Elyasheev Leibtag
Weizmann institute of science

Tutorial

WEEK 9

1. Grasmanian Manifold

Definition 1.0.1. Let V be a n dimensional vector space, let $Gr_k(V)$ denote the set of all k-dimensional linear subspaces of V.

We will see that $Gr_k(V)$ is a manifold of dimension $k(n-k)$ and we will give it a structure of a projective variety.
Let W be a subspace of dimension k, then we have that $V = W \oplus W^{\perp}$ $(\dim(W^{\perp}) = n-k)$.

Exercise 1.0.1. For any $A\colon W \to W^{\perp}$ linear map we get a k dimensional subspace $\Delta(A) := \{x + A(x) \ : \ x \in W\}$

Notice that $\Delta(A) \cap W^{\perp} = \{0\}$, what about the other direction?

Lemma 1.0.1. Any k dimensional subspace U s.t. $U \cap W^{\perp} = \{0\}$ is of the form $\Delta(A)$ for some linear map $A\colon W \to W^{\perp}$

Proof. Let $u \in U$ then u decomposes into $u_W + u_{W^{\perp}}$, we define for such $u_W \in W$

$$A(u_W) = u_{W^{\perp}}$$

Indeed, define $Pr_1 : U \to W$ be the projection and $Pr_2 : U \to W^{\perp}$ then $A = Pr_2 \circ Pr_1^{-1}$ (draw the picture)
We need to show that if $u_1, u_2 \in Pr_1^{-1}(w)$ then $Pr_2(u_1) = Pr_2(u_2)$.
Indeed we get $u_1 - u_2 \in \ker(Pr_1)$ so $u_1 - u_2 \in W^{\perp} \cap U$ so are the same. □

Notation. Denote $L := Hom(W, W^{\perp})$ and denote by $U_W \subset Gr_k(V)$ the collection of subspaces of dim k who intersect $W^{\perp}$ trivially.

Corollary 1.0.1. $L \cong U_W$

Clearly L is $M_{(n-k)\times k}(\mathbb{C})$ so is a manifold of dimension $(n-k)k$. We will give $Gr_k(V)$ the structure of a manifold by making U_W an open cover of the space.

2. Grasmanian Variety

Let V be a n-dimentional vector space and fix a standard basis of V. Now take $S \in Gr_d(V)$ choose a basis for S and write down the basis elements in rows (w.r.t the standard basis we choose), this gives us a $d \times n$ matrix that represents S, call it $A(S)$.
If we choose a different basis for S we get a different matrix $A'(S)$. What is the relation between $A'(S)$ and $A(S)$?
Answer: Just multiplying by some invertable $d \times d$ matrix. This relation affects all d by d minors by multiplication by the determinant of base change matrix.
We have $\binom{n}{d}$ minors in $A(S)$ and at least one is non zero since rank of $A(S)$ is d.
For later convenience we choose some order on the minors as follows: The left most minor and then for each column $i \leq d$ and $d < j$ we change the i and j column then proceed by some arbitrary order. We get a map $A(S) \to \mathbb{C}^{\binom{n}{d}}$, notice that this map is dependent on the basis of S only upto scalar multiplication so we have a well defined map

$$\psi \colon Gr_d(V) \to \mathbb{P}^{\binom{n}{d}-1}$$

Theorem 2.0.1. The map ψ is injective and has a closed image

Proof. Recall from previous section that $U_S \subset Gr_d(V)$ is the subset of all k dimensional subspaces that do not intersect $S^\perp$ and $U_S \cong Hom(S, S^\perp)$.
Let $e_1, ..., e_n$ denote the standard basis for V and choose $S =< e_1, ..., e_d >$, $S^\perp =< e_{d+1}, .., e_n >$
As usual denote the affine open sets of the projective space by $U_i \subset \mathbb{P}^{\binom{n}{d}-1}$.
Claim:

$$\psi(T) \in U_0 \iff T \in U_S$$

Indeed, assume $T \notin U_S$ thus $e \in T \cap S^\perp$ is non zero vector- complete it to a basis of T - the left most $d \times d$ minor has a row of zeroes! thus $\psi(T) \notin U_0$.
Conversely if $\psi(T) \notin U_0$ then the left most minor is 0 we can change a basis to make ot have a row of zeros and get a basis element of T that is in $S^\perp$.
Using this claim it is enough to show that $\psi|_{U_S} \colon U_S \to U_0$ is injective and with closed image.
Recall that $T \in U_S$ was given by $A \in Hom(S, S^\perp)$ by $T =< \{e_i + A(e_i)\} >=< \{e_i + \sum_{j=d+1}^{n} a_{ij} e_j\} >$ where these a_{ij} completely determine T.
So $\psi(T) = (1 : a_{ij} : ... : a_{dn} : f_1(a_{ij}) : ... : f_m(a_{ij}))$ where the f_i are polynomials in a_{ij} whose coefficients in $\mathbb{Z}$. So we get a one to one correspondence from U_S and these a_{ij}.

$\psi(U_L) \subset U_0 \cong \mathbb{C}^{\binom{n}{d}-1}$ and is closed since is the graph of the morphism

$$(..., a_{ij}, ...) \mapsto (..., f(a_{ij}), ...) \quad : \mathbb{C}^{d(n-d)} \to \mathbb{C}^{\binom{n}{d}-1-d(n-d)}$$

And hence is closed □

3. Moromorphic differentials and Riemann-Hurwitz

Let $\{\psi_\alpha : U_\alpha \to V_\alpha \ : \ \alpha \in A\}$ be a holomorphic atlas on Reimannian surface S.
A moromorphic differential η on S is given by a collection

$$\{\eta_\alpha : V_\alpha \to \mathbb{P}^1 \ : \ \alpha \in A\}$$

such that if $u \in U_\alpha \cap U_\beta$ then $\eta_\alpha(\psi_\alpha u) = \eta_\beta(\psi_\beta u) \cdot (\psi_\beta \circ \psi_\alpha^{-1})' (\psi_\alpha(u))$
We denote $\eta = fdg$ if $n_\alpha = (f \circ \psi_\alpha^{-1}) \cdot (g \circ \psi_\alpha^{-1})'$
We then can integrate

$$\int_\gamma \eta = \int_\gamma fdg = \int_a^b f \circ \gamma(t)(g \circ \gamma)'(t)dt$$

We say that η has a pole/zero at $p \in U_\alpha$ if $\eta_\alpha(\psi_\alpha(p))$ is a pole/zero.

Definition 3.0.1. A divisor D is a formal sum $\sum n_p[p]$

This is an abelian group.

Definition 3.0.2. The degree of a divisor is the sum $\sum n_p$. It is a homomorphism from Div to $\mathbb{Z}$.

If $n_p \geq 0$ for any point then the divisor $D \geq 0$ is non negative. And we write $D \geq D'$ if $D - D'$ is non negative.

Definition 3.0.3. A divisor of a meromorphic function f is denoted (f) and it $\sum n_p[p]$ where p is a pole/zero with muliplicity n_p.
(f) is called a principle divisor.

We have an equivalant class $D \sim D' \iff D - D' = (f)$ (we say D and D' are linearly dependent)

Definition 3.0.4. a divisor of a 1 form is called a canonical divisor. It is cananical since $(\eta) = (f\omega) = (f) + (\omega) \sim (\omega)$

Theorem 3.0.1 ((Riemann-Hurwitz))**.** Let η be a meromorphic differential form then $deg(\eta) = 2g - 2$ (g is the genus of the surface)

Proof. . □

Definition 3.0.5. RRD= all meromorphic functions satisfying $(f) + D \geq 0$
These are all f such that f is holomorphic at all p besides when $n_p > 0$ and has bounded order pf pole, also f has zero of deree atleast $|n_p|$ when $n_p < 0$.

We saw in class:

Corollary 3.0.1. If $D < 0$ then $dim(RRD) = 0$

Proof. If f is meromorphic such that $(f) + D \geq 0$ then $\deg(D) = \deg((f) + D) \geq 0$ $\square$

Corollary 3.0.2. If $D \sim D'$ then $\dim(D) = \dim(D')$

Proof. If g is meromorphic such that $(g) + D = D'$ then $f \mapsto fg$ is a isomorphism of the vector spaces $\square$