Elyasheev Leibtag
Weizmann institute of science

Tutorial

WEEK 5

1. K-algebras

Definition 1.0.1. A vector space over a field K together with binary operation $\cdot$ such that for any $\alpha \in K$ and $a, a', b, b' \in A$ we have the following

(1) $(\alpha a) \cdot b = \alpha(a \cdot b)$
(2) $(a + a') \cdot b = ab + a'b$
(3) $a(b + b') = ab + a'b$

If the multiplication is associative or commutative we say that the algebra is associative or commutative. A unit in an algebra is a unit wrt the multiplication.

Remark 1.0.1. Let R be a commutative ring with unit, and let $\varphi : K \to R$ be a morphism of rings (with unit) - this gives us a structure of a associative commutative unital algebra in R.

Example 1.0.1. Ring of matrices - what will be the map from $\mathbb{C}$?
Ring of polynomials

Definition 1.0.2. A algebra A (over a field K) is called finitle generated if there exists a finite set of elements $a_1, ..., a_n$ of A such that every element of A can be expressed as a polynomial in $a_1, ..., a_n$ with coefficients in K.

Exercise 1.0.1. $\mathbb{C}(x)$ is not finitely generated $\mathbb{C}$ algebra.

2. Modules

Definition 2.0.1. Let R be a ring, an abelian group M is a module over R if the ring "acts on M" in a respectful way.
i.e. there is a function $\cdot : R \times M \to M$ with

$$(r_1 + r_2) \cdot m = r_1 \cdot m + r_2 \cdot m$$
$$(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$$
$$r \cdot (m_1 + m_2) = r \cdot m_1 + r \cdot m_2$$
$$1 \cdot m = m$$

Example 2.0.1. A Module over a field is a vector space

Example 2.0.2. Any abelian group is a A Module over the ring $\mathbb{Z}$

Example 2.0.3. For the ring $(R, \cdot, +)$ the additive group of the ring is a module over the ring.
Actually any Ideal $I \lhd R$ is a module over R.
Any quotient ring is a module... (Quotient of modules is is module)

Definition 2.0.2. A module M is said to be finitely generated, if there exist finite amount of elements $m_0, ..., m_n$ such that any $m \in M$ is a sum of these elements with coefficients in R

Example 2.0.4. A basis for a vector space is a (minimal) generating set.

Example 2.0.5. We saw HBT that the ideals in $\mathbb{C}[x_1, ..., x_n]$ are finite generated as modules over $\mathbb{C}[x_1, ..., x_n]$.
What about thinkng at $\mathbb{C}[x]$ a $\mathbb{C}$ module, is it finitely generated?

3. Notherian modules

Definition 3.0.1. M is Notherian R module iff any sub module has the ascending chain condition for submodules. (or equiv is f.g.)

Definition 3.0.2. A ring R is called Notherian if any asending chain of ideals in R stabilizes.
i.e. if $I_1 \leq I_2 \leq .... \leq I_i \leq ....$ then there is k s.t $I_k = I_{k+1}$
i.e. R is Notherian as a module over itself

Exercise 3.0.1 (trivial)**.** Show that any filed in Notherian...

Lemma 3.0.1. Let $0 \to N \to M \to L \to 0$ be a exact chain of R modules then M is Nother iff L and N are

Proof. If M is then clearly any submodule of N is submodule of M and same for quotient...
In the other direction let P be a submodule then $P \cap N$ is finitly generated say by $n_1, ..., n_k$ we have that $P/P \cap N \cong P + N/N$ thus is a submodule of L so is finitly generated by $l_1 + N, ..., l_j + N$
See that l_i, n_j generate P. $\square$

Theorem 3.0.1 (HBT)**.** Let R be a Notherian ring, then the ring of polynomials over R: $R[x]$ is a Notherian ring.

Exercise 3.0.2 (for home)**.** Prove HBT *mutatis mutandis* from proof in class.

4. Notherian in topology

Definition 4.0.1. A topological space is called Notherian if any descending chain of closed subsets stabilizes.

Exercise 4.0.1. Any Notherian space is compact...

Lemma 4.0.1. Let X be a Notherian space- any closed subset of X can be presented a union of irreducible sets, and moreover if we assume the irreducible sets are disjoint this presentation is unique.

Proof. Assume by contradiction that there exist a minimal closed set Y such that Y can not be represented as above. In particular Y is reducible then $Y = Y_1 \cup Y_2$ by minimality of Y, Y_i can be decomposed $Y_1 = \bigcup_{j=0}^{l} Y_1^j$ and $Y_2 = \bigcup_{j=0}^{k} Y_1^j$

$$Y = \bigcup_{i=1,2, \quad j \leq k \wedge l} Y_i^j$$

For uniqueness assume $Y = \bigcup_{j=0}^{l} Y_j = \bigcup_{i=k}^{l} Z_i$ then $Z_i = \cup_j (Z_i \cap Y_j)$. But Z_i is irreducible! hence $Z_i \subset Y_j$ and similarly $Y_j \subset Z_k$ this means $Z_i \subset Z_k$ is contradiction... □

So lets see what this gives us for algebraic varieties...

5. Irreducible topological spaces

Definition 5.0.1. Let X be a topological space. TFAE:

- If there exist Z_1, Z_2 closed in X such that $X = Z_1 \cup Z_2 \Rightarrow Z_1 = X \text{ or } Z_2 = X$
- For any $U \subset X$ open $\overline{U} = X$

Definition 5.0.2. An ideal $I \lhd R$ is called irreducible if it cannot be written as the intersection of two strictly larger ideals.

Lemma 5.0.1. Let X be a algebraic set with $I = \mathcal{I}(X)$, then X is irreducable in $\mathbb{C}^n$ (Zariski) iff $\mathcal{I}(X)$ is prime in $\mathbb{C}[x_2, ..., x_n]$.

Proof. Assume I not prime, then there is $f, g \notin I$ such that $fg \in I$ this means $X = \mathcal{Z}(I) \subset \mathcal{Z}fg = \mathcal{Z}g \cup \mathcal{Z}f$ but $X \not\subseteq \mathcal{Z}(f) \text{ or } \mathcal{Z}(g)$.
If we assume X to be non irreducable $X = \mathcal{Z}(J_1) \cup \mathcal{Z}(J_2) = \mathcal{Z}(J_1 \cdot J_2)$ then there is a $f_i \in J_i, \ f_i \notin I$ with $f_1 f_2 \in I$ □

Lemma 5.0.2. Let $I \subset \mathbb{C}[x_1, ..., x_n]$, if I is prime Ideal then it is irreducible.

Proof. In general Prime implies irreducible since if not irreducable then $I = J_1 \cap J_2$ then there is $a_i \in J_i$ not in I with $a_1 a_2 \in J_1 \cap J_2$ (since are ideals) hence $a_1 a_2 \in I$.

□