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# Tutorial

## WEEK 10

### 1. Open subset which is non affine

First -A union of affine varieties is a variety.

**Lemma 1.0.1.** Let $X := \mathbb{C}^2 \setminus \{0\}$ is an algebraic variety which is not affine.

*Proof.* First observe that $X$ is a variety since $X = Z(x)^c \cup Z(y)^c$. (Denote $D(x) = Z(x)^c = \{(x, y) \ : \ x \neq 0\}$)
A regular function on $X$ is one such that restricted to each affine part is regular. Recall that $\mathcal{O}_{\mathbb{C}^2}(D(x)) = \mathbb{C}[x, y](1/x)$ and $\mathcal{O}_{\mathbb{C}^2}(D(y)) = \mathbb{C}[x, y](1/y)$.
Let $f \in \mathcal{O}_{\mathbb{C}^2}(X)$ then $f|_{D(x)} = \dfrac{p(x, y)}{x^n}$ and $f|_{D(y)} = \dfrac{q(x, y)}{y^m}$ (where we assume $x \nmid p(x, y)$ , $y \nmid q(x, y)$ or is zero polynomial ).
We get that on the open set $U = D(x) \cap D(y) = \{\{(x, y) \ : \ x, y \neq 0\}\}$ $x^n q(x, y) = y^m p(x, y)$, since $U$ is Zariski dense in $\mathbb{C}^2$ we get this identity on the whole plane. Therefore $q(x, y) = p(x, y) = 0$ or $n = m = 0$ , if $p(x, y) = q(x, y) = 0$ then $f = 0$ on all $X$, if not then $f = p(x, y) \in \mathbb{C}[x, y]$. We got that $\mathcal{O}(X) \cong \mathcal{O}(\mathbb{C}^2)$. So assuming $X$ is affine and looking at the inclusion map $i \colon X \to \mathbb{C}^2$ we get that $i^*$ is isomorphism thus $i$ should be as well. □

### 2. Krull dimension

**Definition 2.0.1.** Let $X$ be a variety,

$$\dim(X) = \max_k \{\exists \text{closed irreducable } Z_i \subset X, \ 0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq .... \subsetneq Z_k \subset X\}$$

This is equivalent to the transcendence degree definition, we can show that $\dim(\mathbb{C}^n) = n$ and that the dimension is preserved for finite maps.

**Lemma 2.0.1.** Let $\nu : X \to Y$ be a finite (surjective) map, then $\dim(X) = \dim(Y)$

*Proof.* We showed tat finite maps are closed and since irreducible sets map to irreducible sets we get that $\dim(X) \leq \dim(Y)$ .
On the contrary we prove by induction on $\dim(Y)$, take the chain $0 \subsetneq W_1 \subsetneq W_2 \subsetneq .... \subsetneq W_p \subset Y$, decompose $\nu^{-1}(W_p) = K_1 \cup .... \cup K_n$ so

$W_p = \nu(K_1) \cup ... \cup \nu(K_n)$ by irreducibility of $W_p$ we get that $W_p = \nu(K_{i_p})$ set $Z_p := K_{i_p}$ $\square$

## 3. Princaple Ideal theorem

**Theorem 3.0.1** (Princaple Ideal theorem)**.** Let $X$ be irreducable and $g \in \mathcal{O}(X)$ then each component of $Z(g)$ has dimension $\dim(x) - 1$

This theorem is very useful but we will not prove it in this tutorial. Mainly what we seen in class can be stated

**Lemma 3.0.1.** Let $g = c \prod_i g_i^{n_i}$ where $g_i$ are irreducible polynomials in $\mathbb{C}[x_1, ..., x_n]$, let $X := Z(g)$ then:

1. The irreducible components of $X$ are $Z(g_i)$.
2. $\mathcal{O}(X) = \mathbb{C}[x_1, ..., x_n]/(\prod_i g_i)$.
3. Each component of $X$ has dimension $n - 1$
4. Any closed $Y \subset \mathbb{C}^n$ with the property that any irreducible component is co-dimension 1 is a hyper-surface.

*Proof.*

1. $Z(g) = Z(g_1) \cup Z(g_1) \cup .... \cup Z(g_d)$ so enough to show that $Z(g_i)$ is irreducable and this follows from irreduciblity of the polynomial $g_i$.
   (If $Z(g_i) = X_1 \cup X_2$ choose $f_j|_{X_j} = 0$ get that $f_1 f_2 = 0$ on $X$ thus $g_i \mid f_1 f_2$ by irreduciblity w.l.o.g $g_i \mid f_1$ thus $X_1 = X$)
2. Just need to show $< \prod_i g_i >$ is a radical ideal. First clearly $\prod_i g_i$ belong to the radical $\sqrt{g}$. In the other direction if $f \in \sqrt{< g >}$ then $f^m = gh = c \prod_i g_i^{n_i}$, now assume $f = \prod_j f_j^{k_j}$ irreducible decomposition, then we get
   $$\prod_j f_j^{mk_j} = c \prod_i g_i^{n_i}$$
   By UFD we get that each $g_i$ is some $f_j$ (times constant) so $\prod_i g_i \mid f$ hence $f \in < \prod_i g_i >$ .
3. Shown in class.
4. Let $X$ be irreducible with dimension $n-1$ we show it is a hyper-surface.
   Since $X \neq \mathbb{C}^n$ there is a polynomial $f$ s.t. $X \subset Z(f)$, by (a) assume that $f$ is irreducible, we claim that $X = Z(f)$ this is since $\dim(X) = n-1$ and if $X \subsetneq Z(f)$ then the Krull dimension would be lower.

$\square$

**Corollary 3.0.1.** If $g \in \mathbb{C}[x_1, ..., x_n]$ with $\dim(Z(g)) < n - 1$ then $g$ is constant.

## 4. Dimension of intersection

**Theorem 4.0.1.** Let $X, Y \subset \mathbb{C}^n$ irreducible closed, then **each** component of $X \cap Y$ has dimension $\dim() \geq \dim(X) + \dim(Y) - n$

*Proof.* Observe $X \cap Y \cong \Delta(X \cap Y) = X \times Y \cap \Delta(\mathbb{C}^n)$ thus is given by $n$ equations in $X \times Y$ $\square$

**Corollary 4.0.1.** If $\dim(X) + \dim(Y) > n$ then $X \cap Y \neq \emptyset$

We will show this is also true for irreducible projective variety.

**Definition 4.0.1.** Let $Z \subset \mathbb{P}^n$ define the cone over $Z$ to be $C(Z) \coloneqq \overline{\pi^{-1}(Z)} = \pi^{-1}(Z) \cup \{0\}$

**Lemma 4.0.1.** $\dim(C(Z)) = \dim(Z) + 1$

*Proof.* Define the map $\phi_i \colon Z \cap U_i \times \mathbb{C}^* \to C(Z) \cap \pi^{-1}(U_i)$,

$$((x_0, ...1_i, ..., x_n), \lambda) \mapsto \lambda(x_0, ...1_i, ..., x_n)$$

This is an isomorphism, and since these open sets cover the variety we get that $\dim(Z \times C^*) = \dim(Z) + 1 = \dim(C(Z))$ $\square$