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Adaptive Bathtub Hazard Rate Curve Modelling via Transformed Radial Basis Functions

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***Abstract*— The bathtub curve is essential in interpreting the hazard rate or failure rate function for the engineering reliability analysis. An adaptive bathtub hazard function (ABF) or failure rate function curve modelling is proposed by using the transformed radial basis functions, which includes a type of symmetric ABF (SABF) and a type of asymmetric ABF (AABF). The ABF provides a parametric approach to represent the failure behaviours of various engineering applications and can also be easily utilised as the design objectives for the further accelerated life prediction, parameters determination and reliability based optimisation studies.**

ACRONYM

ABF	the adaptive bathtub failure rate
AABF	the asymmetric ABF
BFR	the bathtub-shaped failure rate
FRF	the failure rate function
RBF	the radial basis function
SABF	the symmetric ABF

NOTATION

μ_i	the radial basis function i
$b(x)$	the segment function for RBF
$f(t)$	the probability density
α	the support factor
β	the boundary factor
γ	the gain factor
ζ	the shape factor

I. Introduction

The bathtub hazard rate curve is a well known concept to represent failure behaviour of various engineering products. In many reliability engineering applications, it is a common situation the failure rate function (FRF), or the hazard rate function has a bathtub shape. For a traditional bathtub-shaped failure rate (BFR), its curve can be divided into three portions as given in Figure 1: 'Infant Mortality', 'Steady-state' and 'Wear-out Failures', which denote three phases that a newly manufactured product passes through during its life cycle[1], [2], [3], [4], [5], [6], [7], [8].

The FRF is essential for reliability analysis of the life cycle modelling of a complex system. A lot of researches dealing with the models of the BFR are reported, Hjorth [9] reported

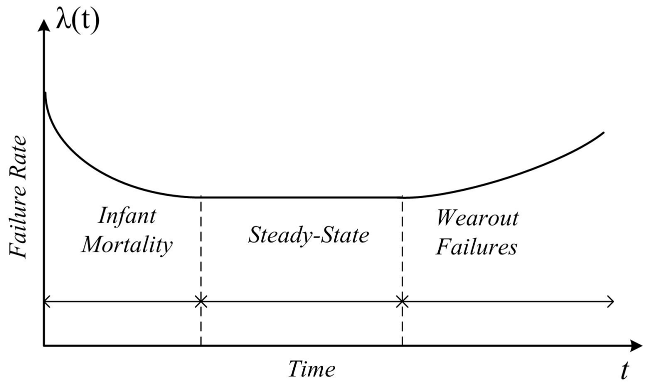

Fig. 1
THE CLASSICAL BATHTUB CURVE[4]

a three parameter distribution with increasing, decreasing, constant or bathtub-shaped failure rate function. Griffith [10] examined the relationship between the exponential distribution and the distributions have monotonic or bathtub shaped failure rates, and he also proposed a method of representing random variables with monotonic or bathtub shapes to transform the exponential and the uniform distributions. Aarset [11] proposed a method for BFR identification based on the total time on test. Wang [12] discussed a model based on adding two Burr XII distributions is presented for modelling mechanical and electronic components using the graphical estimation on probability paper. Jiang and Murthy[13][14] presented a graphical representation for a mixture of two Weibull distributions. Xie et. al [15][16] studied a series of modified Weibull models and their BFR functions. Bebbington, Lai and Zitikis [17] suggested a mathematical definitions for the 'useful period' of lifetime distributions with bathtub shaped hazard rate functions. Barriga, Louzada-Neto and Cancho[18] proposed a lifetime distribution model for the bathtub-shaped, unimodal, increasing and decreasing hazard rate functions.

As the BFR function is widely used, we hence consider to generalise the BFR by the radial basis function (RBF) which provides the shape, position and gain factors with feasibility for engineering applications and further optimisations. This paper is organised as follows, Section I introduces the background of the BFR. Section II gives the parametrisation method via RBF. Section III proposes the adaptive BFR by the parametrisation method in section II. Section IV discusses the reliability modelling based on the adaptive

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BFR. Section V studies the ABF with different parameters and the simulation results are discussed. In section VI, the feasibility of the ABF curve modelling is concluded.

II. Parametrisation Method via Radial Basis Functions

As illustrated in Fig. 2 without loss of generality, the RBF can be defined as μ_1 by equation (1), in which $b(t)$ is a segment function as given by equation (2). Stated in equation (2), the parameters of $b(t)$ are defined as: α is the support factor, β is the boundary factor, γ is the gain factor and ζ is the shape factor [19], [20]. Referring to μ_1 definition above, μ_2 is mirrored as the reflection of μ_1 over the x-axis by equation (3).

Fig.3 demonstrates three of the further transformations of μ_2, in which μ_3 is translated by μ_2 + 1, μ_4 is $\mu_3 \times \eta$ and μ_5 is μ_4 + δ, as expressed by equations (4), (5) and (6). That is, the original RBF function μ_1 is translated by reflection, shifting upward one unit, scaling and shifting upward δ units as stated in the functions μ_2 to μ_5 respectively.

$$\mu_1\left(x\right) = e^{b(x)} \tag{1}$$

$$b(x) = \begin{cases} -\dfrac{(|x-\alpha|-\beta)^{\zeta}}{\gamma} & \text{if } |x-\alpha| \geq \beta \\ 0 & \text{if } |x-\alpha| < \beta \end{cases} \tag{2}$$

$$\mu_2\left(x\right) = -e^{b(x)} \tag{3}$$

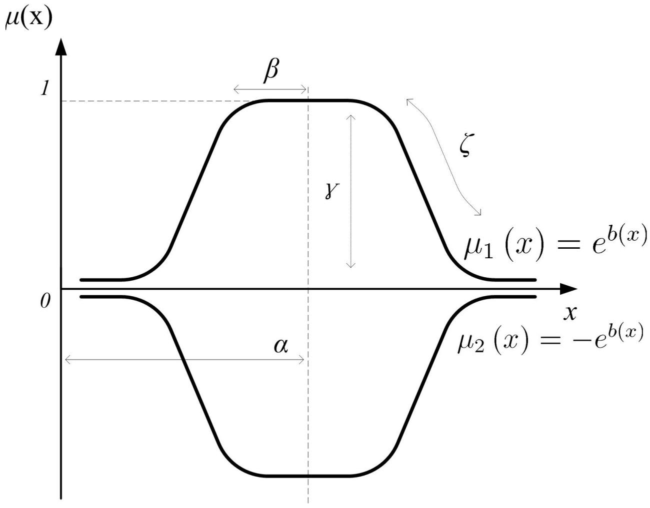

Fig. 2

Radial basis function μ_1 and its reflection μ_2 over the x-axis

$$\mu_3\left(x\right) = 1 - e^{b(x)} \tag{4}$$

$$\mu_4\left(x\right) = \eta\left(1 - e^{b(x)}\right) \tag{5}$$

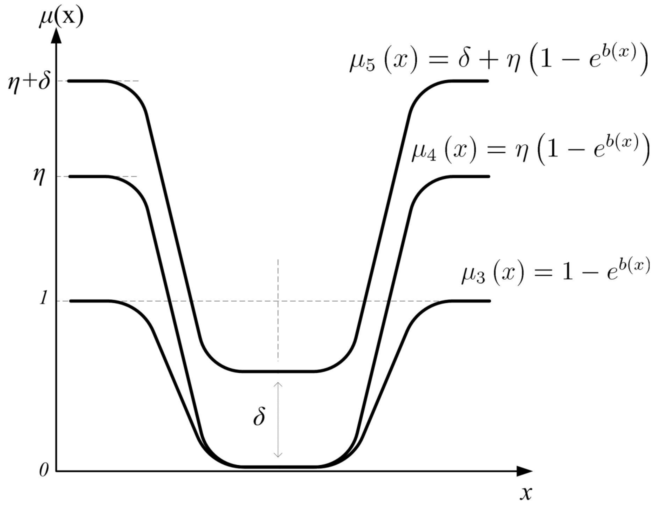

Fig. 3

Transformations of μ_3,μ_4 and μ_5

$$\mu_5\left(x\right) = \delta + \eta\left(1 - e^{b(x)}\right) \tag{6}$$

III. Adaptive Bathtub-shaped Failure Rate Function

By borrowing the form of the transformed RBF μ_5, this paper attempts to propose an adaptive model concerned with the behaviour of the failure rate in the full range of bathtub curve, which is named 'adaptive bathtub-shaped failure rate function' (ABF). Basically, there are two types of the ABF functions, namely, the symmetric ABF (SABF) in Fig. 4 and the asymmetric ABF (AABF) in Fig. 5, as defined by equations (7) and (8) respectively.

$$\lambda_1\left(t\right) = \delta + \eta\left(1 - e^{b(t)}\right), t \geq 0 \tag{7}$$

$$\lambda_2\left(t\right) = \begin{cases} \delta_1 + \eta_1\left(1 - e^{b_1(t)}\right) & \text{if } t \geq \alpha \\ \delta_2 + \eta_2\left(1 - e^{b_2(t)}\right) & \text{if } 0 \leq t < \alpha \end{cases} \tag{8}$$

IV. Adaptive Reliability Function Modelling

The hazard rate function $\lambda\left(t\right)$ is interpreted as a ratio of the probability density $f(t)$ function to the reliability function $R(t)$, as expressed by equation (9), and then by equation (10) as a re-statement[4], [21].

$$\lambda\left(t\right) = \frac{f(t)}{R(t)} = -\frac{1}{R(t)}\frac{dR(t)}{dt} \tag{9}$$

$$-\lambda\left(t\right)dt = \frac{1}{R\left(t\right)}dR(t) \tag{10}$$

Integrating both sides of equation (10) over the time interval [0, t], we can get equation (11).

$$\int_0^t -\lambda\left(t\right)dt = \int_1^{R(t)} \frac{1}{R\left(t\right)}dR(t) \tag{11}$$

Since at $t = 0$, $R(t) = 1$, we are able to evaluate the right hand side of equation (11) and re-arrange the result as equation (12).

$$\ln R\left(t\right) = -\int_0^t \lambda\left(t\right)dt \tag{12}$$

Accordingly, we can get equation (13) from equation (12), which is the general expression of the reliability function of ABF.

Equations (14) and (15) are the reliability functions of SABF and AABF, which can be utilised to obtain the adaptive reliability functions for the engineering applications and optimisations.

$$R\left(t\right) = e^{-\int_0^t \lambda(t)dt} \tag{13}$$

$$R_1\left(t\right) = e^{-\int_0^t \left(\delta+\eta\left(1-e^{b(t)}\right)\right)dt}, t \geq 0 \tag{14}$$

$$R_2\left(t\right) = \begin{cases} e^{-\int_0^t \left(\delta_1+\eta_1\left(1-e^{b_1(t)}\right)\right)dt} & \text{if } t \geq \alpha \\ e^{-\int_0^t \left(\delta_2+\eta_2\left(1-e^{b_2(t)}\right)\right)dt} & \text{if } 0 \leq t < \alpha \end{cases} \tag{15}$$

Specifically, given the conditions $t \geq 0, \frac{(|t-\alpha|-\beta)^\zeta}{\gamma} > 0, \zeta \geq 0$, equation (14) can be re-written as equation (16), and equation (15) can be written as similar form by the given engineering requirements.

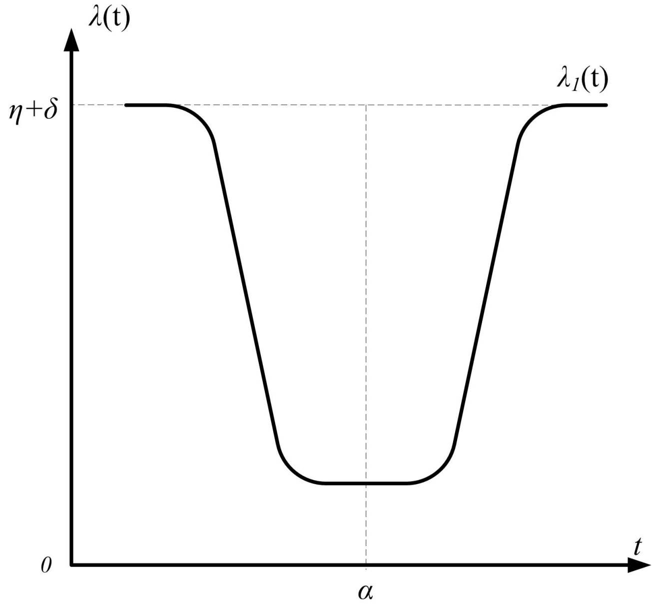

Fig. 4
THE SYMMETRIC ADAPTIVE BATHTUB-SHAPED FAILURE RATE FUNCTION, SABF

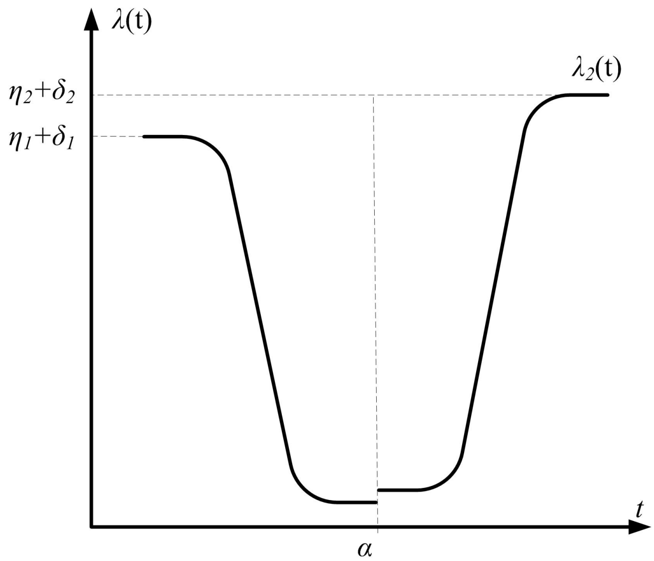

Fig. 5
THE ASYMMETRIC ADAPTIVE BATHTUB-SHAPED FAILURE RATE FUNCTION, AABF

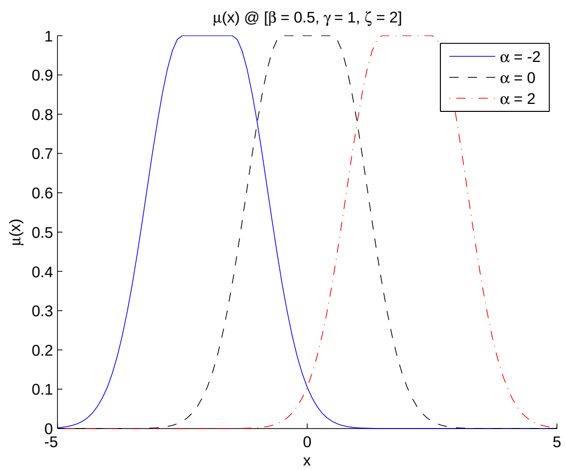

Fig. 6
THE RADIAL BASIS FUNCTION μ_1 WITH α = -2, 0 AND 2

$$R_1(t) = \eta\left(|t-\alpha|-\beta\right)\left(\frac{\left(\Gamma\left(\frac{1}{\zeta},\frac{(|t-\alpha|-\beta)^{\zeta}}{\gamma}\right)-\Gamma\left(\frac{1}{\zeta}\right)\right)\left(\frac{(|t-\alpha|-\beta)^{\zeta}}{\gamma}\right)^{-\frac{1}{\zeta}}}{\zeta}+1\right), t \geq 0 \tag{16}$$

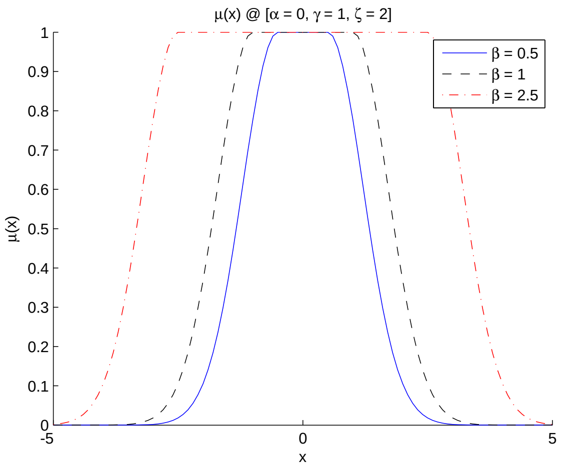

Fig. 7

THE RADIAL BASIS FUNCTION μ_1 WITH β = 0.5, 1 AND 2.5

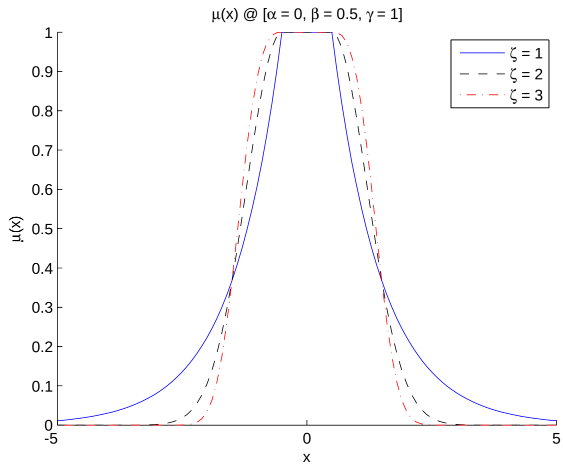

Fig. 9

THE RADIAL BASIS FUNCTION μ_1 WITH ζ = 1, 2 AND 3

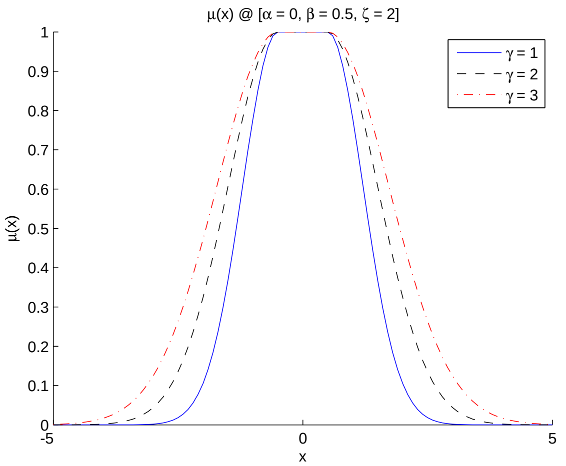

Fig. 8

THE RADIAL BASIS FUNCTION μ_1 WITH γ = 1, 2 AND 3

V. SIMULATIONS

According to the RBF function by equation (1), the behaviours of the RBF functions are defined by the parameters α, β, γ and ζ. To demonstrate the properties of the RBF function, Figs. 6, 7, 8 and 9 describe the RBF behaviours affected by each parameter α, β, γ and ζ respectively.

Given β = 0.5, γ = 1 and ζ = 2, Fig. 6 demonstrates the three RBF curves positioned by α = -2, 0 and 2, which indicate that α locates the position of the RBF curve.

Given α = 0, γ = 1 and ζ = 2, Fig. 7 shows that β set the boundaries (widths) of the RBF curves with different values 0.5, 1 and 2.5.

Fig. 8 expresses γ's gain effectiveness on the RBF curves, by setting γ = 1, 2 and 3 with α = 0, β = 0.5 and ζ = 2.

Fig. 9 demonstrates that ζ can shape the RBF curves with specific values, such as ζ = 1, 2 and 3 with α = 0, β = 0.5 and γ = 1.

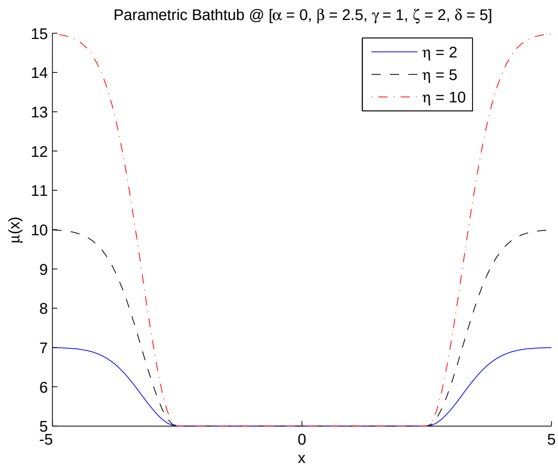

Fig. 10

A CASE OF SABF,α = 0, β = 2.5, γ = 1, ζ = 2, δ = 5, η = [2, 5, 10]

The SABF and AABF are demonstrated in Figs. 10 and 11 with the given parameters, which indicate that the ABF can be parameterised properly via adjusted parameter settings.

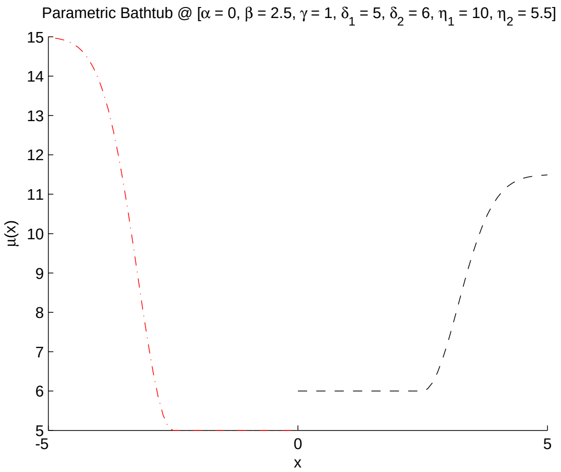

Fig. 11

A CASE OF AABF, $\alpha = 0$, $\beta = 2.5$, $\gamma = 1$, $\zeta = 2$, $\delta_1 = 5$, $\delta_2 = 6$, $\eta_1 = 10$, $\eta_2 = 5.5$

VI. CONCLUSIONS

The adaptive bathtub hazard function modelling is proposed and discussed in this paper, the cases for the two types of ABFs are devised, which demonstrated this proposed parametric approach is able to represent the failure behaviours of bathtub curves according to specific engineering requirements.

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