



M902

Βασικές Μαθηματικές Έννοιες στη Γλωσσική Τεχνολογία

Project 3

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Question **1**

Question **2**

Question **3**

Question 4

Question 5

$$B = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$$

a) Find B^{-1}

b) Verify $B \cdot B^{-1} = B^{-1} \cdot B = I$

Following the formula of the inverse matrix:

$$a) B^{-1} = \frac{1}{|B|} \cdot adj(B) = \frac{1}{|B|} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^T = \frac{1}{|B|} \cdot \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix} = \frac{1}{|B|} \cdot \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}$$

$$det(B) = |B| = \begin{vmatrix} 2 & -3 \\ 4 & -2 \end{vmatrix} = 2 \cdot (-2) - (4 \cdot (-3)) = -4 + 12 = 8 \quad (1)$$

$$adj(B) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^T = \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -4 & 2 \end{bmatrix} \quad (2)$$

$$\stackrel{(1), (2)}{\implies} B^{-1} = \frac{1}{|B|} \cdot adj(B) = \frac{1}{8} \cdot \begin{bmatrix} -2 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{8} & \frac{3}{8} \\ -\frac{4}{8} & \frac{2}{8} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \quad (3)$$

$$b) B \cdot B^{-1} = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{2}{4} + \frac{3}{2} & \frac{6}{8} - \frac{3}{4} \\ -\frac{4}{4} + \frac{2}{2} & \frac{12}{8} - \frac{2}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$B^{-1} \cdot B = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{4} + \frac{12}{8} & \frac{3}{4} - \frac{6}{8} \\ -\frac{2}{2} + \frac{4}{4} & \frac{3}{2} - \frac{2}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Question 6

Let $A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, the unknown vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and the dot products vector

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix}.$$

This system of linear equations in its condensed form, is written as follows:

$$A \cdot x = b$$

The unknown vector x is calculated by:

$$A \cdot x = b \implies A^{-1} \cdot A \cdot x = A^{-1} \cdot b \implies I \cdot x = A^{-1} \cdot b \implies x = A^{-1} \cdot b \quad (1)$$

$$A^{-1} = \frac{1}{|A|} \text{adj}(A) \quad (2)$$

Determinant of matrix A :

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 2 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + (-1)^{1+3} \cdot 0 \cdot \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1 \cdot 1 \cdot (0 \cdot 2 - (-1) \cdot 1) - 1 \cdot 2 \cdot (2 \cdot 2) + 0 = +1 - 8 = -7 \quad (3) \end{aligned}$$

Adjoint matrix (transpose of cofactor matrix) of A :

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix},$$

where $A_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$, i, j the rows and columns of matrix A respectively and M_{ij} the matrix made by removing row i and column j of matrix A . The proper signs of the cofactors in the **adjoint** (adjugate) matrix, are included in the calculation of A_{ij} .

$$A_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 1, \quad A_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} = -4, \quad A_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$$

$$A_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = -4, \quad A_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2, \quad A_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1$$

$$A_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} = -2, \quad A_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 1, \quad A_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

$$\text{adj}(A) = \begin{bmatrix} 1 & -4 & -2 \\ -4 & 2 & 1 \\ 2 & -1 & -4 \end{bmatrix} \quad (4)$$

$$\stackrel{(2), (3), (4)}{\implies} A^{-1} = \frac{1}{-7} \cdot \begin{bmatrix} 1 & -4 & -2 \\ -4 & 2 & 1 \\ 2 & -1 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{2}{7} & -\frac{1}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{4}{7} \end{bmatrix} \quad (5)$$

$$\stackrel{(1), (5)}{\implies} x = A^{-1} \cdot b \implies$$

$$x = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{2}{7} & -\frac{1}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{4}{7} \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \cdot 5 + \frac{4}{7} \cdot 8 + \frac{2}{7} \cdot 9 \\ \frac{4}{7} \cdot 5 - \frac{2}{7} \cdot 8 - \frac{1}{7} \cdot 9 \\ -\frac{2}{7} \cdot 5 + \frac{1}{7} \cdot 8 + \frac{4}{7} \cdot 9 \end{bmatrix} = \begin{bmatrix} \frac{45}{7} \\ -\frac{5}{7} \\ \frac{34}{7} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\implies x_1 = \frac{45}{7}, x_2 = -\frac{5}{7}, x_3 = \frac{34}{7}$$

(coordinates of the previously-but-not-still-known vector x)

Question 7

$$A = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$

a) Find A^2

b) Find A^{-1}

$$a) A^2 = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \cdot \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

where I_2 , the identity matrix of dimension 2×2 .

$$b) A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{|A|} \cdot \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \frac{1}{|A|} \cdot \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \quad (1)$$

$$\det(A) = |A| = \begin{vmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{vmatrix} = -\cos^2 \theta - \sin^2 \theta = -(\cos^2 \theta + \sin^2 \theta) = -1 \quad (2)$$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} -\cos \theta & -\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (3)$$

$$\stackrel{(1), (2), (3)}{\Rightarrow} A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{|A|} \cdot \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = -1 \cdot \begin{bmatrix} -\cos \theta & -\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} = A$$

From questions 7.a and 7.b both, we extract the information that A is an **involutory** matrix, which means it is **its own reverse**, as it complies with the following:

$$A^2 = I_2$$

$$A^{-1} = A$$

Question 8

Vectors: $u = [3,4]$, $v = [4,3]$

The included angle θ of two vectors, is dependent on their respective magnitudes and inner product and is calculated as follows:

$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos \theta_{uv} \implies \cos \theta_{uv} = \frac{u \cdot v}{\|u\| \cdot \|v\|} \quad (1)$$

$$\stackrel{(1)}{\implies} \cos \theta_{uv} = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{[3,4] \cdot [4,3]}{\|[3,4]\| \cdot \|[4,3]\|} = \frac{12 + 12}{(\sqrt{9 + 16}) \cdot (\sqrt{16 + 9})} = \frac{24}{5 \cdot 5}$$

$$= \frac{24}{25} = 0.96 \implies \cos^{-1}(0.96) = 16.26^\circ \implies \theta_{uv} = 16.26^\circ$$

Vectors u and v are **not collinear**, as their included angle θ is neither 0° nor 180° ($\cos \theta = 1$ and $\cos \theta = -1$ respectively). They are also **not orthogonal**, as in that case, their included angle θ would be 90° ($\cos \theta = 0$).

Question 9

Vectors: $u = [a, b]$, $v = 2u$

As described in **Question 8**, the included angle θ of two vectors, is dependent on their respective magnitudes and inner product, calculated by using formula **(1)** :

$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos \theta_{uv} \implies \cos \theta_{uv} = \frac{u \cdot v}{\|u\| \cdot \|v\|} \quad (1)$$

$$\stackrel{(1)}{\implies} \cos \theta_{uv} = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \cos \theta_{uv} = \frac{u \cdot 2u}{\|u\| \cdot \|2u\|} = \frac{[a, b] \cdot [2a, 2b]}{\|[a, b]\| \cdot \|[2a, 2b]\|} =$$

$$\frac{2 \cdot (a^2 + b^2)}{(\sqrt{a^2 + b^2}) \cdot (\sqrt{4a^2 + 4b^2})} = \frac{2 \cdot (a^2 + b^2)}{2 \cdot \left(\sqrt{a^2 + b^2}\right)^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$$

$$\implies \cos^{-1}(1) = 0^\circ \implies \theta_{uv} = 0^\circ$$

In this case, the two vectors are **collinear**, as vector v is a **scalar multiple** of vector u ($v = k \cdot u$), and have the **same direction** as the **scalar** k is a **positive** number ($k = 2$), therefore their included angle θ is 0° (validated by the calculations above).

Question 10

In order to find the similar documents based on the 3-dimensional vector representations, $d_n \in \mathbb{R}^3$, $n = 1, \dots, 5$, a 5x5 matrix was constructed, with each row being each of the 3-D vectors d_1, d_2, d_3, d_4, d_5 , normalised as follows:

$$\sum_{i=1}^5 \sum_{j=1}^3 \frac{d_{ij}}{\|d_i\|}$$

where d_{ij} the j -th element of the i -th vector and $\|d_i\| = \sqrt{d_{i1}^2 + d_{i2}^2 + d_{i3}^2}$ the norm of the respective vector i :

$$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} \Rightarrow D_{5 \times 3} = \begin{bmatrix} 8 & 6 & 0 \\ 0 & 6 & 8 \\ 6 & 0 & 8 \\ 2 & 3 & 0 \\ 9 & 6 & 0 \end{bmatrix} \Rightarrow D_{normal} = \begin{bmatrix} \frac{8}{10} & \frac{6}{10} & \frac{0}{10} \\ \frac{0}{10} & \frac{6}{10} & \frac{8}{10} \\ \frac{6}{10} & \frac{0}{10} & \frac{8}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{0}{10} \\ \frac{9}{10.8166} & \frac{6}{10.8166} & \frac{0}{10.8166} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0.6 & 0.8 \\ 0.6 & 0 & 0.8 \\ 0.5547 & 0.8320 & 0 \\ 0.8320 & 0.5547 & 0 \end{bmatrix}$$

Finally, we multiply **normalized** matrix **D** with its **transpose**, to get the inner product of every pair of vectors and perform the comparisons between the document representations:

$$D_{normal} \cdot D_{normal}^T = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0.6 & 0.8 \\ 0.6 & 0 & 0.8 \\ 0.5547 & 0.8320 & 0 \\ 0.8320 & 0.5547 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0 & 0.6 & 0.5547 & 0.8320 \\ 0.6 & 0.6 & 0 & 0.8320 & 0.5547 \\ 0 & 0.8 & 0.8 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.36 & 0.48 & 0.9429 & 0.9984 \\ \mathbf{0.36} & 1 & 0.64 & 0.4992 & 0.3328 \\ \mathbf{0.48} & \mathbf{0.64} & 1 & 0.3328 & 0.4992 \\ \mathbf{0.9429} & \mathbf{0.4992} & \mathbf{0.3328} & 1 & 0.9230 \\ \mathbf{0.9984} & \mathbf{0.3328} & \mathbf{0.4992} & \mathbf{0.9230} & 1 \end{bmatrix} = D_{normal_dot_product}$$

We keep only the **lower triangular matrix**, where each value represents the dot product of the j -th vector (d_j) with the i -th vector (d_i), where i, j the rows and the columns of the matrix $D_{dot_product}$ respectively.

The formula of the dot product of two vectors, using the included angle θ , is:

$$d_i \cdot d_j = \|d_i\| * \|d_j\| * \cos \theta \implies \cos \theta = \frac{d_i \cdot d_j}{\|d_i\| * \|d_j\|} = \cos \theta_{d_i, d_j}$$

The process we followed above, calculates exactly the value of the cosine of the included angle θ , giving information on the relation between those vectors. If the angle's value is close to 0, then the vectors are orthogonal, if it is close to 1 they are

Therefore, the results are the following:

Inner Product (of normalized vectors)	Value	Similarity
$\cos \theta_{d_1, d_2} = 0.36$		Low
$\cos \theta_{d_1, d_3} = 0.48$		
$\cos \theta_{d_1, d_4} = 0.9429$	$> 0.94 \implies \theta_{d_1, d_4} < 19^\circ$	High
$\cos \theta_{d_1, d_5} = 0.9984$	$\sim 1 \implies \theta_{d_1, d_5} \sim 0^\circ$	High
$\cos \theta_{d_2, d_3} = 0.64$		
$\cos \theta_{d_2, d_4} = 0.4492$		
$\cos \theta_{d_2, d_5} = 0.3328$		Low
$\cos \theta_{d_3, d_4} = 0.3328$		Low
$\cos \theta_{d_3, d_5} = 0.4992$		
$\cos \theta_{d_4, d_5} = 0.9230$	$> 0.92 \implies \theta_{d_4, d_5} < 23^\circ$	High

Table 3.10

The **Documents 1, 4 and 5** are **very similar**.

Notes

For some questions of project 3, some **code** was developed to **experiment** with the given tasks and **validate** the manually calculated results, the file (M902_Project_3_CTKylafi_LT1200012.py) of which is included in the uploaded .zip file (M902_Project3_CTKylafi_LT1200012.zip).