Lecture 5: Mixed strategies and expected payoffs

As we have seen for example for the Matching pennies game or the Rock-Paper-scissor game, sometimes game have no Nash equilibrium. Actually we will see that Nash equilibria exist if we extend our concept of strategies and allow the players to *randomize* their strategies.

In a first step we review basic ideas of probability and introduce notation which will be useful in the context of game theory.

Mini-review of probability:

Probability: Let us suppose that an experiment results in N possible outcomes which we denote by $\{1, \cdots, N\}$. We assign a *probability* $\sigma(i)$ to each possible outcomes

$$\sigma(i) = \text{ Probability that outcome i occurs}$$

The numbers $\sigma(i)$ are such that

$$\sigma(i) \geq 0 \,, \quad \sum_i \sigma(i) = 1 \,.$$

You may think that $\sigma(i)$ as a frequency: if the experiment is repeated many times, say X times with X very large then if you observe that the event i occurs $X(i)$ times out X experiments then $\sigma(i) \approx \frac{X(i)}{X}$ (this is the Law of Large Numbers).

We will represent the probability of all the N outcomes of the experiment by a *probability vector*

$$\boxed{\sigma = (\sigma(1), \cdots, \sigma(N)) \,, \text{ with } \sigma(i) \geq 0 \text{ and } \sum_{i=1}^{N} \sigma(i) = 1 \,, \quad \textbf{Probability vector}}$$

For example if $N = 2$ then the probability vectors can be written as

$$\sigma = (p, 1-p) \text{ with } 0 \leq p \leq 1 \,.$$

For $N = 3$ we the probability vectors can be written as

$$\sigma = (p, q, 1-p-q) \text{ with } 0 \leq p \leq 1 \text{ and } 0 \leq q \leq 1 \,.$$

Expected value: Suppose that the experiment results in a certain reward. Let us say that that the outcome i leads to the reward $f(i)$ for each i. Then if we repeat the experiment many times the average reward is given by

$$f(1)\sigma(1) + f(2)\sigma(2) + \cdots + f(N)\sigma(N)$$

We can think of f as a function with $f(i)$ being the reward for outcome i.

In probability the function f is called a *random variable* and the average reward is called the *expected value* of the random variable f and is denoted by $E[f]$, that is,

$$E[f] \;=\; \sum_{i=1}^{N} f(i)\sigma(i)$$

It will be useful to use a vector notation for random variable and expected value. We form a vector f as

$$f = (f(1), f(2), \cdots, f(N))$$

The expected value can be written using the *scalar product*: if x and y are two vectors in $\mathbf{R}^N$ then the scalar product is given by

$$\langle x, y\rangle \;=\; \sum_{i=1}^{N} x_i y_i \;=\; x_1y_1 + \cdots + x_N y_N$$

We have then

$$\boxed{E[f] = \langle f, \sigma\rangle\,, \quad \textbf{Expected value of the random variable } f}$$

Mixed strategies and expected payoffs: In the context of game we now allow the players to randomize their strategies. Suppose player R has N strategies at his disposal and we will number them $1, 2, 3, \cdots, N$. From now we will call these strategies *pure strategies*. In a pure strategy the player makes a choice which involves no chance or probability.

Definition 1. *A* mixed strategy *for player R is a probability vector $\sigma_R = (\sigma_R(1), \cdots, \sigma_R(N))$*

$$\sigma_R(i) = \text{Probability that R plays strategy } i$$

A pure strategy *for player R is a special case of a mixed strategy*

$$\text{Strategy } i \longleftrightarrow \sigma_R = (0, \cdots, \underbrace{1}_{\text{i}^{\text{th}}}, \cdots, 0)$$

If a player plays a mixed strategy then its opponent's payoff becomes a random variable and we will assume that every player is trying to maximize his *expected payoff*. To compute this it will be useful to use matrix notation.

We assume that player R has N strategies to choose from and that player C has M strategies to choose from, then the payoff matrices P_R has N rows and M columns

$$P_R = \begin{array}{c} \\ 1 \\ 2 \\ \\ N \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \cdots & \quad M \quad \end{array} \\ \begin{pmatrix} P_R(1,1) & P_R(1,2) & \cdots & P_R(1,M) \\ P_R(2,1) & P_R(2,2) & \cdots & P_R(2,M) \\ \vdots & \vdots & & \vdots \\ P_R(N,1) & P_R(N,2) & \cdots & P_R(N,M) \end{pmatrix} \end{array}$$

If we multiply the matrix P_R with the vector σ_C we obtain the vector

$$\begin{aligned} P_R\sigma_C &= \begin{pmatrix} P_R(1,1) & P_R(1,2) & \cdots & P_R(1,M) \\ P_R(2,1) & P_R(2,2) & \cdots & P_R(2,M) \\ \vdots & \vdots & & \vdots \\ P_R(N,1) & P_R(N,2) & \cdots & P_R(N,M) \end{pmatrix} \begin{pmatrix} \sigma_C(1) \\ \sigma_C(2) \\ \vdots \\ \sigma(M) \end{pmatrix} \\ &= \begin{pmatrix} P_R(1,1)\sigma_C(1) + \cdots + P_R(1,M)\sigma_C(M) \\ P_R(2,1)\sigma_C(1) + \cdots + P_R(2,M)\sigma_C(M) \\ \vdots \\ P_R(N,1)\sigma_C(1) + \cdots + P_R(N,M)\sigma_C(M) \end{pmatrix} \end{aligned}$$

and we see that the i^th entry of $P_R\sigma_C$ is

$$\boxed{\begin{aligned} P_R\sigma_C(i) &= P_R(i,1)\sigma_C(1) + \cdots + P_R(i,M)\sigma_C(M) \\ &= \textbf{Expected payoff for R to play i against } \sigma_C \end{aligned}}$$

Further if we think that R is playing a mixed strategy σ_R then his payoff will be then $P_R\sigma_C(i)$ with probability $\sigma_R(i)$ and so we have

$$\boxed{\langle \sigma_R, P_R\sigma_C \rangle = \textbf{Expected payoff for R to play } \sigma_R \textbf{ against } \sigma_C}$$

We can argue in the same way to compute payoffs for C but to do this we should interchange rows and columns or in other words we consider the transpose matrix P_C^T

(which has M rows and N columns). If we multiply this matrix by the vector σ_R we find

$$\begin{aligned}
P_C^T \sigma_R &= \begin{pmatrix} P_C(1,1) & P_C(2,1) & \cdots & P_C(N,1) \\ P_C(1,2) & P_C(2,2) & \cdots & P_C(N,2) \\ \vdots & \vdots & & \vdots \\ P_C(1,M) & P_C(2,M) & \cdots & P_C(N,M) \end{pmatrix} \begin{pmatrix} \sigma_R(1) \\ \sigma_R(2) \\ \vdots \\ \sigma_R(N) \end{pmatrix} \\
&= \begin{pmatrix} P_C(1,1)\sigma_R(1) + P_C(2,1)\sigma_R(2) + \cdots + P_C(N,1)\sigma_R(N) \\ P_C(1,2)\sigma_R(1) + P_C(2,2)\sigma_R(2) + \cdots + P_R(N,2)\sigma_R(N) \\ \vdots \\ P_C(1,M)\sigma_R(1) + P_C(2,M)\sigma_R(2) \cdots + P_R(N,M)\sigma_R(N) \end{pmatrix}
\end{aligned}$$

and we find that

$$\boxed{\begin{aligned} P_C^T \sigma_R(i) &= P_C(1,i)\sigma_R(1) + \cdots + P_C(N,i)\sigma_R(N) \\ &= \textbf{Expected payoff for C to play i against } \sigma_R \end{aligned}}$$

If we average over the strategy of σ_C we find that the expected payoff for C to play σ_C against σ_R is given by $\langle P_C^T \sigma_R, \sigma_C \rangle$. By the property of the transpose matrix we have $\langle P_C^T \sigma_R, \sigma_C \rangle = \langle \sigma_R, P_C \sigma_C \rangle$ and so we have

$$\boxed{\langle \sigma_R, P_C \sigma_C \rangle = \textbf{Expected payoff for C to play } \sigma_C \textbf{ against } \sigma_R}$$

Example: Consider the game with payoff table

		Collin A	Collin B	Collin C
Robert	I	10, 0	5, 1	4, -2
	II	10, 1	5, 0	10, -1

If $\sigma_C = (p, q, 1-p-q)$ then the payoff for R are given by

$$P_R \sigma_C = \begin{pmatrix} 10 & 5 & 4 \\ 10 & 5 & 10 \end{pmatrix} \begin{pmatrix} p \\ q \\ 1-p-q \end{pmatrix} = \begin{pmatrix} 4 + 6p + q \\ 10 - 5q \end{pmatrix} \tag{1}$$

so the payoff for Robert to play I against σ_C is $4 + 6p + q$ and to play II the payoff is $10 - 5q$.

If R has the mixed strategy $(r, 1-r)$ then the payoffs for C are given by

$$P_C^T \sigma_R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} r \\ 1-r \end{pmatrix} = \begin{pmatrix} 1-r \\ r \\ -1-r \end{pmatrix} \tag{2}$$

and so the payoff for Collin to play A against σ_R is $1-r$, to play B it is r and to play C it is $-1-r$.