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Tutorial

WEEK 7

1. A projective non singular curve as a holomorphic surface

Theorem 1.0.1. If C is a projective curve in $\mathbb{P}^2$ then $C \backslash \{\text{singular points}\}$ has a holomorphic atlas.

Proof. Let $[a:b:c] \in C$ non singular, and assume $\frac{\partial P}{\partial y}(a,b,c) \neq 0$. By Euler relation

$$a \cdot \frac{\partial P}{\partial x}(a,b,c) + b \cdot \frac{\partial P}{\partial y}(a,b,c) + c \cdot \frac{\partial P}{\partial z}(a,b,c) = dP(a,b,c) = 0$$

Thus if $a = c = 0$ we get that $b = 0$. Therefore a or c are non zero, assume that $c \neq 0$ and observe that since the derivative sis homogeneous of degree $d-1$ we get

$$\frac{\partial P}{\partial y}(a/c, b/c, 1) = c^{-(d-1)} \cdot \frac{\partial P}{\partial y}(a,b,c) \neq 0$$

so look at the polynomial in two variables $P(x,y,1)$ by the implicit function theorem around the point $x_0 = a/c, y_0 = b/c \in \mathbb{C}$ there are open sets V, W and a holomorphic map $g\colon (V, x_0) \to (W, y_0)$ such that in $V \times W$ $P(x,y,1) = 1 \iff y = g(x)$. We obtain an open set around $[a:b:c] = [a/c : b/c : 1] \in C$

$$U = \{[x:y:1] \in C \ : \ x \in V, y \in W\}$$

where the map $U \to V$ $[x:y:z] \mapsto x/z$ is a homeomorphism with inverse $w \mapsto [w : g(w) : 1]$
We get similar maps for other cases ($[x:y:z] \mapsto z/x, y/z/, z/y, x/y, y/x$) with inverse maps $[1 : g(w) : w]$ exetera, the transition maps will be of the form $w \mapsto w, g(w), 1/w, 1/g(w), w/g(w), g(w)/w$ all defined and holomorphic on there domain $\square$

Definition 1.0.1. We now can say what is a holomorphic map between 2 holomorphic surfaces. Fiven S_1, Ψ_1 and S_2, Ψ_2 a continuous function $f : S_1 \to S_2$ is holomorphic (with respect to the atlas) if

$$\psi^2_\beta \circ f \circ {\psi^1_\alpha}^{-1}|_{\psi^1_\alpha(U_\alpha \cap f^{-1}W_\beta)} \colon \psi^1_\alpha(U_\alpha \cap f^{-1}W_\beta) \to Y_\beta$$

is holomorphic.
(lets not get to specific here)

2. Genus Degree & Branch covering

Lemma 2.0.1. Let C be the projective curve defined by $x^d + y^d + z^d$. The map $C \to \mathbb{P}^1$ taking $[x : y : z]$ to $[x : z]$ is holomorphic.

Proof. Let $[a : b : c] \in C$ then if $b \neq 0$ we know that $\frac{\partial P}{\partial y}(a, b, c) \neq 0$ so assume $c \neq 0$ and we have a chart around $[a/c : b/c : 1]$ going to a/c with inverse $w \mapsto [w : g(w) : 1]$ this goes to $[w : 1]$ which maps to w and is holomorphic with respect to the charts, and so on. $\square$

Remark 2.0.1. For $a \neq 0$ we have $\frac{\partial P}{\partial x}(a, b, c) \neq 0$ we may assume $c \neq 0$ and get a map $[a : b : c] = [a/c : b/c : 1]$ going to b/c with local inverse $w \mapsto [g(w) : w : 1]$ and this is locally looking like $w \mapsto g(w)$ between the manifolds. This map is of the form $z \mapsto \sum_i (z - x_0)^i$ on a small enough neighborhood, let n be the smallest non zero power. then $[a : b : c]$ is a branch point of index $n - 1$.
The point $[\zeta : 0 : 1]$ is a branch point since g as above is the map $z \mapsto \zeta + h(z)$ and we have that $(z)^d + (\zeta + h(z))^d = -1$ thus

$$z^d + \sum_{k=0}^{d} \binom{d}{k} \zeta^{d-k} h(z)^k = -1$$

so

$$z^d + \sum_{k=1}^{d} \binom{d}{k} \zeta^{d-k} h(z)^k = 0$$

hence the lowest term in the taylor expansion is d. (exercise)

Here is a alternative definition, more easier to compute.

Exercise 2.0.1. Let C be a projective curve defined by P and assume $P(0, 1, 0) \neq 0$ the map $[x : y : z] \to [x : z]$ is well defined and the index of a point $[a : b : c]$ is one-minus the order of zero of the polynomial $P(a, y, c)$ at b.

3. Hyperelliptic curves

Let $p(x, y) = y^2 + a(x)$ then the curve defined by p are exactly the points satisfying

$$y^2 = a(x) = \prod_i (x - a_i)$$

This is called a hyperelliptic curve, if the degree of $a(x)$ is 4 we would expect to get that it defines a genus 3 surface.
We may draw the curve and try to glue and see that we get a genus 2 surface and not 3.
The reason is that the projectification of this curve is has singularities at infinity!
We will later in the course how to resolve this singularities

4. SEPARATED DEFINITIONS

Theorem 4.0.1. A topolocial space X is housdorff if and only if ΔX is a closed subspace of $X \times X$

Theorem 4.0.2. A Housdorff topolocial space X is compact if and only if for any other space Y, the projection $\pi_y : X \times Y \to Y$ is a closed map.

Recall that the product of affine algebraic varieties is NOT the Zariski-topology product. So we have analogs of Hausdorff and compact in algebraic varietys.
For our sake an algebraic variety is eater affine or projective

Definition 4.0.1. A algebraic variety X is called seperated if ΔX is a closed subspace of $X \times X$

Exercise 4.0.1. Affine varieties are separated.

Lemma 4.0.1. Protective varieties are separated (this will imply that algebraic ones are since subsets of separated is separated)

Proof. Enough to prove for $\mathbb{P}^n$. By the Segre embedding we get that $\mathbb{P}^n \times \mathbb{P}^n$ is a closed projective subvariety of $\mathbb{P}^{n^2+2n}$ and the digonal are the elements where $z_{ij} = z_{ji}$ (recall the embedding). □

Definition 4.0.2. A Seperated variety X is complete if for any other space Y, the projection $\pi_y : X \times Y \to Y$ is a closed map.

What we prove it class was the following claim:

Lemma 4.0.2. Ptojective varieties are complete.

Proof. We saw the proof in class, here are some remarks.
We saw that it is enough to show for $Z \subset \mathbb{P}^n \times \mathbb{C}^k$.
We will define again exactly what is the topology on this product.
The closed sets are of the form $\{(y_0, ..., y_n, x_1, ..., x_k) \in \mathbb{C}^{n+1} \times \mathbb{C}^k \ : f_i(x, y) = 0\}$ where f_i is homogeneous in $x_0, ..., x_n$.
i.e. adding 0, a subset $A \in \mathbb{P}^n \times \mathbb{C}^k$ is a closed subset in $\mathbb{C}^{n+1} \times \mathbb{C}^k$ that is defined by homethey invariant polynomails in the first $n + 1$ coordinates.
The projection to $\mathbb{C}^K$ is exactly the set

$$\{(z_1, ..., z_k) \in \mathbb{C}^k \ : \ f_i(y_0, ..., y_n, z_1, ..., z_k) \in \mathbb{C}[y_0, ..., y_n] \text{have common zero in } \mathbb{P}^n\}$$

This is where we use the projective NSS: The set $h_i(y_0, ..., y_n)$ has no common zeroes in $\mathbb{C}^{n+1}$ besides 0, that is $1 \in I$ where $I = \mathcal{I}(Z(h_i))$ if has no common zero, or $Y_i \in I$ if 0 is the only common root.
Any way $\{Y_i\} \subset I$ so *there is* some m s.t. $Y_i^m \in < h_i >$
This means

$$\{(z_1, ..., z_k) \in \mathbb{C}^k \ : \ f_i(y_0, ..., y_n, z_1, ..., z_k) \in \mathbb{C}[y_0, ..., y_n] \text{have common zero in } \mathbb{P}^n\}$$

is equal to

$$\{(z_1, ..., z_k) \in \mathbb{C}^k \ : \forall m > 1, < f_i > \subset \mathbb{C}[y_0, ..., y_n] \text{do not contain all monomials of degree m } \}$$

equal to

$$\bigcap_{m>1} \{(z_1, ..., z_k) \in \mathbb{C}^k \ :< f_i > \subset \mathbb{C}[y_0, ..., y_n] \text{do not contain all monomials of degree m } \}$$

And this is an intersection of closed sets by the following argument:
Denote by $< f_i >_m$ all m homogeneous polynomials of the ideal - this is a **sub** vector space of $\mathbb{C}[y_0, ..., y_n]_d$ which is spanned by the monomials. The bases of $< f_i >_m$ is the polynomials $f_i \cdot z^I$ where I is multi index to make it degree m, this spanning set is finite. So we are left with the question does a finite set pf vectors span the whole space?
Write these vectors as columns in the standard (monomial) basis get a matrix and check that the rank is smaller than $\binom{m+n}{n}$ □