Strategic Games and Nash Equilibrium

Algorithmic Game Theory

2012

Next week's lecture will start 15 minutes late!

That is at 16:30.

Selfish Routing and the Price of Anarchy

- Routing games on directed graphs with latency functions
- Flow controlled by infinitely many selfish players
- Wardrop equilibrium
- Pigou's example
- Price of anarchy
- Linear functions $\frac{4}{3}$.

Prisoner's Dilemma

	S	C
S	2, 2	1, 5
C	5, 1	4, 4

- Two criminals interrogated separately.
- Strategies: (C)onfess, remain (S)ilent
- Confessing yields a smaller verdict if the other one is silent
- If both confess, the verdict is larger for both (4 years) compared to when they both remain silent (2 years).

Normal Form Games

Definition

A normal form game is a triple $(\mathcal{N}, (S_i)_{i \in N}, (c_i)_{i \in N})$ where

- $\mathcal{N}$ is the set of players, $n = |\mathcal{N}|$,
- S_i is the set of (pure) strategies of player i,
- $S = S_1 \times \ldots \times S_n$ is the set of states,
- a state is $s = (s_1, \ldots, s_n) \in S$,
- $c_i : S \rightarrow \mathbb{R}$ is the cost function of player $i \in \mathcal{N}$. In state s player i has a cost of $c_i(s)$.

We denote by $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ a state s without the strategy s_i.

Remark: Sometimes games are described using utilities = negative costs.

Prisoner's Dilemma

	S	C
S	2 2	1 5
C	5 1	4 4

- If both players remain (S)ilent, the total cost is smallest.
- If both players (C)onfess, the cost is larger for both of them.
- Still, for each player confessing is always the preference!

Dominant Strategies

Definition

A pure strategy s_i is called a dominant strategy for player $i \in \mathcal{N}$ if $c_i(s_i, s_{-i}) \leq c_i(s_i', s_{-i})$ for every $s_i' \in S_i$ and every s_{-i}.

Definition

A pure strategy s_i is called a dominated strategy for player $i \in \mathcal{N}$ if there exists a $s_i' \in S_i$ with $c_i(s_i, s_{-i}) > c_i(s_i', s_{-i})$ and every s_{-i}.

Definition

A state $s = (s_1, \ldots, s_n)$ is called a dominant strategy equilibrium if for every player $1 \leq i \leq n$ strategy $s_i \in S_i$ is a dominant strategy.

Does every game have a dominant strategy equilibrium? No!

Battle of the Sexes

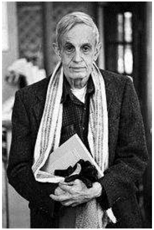
	Shop	Game
Shop	1 2	6 6
Game	5 5	2 1

- In state (Shop,Shop) the preference for both is Shop.
- In state (Game,Game) the preference for both is Game.

$\Rightarrow$ No global preference.

What is a likely outcome in this situation?

John Nash



- Born in 1928
- Obtained a PhD in 1950 for a 23-page thesis
- Subsequently moved to real algebraic geometry
- Received a Nobel prize for his PhD work in 1994
- Is currently a faculty member at Princeton
- Movie "The Beautiful Mind"

Pure Nash Equilibrium

Definition

A strategy s_i is called a best response against a collection of strategies s_{-i} if $c_i(s_i, s_{-i}) \leq c_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

Note: s_i dominant strategy $\Leftrightarrow$ s_i best response for all s_{-i}.

Definition

A state $s = (s_1, \ldots, s_n)$ is called a pure Nash equilibrium if s_i is a best response against the other strategies s_{-i} for every player $1 \leq i \leq n$.

A Nash equilibrium

- ... is a collection of local preferences in the game.
- ... is stable against unilateral deviation.

Does every game have a pure Nash equilibrium? No!

Rock-Paper-Scissors

	R	P	S
R	0 ↓→ 0	-1 ↓ 1	1 ← -1
P	1 → -1	0 ↓→ 0	-1 ↑ 1
S	-1 ↑ 1	1 ← -1	0 ←↑ 0

Mixed Strategies

Definition

A mixed strategy x_i for player i is a probability distribution over the set of pure strategies S_i.

For finite games x_i is such that $x_{ij} \in [0,1]$ and $\sum_{j \in S_i} x_{ij} = 1$.

The cost of a mixed state for player i is

$$c_i(x) = \sum_{s \in S} p(s) \cdot c_i(s) \quad ,$$

where $p(s) = \prod_{i \in \mathcal{N}, j = s_i} x_{ij}$ is the probability that the outcome is pure state s.

Mixed Nash Equilibrium

Definition

A (mixed) best response strategy x_i against a collection of mixed strategies x_{-i} is such that $c(x_i, x_{-i}) \leq c_i(x_i', x_{-i})$ for all other mixed strategies x_i'.

Definition

A mixed state x is called a (mixed) Nash equilibrium if x_i is a best response strategy against x_{-i} for every player $1 \leq i \leq n$.

Note:

- Every pure strategy is also a mixed strategy.
- Every pure Nash equilibrium is also a mixed Nash equilibrium.

Example

	0.3	0.7	
0.2	2 1	3 2	$0.3 \cdot 1 + 0.7 \cdot 2$ $= 0.3 + 1.4$ $= 1.7$
0.8	4 1	2 5	$0.3 \cdot 1 + 0.7 \cdot 5$ $= 0.3 + 3.5$ $= 3.8$
	$0.2 \cdot 2 + 0.8 \cdot 4$ $= 0.4 + 3.2$ $= 3.6$	$0.2 \cdot 3 + 0.8 \cdot 2$ $= 0.6 + 1.6$ $= 2.2$	

- $c_1(x) = 1.7 \cdot 0.2 + 3.8 \cdot 0.8 > 1.7$ – best response is $(1, 0)$
- $c_2(x) = 3.6 \cdot 0.3 + 2.2 \cdot 0.7 > 2.2$ – best response is $(0, 1)$
- State x with $x_1 = (0.2, 0.8)$ and $x_2 = (0.3, 0.7)$ is no mixed Nash equilibrium.

Observation

In the previous example x is not a mixed Nash equilibrium, because players play suboptimal pure strategies with positive probability.

Fact

If a mixed best response x_i against x_{-i} has $x_{ij} > 0$, then j is a pure best response against x_{-i}.

The cost of x_i is a "weighted average" of the cost of the pure strategies. It is minimal if and only if the averaging is just over pure strategies with minimum cost.

Example

	1	0	
1	2 1	3 2	$1 \cdot 1 + 0 \cdot 2$ $= 1$
0	4 1	2 5	$1 \cdot 1 + 0 \cdot 5$ $= 1$
	$1 \cdot 2 + 0 \cdot 4$ $= 2$	$1 \cdot 3 + 0 \cdot 2$ $= 3$	

- State x with $x_1 = (1, 0)$ and $x_2 = (1, 0)$ is a pure (and hence also a mixed) Nash equilibrium.

Example

	1	0	
$\frac{2}{3}$	1 \ 2	2 \ 3	$1 \cdot 1 + 0 \cdot 2 = 1$
$\frac{1}{3}$	1 \ 4	5 \ 2	$1 \cdot 1 + 0 \cdot 5 = 1$
	$\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 4 = \frac{8}{3}$	$\frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 2 = \frac{8}{3}$	

- State x with $x_1 = (\frac{2}{3}, \frac{1}{3})$ and $x_2 = (1, 0)$ is a mixed Nash equilibrium.
- For the row player the upper strategy is a dominant strategy, but in the first column it is not *strictly* better. If it was strictly better in every column, the lower strategy would not be played in any mixed Nash equilibrium.

Nash Theorem

Theorem (Nash Theorem)

Every finite normal form game has a mixed Nash equilibrium.

We will use Brouwer's fixed point theorem to prove it.

Theorem (Brouwer Fixed Point Theorem)

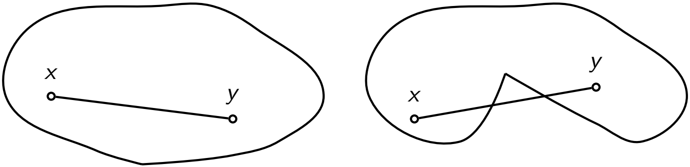
Every continuous function $f : D \to D$ mapping a compact and convex nonempty subset $D \subseteq \mathbb{R}^m$ to itself has a fixed point $x^* \in D$ with $f(x^*) = x^*$.

Brouwer's Theorem: Prerequisites and Definitions

- A set $D \subset \mathbb{R}^m$ is convex if for any $x, y \in D$ and any $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda) y \in D$.
- A subset $D \subset \mathbb{R}^m$ is compact if and only if it is closed and bounded.
- A set $D \subseteq \mathbb{R}^m$ is bounded if and only if there is some integer $M \geq 0$ such that $D \subseteq [-M, M]^m$.
- Consider a set $D \subseteq \mathbb{R}^m$ and a sequence $x_0, x_1, \ldots$, where for all $i \geq 0$, $x_i \in D$ and there is $x \in \mathbb{R}^m$ such that $x = \lim_{i \to \infty} x_i$ (i.e., for all $\epsilon > 0$ there is integer $k > 0$ such that $||x - x_j||_2 < \epsilon$ for all $j > k$). A set D is closed if $x \in D$ for every such sequence.
- A function $f : D \to \mathbb{R}^m$ is continuous at a point $x \in D$ if for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $y \in D$: If $||x - y||_2 < \delta$ then $||f(x) - f(y)||_2 < \epsilon$. f is called continuous if it is continuous at every point $x \in D$.

Brouwer's Theorem: Prerequisites and Examples

- Convex/Non-convex:

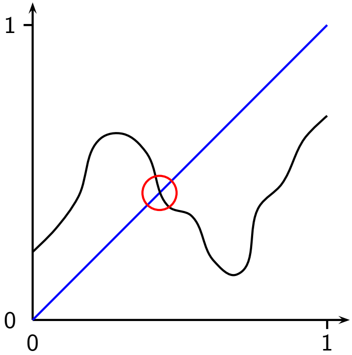


- Closed and bounded:
  $[0,1]^2$ is closed and bounded.
  $[0,1)$ is not closed but bounded.
  $[0,\infty)$ is closed and unbounded.

- Continuous: Clear.

Brouwer's Theorem: Example

Every continuous $f : [0, 1] \to [0, 1]$ has a fixed point:



For $f : [0, 1]^2 \to [0, 1]^2$: Crumpled Sheet Experiment

Nash Theorem

Theorem (Nash Theorem)

Every finite normal form game has a mixed Nash equilibrium.

Proof:

First check the conditions of Brouwer's Theorem.

Fact

The set X of mixed states $x = (x_1, \ldots, x_n)$ of a finite normal form game is a convex compact subset of $\mathbb{R}^m$ with $m = \sum_{i=1}^{n} m_i$ with $m_i = |S_i|$.

We will define a function $f : X \to X$ that transforms a state into another state. The fixed points of f are shown to be the mixed Nash equilibria of the game.

Properties of Nash equilibria

Recall:

- A mixed Nash equilibrium x is a collection of mixed best responses x_i.
- If a best response x_i against x_{-i} has $x_{ij} > 0$, then $j \in S_i$ is pure best response against x_{-i}.
- A collection of best responses (i.e. a mixed Nash equilibrium) $x = (x_1, ..., x_n)$ has

$$c_i(x) - c_i(j, x_{-i}) \leq 0 \qquad \text{for all } j \in S_i \text{ and all } i \in \mathcal{N}$$

Proof of Nash's Theorem: Definition

- For mixed state x let

$$\phi_{ij}(x) = \max\{0, c_i(x) - c_i(j, x_{-i})\} \ .$$

- Define $f : X \to X$ with $f(x) = x' = (x'_1, ..., x'_n)$ by

$$x'_{ij} = \frac{x_{ij} + \phi_{ij}(x)}{1 + \sum_{k=1}^{m_i} \phi_{ik}(x)}$$

  for all $i = 1, \ldots, n$ and $j = 1, ..., m_i$.

Fact

f satisfies the prerequisites of Brouwer's Theorem: f is continuous and if $x \in X$, then $f(x) = x' \in X$ is a state.

(Check as an exercise.)

Example

- Player i has 3 pure strategies
- Current mixed strategy $x_i = (0.2, 0.5, 0.3)$
- Current costs for strategies $c_i(\cdot, x_{-i}) = (2.2, 4.2, 2.2)$
- Current cost $c(x_i, x_{-i}) = 3.2$
- Under these conditions strategy x_i is mapped to x_i' as follows:

x_{ij}	$c_i(j, x_{-i})$	$\phi_{ij}(x)$	x_{ij}'
0.2	2.2	1	$\frac{0.2+1}{1+2} = 0.4$
0.5	4.2	0	$\frac{0.5+0}{1+2} \approx 0.166$
0.3	2.2	1	$\frac{0.3+1}{1+2} \approx 0.434$

Fixed Points

Brouwers Theorem tells us that there is x^* with $f(x^*) = x^*$. Show that $f(x) = x$ if and only if x is mixed Nash equilibrium.

- Easy: Every Nash equilibrium x has $f(x) = x$, because all $\phi_{ij}(x) = 0$.
- To show: Every fixed point $x^* = f(x^*)$ is a Nash equilibrium.

Fixed Points as Nash equilibria

For each $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$ we have

$$x^*_{ij} = \frac{x^*_{ij} + \phi_{ij}(x^*)}{1 + \sum_{k=1}^{m_i} \phi_{ik}(x^*)} \ ,$$

so

$$x^*_{ij} \cdot \left(1 + \sum_{k=1}^{m_i} \phi_{ik}(x^*)\right) = x^*_{ij} + \phi_{ij}(x^*) \ ,$$

and

$$x^*_{ij} \sum_{k=1}^{m_i} \phi_{ik}(x^*) = \phi_{ij}(x^*) \ .$$

We will show that $\sum_{k=1}^{m_i} \phi_{ik}(x^*) = 0$. This means that x^*_i chooses only pure best responses and implies that it is a mixed best response.

Fixed Points as Nash equilibria

Claim

For every mixed state x and every player $i \in \mathcal{N}$, there is some pure strategy $j \in S_i$ such that $x_{ij} > 0$ and $\phi_{ij}(x) = 0$.

Proof of Claim:

Note that $c_i(x) = \sum_{j=1}^{m_i} x_{ij} \cdot c_i(j, x_{-i})$, so there must be some j with $x_{ij} > 0$ and cost no less than this "weighted average".

More formally, there is j with $x_{ij} > 0$ and

$$c_i(x) - c_i(j, x_{-i}) \leq 0 \ .$$

Therefore, $\phi_{ij}(x) = \max\{0, c_i(x) - c_i(j, x_{-i})\} = 0$. □

Fixed Points as Nash equilibria

For every player i we consider strategy j from the claim. This implies $x_{ij}^* > 0$ and

$$x_{ij}^* \cdot \sum_{k=1}^{m_i} \phi_{ik}(x^*) = \phi_{ij}(x^*) = 0 \ .$$

Since $x_{ij}^* > 0$ it must hold that

$$\sum_{k=1}^{m_i} \phi_{ik}(x^*) = 0 \ ,$$

so $\phi_{ik}(x^*) = 0$ for all $k = 1, \ldots, m_i$. Therefore

$$c_i(x^*) \leq c_i(j, x_{-i}^*) \qquad \text{for all } j \in S_i.$$

Hence, x_i^* is a best response. This proves Nash's Theorem. $\square$

Computing Nash equilibria

How can we compute a mixed Nash equilibrium?
What is the complexity of computing a Nash equilibrium?

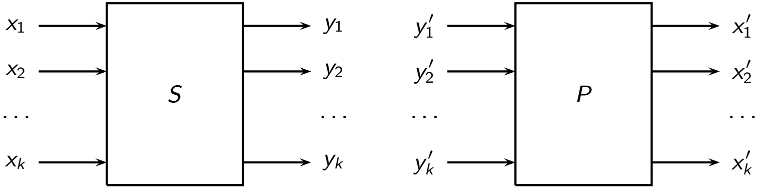
Different from problems we usually encounter:

- Not an optimization or decision problem, existence guaranteed
- Search problem, find Nash equilibrium.
- Different Complexity Class: PPAD
  (polynomial parity argument, directed case)
- Completeness Idea as for NP:
  Define PPAD-complete problem, construct polynomial reductions

There are 3-player games with rational payoffs where all mixed Nash equilibria are irrational. Thus, we can only hope to obtain approximations to mixed Nash equilibria or Brouwer fixed points.

A PPAD-complete problem

vb

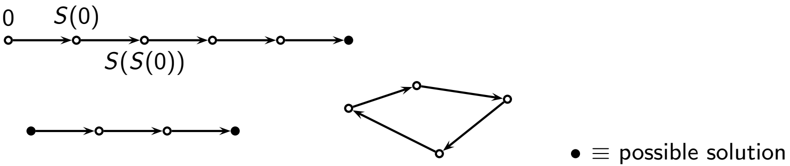


An instance of the END-OF-LINE search problem is given by

- Two circuits S and P, same number of inputs and outputs
- S and P define a directed graph:
  Vertices: k-bit vectors
  Edges: There is a directed edge (x, y) if $S(x) = y$ and $P(y) = x$
- Exception: The all-0-vector has no predecessor!

Problem: Find a different source or sink in the graph.

END-OF-LINE



Observations:

- Every vertex in the graph has indegree and outdegree at most 1.
- By parity argument END-OF-LINE always admits a solution.
- Not necessarily the end of the line from 0, finding this specific sink is PSPACE-complete.
- Only circuits are given, graph is exponentially large.

Computing a solution to END-OF-LINE is PPAD-complete.
It is believed that there is no efficient algorithm for this problem.

Finding (Approximate) Brouwer Fixed Points

Lemma

Finding an (approximate) mixed Nash equilibrium is in PPAD.

Proof Sketch:

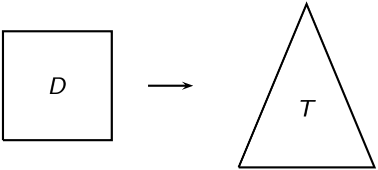
- Reduction: Finding fixed points with END-OF-LINE
- Subdivide the space into finite number of smaller areas
- Find an area close to a fixed point (Approximation)
- By continuity: Finer granularity yields more precise approximation.

Divide the space into simplices ("multidimensional triangles") and color vertices according to direction of Brouwer function

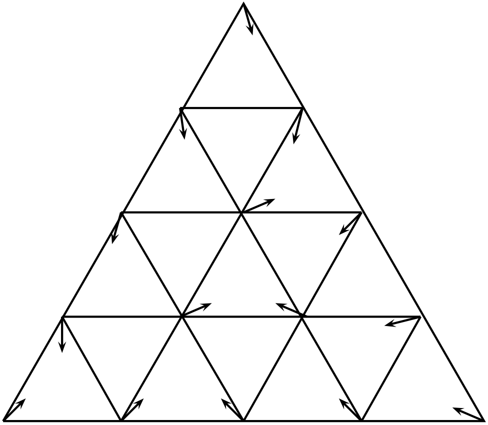
For simplicity of presentation we here consider only problems with $D \subseteq \mathbb{R}^2$, e.g., $f : [0,1]^2 \to [0,1]^2$.

Triangles

For simplicity we transform representation of $[0,1]^2$ to a triangle T. Equivalent fixed point problem with $f' : T \to T$.

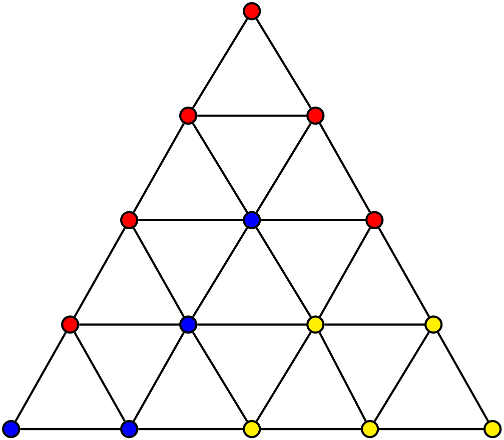


Subdivision and Coloring



- The trianlge space T is subdivided into smaller triangles
- For each vertex consider the direction, in which f' maps the point
- Depending on the direction the vertex receives a color.

Subdivision and Coloring



- The trianlge space T is subdivided into smaller triangles
- For each vertex consider the direction, in which f' maps the point
- Depending on the direction the vertex receives a color.
- With increasing granularity trichromatic triangles become the fixed points of f'.

Sperner Coloring

Definition

A subdivided triangle is a division of a triangle into smaller triangles.

Definition

A Sperner coloring of the vertices of a subdivided triangle satisfies:

- Each extremal vertex gets a different color.
- A vertex on a side of the largest triangle gets a color of one of the corresponding endpoints.
- Other vertices are colored arbitrarily.

Verify that our coloring based on directions of f' yields a Sperner coloring.

Sperner's Lemma

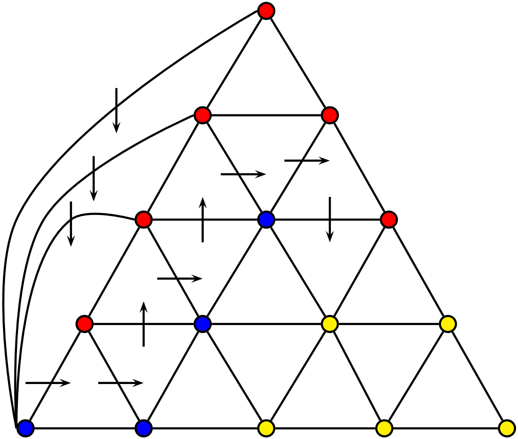
Lemma (Sperner's Lemma)

Every Sperner coloring of a subdivided triangle contains a trichromatic triangle.

Proof:

- Connect all vertices on the outer blue/red edge to the blue vertex. Start at the outside face and move over lines connecting a red and a blue vertex. There are at most 2 such lines in each triangle, never visit a triangle twice.
- This implies an instance of END-OF-LINE:
  Vertices: Small triangles
  Edges: There is an edge if two triangles share a line between a red and blue vertex.
- By construction indegree and outdegree at most 1
- There is a starting point by creation, other sources/sinks are the trichromatic triangles. □

Proof by END-OF-LINE



Implications and Results

Sperner's Lemma is a *discretized version of Brouwer's fixed point theorem*. The proof of the lemma...

- shows that Sperner colorings create an instance of END-OF-LINE.
- can be generalized to more dimensions and simplicies instead of triangles. Then trichromatic triangles correspond to simplicies with maximum number of colors.
- with "infinite granularity" implies maximally colored simplicies are Brouwer fixed points.

This proves that finding a Brouwer fixed point and, hence, a mixed Nash equilibrium in a finite game is in PPAD. □

Recent fundamental result in the literature:

Theorem

Finding a mixed Nash equilibrium in a finite 2-player game is PPAD-complete.

Recommended Literature

- G. Owen. Game Theory. Academic Press, 2001. (Chapter 1)
- Chapters 1 and 2 in the AGT book
- J. Nash. Non-cooperative Games. Annals of Mathematics 54, pp. 286–295, 1951.
- P. Goldberg, C. Daskalakis, C. Papadimitriou. The Complexity of Computing a Nash Equilibrium. SIAM Journal on Computing, 39(1), pp. 195–259, 2009.
- X. Chen, X. Deng, S.-H. Teng. Settling the Complexity of Computing Two-Player Nash Equilibria. Journal of the ACM, 56(3), 2009.