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Tutorial

WEEK 13

We will answer some questions from the problem set

1. Question 1

1.1. assumptions that should be made in this question. First we assume that D is an effective divisor that is "base-point free" i.e: For any $p \in C$ $L(D-[P]) \subsetneq L(D)$.
Notation: Given a principal divisor (f) (for $f \in Rat(C)$) we may represent (f) as the difference of 2 effective divisors $(f)_0 - (f)_\infty$ (poles minus zeros).
Secondly, given a hyperplane $H = \mathcal{Z}(\sum_{i=0}^n \lambda_i X_i) \subset \mathbb{P}^n$ we define the divisor $f^{-1}(H)$ to be the effective divisor defined by $(\sum_{i=0}^n \lambda_i f_i)_0$. (Try playing around with examples and see why this makes scene).
Solutions

(1) If f would be contained in hyperplane then this would define a linear dependence of the bases f_i.
(2) First we remark that for any two hyperplanes H, H' defined by linear forms S, S' (respectively) the divisor $f^{-1}(H)$ is equivalent so the divisor $f^{-1}(H')$ since $f^{-1}(H) - f^{-1}(H') = (S/S')$.
Now by the assumption that D is base-point free we know that $\cup_i L(D-[p_i]) \subsetneq L(D)$ (union over points in D). hence there exist $g \in L(D) \setminus cup_i L(D-[p_i])$ therefore we get the condition (*)
$$(g)_\infty \nleq (D-[p_i]).$$
Since $(g)_0$ and $(g)_\infty$ have disjoint support we obtain that $(g) + D = (g)_0 - (g)_\infty \geq 0 \Rightarrow D \geq (g)_\infty$ but together with (*) we obtain that $(g)_\infty = D$ thus $(g)_0 \sim (g)_\infty \sim D$.
Now since all pullback of hyperplane sections are equivalent what we do now is find the "correct" hyperplane. Since $g \in L(D)$ $g = \sum_{i=0}^n \lambda_i f_i$ so we just take the hyperplane to be $H = \mathcal{Z}(\sum_{i=0}^n \lambda_i X_i)$ and we get by definition that $f^{-1}(H) = (g)_0$.
(3) The bijection: Given a divisor D as above we define $f\colon C \to \mathbb{P}(L(D))$ as defined in the question.
We show this map is well defined : Let $\{f_i\}_{i=0}^n$, be a basis for $L(D)$, assume $D \sim D'$ then $D - D' = (h)$, define the map $\hat{h} : L(D) \to L(D')$ $\hat{h} : g \mapsto hg$ is a linear bijection since

$(g) + D \geq 0 \iff (g) + (h) + D' = (gh) + D' \geq 0$. (we get that $L(D) \cong L(D')$)
So regarding $L(D)$ with basis $\{f_i\}_{i=0}^n$ and $L(D')$ with basis $\{hf_i\}_{i=0}^n$ as $n+1$ dimensional vector spaces we get a matrix $A \in GL(n+1)$ (that satisfies $hf_i = \sum_j A_{i,j} f_j$ as functions on C) let $\bar{A} \in PGL$ be its projective class. We obtain the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\Phi_D"] \arrow[dr, "\Phi_{D'}"'] & \mathbb{P}^n \arrow[d, "\bar{A}"] \\ & \mathbb{P}^n \end{tikzcd}$$

So the map is well defined.
Injective:
Assume by contradiction that $D \nsim D'$ but there exist $\bar{A} \in PGL(n+1)$ such that $\bar{A} \begin{bmatrix} f_0(p) \\ \vdots \\ f_n(p) \end{bmatrix} = \begin{bmatrix} f_0'(p) \\ \vdots \\ f_n(p) \end{bmatrix}$ Then we obtain that $D' \sim f'^{-1}(X_0 = 0) = (f_0')_0 = (\sum_j A_{i,j} f_j)_0 = f^{-1}(\sum_j A_{i,j} X_j = 0) \sim D$ in contradiction. **surjective** Let $f : C \to \mathbb{P}^n$ be a map that does not intersect any hyperplane. $f(p) = [f_0(p) : f_1(p) : ... : f_n(p)]$ we show that $f_i \in L(D)$ for the divisor $(f_0 + f_1 + ... + f_n)_\infty$, this is just since $(f_i) + D = (f_i)_0 + (f_0 + .. + f_{i-1} + f_{i+1} + ... + f_n)_\infty \geq 0$. The fact that f_i are a basis comes from the assumption that the map does not intersect any hyperplane.