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# Tutorial

## WEEK 13

We will answer some questions from the problem set

## 1. Question 1

### 1.1. assumptions that should be made in this question.

First we assume that $D$ is an effective divisor that is "base-point free" i.e: For any $p \in C$ $L(D - [P]) \subsetneq L(D)$.

Notation: Given a principal divisor $(f)$ (for $f \in Rat(C)$) we may represent $(f)$ as the difference of 2 effective divisors $(f)_0 - (f)_\infty$ (poles minus zeros).

Secondly, given a hyperplane $H = \mathcal{Z}(\sum_{i=0}^n \lambda_i X_i) \subset \mathbb{P}^n$ we define the divisor $f^{-1}(H)$ to be the effective divisor defined by $(\sum_{i=0}^n \lambda_i f_i)_0$. (Try playing around with examples and see why this makes scene).

**Solutions**

(1) If $f$ would be contained in hyperplane then this would define a linear dependence of the bases $f_i$.

(2) First we remark that for any two hyperplanes $H, H'$ defined by linear forms $S, S'$ (respectively) the divisor $f^{-1}(H)$ is equivalent so the divisor $f^{-1}(H')$ since $f^{-1}(H) - f^{-1}(H') = (S/S')$.

Now by the assumption that $D$ is base-point free we know that $\cup_i L(D - [p_i]) \subsetneq L(D)$ (union over points in $D$). hence there exist $g \in L(D) \setminus cup_i L(D - [p_i])$ therefore we get the condition (*)

$$(g)_\infty \nleq (D - [p_i]).$$

Since $(g)_0$ and $(g)_\infty$ have disjoint support we obtain that $(g) + D = (g)_0 - (g)_\infty \geq 0 \Rightarrow D \geq (g)_\infty$ but together with (*) we obtain that $(g)_\infty = D$ thus $(g)_0 \sim (g)_\infty \sim D$.

Now since all pullback of hyperplane sections are equivalent what we do now is find the "correct" hyperplane. Since $g \in L(D)$ $g = \sum_{i=0}^n \lambda_i f_i$ so we just take the hyperplane to be $H = \mathcal{Z}(\sum_{i=0}^n \lambda_i X_i)$ and we get by definition that $f^{-1}(H) = (g)_0$.

(3) The bijection: Given a divisor $D$ as above we define $f\colon C \to \mathbb{P}(L(D))$ as defined in the question.

We show this map is well defined : Let $\{f_i\}_{i=0}^n$, be a basis for $L(D)$, assume $D \sim D'$ then $D - D' = (h)$, define the map $\hat{h} : L(D) \to L(D')$ $\hat{h} : g \mapsto hg$ is a linear bijection since

$(g) + D \geq 0 \iff (g) + (h) + D' = (gh) + D' \geq 0$. (we get that $L(D) \cong L(D')$)
So regarding $L(D)$ with basis $\{f_i\}_{i=0}^n$ and $L(D')$ with basis $\{hf_i\}_{i=0}^n$ as $n+1$ dimensional vector spaces we get a matrix $A \in GL(n+1)$ (that satisfies $hf_i = \sum_j A_{i,j} f_j$ as functions on $C$) let $\bar{A} \in PGL$ be its projective class. We obtain the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\Phi_D"] \arrow[dr, "\Phi_{D'}"'] & \mathbb{P}^n \arrow[d, "\bar{A}"] \\ & \mathbb{P}^n \end{tikzcd}$$

So the map is well defined.
**Injective**:
Assume by contradiction that $D \nsim D'$ but there exist $\bar{A} \in PGL(n+1)$ such that $\bar{A} \begin{bmatrix} f_0(p) \\ \vdots \\ f_n(p) \end{bmatrix} = \begin{bmatrix} f'_0(p) \\ \vdots \\ f_n(p) \end{bmatrix}$ Then we obtain that $D' \sim f'^{-1}(X_0 = 0) = (f'_0)_0 = (\sum_j A_{i,j} f_j)_0 = f^{-1}(\sum_j A_{i,j} X_j = 0) \sim D$ in contradiction.
**surjective**
Let $f : C \to \mathbb{P}^n$ be a map that does not intersect any hyperplane. $f(p) = [f_0(p) : f_1(p) : ... : f_n(p)]$ we show that $f_i \in L(D)$ for the divisor $(f_0)_\infty + (f_1)_\infty + ... + (f_n)_\infty$, this is just since $(f_i) + D = (f_i)_0 + (f_0)_\infty + .. + (f_{i-1})_\infty + (f_{i+1})_\infty + ... + (f_n)_\infty \geq 0$.
The fact that $f_i$ are a basis comes from the assumption that the map does not intersect any hyperplane.