4 Theory of Relativity

In this section we will discuss how to treat gravity. But first, lets discuss the Special Theory of Relativity

4.1 Special Theory of Relativity (STR)

The Galilean Relativity states *the laws of motion are same in all inertial frames.* Maxwell theory of light, in which the speed of light is constant in vacuum, do not refer to any particular frame of reference. Therefore, the Maxwell's theory is in direct conflict with the Galilean relativity. For example, the speed of light is constant irrespective of the relative velocity between the observer and the source. So either the Maxwell's theory of light is wrong or the Galilean transformations are not the correct transformations. In 1902-1905 Michaelson and Morley through an experiment found that the *speed of light in inertial frames is indeed independent of the relative velocity between source and observer.* Taking the result of the Michaelson and Morley experiment at face value, Einstein proposed a new theory, the Special Theory of Relativity, to make sense of all these. The STR is based on the following two postulates by Einstein

- *laws of physics have the same form (invariant) in all inertial frames*
- *for all inertial observers, the speed of light in vacuum is a constant c and is independent of the relative velocity with the source of the light*

Einstein's task was to find a set of transformation rules between inertial frames that satisfy the above two statements. Such transformations were already worked out by Hendrik Lorentz.

4.2 Lorentz Transformations

With reference to the figure 8, consider two frames $S \equiv (x, y, z)$ and $S' \equiv (x', y', z')$, such that the axes of the two frames are parallel. Assume that at $t = 0$ the origins of the two frames coincide. At some other time the relative velocity between the frames is v along the x-axis. In the S frame, an event is observed to takes place at time t and location (x, y, z). In the S' frame the same event is observed to take place at time t' at the location (x', y', z') . The Lorentz Transformation (LT) relations between the space and time coordinates are

$$\begin{aligned} t' &= \gamma\big(t - \frac{v}{c}\frac{x}{c}\big) = \gamma\big(t - \beta\frac{x}{c}\big)\,, \\ x' &= \gamma\big(x - vt\big) = \gamma\big(x - \beta ct\big)\,, \\ y' &= y\,, \quad z' = z \end{aligned}$$

where

$$\beta = v/c\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,.$$

A set of reverse transformation (from $S' \to S$) also exists – they can be obtained from the above set by interchanging the primed and unprimed coordinates, and changing the direction of the velocity. For the above set of relations, we have assumed the velocity along the x direction. For velocity in arbitrary directions the relations are a little more complicated, but we do not need them for our discussions.

For convenience, we introduce the following notations:

$$x^0 \equiv ct, x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$$

and, similarly for x'^1, x'^2, x'^3, x'^4. So the LT relations can be written as

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}\,, \tag{10}$$

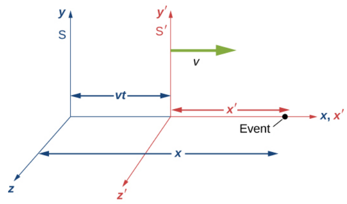

Figure 8: Two inertial frames with relative speed v.

Lets define two column vectors

$$x'^{\mu} \equiv \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}, \qquad x^{\mu} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \qquad \Lambda^{\mu}{}_{\nu} \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

where $\mu, \nu = 0, 1, 2, 3$ are called the Lorentz indices. Therefore, one can write (10) as

$$\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu}} \tag{12}$$

Using the Einstein summation convention (repeated indices are summed over) we can write LT (10) as

$$\boxed{x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} = \Lambda^{\mu}{}_{\nu} x^{\nu}\,, \qquad \mu, \nu = 0, 1, 2, 3} \tag{13}$$

One crucial observation is to be made here – the space and the time coordinates are no longer separate entities – they transform together as single entity. It is called the *spacetime*. In analogy with the location of a point in three dimensional space, lets introduce the concept of an *event*, defined as a set of four coordinates $x^{\mu} \equiv (t, x, y, z)$ in the 4-dimensional spactime. The x^{μ} is a vector in 4D spacetime that transforms like (13) under a LT. It is noting but a location of an event in 4D spacetime. And it is analogous to specifying the location of a point in a 3D space. Therefore, x^{μ} is called a 4-vector

$$x^{\mu} = (x^0, x^1, x^2, x^3) = (x^0, \vec{x}),$$

where $\vec{x}$ is the usual three vector. Conventionally, the x^0 is called the time component and x^1, x^2, x^3 are the space components.