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Question **1**

The composite function $S(f(x))$, where $S(x) = \dfrac{1}{1+e^{-x}}$ and $f(x) = ax + b$, is calculated as follows:

$$S(f(x)) = \frac{1}{1+e^{-f(x)}} = \frac{1}{1+e^{-(ax+b)}} = \frac{1}{1+\frac{1}{e^{(ax+b)}}} = \frac{1}{\frac{e^{(ax+b)}+1}{e^{(ax+b)}}} = \frac{e^{(ax+b)}}{e^{(ax+b)}+1} = \frac{e^{ax}e^{b}}{e^{ax}e^{b}+1}$$

where:

- $x \in Dom(f) = \mathbb{R}$
- $f(x) \in Im(f) \subseteq Dom(S) = \mathbb{R}$
- $S(f(x)) \in Im(S)$

Question **2**

A function is **invertible** only if each input has a unique output, which means each output is paired with exactly one input. That way, when the mapping is reversed, it will still be a function.

Definition

Let f be a function whose domain is the set A and whose codomain is the set B ($f : A \to B$). Then we say that f is invertible if f is **one-to-one** mapping and there is a function g with domain $Im(f) \subseteq B$ and image (range) A ($g : Im(f) \to A$) such that:

$$f(x) = y \iff g(y) = x$$

In this case, we call g the inverse of f and denote it by f^{-1}.

i) $\mathbf{f(x) = \sqrt{x-3}}$

• $x - 3 \geq 0 \iff x \geq 3 \iff Dom(f) = [\ 3,\ +\infty\ )$ **(1)**

• f is **continuous** on $Dom(f) = [\ 3,\ +\infty\ )$ [1] **(2)**

• f is **differentiable** on $A = \{\ Dom(f) - \{3\}\ \} = \{\ [\ 3,\ +\infty\ ) - \{3\}\ \}$

$= \{\ (\ 3,\ +\infty\ )\ \}$ [2] **(3)**

First, we have to prove that f is **one-to-one** mapping:

$$\overset{(2),\ (3)}{\Longrightarrow} f'(x) = (\ \sqrt{x-3}\ )' = \frac{1}{2\sqrt{x-3}} > 0,\ \forall\ x \in A = (3,\ +\infty) \qquad \textbf{(4)}$$

$\overset{(4)}{\Longrightarrow} f$ is **strictly monotonic** (strictly inscreasing), therefore it is **one-to-one** mapping

One approach for finding a formula for f^{-1} is to solve $f(x) = y$ for x.

$$f(x) = \sqrt{x-3} = y \overset{y \geq 0,\ x \geq 3}{\iff} x - 3 = y^2 \iff x = y^2 + 3 = f^{-1}(y) \qquad \textbf{(5)}$$

$$\overset{(1)}{\Longrightarrow} x \geq 3 \iff y^2 + 3 \geq 3 \iff y^2 \geq 0,\ \ true\ \forall\ y \in \mathbb{R} \qquad \textbf{(6)}$$

---

[1] as a composition of continuous functions

[2] as a composition of differentiable functions

$$\xrightarrow{(5),\,(6)} Im(f) = \{\ \mathbb{R}\ \cap\ [\ 0,\ +\infty\ )\ \} = [\ 0,\ +\infty\ ) = Dom(f^{-1})$$

• $f : [\ 3,\ +\infty\ ] \to \mathbb{R}$

$f(x) = \sqrt{x-3}$

• $f^{-1} : [\ 0,\ +\infty\ ] \to \mathbb{R}$

$f^{-1}(x) = x^2 + 3$

ii) f(x) = log(x − 2)

(Let $f(x)$ logarithmic function with base 2, as 2^x which is $log_2(x)$ function's inverse, is also used in **Question 4**)

• $x - 2 > 0 \iff x > 2 \iff Dom(f) = (\ 2,\ +\infty\ )$ $\quad$ (**7**)

First, we have to prove that f is **one-to-one** mapping:

$Let\ x_1,\ x_2\ \in\ Dom(f) = (\ 0,\ +\infty\ ),\ x_1 \neq x_2$

$f(x_1) = f(x_2) \iff log_2(x_1 - 2) = log_2(x_2 - 2) \iff x_1 - 2 = x_2 - 2 \iff x_1 = x_2$

$\iff$ f is an **one-to-one** mapping of $Dom(f)$ to $Im(f)$

As in i above, we will solve $f(x) = y$ for x.

$f(x) = \log_2(x-2) = y \iff 2^{log_2(x-2)} = 2^y \iff x - 2 = 2^y$

$\iff x = 2^y + 2 = f^{-1}(y)$ $\quad$ (**8**)

$\xrightarrow{(7)} x > 2 \iff 2^y + 2 > 2 \iff 2^y > 0,\ true\ \forall\ y \in \mathbb{R}$ $\quad$ (**9**)

$\xrightarrow{(9)} Im(f) = \mathbb{R} = Dom(f^{-1})$

• $f : (\ 2,\ +\infty\ ] \to \mathbb{R}$

$f(x) = log(x-2)$

• $f^{-1} : \mathbb{R} \to \mathbb{R}$

$f^{-1}(x) = 2^x + 2$

Question **3**

$y = A\ cos(\ 2\pi f x\ )$

(a) $A_1 = 1, \quad f_1 = 1, \quad \theta_1 = 0, \quad y_1 = cos(\ 2\pi x\ )$

(b) $A_2 = 2, \quad f_2 = 3, \quad \theta_2 = 0, \quad y_2 = 2\ cos(\ 6\pi x\ )$

(c) $A_3 = 1.5, \quad f_3 = 2, \quad \theta_3 = \pi, \quad y_3 = 1.5\ cos(\ 4\pi x\ + \ \pi)$

(d) $A_4 = 2, \quad f_4 = 0.5, \quad \theta_4 = 0, \quad y_4 = 2\ cos(\ \pi x\ )$

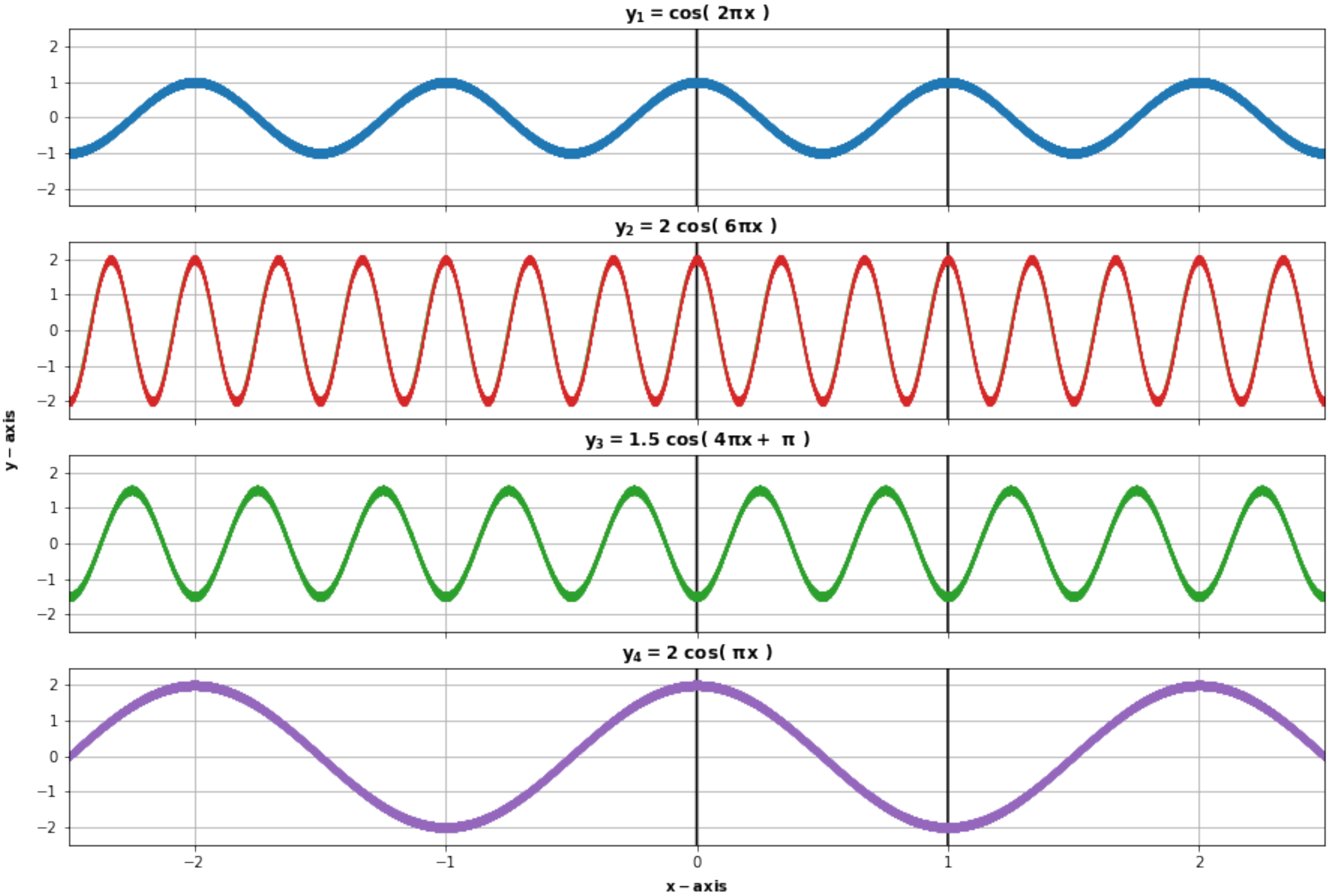

Figure 1

Question 4

Following the steps in **Question 2**, the inverse function of $f(x) = 2^x$ is calculated similarly:

$$f(x) = 2^x = y \overset{y>0}{\iff} log_2(\ 2^x\ ) = log_2(\ y\ ) \iff x = log_2(y) = f^{-1}(y)$$

• $f^{-1} : \ (\ 0,\ +\infty\ ) \rightarrow \mathbb{R}$
$f^{-1}(x) = log_2(\ x\ )$

In **Figure 2**, functions $f,\ f^{-1},\ y = x$ and the respective points are depicted.

Points of f : $(0,1),\ (1,2),\ (2,4),\ (3,8)$
Points of f^{-1} : $(1,0),\ (2,1),\ (4,2),\ (8,3)$

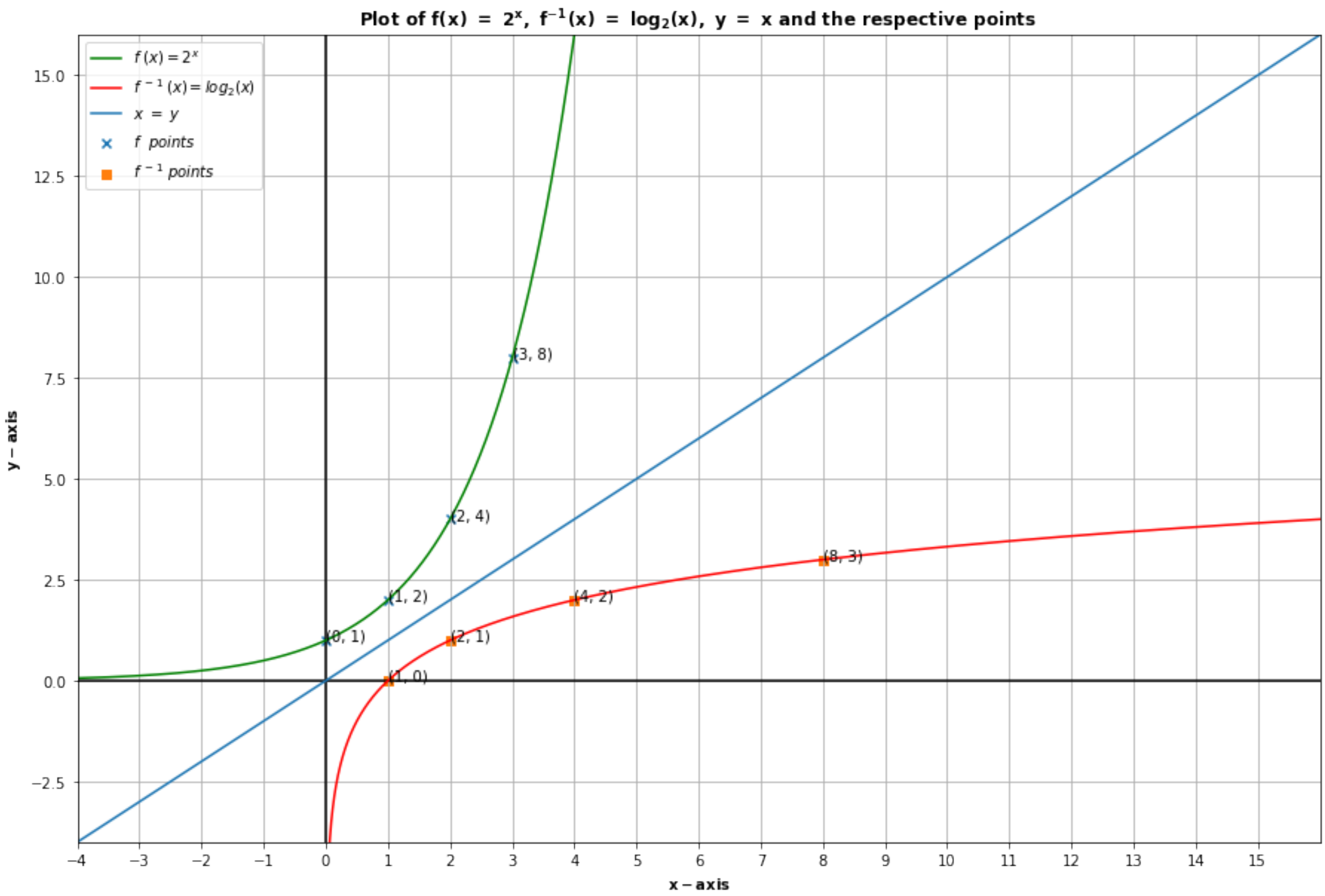

Figure 2

Question **5**

(a) The derivative of function $f(x) = ax^2$ is $f'(x) = 2ax$ ($\mathbf{a} \rightarrow \mathbf{4}$)

(b) The derivative of function $f(x) = cos(2\pi fx)$ is $f'(x) = -sin(2\pi fx)$ ($\mathbf{b} \rightarrow \mathbf{1}$)

(c) The derivative of function $f(x) = bx^3$ is $f'(x) = 3bx^2$ ($\mathbf{c} \rightarrow \mathbf{2}$)

(d) The derivative of function $f(x) = e^{cx}$ is $f'(x) = ce^{cx}$ ($\mathbf{d} \rightarrow \mathbf{3}$)

Question **6**

$S(x)$ is differentiable in $Dom(S)$, as a composition of differentiable functions with $S'(x)$ as follows:

$$S'(x) = \left(\frac{1}{1+e^{-x}}\right)' = [(1+e^{-x})^{-1}]' = (-1)(1+e^{-x})^{-2}(1+e^{-x})'$$

$$= -\frac{(e^{-x})'}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2} \qquad (\ \mathbf{1}\ )$$

$$S(x)\ (1 - S(x)) = \left(\frac{1}{1+e^{-x}}\right)\left(1 - \frac{1}{1+e^{-x}}\right) = \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2} \qquad (\ \mathbf{2}\ )$$

$$\overset{(1)\ ,\ (2)}{\Longrightarrow}\quad S'(x) = S(x)\ (\ 1 - S(x)\ )$$

Question **7**

$$S(x) = \frac{1}{1+e^{-x}}, \quad f(x) = ax + b, \quad S(f(x)) = \frac{1}{1+e^{-f(x)}}$$

$S(f(x))$ is differentiable in $Dom(S)$, as a composition of differentiable functions $S,\ f,$ with $S'(f(x))$ as follows:

$$S'(f(x)) = \left(\frac{1}{1+e^{-f(x)}}\right)' \overset{Question\ 1}{=} \left(\frac{e^{ax}e^{b}}{e^{ax}e^{b}+1}\right)'$$

$$= \frac{(e^{ax}e^{b})'(e^{ax}e^{b}+1)\ -\ (e^{ax}e^{b})(e^{ax}e^{b}+1)'}{(e^{ax}e^{b}+1)^2}$$

$$= \frac{(ae^{ax}e^{b})(e^{ax}e^{b}+1)\ -\ (e^{ax}e^{b})(ae^{ax}e^{b})}{(e^{ax}e^{b}+1)^2}$$

$$= \frac{(ae^{ax}e^{b})(e^{ax}e^{b}+1\ -\ e^{ax}e^{b})}{(e^{ax}e^{b}+1)^2}$$

$$= \frac{ae^{ax+b}}{(e^{ax}e^{b}+1)^2}$$

Question **8**

Derivatives of f_n with respect to μ (in the following functions, let $log \equiv log_e \equiv ln$) :

$$i)\ f_1(\mu) = log\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)$$

$$\frac{df_1(\mu)}{d\mu} = f_1'(\mu) = \left(log\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)\right)' = \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)\right)'$$

$$= \left(\frac{-(x_1-\mu)^2}{2\sigma^2}\right)' = \left(-\frac{2\ (x_1-\mu)\ (x_1-\mu)'}{2\sigma^2}\right)$$

$$= \frac{x_1-\mu}{\sigma^2}$$

$$ii)\ f_2(\mu) = log\left(\frac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)$$

$$\frac{df_2(\mu)}{d\mu} = f_2'(\mu) = \left(log\left(\frac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right) \right)'$$

$$= \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)' = \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \right)' + \left(e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \frac{\left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\frac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\left(-\frac{2\ (x_1-\mu)\ (x_1-\mu)'}{2\sigma^2} \right) e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(-\frac{2\ (x_2-\mu)\ (x_2-\mu)'}{2\sigma^2} \right) e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\frac{x_1-\mu}{\sigma^2}\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \frac{x_2-\mu}{\sigma^2}\ e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \frac{\frac{(x_1-\mu)\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}}{\sigma^2} + \frac{(x_2-\mu)\ e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \frac{\frac{(x_1-\mu)\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + (x_2-\mu)\ e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \frac{(x_1-\mu)\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + (x_2-\mu)\ e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)}$$

Question **9**

Derivatives of f_n with respect to σ (in the following functions, let $log \equiv log_e \equiv ln$) :

$$i)\ f_1(\sigma) = log\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)$$

$$\frac{df_1(\sigma)}{d\sigma} = f_1'(\sigma) = \left(log\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)\right)'$$

$$= \left(\ log(1)\ \right)' - \left(\ log(\sigma\sqrt{2\pi})\ \right)' + \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\ \right)\right)'$$

$$= -\ \frac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}}\ + \left(\ \frac{-(x_1-\mu)^2}{2\sigma^2}\ \right)' = -\ \frac{1}{\sigma}\ -\frac{(x_1-\mu)^2}{2}\left(\ \frac{1}{\sigma^2}\right)'$$

$$= -\ \frac{1}{\sigma}\ + \frac{(x_1-\mu)^2}{2}\left(\frac{2}{\sigma^3}\right) = -\ \frac{1}{\sigma}\ + \frac{(x_1-\mu)^2}{\sigma^3} = \frac{-\sigma^2+(x_1-\mu)^2}{\sigma^3}$$

$$= \frac{x_1^2+\mu^2-2x_1\mu-\sigma^2}{\sigma^3}$$

$$ii)\ f_2(\sigma) = log\left(\frac{1}{\sigma\sqrt{2\pi}}\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)$$

$$\frac{df_2(\sigma)}{d\sigma} = f_2'(\sigma) = log\left(\frac{1}{\sigma\sqrt{2\pi}}\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= \left(log\left(\frac{1}{\sigma\sqrt{2\pi}} \right) \right)' + \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$\overset{i}{=} -\frac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}} + \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= -\frac{1}{\sigma} + \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = -\frac{1}{\sigma} + \frac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \right)' + \left(e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\frac{1}{\sigma} + \frac{\left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\frac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\frac{1}{\sigma} + \frac{\frac{(x_1-\mu)^2}{\sigma^3}\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \frac{(x_2-\mu)^2}{\sigma^3}\ e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \frac{-e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} - e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} + \frac{(x_1-\mu)^2}{\sigma^2}\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \frac{(x_2-\mu)^2}{\sigma^2}\ e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma\ \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)}$$

$$= \frac{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\left(\frac{(x_1-\mu)^2}{\sigma^2} - 1 \right) + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}\left(\frac{(x_2-\mu)^2}{\sigma^2} - 1 \right)}{\sigma\ \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)}$$

Question **10**

Mean value of random variable X :

$$E[X] = \int_a^b x\ f_x(x)\ dx = \int_a^b x\ \frac{1}{b-a}\ dx = \frac{1}{b-a}\ \int_a^b x\ dx = \frac{1}{b-a}\ \int_a^b \left(\frac{x^2}{2}\right)'\ dx$$

$$= \frac{1}{b-a}\ \left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\ \left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2\ (b-a)}$$

$$= \frac{a+b}{2}$$

Second moment of random variable X :

$$E[X^2] = \int_a^b x^2\ f_x(x)\ dx = \int_a^b x^2\ \frac{1}{b-a}\ dx = \frac{1}{b-a}\ \int_a^b x^2\ dx = \frac{1}{b-a}\ \int_a^b \left(\frac{x^3}{3}\right)'\ dx$$

$$= \frac{1}{b-a}\ \left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a}\ \left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{b^3-a^3}{3\ (b-a)} = \frac{(b-a)(a^2+ab+b^2)}{3\ (b-a)}$$

$$= \frac{a^2+ab+b^2}{3}$$