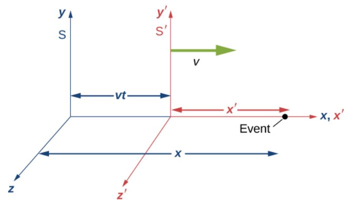

Figure 8: Two inertial frames with relative speed v.

Lets define

$$x'^{\mu} \equiv \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}, \qquad x^{\mu} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad \Lambda^{\mu}{}_{\nu} \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

where $\mu, \nu = 0, 1, 2, 3$ are called the Lorentz indices. Therefore, one can write (10) as

$$\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu}} \tag{12}$$

Using the Einstein summation convention (repeated indices are summed over) we can write LT (10) as

$$\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \qquad \mu, \nu = 0, 1, 2, 3} \tag{13}$$

One crucial observation is to be made here – the space and the time coordinates are no longer separate entities – they transform together as single entity. It is called the *spacetime*. In analogy with the location of a point in three dimensional space, lets introduce the concept of an *event*, defined as a set of four coordinates $x^{\mu} \equiv (t, x, y, z)$ in the 4-dimensional spactime. The x^{μ} is a vector in 4D spacetime that transforms like (13) under a LT. It is noting but a location of an event in 4D spacetime. And it is analogous to specifying the location of a point in a 3D space. Therefore, x^{μ} is called a 4-vector

$$x^{\mu} = (x^0, x^1, x^2, x^3) = (x^0, \vec{x}),$$

where $\vec{x}$ is the usual three vector. Conventionally, the x^0 is called the time component and x^1, x^2, x^3 are the space components.

4.3 Realtivistic Velocity Addition

Here, we demonstrate that if the set of equations (10) represent the transformation relations between two frames, then the speed of light remains constant, regardless of the relative velocity between the light source and the observer. The proof is as follows: Consider a particle moving along the X-direction (refer to figure 8). The velocities observed in the S- and S'-frames are given by:

$$u_x = \frac{dx}{dt}, \quad u'_x = \frac{dx'}{dt'}.$$

Using the Lorentz transformation relations, we can write:

$$dx = \gamma\left(dx' + \beta c dt\right), \quad dt = \gamma\left(dt' + \frac{\beta}{c} dx'\right).$$

Thus, the velocity in the S-frame is:

$$u_x = \frac{dx}{dt} = \frac{dx' + \beta c dt'}{dt' + \frac{\beta}{c} dx'} = \frac{u_{x'} + v}{1 + \frac{v}{c^2} u_{x'}}.$$

This is the relativistic velocity addition formula when the relative velocity between the two frames is significant. If the relative velocity is small, i.e., $v \ll c$, then the velocity in the S-frame reduces to the Galilean form:

$$u_x = u'_x + v.$$

Now, assume that the particle in question is light, so its velocity in the S'-frame is $u'_x = c$. Substituting this into the relativistic velocity addition formula gives:

$$u_x = c.$$

Therefore, the speed of light remains the same in all inertial frames, regardless of their relative velocities.

4.4 Length Contraction

In this section, we discuss an interesting consequence of Lorentz transformation (LT). With reference to figure 9, consider two reference frames S and S', where their axes are aligned, and S' moves with velocity v relative to S along the positive x-direction. Suppose there is a stick lying along the x'-axis in the S'-frame. The observer in the S-frame measures the endpoints of the stick at positions x_1 and x_2. The length measured in the S-frame is:

$$L = x_2 - x_1.$$

Suppose the observer in the S-frame also knows the velocity with which S' is moving. Using this information, along with his measurements, he can predict the actual length of the stick. By "actual length," we mean the length that the observer in the S'-frame would measure, which is the rest length of the stick. It can be computed as follows:

$$x'_1 = \gamma(x_1 - \beta ct), \quad x'_2 = \gamma(x_2 - \beta ct),$$

where the same time t is used in both equations, because the measurements in the S-frame must be simultaneous, as the object is in motion. The length measured in the S'-frame is:

$$L_0 = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma L.$$

Rearranging this, we get the relation between the lengths:

$$L = \frac{L_0}{\gamma} = \sqrt{1 - \beta^2} L_0.$$

Thus, $L < L_0$, where L_0 is the proper length of the object — the length measured in a frame where the object is at rest. Therefore, the length of a moving object appears contracted when measured from a frame in which it is in motion.

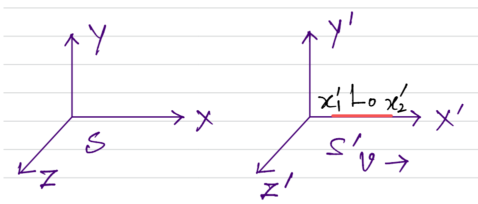

Figure 9: Two inertial frames with relative speed v.

4.5 Time Dilation

Now consider a clock located at position x' in the S'-frame. We will show that an observer in the S-frame perceives the clock to be running slower. In the S'-frame, two consecutive ticks of the clock occur at times t'_1 and t'_2. Using the Lorentz transformation to convert from the S'-frame to the S-frame, the times observed by an observer in the S-frame can be written as:

$$t_1 = \frac{t'_1 + \frac{v}{c^2}x'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad t_2 = \frac{t'_2 + \frac{v}{c^2}x'}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Thus, the time difference between the two ticks as observed in the S-frame is:

$$t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

This shows that the time interval between the ticks is longer in the S-frame than in the S'-frame, where the clock is at rest. Hence, to an observer in the S-frame, the moving clock appears to be running slower.

This phenomenon is known as **time dilation**, which leads us to the concept of **proper time**. Proper time is defined as the time measured by an observer whose clock is at rest relative to them.

4.6 Proper time

It is annoying that the timelike worldline of an ordinary particle is negative. It is customary to define a quantity

$$\boxed{(d\tau)^2 = -(ds)^2 = -\eta_{\mu\nu}dx^\mu dx^\nu}. \tag{14}$$

Suppose there is a clock attached to the rest frame of an observer. The observer observes two consecutive ticks in the clock, $t_1 = 0$ and $t_2 = 0 + dt$. Since the clock is at rest $dx^1 = 0, dx^2 = 0, dx^3 = 0$, and $dx^0 = t_2 - t_1 = dt$. So, the interval between the two ticks is

$$(d\tau)^2 = -(ds)^2 = (dx^0)^2 = dt^2\,, \quad d\tau = dt\,, \tag{15}$$

i.e., $d\tau$ is the measure of the ellapsed time dt in the clock of the observer. For this reason, $d\tau$ is known as the *proper time* of an observer. Proper time is invariant, *i.e.,* $d\tau' = d\tau$.

4.7 4-vectors

The concept of 4vector extends beyond the location of an event in spacetime. Any four component object $A^\mu \equiv (A^0, A^1, A^2, A^3) = (A^0, \vec{A})$ that transforms like (13)

$$\boxed{A'^\mu = \Lambda^\mu{}_\nu A^\nu\,,} \tag{16}$$

is called a four-vector. Remember that any set of four objects do not constitute a 4vector unless they transform like the above.

In analogy with x^μ, the 0^{th} component A^0 is called the *time component* and the other three components are called the *space components*. In addition to x^μ another 4vector that you should remember by heart is the four momentum of a particle denoted as p^μ. The time component of p^μ is its energy E, and the space components are just the components of its ordinary three-momentum $\vec{p}$. So

$$p^\mu = (E/c, \vec{p})\,. \tag{17}$$

If m is the mass of the particle then $E = \sqrt{\vec{p}^2 + m^2}$. Hence the squared of the 4momentum is

$$\boxed{p^\mu p_\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2}. \tag{18}$$

For massless particle, for example, photon of frequency ν the four-momentum is

$$p^\mu = \left(\frac{h\nu}{c}, \frac{h\nu}{c}\hat{k}\right),$$

where $\hat{k}$ is the unit vector in the direction of propagation. In terms of the wave vector $\mathbf{k}$ it can be written as

$$p^\mu = \hbar\mathbf{k}, \quad \text{where, } k^\mu = (\frac{\omega}{c}, \mathbf{k}) \tag{19}$$

where $\omega = 2\pi\nu$ is the angular frequency, and $|\mathbf{k}| = 2\pi/\lambda$ is the wave vector. In case of massless particles, it can be shown that

$$p^\mu p_\mu = 0$$

4.8 Minkowski metric

The distance between two nearby points, (x, y, z) and $(x + dx, y + dy, z + dz)$, in a three-dimensional Cartesian coordinate is $\sqrt{dx^2 + dy^2 + dz^2}$. The distance remains same (or invariant) when measured from a different coordinate system that is translated, or rotated about any axis. This motivates us to look for an expression of *distance between two events*, (x^0, x^1, x^2, x^3) and $(x^0 + dx^0, x^1 + dx^2, x^2 + dx^2, x^3 + dx^3)$, in the spacetime that remains *invariant* under the LT. Consider the quantity

$$\boxed{ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2}.$$

measured from a coordinate system S (you can refer to the previous figure.) Suppose, we perform a LT of the coordinate system to S' and the events are x'^μ and $x'^\mu + dx'^\mu$. Let say the quantity ds^2 is transformed to $(ds')^2$

$$\boxed{ds'^2 = -(dx'^0)^2 + (dx'^1)^2 + (dx'^2)^2 + (dx'^3)^2}.$$

Using LT relation $dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$ we can show (try to do it yourself – will be roughly done in the class)

$$ds'^2 = ds^2,$$

So $ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is an invariant quantity. The ds is called *spacetime interval*, or *line element*, or *metric*. This is analogous to the concept of length in a 3D space.

With the aid of the summation convention, the spacetime interval can be written in a compact form as

$$\boxed{ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \eta_{\mu\nu}dx^\mu dx^\nu}, \tag{20}$$

where in the last line we have introduced the *Minkowski metric* or simply the *metric of spacetime*

$$\boxed{\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}. \tag{21}$$

4.9 Contravariant and covariant vectors

The dot product or scalar product of a Cartesian 3-vector $\vec{A}$ is $\vec{A}.\vec{A}$. The analog of a scalar product for a 4-vector A^μ is $\eta_{\mu\nu}A^\mu A^\nu$. By defining

$$A_\mu \equiv \eta_{\mu\nu}A^\nu \tag{22}$$

we can express the scalar $\eta_{\mu\nu}A^\mu A^\nu$ as

$$\eta_{\mu\nu}A^\mu A^\nu = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2 = A^\mu A_\mu. \tag{23}$$

The $A^\mu A_\mu$ form is analogous to $\vec{A}.\vec{A}$.

The vector with the upper indices is called *contravariant* 4-vector: $A^\mu = (A^0, A^1, A^2, A^3)$, and the one with the lower indices $A_\mu = (A_0, A_1, A_2, A_3)$ is called the *covarint* 4-vector. Note that we can write the components of a covariant vector in terms of the components of contravarint 4-vector as

$$A_\mu \equiv \eta_{\mu\nu} A^\nu = (-A^0, A^1, A^2, A^3)\,.$$

The time component of contravariant vector is negative of the time component of its covariant form. For two different 4-vectors A^μ and B^μ we can write scalar product as

$$\eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 = -A^0 B^0 + \vec{A}.\vec{B} = A^\mu B_\mu\,. \tag{24}$$

The metric $\eta_{\mu\nu}$ lowers the index of a covarint vector and makes it contravarint. The inverse of $\eta_{\mu\nu}$ is $\eta^{\mu\nu}$ defined as

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho\,, \qquad \delta^\mu_\rho = 1 \quad \text{for } \mu = \rho \text{, or 0 otherwise} \tag{25}$$

$\eta^{\mu\nu}$ can be used to raise the index of a contravarint vector

$$\eta^{\mu\nu} A_\nu = A^\mu\,. \tag{26}$$

From the diagonal expression of $\eta_{\mu\nu}$ it is obvious that

$$\eta_{\mu\nu} = \eta^{\mu\nu}\,. \tag{27}$$

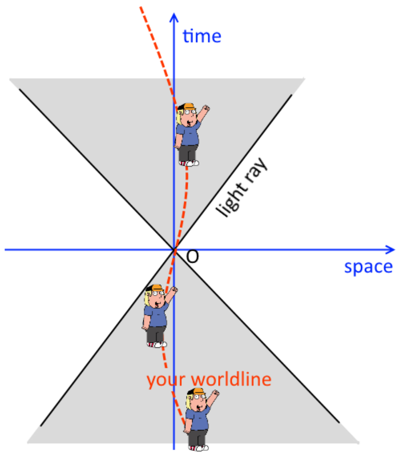

Figure 10: Light cone and your world line.

4.10 Light-cone

The trajectories of particles in 4D spacetime are called *world lines*. The world line of light is special. Consider the line element

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \equiv -(dx^0)^2 + d\vec{x}^2\,.$$

The wavefront of light has velocity $c = |d\vec{x}/dx^0|$ which means the worldline of light corresponds to

$$ds^2 = 0\,. \tag{28}$$

One can draw the world line of a light ray in a space vs time plot. Since it is impossible to plot in four dimension, we will resort to a two dimensional plot where the horizontal axis corresponds to space and the vertical axis is time. The world line of a light ray makes 45^o angle with respect to the space axis as shown in 10. This is also called the *lightlike world line*. In 4-dimension this would actually be a cone, hence 10 is called the *light cone*. For ordinary particles $|dx/dt| < c$, therefore, the world lines for any massive particles makes larger angle with the space axis.

Due to the relative negative sign between the time and space components, ds^2 in eq.(20) can be of three types

- *timlike*, $ds^2 < 0$: Two events separated by $ds^2 < 0$ are causally connected, *i.e.,* information can travel from one event to the other. An example is two events taking place at the same spatial position but at different times.

- *spacelike*, $ds^2 > 0$: Spacelike event are causally disconnected. Example would be two events are simultaneous in time but separated in space.

- *null or lightlike*, $ds^2 = 0$: Causally connected and lie in the world line of light.