Algebraic Geometry
Tutorial notes

1. The fiber dimension theorem

In this part of the tutorial we will reprove the following theorem.

THEOREM 1.1 (Chevalley). *Let X, Y be varieties over $\mathbb{C}$, with X irreducible and $\varphi : X \to Y$ a morphism.*

(1) *If φ is surjective, then, for any $y \in Y$,*

$$\dim(\varphi^{-1}(y)) \geq \dim Y - \dim X.$$

(2) *If φ is dominant, then there exists an open subset $U \subseteq Y$ such that*

$$\dim(\varphi^{-1}(y)) = \dim Y - \dim X,$$

for all $y \in Y$.

LEMMA 1.2. *Let $X \subseteq \mathbb{C}^n$ be irreducible affine variety and $\pi : \mathbb{C}^n \to \mathbb{C}^{n-1}$ the projection onto the last $n-1$ coordinates. Assume $\pi \mid_X$ is finite. Then, for any $f \in \mathbb{C}[x_1, \ldots, x_n]$, if $f \mid_X$ is non-constant then there exists a polynomial $\varphi(t_1, \ldots, t_{n-1})$ such that $\pi(X \cap Z(f)) = \pi(X) \cap Z(\varphi)$.*

Let us consider a concrete example.

EXAMPLE 1.3. Consider the variety $X = V(x^2 - y^3) \subseteq \mathbb{C}^2$. The algebra of functions for this variety is $\mathbb{C}[x, y]/(x^2 - y^3)$, which is generated over $\mathbb{C}[y]$ by $1, x$, and hence the projection map $\mathbb{C}[x, y] \to \mathbb{C}[y]$ is finite. The function $x \in \mathbb{C}[x]$, when restricted to X, satisfies the equation

$$x^2 + a_0(y) \cdot 1 = 0, \quad \text{with } a_0(y) = -y^3.$$

Given $t \in \mathbb{C}$, we defined

$$V_t = \mathbb{C}[x]/(x^2 - t^3)$$

which are all 2-dimensional vector spaces, with bases $\{1, x\}$, and equal the algebra of functions on $\pi \mid_X{}^{-1}(t) \subseteq X$. Let $f(x, y) = x^2 - 1$ be a non-constant polynomial on X. Note:

$$X \cap Z(f) = \left\{(1,1), (1,\zeta), (1,\zeta^2), (-1,1), (-1,\zeta), (-1,\zeta^2)\right\},$$

where $\zeta^3 = 1$ and $\zeta \neq 1$, and

$$\pi(X \cap Z(f))) = \left\{1, \zeta, \zeta^2\right\}.$$

Now, the polynomial f defines a map $T_t^f : V_t \to V_t$ given by

$$g(x) + (x^2 - t^3) \overset{T_t^f}{\mapsto} f(x,t)g(x) + (x^2 - t^3).$$

Note: if $t \in \pi(X \cap Z(f))$ then there exists $s \in \mathbb{C}$ such that $(s, t) \in X$ (i.e. $s^2 - t^3 = 0$) and $f(s, t) = 0$. In particular, the polynomial $f(x, t)$ is divisible by a prime factor of

$s^2 - t^3$ and hence there exists $g(x)$ such that $g(x)f(x,t) \in (x^2 - t^3)$, and therefore T_t^f is not-invertible. Conversely, if $t \in \pi(X) \setminus \pi(X \cap Z(f))$, then $f(x,t)$ is non-vanishing on X and the map T_t^f is invertible. We deduce:

$$\pi(X \cap Z(f)) = \pi(X) \cap \left\{ t \in \mathbb{C} : \det(T_t^f) = 0 \right\}.$$

Finally, we notice that the matrix representing T_t^f in the basis $\{1, x\}$ varies *polynomially* with t. We can compute:

$$[T_t^f]_{\{1,x\}} = \begin{pmatrix} [f(x,t)] & [xf(x,t)] \end{pmatrix} = \begin{pmatrix} -1+t^3 & 0 \\ 0 & t^3-1 \end{pmatrix}$$

since $f(x,t) = x^2 - 1 = t^3 - 1$ in V_t. Indeed, $\det(T_t^f) = (t^3-1)^2$, and its vanishing set is $\{1, \zeta, \zeta^2\}$.

As a corollary of Lemma 1.2, we deduced that $\dim(X \cap Z(f)) = \dim(X) - 1$, whenever $f \in \mathbb{C}[\mathbf{x}]$ is non-constant on X and X is irreducible. In general, if X is not assumed irreducible, we get $\dim(X \cap Z(f)) \geq \dim X - 1$.

Proof of Theorem 1.1. (1) To begin with, it's harmless to assume Y is affine, since given $y \in Y$ we can find $y \in U \subseteq Y$ affine and open, and we can restrict φ to $\varphi^{-1}(U) \subseteq X$. This restriction makes no difference in terms of dimension, i.e. $\dim X = \dim \varphi^{-1}(U)$, $\dim Y = \dim U$. Write $d = \dim Y$. Then we can choose a finite map $\psi : Y \to \mathbb{C}^d$ with $\psi^* : \mathbb{C}[t_1, \dots, t_d] \to \mathcal{O}_Y(Y)$.

Claim. There exist $g_1, \dots, g_d \in \mathcal{O}_Y(Y)$ such that $Z(g_1, \dots, g_d)$ is finite and contains y.

Specifically, we may take $g_i = \psi^*(t_i - y_i')$, where $(y) = (y_1', \dots, y_d')$. Indeed, $y \in Z(g_1, \dots, g_d)$, by definition of ψ^*, and is finite, since $\mathcal{O}_Y(Y)/(g_1, \dots, g_d)$ is finite dimensional over $\mathbb{C} = \mathbb{C}[t_1, \dots, t_d]/(t_1, \dots, t_d)$.

Now, recall that φ denotes our map $X \to Y$, and consider the functions $\varphi^* g_1, \dots, \varphi^* g_d$ (here $\varphi^* g_i = g_i \circ \varphi$). Put $Z = Z(\varphi^* g_1, \dots, \varphi^* g_d)$. By definition of φ^*, we have that $x \in Z$ if and only if $g_i(\varphi(x)) = 0$ for all $i = 1, \dots, d$, and, in particular, that Z is the union of fibers of points in $Z(g_1, \dots, g_d)$ under φ. In particular, $\varphi^{-1}(y)$ is a union of some of the irreducible component of Z and $\dim \varphi^{-1}(y) = \dim Z$. On the other hand, we know that φ^* is an injective algebra homomorphism, since φ is surjective (dominant would be enough for this), and none of the g_i's was constant, so all $\varphi^* g_i$ are also non-constant. By the remark above, since Z is not necessarily irreducible, we have that

$$\dim \varphi^{-1}(y) = \dim Z \geq \dim Z(\varphi^* g_1, \dots, \varphi^* g_{d-1}) - 1 \geq \ldots \geq \dim X - d,$$

as wanted.

(2) Now we assume φ is dominant. We may restrict both the domain and range to be irreducible and affine. Then φ^* induces an injective map of fields $\mathrm{Rat}(Y) \to \mathrm{Rat}(X)$, which we think of as an inclusion. Since dimension is given by the transcendence degree of the field of rational functions, writing $c = \dim X - \dim Y$, by Nöther normalization, there exist $t_1, \dots, t_c \in \mathrm{Rat}(X)$ such that $\mathrm{Rat}(X)$ is algebraic over $\mathrm{Rat}(Y)[t_1, \dots, t_c]$. Moreover, by multiplying by a common denominator, we may assume $t_1, \dots, t_c \in \mathcal{O}_X(X)$.

Pick generators $f_1, \dots, f_N \in \mathcal{O}_X(X)$. Since the f_i's are algebraic over $\mathrm{Rat}(Y)[t_1, \dots, t_c]$, there exist $a_{i,j}(t_1, \dots, t_c)$ $(i = 1, \dots, N,\ j = 1, \dots, d_i)$ such that

$$a_{i,0}(\mathbf{t}) + a_{i,1}(\mathbf{t}) f_i + \cdots + a_{i,d_i}(\mathbf{t}) f_i^{d_i} = 0, \tag{1.1}$$

and such that the $a_{i,j}$'s are elements of $\mathcal{O}_Y(Y)$. Put $U = \{y \in Y : a_{i,d_i}(y) \neq 0\}$ and let $y \in U$.

Since the ring of functions on $\varphi^{-1}(y)$ is a quotient of $\mathbb{C}[X]$, we have that the $f_i \mid_{\varphi}^{-1} (y)$'s generate $\mathcal{O}_{f^{-1}(y)}(f^{-1}(y))$. Furthermore, (1.1) implies that the $f_i \mid_{\varphi^{-1}(y)}$'s satisfy polynomial equations whose coefficients are polynomials in $t_i \mid_{\varphi^{-1}(y)}$ over $\mathbb{C}$. In particular, we have that the fraction field $\mathrm{Rat}(\varphi^{-1}(y))$ is algebraic over $\mathbb{C}[t_1 \mid_{\varphi^{-1}(y)}, \ldots, t_c \mid_{\varphi^{-1}(y)}]$, and hence of transcendence degree $\leq c$. That is $\dim \varphi^{-1}(y) \leq \dim X - \dim Y$. The equality follows from the previous item.

$\square$

2. Examples

EXAMPLE 2.1. Consider $\varphi = ((x, y) \mapsto (x, xy)) : \mathbb{C}^2 \to \mathbb{C}^2$. Then, for any $(s, t) \in \mathbb{C}^2$ we have

$$\varphi^{-1}(s,t) = \begin{cases} \{(s, t/s)\} & \text{if } s \neq 0 \\ \{0\} \times \mathbb{C} & \text{if } (s,t) = (0,0) \\ \varnothing & \text{otherwise.} \end{cases}$$

In particular, on the open set $U = \mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$ the map φ has 0-dimensional fibres and $\varphi^{-1}(0, 0)$ is one dimensional.

EXAMPLE 2.2. What is the dimension of the variety of nilpotent 2×2 matrices over $\mathbb{C}$?

PROOF. Let $\mathcal{N}$ denote the variety of nilpotent 2×2 matrices. Note that, as a matrix $x \in \mathrm{M}_2(\mathbb{C})$ is nilpotent if and only if $x^2 = 0$, we have that $\mathcal{N}$ is the vanishing set of four polynomials in the coefficients of x, and hence indeed a closed subset of $\mathrm{M}_2(\mathbb{C})$.

Recall that a matrix $x \in \mathrm{M}_2(\mathbb{C})$ is nilpotent if and only if its characteristic polynomial is t^2. In the case of 2×2 matrices, the characteristic polynomial is given by $c_x(t) = t^2 - \mathrm{Tr}(x)t + \det(x)$, and its coefficients are given by polynomials in the entries of x. In particular, setting $X = \mathrm{M}_2(\mathbb{C})$ and $Y = \mathbb{A}^2_{\mathbb{C}}$, we have a morphism $\psi(x) = (\mathrm{Tr}(x), \det(x))$, which is surjective (Exercise, e.g. using companion matrices). The set of nilpotent matrices is precisely the fiber over the point $(0, 0)$, and has dimension $\geq \dim(X) - \dim(Y) = 4 - 2 = 2$.

To show that this is precisely the dimension, we can use the fiber dimension formula again. Define a map $\mathcal{N} \to \mathbb{C}$ by mapping $\left(\begin{smallmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{smallmatrix}\right) \mapsto t_{2,1}$. This is just a coordinate projection, so it is a morphism, and is surjective (consider strictly lower-triangular matrices). What is the fiber over 0? It is precisely the set of nilpotent upper triangular matrices. But such a matrix is necessarily of the form $\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$ (compute the characteristic polynomial to verify this). Hence, both the image and the fiber over 0 are one-dimensional, and

$$\dim \mathcal{N} \leq \dim \mathbb{C} + \dim(\text{fiber over } 0) = 2.$$

$\square$

REMARK 2.3. As noted in class, we can also prove the lower bound $\dim \mathcal{N} \geq 2$ by considering the following chain of closed irreducible subvarieties:

$$\{0\} \subseteq \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\} \subseteq \mathcal{N}.$$

EXAMPLE 2.4. Let C be a smooth curve in $\mathbb{P}^{5780}$. Then C embeds in $\mathbb{P}^3$.

Proof. Of course, 5780 is a joke. We will show that given a smooth curve in $\mathbb{P}^N$ and $N > 3$, we can find $\xi \in \mathbb{P}^N$ such that the projection $\mathbb{P}^N \to \mathbb{P}^{N-1}$ in the direction of ξ restricts to an isomorphism of C onto its image. This is equivalent to the condition that any line through ξ meets C in at most one point, and is not tangential to C (Proving this is beyond the scope of this tutorial, we saw some motivating examples).

Recall the definition of a projection in the direction of $\xi \in \mathbb{P}^n$: any such ξ corresponds to an affine line in $\mathbb{A}^{n+1}$ and is defined by the vanishing of n linear equations $L_0, \dots, L_{n-1}$. Then we can define

$$\pi_\xi([a_0 : a_1 : \dots : a_n]) = [L_0(\mathbf{a}) : \dots : L_{n-1}(\mathbf{a})].$$

For example, if $\xi = [1 : 0 : \dots : 0]$, then the L_i's are just the projections onto the last $n-1$ coordinates and π is the projection on these coordinates. Consider the set

$$X = \left\{(p, q, \xi) \in C \times C \times \mathbb{P}^N : p, q, \xi \text{ are collinear or } p = q \text{ and } \xi \text{ is tangential to } C \text{ at } p\right\}$$

. Note that $X \subseteq \mathbb{P}^N$ is closed. To prove this, it is enough to show that $(\mathbb{P}^N \setminus X) \cap U_i$ is open for any affine chart $\mathbb{C}^N \simeq U_i \subseteq \mathbb{P}^N$. This holds since 3 points $x, y, z \in \mathbb{C}^N$ are not-collinear if and only if the set $\{x - z, y - z\}$ is linearly independent, which holds if and only if the $N \times 2$ with columns $x - z$ and $y - z$ is of full rank. The last condition is equivalent to the non-vanishing of all 2×2 of the matrix, which defines an open subset.

Let us show that X has dimension at most 3, and therefore the projection $(p, q, \xi) \mapsto \xi : X \to \mathbb{P}^N$ is not surjective. Indeed, we can write $X = U_1 \cup U_2$, with

$$U_1 = \{(p, q, \xi) : p, q, \xi \text{ are collinear}\} \quad \text{and } U_2 = \{(p, p, \xi) : \xi \in T_p(C)\}.$$

Then the projection onto the first two coordinates $U_1 \to C \times C$ is a dominant map whose fibers are one-dimensional, hence $\dim U_1 \leq \dim C \times C + \dim\ \text{fibers} = 2 + 1 = 3$. Similarly, by the assumption that C is smooth, all tangent spaces are one-dimensional and hence the projection map $(p, p, \xi) \mapsto p : U_2 \to C$ has one-dimensional image and one-dimensional fibers. Thus, $\dim U_2 \leq 2$. In particular, $\dim X \leq \max\{\dim U_1, \dim U_2\} = 3$.

Pick $\xi \in \mathbb{P}^N$ which is not in the image of the map $(p, q, \xi) \mapsto \xi : X \to \mathbb{P}^N$. Then the projection in the direction of ξ restricts to and isomorphism of C onto its image. $\square$