

CS 206: Practice problems - Set I

Summer 2018

(Based on lectures 1-4)

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Problem 1

Answer the following questions. Explain in one line how you got the answer.

1. Consider the word “OHMYGODIMONFIRE”. How many distinguishable ways are there to rearrange the letters? 3 O's , 2 M's and 2 I's are the repeated symbols

The answer using 'bookkeeper' rule is $\frac{15!}{3!2!2!}$

2. How many sequences of 1s and -1s of length 10 are there that sum up to 2?

Let's find the number of +1 s and -1s that can be there. There have to be two more 1s than zeros. Thus, 6 1s and 4 -1s have to be there.

The answer is $\binom{10}{4}$.

3. How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition of letters is allowed?

Straight forward : 26 choices for each of the four places to be filled in 26^4

4. How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition is not allowed?

Four places need to be filled in, so pick 4 letters. $\binom{26}{4}$. Now permute them and place them - $4!$ ways to do it. Thus, the answer is $\binom{26}{4} \cdot 4!$.

5. How many 4-letter words contain at least one repeated letter?

The answer is the total number of words that can be formed (repetition allowed) minus the ones which have no repeated letters. Difference rule.

$$26^4 - \binom{26}{4} \cdot 4!$$

6. How many 4-letter words contain the letter X?

Answer is the total number of words minus the ones which don't contain the letter X. Difference rule.

$$26^4 - 25^4$$

7. How many 4-letter words consist of only the letters X and/or Y? (The words XXXX and YYYYY are included in this count.)

Four places need to be filled with 2 choices for each of them. 2^4

8. How many ways are there of forming a committee of 10 members that has at least 4 women if you have to choose from 30 men and 20 women?

Easier way to do this is using difference method.

The number of ways to choose a committee of 10 members from 30 men and 20 women is $\binom{50}{10}$.

The number of ways to choose a committee of 10 members from 30 men and 20 women such that there are zero women is $\binom{30}{10} * \binom{20}{0}$.

The number of ways to choose a committee of 10 members from 30 men and 20 women such that there is 1 woman is $\binom{30}{9} * \binom{20}{1}$.

The number of ways to choose a committee of 10 members from 30 men and 20 women such that there are 2 women is $\binom{30}{8} * \binom{20}{2}$.

The number of ways to choose a committee of 10 members from 30 men and 20 women such that there are 3 women is $\binom{30}{7} * \binom{20}{3}$.

Thus, the answer is $\binom{50}{10} - \binom{30}{10} * \binom{20}{0} - \binom{30}{9} * \binom{20}{1} - \binom{30}{8} * \binom{20}{2} - \binom{30}{7} * \binom{20}{3}$.

9. A group of n friends are all about to go separate ways in life and so they all hug each other. If one hug involves exactly 2 people and every pair of friends hugged exactly once, how many hugs were there in total?

Two ways to do this :

1. If you notice that what we are trying to count here is the number of ways pairs can be picked up from n people. The answer is thus $\binom{n}{2}$

2. Look at every person, he/she/it hugs all the $n-1$ others. So the answer seems to be $n * (n - 1)$. But that's wrong. There is overcounting if we count this way, since every pair AB is counted twice - once when you are fixed on A and count B among the other $n-1$ people, and the other time when you are fixed on B and count A among ther others. Thus, the correct answer for the total number of (unordered) pairs is $\frac{n(n-1)}{2}$

10. A group of n friends are all about to go separate ways in life and so they all give each other gifts. If every pair of friends exchanged gifts exactly once, how many gifts were there in total?

The answer follows from the previous question. For every pair, there are two gifts involved - one that A gives B and one that B gives A. So the number of gifts exchanged is twice the number of pairs. Ans: $n(n - 1)$

Problem 2

Given a binary string S , let $\Delta(S)$ = number of 1's in S - number of 0's in S . S is balanced if $-2 \leq \Delta(S) \leq 2$. How many balanced binary strings of length n are there. Provide a formal proof for your answer.

Use the partition method. Count strings for each possible value of $\Delta(S)$.

For smaller values of n , the answers will be trivial. Let's assume that n is sufficiently large so that the problem is interesting.

First case : let n be even; that means $\frac{n}{2}$ is legit.

- Number of strings with $\Delta(S) = 0$ is $\binom{n}{\frac{n}{2}}$
- Number of strings with $\Delta(S) = 1$ is 0.
- Number of strings with $\Delta(S) = 2$ is the number of strings with $n/2+1$ ones and $n/2-1$ zeroes. Thus, $\binom{n}{\frac{n}{2}-1}$.
- Number of strings with $\Delta(S) = -1$ is 0.
- Number of strings with $\Delta(S) = -2$ is the number of strings with $n/2-1$ ones and $n/2+1$ zeroes. Thus, $\binom{n}{\frac{n}{2}-1}$.

Thus, when n is even, the answer is $\binom{n}{\frac{n}{2}} + 2 \cdot \binom{n}{\frac{n}{2}-1}$.

Second case: n is odd.

- Number of strings with $\Delta(S) = 0$ is 0.
- Number of strings with $\Delta(S) = 1$ is the number of strings with $\lfloor n/2 \rfloor$ zeroes. Thus, $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.
- Number of strings with $\Delta(S) = 2$ is 0.
- Number of strings with $\Delta(S) = -1$ is the number of strings with $\lfloor n/2 \rfloor$ ones. Thus, $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.
- Number of strings with $\Delta(S) = -2$ is 0.

Thus, when n is even, the answer is $2 \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Problem 3

In each of the following cases, how many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 17$ where x_1, x_2, x_3, x_4, x_5 are non-negative integers.

1. $x_1 \geq 1$.

Distribute 1 to x_1 . 16 remain. Thus the answer is $\binom{16+4}{4}$

2. $x_i \geq 2$ for $i = 1, 2, \dots, 5$.

Distribute 2 to each person. 7 remain. Thus, the answer is $\binom{7+4}{4}$

3. $0 \leq x_1 \leq 10$. We will look at the case where $x > 10$ and subtract that from the number of solutions when we don't have any upper bound on x_1 (think difference method). For the case when $x_1 > 10$ or $x_1 \geq 11$: Give 11 to x_1 6 remain. The answer in this case is $\binom{6+4}{4}$

For the number of solutions when there is no upper bound on x_1 , the solution is just $\binom{17+4}{4}$, so the final answer is $\binom{21}{4} - \binom{10}{4}$.

4. $0 \leq x_1 \leq 3, 1 \leq x_2 \leq 4, x_3 \geq 15$.

Give fifteen to the 3rd variable. Give one to the 2nd variable.

Thus, the problem we want to solve now becomes

$$x_1 + x'_2 + x'_3 + x_4 + x_5 = 1$$

where $x_1, x'_2, x'_3, x_4, x_5$ are integers such that

$$0 \leq x_1 \leq 3,$$

$$0 \leq x'_2 \leq 3,$$

$$0 \leq x'_3$$

Note that the upper bounds on the variables don't matter now, since we have only one object to distribute.

i.e., the number of solutions possible is equivalent to those of the following problem :

$$x_1 + x'_2 + x'_3 + x_4 + x_5 = 1$$

where $x_1, x'_2, x'_3, x_4, x_5$ are integers such that

$$0 \leq x_1,$$

$$0 \leq x'_2,$$

$$0 \leq x'_3$$

The number of ways to do this is $5 = \binom{5}{1}$ - another way to think is - distributing one object among five people can be done in five ways, depending on who gets it.

5. $x_1 \geq x_2$

Let us write $x_1 = x_2 + y$ where $y \geq 0$. Then we are now looking at solutions to the equation

$$2x_2 + y + x_3 + x_4 + x_5 = 17,$$

under the constraints that $y, x_2, \dots, x_5 \geq 0$. Notice that x_2 cannot be more than 8 because the sum of ALL the variables must be 17, and if $x_2 \geq 9$ then $2x_2 > 17$. So we have that $0 \leq x_2 \leq 8$.

We are in an unusual situation because we have the term " $2x_2$ ". How do we deal with it? We will do a case analysis. Let us look at the case when $x_2 = i$

for some $0 \leq i \leq 8$. In that case, we see that we are counting solutions to $y + x_3 + x_4 + x_5 = 17 - 2i$ with $y, x_3, x_4, x_5 \geq 0$, and thus the number of solutions is

$$\binom{(17 - 2i) + 3}{3} = \binom{20 - 2i}{3}.$$

Since $0 \leq i \leq 8$, we must sum over all these possible values of i (sum rule: all these 9 cases are disjoint) and thus get that the total number of solutions is $\sum_{i=0}^8 \binom{20-2i}{3}$.

Justify your answers.

Problem 4

What is the number of binary strings of length 100 that contain exactly three occurrences of the string 01?

Proof. One way to specify a string of length 100 with exactly three occurrences of 01 is the following. Let R_1 be the part of the string that occurs to the left of the first 01, let R_2 be the part that occurs between the first and the second occurrence of 01 (not including the 01's themselves), R_3 be the part of the string occurs between the second and third occurrence of 01 (again, not inclusive of the 01's), and finally let R_4 be everything to the right of the last occurrence of 01. Specifying what R_1, \dots, R_4 are, uniquely specifies a string of length 100 with exactly three occurrences of 01: the string must look like $R_1 01 R_2 01 R_3 01 R_4$.

To count the number of such strings, all we need to count is the number of possibilities for R_1, \dots, R_4 . The question now is, what kind of strings can we expect R_1, \dots, R_4 to be? Notice that R_1 must be of the form: 0 or more ones followed by 0 or more zeros, i.e. R_1 can be 11110000 or 00000000 or 11111111, but not 1000011 or 0110111. Why? Because the last two strings both contain occurrences of 01 but there cannot be any occurrences of 01 in R_1 , since it's everything to the left of the first occurrence of 01. So R_1 must consist of x_1 ones followed by x_2 zeros where $x_1 \geq 0, x_2 \geq 0$.

It's not hard to see that the R_2, R_3, R_4 also cannot contain any occurrence of 01, and so they must look like a bunch of zero or more 1s followed by a bunch of zero or more 0s. Let us assume that R_2 is x_3 ones followed by x_4 zeros, R_3 is x_5 ones followed by x_6 zeros, R_4 is x_7 ones followed by x_8 zeros, where $x_3, \dots, x_8 \geq 0$.

Now notice that the three occurrences of 01 take up 6 out of the 100 possible positions in the string, so we are left with 94 positions for R_1, \dots, R_4 , and so we must have that

$$\text{length}(R_1) + \text{length}(R_2) + \text{length}(R_3) + \text{length}(R_4) = 94.$$

But, $\text{length}(R_1)$ is just $x_1 + x_2$, the length of R_2 is just $x_3 + x_4$, the length of R_3 is just $x_5 + x_6$, and the length of R_4 is just $x_7 + x_8$, so we get

$$x_1 + x_2 + \dots + x_8 = 94,$$

under the condition that $x_1, \dots, x_8 \geq 0$. Thus, the number of binary strings of length 100 with three occurrences of 01 is the same as the number of solutions to the aforementioned equation with the given constraints. We know that the number of solutions is just $\binom{94+7}{7} = \binom{101}{7}$. \square

Problem 5

Suppose you have 10 bins, all of different color (they are identical except for their distinct colors), and 30 balls, each ball having a unique number painted on it (they are identical except for the distinct numbers on them).

1. In how many ways can you distribute the balls into the bins?

The balls are not identical objects for now, since they have numbers written on them. Thus, each ball has 10 different ways it can go. The answer 10^{30} .

2. Suppose you only want to put exactly one ball in every bin (and discard the rest). How many ways of doing that are there?

First pick the ten balls that we are doing to use. That's $\binom{30}{10}$. Now place them, first ball has 10 different ways to go into the bins, next ball has 9, and so on. That's $10!$

Answer is $\binom{30}{10} * 10!$

3. Now suppose that all the bins are painted with the same color so that they all look the same. What will your answer to the second question be?

As in the previous part, we still need to choose 10 out of 30 balls that we are going to put into the bins. So there are $\binom{30}{10}$ ways of doing that. However, unlike in the previous part where we had to worry about deciding which ball goes into which bin, we don't need to do that here because all the bins are identical. For example, say we choose balls with the numbers 1, 23, 3, 6, 7, 9, 4, 30, 24, 2. We have to put one ball in each bin. The point is that it doesn't matter which bin 1 goes to or which bin 23 goes to, and so on. You just arbitrarily place the 10 chosen balls into the bins, and in some sense there is only one way to put 10 distinct balls into 10 identical bins if every bin must get exactly one ball. Thus, the overall answer is just $\binom{30}{10}$ — the actual step of putting 10 balls into 10 bins with one ball in each bin can only be done in one way!

This might sound a bit weird but what's happening here is that in combinatorics what matters is the number of *visually* distinct outcomes. Once you have chosen 10 balls out of 30 (say the ones I listed above), it doesn't matter how I place them in the bins because all the configurations will be identical, visually speaking.

Of course, in reality, there is no such thing as 10 identical bins because you will always be able to distinguish between the bins one way or another (even if they look exactly similar, you could distinguish, say, by the way are arranged on the floor etc.), so this is a purely theoretical construct.

Don't worry about this problem too much though. It's just to illustrate how drastically things can change when we shift from distinct bins to identical bins.

4. Now suppose the numbers painted on the balls get erased and the balls look identical (but the bins still have their colors): what will your answer to the first question be?

The balls are identical objects now.

The answer is the number of non-negative integral solutions to $x_1 + x_2 + \cdots + x_{10} = 30$.

$$\binom{30+9}{9}$$

5. Now suppose that the numbers on the balls are erased, and someone paints a third of the balls white, a third of the balls black, and the remaining third blue, while the bins still retain their original color: what's your answer to the first question? What about the second question?

Answer to the second question would now be 3^{10} . Think of the bins now, each bin has 3 ways that it can be filled in (since there are three different colors), and there are 10 bins in all.

6. Assuming the setting of the previous part, how many ways are there of distributing all the balls so that every bin contains at least one ball of every color?

One. 3 balls each for ten bins. We have no more balls left to fill in the bins.

7. Now suppose that the distinct numbers are painted back onto the balls while still retaining their color (as in the previous two parts), and the bins still have their color. How many ways of distributing the balls are there so that every bin gets at least one ball of every color?

Essentially, all balls of the same colour are different. Think algorithmically here. Distribute all balls of the same colour one by one, and doing the same with different colours, in succession. i.e First pick all the reds, distribute them, pick all the blues, distribute them and then the greens.

First distribute all the balls of the 1st colour into the bins. That's $10!$ ways to do it. Now think of the second colour. Independent of how the balls of the first colour are distributed in the bins, there are $10!$ ways to distribute those of the second. Same argument holds for the third colour and we can thus use the product rule.

Answer is $(10!)^3$.

8. Assume that the balls still have their colors and the distinct numbers on them from the previous part, however, now, the bins are all painted with the same color so that they become identical. What is your answer to the second question if you want to ensure that at least 8 white balls and at least 7 blue balls are discarded while distributing?

Partition method. The number of white balls possible is $\{0, 1, 2\}$ and that of blue colour is $\{0, 1, 2, 3\}$

First pick the balls you want to place in the bins. Next place the balls and count the number of ways to do it. Note that since the bins are identical, the number of

ways to place any set of picked balls is exactly one. Thus, this question is more about counting the number of possible subsets of these (distinct) balls according to the constraints.

- We pick 0 white, 0 blue, 10 black. The number of ways to pick 0 white balls is $\binom{10}{0}$. The number of ways to pick the blue ball is $\binom{10}{0}$, since all blue balls are distinct. Number of ways to pick the black balls is $\binom{10}{10}$. Only one way to place these. Thus, $\binom{10}{0} * \binom{10}{0} * \binom{10}{10} * 1$
- We pick 0 white, 1 blue, 9 black. The number of ways to pick 0 white balls is $\binom{10}{0}$. The number of ways to pick the blue ball is $\binom{10}{1}$, since all blue balls are distinct. Number of ways to pick the black balls is $\binom{10}{9}$. Only one way to place these. Thus, $\binom{10}{0} * \binom{10}{1} * \binom{10}{9} * 1$
- We pick 0 white, 2 blue, 8 black. The number of ways to pick 0 white balls is $\binom{10}{0}$. The number of ways to pick the blue ball is $\binom{10}{2}$, since all blue balls are distinct. Number of ways to pick the white balls is $\binom{10}{8}$. Only one way to place these. Thus, $\binom{10}{0} * \binom{10}{2} * \binom{10}{8} * 1$
- We pick 0 white, 3 blue, 7 black. $\binom{10}{0} * \binom{10}{3} * \binom{10}{7} * 1$
- We pick 1 white, 0 blue, 9 black. $\binom{10}{1} * \binom{10}{0} * \binom{10}{9} * 1$
- We pick 1 white, 1 blue, 8 black. $\binom{10}{1} * \binom{10}{1} * \binom{10}{8} * 1$
- We pick 1 white, 2 blue, 7 black. $\binom{10}{1} * \binom{10}{2} * \binom{10}{7} * 1$
- We pick 1 white, 3 blue, 6 black. $\binom{10}{1} * \binom{10}{3} * \binom{10}{6} * 1$
- We pick 2 white, 0 blue, 8 black. $\binom{10}{2} * \binom{10}{0} * \binom{10}{8} * 1$
- We pick 2 white, 1 blue, 7 black. $\binom{10}{2} * \binom{10}{1} * \binom{10}{7} * 1$
- We pick 2 white, 2 blue, 6 black. $\binom{10}{2} * \binom{10}{2} * \binom{10}{6} * 1$
- We pick 2 white, 3 blue, 5 black. $\binom{10}{2} * \binom{10}{3} * \binom{10}{5} * 1$

Thus, the answer is the sum of all the parts above.

$$\begin{aligned} & \binom{10}{0} * \binom{10}{0} * \binom{10}{10} * 1 + \binom{10}{0} * \binom{10}{1} * \binom{10}{9} * 1 + \binom{10}{0} * \binom{10}{2} * \binom{10}{8} * 1 + \binom{10}{0} * \binom{10}{3} * \binom{10}{7} * 1 \\ & + \binom{10}{1} * \binom{10}{0} * \binom{10}{9} * 1 + \binom{10}{1} * \binom{10}{1} * \binom{10}{8} * 1 + \binom{10}{1} * \binom{10}{2} * \binom{10}{7} * 1 + \binom{10}{1} * \binom{10}{3} * \binom{10}{6} * 1 \\ & + \binom{10}{2} * \binom{10}{0} * \binom{10}{8} * 1 + \binom{10}{2} * \binom{10}{1} * \binom{10}{7} * 1 + \binom{10}{2} * \binom{10}{2} * \binom{10}{6} * 1 + \binom{10}{2} * \binom{10}{3} * \binom{10}{5} * 1 \end{aligned}$$

Could you have written this as a summation with a general term?