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Article *in* Structural and Multidisciplinary Optimization · March 2013

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RESEARCH PAPER

An adaptive dimension decomposition and reselection method for reliability analysis

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Received: 11 January 2012 / Revised: 3 July 2012 / Accepted: 23 July 2012

Abstract Recently, the research community in reliability analysis has seen a strong surge of interest in the dimension decomposition approach, which typically decomposes a multi-dimensional system response into a finite set of low-order component functions for more efficient reliability analysis. However, commonly used dimension decomposition methods suffer from two limitations. Firstly, it is often difficult or impractical to predetermine the decomposition level to achieve sufficient accuracy. Secondly, without an adaptive decomposition scheme, these methods may unnecessarily assign sample points to unimportant component functions. This paper presents an adaptive dimension decomposition and reselection (ADDR) method to resolve the difficulties of existing dimension decomposition methods for reliability analysis. The proposed method consists of three major components: (i) an adaptive dimension decomposition and reselection scheme to automatically detect the potentially important component functions and adaptively reselect the truly important ones, (ii) a test error indicator to quantify the importance of potentially important component functions for dimension reselection, and (iii) an integration of the newly developed asymmetric dimension-adaptive tensor-product (ADATP) method into the adaptive scheme to approximate the reselected component functions. The merits of the proposed method for reliability analysis are three-fold: (a) automatically detecting and adaptively representing important component functions, (b) greatly alleviating the curse of dimensionality, and (c) no need of response sensitivities. Several mathematical and engineering high-dimensional problems are used to demonstrate the effectiveness of the ADDR method.

Keywords Adaptive dimension decomposition · Dimension reselection · Variate interaction · Reliability analysis

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Nomenclature

$\mathbf{e}_k$	kth unit vector
$\mathbf{e}_k^{+/-}$	kth directional unit vector
$\mathbf{i}$	multi-index
g	performance function
l	interpolation level
M	number of collocation points
N	number of input random variables
$\mathbf{x}$	vector of input random variables
ADATP	asymmetric dimension-adaptive tensor-product
BDR	bivariate dimension reduction
DI	directional index
DR	dimension reduction
DSG	directional sparse grid
FORM	first order reliability method
MCS	Monte Carlo simulation
MPP	most probable point
PCE	polynomial chaos expansion
PDF	probability density function
SORM	second order reliability method
UDR	univariate dimension reduction

Published online: 04 September 2012

1 Introduction

In the past few decades, tremendous research efforts have been devoted to reliability analysis of an engineered system during the design and development process. The reliability analysis problem is concerned with determining the probability that the performance (e.g., fatigue, corrosion, fracture) of an engineered system meets its marginal value while taking into account various uncertainty sources (e.g., material properties, loads, geometries). In order to formulate reliability analysis in a mathematical framework, random variables are often used to model uncertainty sources in engineered systems. Reliability analysis can then be formulated as a multi-dimensional integration of the probability density function $f(\mathbf{x})$ over a safety region

$$R = \int_{\Omega^S} f(\mathbf{x})\, d\mathbf{x} \tag{1}$$

where R denotes the reliability; $\mathbf{x} = \left(x^1, x^2, \ldots, x^N\right)^{\mathrm{T}}$ denotes an N-dimensional random vector that models uncertainty sources such as material properties, loads, geometric tolerances; $f(\mathbf{x})$ denotes the joint probability density function (PDF) of the vector of random variables; the safety domain Ω^{S} is defined as $\Omega^{\mathrm{S}} = \{\mathbf{x}: g(\mathbf{x}) < 0\}$; $g(\mathbf{x})$ is a system performance (or response) function.

In practice, however, it is extremely difficult to perform the multi-dimensional numerical integration when the number of random variables is relatively large. The search for feasible computational procedures to estimate the reliability has resulted in a variety of computational statistical analysis methods such as the first- or second-order reliability method (FORM/SORM) (Hasofer and Lind 1974; Breitung 1984; Tvedt 1984), direct or smart Monte Carlo simulation (MCS) (Rubinstein 1981; Fu and Moses 1988; Au and Beck 1999; Hurtado 2007; Naess et al. 2009), stochastic spectral method (Ghanem and Spanos 1991; Wiener 1938; Xiu and Karniadakis 2002; Paffrath and Wever 2007; Wan and Karniadakis 2006; Foo et al. 2008; Foo and Karniadakis 2010), stochastic collocation method (Smolyak 1963; Gerstner and Griebel 1998; Barthelmann et al. 2000; Griebel 1998; Xiu and Hesthaven 2005; Xiu 2007; Nobile et al. 2008; Grestner and Griebel 2003; Klimke 2006; Ganapathysubramanian and Zabaras 2007; Ma and Zabaras 2009; Eldred and Burkardt 2009; Eldred et al. 2008; Xiong et al. 2010; Hu and Youn 2011), and dimension reduction (DR) method (Rabitz et al. 1999; Rabitz and Alis 1999; Alis and Rabitz 2001; Li et al. 2001a, b; Sobol 2003; Rahman and Xu 2004; Xu and Rahman 2004; Youn and Wang 2008; Youn et al. 2008).

Among many reliability analysis methods, the first- or second-order reliability method (FORM (Hasofer and Lind 1974) or SORM (Breitung 1984; Tvedt 1984)) is most commonly used. The FORM/SORM uses the first- or second-order Taylor expansion to approximate a limit-state function at the most probable point (MPP) where the limit-state function separates failure and safety regions of a system response. Some major challenges of the FORM/SORM include: (i) a high computational cost to build the probability density function (PDF) of the response and (ii) sensitivity requirement for reliability analysis.

The direct or smart MCS provides an alternative way for multi-dimensional integration (Rubinstein 1981; Fu and Moses 1988; Au and Beck 1999; Hurtado 2007; Naess et al. 2009). Although the direct MCS (Rubinstein 1981) produces accurate results for reliability analysis and allows for relative ease in the implementation, it demands a prohibitively large number of simulation runs. Thus, it is often used for the purpose of a benchmarking in reliability analysis. To alleviate the computational burden of the direct MCS, researchers have developed various smart MCS methods, such as the (adaptive) importance sampling methods (Fu and Moses 1988; Au and Beck 1999; Hurtado 2007) and the enhanced MCS method with an optimized extrapolation (Naess et al. 2009). Despite the improved efficiency than the direct MCS, these methods are still computationally expensive.

The stochastic spectral method (Ghanem and Spanos 1991) is an emerging technique for reliability analysis of complex engineering problems. This method uses a number of response samples and generates a stochastic response surface approximation with multi-dimensional polynomials over a random space. Therefore, this method can be broadly classified as the stochastic response surface method. Once the explicit response surface is constructed, MCS is often used for reliability analysis due to its convenience. The most popular stochastic spectral method is the polynomial chaos expansion (PCE) method. The original Hermite polynomial chaos basis was proposed by Wiener (1938) for modeling stochastic responses with Gaussian input random variables. Xiu and Karniadakis (2002) extended the method under the Askey polynomial scheme to non-Gaussian random variables (e.g., Gamma, Uniform, and Beta). For the estimation of a small failure probability, shifted and windowed Hermite polynomial chaos were proposed to enhance the accuracy of a response surface in the failure region (Paffrath and Wever 2007). Although the PCE method is considered to be accurate, the primary drawback of the method is the curse of dimensionality, which substantially increases the computational cost as the number of random variables increases. To alleviate the difficulty, many adaptive algorithms (Wan and Karniadakis 2006; Foo et al. 2008; Foo and Karniadakis 2010) were recently developed. Although these adaptive algorithms alleviate the curse of dimensionality to some degree, more research efforts are still needed to fully resolve this difficulty.

The stochastic collocation (SC) method is another stochastic response surface technique that approximates a multi-dimensional random function using function values given at a set of collocation points. A comparison between the SC and PCE methods for uncertainty quantification (UQ) was discussed in Eldred and Burkardt (2009), where the SC method was reported to consistently outperform the PCE method. In the SC method, the great improvement in reducing the curse of dimensionality in numerical integration was accomplished by Smolyak (1963), who introduced the concept of the sparse grid. Since then, the sparse grid has been applied to high dimensional integration (Gerstner and Griebel 1998) and interpolation (Barthelmann et al. 2000), UQ in reliability analysis (Eldred and Burkardt 2009; Xiong et al. 2010) and design (Eldred et al. 2008), and PDEs with deterministic inputs (Griebel 1998) and random inputs (Xiu and Hesthaven 2005; Xiu 2007; Nobile et al. 2008). Compared to a full grid, the sparse grid achieves the same accuracy level for integration and interpolation but with a much smaller number of collocation points. Recently, the so called dimension-adaptive tensor-product (DATP) quadrature method introduced the concept of the generalized sparse grid and considered the dimensional importance indicated by an error estimator to adaptively refine the collocation points for efficient multi-dimensional integration (Grestner and Griebel 2003). Klimke (2006) further developed this work for hierarchical interpolation by using either piecewise multi-linear basis functions or Lagrangian polynomials. In this method, all the dimensions in the random space are not considered as of equal importance and the adaptive sampling scheme automatically detects the highly nonlinear dimensions and adaptively refines the collocation points in those dimensions. The DATP method showed a promising application in stochastic natural convector problems (Ganapathysubramanian and Zabaras 2007). The further improvement of the adaptive capability of the DATP method was achieved in the most recent works, where the authors in Ma and Zabaras (2009) developed an adaptive sparse grid collocation method to resolve local discontinuity with a successful application to natural convector problems and the authors in Hu and Youn (2011) developed an asymmetric dimension-adaptive tensor-product (ADATP) method to detect both dimensional and directional importance with a promising application to reliability analysis. Although these methods are equipped with a strong adaptive capability and greatly alleviate the curse of dimensionality, they still cannot achieve a satisfactory error decay rate for high-dimensional problems with rapidly diminishing importance of component functions and/or equal importance in each dimension (Ma and Zabaras 2009). In particular, it was reported in Ma and Zabaras (2009) that, in cases where the importance of each dimension weighs equally, the adaptive sparse grid collocation method could not provide good accuracy and satisfactory error decay rate even for stochastic problems with moderate dimensionality ($N = 25$).

These difficulties in dealing with high-dimensional problems have drawn tremendous research efforts devoted to the development of the so-called dimension reduction (DR) method (Rahman and Xu 2004; Xu and Rahman 2004, 2005; Youn and Wang 2008; Youn et al. 2008). This method approximates a multi-dimensional response function as a hierarchical superposition of component functions with increasing numbers of random variables ranging from 0 (i.e., the component function being a constant) to the total number of random variables (i.e., the component function being a multi-dimensional function).. Specialized versions of this method include the univariate dimension reduction (UDR) method that simplifies one multi-dimensional response function to multiple one-dimensional component functions (Rahman and Xu 2004; Youn et al. 2008) and the bivariate dimension reduction (BDR) method that simplifies one multi-dimensional response function to multiple one- and two-dimensional integrations (Xu and Rahman 2004, 2005). The eigenvector dimension reduction (EDR) method (Youn et al. 2008) improves numerical efficiency and stability of the UDR method with the ideas of eigenvector samples and stepwise moving least squares method. To further predict the reliability or PDF of the response, the DR method employs a PDF generation technique in the case of integration (Xu and Rahman 2004) and the direct MCS in the case of interpolation (Xu and Rahman 2005). Out of the engineering design society, the DR method is widely known as the high-dimensional model representation (HDMR) method that was originally developed for efficient multivariate model representation in chemical system modeling (Rabitz et al. 1999; Rabitz and Alis 1999; Alis and Rabitz 2001; Li et al. 2001a, b). If high-order variate interactions in a system response are negligibly weak (i.e., high-order interactions of random variables have negligible effect on the system response), the HDMR method enables an efficient yet accurate formulation of this response function with the low-order component functions—usually second-order or bivariate being sufficient. In fact, the responses of most practical engineered systems are significantly affected by only low-order interactions of the random input variables. Depending on the way to determine the component functions, the HDMR method can be categorized into two types: ANOVA-HDMR and Cut-HDMR (Rabitz et al. 1999). ANOVA-HDMR follows exactly the way of the analysis of variance (ANOVA) and is useful for measuring the contributions of the variance of each component function to the output variance (Sobol 2003). However, multi-dimensional integrations involved in ANOVA-HDMR make this expansion computationally rather unattractive. On the other hand, Cut-HDMR expansion exactly represents

the response function in the hyperplane that passes through a reference point in the input random space. This expansion does not require multi-dimensional integrations and is computationally much more efficient than ANOVA-HDMR. We note, in fact, that the DR method is essentially Cut-HDMR designated for the purpose of reliability analysis. It is also noted that reliability analysis methods with the name HDMR employ the moving least squares interpolation to approximate component functions (Chowdhury et al. 2009; Rao and Chowdhury 2009).

Most recently, there has been an increase of interest in fusing the stochastic collocation method and Cut-HDMR (Foo and Karniadakis 2010; Griebel and Holtz 2010; Ma and Zabaras 2010) with an aim to thoroughly resolve the curse of dimensionality. This wave of interest originated from Foo and Karniadakis (2010) where the authors first use the Cut-HDMR to decompose a high-dimensional function to a set of low-dimensional functions and then employ the multi-element probabilistic collocation method to integrate the resulting low-dimensional functions. Later, the authors in Griebel and Holtz (2010) developed the first adaptive version of Cut-HDMR, namely the dimension-wise integration method, coupled with sparse grid methods as numerical solvers for the computation of high-dimensional integrals and applied their developments to high-dimensional integration problems in finance. Following the work in Griebel and Holtz (2010), the authors in Ma and Zabaras (2010) developed an adaptive HDMR method which adaptively decomposes the multi-dimensional stochastic response into important low-order component functions and approximate each component function with the adaptive sparse grid collocation method. A new error indicator is defined as the integral of the multiplication of the hierarchical surplus and the basis interpolation function. This new definition incorporates the probabilistic characteristics of the random input variables and measures the lack of accuracy in the integration rather than the interpolation. After constructing the first-order (univariate) component functions and assigning an importance weight to each component function, the method proceeds by identifying potentially important higher-order component functions based on the weights of lower-order component functions and constructing each component function (Ma and Zabaras 2010).

Compared to the direct use of a sparse grid collocation method, the dimension-wise integration method (Griebel and Holtz 2010) and the adaptive HDMR method (Ma and Zabaras 2010) achieves a higher convergence rate by regulating the integration (for the former) and interpolation process (for the latter) from low-order component functions to high-order ones and constructing only potentially important component functions. In the adaptive component function integration or interpolation, these methods selects the component functions whose lower-order sub-functions have been detected as important component functions. However, in many engineering problems, even though a set of lower-order component functions or dimensions are important, their higher-order combinatory component functions may be unimportant. In such cases, the use of only lower-order component functions already achieves sufficiently high accuracy it is undesirable and wasteful to consider these potentially important but actually unimportant component functions. For this very reason, we present an adaptive dimension decomposition and reselection (ADDR) method for reliability analysis. The proposed method consists of three major components: (i) an adaptive dimension decomposition and reselection scheme to automatically detect the potentially important component functions and adaptively reselect the truly important ones, (ii) a component error indicator to quantify the importance of potentially important component functions for dimension reselection, and (iii) an integration of the newly developed asymmetric dimension-adaptive tensor-product (ADATP) method into the adaptive scheme to build the resulting component functions. To guide the adaptive sampling process toward accurate uncertainty quantification rather than accurate interpolation, we employed the definition of the error indicator in Ma and Zabaras (2010) and extends the use of this definition to non-uniform random inputs and higher-order basis interpolation functions other than the piecewise linear spline.

This paper is organized as follows. Section 2 gives a brief review on the stochastic collocation methods, including the classical tensor-product grid and the adaptive tensor-product grid or the ADATP method. Section 3 presents the adaptive dimension decomposition and reselection method. The effectiveness of the proposed method is demonstrated using several case studies in Section 4. Section 5 concludes this paper.

2 Review of stochastic collocation methods

This section briefly reviews the stochastic collocation methods using the classical tensor-product grid and the adaptive tensor-product grid or the ADATP method. We will closely follow the description in Hu and Youn (2011) where more detailed information can be found.

In what follows, we will model the N-dimensional real random variables $\mathbf{x} = \left(x^1, x^2, \ldots, x^N\right)^{\mathrm{T}}$ in a complete probability space $(\Omega, A, \mathcal{P})$, where Ω is a sample space, $\mathcal{A}$ is a σ-algebra on Ω, and $\mathcal{P}$ is a probability measure function $\mathcal{P} : \mathcal{A} \rightarrow [0, 1]$. Then the probability density function (PDF) of the random variable x^i defines a probability mapping $f_i\left(x^i\right) : \Pi_i \rightarrow \mathbb{R}^+$, where the support Π_i is a

one-dimensional random space of x^i. Under the assumption of statistical independence, the probabilistic characteristics of the random variables $\mathbf{x}$ can then be completely defined by the joint PDF $f(\mathbf{x}) = f_1(x^1) \cdot f_2(x^2) \cdot \cdots \cdot f_N(x^N)$ with the support $\Pi = \Pi_1 \cdot \Pi_2 \cdot \cdots \cdot \Pi_N$. If the assumption of statistical independence does not hold, that is, the random variables such as fatigue material properties (fatigue ductility coefficient and exponent) are statistically dependent, a copula (Roser 1999; Noh et al. 2008) can be employed to select an appropriate dependence structure and formulate a joint CDF of the random variables based on available input data, which then allows the use of the Rosenblatt transformation (Rosenblatt 1952) to transform the dependent random variables into independent standard normal random variables. Since the construction of an interpolation in the stochastic collocation method often requires a specially bounded support $\Gamma = [0, 1]^N$ of the random variables $\mathbf{x}$, we first truncate any unbounded one-dimensional random space Π_i (e.g. in the case of a Gaussian random variable) to a bounded one $\Gamma_i^* = [c_i, d_i]$ that achieves a nearly full coverage of Π_i and then map any truncated one-dimensional support $[c_i, d_i]$ to $[0, 1]$, resulting in a bounded hypercube $\Gamma = [0, 1]^N$. Let $g(\mathbf{x})$ denote a smooth, measurable performance function on (Ω, A), which can be treated as a one-to-one mapping between the transformed N-dimensional random space and one-dimensional space $g : [0, 1]^N \rightarrow \mathrm{R}$. In general, the performance function $g(\mathbf{x})$ cannot be analytically obtained, and the function evaluation of g for a given input $\mathbf{x}$ requires an expensive computer simulation. Therefore, it is important to employ a numerical method for reliability analysis that is capable of producing accurate probabilistic characteristics of $g(\mathbf{x})$ with an acceptably small number of function evaluations.

2.1 Classical stochastic collocation: tensor-product grid

The stochastic collocation method basically approximates the performance function g using N-dimensional interpolating functions with performance function values at a finite number of collocation points $\mathbf{\Theta} = \{\mathbf{x}_j \,|\, \mathbf{x}_j \in \Gamma,\ j = 1, \ldots, M_T\}$. Suppose that we can obtain the performance function value $g(\mathbf{x}_j)$ at each collocation point $\mathbf{x}_j$. We then aim at building an interpolation or surrogate model of the original performance function g by using the linear combinations of these function values $g(\mathbf{x}_j)$. We begin by constructing the interpolation with the tensor-product grid, or the tensor-product of one-dimensional interpolation formulas.

In the one-dimensional case ($N = 1$), we can construct the following one-dimensional interpolation

$$U^i(g) = \sum_{j=1}^{m_i} a_j^i \cdot g\left(x_j^i\right) \tag{2}$$

with a set of support nodes

$$X^i = \left\{ x_j^i \,\middle|\, x_j^i \in [0, 1],\ j = 1, 2, \ldots, m_i \right\} \tag{3}$$

where $i \in \mathbb{N}$ is the interpolation level that determines the number of support nodes and thus the interpolation resolution along this dimension, $a_j^i \in C([0, 1])$ the jth interpolation nodal basis functions, x_j^i the jth support nodes and m_i the number of support nodes in the interpolation level i. By following the descriptions in Xiu (2007), Klimke (2006) and Ganapathysubramanian and Zabaras (2007), we use the superscript i to denote the interpolation level during the development of stochastic collocation methods. Two widely used nodal basis functions are piecewise multi-linear basis functions and Lagrange polynomials (Klimke 2006).

Applying a sequence of formulas in (2) on the original performance function g in a nested form for all N dimensions (Hu and Youn 2011), we can easily derive the tensor-product of multiple one-dimensional interpolation formulas as the following multi-dimensional interpolation formula

$$\left(U^{i_1} \otimes \cdots \otimes U^{i_N}\right)(g) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(a_{j_1}^{i_1} \otimes \cdots \otimes a_{j_N}^{i_N}\right) \cdot g\left(x_{j_1}^{i_1}, \ldots, x_{j_N}^{i_N}\right) \tag{4}$$

where the superscript $i_k, k = 1, ..., N$, denotes the interpolation level along the kth dimension, U^{i_k} are the interpolation functions with the interpolation level i_k along the kth dimension and the subscript $j_k, k = 1, ..., N$, denotes the index of a given support node in the kth dimension. The number of function evaluations required by the tensor-product formula reads

$$M_T = m_1 \cdot m_2 \cdots m_N \tag{5}$$

which becomes intolerably large for high-dimensional problems. The search for more efficient sampling schemes than the tensor-product grid has resulted in an adaptive tensor-product method of which the fundamentals will be briefly introduced in the subsequent section.

2.2 Asymmetric dimension-adaptive tensor-product method

With an aim to incorporate an adaptive feature into the tensor-product grid, the concept of the directional sparse grid (DSG) was introduced which facilitated the development of an adaptive tensor-product grid, namely the asymmetric dimension-adaptive tensor-product (ADATP) method (Hu and Youn 2011). This section briefly reviews

the ADATP method to prepare for our further development in the subsequent section.

2.2.1 Directional sparse grid (DSG)

For the construction of the directional sparse grid (DSG), a conventional index i in the case of the univariate interpolation is decomposed into positive and negative directional index (DI) sets as (Hu and Youn 2011)

$$\mathbf{I}^{\mathrm{D}} = \left\{i^{+},\, i^{-}\right\} \tag{6}$$

where the positive DI i^{+} corresponds to the DSG which belong to the index i and whose values are greater than the value (0.5) of the center grid point, and the negative DI i^{-} corresponds to the DSG which belong to the index $i$and whose values are smaller than 0.5. For the multivariate case ($N > 1$), a tensor-product formula of DI sets for a multi-index $\mathbf{i}$ can be obtained as

$$\mathbf{I}^{\mathrm{D}} = \mathbf{I}_{1}^{\mathrm{D}} \times \cdots \times \mathbf{I}_{N}^{\mathrm{D}} \tag{7}$$

where, $\mathbf{I}_{k}^{D} = \left\{i_{k}^{+}, i_{k}^{-}\right\}$, $1 \le k \le N$. Here, the forward neighborhood of a multi-dimensional DI $\mathbf{i}^{\mathrm{d}} \in \mathbf{I}^{\mathrm{D}}$ is defined as the N indices $\left\{\mathbf{i}^{\mathrm{d}} + \mathbf{e}_{k}^{+/-}\right\}$, $1 \le k \le N$, and the sign of kth directional unit vector $\mathbf{e}_{k}^{+/-}$ is the same with that of the kth index element $i_{\mathrm{k}}^{\mathrm{d}}$ $\left(i_{\mathrm{k}}^{+} \text{ or } i_{\mathrm{k}}^{-}\right)$ of $\mathbf{i}^{\mathrm{d}}$. It is noted that the DI divides the conventional index space into the four quadrants. This division allows for an adaptive refinement of the collocation points in these quadrants.

2.2.2 Hierarchical interpolation scheme using multivariate hierarchical basis functions

For the adaptive interpolation, the hierarchical interpolation scheme provides a more convenient way for error estimation than the nodal interpolation scheme (Klimke 2006; Ma and Zabaras 2009). If we apply a Smolyak sparse grid formula (Smolyak 1963), we can obtain a multivariate hierarchical interpolation formula, expressed as (Klimke 2006)

$$\begin{aligned} &A_{q,N}(g) \\ &\quad = A_{q-1,N}(g) + \Delta A_{q,N}(g) \\ &\quad = A_{q-1,N}(g) + \sum_{|\mathbf{i}|=q} \sum_{\mathbf{j}} \underbrace{\left(a_{j_1}^{i_1} \otimes \cdots \otimes a_{j_N}^{i_N}\right)}_{\mathbf{a}_{\mathbf{j}}^{\mathbf{i}}} \\ &\qquad \cdot \underbrace{\left(g\left(x_{j_1}^{i_1}, \cdots, x_{j_N}^{i_N}\right) - A_{q-1,N}(g)\left(x_{j_1}^{i_1}, \cdots, x_{j_N}^{i_N}\right)\right)}_{\mathbf{w}_{\mathbf{j}}^{\mathbf{i}}} \end{aligned} \tag{8}$$

Here, $\mathbf{i} = (i_1, \ldots, i_N)$ is the multi-index denoting the interpolation levels along all N dimensions; $\mathbf{j} = (j_1, \ldots, j_N)$ is the multi-index denoting the indices of support nodes in all N dimensions; $\mathbf{w}_{\mathbf{j}}^{\mathbf{i}}$ is defined as the vector of hierarchical surpluses, which indicates the interpolation error of a previous interpolation at the node $\mathbf{x}_{\mathbf{j}}^{\mathbf{i}}$ of the current interpolation level $\mathbf{i}$. Basically, Based on the Smolyak algorithm, (8) builds the multi-dimensional hierarchical interpolation by considering one-dimensional functions of interpolation levels $i_1,\cdots,i_N$ under the constraint that the sum of these interpolation levels lies within the range $[q - N + 1, q]$. Figure 1 illustrates the comparison between the hierarchical and nodal interpolations with the piecewise linear spline and Clenshaw–Curtis grid (Klimke 2006). It can be observed that, compared to the nodal interpolation, its hierarchical counterpart builds an interpolation based on the interpolation at a previous level and the interpolation at the current level in a recursive level-by-level form. Furthermore, the hierarchical surplus w_j^i in the hierarchical interpolation, which indicates the interpolation error of a previous interpolation at the node x_j^i of the current interpolation level i, can be used as a natural candidate for error estimation and control, since, for smooth performance functions, the hierarchical surpluses approach zero as the interpolation level goes to infinity.

2.2.3 Asymmetric dimension-adaptive tensor-product (ADATP) interpolation

The ADATP interpolation proceeds by selecting all the indices (DIs) whose relative error indicators are greater than a predefined error threshold ε_C. The relative error indicator used in the interpolation scheme is defined for a DI $\mathbf{i}$ as (Hu and Youn 2011)

$$\varepsilon_r(\mathbf{i}) = \frac{1}{(g_{\max} - g_{\min})\, M_{\mathbf{i}}} \sum_{\mathbf{j}} \left| w_{\mathbf{j}}^{\mathbf{i}} \right| \tag{9}$$

where $w_{\mathbf{j}}^{\mathbf{i}}$ are the hierarchical surpluses of the collocation points $\mathbf{X}^{\mathbf{i}} = X_{\Delta}^{i_1} \times \cdots \times X_{\Delta}^{i_N}$, with $\mathbf{j} = (j_1, \ldots, j_N)$, $j_k = 1, \ldots, m_{\Delta}^{i_k}$, $1 \le k \le N$, and $M_{\mathbf{i}} = m_{\Delta}^{i_1} \cdot m_{\Delta}^{i_2} \cdot \cdots \cdot m_{\Delta}^{i_N}$.

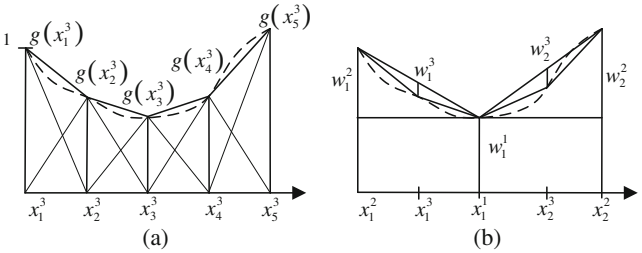


Fig. 1 Nodal (**a**) and hierarchical (**b**) interpolations in 1D with the piecewise linear spline and Clenshaw–Curtis grid

3 Adaptive dimension decomposition and reselection method

As an attempt to enhance the adaptive feature of the ADATP algorithm for high-dimensional problems, we develop an adaptive dimensional decomposition algorithm with a dimension reselection scheme to accurately represent the underlying response function with, to date, the minimum number of component functions. The component functions are treated as sub-problems which are resolved with the ADATP method.

3.1 Generalized dimension decomposition

In the classical ANOVA decomposition, an N-dimensional real-valued smooth stochastic response can be decomposed in a hierarchical and convergent manner as (Rabitz and Alis 1999; Xu and Rahman 2004)

$$g(\mathbf{x}) = g_0 + \sum_{i=1}^{N} g_i(x_i) + \sum_{1\le i_1<i_2\le N} g_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{1\le i_1<\cdots<i_s\le N} g_{i_1\cdots i_S}(x_{i_1}, \cdots, x_{i_s}) + \cdots + g_{1\cdots N}(x_1, \cdots, x_N) \tag{10}$$

Here, g_0 is a constant acting as the zeroth-order component function which represents the mean effect; $g_i(x_i)$ is a univariate function acting as the first-order component function which expresses the individual effect of x_i on the response $g(\mathbf{x})$, $g_{i_1 i_2}(x_{i_1}, x_{i_2})$ is a bivariate function acting as the second-order component function which describes the interactive effects of x_{i_1} and x_{i_2} on the response; the higher order terms gives the interactive effects of increasing numbers of input random variables acting together to contribute to the response; and the last term accounts for any residual dependence of all the input random variables cooperatively locked together to affect the response. It is noted that, for notational convenience, we use the notation x_i instead of x^i to denote the ith random variable for $i = 1, ..., N$ from this section. Once we suitably determine all the important component functions, the resulting decomposed model can be used as a stochastic response surface model to efficiently compute the response. Here, two key issues are which component functions should be selected for inclusion in the decomposed model and how to represent or approximate these selected component functions, and this is precisely what we attempt to resolve in the subsequent sections.

If we define a set of dimensional indices $\mathbf{u} \subseteq \mathcal{D}$ where $\mathcal{D} := \{1, \ldots, N\}$ denotes a set of all dimensional indices, we can obtain a more compact notation of the generalized dimension decomposition, expressed as (Griebel and Holtz 2010)

$$g(\mathbf{x}) = \sum_{\mathbf{u}\in\mathcal{D}} g_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \tag{11}$$

Here, $g_{\mathbf{u}}$ denotes a $|\mathbf{u}|$-dimensional component function whose random dimensions correspond to the dimensional indices belonging to $\mathbf{u}$, where $|\mathbf{u}|$ is the number of indices in the set $\mathbf{u}$. For example, if we have $\mathbf{u} = \{1, 2, 4\}$, then $g_{\mathbf{u}} = g_{124}(x_1, x_2, x_4)$.

The component functions can be obtained by defining an appropriate product measure and an error functional and minimizing this error functional (Rabitz et al. 1999; Rabitz and Alis 1999). An efficient way is to choose the measure as the Dirac measure at a reference point $\boldsymbol{\mu}_{\mathbf{x}}$, leading to the Cut-HDMR decomposition as (Rabitz et al. 1999; Griebel and Holtz 2010)

$$g(\mathbf{x}) = \sum_{\mathbf{u}\in\mathcal{D}} g^C_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) \tag{12}$$

where the component functions are explicitly given as

$$g^C_0 = g(\boldsymbol{\mu}_{\mathbf{x}}),\ g^C_i = g(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}}\backslash x_i} - g^C_0,$$
$$g^C_{i_1 i_2} = g(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}}\backslash(x_{i_1}, x_{i_2})} - g^C_{i_1} - g^C_{i_2} - g^C_0, \cdots \tag{13}$$

Here, the notation $\mathbf{x} = \boldsymbol{\mu}_{\mathbf{x}}\backslash \mathrm{x_i}$ denotes the vector $\mathbf{x}$ with its components other than x_i being set equal to the corresponding components of the reference vector $\boldsymbol{\mu}$. We can observe that those component functions can be recursively derived as (Rabitz et al. 1999)

$$g^C_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) = g(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}}\backslash\mathbf{x}_{\mathbf{u}}} - \sum_{\mathbf{v}\subset\mathbf{u}} g^C_{\mathbf{v}}(\mathbf{x}_{\mathbf{v}}) \tag{14}$$

and can also be more conveniently expressed as (Kuo et al. 2010)

$$g^C_{\mathbf{u}}(\mathbf{x}_{\mathbf{u}}) = \sum_{\mathbf{v}\subseteq\mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|}\, g(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}}\backslash\mathbf{x}_{\mathbf{v}}} \tag{15}$$

where the notation $\mathbf{x} = \boldsymbol{\mu}_{\mathbf{x}}\backslash\mathbf{x}_{\mathbf{u}}$ denotes the vector $\mathbf{x}$ with its components other than those indices that belong to the set $\mathbf{u}$ being set equal to the corresponding components of the reference vector $\boldsymbol{\mu}$.

It was reported in Rabitz et al. (1999), Rabitz and Alis (1999) and numerous reliability analysis papers that the responses of most practical engineered systems are significantly affected by only low-order interactions (usually up to second-order) of the random input variables while the high-order interactions of these variables are often

very weak. In these systems, a few lower-order component functions are sufficient to accurately capture the response uncertainty. It is also worth noting that any component function in the CUT-HDMR expansion accounts for an infinite number of Taylor series terms containing the same set of random variables as that component function. For example, the univariate decomposed component function $g_i^C(x_i)$ in (12) contains the univariate terms with x_i of any order in the Taylor series expansion and so on. Thus, the dimension decomposition of any order in (12) should not be viewed as a Taylor series expansion of the same order nor do they represent a limited degree of nonlinearity in $g(\mathbf{x})$. In fact, the dimension decomposition provides higher accuracy a Taylor series expansion of the same or even higher order. In particular, the residual error in a univariate approximation to a multidimensional integration of a system response over a symmetric domain was reported to be far less than that of a second-order Taylor expansion method for probability analysis (Xu and Rahman 2004).

Finally, to construct the dimension decomposition of a response function, or the Cut-HDMR, we need to first define a reference point $\boldsymbol{\mu}_{\mathbf{x}} = (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_N})$ in the input random space. Regarding this issue, the works by Sobol (2003) suggested that it is optimum to define the reference point as the mean values of the input random variables. But we also note that, to make more collocation points relevant to reliability analysis, we could interpolate the limit-state function in the vicinity of the most probable point (MPP) instead of the mean point of random inputs. However, this MPP-based approach suffers from the following two shortcomings: (i) the MPP search requires the sensitivity information of the response function and can be very expensive for high dimensional problems; and (ii) for design problems that require both moment estimation and reliability analysis such as reliability-based robust design optimization (Youn et al. 2005, 2007; Lee et al. 2009), the collocation points that are used to approximate the response function over the truncated input domain for uncertainty quantification (e.g., PDF and moment estimation) cannot be reused for reliability analysis, resulting in an added cost for reliability analysis. In contrast to the second aforementioned shortcoming of the MPP-based approach, the non-MPP-based approach to approximate the response function over the entire (truncated) input domain allows for the derivation of any probabilistic characteristics (e.g., statistical moments, reliability, and PDF) based on the same set of collocation points and can be used for design problems. Considering the aforementioned aspects, we intended to employ the proposed non-MPP-based method with the mean point as the reference point for reliability analysis which we expect to achieve a much better efficiency with satisfactory accuracy in most problems.

3.2 Component function approximation

Now we need to approximate the component functions in the generalized dimension decomposition. A desirable way is to construct an explicit function approximation to the component functions that can be used as a stochastic response surface model. This can be achieved by using the ADATP method with the relative merits of dimensional and directional adaptivity (Hu and Youn 2011).

Combining (12) and (15) gives us the following formula for the generalized dimension decomposition as

$$g(\mathbf{x}) = \sum_{\mathbf{u}\in\mathcal{D}} \sum_{\mathbf{v}\subseteq\mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|}\, g(\mathbf{x})|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}}\setminus\mathbf{x}_{\mathbf{v}}} \tag{16}$$

The lower-order terms in the right-hand side of the above equation can be treated as $|\mathbf{v}|$-dimensional sub-problems which can be solved by the ADATP method in the following manner:

$$g(\mathbf{x}) = \sum_{\mathbf{u}\in\mathcal{D}} \sum_{\mathbf{v}\subseteq\mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|} \sum_{\mathbf{i}\in\mathbf{I}_{\mathbf{v}}} \sum_{\mathbf{j}} a_{\mathbf{j}}^{\mathbf{i}}(\mathbf{x}_{\mathbf{v}}) \cdot w_{\mathbf{j}}^{\mathbf{i}} \tag{17}$$

where $a_{\mathbf{j}}^{\mathbf{i}}$ and $w_{\mathbf{j}}^{\mathbf{i}}$ are the hierarchical basis functions and hierarchical surpluses for the DI $\mathbf{i}$, respectively, and $\mathbf{I}_{\mathbf{v}}$ denotes the set of DIs obtained by the ADATP method for the dimension index set $\mathbf{v}$. Note that the hierarchical basis functions $a_{\mathbf{j}}^{\mathbf{i}}$ only contain random variables $\mathbf{x}_{\mathbf{v}}$ whose dimensions are contained in the set $\mathbf{v}$.

Since we are interested in quantifying the response uncertainty, an appropriate error indicator associated with a DI should reflect the contribution of that DI to the integration rather than its potential to enhance the accuracy in the interpolation. A new error indicator is thus defined in this study as the integral of the multiplication of the hierarchical surplus and the basis interpolation function, as motivated by the work in Ma and Zabaras (2010). This new definition, which incorporates the probabilistic characteristics of the random input variables, is defined for a DI $\mathbf{i}$ as

$$\varepsilon_r(\mathbf{i}) = \frac{1}{g_{\max} - g_{\min}} \sum_{\mathbf{j}} \int_{\Gamma} a_{\mathbf{j}}^{\mathbf{i}}(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x} \cdot w_{\mathbf{j}}^{\mathbf{i}} \tag{18}$$

where $w_{\mathbf{j}}^{\mathbf{i}}$ are the hierarchical surpluses of the collocation points $\mathbf{X}^{\mathbf{i}} = X_{\Delta}^{i_1} \times \cdots \times X_{\Delta}^{i_N}$, with $\mathbf{j} = (j_1, \ldots, j_N)$, $j_k = 1, \ldots, m_{\Delta}^{i_k}$, $1 \le k \le N$. It is noted that, for simplicity, we use $\mathbf{i} = (i_1, \ldots, i_N)$ instead of $\mathbf{i}^{\mathrm{d}} = (i_1^{\mathrm{d}}, \ldots, i_N^{\mathrm{d}})$ to denote a multi-dimensional DI and that the term "index" in the description of the ADATP method refers to the DI. Also note that, a DI may have a relatively large interpolation error locally but may not significantly affect the response globally. With the following definition,

$$Q_{\mathbf{j}}^{\mathbf{i}} = \int_{\Gamma} a_{\mathbf{j}}^{\mathbf{i}}(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x} \tag{19}$$

we can rewrite (18) in a simpler form as

$$\varepsilon_r(\mathbf{i}) = \frac{1}{g_{\max} - g_{\min}} \sum_{\mathbf{j}} Q_{\mathbf{j}}^{\mathbf{i}} \cdot w_{\mathbf{j}}^{\mathbf{i}} \tag{20}$$

If the random input variables follow uniform distribution and the piecewise multi-linear spline is employed as the hierarchical interpolation basis function, the 1D integral can be analytically obtained as

$$Q_j^i = \int_0^1 a_j^i dx = \begin{cases} 1, & \text{if } i = 1 \\ \frac{1}{4}, & \text{if } i = 2 \\ 2^{1-i}, & \text{otherwise} \end{cases} \tag{21}$$

For cases where we have non-uniform random inputs or higher-order basis interpolation functions other than the piecewise linear spline, the integral can be computed by using the Gaussian quadrature technique. The M-node quadrature formulae for computing the above integral can be expressed as

$$Q_j^i = \int_0^1 a_j^i(x)\, f(x)\, dx = \sum_{k=1}^{M} w_k^q a_j^i\left(x_k^i\right) \tag{22}$$

where x_k^i and w_k^q are the location and weight of the kth quadrature node, respectively. The selection of M depends on the interpolation level i and, for a high interpolation level i, a sufficiently large M should be used to achieve good accuracy since the basis function a_j^i vanishes to zero in a very narrow neighborhood of x_j^i which may not include any quadrature point. Since the random variables are either independent or have been transformed to independent random variables, the multi-dimensional integral in (18) can be obtained as the product of multiple one-dimensional integrals.

3.3 Adaptive dimension decomposition and reselection (ADDR) interpolation

Theoretically speaking, the important component functions among all 2^N ones in the generalized dimension decomposition (see (12)) can be either known as a priori, based on our prior knowledge on the performance function, or detected as a posteriori, based on an adaptive algorithm. However, in engineering practice, the prior information regarding the performance function is rarely known. Thus, we need to build the decomposition in an adaptive fashion where important terms (or dimensions) are automatically detected with the minimum computational cost. This section is devoted to presenting our attempt to develop such an adaptive scheme.

Our presentation will go along with a hypothetical ten-dimensional example of which the ADDR interpolation is graphically shown in Fig. 2. The adaptive process consists of three iteratively executed steps: (i) evaluation of component functions with the ADATP method, (ii) selection of potentially important dimensions based on the dimensional weights and (iii) reselection of important dimensions based on the test error indicators. The detailed procedure of the proposed ADDR interpolation is given in Table 1.

Evaluating important dimensions Initially, we construct all the first-order component functions with the ADATP

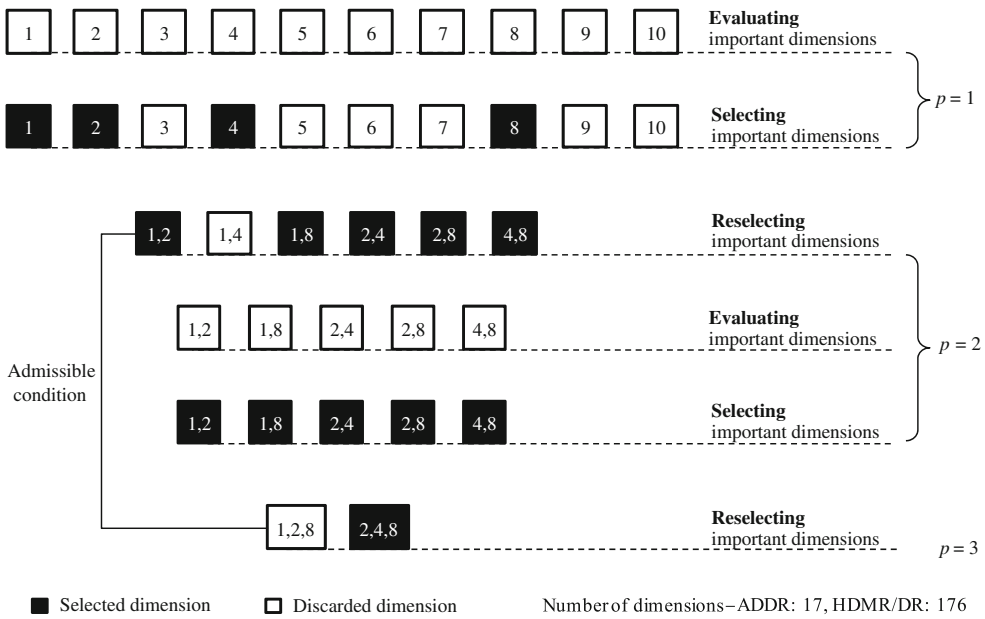


Fig. 2 Adaptive process of dimension decomposition and reselection

Table 1 Procedure of the proposed ADDR interpolation

STEP 1	Set the initial decomposition order $p = 1$; set the initial important dimension set $\mathcal{S} = \emptyset$, the initial constructed dimension set $\mathcal{C} = \emptyset$ and the initial selected dimension set $\mathcal{R} = \emptyset$.
STEP 2	Construct the zeroth- and first-order component functions using the ADATP method with the relative error threshold ε_C using (18); add the dimensional indices $\{1, 2,..., N\}$ to the set $\mathcal{C}$; compute the dimensional weight of each first-order component function using (23) and add those terms whose dimensional weights $\gamma \geq \gamma_C$ to the set $\mathcal{S}$.
STEP 3	Set $p = p + 1$; construct the set $\mathcal{R}$ with those pth order dimensional indices passing the admissibility test in (24). If $\mathcal{R} = \emptyset$, go to STEP 7.
STEP 4	Select and remove a dimensional index $\mathbf{u}$ from R. If $\mathcal{R} = \emptyset$, go to STEP 3. If the number of the collocation points M exceeds the maximum number $M_{\max}$, go to STEP 7.
STEP 5	Compute its test error indicator $\eta_{\mathbf{u}}$ using (25). If $\eta_{\mathbf{u}} \geq \eta_C$, go to STEP 6; otherwise, go to STEP 4.
STEP 6	Construct the component function for $\mathbf{u}$ using the ADATP method with the relative error threshold ε_C using (18); add the dimensional indices $\mathbf{u}$ to the set $\mathcal{C}$; compute the dimensional weight of $\mathbf{u}$ using (23). If $\gamma_{\mathbf{u}} \geq \gamma_C$, add $\mathbf{u}$ to the set $\mathcal{S}$. Go to STEP 4.
STEP 7	Construct an explicit interpolation $\hat{g}$ of the performance function g.

method. Since we only deal with one-dimensional functions, the computational cost only linearly increases with the number of dimensions, resulting in a relatively small amount of function evaluations. Upon the construction of the one-dimensional functions, we then adaptively construct important second- and higher-order component functions that can be identified through the selection and reselection steps detailed below.

Selecting important dimensions We adaptively construct higher-order component functions by first selecting potentially important component functions at the current decomposition order. In order to quantify the contribution of each component function (or each dimension) to the integration, we define a relative error indicator (or dimensional weight) for a component function $g_{\mathbf{u}}^C$ as the integral of that component function, expressed as

$$\gamma_{\mathbf{u}} = \frac{1}{g_{\max} - g_{\min}} \sum_{\mathbf{v} \subseteq \mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|} \sum_{\mathbf{i} \in \mathbf{I}_{\mathbf{v}}} \sum_{\mathbf{j}} Q_{\mathbf{j}}^{\mathbf{i}} \cdot w_{\mathbf{j}}^{\mathbf{i}} \tag{23}$$

This error indicator quantifies the contribution of the component function $g_{\mathbf{u}}^C$ to the response in a statistical sense. It can be treated as the first-order sensitivity of the response to that particular component function. The component functions whose weights are larger than a predefined threshold γ_C are identified as important dimensions. In the example shown in Fig. 2, the important component functions are those with the dimensions 1, 2, 4, and 8. These dimensions are then put into an important dimension set $\mathcal{S}$. Their higher-order forward dimensions {1,2}, {1,4}, {1,8}, {2,4}, {2,8} and {4,8} are then selected as potentially important dimensions for dimension reselection later. When selecting potentially important dimensions for a higher expansion order, we need to conduct an admissibility test on all the possible higher-order dimensions. For a higher-order dimension with the index $\mathbf{u}$, the admissibility test requires that any subset of $\mathbf{u}$ belongs to the important dimension set $\mathcal{S}$, expressed as

$$\mathbf{u} \in \mathcal{D} \text{ and } \forall \mathbf{v} \subset \mathbf{u},\ \mathbf{v} \in \mathcal{S} \tag{24}$$

This test is to ensure that the potentially important dimension $\mathbf{u}$ can be recursively computed with the already computed dimensions in $\mathcal{S}$ by using (15). For the dimension selection in the second-order decomposition, among the possible forward dimensions {1,2,4}, {1,2,8}, {2,4,8}, only the latter two {1,2,8}, {2,4,8} are selected while the first one {1,2,4} is discarded. This is due to the fact that a subset {1,4} in {1,2,4} does not belong to the important set $\mathcal{S}$ and thus {1,2,4} does not pass the admissibility test.

Reselecting important dimensions In this step, we reselect the truly important dimensions among the potentially important ones obtained in the dimension selection step. For this purpose, we define a test sample point $\mathbf{x}_{\mathbf{u}}^T$ for each potentially important dimension to test the strength of interaction among the dimensions in $\mathbf{u}$. This test sample point takes the value 1.0 along any dimension in $\mathbf{u}$ and the value 0.5 along any other dimension. We then define a test error indicator for $\mathbf{u}$ as

$$\eta_{\mathbf{u}} = \frac{1}{g_{\max} - g_{\min}} \sum_{\mathbf{v} \subseteq \mathbf{u}} (-1)^{|\mathbf{u}|-|\mathbf{v}|}\, g\left(\mathbf{x}_{\mathbf{v}}^T\right)\Big|_{\mathbf{x}=\boldsymbol{\mu}_{\mathbf{x}} \setminus \mathbf{x}_{\mathbf{v}}} \tag{25}$$

where $\mathbf{x}_{\mathbf{v}}^T$ is the test sample associated with the index $\mathbf{v}$ and has been computed in a previous reselection step if $\mathbf{v} \neq \mathbf{u}$. According to (25), the test error indicator $\eta_{\mathbf{u}}$ can be viewed

as a measurement of the interactive effects of the dimensions $\mathbf{u}$ on the output at the corner point. Then we define the truly important dimensions as those whose test error indicators are larger than a predefined test error threshold η_C. Thus, the amount of information that can be kept after reselecting the truly important component functions depends on the choice of the test error threshold η_C. With decreasing η_C, more component functions become important and therefore more terms (or information) will be included in the ADDR. The logic behind the selection of the corner point to compute $\eta_{\mathbf{u}}$ lies in the fact that this point can be treated as the most probable point with possibly the largest interaction. It follows that the integral of the component function over the whole input domain should be most likely less than its absolute point value at the corner point with potentially stronger interaction than points in other regions. Therefore, if the point value of the component function at the corner point is smaller than a predefined test error threshold η_C, we conjecture that the dimensional weight in (23) should also be smaller than the threshold and this component function should be disregarded. We note that, even if the test error indicator of a component function is larger than the threshold, its dimensional weight may still be smaller than the threshold and, strictly speaking, we should discard this component function. However, our reselection scheme still selects this component function so as to achieve conservativeness.

3.4 Uncertainty quantification (UQ) and reliability analysis

Once the adaptive dimension decomposition and reselection sampling procedure is completed, an approximate function $\hat{g}$ of the original performance function g can be obtained by interpolation using hierarchical basis functions at collocation points. Thus, any probabilistic characteristics of $g(\mathbf{x})$, including statistical moments, reliability, and PDF, can be easily estimated by performing MCS. For example, any rth moment can be calculated as

$$\begin{aligned}\beta_r &\cong \int \hat{g}^r(\mathbf{x})\, f(\mathbf{x}) d\mathbf{x} \\ &= E\left(\hat{g}^r(\mathbf{x})\right) = \lim_{ns\to\infty} \frac{1}{ns} \sum_{j=1}^{ns} \hat{g}^r\left(\mathbf{x}_j\right)\end{aligned} \tag{26}$$

where β_r is the rth moment of the performance function $g(\mathbf{x})$; $f(\mathbf{x})$ is the joint PDFs; $\mathbf{x}_j$ is the jth realization of $\mathbf{x}$; and ns is the sampling size. For reliability estimation, let us define an approximate safe domain for the performance function g as

$$\hat{\Omega}^S = \left\{\mathbf{x} : \hat{g}(\mathbf{x}) < 0\right\} \tag{27}$$

Therefore, the reliability R can also be estimated by performing MCS as

$$\begin{aligned}R &\cong \int I_{\hat{\Omega}^S}(\mathbf{x})\, f(\mathbf{x}) d\mathbf{x} \\ &= E\left(I_{\hat{\Omega}^S}(\mathbf{x})\right) = \lim_{ns\to\infty} \frac{1}{ns} \sum_{j=1}^{ns} I_{\hat{\Omega}^S}\left(\mathbf{x}_j\right)\end{aligned} \tag{28}$$

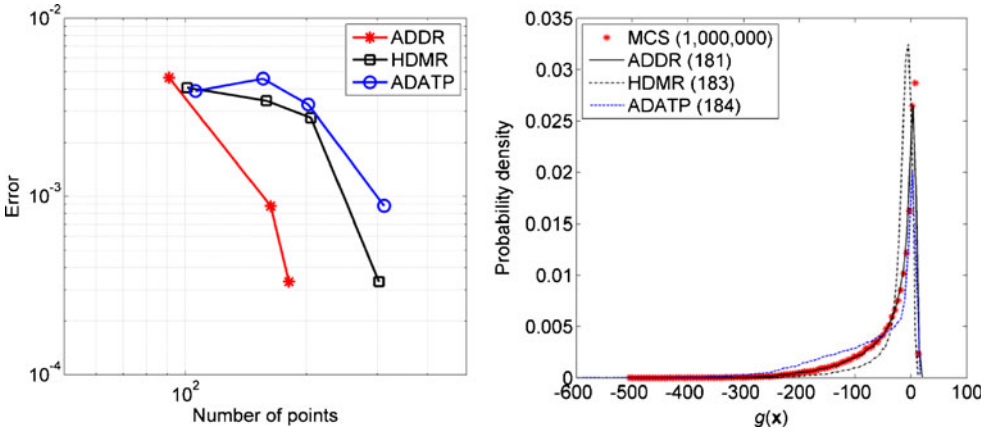
where $I[\cdot]$ is an indicator function of safe or fail state such that

$$I_{\hat{\Omega}^S}\left(\mathbf{x}_j\right) = \begin{cases} 1, & \mathbf{x}_j \in \hat{\Omega}^S \\ 0, & \mathbf{x}_j \in \Omega \backslash \hat{\Omega}^S \end{cases} \tag{29}$$

It should be noted that the MCS performed here employs the explicit interpolation $\hat{g}$ instead of the original performance function g and is thus inexpensive. We also note that the way of approximating the response function over the entire (truncated) input domain allows for the derivation of any probabilistic characteristics (e.g., statistical moments, reliability, and PDF) based on the same set of collocation points and can be used for design problems that require both moment estimation and reliability analysis such as reliability-based robust design optimization (Youn et al. 2005, 2007; Lee et al. 2009). Lastly, it is worthy to note that, since the proposed method essentially builds an approximate stochastic response surface (an explicit mathematical function) for the performance function g, we can directly apply the MCS with little additional computational cost to obtain the full probabilistic characteristics (e.g., statistical moments, reliability, and PDF) of g regardless of the distribution types (e.g., normal, lognormal and Weibull) of the input random variables. Specifically, we can first generate MCS samples for Gaussian and/or non-Gaussian input random variables and then evaluate the performance function at these MCS sample points. As long as we obtain a sufficiently accurate response surface, the estimated performance function values at the randomly generated sample points will be sufficiently accurate, leading to accurate probability analysis results.

4 Case studies

This section demonstrates the effectiveness of the ADDR method with five mathematical and engineering examples. The first three mathematical examples involve high-dimensional response surfaces with considerable bivariate interaction, trivariate interaction and anisotropic nonlinearity, respectively. The subsequent two engineering examples were dedicated to high-dimensional reliability analysis with an aim to investigate the computational accuracy and efficiency of the proposed method for reliability analysis. To



(a) (b)

Fig. 3 Error decay pathways (**a**) and PDF approximations (**b**) of the ADDR, HDMR and ADATP methods for example I

investigate the relative merits of the ADDR method, we conducted comparative studies between the generalized dimension decomposition method (Xu and Rahman 2004) or the high-dimensional model representation (HDMR) method (Rabitz et al. 1999), the asymmetric dimension-adaptive tensor-product method (ADATP) (Hu and Youn 2011), the FORM/SORM (Hasofer and Lind 1974; Breitung 1984), and the ADDR method, whenever possible. The component functions in both the HDMR and ADDR methods were computed with the ADATP method.

4.1 Mathematical example I: bivariate interaction

We first consider the following mathematical function with several bivariate interaction terms

$$g\left(\mathbf{x}\right) = \sum_{k=1}^{N} x_k^2 - \sum_{k=1}^{N/2} x_{2k-1}^4 x_{2k}^4 \tag{30}$$

where the number of random variables $N = 10$, the ten random variables were independently and uniformly distributed between 0 and 2. Since we intended to evaluate the accuracy of the approximate stochastic response surfaces in UQ, we defined a normalized L_2 error as

$$\varepsilon_{L_2} = \frac{\sqrt{\sum_{j=1}^{ns}\left(g\left(\mathbf{x}_j\right) - \hat{g}\left(\mathbf{x}_j\right)\right)^2}}{\sqrt{\sum_{j=1}^{ns}\left(g\left(\mathbf{x}_j\right)\right)^2}} \tag{31}$$

where ns denotes the number of random samples generated based the input distributions and was set to 1,000,000 in this study. A relative error threshold $\varepsilon_C = 0.01$ was used in the ADATP method which employed the piecewise multi-linear basis functions as the hierarchical basis functions and the Clenshaw-Curtis grid as the grid type. The error decay pathways and PDF approximations of the ADDR, HDMR and ADATP methods are graphically shown in Fig. 3. The error decay points of the ADDR method were obtained by varying the error threshold γ_C; the error decay pathway of the HDMR method was plotted by increasing the decomposition order and varying the maximum number of collocation points $M_{\max}$; the points for the ADATP method were obtained by varying the maximum number of collocation points $M_{\max}$. It is noted that in this example and all the subsequent examples, the test error threshold η_C is set equal to γ_C, since both the test error indicator $\eta_{\mathbf{u}}$ and the relative error indicator (or dimensional weight) $\gamma_{\mathbf{u}}$ quantifies the importance of a component function in the first-order forms. In Fig. 3a, the ADDR method shows faster error decay than any of the HDMR and ADATP methods. This can be attributed to the fact that the ADDR method identified important bivariate component functions, {1,2}, {3,4}, {5,6}, {7,8} and {9,10}, and only evaluated those bivariate functions while discarding the others. In contrast, the lack of adaptivity in the HDMR method led to the consideration of all bivariate and even higher-order component functions. Table 2 compares the accuracy and efficiency of the ADDR method, the adaptive dimension decomposition (ADD) method (Griebel and Holtz 2010; Ma and Zabaras 2010) without dimension reselection and the second-order HDMR method. We observe that the ADDR method requires far less component functions and thus less collocation points than the ADD and second-order HDMR methods. It is noted that the ADD method produced

Table 2 Comparison of the ADDR, ADD and HDMR methods for example I

Method	No. component functions	No. function evaluations	ε_{L_2}
ADDR ($\gamma_C = 0.01$)	16	181	3.33E–4
ADD ($\gamma_C = 0.01$)	56	261	3.33E–4
HDMR ($p = 2$)	56	261	3.33E–4

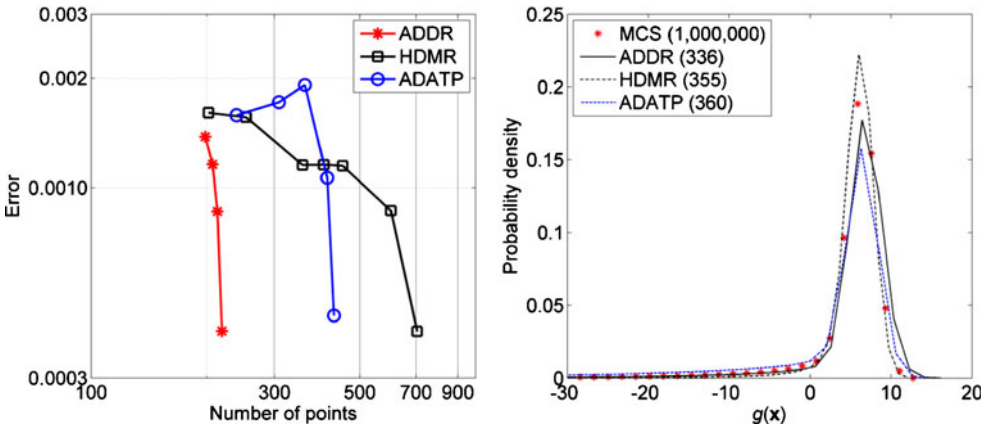


Fig. 4 Error decay pathways (**a**) and PDF approximations (**b**) of the ADDR, HDMR and ADATP methods for example II

exactly the same response surface as that by the second-order HDMR method. The results suggest that, without the dimension reselection, the adaptive dimension decomposition may unnecessarily consider unimportant high-order component functions and may even lose the relative merits over the HDMR method when component functions at the same decomposition level are equally important, as is the case in this example. Regarding the PDF approximation, the proposed ADDR method again produced better performance (see Fig. 3b), compared to the HDMR and ADATP methods. This example verifies that, for a performance function with only a part of low-order component functions being significant, the ADDR method is advantageous over the HDMR and ADATP methods.

4.2 Mathematical example II: trivariate interaction

Let us then consider the following mathematical function with several trivariate interaction terms

$$g(\mathbf{x}) = \sum_{k=1}^{N} x_k^2 - \sum_{k=1}^{(N-1)/3} x_{3k-2}^4 x_{3k-1}^4 x_{3k}^4 \tag{32}$$

where the number of random variables $N = 10$, the ten random variables were assumed to be statistically independent and uniformly distributed between 0 and 1.5. This nonlinear function contains three important trivariate component functions {1,2,3}, {4,5,6}, {7,8,9}, among all $N!/(N-3)!/3! = 120$ trivariate component functions. A relative error threshold $\varepsilon_C = 0.01$ and the piecewise multi-linear basis functions were again used in the ADATP method to directly conduct UQ and reliability analysis as well as to construct component functions for the ADDR and HDMR methods. The performance of these methods in terms of the L_2 error decay and PDF approximation is compared in Fig. 4. The error decay points were obtained in the same manner as the first example. In Fig. 4a, we again observe that the ADDR method exhibits faster error decay than any of the HDMR and ADATP methods. It is noted that, for the ADDR and ADATP method, the error decay becomes very sharp when the construction of bivariate component functions has been completed and the construction of the three important trivariate component functions is in progress. Compared to the ADATP method, the ADDR method detects and constructs the important trivariate component functions in a much earlier stage due to the dimension-wise regulation (initially starting from low-order component functions and gradually increasing the decomposition order) imposed by the HDMR. In contrast, the HDMR method gives the slowest error decay rate due to the lack of adaptivity in constructing bivariate and higher-order component functions. Table 3 compares the performance of the ADDR, ADD and third-order HDMR methods. For the ADDR method, we require 23 component functions and 219 function evaluations to achieve an accurate integration. In contrast, to achieve the same integration error, we need far more component functions (and thus function evaluations) for the ADD and third-order HDMR methods. The results again suggest that the dimension reselection plays an essential role in screening out the potentially important but actually unimportant dimensions to derive the minimum number of component functions. The PDF approximation by the proposed ADDR method exhibits better accuracy (see Fig. 4b) in the left tail region, compared to those by the HDMR and ADATP methods, and comparable accuracy to the HDMR method in the middle peak region (slightly

Table 3 Comparison of the ADDR, ADD and HDMR methods for example II

Method	No. component functions	No. function evaluations	ε_{L_2}
ADDR ($\gamma_C = 0.01$)	23	219	4.03E–4
ADD ($\gamma_C = 0.01$)	59	327	4.03E–4
HDMR ($p = 3$)	176	705	4.03E–4

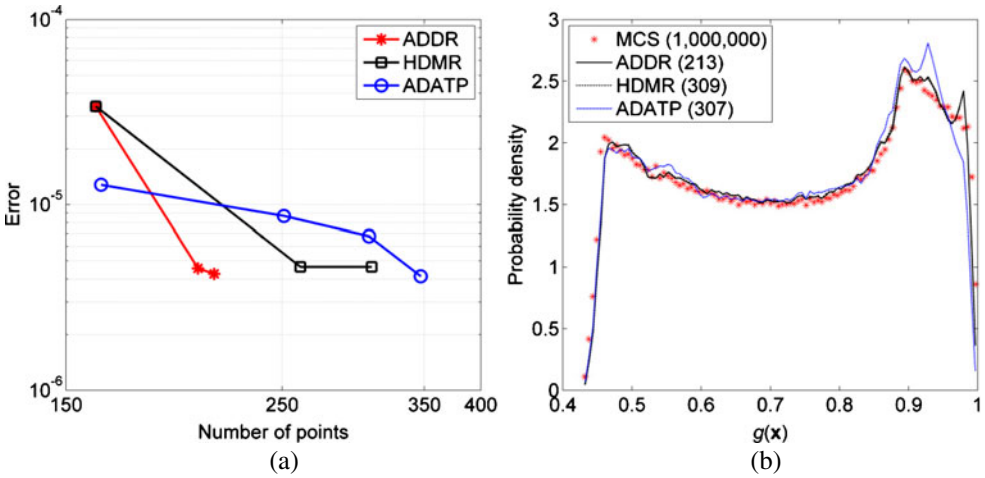


Fig. 5 Error decay pathways (**a**) and PDF approximations (**b**) of the ADDR, HDMR and ADATP methods for example III

better accuracy than the ADATP method). This example further verifies that, for a performance function with only a part of low-order component functions being significant, the ADDR method yields better efficiency with comparable accuracy to the HDMR and ADATP methods.

4.3 Mathematical example III: anisotropic nonlinearity

Consider the following mathematical function with anisotropic nonlinearity

$$g(\mathbf{x}) = \left(1 + \sum_{k=1}^{N} \alpha_k x_k^2\right)^{-1} \tag{33}$$

where the number of random variables $N = 10$, the ten random variables were assumed to be statistically independent and uniformly distributed between $-2\sqrt{3}$ and $2\sqrt{3}$ (i.e., with the means being 0 and standard deviations being 2), and $\alpha_k = 1/10^k$, for $k = 1, 2, ..., 10$. This example was modified based on the one in Ma and Zabaras (2010). Since the ratio of the smallest coefficient to the largest one is about 10^{-9}, this mathematical function exhibits highly anisotropic nonlinearity with only the first few dimensions being important. The ADATP method used a relative error threshold $\varepsilon_C = 0.005$ and the piecewise multi-linear basis functions. The results in Fig. 5 suggest that, regarding the L_2 error decay (see Fig. 5a), the ADDR method performs better than any of the HDMR and ADATP methods and, regarding the PDF approximation (see Fig. 5b), the ADDR method yielded an almost identical PDF approximation compared to the HDMR method and a more accurate PDF approximation compared to the ADATP method but with much less collocation points than both methods. Table 4 compares the number of component functions as well as the number of function evaluations used in the ADDR and second-order HDMR method. It can be seen that the ADDR method built much less component functions and thus required much less function evaluations than the HDMR method. In fact, due to the highly anisotropic weights (only the first three dimensions are regarded as important), the ADDR method only built a few important component functions at the second decomposition order, i.e., {1,2}, {1,3} {2,3}, and only one important component function {1,2,3} at the third decomposition order. In contrast, the HDMR method evaluated all the component functions at the second decomposition order and is expected to evaluate all the component functions at the third- and higher-orders. Overall speaking, the ADDR method is capable of significantly enhancing the efficiency of the HDMR method without any loss of accuracy for cases where only a few low-order component functions are sufficient to approximate the performance function with a good accuracy, as shown in this example.

4.4 Ten-bar truss structure: linear elastic reliability

This example considers a linear-elastic, 10-bar truss structure (see Fig. 6) to demonstrate the accuracy and efficiency of the proposed method for reliability analysis. The truss structure is supported at nodes 5 and 6, and is subjected to two vertical loads with the magnitudes 10^5 lb at nodes 2 and 4. The Young's modulus of the truss material is assumed to be 10^7 psi. The random variables are ten cross-sectional areas $[x_1, x_2, ..., x_{10}]$ for each bar, which follow normal

Table 4 Comparison of the ADDR, ADD and HDMR methods for example III

Method	No. component functions	No. function evaluations	ε_{L_2}
ADDR ($\gamma_C = 0.001$)	15	213	4.26E–6
HDMR ($p = 2$)	56	401	4.95E–6

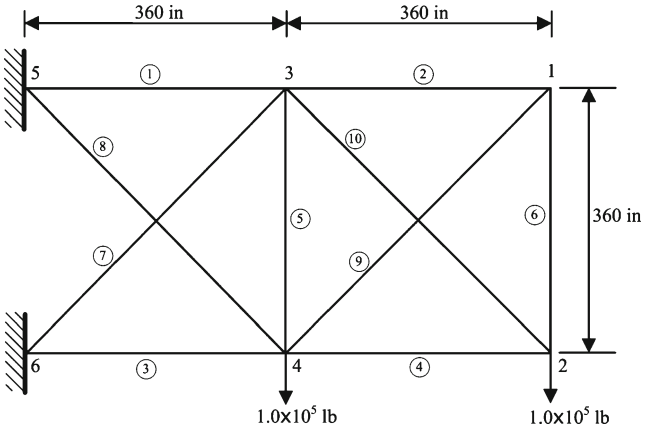


Fig. 6 A ten-bar truss structure

distributions with means $\mu = 2.5$ in^2 and standard deviations $\sigma = 0.5$ in^2. We consider the maximum displacement u_2 at node 2 for reliability analysis. The performance function is defined as (Wei and Rahman 2007)

$$g(\mathbf{x}) = U_2(x_1, x_2, \ldots, x_{10}) - 18 \tag{34}$$

The ADATP method which solve each component function selected by the ADDR method used a relative error threshold $\varepsilon_C = 0.005$ and the cubic Lagrange splines as the hierarchical interpolation basis functions. The Gaussian Hermite quadrature was employed to compute the first-order moment of any interpolation basis function according to (22). Table 5 summarizes the uncertainty analysis results of the ADDR method with comparison to the MCS, FORM and SORM. To reflect the variation in the probability estimate by MCS, we computed the error bounds with a 95 % confidence level. The error bound with a 100(1-α)% confidence can be computed as (Law and Kelton 1982)

$$\varepsilon_S = z_{1-\alpha/2}\sqrt{\frac{P(1-P)}{N_S}} \tag{35}$$

where $z_{1-\alpha/2}$ is the 100(1−α/2)th percentile of the standard normal distribution, P is the probability estimate by MCS and N_S is the number of MCS samples. For a 95 % confidence level, $\alpha = 0.05$ and $z_{1-\alpha/2} = 1.96$. As can be seen from Table 6, the ADATP method produced more accurate reliability estimates than the FORM and SORM. This is due to a more accurate performance function approximation by the ADDR method than by the FORM and SORM. Regarding the efficiency, the ADDR method requires more function evaluations than the FORM but comparable function evaluations to the SORM. The PDF approximations by the MCS and the ADDR method are compared in Fig. 7, where we observe a fairly good agreement between the two approximations.

Table 5 Reliability analysis results for the ten-bar truss example

Method	Pr($g < 0$)	No. function evaluations
ADDR ($\gamma_C = 0.1$)	0.864871	101
MCS	0.860221 (±0.000680[a])	1,000,000
FORM	0.913755	55
SORM	0.871424	100

[a]Error bounds computed with a 95 % confidence level

Table 6 Input random variables for the lower control A-arm example

Component	Distri. type	Mean (in)	Std. dev. (in)
x_1	Normal	0.157	0.006
x_2	Normal	0.183	0.006
x_3	Normal	0.178	0.009
x_4	Normal	0.200	0.007
x_5	Normal	0.312	0.013
x_6	Normal	0.250	0.009
x_7	Normal	0.200	0.007
x_8	Normal	0.201	0.009

4.5 Lower control A-arm: nonlinear fatigue reliability

Vehicle suspension systems experience intense loading conditions throughout their service lives. Control arms act as the backbone of the suspension system, through which the majority of these loads are transmitted (Youn et al. 2007). Therefore, it is crucial that the fatigue life of control arms be high enough to fulfill the design requirement. A HMMWV lower control-arm was employed to conduct fatigue reliability analysis using the ADDR method.

The simulation consists of three steps. Firstly, we modeled the lower control-arm with plane stress elements using 54,666 nodes, 53,589 elements, and 327,961 DOFs, where

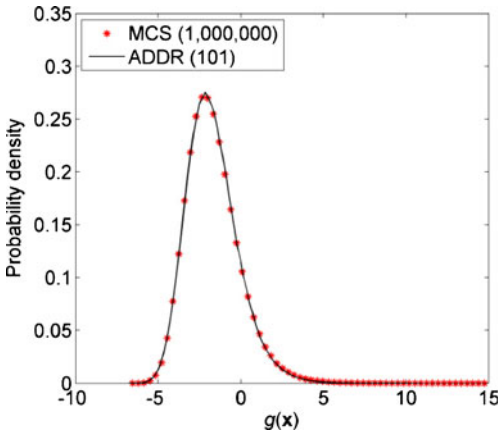


Fig. 7 PDF approximations for the ten-bar truss example

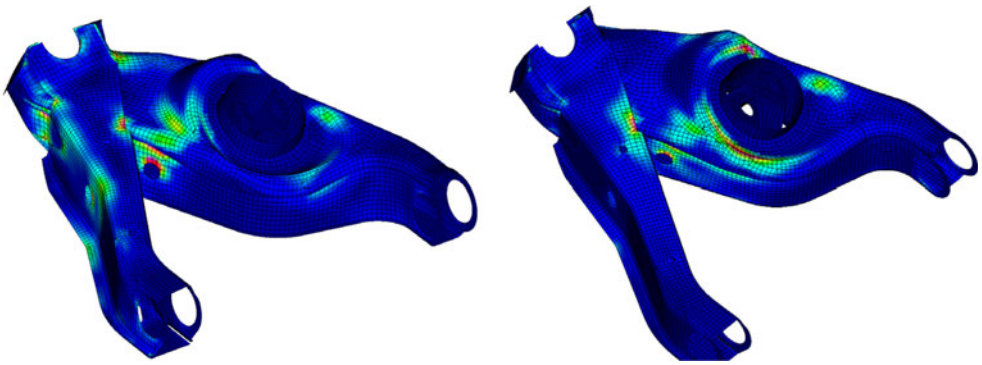


Fig. 8 Stress contours for load case 2 (**a**) and load case 8 (**b**)

(a) (b)

all welds were modeled using rigid beam elements. HyperWorks 8.0 was used for finite element modeling and design parameterization. Secondly, we used ANSYS 10.0 to conduct stress analyses for 14 load cases at four joints of the A-arm: a ball joint, a spring-damper joint and front and rear pivot bushing joints, respectively. The stress contours for two loading cases are shown in Fig. 8. Lastly, we employed the fe-safe 5.0 to carry out durability analyses based on the dynamic stress results from ANSYS. A preliminary durability analysis was executed in fe-safe to estimate the fatigue life of the HMMWV A-Arm and to predict the critical regions that experience a low fatigue life. For this preliminary durability analysis, the fatigue life for crack initiation was calculated using the equivalent von Mises stress-life approach at all surface nodes of the mechanical component (i.e., A-arm) in order to predict the critical regions. More accurate durability analysis was then carried using the strain-life method at the selected critical regions of the A-arm that experience short life spans.

This study treated the thicknesses of the eight major components of the control arm (see Fig. 9) as random input variables. The statistical information of these random variables is summarized in Table 7. From a worst-case scenario analysis, we identified one hotspot with the smallest fatigue life at the rear pivot bushing joint and selected this hotspot for fatigue reliability analysis. In this study, the fatigue reliability is defined as $R = \Pr\left(L > L^t\right)$, where L^t denotes the target fatigue life. Thus, the system performance function can be expressed as $g = L^t - L$, the computation of which relies on the aforementioned finite element simulation (ANSYS for stress analysis and fe-safe for durability analysis).

The ADDR method used $\varepsilon_C = 0.01$ as the relative error threshold and the cubic Lagrange splines as the hierarchical interpolation basis functions for the component function construction with the ADATP method. The fatigue reliability at the selected hotspot was evaluated with the ADDR method which allows for a stochastic response surface approximation from a small number of deterministic finite element and fatigue analyses through the construction of an explicit hierarchical interpolation formula with respect to the random inputs. Conducting the MCS on the explicit interpolation formula gives the full probability information (i.e., moments, reliability and PDF) of the fatigue life. For benchmarking, we carried out a direct MCS with 1,000 random samples. The error decay pathway of the ADDR method and the PDF approximations by the ADDR method and MCS are shown in Fig. 10a and b, respectively. We can observe fast error decay in Fig. 10a and a good agreement between the PDF approximations by the ADDR method and MCS in Fig. 10b. The uncertainty analysis results are

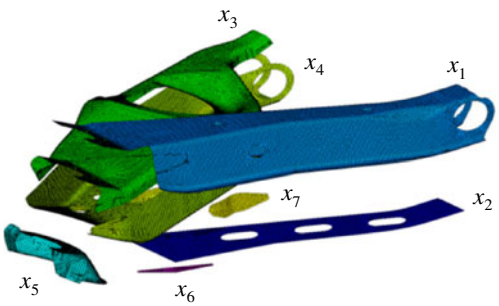


Fig. 9 Seven thickness variables (x_8 not shown)

Table 7 Uncertainty analysis results for the lower control A-arm example

	ADDR	MCS
Mean (blocks)	2.8819E+6	2.8866E+6
Std. dev. (blocks)	1.1318E+6	1.1612E+6
Skewness	8.8198E–1	1.2608E+0
Kurtosis	3.9938E+0	5.8083E+0
$R = \Pr\left(L > 2.5 \times 10^6\right)$	0.579	0.573 ($\pm$0.031[a])
$R = \Pr\left(L > 2.0 \times 10^6\right)$	0.762	0.774 ($\pm$0.026[a])
$R = \Pr\left(L > 1.5 \times 10^6\right)$	0.921	0.930 ($\pm$0.016[a])
$R = \Pr\left(L > 1.0 \times 10^6\right)$	0.994	0.993 ($\pm$0.005[a])
No. FE	45	1,000

[a]Error bounds computed with a 95 % confidence level

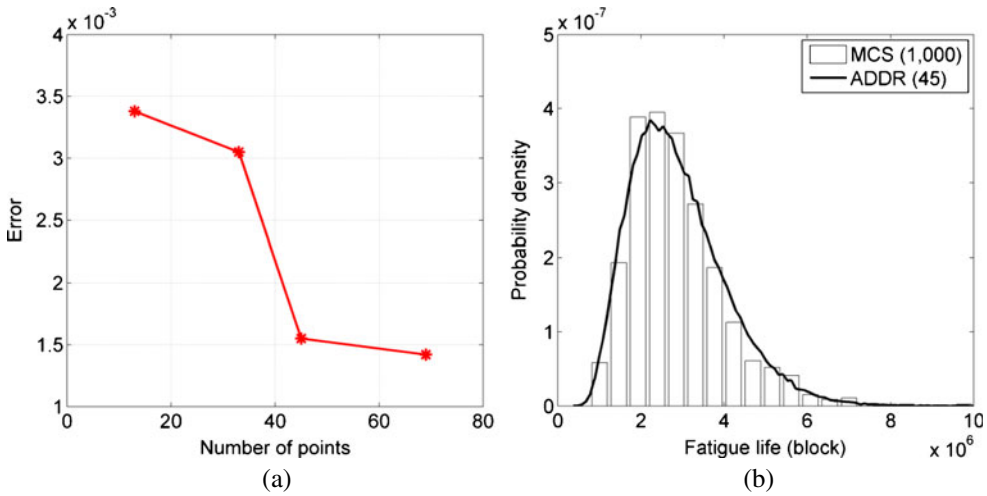


Fig. 10 Error decay pathway (**a**) and PDF approximation (**b**) of the ADDR method for the lower control A-arm example

summarized in Table 7, where the ADDR method produced good accuracy for low (between 0.50 and 0.60), moderate (between 0.70 and 0.80), high (between 0.90 and 0.95) and very high (above 0.99) reliability levels. At all four reliability levels, the ADDR reliability estimates fall into the 95 % confidence intervals of the corresponding MCS reliability estimates.

5 Conclusion

This paper proposes an adaptive dimension decomposition and reselection (ADDR) method for efficient high-dimensional reliability analysis involving high nonlinearity. The contributions of the ADATP method are three-fold, namely an adaptive dimension decomposition and reselection scheme to automatically detect the potentially important component functions and adaptively reselect the truly important ones, a test error indicator to quantify the importance of potentially important component functions for dimension reselection, and an integration of the newly developed asymmetric dimension-adaptive tensor-product (ADATP) method into the adaptive scheme to build the resulting component functions.

Results from high-dimensional mathematical and engineering problems suggest that the proposed ADDR method achieves better accuracy and efficiency than the HDMR and ADATP methods for performance functions with only a part of low-order component functions (up to the third-order) being significant. We note that, the responses of most practical engineered systems belong to the aforementioned cases with only low-order interactions of the random input variables being important, thus implying the potentially wide application of the proposed method. Among the state-of-the-art adaptive HDMR methods, the ADDR method, to the best of our knowledge, builds the minimum number of component functions for an accurate representation of the performance function. This supreme performance can be attributed to the adaptive dimension selection and reselection scheme to select potentially important dimensions and reselect truly important dimensions with virtually no extra cost.

However, we also note that, if a performance function is greatly influenced by high-order interactions of random input variables which are even stronger than lower-order interactions, the HDMR method as well as the ADDR method is no longer attractive. For the extremely case where all 2^N component functions must be used, we cannot gain any benefit by decomposing the performance function in a dimension-wise manner.

Acknowledgments This work was partially supported by a grant from the Energy Technology Development Program (2010101010027B) and International Collaborative R&D Program (0420-2011-0161) of Korea Institute of Energy Technology Evaluation and Planning (KETEP), funded by the Korean government's Ministry of Knowledge Economy, the National Research Foundation of Korea (NRF) grant (No. 2011-0022051) funded by the Korea government, the Basic Research Project of Korea Institute of Machinery and Materials (Project Code: SC0830) supported by a grant from Korea Research Council for Industrial Science & Technology, and the Institute of Advanced Machinery and Design at Seoul National University (SNU-IAMD).

References

Alis OF, Rabitz H (2001) Efficient implementation of high dimensional model representations. J Math Chem 29(2):127–142

Au SK, Beck JL (1999) A new adaptive importance sampling scheme for reliability calculations. Struct Saf 21(2):135–158

Barthelmann V, Novak E, Ritter K (2000) High dimensional polynomial interpolation on sparse grids. Adv Comput Math 12(4):273–288

Breitung K (1984) Asymptotic approximations for multinormal integrals. ASCE J Eng Mech 110(3):357–366

Chowdhury R, Rao BN, Prasad AM (2009) High dimensional model representation for structural reliability analysis. Commun Numer Methods Eng 25(4):301–337

Eldred MS, Burkardt J (2009) Comparison of non-intrusive polynomial chaos and stochastic collocation methods for uncertainty quantification. In: Proceedings of the 47th AIAA aerospace sciences meeting, Orlando, FL

Eldred MS, Webster CG, Constantine P (2008) Design under uncertainty employing stochastic expansion methods. In: Proceedings of the 12th AIAA/ISSMO multidisciplinary analysis and optimization conference, Victoria, British Columbia, Canada

Foo J, Karniadakis GE (2010) Multi-element probabilistic collocation method in high dimensions. J Comput Phys 229:1536–1557

Foo J, Wan X, Karniadakis GE (2008) The multi-element probabilistic collocation method (ME-PCM): error analysis and applications. J Comput Phys 227:9572–9595

Fu G, Moses F (1988) Importance sampling in structural system reliability. In: Proceedings of ASCE joint specialty conference on probabilistic methods, Blacksburg, VA, pp 340–343

Ganapathysubramanian B, Zabaras N (2007) Sparse grid collocation schemes for stochastic natural convection problems. J Comput Phys 225(1):652–685

Gerstner T, Griebel M (1998) Numerical integration using sparse grids. Numer Algorithm 18(3–4):209–232

Grestner T, Griebel M (2003) Dimension-adaptive tensor-product quadrature. Computing 71(1):65–87

Ghanem RG, Spanos PD (1991) Stochastic finite elements: a spectral approach. Springer, New York

Griebel M (1998) Adaptive sparse grid multilevel methods for elliptic PDEs based on finite differences. Computing 61(2):151–179

Griebel M, Holtz M (2010) Dimension-wise integration of high-dimensional functions with applications to finance. J Complex 26(5):455–489

Hasofer AM, Lind NC (1974) Exact and invariant second-moment code format. ASCE J Eng Mech 100(1):111–121

Hu C, Youn BD (2011) An asymmetric dimension-adaptive tensor-product method for reliability analysis. Struct Saf 33(3): 218–231

Hurtado JE (2007) Filtered importance sampling with support vector margin: a powerful method for structural reliability analysis. Struct Saf 29(1):2–15

Klimke A (2006) Uncertainty modeling using fuzzy arithmetic and sparse grids. PhD thesis, Universität Stuttgart, Shaker, Aachen

Kuo FY, Sloan IH, Wasilkowski GW, Wozniakowski H (2010) On decompositions of multivariate functions. Math Comput 79: 953–966

Law AM, Kelton WD (1982) Simulation modeling and analysis. McGraw-Hill, New York

Lee SH, Chen W, Kwak BM (2009) Robust design with arbitrary distributions using Gauss-type quadrature formula. Struct Multidisc Optim 39(3):227–243

Li G, Rosenthal C, Rabitz H (2001a) High dimensional model representations. J Phys Chem A 105:7765–7777

Li G, Wang SW, Rabitz H (2001b) High dimensional model representations generated from low dimensional data samples—I: mp-Cut-HDMR. J Math Chem 30(1):1–30

Ma X, Zabaras N (2009) An adaptive hierarchical sparse grid collocation algorithm for the solution of stochastic differential equations. J Comput Phys 228(8):3084–3113

Ma X, Zabaras N (2010) An adaptive high-dimensional stochastic model representation technique for the solution of stochastic partial differential equations. J Comput Phys 229:3884–3915

Naess A, Leira BJ, Batsevych O (2009) System reliability analysis by enhanced Monte Carlo simulation. Struct Saf 31(5): 349–355

Nobile F, Tempone R, Webster C (2008) A sparse grid collocation method for elliptic partial differential equations with random input data. SIAM J Numer Anal 46(5):2309–2345

Noh Y, Choi KK, Du L (2008) Selection of copula to generate input joint CDF for RBDO. In: Proceedings of ASME international design engineering technical conferences (IDETC) and computers and information in engineering conference (CIE), IDETC2008-49494, Brooklyn, New York, United States

Paffrath M, Wever U (2007) Adapted polynomial chaos expansion for failure detection. J Comput Phys 226(1):263–281

Rabitz H, Alis OF (1999) General foundations of high dimensional model representations. J Math Chem 25(2–3):197–233

Rabitz H, Alis OF, Shorter J, Shim K (1999) Efficient input–output model representations. Comput Phys Commun 117(1–2):11–20

Rahman S, Xu H (2004) A Univariate dimension-reduction method for multi-dimensional integration in stochastic mechanics. Probab Eng Mech 19(4):393–408

Rao BN, Chowdhury R (2009) Enhanced high dimensional model representation for reliability analysis. Int J Numer Methods Eng 77(5):719–750

Rosenblatt M (1952) Remarks on a multivariate transformation. Ann Math Stat 23:470–472

Roser BN (1999) An introduction to copulas. Springer, New York

Rubinstein RY (1981) Simulation and the Monte Carlo method. Wiley, New York

Smolyak S (1963) Quadrature and interpolation formulas for tensor product of certain classes of functions. Soviet Math Dokl 4: 240–243

Sobol IM (2003) Theorems and examples on high dimensional model representations. Reliab Eng Syst Saf 79(2):187–193

Tvedt L (1984) Two second-order approximations to the failure probability. Section on structural reliability, A/S Vertas Research, Hovik, Norway

Wan X, Karniadakis GE (2006) Multi-element generalized polynomial chaos for arbitrary probability measures. SIAM J Sci Comput 28:901–928

Wei D, Rahman S (2007) Stuctural reliability analysis by univariate decomposition and numerical integration. Probab Eng Mech 22: 27–38

Wiener N (1938) The homogeneous chaos. Am J Math 60(4):897–936

Xiong F, Greene S, Chen W, Xiong Y, Yang S (2010) A new sparse grid based method for uncertainty propagation. Struct Multidisc Optim 41(3):335–349

Xiu D (2007) Efficient collocational approach for parametric uncertainty analysis. Commun Comput Phys 2(2):293–309

Xiu D, Hesthaven JS (2005) High order collocation methods for the differential equation with random inputs. SIAM J Sci Comput 27(3):1118–1139

Xiu D, Karniadakis GE (2002) The Wiener–Askey polynomial chaos for stochastic differential equations. SIAM J Sci Comput 24(2):619–644

Xu H, Rahman S (2004) A generalized dimension-reduction method for multi-dimensional integration in stochastic mechanics. Int J Numer Methods Eng 61(12):1992–2019

Xu H, Rahman S (2005) Decomposition methods for structural reliability analysis. Probab Eng Mech 20:239–250

Youn BD, Wang P (2008) Bayesian reliability-based design optimization using eigenvector dimension reduction (EDR) method. Struct Multidisc Optim 36(2):107–123

Youn BD, Choi KK, Yi K (2005) Performance moment integration (PMI) method for quality assessment in reliability-based robust design optimization. Mech Base Des Struct Mach 33:185–213

Youn BD, Zhimin X, Wang P (2007) Reliability-based robust design optimization using the eigenvector dimension reduction (EDR) method. Struct Multidisc Optim 37(5):475–492

Youn BD, Zhimin X, Wang P (2008) Eigenvector dimension reduction (EDR) method for sensitivity-free uncertainty quantification. Struct Multidisc Optim 37(1):13–28