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Tutorial

WEEK 8

1. Rabinowitz trick and ring of functions on basic open set

We saw that there is a one-to-one correspondence between affine varieties and reduced f.g. rings.
Let X be an affine variety and $f \in \mathcal{O}(X) = \mathbb{C}[x_1, ..., x_n]/\mathcal{I}(X)$ we may give $X \setminus \mathcal{Z}(f)$ a structure of an affine variety embedded in $\mathbb{C}^{n+1}$ by noticing that $Y = X \setminus \mathcal{Z}(f) \cong \mathcal{Z}(\{\mathcal{I}(X), yf(x_1, .., x_n) - 1\})$ in this case

$$\mathcal{O}(Y) = \mathcal{O}(X)[1/f]$$

(since it is $\mathbb{C}[x_1, ..., x_n, y]/\sqrt{\mathcal{I}(X), y \cdot f - 1}$ and we claim is equal $\mathcal{O}(X)[1/f]$ just be taking y to $1/f$.).
From here we can try and generalize what is a regular function on a open set

2. Rational functions

Definition 2.0.1. Let X be an affine variety and $U \subset X$ be a open set. A function $f \colon U \to \mathbb{C}$ is called *regular* if there exist a open cover $U = \bigcup_i \mathcal{Z}(f_i)^c$ such that $f|_{\mathcal{Z}(f_i)^c}$ is regular i.e. in $\mathcal{O}(X)[1/f_i]$.
We denote $\mathcal{O}_x(U)$ the ring of regular functions.

Definition 2.0.2. Let X be an irreducible variety A rational function on X is an equivalence class of (U, g) (Open subset of X, a function in $\mathcal{O}_x(U)$) where the equivalence is given by

$$(U_1, g_1) \sim (U_2, g_2) \iff g_1|_{U_1 \cap U_2} = g_2|_{U_1 \cap U_2}$$

Remark 2.0.1. This is a field

Exercise 2.0.1. Let $X = \mathcal{Z}(f(x, y))$ be a irreducable affine algebraic curve, then the field of rational functions $Rat(X)$ is isomorphic to the field given by the equivalence relation on polynomials in two variables

$$\{\frac{p(x, y)}{q(x, y)} \ : \ f \nmid q\}/\sim, \text{where} \quad \frac{p_1}{q_1} \sim \frac{p_2}{q_2} \iff f \mid q_2 p_1 - q_2 p_1$$

Hint:Look at $p/q \mapsto (\mathcal{Z}(q)^c, p/q)$
This is homomorphism of rings, is injective since if $p(U) = 0$ then $p(\overline{U}) = p(X) = 0$ and is onto since $(U, f) \sim (\mathcal{Z}(h)^c, f|_{\mathcal{Z}(h)^c})$

We will denote this field by $\mathbb{C}(X)$.
The field $\mathbb{C}(X)$ can be seen simply as the field of rational functions on X (the equivalence relation means equivalence as function on X). And notice each function is defined at all but maybe finitely many points i.e. *defined on Zariski open sets of the affine curve.*

Remark 2.0.2. It may be that $\frac{p_1}{q_1} \sim \frac{p_2}{q_2}$ and for some $(x_0, y_0) \in X$ $q_1(x_0, y_0) = 0 \neq q_2(x_0, y_0)$ for example on the circle $x^2 + y^2 = 1$ we get that the rational function $\frac{1-y}{x}$ is similar to $\frac{x}{1+y}$.
(If the function has an expression such that is defined at p we say the function is regular at p)

Exercise 2.0.2. The field $\mathbb{C}(X)$ is has transcendence degree 1 over $\mathbb{C}$

Example 2.0.1. If $X = \mathcal{Z}(y)$ then any rational function on X is a rational function in one variable...

3. Rational curves

(Not to be confused with rational functions)

Example 3.0.1. Let X be the curve defined by $y^2 = x^2 + x^3$, notice that the line $y = tx$ intersects X at a single besides out of $(0, 0)$.
Indeed if $y = xt$ then assuming $x^2t^2 = x^2 + x^3$ yields $\cancel{x^2}(t^2 - x - 1) = 0$.
So $x = t^2 - 1$ and $y = t(t^2 - 1)$
The geometric meaning is as follows "t is the slop of the line throw (x, y) and this line intersecs the curve at $(t^2 - 1, t^3 - t)$

Definition 3.0.1. An irreducible algebraic curve X defined by f is called rational if there exist $\phi, \psi \in \mathbb{C}(t)$ rational functions (at least one non constant) such that $f(\phi(t), \psi(t)) = 0$

Remark 3.0.1. Given a rational curve, we will show soon that the correspondence $t \mapsto f(\phi(t), \psi(t))$ is "almost bijective" (misses finite amount of points).

These curves are nice since if for example f, ϕ, ψ are defined over $\mathbb{Q}$ then we can find all rational points (but finitely many) on the curve by just plugging in rational number for t.
Also for integrating $\int f(x, y)dx$ can be substituted by a rational expression

Example 3.0.2. Any conic (deg 2) irreducible curve is rational.
Prove if we have time

Question: When are curves rational?

4. From rational curve to rational functions

Assume that the irreducable curve X is rational with parameterization $(\phi(t), \psi(t))$. Given $\frac{p(x,y)}{q(x,y)} \in \mathbb{C}(X)$ we may map it to the element $\frac{p(\phi(t),\psi(t))}{q(\phi(t),\psi(t))} \in \mathbb{C}(t)$ This is well defined. First of all if $q(\phi(t), \psi(t)) = 0$ we get that $(q, f) \neq 1$ so they have common factor, second of all if $\frac{p_1}{q_1} \sim \frac{p_2}{q_2}$ then

$$\frac{p_1}{q_1}(\phi(t), \psi(t)) = \frac{p_2}{q_2}(\phi(t), \psi(t))$$

Hence we get a field morphism

$$\mathbb{C}(X) \subseteq \mathbb{C}(t)$$

By **Lurth theorem** $\mathbb{C}(X)$ is itself a field of rational functions.
In the other direction (this gives tha "almost bijection in the previous remark), if $\mathbb{C}(X)$ is isomorphic to $\mathbb{C}(t)$ then under the isomorphism the image of x denoted $\phi(t)$ and the image of y denoted by $\psi(t)$ gives a rational structure to the curve.
We get a reformulation to our question.
Question: When is a 2 generated transcendence degree 1 extension of $\mathbb{C}$ isomorphic to the field of rational functions ?

5. Birational

Recall the definition of a bi rational map

Definition 5.0.1. A rational map between irreducible variety X, Y is an equivalence class of (U, f) where U is open in X and f is a morphism from U to Y. (same equivalence as before)

Exercise 5.0.1. Let X, Y be algebraic curves , a rational morphism from X to Y is a map defined by $u_1, u_2 \in \mathbb{C}(X)$ such that $p \mapsto (u_1(p), u_2(p)) \in Y$ whenever u_1, u_2 are defined.

Example 5.0.1. A rational morphism from the line to the rational curve X. (Via Lurth theorem)

Definition 5.0.2. A rational map (W, f) is called bi-rational if for some $U \subset X$ and $V \subset Y$ if (W, f) is equivalant to (U, g) and g is isomorphism of U and V.

Exercise 5.0.2. Between curves, A rational map $\varphi = (u_1, u_2)$ is birational if is has a rational inverse i.e. there is $\psi = (v_1, v_2) : Y \to X$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity when they are defined.

$$\psi \circ \varphi(p) = \psi(u_1(p), u_2(p)) = (v_1(u_1(p), u_2(p)), v_2(u_1(p), u_2(p)))$$

We saw in class that $\mathbb{C}(X) \cong \mathbb{C}(Y) \iff X \text{birational} Y$

Corollary 5.0.1. Rational curves are ones that are birational to the line (Bi-rational is what we called "almost bijective")

6. LOCAL PARAMETER

Assume f defines a curve X that is non singular at $(0,0)$ and $df_x(0,0) = 1 \neq 0$ thus $f = x + \alpha y + ...$
We gather all y elements together and get that

$$f = yg(y) + x = yg(y) + x + xh(x,y) = yg(y) + x(1 + h(x,y))$$

Since $df_x(0,0) = 1 + h(0,0) = 1$ so $h(0,0) = 0$ thus around the open set $U_h = \mathcal{Z}(1-h)^c$, as function on X we get $x = \frac{yg(y)}{1+h(x,y)} = y^k \underbrace{\frac{a(y)}{1+h(x,y)}}_{r}(x,y)$ where $g(y) = y^{k-1}a(y)$
Thus given a non zero rational function on X $u = p/q \in \mathbb{C}(X)$ we may write $u = y^k v$ $(k \in \mathbb{Z})$.
And moreover $v = a/b$ with $a(0,0), b(0,0) \neq 0$.
Since we may write any polynomial $p(x,y) \in \mathbb{C}(X)$ as $p(y^k r(x,y), y) = y^k r(x,y)$ where $r(x,y)$.

7. Extensions

Let $f = a/b$ a polynomial rational function in one variable, so f is defined from $\mathcal{Z}(b)^c$ to $\mathbb{C}$, can we extend this to be defined on all of $\mathbb{C}$??
Answer: NO!
Since we may have points going to infinity.
Lucky for us, this is the only thing that can happen, so we may extend the function from $\mathbb{C}$ to the projective plane! (Yayy)
Now. Assume $U \subset \mathbb{C}$ is open and we have 2 rational function $U \to \mathbb{C}^2$ one for each coordinate $z \mapsto (f_1, f_2) = (\frac{a_1(x)}{b_1(z)}, \frac{a_2(z)}{b_2(z)})$. Can we extent this from $\mathbb{C}$ to all of $\mathbb{C}^2$?
Answer: Of course NOT!
Lucky for us, we may define $t \mapsto [1 : f_1 : f_2] = [b_1 b_2 : a_1 b_2 : a_2 b_1]$ this is well defined and agrees on U
We get the following lemma

Lemma 7.0.1. Any rational function from $\mathbb{P}^1$ to $\mathbb{P}^2$ comes from a regular function.

Proof. We have a rational function, this gives us a regular function $f : U \to \mathbb{P}^2$ we may assume U is open affine with image in $\mathbb{C}^2$ and this extends to the affine piece of U □

Theorem 7.0.1. Let X be a non singular irreducible algebraic curve, then any rational function into $\mathbb{P}^2$ comes from a regular one.

Proof. Let $f : U \to \mathbb{P}^2$ be a regular function, then U is a co-finite set, we will extend for any point not in U w.l.o.g U is affine and $(0,0) \notin U$ and image of U is in $\mathbb{C}^2$.
This regular function is regular on each coordinate and it defines on

each coordinate a rational function from the affine part of the curve so is of the form

$$(x,y) \mapsto [1 : y^{k_1} v(x,y), y^{k_2} v(x,y)] = [y^{k_0} : y^{k_0-k_1} v(x,y), y^{k_0-k_2} v(x,y)]$$

Now having all powers of y non negative, so is well defined map hat extends the rational function. $\square$