ΤΜΗΜΑ ΠΛΗΡΟΦΟΡΙΚΗΣ & ΤΗΛΕΠΙΚΟΙΝΩΝΙΩΝ

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# M902
# Βασικές Μαθηματικές Έννοιες στη Γλωσσική Τεχνολογία

## Project 2


Κυλάφη Χριστίνα-Θεανώ

LT1200012


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# Question **1**

The composite function $S(f(x))$, where $S(x) = \dfrac{1}{1+e^{-x}}$ and $f(x) = ax + b$, is calculated as follows:

$$S(f(x)) = \frac{1}{1+e^{-f(x)}} = \frac{1}{1+e^{-(ax+b)}} = \frac{1}{1+\frac{1}{e^{(ax+b)}}} = \frac{1}{\frac{e^{(ax+b)}+1}{e^{(ax+b)}}} = \frac{e^{(ax+b)}}{e^{(ax+b)}+1} = \frac{e^{ax}e^{b}}{e^{ax}e^{b}+1}$$

where:

- $x \in Dom(f) = \mathbb{R}$
- $f(x) \in Im(f) \subseteq Dom(S) = \mathbb{R}$
- $S(f(x)) \in Im(S)$

## Question **2**

A function is **invertible** only if each input has a unique output, which means each output is paired with exactly one input. That way, when the mapping is reversed, it will still be a function.

### Definition

Let $f$ be a function whose domain is the set $A$ and whose codomain is the set $B$ ($f : A \to B$ ). Then we say that $f$ is invertible if $f$ is **one-to-one** mapping and there is a function $g$ with domain $Im(f) \subseteq B$ and image (range) $A$ ( $g : Im(f) \to A$) such that:

$$f(x) = y \iff g(y) = x$$

In this case, we call $g$ the inverse of $f$ and denote it by $f^{-1}$.

**i) $\mathbf{f(x) = \sqrt{x - 3}}$**

- $x - 3 \geq 0 \iff x \geq 3 \iff Dom(f) = [\ 3,\ +\infty\ )$ **(1)**
- $f$ is **continuous** on $Dom(f) = [\ 3,\ +\infty\ )$ [1] **(2)**
- $f$ is **differentiable** on $A = \{\ Dom(f) - \{0\}\ \} = \{\ [\ 3,\ +\infty\ ) - \{0\}\ \}$
$= \{\ (\ 3,\ +\infty\ )\ \}$ [2] **(3)**

First, we have to prove that $f$ is **one-to-one** mapping:

$$\overset{(2),\ (3)}{\Longrightarrow} f'(x) = (\ \sqrt{x-3}\ )' = \frac{1}{2\sqrt{x-3}} > 0,\ \forall\ x \in A = (3,\ +\infty) \qquad (\mathbf{4})$$

$\overset{(4)}{\Longrightarrow} f$ is **strictly monotonic** (strictly inscreasing), therefore it is **one-to-one** mapping

One approach for finding a formula for $f^{-1}$ is to solve $f(x) = y$ for $x$.

$$f(x) = \sqrt{x-3} = y \overset{y\geq 0,\ x\geq 3}{\iff} x - 3 = y^2 \iff x = y^2 + 3 = f^{-1}(y) \qquad (\mathbf{5})$$

$$\overset{(1)}{\Longrightarrow} x \geq 3 \iff y^2 + 3 \geq 3 \iff y^2 \geq 0,\ \ true\ \forall\ y \in \mathbb{R} \qquad (\mathbf{6})$$

[1] as a composition of continuous functions

[2] as a composition of differential functions

$$\overset{(5),\,(6)}{\Longrightarrow} Im(f) = \{\ \mathbb{R}\ \cap\ [\ 0,\ +\infty\ )\ \} = [\ 0,\ +\infty\ ) = Dom(f')$$

• $f : [\ 3,\ +\infty\ ] \to \mathbb{R}$

$f(x) = \sqrt{x-3}$

• $f^{-1} : [\ 0,\ +\infty\ ] \to \mathbb{R}$

$f^{-1} = x^2 + 3$

**ii) $\mathbf{f(x) = \log_2(x-2)}$**

( Let $f(x)$ logarithmic function with base 2, as $2^x$ which is $log_2(x)$ function's inverse, is also used in **Question 4** )

• $x - 2 > 0 \iff x > 2 \iff Dom(f) = (\ 2,\ +\infty\ )$ **( 7 )**

First, we have to prove that $f$ is **one-to-one** mapping:

$Let\ x_1,\ x_2\ \in\ Dom(f) = (\ 0,\ +\infty\ ),\ x_1 \neq x_2$

$f(x_1) = f(x_2) \iff log_2(x_1 - 2) = log_2(x_2 - 2) \iff x_1 - 2 = x_2 - 2 \iff x_1 = x_2$

$\iff x_1 = x_2 \iff f$ is an **one-to-one** mapping of $Dom(f)$ to $Im(f)$

As in $i$ above, we will solve $f(x) = y$ for $x$.

$f(x) = \log_2(x-2) = y \iff 2^{log_2(x-2)} = 2^y \iff x - 2 = 2^y$

$\iff x = 2^y + 2 = f^{-1}(y)$ **( 8 )**

$\overset{(7)}{\Longrightarrow} x > 2 \iff 2^y + 2 > 2 \iff 2^y > 0,\ true\ \forall\ y \in \mathbb{R}$ **( 9 )**

$\overset{(9)}{\Longrightarrow} Im(f) = \mathbb{R} = Dom(f')$

• $f : (\ 2,\ +\infty\ ] \to \mathbb{R}$

$f(x) = log(x-2)$

• $f^{-1} : \mathbb{R} \to \mathbb{R}$

$f^{-1} = 2^y + 2$

# Question **3**

$s = A\ cos(\ 2\pi ft\ )$

(a) $A_1 = 1, \quad f_1 = 1, \quad \theta_1 = 0, \quad s_1 = cos(\ 2\pi t\ )$
(b) $A_2 = 2, \quad f_2 = 3, \quad \theta_2 = 0, \quad s_2 = 2\ cos(\ 6\pi t\ )$
(c) $A_3 = 1.5, \quad f_3 = 2, \quad \theta_3 = \pi, \quad s_3 = 1.5\ cos(\ 4\pi t\ +\ \pi)$
(d) $A_4 = 2, \quad f_4 = 0.5, \quad \theta_4 = 0, \quad s_4 = 2\ cos(\ \pi t\ )$

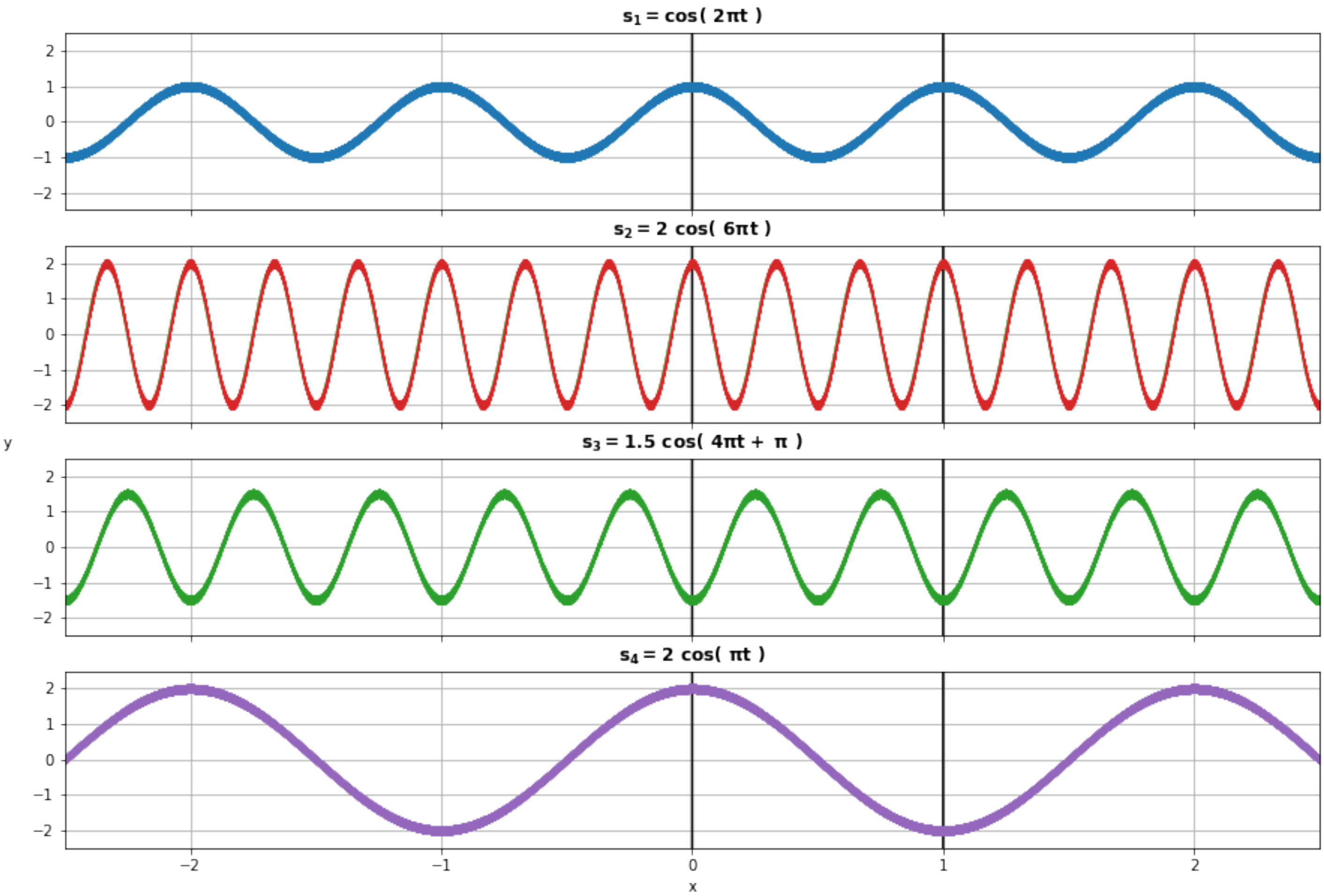


**Figure 1**

# Question 4

Following the steps in **Question 2**, the inverse function of $f(x) = 2^x$ is calculated similarly:

$$f(x) = 2^x = y \overset{y>0}{\Longleftrightarrow} log_2(\ 2^x\ ) = log_2(\ y\ ) \ \Longleftrightarrow\ x = log_2(y) = f^{-1}(y)$$

• $f^{-1} :\ (\ 0,\ +\infty\ ) \rightarrow \mathbb{R}$
$f^{-1}(x) = log_2(\ x\ )$

In **Figure 2**, functions $f,\ f^{-1},\ y = x$ and the respective points are depicted.

Points of $f$ $\ :\ (0{,}1),\ (1{,}2),\ (2{,}4),\ (3{,}8)$
Points of $f^{-1}$ $:\ (1{,}0),\ (2{,}1),\ (4{,}2),\ (8{,}3)$

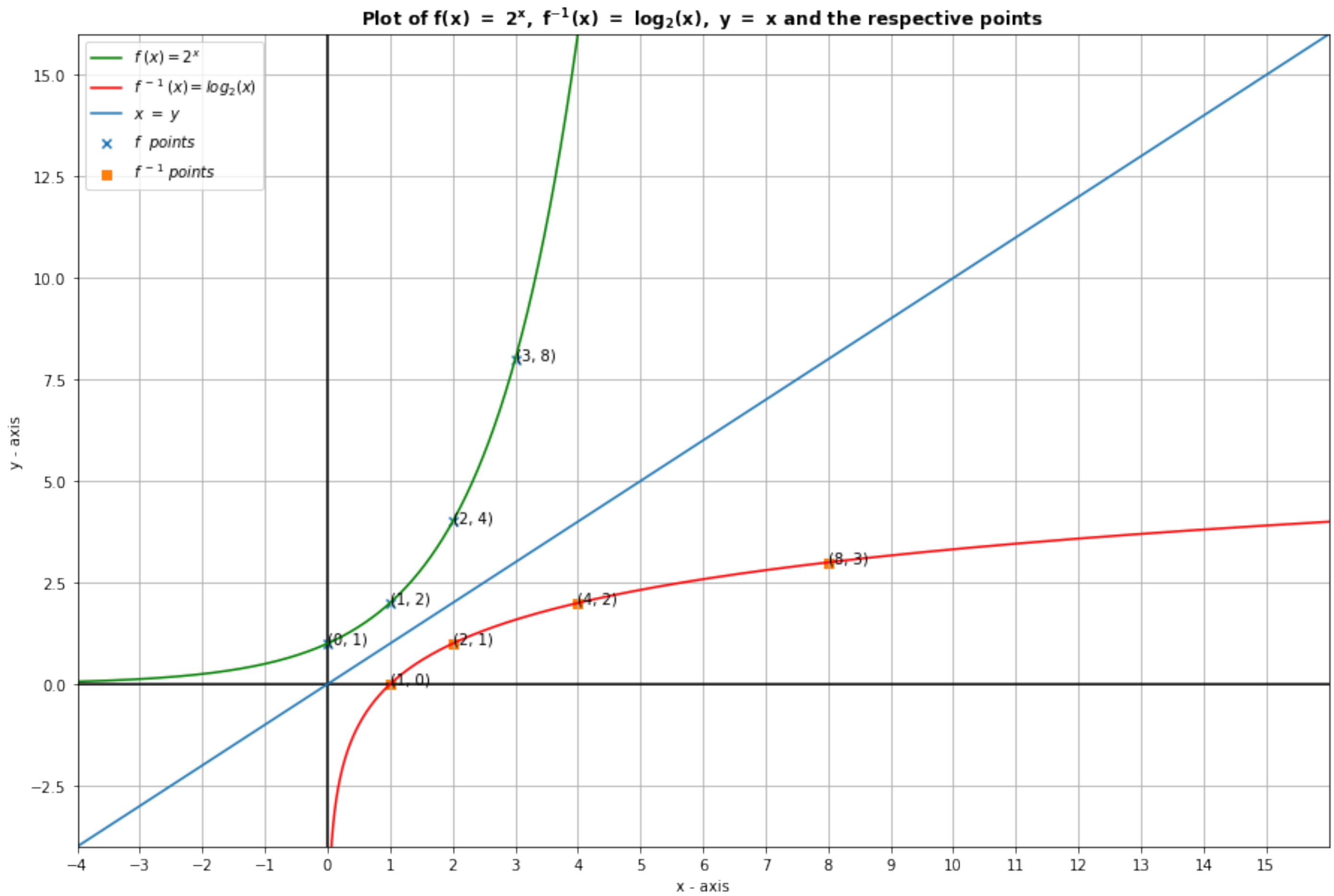


**Figure 2**

## Question **5**

(a) The derivative of function $f(x) = ax^2$ is $f'(x) = 2ax$ ( $\mathbf{a} \rightarrow \mathbf{4}$ )

(b) The derivative of function $f(x) = cos(2\pi ft)$ is $f'(x) = -sin(2\pi ft)$ ( $\mathbf{b} \rightarrow \mathbf{1}$ )

(c) The derivative of function $f(x) = bx^3$ is $f'(x) = 3bx^2$ ( $\mathbf{c} \rightarrow \mathbf{2}$ )

(d) The derivative of function $f(x) = e^{cx}$ is $f'(x) = ce^{cx}$ ( $\mathbf{d} \rightarrow \mathbf{3}$ )

# Question **6**

$S(x)$ is differentiable in $Dom(S)$ , as a composition of differentiable functions with $S'(x)$ as follows:

$$S'(x) = \left(\frac{1}{1+e^{-x}}\right)' = [(1+e^{-x})^{-1}]' = (-1)(1+e^{-x})^{-2}(1+e^{-x})' = -\frac{(e^{-x})'}{(1+e^{-x})^2}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} \quad (\mathbf{1})$$

$$S(x)\ (1-S(x)) = \left(\frac{1}{1+e^{-x}}\right)\left(1-\frac{1}{1+e^{-x}}\right) = \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right) = \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right)$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} \quad (\mathbf{2})$$

$$\overset{(1)\ ,\ (2)}{\Longrightarrow} \quad S'(x) = S(x)\ (\ 1 - S(x)\ )$$

# Question **7**

$$S(x) = \frac{1}{1+e^{-x}}, \quad f(x) = ax + b, \quad S(f(x)) = \frac{1}{1+e^{-f(x)}}$$

$S(f(x))$ is differentiable in $Dom(S)$, as a composition of differentiable functions $S,\ f,$ with $S'(f(x))$ as follows:

$$S'(f(x)) = \left(\frac{1}{1+e^{-f(x)}}\right)' \overset{Question\ 1}{=} \left(\frac{e^{ax}e^{b}}{e^{ax}e^{b}+1}\right)' = \frac{(e^{ax}e^{b})'(e^{ax}e^{b}+1)\ -\ (e^{ax}e^{b})(e^{ax}e^{b}+1)'}{(e^{ax}e^{b}+1)^2}$$

$$= \frac{(ae^{ax}e^{b})(e^{ax}e^{b}+1)\ -\ (e^{ax}e^{b})(ae^{ax}e^{b})}{(e^{ax}e^{b}+1)^2} = \frac{(ae^{ax}e^{b})(e^{ax}e^{b}+1\ -\ e^{ax}e^{b})}{(e^{ax}e^{b}+1)^2}$$

$$= \frac{ae^{ax+b}}{(e^{ax}e^{b}+1)^2}$$

# Question **8**

Lala

# Question **9**

Derivatives of $f_n$ with respect to $\sigma$ :

$$i)\ f_1(\sigma) = log\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)$$

$$i)\ \frac{df_1(\sigma)}{d\sigma} = f_1'(\sigma) = \left(log\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)\right)' = (\ log(1)\ )' - (\ log(\sigma\sqrt{2\pi})\ )' + \left(\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\ \right)'$$

$$= -\ \frac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}}\ + \left(\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\ \right)'$$

$$ii)\ \frac{df_2(\sigma)}{d\sigma} = f_2(\sigma) = log\left(\ \frac{1}{\sigma\sqrt{2\pi}}\left(\ e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}\ \right)\ \right)$$

## Question **10**

**Mean** value of random variable $X$ :

$$E[X] = \int_a^b x\ f_x(x)\ dx = \int_a^b x\ \frac{1}{b-a}\ dx = \frac{1}{b-a}\ \int_a^b x\ dx = \frac{1}{b-a}\ \int_a^b \left(\frac{x^2}{2}\right)'\ dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left(\frac{b^2}{2}-\frac{a^2}{2}\right) = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2\ (b-a)}$$

$$= \frac{a+b}{2}$$

**Second moment** of random variable $X$ :

$$E[X^2] = \int_a^b x^2 f_x(x)\ dx = \int_a^b x^2\ \frac{1}{b-a}\ dx = \frac{1}{b-a}\int_a^b x^2\ dx = \frac{1}{b-a}\int_a^b \left(\frac{x^3}{3}\right)'\ dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a}\left(\frac{b^3}{3}-\frac{a^3}{3}\right) = \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(a^2+ab+b^2)}{3\ (b-a)}$$

$$= \frac{a^2+ab+b^2}{3}$$