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Tutorial

WEEK 2

1. Gauss lemma

Lemma 1.0.1 (Gauss)**.** If $f \in \mathbb{Z}[x]$ is irreducible and gcd of coefficient is 1 then f is irreducible over $\mathbb{Q}$.
In general this applies to UFD and ring of fractions.

Corollary 1.0.1. If $p(\pi, y) \in \mathbb{Q}[\pi][y]$ is irreducable then it is irreducable over $\mathbb{Q}(\pi)[y]$. Since irreducibility in this case implies primitive.

2. Complex analysis

First we will define what is a holomorphic and meromorphic function, and then sate important definitions/results.

Definition 2.0.1. Let U be a open subset in $\mathbb{C}$ A function $f\colon U \to \mathbb{C}$ is holomorpic if its derivative exist for any point $w \in U$.
A function on the open disk of radius r around a: $B_r(a)$ is holomorphic if $f(x) = \sum_n c_n(x-a)^n$ (this is given by the Taylor series)

Definition 2.0.2. Let U be a open subset in $\mathbb{C}$ A function $f\colon U \to \mathbb{C} \cup \infty$ is moromorpic if $f|_{U \setminus f^{-1}(\infty)}$ is holomorphic and each $a \in f^{-1}(\infty)$ is a pole.
i.e. in some neighborhood of a, $f(x) = \frac{g(x)}{(x-a)^m}$ for $g(a) \neq 0$ holomorphic and $m \in \mathbb{N}^+$
So in a neighborhood of a we get that

$$f(x) = \sum_{n \geq -m} c_n(x-a)^n = \frac{1}{(x-a)^m} \underbrace{\sum_k c_{k-m}(x-a)^k}_{g(x)}$$

We obtain that $g(a) = c_{-m}$ and define the **residue** of f to be $resf(x); a \coloneqq c_{-1}$

Example 2.0.1. The function $\sqrt{z}$ is not holomorophic on $\mathbb{C}$, first of all, it is not well defined
But even choosing some sign$\pm\sqrt{z}$: $re^{i\theta} \mapsto \sqrt{r}e^{\frac{i\theta}{2}}$ is not holomorphic. Since for points on the non negative real axis $r \in [0, \infty]$ we get that for $z_n = r \cdot e^{ia_n},\ w_n = r \cdot e^{2\pi i - ia_n}$ with $a_n \to 0$ that

$$\lim_{n \to \infty} z_n, w_n \to r,$$

but $\sqrt{z_n} \to \sqrt{r}$ and $\sqrt{w_n} \to -r$, so is not even contiuous.

Exercise 2.0.1. $\pm\sqrt{z}$ is holomorphic on $\mathbb{C} \setminus [0, \infty)$

Theorem 2.0.1 (The implicit function theorem)**.** Let $A(x, y) \in \mathbb{C}[x, y]$ and let $x_0, y_0 \in \mathbb{C}$ such that

$$A(x_0, y_0) = 0, \quad \frac{\partial A}{\partial y}(x_0, y_0) \neq 0$$

Then there are open sets $x_0 \in U,\ y_0 \in V$ and a **holomorphic** function $f\colon U \to V$, such that $f(x_0) = y_0$.
And if $(x, y) \in U \times V$ with $f(x) = y$ then $A(x, y) = 0$ (write$A(x, f(x)) = 0$).

Theorem 2.0.2 (Inverse function theorem)**.**
(1) Let $d\colon U \to V$ be a holomorphic bijection between open sets. Then $\forall u \in U f'(u) \neq 0$ and f^{-1} is holomorphic
(2) Let $d\colon U \to \mathbb{C}$ be holomorphic and $u \in U$ with $f'(u) \neq 0$, then there exist an open set $U' \subset U$ containing a and open V containing $f(u)$ such that f is bijective holomorphic between U' and V.

3. Surfaces

Definition 3.0.1 (surface)**.** A surface is a Housdorf topological space locally homoemorphic to $\mathbb{C}$ (or $\mathbb{R}^2$), i.e. for every $x \in S$ there is a open $U \subset S$ containing S such that U is homeomorphic to $\mathbb{C}$.
Such a homoeomorphisim is called a chart

Definition 3.0.2 (atlas)**.** An atlat Φ on a surface S is a collection of charts $\phi_\alpha : U_\alpha \to V_\alpha$ such that $S = \bigcup_\alpha U_\alpha$, together with that property that the image under any chart of $U\alpha \cap U_\beta$ is open.

Definition 3.0.3 (translation function)**.** Given an atlas S, a translation function $\phi_{\alpha\beta} : \phi_\beta(U\alpha \cap U_\beta) \to \phi_\alpha(U\alpha \cap U_\beta)$ defined by $\phi_\alpha \circ \phi_\beta^{-1}$. is a map between open subsets on $\mathbb{C}$.
The atlas is called **Holomorphic** if translation functions are holomorphic.

4. Picturing the curve

Recall that we had $\sqrt{z}$ a non holomorphic function.
What about the function $\sqrt{z(z-1)(z+1)}$
We can thik as a curve $y^2 = x(x+1)(x-1)$ - what does this curve look like? (maybe 2 copys of C with 3 glues points).
The each copy for a y point - the plane itself is x.
what does it really look like? since if we go around a point - we get to its minus - recall the sqrt(Z) and draw from there

5. Resultant of polynomials

Definition 5.0.1. Let $P(x), Q(x) \in \mathbb{C}[x]$ write $P(x) = \sum_{i=1}^{n} a_i x^i$ $Q(x) = \sum_{i=1}^{m} b_i x^i$ with $a_n, b_m \neq 0$.
The *Resultant* on P and Q, denoted by $\mathcal{R}(P, Q)$ is the determinant of the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_n & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ \vdots & & & & \cdots & & & \vdots \\ 0 & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_0 & b_1 & & \cdots & b_m \end{bmatrix}$$

Lemma 5.0.1. Let $P(x), Q(x) \in \mathbb{C}[x]$ then P, Q have non constant common factor (hence a common zero) iff $\mathcal{R}(P, Q) = 0$

Lemma 5.0.2. If $P(x) = \prod_{i=1}^{n}(x - \lambda_i)$ and $Q(x) = \prod_{i=1}^{m}(x - \mu_i)$ then $\mathcal{R}(P, Q) = \prod_{i,j}(\mu_j - \lambda_i)$.
In particular for polynomails $P, Q, A \in \mathbb{C}[x]$ we get that

$$\mathcal{R}(P, QA) = \mathcal{R}(P, Q)\mathcal{R}(P, A)$$

6. Examples of Zariski closed sets

Recall the Zariski correspondence and NSS

Exercise 6.0.1. Show that any set in $\mathbb{C}^n$ containing $\{(t, \sin(t)\}$ is Zariski dense.

HInt: Plug in $f(\pi/4, 1/\sqrt{2})$ and play with polynomial...

Exercise 6.0.2. Let $a, b \in K$ what is the Zariski closer of the set $\{(a^n, b^n)\ n \in \mathbb{N}\}$.
Show that the set is dense iff $a^{n_0} \neq b^{m_0}$ for any n_0, m_0.
Hint: If $a^{n_0} = b^{m_0}$ then we have for large enough n that (a^n, b^n) is satisfied by $x^{m_0} = y^{n_0}$.

Exercise 6.0.3. Show that set $\{t, t^2, t^3\}$ is Zariski closed, what is its Ideal?