Elyasheev Leibtag
Weizmann institute of science

**Tutorial**

**WEEK 12**

## 1. The co-tangent space

Let $X = \mathcal{Z}(f_i)$ recall that in the lecture we defined $T_0(X) = \mathcal{Z}\{f_{i,1}\}$ the zeros locus of the linear part of $f_i \in I$. Notice that this indeed is a linear subspace since $\mathcal{Z}\{f_{i,1}\} = \bigcap_i \mathcal{Z}\{f_{i,1}\}$ is intersection of hyper-surfaces defined by degree 1 homogeneous polynomial thus are linear sub spaces.
Let $m_0 \lhd \mathcal{O}(X) = \mathbb{C}[x_1, .., x_n]/I$ be the ideal of functions vanishing at the point 0 these are all regular functions of degree greater than 0.
This is a maximal ideal (NSS)
The quotient ideal $m_0/(m_0)^2$ is a $\mathbb{C}$ module as well, notice that $m_0^2$ are all functions of degree greater than 1 so the this quotient identifies regular functions on $X$ which have the same linear part.

**Lemma 1.0.1.** $T_0X^* \cong m_0/m_0^2$

*Proof.* We have that $T_0X \cong \mathbb{C}^k$ is a linear subspace of $\mathbb{C}^n$ so functional on this space is a restriction of a functional on $\mathbb{C}^n$ which is given by a linear function $\mathbb{C}^n \to \mathbb{C}$ i.e. be a homogeneous degree 1 polynomial.
We therefore get a $\mathbb{C}$ module map $\Psi : m_0 \to (T_0X)^*$ $\Psi(g) = g_1$ taking the linear part of the polynomial. This is a $\mathbb{C}$ linear map and is clearly surjective, we show $\ker(\Psi) = m_0^2$ it is easy ti check that if $g \in m_0^2$ then $\Psi(g) = 0$. In the other direction assume $\Psi(g) = g_1 = 0$ this means that $g_1 \in < f_{i,1} >$ thus $g_1 = \sum a_i f_{i,1}$ so as a regular function $g$ is equivalent to $g - \sum a_i f_i$ which has degree larger that 1. $\square$

**Definition 1.0.1.** For a general point $x \in X$ define the tangent space $T_xX = (m_x/m_x^2)^*$

### 1.1. The differential map.

Gven a map $\psi : X \to Y$ we get a map $\psi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ this map restricts to map $\psi^* : m_{\psi(x)} \to m_x$ since if $h \in m_{f(x)}$ then $\psi * h(x) = h(\psi(x)) = 0$, we can compose the map

$$m_{\psi(x)} \xrightarrow{\psi^*} m_x \xrightarrow{\pi} m_x/m_x^2$$

Notice that $m_{\psi(x)}^2 \mapsto m_x^2$ thus we get an induced map

$$d\psi^* \colon m_{\psi(x)}/m_{\psi(x)}^2 \to m_x/m_x^2$$

This map gives a dual map $d\psi : T_xX \to T_{\psi(x)}Y$.

## 2. A nice finite map with inverse

We saw in previous tutorial that given this lemma we can embed any curve into $P^3$

**Lemma 2.0.1.** (Local condition for isomorphism)
Let $f : X \to Y$ be a finite map. $f$ is isomorphism iff $f$ is bijection and $T(f) : T_x(X) \to T_{f(x)}Y$ is injective for all $x \in X$.

We will need the following Fact do to Nakayama from commutative algebra

**Fact 2.0.1** (Nakayama)**.** Let $M$ be a f.g. $A$-Module and $I \lhd A$ s.t. $I \subset J(A)$, if $IM = M$ then $M = 0$ ($J(A) = \cap_{m \lhd A} m$ intersection of maximal ideals)

*Proof.* (Of Isomorphism Lemma.) First by finitness we may assume $X$ any $Y$ to be affine and therefore get that $f$ induces $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ since $f$ is surjective $f^*$ is injective and we denote $\mathcal{O}(Y) = A \subset B = \mathcal{O}(X)$ to be the integral extension. We need to show $f^*$ is isomorphism. The fact that $f$ is bijection gives us bijection between maximal ideals in $A$ to maximal ideals in $B$, taking the ideal $m \lhd A$ to maximal ideal containing $mB$, and for $n \lhd B$ we get $n \cap A \lhd A$ maximal.

**Claim: enough to show for localization at maximal ideal**
*We will how this claim later- these locallizations are exactly the local rings* $\mathcal{O}_{X,x}$

So assume now that $A$ and $B$ are local with maximal ideals $m$ and $n$.
Since $d\psi : T_xX \to T_{f(x)}Y$ is injection we get a surjection $m/m^2 \to n/n^2$ thus $mB + n^2 = n$ as ideals in $B$ (As $B$-modules).
Now apply Nakayama lemma applied to the $B$-mod $n/(mB)$
(Take $n/mB$ to be the module and $n$ to be the ideal)
we get that $mB = n$. Since the map is finite $B$ is a finitely generated $A$-module and since $mB = n$ we get that $B/mB = B/n = \mathbb{C} = A/m$ as $A$ module thus $A/m = A/(A \cap n) = (A + n)/n = \mathbb{C} = B/n$ thus $n + A = mB + A = B$ and we get by Nakayama on$B/A$ observing $m \cdot B/A = mB/A = B/A$ that $A = B$. □

Remarks:

**Remark 2.0.1.** FACT: If $X$ and $Y$ are projective, finite fibers imply finite map, so the condition on the map being isomorphism is redundant.

Why is it enough to show for local rings?
We want to show $B = A$, so we show $B/A = 0$ as a $A$ module. We do this by showing that $B_m/A_m = 0$ as $A_m$ module. Following 2 lemmas:

**Lemma 2.0.2.** Let $N \to M \to L$ be exact sequence of $A$ modules then $S^{-1}N \to S^{-1}M \to S^{-1}L$ is an exact $S^{-1}A$ module.

**Corollary 2.0.1.** $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$

we get the conclusion by:

**Exercise 2.0.1.** If $M_p = 0$ for any $p \lhd A$ maximal, then $M = 0$.

*Proof.* Indeed, assume $M \neq 0$ and $x \in M$ then $Ann(x) \subset q \lhd A$ for some maximal $q$, then since $M_q = 0$ we get that $x/1 = 0$ thus $\exists t \notin q$ s.t. $tq = 0$ hence $t \in Ann(x) \subset q$ in contradiction. $\square$