M902

Βασικές Μαθηματικές Έννοιες στη Γλωσσική Τεχνολογία

Project 2

Κυλάφη Χριστίνα-Θεανώ

LT1200012

November, 2020

TABLE OF CONTENTS

Question 1	3
Question 2	4
Question 3	6
Question 4	7
Question 5	8
Question 6	9
Question 7	10
Question 8	11
Question 9	13
Question 10	15

Question 1

The composite function $S(f(x))$, where $S(x) = \frac{1}{1 + e^{-x}}$ and $f(x) = ax + b$, is calculated as follows:

$$S(f(x)) = \frac{1}{1 + e^{-f(x)}} = \frac{1}{1 + e^{-(ax+b)}} = \frac{1}{1 + \frac{1}{e^{(ax+b)}}} = \frac{1}{\frac{e^{(ax+b)} + 1}{e^{(ax+b)}}} = \frac{e^{(ax+b)}}{e^{(ax+b)} + 1} = \frac{e^{ax}e^b}{e^{ax}e^b + 1}$$

where:

- $x \in \text{Dom}(f) = \mathbb{R}$
- $f(x) \in \text{Im}(f) \subseteq \text{Dom}(S) = \mathbb{R}$
- $S(f(x)) \in \text{Im}(S)$

Question 2

A function is **invertible** only if each input has a unique output, which means each output is paired with exactly one input. That way, when the mapping is reversed, it will still be a function.

Definition

Let f be a function whose domain is the set A and whose codomain is the set B ($f : A \rightarrow B$). Then we say that f is invertible if f is **one-to-one** mapping and there is a function g with domain $Im(f) \subseteq B$ and image (range) A ($g : Im(f) \rightarrow A$) such that:

$$f(x) = y \iff g(y) = x$$

In this case, we call g the inverse of f and denote it by f^{-1} .

i) $f(x) = \sqrt{x-3}$

• $x-3 \geq 0 \iff x \geq 3 \iff Dom(f) = [3, +\infty)$ (1)

• f is **continuous** on $Dom(f) = [3, +\infty)$ ¹ (2)

• f is **differentiable** on $A = \{ Dom(f) - \{3\} \} = \{ [3, +\infty) - \{3\} \}$
 $= \{ (3, +\infty) \}$ ² (3)

First, we have to prove that f is **one-to-one** mapping:

$$\stackrel{(2), (3)}{\implies} f'(x) = (\sqrt{x-3})' = \frac{1}{2\sqrt{x-3}} > 0, \forall x \in A = (3, +\infty) \quad (4)$$

$$\stackrel{(4)}{\implies} f \text{ is **strictly monotonic** (strictly inscreasing), therefore it is **one-to-one** mapping}$$

One approach for finding a formula for f^{-1} is to solve $f(x) = y$ for x .

$$f(x) = \sqrt{x-3} = y \stackrel{y \geq 0, x \geq 3}{\iff} x-3 = y^2 \iff x = y^2 + 3 = f^{-1}(y) \quad (5)$$

$$\stackrel{(1)}{\implies} x \geq 3 \iff y^2 + 3 \geq 3 \iff y^2 \geq 0, \text{ true } \forall y \in \mathbb{R} \quad (6)$$

¹ as a composition of continuous functions

² as a composition of differentiable functions

$$\stackrel{(5), (6)}{\implies} Im(f) = \{ \mathbb{R} \cap [0, +\infty) \} = [0, +\infty) = Dom(f')$$

$$\bullet f : [3, +\infty) \rightarrow \mathbb{R}$$

$$f(x) = \sqrt{x-3}$$

$$\bullet f^{-1} : [0, +\infty) \rightarrow \mathbb{R}$$

$$f^{-1}(x) = x^2 + 3$$

$$\text{ii) } f(x) = \log(x-2)$$

(Let $f(x)$ logarithmic function with base 2, as 2^x which is $\log_2(x)$ function's inverse, is also used in **Question 4**)

$$\bullet x - 2 > 0 \iff x > 2 \iff Dom(f) = (2, +\infty) \quad (7)$$

First, we have to prove that f is **one-to-one** mapping:

$$Let \ x_1, x_2 \in Dom(f) = (0, +\infty), \ x_1 \neq x_2$$

$$f(x_1) = f(x_2) \iff \log_2(x_1 - 2) = \log_2(x_2 - 2) \iff x_1 - 2 = x_2 - 2 \iff x_1 = x_2$$

$$\iff x_1 = x_2 \iff f \text{ is an } \textbf{one-to-one} \text{ mapping of } Dom(f) \text{ to } Im(f)$$

As in i above, we will solve $f(x) = y$ for x .

$$f(x) = \log_2(x-2) = y \iff 2^{\log_2(x-2)} = 2^y \iff x-2 = 2^y$$

$$\iff x = 2^y + 2 = f^{-1}(y) \quad (8)$$

$$\stackrel{(7)}{\implies} x > 2 \iff 2^y + 2 > 2 \iff 2^y > 0, \text{ true } \forall y \in \mathbb{R} \quad (9)$$

$$\stackrel{(9)}{\implies} Im(f) = \mathbb{R} = Dom(f')$$

$$\bullet f : (2, +\infty) \rightarrow \mathbb{R}$$

$$f(x) = \log(x-2)$$

$$\bullet f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$$

$$f^{-1}(x) = 2^x + 2$$

Question 3

$$s = A \cos(2\pi f t)$$

- (a) $A_1 = 1, \quad f_1 = 1, \quad \theta_1 = 0, \quad s_1 = \cos(2\pi t)$
(b) $A_2 = 2, \quad f_2 = 3, \quad \theta_2 = 0, \quad s_2 = 2 \cos(6\pi t)$
(c) $A_3 = 1.5, \quad f_3 = 2, \quad \theta_3 = \pi, \quad s_3 = 1.5 \cos(4\pi t + \pi)$
(d) $A_4 = 2, \quad f_4 = 0.5, \quad \theta_4 = 0, \quad s_4 = 2 \cos(\pi t)$

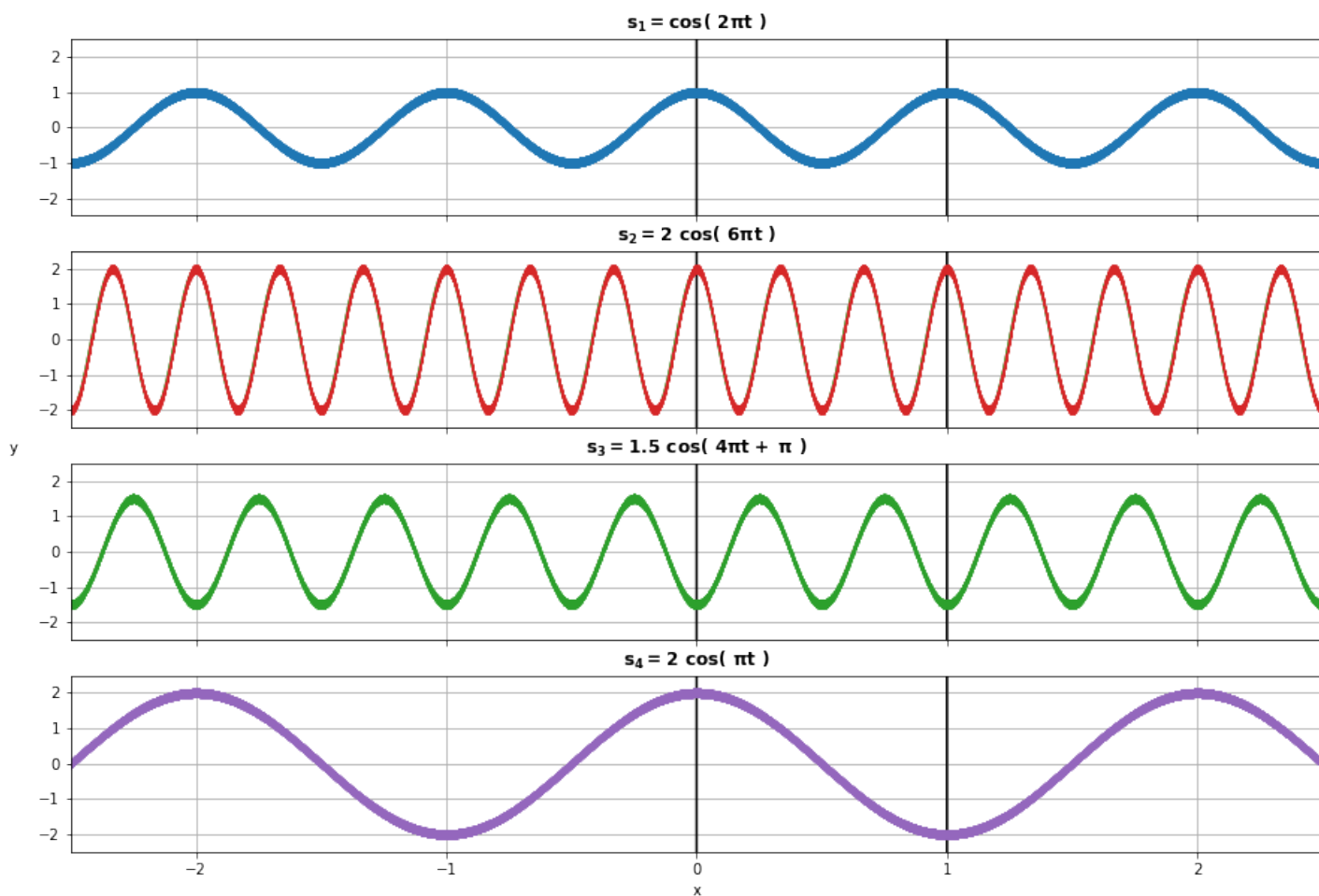


Figure 1

Question 4

Following the steps in **Question 2**, the inverse function of $f(x) = 2^x$ is calculated similarly:

$$f(x) = 2^x = y \stackrel{y>0}{\iff} \log_2(2^x) = \log_2(y) \iff x = \log_2(y) = f^{-1}(y)$$

$$\bullet f^{-1} : (0, +\infty) \rightarrow \mathbb{R}$$

$$f^{-1}(x) = \log_2(x)$$

In **Figure 2**, functions f , f^{-1} , $y = x$ and the respective points are depicted.

Points of f : (0,1), (1,2), (2,4), (3,8)

Points of f^{-1} : (1,0), (2,1), (4,2), (8,3)

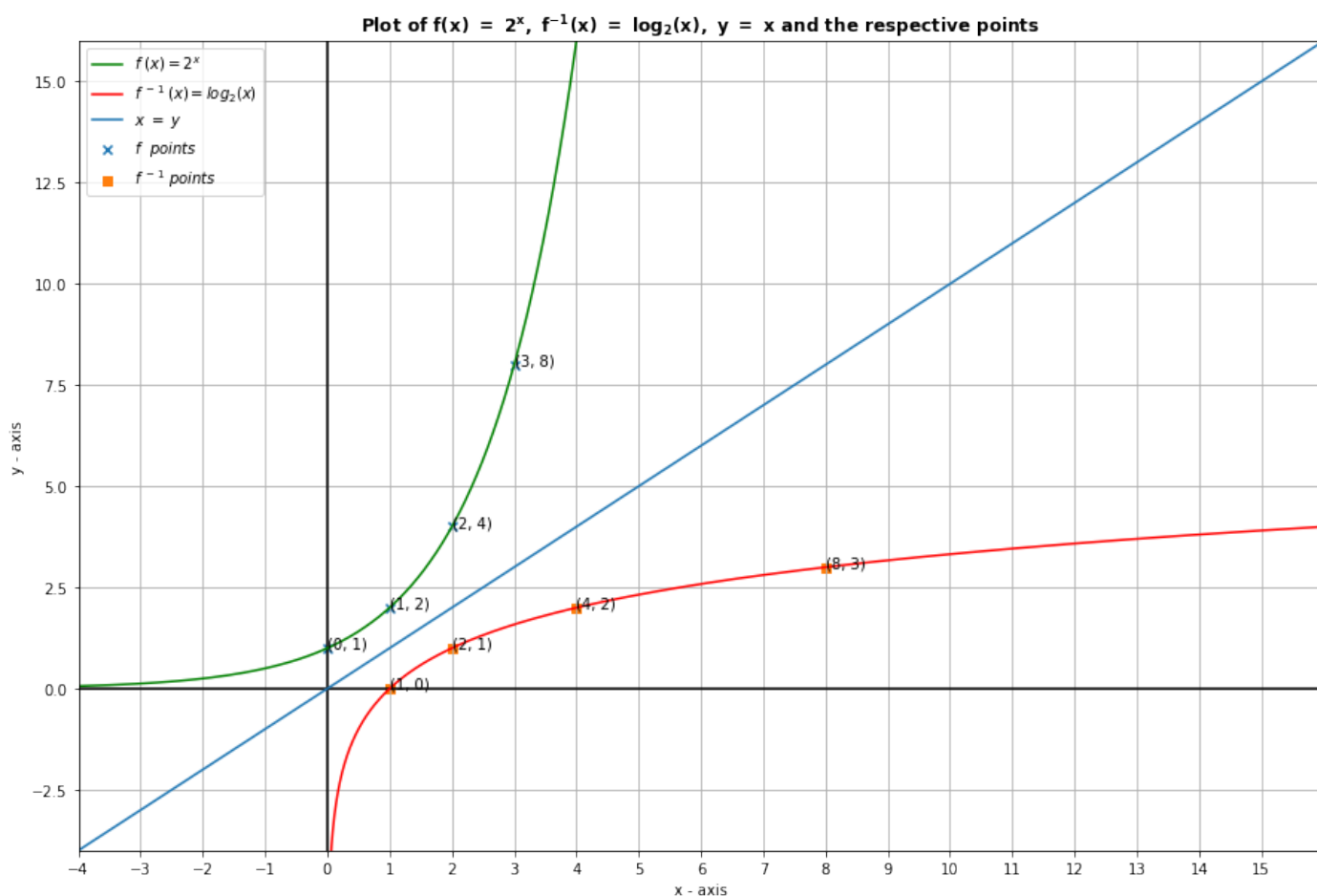


Figure 2

Question 5

- (a) The derivative of function $f(x) = ax^2$ is $f'(x) = 2ax$ (**a** → **4**)
- (b) The derivative of function $f(x) = \cos(2\pi ft)$ is $f'(x) = -\sin(2\pi ft)$ (**b** → **1**)
- (c) The derivative of function $f(x) = bx^3$ is $f'(x) = 3bx^2$ (**c** → **2**)
- (d) The derivative of function $f(x) = e^{cx}$ is $f'(x) = ce^{cx}$ (**d** → **3**)

Question 6

$S(x)$ is differentiable in $Dom(S)$, as a composition of differentiable functions with $S'(x)$ as follows:

$$\begin{aligned} S'(x) &= \left(\frac{1}{1+e^{-x}} \right)' = [(1+e^{-x})^{-1}]' = (-1)(1+e^{-x})^{-2}(1+e^{-x})' = -\frac{(e^{-x})'}{(1+e^{-x})^2} \\ &= \frac{e^{-x}}{(1+e^{-x})^2} \quad (1) \end{aligned}$$

$$\begin{aligned} S(x) (1 - S(x)) &= \left(\frac{1}{1+e^{-x}} \right) \left(1 - \frac{1}{1+e^{-x}} \right) = \left(\frac{1}{1+e^{-x}} \right) \left(\frac{1+e^{-x}-1}{1+e^{-x}} \right) = \left(\frac{1}{1+e^{-x}} \right) \left(\frac{e^{-x}}{1+e^{-x}} \right) \\ &= \frac{e^{-x}}{(1+e^{-x})^2} \quad (2) \end{aligned}$$

$$\stackrel{(1), (2)}{\implies} S'(x) = S(x) (1 - S(x))$$

Question 7

$$S(x) = \frac{1}{1 + e^{-x}}, \quad f(x) = ax + b, \quad S(f(x)) = \frac{1}{1 + e^{-f(x)}}$$

$S(f(x))$ is differentiable in $Dom(S)$, as a composition of differentiable functions S , f , with $S'(f(x))$ as follows:

$$\begin{aligned} S'(f(x)) &= \left(\frac{1}{1 + e^{-f(x)}} \right)' \stackrel{\text{Question 1}}{=} \left(\frac{e^{ax}e^b}{e^{ax}e^b + 1} \right)' = \frac{(e^{ax}e^b)'(e^{ax}e^b + 1) - (e^{ax}e^b)(e^{ax}e^b + 1)'}{(e^{ax}e^b + 1)^2} \\ &= \frac{(ae^{ax}e^b)(e^{ax}e^b + 1) - (e^{ax}e^b)(ae^{ax}e^b)}{(e^{ax}e^b + 1)^2} = \frac{(ae^{ax}e^b)(e^{ax}e^b + 1 - e^{ax}e^b)}{(e^{ax}e^b + 1)^2} \\ &= \frac{ae^{ax+b}}{(e^{ax}e^b + 1)^2} \end{aligned}$$

Question 8

Derivatives of f_n with respect to μ :

$$i) f_1(\mu) = \log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1 - \mu)^2}{2\sigma^2}}\right)$$

$$\begin{aligned}\frac{df_1(\mu)}{d\mu} &= f_1'(\mu) = \left(\log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1 - \mu)^2}{2\sigma^2}}\right)\right)' = \left(\log\left(e^{\frac{-(x_1 - \mu)^2}{2\sigma^2}}\right)\right)' \\ &= \left(\frac{-(x_1 - \mu)^2}{2\sigma^2}\right)' = \left(-\frac{2(x_1 - \mu)(x_1 - \mu)'}{2\sigma^2}\right) \\ &= \frac{x_1 - \mu}{\sigma^2}\end{aligned}$$

$$ii) f_2(\mu) = \log \left(\frac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)$$

$$\begin{aligned} \frac{df_2(\mu)}{d\mu} &= f_2'(\mu) = \left(\log \left(\frac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right) \right)' \\ &= \left(\log \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)' = \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} \\ &= \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \right)' + \left(e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \frac{\left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\frac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} \\ &= \frac{\left(-\frac{2(x_1-\mu)(x_1-\mu)'}{2\sigma^2} \right) e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(-\frac{2(x_2-\mu)(x_2-\mu)'}{2\sigma^2} \right) e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} \\ &= \frac{\frac{x_1-\mu}{\sigma^2} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \frac{x_2-\mu}{\sigma^2} e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \frac{(x_1-\mu) e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}}{\sigma^2} + \frac{(x_2-\mu) e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2} \\ &= \frac{\frac{(x_1-\mu) e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + (x_2-\mu) e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \frac{(x_1-\mu) e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + (x_2-\mu) e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2 \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)} \end{aligned}$$

Question 9

Derivatives of f_n with respect to σ (in the following functions, let $\log \equiv \log_e \equiv \ln$) :

$$i) f_1(\sigma) = \log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)$$

$$\frac{df_1(\sigma)}{d\sigma} = f'_1(\sigma) = \left(\log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right) \right)'$$

$$= \left(\log(1) \right)' - \left(\log(\sigma\sqrt{2\pi}) \right)' + \left(\log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right) \right)'$$

$$= -\frac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}} + \left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' = -\frac{1}{\sigma} - \frac{(x_1-\mu)^2}{2} \left(\frac{1}{\sigma^2} \right)'$$

$$= -\frac{1}{\sigma} + \frac{(x_1-\mu)^2}{2} \left(\frac{2}{\sigma^3} \right) = -\frac{1}{\sigma} + \frac{(x_1-\mu)^2}{\sigma^3} = \frac{-\sigma^2 + (x_1-\mu)^2}{\sigma^3}$$

$$= \frac{x_1^2 + \mu^2 - 2x_1\mu - \sigma^2}{\sigma^3}$$

$$ii) f_2(\sigma) = \log \left(\frac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)$$

$$\frac{df_2(\sigma)}{d\sigma} = f'_2(\sigma) = \log \left(\frac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= \left(\log \left(\frac{1}{\sigma\sqrt{2\pi}} \right) \right)' + \left(\log \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= -\frac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}} + \left(\log \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= -\frac{1}{\sigma} + \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = -\frac{1}{\sigma} + \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \right)' + \left(e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\frac{1}{\sigma} + \frac{\left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\frac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\frac{1}{\sigma} + \frac{\left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\frac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\frac{1}{\sigma} + \frac{\frac{(x_1-\mu)^2}{\sigma^3} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \frac{(x_2-\mu)^2}{\sigma^3} e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

Question 10

Mean value of random variable X :

$$\begin{aligned} E[X] &= \int_a^b x f_x(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \int_a^b \left(\frac{x^2}{2}\right)' dx \\ &= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} \\ &= \frac{a+b}{2} \end{aligned}$$

Second moment of random variable X :

$$\begin{aligned} E[X^2] &= \int_a^b x^2 f_x(x) dx = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \int_a^b \left(\frac{x^3}{3}\right)' dx \\ &= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(a^2 + ab + b^2)}{3(b-a)} \\ &= \frac{a^2 + ab + b^2}{3} \end{aligned}$$