Homework 3 (Solutions)

CS 206: Discrete Structures II
Summer 2018

Due: 5:00 PM EDT, Thursday, July 19th, 2018

Total points: 60

Name:

NetID:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided for each problem. This is very important! Even if you are using LaTeX, make sure your solutions fit into the given space.
2. Make sure you write your name and NetID in the space provided above.
3. After you are done writing, scan the sheets in the correct order into a PDF, and upload the PDF to Gradescope before the deadline mentioned above. No late submissions barring exceptional circumstances! **The submitted PDF should have all the 9 pages in the correct order.**
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [10 pts]
How many strings of length 10 are there that consist only of lower case letters (a-z) and contain at least one occurrence of every vowel?

We will use the inclusion-exclusion principle.
Let S be the set consisting of all length 10 strings which have size 10, and are made using lower case characters of the alphabet, i.e. a-z. Thus, $|S| = 26^{10}$

Let A_1 be the set of length 10 strings containing at least one 'a'
Let A_2 be the set of length 10 strings containing at least one 'e'
Let A_3 be the set of length 10 strings containing at least one 'i'
Let A_4 be the set of length 10 strings containing at least one 'o'
Let A_5 be the set of length 10 strings containing at least one 'u'
A is the set of interest - all strings containing at least one occurrence of each vowel.

Thus, $A = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$. Since we want to deal with unions instead of intersections, we will use the difference method:

$$|A| = |S| - |S/A| = |S| - |(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)^c| = |S| - |A_1^c \cup A_2^c \cup A_3 \cup^c A_4^c \cup A_5^c|,$$

where the last step follows from De Morgan's law.

Now, in order to apply the inclusion-exclusion principle, we want to compute the terms at every level of the formula:
Level 1 terms : $+\binom{5}{1}25^{10}$: we can use any letter except for the chosen vowel.
Level 2 terms : $-\binom{5}{2}24^{10}$: we can use any letter except for the two chosen vowels.
Level 3 terms : $+\binom{5}{3}23^{10}$: we can use any letter except for the three chosen vowels.
Level 4 terms : $-\binom{5}{4}22^{10}$: we can use any letter except for the four chosen vowels.
Level 5 terms : $+\binom{5}{5}21^{10}$: we can use any letter except for any of the vowels.

Thus, using the inclusion-exclusion formula, we get that:

$$|(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)^c| = \binom{5}{1}25^{10} - \binom{5}{2}24^{10} + \binom{5}{3}23^{10} - \binom{5}{4}22^{10} + \binom{5}{5}21^{10},$$

and thus the answer is

$$|A| = 26^{10} - [\binom{5}{1}25^{10} - \binom{5}{2}24^{10} + \binom{5}{3}23^{10} - \binom{5}{4}22^{10} + \binom{5}{5}21^{10}].$$

Problem 2. [20 pts]
There are 100 new teachers in the state of NJ who just finished their training and 21 high schools (say one from each of the 21 counties of NJ) in need of teachers. Of course, if the teachers are left to their own devices, they will all flock to the schools in the good counties, and so the Board of Education wants to make sure that every school gets at least one teacher assigned to it. The Board of Education wants to know in how many ways can they assign teachers to schools so that every school gets at least one new teacher assigned to it. Can you help them? You can leave you answer in the form of a sum if needed.

We want to distribute 100 teachers to 21 schools with each school getting at least one teacher assigned to it.

First attempt. On the surface, the answer is the number of solutions to:

$$x_1 + x_2 + \cdots + x_{21} = 100, \ (x_i \geq 1, \ \forall i = 1, 2, ..., 21)$$

But this is <u>INCORRECT</u> since we did not account for different assignments of the teachers. ("Alice goes to Middlesex County" and "Bob goes to Middlesex County" are two different assignments; though on both the Middlesex County got one teacher.)

Second attempt. We can first pick the first 21 teachers and assign them to the 21 schools (that is $\binom{100}{21} \times 21!$), and for the next 79 teachers, we can distribute them to any school, so the answer would be:

$$\binom{100}{21} \times 21! \times 21^{79}$$

This is also <u>INCORRECT</u>, since we are over-counting. (Why?)

Correct Answer. In an easier case, for three teachers an two schools, instead of counting how many ways we can distribute, we can assume that, first, $School_1$ gets no teachers, and then assume $School_2$ gets no teachers; and use the difference method.
We are going to do the same thing here.

Let A_i be a set representing all the assignments where $School_i$ gets no teachers. Then, A_i^c is all the assignments such that $School_i$ has at least one teacher. Therefore, the correct answer will be the size of $\left|\bigcap_{i=1}^{21} A_i^c\right|$

We can see that:

- The number of ways that *one* of the schools is empty (A_i), and the teachers are distributed arbitrarily to the other schools is: $\binom{21}{1} \times 20^{100}$.
- The number of ways that *two* of the schools are empty ($A_i \cap A_j$), and the teachers are distributed arbitrarily to the other schools is: $\binom{21}{2} \times 19^{100}$.

- The number of ways that *three* of the schools are empty ($A_i \cap A_j \cap A_k$), and the teachers are distributed arbitrarily to the other schools is: $\binom{21}{3} \times 18^{100}$.
- $\cdots$
- The number of ways that *twenty* of the schools are empty, and the teachers are distributed arbitrarily to the other schools is: $\binom{21}{20} \times 1^{100}$.

For example, $A_3 \cap A_4 \cap A_{19}$ represent all the possible assignments such that schools 3, 4 and 19 are empty.

This pattern is very similar to what we had in the inclusion-exclusion principle. If we calculate $|\bigcup A_i|$, which is all the possible assignments that some schools would be empty, then using the difference method we will get the desired answer.

We have:

$$\begin{aligned}\left|\bigcup_{i=1}^{21} A_i\right| &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots + |A_1 \cap \cdots \cap A_{21}| \\ &= \binom{21}{1} \times 20^{100} - \binom{21}{2} \times 19^{100} + \cdots + \binom{21}{20} \times 1^{100}\end{aligned}$$

And applying the difference method:

$$\left|\bigcap_{i=1}^{21} A_i^c\right| = |S| - \left|\bigcup_{i=1}^{21} A_i\right| \quad = 21^{100} - \binom{21}{1} \times 20^{100} + \binom{21}{2} \times 19^{100} - \cdots - \binom{21}{20} \times 1^{100}$$

Problem 3. [10 + 10 + 10 = 30 pts]
In one of the lectures we saw that when n is odd

$$\sum_{0\le k<\frac{n}{2}} \binom{n}{k} = 2^{n-1}.$$

We want to understand how large can the sum on the left hand side be when n is even. In this problem you will show (in three steps) that:

$$\sum_{0\le k<\frac{n}{2}} \binom{n}{k} \le \frac{2^{n-1}}{1+\frac{1}{n}}$$

1. Prove that, when n is even,

$$\sum_{0\le k<\frac{n}{2}} \binom{n}{k} = \frac{2^n - \binom{n}{n/2}}{2}.$$

   Recall that $\binom{n}{k} = \binom{n}{n-k}$

   We also know that the sum of the binomial coefficients is 2^n (Was done in class/recitations - please revise if you don't know this).

   i.e.

   $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n/2-1} + \binom{n}{n/2} + \binom{n}{n/2+1} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = 2^n$

   Observe that the first and the last terms are equal since $\binom{n}{0} = \binom{n}{n}$. Similarly, the k-th term from the beginning and that from the end in the above summation are equal.

   i.e. $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$ and so on.

   Thus, the summation becomes

   $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n/2-1} + \binom{n}{n/2} + \binom{n}{n/2-1} + \cdots + \binom{n}{2} + \binom{n}{1} + \binom{n}{0} = 2^n$

   Combine the similar terms together

   $\implies 2\times\binom{n}{0} + 2\times\binom{n}{1} + 2\times\binom{n}{2} + \cdots + 2\times\binom{n}{n/2-1} + \binom{n}{n/2} = 2^n$

   $\implies 2\times\sum_{0\le k<\frac{n}{2}} \binom{n}{k} + \binom{n}{n/2} = 2^n$

   Rearranging the terms on either sides of the equality gives you the desired result.

2. Prove that, when n is even,

$$\binom{n}{n/2} \ge \frac{2^n}{n+1}.$$

   [Hint: Try to use the fact that, as function of k, $\binom{n}{k}$ has its maxima at $n/2$ when n is even.]

   As a function of k, $\binom{n}{k}$ has its maxima at $n/2$ when n is even. This means that

$\binom{n}{k} \leq \binom{n}{n/2}$ for all k

Now, consider the summation from the previous part.

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n/2-1} + \binom{n}{n/2} + \binom{n}{n/2-1} + \cdots + \binom{n}{2} + \binom{n}{1} + \binom{n}{0} = 2^n$

Each of the terms on the left hand side of the equality is less than or equal to the maximum $\binom{n}{n/2}$

i.e. $\binom{n}{0} \leq \binom{n}{n/2}$,

$\binom{n}{1} \leq \binom{n}{n/2}$,

$\binom{n}{2} \leq \binom{n}{n/2}$,

$\vdots$

$\binom{n}{n/2-1} \leq \binom{n}{n/2}$,

$\binom{n}{n/2} \leq \binom{n}{n/2}$,

$\binom{n}{n/2+1} \leq \binom{n}{n/2}$,

$\vdots$

$\binom{n}{n-2} \leq \binom{n}{n/2}$,

$\binom{n}{n-1} \leq \binom{n}{n/2}$,

and $\binom{n}{n} \leq \binom{n}{n/2}$

Add all the terms.

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n/2-1} + \binom{n}{n/2} + \binom{n}{n/2-1} + \cdots + \binom{n}{2} + \binom{n}{1} + \binom{n}{0} \leq (n+1) \times \binom{n}{n/2}$

Since there are n+1 terms in all.

Note that the summation equals 2^n.

Thus, $2^n \leq (n+1) \times \binom{n}{n/2}$.

Rearranging gives the desired answer.

3. Combine the previous parts to prove the desired upper bound on $\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}$.

   From part a

   $2 \times \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} + \binom{n}{n/2} = 2^n$

   $\implies \binom{n}{n/2} = 2^n - 2 \times \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}$

   From part b $\binom{n}{n/2} \geq \frac{2^n}{n+1}$

   Thus, $2^n - 2 \times \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} \geq \frac{2^n}{n+1}$

   Rearranging,

   $2^n - \frac{2^n}{n+1} \geq 2 \times \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}$

Dividing both sides by 2

$\frac{2^n - \frac{2^n}{n+1}}{2} \geq \sum_{0 \leq k < \frac{n}{2}} \binom{n}{k}$

$\sum_{0 \leq k < \frac{n}{2}} \binom{n}{k} \leq 2^{n-1} - \frac{2^{n-1}}{n+1} = 2^{n-1} \left(\frac{n}{n+1}\right)$

Divide the numerator and denominator by n to get the desired answer.

P.S. : In questions like these where every part is dependent on the ones before it - you can earn credit for the latter parts if you solve them assuming the ones before it.