Lecture 3: Division rule, subset rule and binomial coefficients, permutations with repetitions and multinomial coefficients

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References: Relevant parts of chapter 15 of the Math for CS book.

1 Recap of the two product rules

Some things to keep in mind:

- If you are counting the number of possible sequences where the choices available at the different positions in the sequence are all independent, then one can try to use the product rule. e.g., what is $|\{0,1,2\}^n|$?
- If the choices are dependent, then you cannot use the simple product rule, and might have to do a nuanced analysis. (You can't always use the generalized product rule in this case!)
- One of the scenarios where the choices are dependent in which you CAN use the generalized product rule is when you want to find the number of ways of arranging n DISTINCT objects. The answer (using the generalized product rule) is $n!$.
- In general, if you have a scenario of dependent choices where:
  - there are P_1 choices for the first entry in the sequence,
  - P_2 choices for second entry *for every choice of the first entry*,
  - P_3 choices for second entry *for every choice of the first and second entry*,
  - .
  - .
  - .
  - P_n choices for the n^{th} entry *for every choice of the first $n-1$ entries.*

  Then the total number of sequences is $P_1 \cdot P_2 \cdot \ldots \cdot P_n$.
- If the n objects are not all distinct then you CANNOT use the generalized product rule.

Question . There are 10 walls in a row. You have 15 buckets of paint, each of a different color, and a single bucket is good for coloring exactly one wall. If you want every wall to be monochromatic (i.e., a single wall should be painted using exactly one color), then how many ways are there to color the walls?

Proof. There are 15 choices for the color of the first wall, 14 choices for the color of the second wall *for every choice of the color of the first wall*, 13 choices for the third wall for every choice of the colors of the first two walls, ..., and so on, and so using the generalized product rule, we have that the number of ways to color the walls is $15 \times 14 \times 13 \ldots \times 6$. Observe that this is equal to $15!/5!$. $\square$

In general, if you have n distinct objects in total, and want to choose and arrange r objects out of the n objects, then the number of ways of doing so is

$$n \times (n-1) \times (n-2) \times \ldots (n-(r-1)) = \frac{n!}{(n-r)!}.$$

The proof is basically a generalization of the solution to the above question: there are n choices for the first position, $n-1$ for the second for every choice of the first, and so on. We will see yet another proof in a later section.

2 Division rule/Generalized bijection method

Definition 1. *A function $f : A \to B$ is called a k-to-1 function if for every $b \in B$ there exist k distinct elements $a_1, \ldots, a_k$ such that all of them are mapped to b by f, i.e. $f(a_1) = f(a_2) = \ldots = f(a_k) = b$.*

If a function is k-to-1 with $k = 1$, then it is a bijection!

Consider the function $f : \mathbb{Z} \setminus 0 \to \mathbb{N} \setminus \{0\}$, i.e. the domain is all integers except 0 and the codomain is all natural numbers except 0, such that $f(x) = abs(x)$ where $abs(x)$ denotes the absolute value of x. Then f is a 2-to-1 function: for every non-zero natural number n, both n and $-n$ are mapped to n by f!

Lemma 2. *If A and B are finite sets, and $f : A \to B$ is a k-to-1 map, then $|A| = k \cdot |B|$.*

Proof. Let us assume that $|B| = m$ and $B = \{b_1, \ldots, b_m\}$. Note that by the definition of a function, every element in A must be map to exactly one element in B. This means we can partition the elements of A into m parts $A_1, A_2, \ldots, A_m$ where the part A_i is the set of all elements in A that get mapped to b_i by f. By the partition rule, we have that $|A| = \sum_{i=1}^{n} |A_i|$. What are the sizes of the A_is? We know that exactly k elements get mapped to every element in B, and this means that for every i, $|A_i| = k$. This means that $|A| = k \cdot m = k \cdot |B|$. This finishes the proof. $\square$

This suggests a new method for finding the size of set B that might be hard to count directly. Find an easy to count set A such that there is a k-to-1 function from A to B for some positive integer k, and then using the above lemma, we have that $|B| = |A|/k$. This method is called the division rule or the generalized bijection method.

Question . Suppose five knights are to be seated around a round table. How many distint ways of seating them are there? Two arrangements are considered to be the same if every knight has the same two knights seated to next to them in both the arrangements.

Proof. Let B be the set of ways of seating the 5 knights around the table. Let A be the number of ways of making the 5 knights stand in a line, one behind the other. We want to find $|B|$, but instead we will find $|A|$, set up a k-to-1 function between A and B, and then use the division rule to find $|B|$.

First note that $|A| = 5!$ since it's just the number of ways of permuting 5 distinct objects. There is a natural way of converting an arrangement of the 5 knights in a line to a seating plan around the table: if the knights are standing in the order $(k_1, k_2, k_3, k_4, k_5)$, then we seat knight k_1 next to knights k_2 and k_5, knight k_2 next to knights k_3 and k_1, knight k_3 next to k_4 and k_2, knight k_4 next to k_5 and k_4, and knight k_5 next to $k1$ and $k4$. Let's call this mapping from permutations to seating arrangements $f : A \to B$.
Notice that all of the following permutations will get mapped to exactly the same seating arrangement: $(k_1, k_2, k_3, k_4, k_5)$, $(k_2, k_3, k_4, k_5, k_1)$, $(k_3, k_4, k_5, k_1, k_2)$, $(k_4, k_5, k_1, k_2, k_3)$, and $(k_5, k_1, k_2, k_3, k_4)$. In general, you can convince yourself that for every seating arrangement in B, there will be exactly 5 permutations in A that will map into it. This means that f is a 5-to-1 map and thus, using the division rule, we have that

$$|B| = \frac{|A|}{k} = 4!.$$

□

3 Subset rule and counting subsets of a set

The basic question we want to answer is as follows: suppose I have the set $\{1, 2, \ldots, n\}$, then how many ways are there to choose k numbers from this set, or, in other words, how many subsets of the given set are there that have exactly k elements? Note that it doesn't matter what order we choose the k objects in. We can assume that we are selecting all k at once so there is no inherent order in the process. Before we state the general formula, let's warm up with an example.

Question . Let $S = \{1, 2, 3, 4, 5\}$. How many subsets of S are there that have exactly 3 elements. In other words, in how many ways can you choose 3 numbers from S?

Proof. Let B be the set of subsets of S that have exactly 3 elements. Let A be the set of permutations of the elements in S. As in the last question, we will first find $|A|$, set up a function k-to-1 function f from A to B, and then find $|B|$ using the division rule.

Once again, $|A|$ is easy to find, and it is exactly 5!. Let us now give a map/function $f : A \to B$ that converts a permutation of the numbers in S to a subset of S of size 3. Here is what the map does:

- Let's say we have a permutation $(a_1, a_2, a_3, a_4, a_5)$.
- f first chops off the last 2 elements from the permutation, so we are left with (a_1, a_2, a_3).
- f then converts the remaining sequence of three elements into a set of size 3 by getting rid of the order: (a_1, a_2, a_3) becomes $\{a_1, a_2, a_3\}$.

For example, $f((1, 2, 4, 5, 3)) = \{1, 2, 4\}$. Notice that lots of different permutations will end up mapping to the same set. For example, we also have that $f(\{4, 1, 2, 3, 5\}) = \{1, 2, 4\}$, same as what we got before. The question is, how many permutations get mapped to a fixed size 3 subset in B? Let's try to compute the number of permutations that get mapped to $\{1, 2, 4\}$. Any permutation that is mapped by f to $\{1, 2, 4\}$ must have some arrangement of the numbers $1, 2, 4$ in its first three positions, and the last two positions have some arrangement of the numbers 3 and 5. Thus, the number of possible options for the first three positions are 3!, and those for the last positions are 2!, and so $3! \times 2!$ permutations get mapped to $\{1, 2, 4\}$ by f.

Note that in the argument above, there was nothing special about $\{1, 2, 4\}$ that we used, and the same argument would work for any size three subset. Thus, for every set in B, there are 2!3! permutations mapping to it, and thus f is a k-to-1 map with $k = 3!2!$. This means that

$$|B| = \frac{|A|}{k} = \frac{5!}{2!3!}.$$

□

We can use the exact same idea for deriving a general formula for the number of size k subsets of $S = \{1, \ldots, n\}$:

As before, let B be the set of all subsets of S of size k, let A be all permutations of the elements of S, and let $f : A \to B$ be a function that converts a permutation into a subset of size k in the following manner:

- Let's say we have a permutation $(a_1, \ldots, a_n)$ of S.
- f first chops off the last $n - k$ elements from the permutation, so we are left with $(a_1, \ldots, a_k)$.
- f then converts the remaining sequence of k elements into a set of size k by getting rid of the order: $(a_1, \ldots, a_k)$ becomes $\{a_1, \ldots, a_k\}$.

Now, using the same ideas as in the above proof, you can convince yourself that there are exactly $k!(n - k)!$ permutations of S that get mapped to any given size k subset of S, and thus f is a k-to-1 map. Using the division rule, and the fact that $|A| = n!$, we get that

$$|B| = \frac{n!}{k!(n-k)!}.$$

Note that there is nothing special about the objects being considered being numbers: the entire argument will go through if we were choosing, say, k students from a group of n students, or k candy bars from a set of n distinct candy bars (say each of a different brand), etc. In general:

Lemma 3. *The number of ways of choosing k objects (not giving any regard to the order) from n distinct objects is $\frac{n!}{k!(n-k)!}$.*

Definition 4. $\binom{n}{k}$ *(pronounced as "n choose k") is a **binomial coefficient** and represents the number of ways of choosing k objects from n distinct elements (without regard for order), and is equal in value to $\frac{n!}{k!(n-k)!}$.*

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ are all binomial coefficients (why didn't we also mention $\binom{n}{n+1}$?) It's not hard to see why the following is true about them (left as an exercise):

Claim 5. $\binom{n}{k} = \binom{n}{n-k}$

Question . How many binary strings of length n are there that contain exactly k ones?

Proof. To specify a binary string of length n with exactly k zeros, all I need to tell you is what positions are the k ones present at. So the number of binary strings of length n with k ones is exactly equal to the number of ways of choosing k positions from n positions, which is $\binom{n}{k}$. □

Question . How many ways are there to draw 5 cards from a deck of 52 cards so that there are at least 3 aces in the selection?

Proof. Here is a wrong way to do it. My selection must definitely have 3 aces. Let me first pick three aces that will definitely be part of my selection. The number of ways of doing that are $\binom{4}{3}$. Once I have picked three aces, I can now freely pick whatever cards I want from the remaining 49 cards, and so the total is $\binom{4}{3}\binom{49}{2}$. Why is this wrong?

To do this correctly, it's better to use the partition method: consider two cases, one when there are exactly 3 aces in the selection, and the other when all four aces are chosen. For the first cases, there are $\binom{4}{3}$ ways of choosing 3 aces from all four aces, and $\binom{48}{2}$ ways of choosing the remaining cards, and so the total is $\binom{4}{3}\binom{48}{2}$. For the second case, there is only one way to choose all four aces, are there are $\binom{48}{1} = 48$ ways of choosing the one card from the non-ace cards. The the final answer is $48 + \binom{4}{3}\binom{48}{2}$. □

3.1 Choose and permute

Recall we computed the number of ways of choosing and arranging r objects out of n distinct objects. We found it to be $\frac{n!}{(n-r)!}$, and we derived it using the generalized product rule. Here is another way to understand how to get to that number. If you have n distinct objects, and want to arrange r objects chosen from these n objects, you can think of it in two steps: first choose r objects out of n, and there are $\binom{n}{r}$ ways of doing so, and having selected the r objects you can now arrange them, and there are $r!$ ways of doing that. So, the total number is

$$\binom{n}{r} \cdot r! = \frac{n!}{r!(n-r)!} \cdot r! = \frac{n!}{(n-r)!}.$$

4 Permutations with repetitions

Question . How many distinct permutations of the word SYSTEMS are there?

Proof. Notice that the answer isn't simply 7!. The number of ways of permuting n *distinct* objects is $n!$, but we have to be more careful when we have repetitions. In this case, we have a repeat: there are three S!. We can use the division rule to solve this problem. Let B be the set of distinct permutations of the $SYSTEMS$. Let A be the set of permutations of the string $S_1YS_2TEMS_3$ (here the three Ss are distinguishable from each other). We know that $|A| = 7!$ because we are simply arranging 7 distinct objects in all possible ways.

Let us now give a map $f : A \to B$ that is k-to-1 for some value of k. We can then find $|B|$ using the division rule (so f maps a permutation of $S_1YS_2TEMS_3$ to an permutation of the letters of $SYSTEMS$). Here is how f is defined: given a permutation of $S_1YS_2TEMS_3$, f simply drops the subscripts of the three Ss in the permutation to get a permutation of $SYSTEMS$. e.g., $S_1TMS_3EYS_2$ is mapped to $STMSEYS$ by f. It's easy to see that exactly 3! different permutations of $S_1YS_2TEMS_3$ in A will map to any given permutation of $SYSTEMS$ in B — the order in which S_1, S_2, S_3 occur in a permutation does not make a difference! For example, all six of the following will map to $STMSEYS$:

- $S_1TMS_3EYS_2$
- $S_1TMS_2EYS_3$
- $S_2TMS_3EYS_1$
- $S_2TMS_1EYS_3$
- $S_3TMS_1EYS_2$
- $S_3TMS_2EYS_1$

Thus, f is a k-to-1 map with $k = 3!$, and so

$$|B| = \frac{|A|}{k} = \frac{7!}{3!}.$$

□

One can generalize this idea to derive an expression for counting permutations when there are multiple repetitions (we will omit the proof here, but the proof is a pretty straightforward extension of what we did above. Also, we will see a different and a cleaner way to derive the same expression.)

Lemma 6. *If there are n objects in total such that there are r_1 objects of type 1, r_2 objects of type 2, ..., r_k objects of type k, and $r_1 + r_2 + \ldots + r_k = n$, then the number of distinct permutations of these n objects is given by*

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}.$$

$\binom{n}{r_1, r_2, \ldots, r_k}$ *is called a* ***multinomial coefficient****.*

Notice that binomial coefficients are just a special case of multinomial coefficients — they are exactly the multinomial coefficients in the case when $k = 2$ and there are r objects of type 1 and $n - r$ objects of 2.

Using this, we can now compute the number of distinct permutations of words such as $ARRANGE$. There are 2 As, 2 Rs, and one copy each of N, G, and E, and so the total number is $\frac{7!}{2!2!}$.