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Tutorial

WEEK 6

1. Tensor products

Definition 1.0.1. Given V, W vector spaces we define $V \otimes W$ to be a vector space satisfying the following universal property:
There exist bi-linear form $\otimes : V \times W \to V \otimes W$ s.t. for any other vector space U and a bi-linear form $\varphi : V \times W \to U$ there is a unique linear map $\varphi' : V \otimes W \to U$ s.t $\varphi' \circ \otimes = \varphi$ (Draw the diagram).

Exercise 1.0.1. As sets $Bil(V \times W, U) \cong Hom(V \otimes W, U)$

Corollary 1.0.1. $Bil(V \times W, \mathbb{C})) \cong (V \otimes W)^*$

Construction:

Definition 1.0.2. Let V be n dimensional and W be m dimensional vector spaces, thee tensor product $V \times W$ is the free vector space generated by elements $\{v_i \otimes w_j\}$ for $\{v_i\}$ basis of V and $\{w_j\}$ basis of W.

In this case $\otimes : V \times W \to V \otimes W$ is just defined on the bases and extends linearly.

Corollary 1.0.2. $\dim(V \otimes W) = n \cdot m$

Another construction

Definition 1.0.3. Consider $F(V \times W)$ the vector space constructed by all elements in the product- Thats huge!
We define $V \otimes W := F(V \times W)/ \sim$ with the relations $(v_1 + v_2, w_1) \sim (v_1, w_1) + (v_2, w_1)$, $(v_1, w_1 + w_2) \sim (v_1, w_2) + (v_1, w_1)$, $\alpha(v, w) \sim (\alpha v, w) \sim (v, \alpha w)$.

Remark 1.0.1. Not all tensors are pure!

Exercise 1.0.2. $O(X) \otimes O(Y) = O(X \times Y)$.

Definition 1.0.4. The tensor products of algebras is also an algebra, let A, B be algebras then $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1 a_2 \otimes b_1 b_2)$ on pure tensors and extend linearly...

Lemma 1.0.1. Let V be finite dimensional vector space and W any vector space then $Hom(V, W) = V^* \otimes W$

Proof. we have a map $f : V^* \otimes W \to Hom(V, W)$ defined on pure tensors $f(\zeta \otimes w)(v) = \zeta(v) \cdot w$
Define $g : Hom(V, W) \to V^* \otimes W$ by $g(L) = \sum_i \zeta_i \otimes L(v_i)$
These two constructions are inverse one to another

$$f \circ g(L)[v] = \sum_i f(\zeta_i \otimes L(v_i)[v] = \sum_i \zeta_i(v) L(v_i) = L(\sum_i \zeta_i(v)(v_i)) = L(v)$$

$$g \circ f(\zeta \otimes w) = g(v \mapsto \zeta(v) \cdot w) = \sum_i \zeta_i \otimes \zeta(v_i) w = (\sum_i \zeta(v_i)\zeta_i) \otimes w$$

□

2. More on modules

Recall the definition of a module....
A finitely generated module is also called "module of finite type"

Definition 2.0.1. Let $A \subset B$ be algebras, an element $b \in B$ is said to be integral over A if it satisfies a monic polynomial

Exercise 2.0.1. The integral elements in $\mathbb{Q}$ over $\mathbb{Z}$ are $\mathbb{Z}$

Proof. Let $p/q \in \mathbb{Q}$ with $(p, q) = 1$ and assume is integral.
we get $\sum_{i=0}^{n} (\frac{p}{q})^i a_i = 0$ $(a_n = 1)$. hence $p^n = q(...)$ which is a contradiction. □

Corollary 2.0.1. $\sqrt{2}$ is not rational.

The **IOU** from class

Lemma 2.0.1. Let A, B be algebras and $A \subset B$ TFAE:
1. (1) b is integral over A.
2. (2) $A[b]$ is f.g. as A module.
3. (3) $A[b] \leq C$, where C is a f.g. sub A-module of of B.
4. (4) There exist an faithful $A[b]$-module that is f.g. over A.

* A module is faithfull if is has trivial annihilator. i.e

$$Ann(M) = \{r \in R \ : \ rM = 0\} = \{0\}$$

This insures that the module is realy over R and not over R/I.

Proof. Obviously $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$.
We show $4 \Rightarrow 1$: Let M be the fathfull $A[b]$ module generated by $m_1, ..., m_n$. Write down

$$b \cdot m_i = \sum_j d_{ij} m_j$$

Write down the matrix $D = (d_{ij})$ the vector

$$\begin{bmatrix} bm_1 \\ \vdots \\ bm_n \end{bmatrix} = \begin{bmatrix} \sum_j d_{1j} m_j \\ \vdots \\ \sum_j d_{nj} m_j \end{bmatrix} = D \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$

so look at the monic polynomail $p(b) = \det(bId - D) = b^n + \sum_{i=0}^{n-1} a_i b^i$ it represents the zero endomorphsim of M over $A[b]$ so is 0 by faithful. $\square$

Lemma 2.0.2. Let $A \subset B$ be two finitely generated algebras (i.e. finitely generated as algebra) $B = A[b_1, ...b_n]$. Of course A is f.g. A module, we get that B is f.g. as A-Mod $\iff$ for any $b \in B$ $b^n = \sum_{i=1}^{n-1} a_i b^i$ i.e. B is integral over A.

Proof. If integral then f.g.:
By induction on n, for $n = 0$ is trivial.
Assume $C = A[b_1, ..., b_{n-1}]$ is f.g. A module with generators $c_1, ..., c_k$. Since b_n is integral over A we have that $C[b_n] = A[b_1, ..., b_n]$ is finitely generated as C module hence as a A module.

We leave for exercise: If C f.g. over B and B f.g. over A then C f.g. over A.

And exercise: the other direction.. $\square$

Corollary 2.0.2. IF $A \leq B$ then $C := int(B)$ is a submodule of B

Proof. If $b_{1,2} \in C$ then $A[b_1, b_2]$ is f.d. A module hence $b_1 - b_2, b_1 b_2$ are in this module and are thus integral. $\square$

Definition 2.0.2. If $A = int(A)$ we say A is integrally closed. If $B = int(A)$ we say B is integral extension

Exercise 2.0.2. $Int(\mathbb{Z})$ in $\mathbb{Q}[i]$ is $\mathbb{Z}[i]$

Exercise 2.0.3. $Int(\mathbb{Z})$ in $\mathbb{Q}[\sqrt{5}]$ is larger that $\mathbb{Z}[\sqrt{5}]$ (it contains $\frac{\sqrt{5}+1}{2}$)

Exercise 2.0.4. C integral over B and B integral over A than C integral over A.

Lemma 2.0.3. Let $A \leq B$ be an integral extension, then for any prime $p \lhd A$ there is a prime $q \lhd B$ s.t. $p = q \cap A$.

Proof. This is out of context already althou not complicated, it is on the way to Nother normalization lemma! $\square$

Theorem 2.0.1 (Nother normalization)**.** Let A be a finitly generated algebra. then there are $x_1, ..., x_n \in A$ algebraically independent (dont satisfy a polynomial with n variables). s.t

$$\mathbb{C}[x_1, ..., x_n] \leq A$$

is integral extension.
i.e. any f.g. algebra is a integral extension of a polynomial ring.

Proof. Actually this proof is to involved $\square$

3. Finite morphisms

Definition 3.0.1. A morphism of affine varieties $\nu : X \to Y$ is called finite if $\mathcal{O}(X)$ is a finite generated $\nu^*(\mathcal{Y})$-Module.

We did not assume it to be injective!
Explain the terminology

Exercise 3.0.1. If $Y = Pt$, ν is finite if and only if X is a finite set.

Exercise 3.0.2. let $\nu : X \to Y$ be finite and $Z \subset X$ be closed (Zariski) , then $\nu|_Z : Z \to Y$ is finite.

Proof. We fisrt claim that: Any closed embedding is finite- clearly and then composetion of finite morphism is finite □

Recall the Nother normalization lemma

Corollary 3.0.1 (of Nother)**.** Let X be an affine variety then there is $n \in \mathbb{N}$ and a finite morphism $\nu : X \to \mathbb{C}^n$ (surjective!)

Proof. We know that injective implies dense image so is finite onto its closure but then is closed map... □

Corollary 3.0.2. Any finite morphism has finite fibers!

We want to define a finite map on protective varieties. Any $Y \subset \mathbb{P}^m$ an be decomposed into affine open cover $Y_i = Y \cap \mathbb{P}^m_i$.
The following definiton is Ad-Hok for our porpase of discussion.

Definition 3.0.2 (affine map)**.** Let $\nu : X \to Y$ be a map of projective spaces we say that ν is an *affine map* if $\nu^{-1}(Y_i)$ is a affine variety.

We can now say what a finite map is

Definition 3.0.3. A affine morphism $\nu : X \to Y$ between projective varieties is called finite if for any i $\nu|_{\{}\nu^{-1}(Y_i)\}$ is finite.

Show that the map defined is class is finite!