Final exam

February 16, 2020

Each question below is worth 18 points. If you have a partial solution (for example, proof assuming the variety is a curve) or even just an idea for a question, write it down to get partial credit. Unless otherwise stated, all varieties are over the complex numbers.

1. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension d.
   (a) Show that there is an $(n-d)$-dimensional projective subspace $V \subset \mathbb{P}^n$ such that V intersects X transversally.
   (b) Show that if V_1, V_2 are $(n-d)$-dimensional projective subspaces that intersect X transversally, then $|X \cap V_1| = |X \cap V_2|$. This number is called the degree of X and denoted by $\deg(X)$.
   (c) In the case where $X = Z(f)$ is a hypersurface, show that $\deg(X) = \deg(f)$.
2. Suppose that $C \subset \mathbb{P}^3$ is an irreducible projective curve and that $S \subset \mathbb{P}^3$ is an irreducible projective surface (i.e., $\dim(S) = 2$). Assume that C intersects S transversally. Show that $|C \cap S| = (\deg C)(\deg S)$.
3. Let $X \subset \mathbb{P}^n$ be a projective variety defined as the vanishing locus of a bunch of polynomials over $\mathbb{Q}$. Let $X(\overline{\mathbb{Q}}) \subset X$ be the subset consisting of points whose coordinates are in the algebraic closure of $\mathbb{Q}$. Show that $X(\overline{\mathbb{Q}})$ is dense in the usual (i.e., not Zariski) topology.
4. Let C_t be the elliptic curve given by the equation $y^2 = x(x-1)(x-t)$. Show that C_2 and C_3 are not birational.

5. Let N be the collection of 3-by-3 nilpotent matrices. Show that N is a Zariski closed subset of $\mathrm{Mat}_3(\mathbb{C})$ and compute the tangent cones to N at the points $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

6. Let $X = Z(x^2 + y^4 + z^4) \subset \mathbb{C}^3$, and let $\overline{X} \subset \mathbb{P}^3$ be the closure of X in $\mathbb{P}^3$.

   (a) Find the singular points of $\overline{X}$.

   (b) Identifying $\mathbb{P}^3 \smallsetminus \mathbb{C}^3$ with $\mathbb{P}^2$, find the curve $\overline{X} \cap \mathbb{P}^2$.

7. Let $\ell_1, \ell_2, \ell_3 \subset \mathbb{P}^3$ be skew (i.e., disjoint) projective lines. Let X be the union of all lines ℓ such that $\ell \cap \ell_i \neq \emptyset$ for $i = 1, 2, 3$. Show that X is a Zariski closed subset and, as a projective variety, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.