Exercises: Dimension and Riemann Roch

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1. Let C be a non-singular projective curve and let D be a divisor on C. Choose a basis $f_0, \dots, f_n$ for $L(D)$ and consider the rational map from C to $\mathbb{P}(L(D)) = \mathbb{P}^n$ given by $p \mapsto [f_0(p) : \dots : f_n(p)]$ (not defined if $f_i(p) = \infty$ or all f_i vanish at p), and extend it to a regular map $f : C \to \mathbb{P}^n$.

   (a) Show that the image of f is not contained in a hyperplane.

   (b) Show that, if $H \subseteq \mathbb{P}^n$ is a hyperplane, then $f^{-1}(H)$ is a divisor equivalent to D.

   (c) Find a bijection between the set of divisors on C with $l(D) = n + 1$ modulo rational equivalence, and the set of regular maps $f : C \to \mathbb{P}^n$ such that $f(C)$ is not contained in a hyperplane, modulo the action of $\mathrm{PGL}(n + 1, \mathbb{C})$.

2. Let C be a non-singular projective curve, let D be a divisor on C, and let $f : C \to \mathbb{P}(L(D))$ be the map from the previous problem. Show that:

   (a) f is one-to-one iff $l(D + p + q) = l(D) + 2$ for all distinct points p, q.

   (b) f is an embedding iff $l(D + p + q) = l(D) + 2$ for all pairs of points p, q.

3. For $i = 1, 2$, assume that C_i are non-singular projective curves and $f_i : C_i \to \mathbb{P}^1$ are regular maps. Assume that the degree of f_i is two and that f_i are branched over the same set in $\mathbb{P}^1$. Prove that C_1 is isomorphic to C_2. (Hint: consider the set $X = \{(x, y) \in C_1 \times C_2 \mid f_1(x) = f_2(y)\}$).

4. Prove that any non-singular curve of genus 0 is isomorphic to $\mathbb{P}^1$.

5. Prove that any non-singular curve of genus 1 is isomorphic to a planar cubic curve (i.e., the zero locus of a cubic polynomial in $\mathbb{P}^2$).

6. Prove that any non-singular curve of genus 2 is a branched double cover of $\mathbb{P}^1$.
7. Let C be a non-singular curve of genus 2. Assume that $f : C \to \mathbb{P}^1$ is a branched double cover of $\mathbb{P}^1$. Let D be the divisor attached to f, and let $B \subseteq \mathbb{P}^1$ be the set of points over which f is branched.
   (a) Show that $|B| = 6$.
   (b) Show that $D \sim K$.
   (c) Show that B is determined by C up to the action of $\mathrm{PGL}(3, \mathbb{C})$.

   Remark 0.1. *This means that the space of all non-singular curves of genus 2 is the quotient of the collection of all six-tuples of distinct points of $\mathbb{P}^1$, modulo the action of $S_6 \times \mathrm{PGL}(3, \mathbb{C})$.*
8. Let $X, Y \subset \mathbb{C}^n$ be algebraic sets.
   (a) Prove that $\dim(X \times Y) = \dim(X) + \dim(Y)$.
   (b) Assume that $X \cap Y \neq \emptyset$. Prove that $\dim(X \cap Y) \geq \dim(X) + \dim(Y) - n$. (Hint: identify the intersection $X \cap Y$ with the intersection of $X \times Y \subset \mathbb{C}^{2n}$ and the diagonal $\{(x, y) \in \mathbb{C}^{2n} \mid x = y\} \subset \mathbb{C}^{2n}$)
9. Let $X, Y \subset \mathbb{P}^n$ and assume that $\dim(X) + \dim(Y) \geq n$. Prove that $X \cap Y \neq \emptyset$. (Hint: use the previous problem)