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Tutorial

WEEK 3

1. Bezout thm

Theorem 1.0.1 (Bezout)**.** If $f, g \in \mathbb{C}[x, y]$ are relatively prime, then

$$|Z(f) \cap Z(g)| \leq (\deg f)(\deg g)$$

What we saw in class is that actually that the projection to the $x-$axis of $Z(f) \cap Z(g)$ has size at most $n \cdot m$.
We fix this by the following exercise:

Exercise 1.0.1. There exist a change of coordinates such that no two solutions lie over the same x point.

2. Curve minus singular points is a holomorphic surface

Lemma 2.0.1. Any complex curve is a surface and actually minus the singular points, it is has a holomorphic atlas.

Proof. Let C by a curve efined by polynomial P ($C = \mathcal{Z}(P)$) and $(a, b) \in \mathbb{C}$ such that $\frac{\partial P}{\partial y}(a, b) \neq 0$
By the imlicit function theorem there is open set $z \in V,\ b \in W$ and hlomorphic function $g \colon V \to W$ with $g(a) = b$ and for $(x, y) \in V \times W$.

$$P(x, y) = 0 \iff g(x) = y$$

We may choose V, W small enough such that $U := C \cap V \times W$ has no singular points.
We get that the projection $(x, y) \mapsto x$ from U is into V and that $(x) \mapsto (x, g(x))$ is a holomorphic inverse.
Same for points with $\frac{\partial P}{\partial x}(a, b) \neq 0$ we get a open set U' in which $(x, y) \mapsto y$ has inverse $(y) \mapsto (h(y), y)$.
If we compute the transition maps we gell get that they may be Id, g, h which are all holomorphic. $\square$

3. Curve next to non singular points

Definition 3.0.1. A point $(a, b) \in \mathbb{C}^2$ is singular point of $f \in \mathbb{C}[x, y]$ if $f(a, b) = \frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$.

Exercise 3.0.1. What are the singular points of the curve $y^2 - x^2 - x^3$. (notice that $(0, -2/3)$ is not on the curve)

We want to get a feel of how curves look at singular points. First we notice :

Lemma 3.0.1. Let p be a singular point of the curve C, the projection map π_x from $C \cap B_\epsilon(p) \to \mathbb{C}$ is a covering map (for small enough ϵ).

Proof. Let $f(x, y) = \sum_{i=0}^{n} a_i(x)y^i$ then for all but finitely many points $|\pi_X^{-1}(x)| = n$, the points where this does not happen are when $a_n(x_0) = 0$ since then by fundamental theorem pf algebra there are at most $n-1$ roots, and at points where $Res(f(x_0, y), \frac{\partial f}{\partial y}(x_0, y)) = 0$ which account for multiple roots.
The point p is a special point as above and thus there is a small enough neighborhood around it that is a cover. $\square$

But how does the curve look near the singularity p? , we look at the curve intersected with the sphere of radios ϵ in $\mathbb{C}^2$.

$$\{(x, y) \in \mathbb{C}^2 \ : \ |x - p_x|^2 + |y - p_y|^2 = \epsilon^2\}$$

For simplicity and w.l.o.g from now on assume that $p = (0, 0)$.

Example 3.0.1. Let $f(x, y) = xy$, we get that

$$\{(x, y) \in \mathbb{C}^2 \ : \ |x|^2 + |y|^2 = \epsilon^2\} = \{(0, \epsilon e^{it})\} \cup \{(\epsilon e^{it}, 0)\}$$

Which is 2 distinct loops is the 3 dimensional sphere

Example 3.0.2. Let $f(x, y) = x^2 - y^3$, we get that

$$\{(x, y) \in \mathbb{C}^2 \ : \ |x|^2+|y|^2 = \epsilon^2\} = \{(\delta^3 e^{3it}, \delta^2 e^{2it}) : \ t \in (0, 2\pi)\} = \bigcup_{k=0}^{2} \{(\delta^3 e^{it}, \delta^2 e^{i\frac{2t}{3} + \frac{i2\pi \cdot k}{3}})\}$$

and δ is the unique positive solution for $\delta^4 + \delta^6 = \epsilon^2$.
This is loop circle parameterized by t.
For example take $\epsilon = 1$ and look at the pre-image of the circle.
Starting from the point $(1, 1)$ and going with t from 0 to 2π we get that the second coordinate travels from one solution to the next. This makes one loop.

Notice that the curve $xy = 0$ is the same (after change of basis $(x - y, x + y)$) as $x^2 = y^2$ which has 2 loops on the sphere.

Exercise 3.0.2. Find a curve with 7 loops on the sphere

Proof. Take $f = x^7 - y^7$, the points on the sphere are

$$\{(x, y) \in C \ : \ |x|^2 + |y|^2 = \epsilon^2\} = \bigcup_{k=0}^{7} \{(\delta e^{it}, \delta e^{it + \frac{i2\pi \cdot k}{7}})\}$$

and δ is positive solution for $2\delta^7 = (\epsilon^2 - \delta^2)^{\frac{7}{2}}$.
The loop starting at $(1, 1)$ goes back to $(1, 1)$ after 2π and does not hit another element in the pre-image of 1. $\square$

Lemma 3.0.2. Let $n \leq m$ the amount of loops in the sphere around $(0,0)$ at the curve $x^n - y^m = 0$ is $gcd(m,n)$.

Recall that this intersection is in a 3 dimensional sphere.
We may do a stereo-graphic projection to $\mathbb{R}^3 + \infty$ and try to imagine the curve.
FACT: The stereo-graphic projection from the ϵ sphere to $\mathbb{R}^3$ can be given by

$$(Re(x), Im(x), Re(y), Im(y)) \mapsto \begin{cases} \frac{\epsilon}{\epsilon - Re(y)}(Re(x), Im(x), Im(y)) & Re(y) \neq \epsilon \\ \infty & Re(y) = \epsilon \end{cases}$$

Lets project some curves and see what we get!

Example 3.0.3. Let $f = xy$ (same example as before) we had that $C \cap S = \{(\cos(t), \sin(t), 0, 0) \,:\, t \in [2\pi]\} \cup \{(0, 0, \cos(s), \sin(s)) \; s \in [2\pi]\}$ under the projection we get the union

$$\{(\cos(t), \sin(t), 0)\} \cup \{\frac{\epsilon}{\epsilon - \sin(s)}(0, 0, \cos(s))\}$$

Which is a circle and a line going throw the circle! we get that the two circles we got above are linked in the sphere

Example 3.0.4. Let $f(x,y) = y^2 - x^3$, (same as before switch x,y) we got that

$$C \cap S = \{(\delta^2 e^{2it}, \delta^3 e^{3it}) \,:\, t \in (0, 2\pi)\} = \{(\delta^2 \cos(2t), \delta^2 \sin(2t), \delta^3 \cos(3t), \delta^3 \sin(3t),)\}$$

and δ is the unique positive solution for $\delta^4 + \delta^6 = \epsilon^2$.
This is a subset of $\{(x,y) \,:\, |x| = \delta^2, |y| = \delta^3\}$ and this maps under the projection to a points $(a,b,c) \in \mathbb{R}^3$ satisfying

$$2\epsilon^2\sqrt{a^2+b^2} = \delta^2(a^2+b^2+c^2+\epsilon^2)$$

which is

$$(\sqrt{a^2+b^2} - \epsilon^2\delta^{-2})^2 + c^2 = \epsilon^2\delta^2$$

which is a surface of revolution of a circle around the c axis, i.e. a torus!
Thus $C \cap S$ maps to a knot on the surface of the torus!

4. Constructable sets

The notion of constructability is purely topological.

Definition 4.0.1. Let X, τ be a topological space, a set $Z \subset X$ in called locally-closed if $Z = U \cap B$ where U is open and B is closed.

Definition 4.0.2. Let X, τ be a topological space, a set $C \subset X$ in called construcable if $C = \bigcup_{i=0}^n Z_i$ where Z_i is constructable.

Notice that this is equivalent to the definition that was given in class when we look at the Zariski-topology of the space.

Exercise 4.0.1. Let $M_{n,m}(\mathbb{C})$ denote the vector space of $n \times m$ matrices. Show that the matrices of rank r is a constructable set.
Hint: Look at condition on the $r+1$ an r minor of the matrix.