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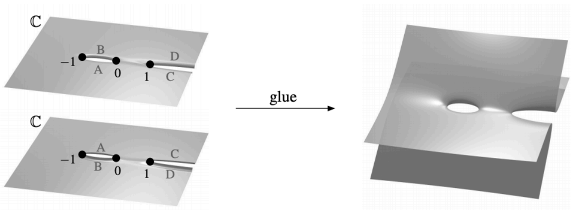
Tutorial

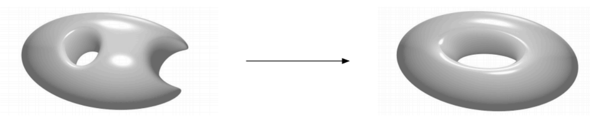
WEEK 4

1. Shape of the curve

Let
$$y^2 = x^3 - x = x(x-1)(x+1)$$
Define an algebraic curve C, what does C look like?
It is very easy to see that plugging in a complex number for x yields two possible values for y, the positive and negative square roots. This is for all x but $-1, 0, 1$.
So how does this curve look like?
Naively it looks like two copy's of the place (one for each root) and over 3 selected points it is smashed to just one point above the (with value 0).
But what happens to the square root as we go around branch point? It moves from one copy to the next, to see this look at a path around the origin $\gamma(t) = re^{it}$ in the "positive sheet" the square root is $\sqrt{r}e^{it/2}$. It gives opposite values in $t = 0, t = 2\pi$, In general as we go around one copy of a branch point we move between the "positive" and the "negetive"" sheet. We may picture it like this:



Adding a point at infinity - which is what we do when we projectify we see this surface become a Torus!



* figures from "Plane Algebraic Curves" by Andreas Gathmann

2. The topology of the projective space

We define a topology on the protective space to be the quotient topology of the map

$$\pi\colon \mathbb{C}^{n+1} \setminus 0 \to P^n, \quad (x_0, ..., x_n) \mapsto [x_0, ..., x_n]$$

i.e. $A \subset P^n$ is open iff $\pi^{-1}(A) \subset \mathbb{C}^n$ is open (In the Housdorff topology) Notice that $\pi(S^{2n+1}) = P^n$ so it is a compact space. And that $U_j = [x_0 : ... : 1 :, ..x_n]$ is homeomorphic to $\mathbb{C}^n$

Lemma 2.0.1. A projective curve $C = \mathcal{Z}(f(x, y, z))$ (f homogeneous) is compact subset of P^2

Proof. We need to show it is close i..e that $\pi^{-1}(C)$ is closed- this is obvious since polynomials are continuous. □

3. points at infinity

We may think of the affine space living in the projective space for example $\mathbb{C}^2 \cong U = \{[x : y : z] \in P^2 \ : \ z \neq 0\}$ where $\psi\colon U \to \mathbb{C}^2$, $[x : y : z] \mapsto (x/z, y/z)$ and the inverse map $(x, y) \mapsto [x : y : 1]$. The complement to U is P^2 is the projective line $P^1 \cong [x : y : 0]$ - we this of this projective line as "the infinity".
Given any polynomail $Q(x, y) = \sum_{i+j\leq d} a_{i,j}x^iy^j \in \mathbb{C}[x, y]$ we want to investigate its points "at infinity", for this we think of $C = \mathcal{Z}(Q)$ as lying inside U as zeros of the homogeneous polynomial $z^dQ(x/z, y/z) = \sum_{i+j\leq d} a_{i,j}x^iy^jz^{d-i-j}$ the projective curve $\tilde{C}$ is such that $\tilde{C} \cap U = C$. The points at infinity of this curve is exactly the points where $z = 0$ i.e the set of points$\{[x : y : 0] \ \ \sum_{i+j=d} a_{i,j}x^iy^j = 0\}$.
A point is this set is a "line" (lemma- homogeneous 2 var= product of lines) these are exactly the asymptotic s of the curve C we started with.

Example 3.0.1. Let $Q(x, y) = x^2+xy+y^2+x^3-y^3 = (x-y+1)(x^2+xy+y^2)$ we get the projectafication $P(x, y, z) = x^2z+xyz+y^2z+x^3-y^3$, its point at infinity are $\{[x : y : 0] \ \ x^3 - y^3 = 0\}$ which are $[1 : 1], [1, e^{\frac{2\pi i}{3}}], [1 : 1, e^{\frac{4\pi i}{3}}]$

4. Intersection of points

Exercise 4.0.1. Any homogeneous polynomials in two variables may be factored into linear components

$$p(x, y) = \prod(a_ix + b_iy)$$

Exercise 4.0.2. Let P and Q be homogeneous polynomials of degree n, m then $Res_{P,Q}(y, z)$ is homogeneous of degree mn

Lemma 4.0.1. Any 2 projective curves in P^2 intersect at least at one point

Proof. Let $C = \mathcal{Z}(P), \quad D = \mathcal{Z}(Q)$ of degrees n, m. Observe that $Res_{Q,P}(y,z)$ is homogeneous in two variables so there is a line $[b : c]$ that vanishes for this polynomial, any way $Res_{Q(x,b,c),P(x,b,c)} = 0$ so there is a point a s.t $P(a,b,c) = Q(a,b,c) = 0$ as needed $\square$

Corollary 4.0.1. Any non-singular projective curve is irreducible.

5. Investigating singular points

Back to the polynomial

$$Q(x,y) = x^2 + xy + y^2 + x^3 - y^3$$

and $C = \mathcal{Z}(Q)$. Notice the curve has singularity at $(0,0)$. We are interested in understanding how the curve looks like near this singularity, for this we look at the sphere of radius ϵ around the point $(0,0)$ and intersect the curve.
Close enough to $(0,0)$ in this sphere, the curve looks like the curve defined by $x^2 + xy + y^2$.
One way to see this is by observing that $x^2 + xy + y^2 + x^3 - y^3 = (x - y - 1)(x^2 + xy + y^2)$. Furthermore we have a change of coordinates $x \mapsto x' = x + \frac{y}{2}, \quad y \mapsto y' = \frac{i\sqrt{3}}{2}y$ that maps $C' = \mathcal{Z}(x^2 + xy + y^2) = \mathcal{Z}(x'^2 - y'^2)$ and around $(0,0)$ we know that curve looks like two cones.

Take home message (That we will make more precise in the next few weeks): The behavior around the singular point is to due with the lowest order terms - these are homogeneous

6. Projective Algebraic sets

Lemma 6.0.1. A closed subset A of $\mathbb{C}^n$ is homothety-invariant ($x \in A \iff \lambda x \in A$) if and only if $\mathcal{I}(A)$ is generated by **finitely** many homogeneous polynomials.

Proof. If $\mathcal{I}(A)$ is generated by **finitely** many homogeneous polynomials $\{f_i\}$ of degree d_i we get that $f_i(x) = 0 \iff \lambda_i^d f(x) = f(\lambda x) = 0$.
On the other hand, assume that A is homothety invariant and look at $f \in \mathcal{I}(A)$ write $f = \sum_{i=1}^{l} f_i$ where all f_i are homogeneous of degree i. We claim that $f_i \in \mathcal{I}(A)$. (Notation: $f_\lambda(x) = f(\lambda x)$)
This follows by induction on l. Base case if $l = 1$ the is is clear since is already homogeneous.
For a general l and λ we get that $\lambda^l \cdot -f_\lambda \in I$ thus

$$\sum_{i=1}^{l} \lambda^l f_i - \sum_{i=1}^{l} \lambda^i f_i = \sum_{i=1}^{l-1} (\lambda^l - \lambda^i) f_i \in I$$

so by induction $(\lambda^l - \lambda^i) f_i \in I$ hence $f_i \in I$.
Finite follows from HBT taking the f_i homogeneous components of a generating set. $\square$

7. Algebraic sets and Coefficient rings

We saw in class that any algebraic set can be paired with a reduces finitely generated algebra.
Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$, we saw in class the two constructions:

(1) Given a morphism $\varphi : X \to Y$ we constructed a morphism $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ by defining $\varphi^*(p)(x) := p(\varphi(x))$.
(2) Given a morphism $\nu : \mathbb{C}[Y] \to \mathbb{C}[X]$ we constructed a morphism $\tilde{\nu} : X \to Y$ by defining $\tilde{\nu}(x) = (\nu(\overline{y_i})(x))_i$.

We left as an exersice in class to show that these constructions are inverse one another i.e. that there is a bijection as sets $Mor(X,Y) \cong Mor(\mathbb{C}[Y], \mathbb{C}[X])$

Exercise 7.0.1. Show that $X \cong Y$ iff $\mathbb{C}[X] \cong \mathbb{C}[Y]$

Proof. This can be cumbersome, try to prove it today and tomarrow we show the nice idea behind the proof.

Claim 1:
Let $Id : X \to X$ be the identity morphism, then $Id^* : \mathbb{C}[X] \to \mathbb{C}[X]$ is the identity morphism.
Claim 2:
Let $Id : \mathbb{C}[X] \to \mathbb{C}[X]$ be the identity morphism, then $\tilde{Id} : X \to X$ is the identity morphism.
Claim 3:
Let X, Y, Z be affine algebraic varieties and $\varphi : X \to Y$, $\psi : Y \to Z$ then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.
Claim 4:
Let X, Y, Z be affine algebraic varieties and $\nu : \mathbb{C}[Z] \to \mathbb{C}[Y]$, $\mu : \mathbb{C}[Y] \to \mathbb{C}[X]$ then $\widetilde{\mu \circ \nu} = \tilde{\nu} \circ \tilde{\mu}$.
Now prove:
If $\varphi : X \to Y$ is isomorphism then $Id_X = \varphi \circ \varphi^{-1}$ hence by claim 1 $Id_{\mathbb{C}[X]} = Id_X^* = (\varphi \circ \varphi^{-1})^*$ which is $(\varphi^{-1})^* \circ \varphi^*$ i.e. $\varphi^* : m\mathbb{C}[Y] \to \mathbb{C}[X]$ is an isomorphsim.
Same for the "tilde"... $\square$

Exercise 7.0.2. Show that $H = \{(x, y) \ : \ xy = 1\} \ncong \mathbb{C}$

Exercise 7.0.3. Show that $P = \{(x, y) \ : \ x - y^2 = 0\} \cong \mathbb{C}$