M902
Βασικές Μαθηματικές Έννοιες
στη
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Project 2

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Question **1**

The composite function $S(f(x))$, where $S(x) = \dfrac{1}{1 + e^{-x}}$ and $f(x) = ax + b$, is calculated as follows:

$$S(f(x)) = \frac{1}{1 + e^{-f(x)}} = \frac{1}{1 + e^{-(ax+b)}} = \frac{1}{1 + \frac{1}{e^{(ax+b)}}} = \frac{1}{\frac{e^{(ax+b)} + 1}{e^{(ax+b)}}} = \frac{e^{(ax+b)}}{e^{(ax+b)} + 1} = \frac{e^{ax}e^{b}}{e^{ax}e^{b} + 1}$$

where:

- $x \in Dom(f) = \mathbb{R}$

- $f(x) \in Im(f) \subseteq Dom(S) = \mathbb{R}$

- $S(f(x)) \in Im(S)$

Question **2**

A function is **invertible** only if each input has a unique output, which means each output is paired with exactly one input. That way, when the mapping is reversed, it will still be a function.

Definition
Let f be a function whose domain is the set A and whose codomain is the set B ($f : A \to B$). Then we say that f is invertible if f is **one-to-one** mapping and there is a function g with domain $Im(f) \subseteq B$ and image (range) A ($g : Im(f) \to A$) such that:

$$f(x) = y \iff g(y) = x$$

In this case, we call g the inverse of f and denote it by f^{-1}.

i) $\mathbf{f(x) = \sqrt{x-3}}$

• $x - 3 \geq 0 \iff x \geq 3 \iff Dom(f) = [\, 3, \ +\infty \,)$ (**1**)

• f is **continuous** on $Dom(f) = [\, 3, \ +\infty \,)$ [1]   (**2**)

• f is **differentiable** on $A = \{\, Dom(f) - \{3\} \,\} = \{\, [\, 3, \ +\infty \,) - \{3\} \,\}$

$= \{\, (\, 3, \ +\infty \,) \,\}$ [2]   (**3**)

First, we have to prove that f is **one-to-one** mapping:

$$\overset{(2),\ (3)}{\implies} f'(x) = (\, \sqrt{x-3} \,)' = \frac{1}{2\sqrt{x-3}} > 0, \ \forall \ x \in A = (3, \ +\infty) \qquad (\ \mathbf{4}\ )$$

$\overset{(4)}{\implies} f$ is **strictly monotonic** (strictly inscreasing), therefore it is **one-to-one** mapping

One approach for finding a formula for f^{-1} is to solve $f(x) = y$ for x.

$$f(x) = \sqrt{x-3} = y \overset{y \geq 0,\ x \geq 3}{\iff} x - 3 = y^2 \iff x = y^2 + 3 = f^{-1}(y) \qquad (\ \mathbf{5}\ )$$

$$\overset{(1)}{\implies} x \geq 3 \iff y^2 + 3 \geq 3 \iff y^2 \geq 0, \ true \ \forall \ y \in \mathbb{R} \qquad (\ \mathbf{6}\ )$$

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[1] as a composition of continuous functions

[2] as a composition of differentiable functions

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$$\overset{(5),\ (6)}{\Longrightarrow} Im(f) = \{\ \mathbb{R} \cap [\ 0,\ +\infty\ )\ \} = [\ 0,\ +\infty\ ) = Dom(f')$$

- $f : [\ 3,\ +\infty\ ] \rightarrow \mathbb{R}$

$$f(x) = \sqrt{x-3}$$

- $f^{-1} : [\ 0,\ +\infty\ ] \rightarrow \mathbb{R}$

$$f^{-1}(x) = x^2 + 3$$

ii) f(x) = log(x − 2)

(Let $f(x)$ logarithmic function with base 2, as 2^x which is $log_2(x)$ function's inverse, is also used in **Question 4**)

- $x - 2 > 0 \iff x > 2 \iff Dom(f) = (\ 2,\ +\infty\ )$ (**7**)

First, we have to prove that f is **one-to-one** mapping:

$$Let\ x_1,\ x_2\ \in\ Dom(f) = (\ 0,\ +\infty\ ),\ x_1 \neq x_2$$

$$f(x_1) = f(x_2) \iff log_2(x_1 - 2) = log_2(x_2 - 2) \iff x_1 - 2 = x_2 - 2 \iff x_1 = x_2$$

$$\iff f\ is\ an\ \textbf{one-to-one}\ mapping\ of\ Dom(f)\ to\ Im(f)$$

As in i above, we will solve $f(x) = y$ for x.

$$f(x) = log_2(x-2) = y \iff 2^{log_2(x-2)} = 2^y \iff x - 2 = 2^y$$
$$\iff x = 2^y + 2 = f^{-1}(y)$$ (**8**)

$$\overset{(7)}{\Longrightarrow} x > 2 \iff 2^y + 2 > 2 \iff 2^y > 0,\ true\ \forall\ y \in \mathbb{R}$$ (**9**)
$$\overset{(9)}{\Longrightarrow} Im(f) = \mathbb{R} = Dom(f')$$

- $f : (\ 2,\ +\infty\ ] \rightarrow \mathbb{R}$

$$f(x) = log(x-2)$$

- $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$

$$f^{-1}(x) = 2^x + 2$$

Question **3**

$$y = A\ cos(\ 2\pi f x\ )$$

(a) $A_1 = 1,\quad f_1 = 1,\quad \theta_1 = 0,\quad s_1 = cos(\ 2\pi x\ )$

(b) $A_2 = 2,\quad f_2 = 3,\quad \theta_2 = 0,\quad s_2 = 2\ cos(\ 6\pi x\ )$

(c) $A_3 = 1.5,\quad f_3 = 2,\quad \theta_3 = \pi,\quad s_3 = 1.5\ cos(\ 4\pi x\ +\ \pi)$

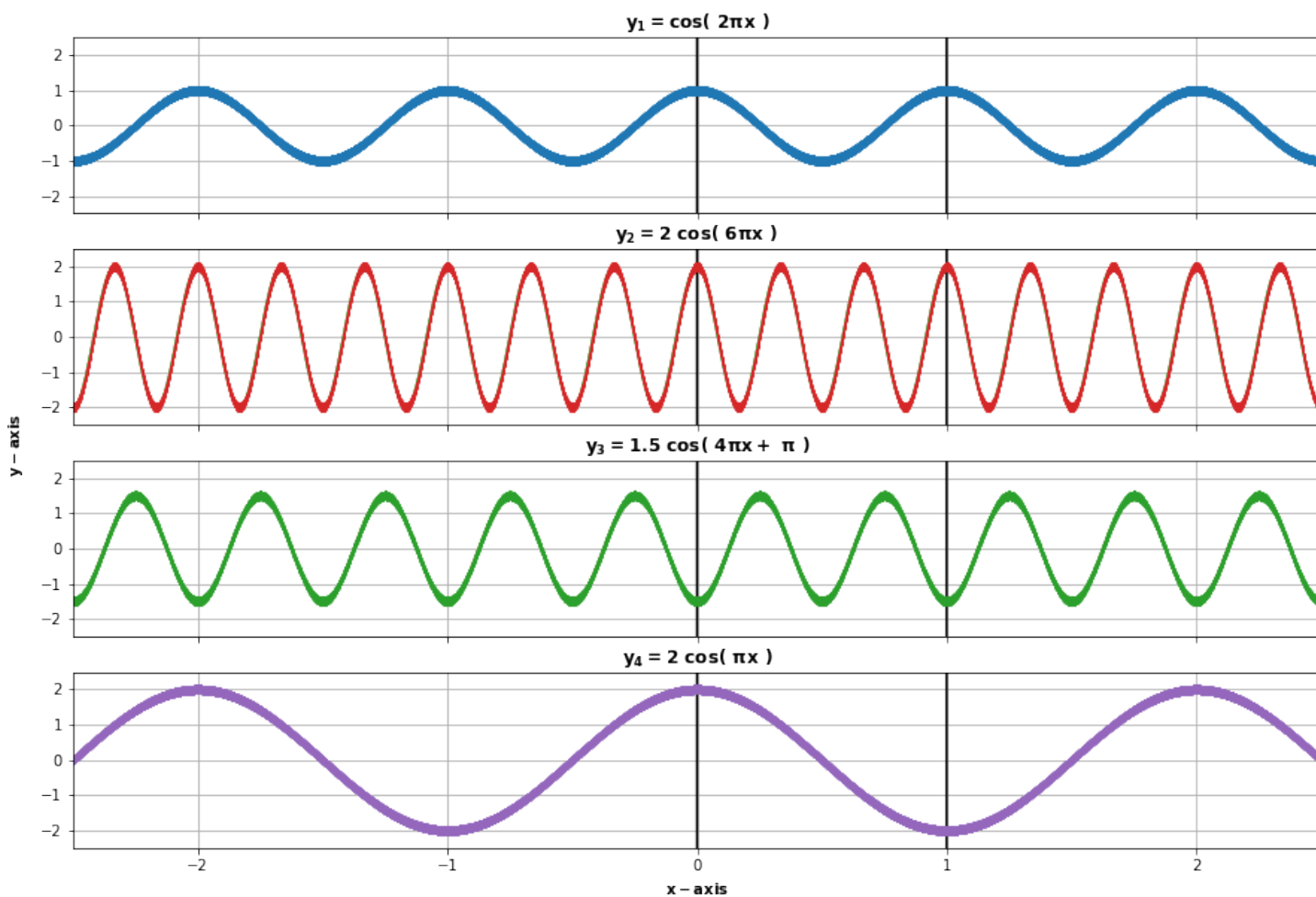(d) $A_4 = 2,\quad f_4 = 0.5,\quad \theta_4 = 0,\quad s_4 = 2\ cos(\ \pi x\ )$

Figure 1

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Question **4**

Following the steps in **Question 2**, the inverse function of $f(x) = 2^x$ is calculated similarly:

$$f(x) = 2^x = y \overset{y>0}{\Longleftrightarrow} log_2(\ 2^x \ ) = log_2(\ y \ ) \iff x = log_2(y) = f^{-1}(y)$$

- $f^{-1} : \ (\ 0, \ +\infty \ ) \to \mathbb{R}$
$f^{-1}(x) = log_2(\ x \ )$

In **Figure 2**, functions f, f^{-1}, $y = x$ and the respective points are depicted.

Points of f : $(0,1), (1,2), (2,4), (3,8)$
Points of f^{-1} : $(1,0), (2,1), (4,2), (8,3)$
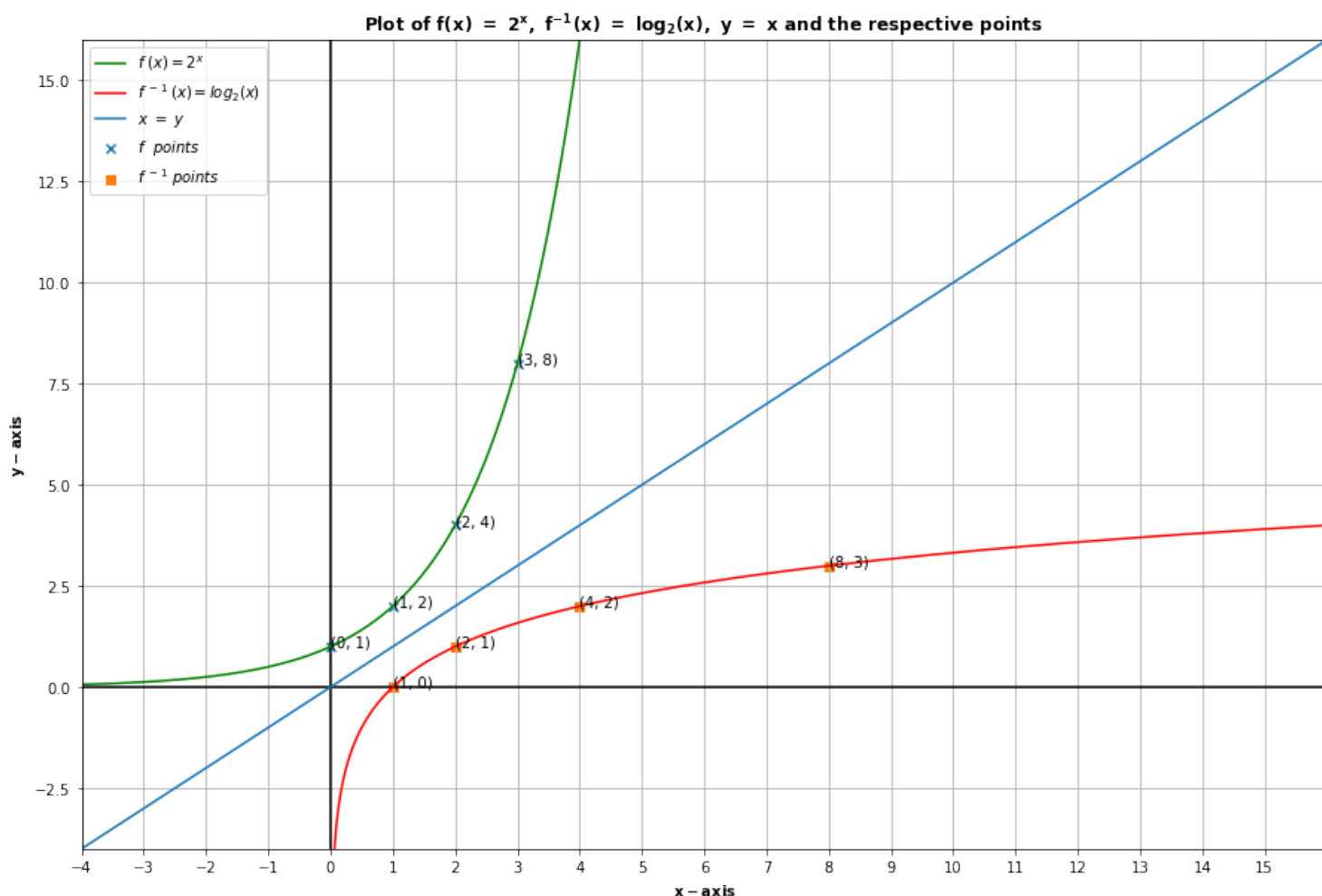

Figure 2

Question 5

(a) The derivative of function $f(x) = ax^2$ is $f'(x) = 2ax$ (**a** → **4**)

(b) The derivative of function $f(x) = cos(2\pi fx)$ is $f'(x) = -sin(2\pi fx)$ (**b** → **1**)

(c) The derivative of function $f(x) = bx^3$ is $f'(x) = 3bx^2$ (**c** → **2**)

(d) The derivative of function $f(x) = e^{cx}$ is $f'(x) = ce^{cx}$ (**d** → **3**)

Question **6**

$S(x)$ is differentiable in $Dom(S)$, as a composition of differentiable functions with $S'(x)$ as follows:

$$S'(x) = \left(\frac{1}{1 + e^{-x}} \right)' = [(1 + e^{-x})^{-1}]' = (-1)(1 + e^{-x})^{-2}(1 + e^{-x})'$$

$$= -\frac{(e^{-x})'}{(1 + e^{-x})^2} = \frac{e^{-x}}{(1 + e^{-x})^2} \qquad (\textbf{1})$$

$$S(x)\,(1 - S(x)) = \left(\frac{1}{1 + e^{-x}} \right)\left(1 - \frac{1}{1 + e^{-x}} \right) = \left(\frac{1}{1 + e^{-x}} \right)\left(\frac{1 + e^{-x} - 1}{1 + e^{-x}} \right)$$

$$= \left(\frac{1}{1 + e^{-x}} \right)\left(\frac{e^{-x}}{1 + e^{-x}} \right) = \frac{e^{-x}}{(1 + e^{-x})^2} \qquad (\textbf{2})$$

$$\overset{(1)\,,\,(2)}{\Longrightarrow} \quad S'(x) = S(x)\,(\,1 - S(x)\,)$$

Question **7**

$$S(x) = \frac{1}{1 + e^{-x}}, \quad f(x) = ax + b, \quad S(f(x)) = \frac{1}{1 + e^{-f(x)}}$$

$S(f(x))$ is differentiable in $Dom(S)$, as a composition of differentiable functions $S, f,$ with $S'(f(x))$ as follows:

$$S'(f(x)) = \left(\frac{1}{1 + e^{-f(x)}} \right)' \overset{Question\ 1}{=} \left(\frac{e^{ax}e^b}{e^{ax}e^b + 1} \right)'$$

$$= \frac{(e^{ax}e^b)'(e^{ax}e^b + 1) - (e^{ax}e^b)(e^{ax}e^b + 1)'}{(e^{ax}e^b + 1)^2}$$

$$= \frac{(ae^{ax}e^b)(e^{ax}e^b + 1) - (e^{ax}e^b)(ae^{ax}e^b)}{(e^{ax}e^b + 1)^2}$$

$$= \frac{(ae^{ax}e^b)(e^{ax}e^b + 1 - e^{ax}e^b)}{(e^{ax}e^b + 1)^2}$$

$$= \frac{ae^{ax+b}}{(e^{ax}e^b + 1)^2}$$

Question **8**

Derivatives of f_n with respect to μ :

i) $f_1(\mu) = log\left(\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)$

$$\dfrac{df_1(\mu)}{d\mu} = f_1'(\mu) = \left(log\left(\dfrac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)\right)' = \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\right)\right)'$$

$$= \left(\dfrac{-(x_1-\mu)^2}{2\sigma^2}\right)' = \left(-\dfrac{2\,(x_1-\mu)\,(x_1-\mu)'}{2\sigma^2}\right)$$

$$= \dfrac{x_1-\mu}{\sigma^2}$$

$ii)\ f_2(\mu) = log\left(\dfrac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)$

$$\dfrac{df_2(\mu)}{d\mu} = f_2'(\mu) = \left(log\left(\dfrac{1}{\sigma\sqrt{2\pi}} \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right) \right)'$$

$$= \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)' = \dfrac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \dfrac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \right)' + \left(e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \dfrac{\left(\frac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\frac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \dfrac{\left(-\frac{2\,(x_1-\mu)\,(x_1-\mu)'}{2\sigma^2} \right) e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(-\frac{2\,(x_2-\mu)\,(x_2-\mu)'}{2\sigma^2} \right) e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \dfrac{\frac{x_1-\mu}{\sigma^2}\, e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \frac{x_2-\mu}{\sigma^2}\, e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \dfrac{\frac{(x_1-\mu)\, e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}}{\sigma^2} + \frac{(x_2-\mu)\, e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \dfrac{\frac{(x_1-\mu)\, e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + (x_2-\mu)\, e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = \dfrac{(x_1-\mu)\, e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + (x_2-\mu)\, e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma^2 \left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)}$$

Question **9**

Derivatives of f_n with respect to σ (in the following functions, let $log \equiv log_e \equiv ln$) :

$$i) \; f_1(\sigma) = log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1 - \mu)^2}{2\sigma^2}} \right)$$

$$\frac{df_1(\sigma)}{d\sigma} = f_1'(\sigma) = \left(log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1 - \mu)^2}{2\sigma^2}} \right) \right)'$$

$$= \left(log(1) \right)' - \left(log(\sigma\sqrt{2\pi}) \right)' + \left(log\left(e^{\frac{-(x_1 - \mu)^2}{2\sigma^2}} \right) \right)'$$

$$= -\frac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}} + \left(\frac{-(x_1 - \mu)^2}{2\sigma^2} \right)' = -\frac{1}{\sigma} - \frac{(x_1 - \mu)^2}{2}\left(\frac{1}{\sigma^2} \right)'$$

$$= -\frac{1}{\sigma} + \frac{(x_1 - \mu)^2}{2}\left(\frac{2}{\sigma^3} \right) = -\frac{1}{\sigma} + \frac{(x_1 - \mu)^2}{\sigma^3} = \frac{-\sigma^2 + (x_1 - \mu)^2}{\sigma^3}$$

$$= \frac{x_1^2 + \mu^2 - 2x_1\mu - \sigma^2}{\sigma^3}$$

$ii)\ f_2(\sigma) = log\left(\dfrac{1}{\sigma\sqrt{2\pi}}\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)$

$$\dfrac{df_2(\sigma)}{d\sigma} = f_2'(\sigma) = log\left(\dfrac{1}{\sigma\sqrt{2\pi}}\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= \left(log\left(\dfrac{1}{\sigma\sqrt{2\pi}} \right) \right)' + \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= -\dfrac{\sqrt{2\pi}}{\sigma\sqrt{2\pi}} + \left(log\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right) \right)'$$

$$= -\dfrac{1}{\sigma} + \dfrac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}} = -\dfrac{1}{\sigma} + \dfrac{\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \right)' + \left(e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)'}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\dfrac{1}{\sigma} + \dfrac{\left(\dfrac{-(x_1-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \left(\dfrac{-(x_2-\mu)^2}{2\sigma^2} \right)' e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= -\dfrac{1}{\sigma} + \dfrac{\dfrac{(x_1-\mu)^2}{\sigma^3} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \dfrac{(x_2-\mu)^2}{\sigma^3} e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}$$

$$= \dfrac{-e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} - e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} + \dfrac{(x_1-\mu)^2}{\sigma^2} e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + \dfrac{(x_2-\mu)^2}{\sigma^2} e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}}{\sigma\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)}$$

$$= \dfrac{e^{\frac{-(x_1-\mu)^2}{2\sigma^2}}\left(\dfrac{(x_1-\mu)^2}{\sigma^2} - 1 \right) + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}}\left(\dfrac{(x_2-\mu)^2}{\sigma^2} - 1 \right)}{\sigma\left(e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} + e^{\frac{-(x_2-\mu)^2}{2\sigma^2}} \right)}$$

Question **10**

Mean value of random variable X :

$$E[X] = \int_a^b x \, f_x(x) \, dx = \int_a^b x \, \frac{1}{b-a} \, dx = \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \int_a^b \left(\frac{x^2}{2}\right)' dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a} \left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2\,(b-a)}$$

$$= \frac{a+b}{2}$$

Second moment of random variable X :

$$E[X^2] = \int_a^b x^2 \, f_x(x) \, dx = \int_a^b x^2 \, \frac{1}{b-a} \, dx = \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \int_a^b \left(\frac{x^3}{3}\right)' dx$$

$$= \frac{1}{b-a} \left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(a^2 + ab + b^2)}{3\,(b-a)}$$

$$= \frac{a^2 + ab + b^2}{3}$$