Homework 1 Solutions

CS 206: Discrete Structures II
Summer 2018

Due: 5:00 PM EDT, Thursday, July 5th 2018

Total points: 30 (10 for easy + 20 for hard)

Name:

NetID:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name and NetID in the space provided above.
3. After you are done writing, scan the sheets in the correct order into a PDF, and upload the PDF to Gradescope before the deadline mentioned above. No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.
5. As for grading, after the deadline, the grader will randomly pick one easy and one hard problem to grade. Needless to say the same problems will be graded in every student's submission. For e.g., the grader may randomly decide to grade the first easy problem and the second hard problem, and so all the submissions will be evaluated on the basis of those two problems.

Part I: The easy stuff

Problem 1. [10 pts]
Let S be defined as follows:

$$S = \{(A, B) \mid A \subseteq \{1, 2, \ldots, n\}, B \subseteq \{1, 2, \ldots, n\},\ |A \cap B| \geq 1\}.$$

What is the cardinality of S? Justify your answer.

Proof. Since $|S|$ seems to be tricky to compute in a direct manner, we will appeal to the difference method. Let $T = \{(A, B) \mid A \subseteq \{1, \ldots, n\}, B \subseteq \{1, \ldots, n\}\}$. First note that $S \subset T$.

It turns out that $|T|$ is easy to compute: T contains all length 2 sequences (A, B), and there are 2^n choices for A, and 2^n choices of B, and these choices are independent. Hence, using the product rule, we have that $|T| = 2^{2n} = 4^n$.

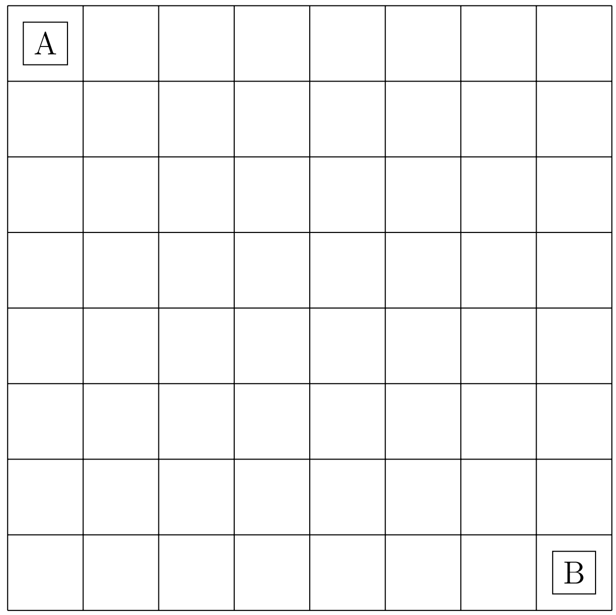
We will now compute $|T \setminus S|$. Notice that

$$T \setminus S = \{(A, B) \mid A \subseteq \{1, 2, \ldots, n\}, B \subseteq \{1, 2, \ldots, n\},\ A \cap B = \emptyset\}.$$

We can think of it the following way: we have n elements, i.e. $\{1, 2, \ldots, n\}$, that we want to distribute between A and B, and it is possible that we don't give some of these elements to either A or B. Thus, for each of the n elements, we have three choices: put the element in set A, put the element in set B, don't put the element in either of A or B. This means that the total number of ways of distributing these n elements is 3^n, and so $|T \setminus S| = 3^n$.

Using the difference method, we have that $|S| = |T| - |T \setminus S| = 4^n - 3^n$. This finishes the proof. $\square$

Problem 2. [10 pts]
A binary string is called *balanced* if it has equal number of ones and zeros. Let $\mathcal{S}$ be the set of balanced binary strings of length 14, and let $\mathcal{P}$ be the set of paths from A to B in the grid below that use only downward and rightward steps (i.e., a path can never go up, left, or diagonal; only down or right). Give a bijection between $\mathcal{P}$ and $\mathcal{S}$. Justify your answer.



Proof. Notice that any path from A to B that is only allowed to go down or right in every step must consist of exactly 7 steps down and 7 steps to the right, in some order. Thus, a path can be represented by a string of length 14 consisting of Ds and Rs. For example, a possible path from A to B is $DRDRDRDRDRDRDR$, which basically means *"Go one step down, then one step right, then one step down,…"*. Thus, the set $\mathcal{P}$ consists of strings of length 14 made up from the letters D and R.

There is a simple bijection between $\mathcal{P}$ and $\mathcal{S}$, namely, given a string in $\mathcal{P}$, replace the Ds by ones and the Rs by zeros. It is easy to see that two distinct paths in $\mathcal{P}$ will map to distinct strings in $\mathcal{S}$ (this shows injectivity). To show surjectivity, every string in $\mathcal{S}$, has a path in $\mathcal{P}$ mapping to it — if x is a string in $\mathcal{S}$, then the path mapping to it is the one that can be obtained by replacing the ones in x by Ds and the zeros by Rs. This completes the proof. $\square$

Part II: The hard stuff

Problem 3. [20 pts]
Let $a_1, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$. We say that this sequence is a *zig-zag* sequence if, for every $2 \leq i \leq n$, either:

- a_i is greater than all the numbers occurring before it in the sequence, i.e. greater than each of $a_{i-1}, a_{i-2}, \ldots, a_1$, or
- a_i is less than all the numbers occurring before it in the sequence, i.e. less than each of $a_{i-1}, a_{i-2}, \ldots, a_1$.

How many of the $n!$ permutations of $1, 2, \ldots, n$ are zig-zag sequences?

Proof. Notice that any zig-zag sequence made up of n distinct numbers $\{a_1, \ldots, a_n\}$ must either end in $min_i\ a_i$ or $max_i\ a_i$. To see this, assume for contradiction that there was a zig-zag sequence that ended in some number a that is neither the maximum nor the minimum of the set $\{a_1, \ldots, a_n\}$. Since it is a zig-zag sequence, a necessary condition is that either of the following two must be true:

- all the numbers that occur before a (the last element in the sequence) must ALL be less than a, or
- all the number that occur before a must ALL be greater than a.

Notice that the numbers $min_i\ a_i$ and $max_i\ a_i$ must both occur somewhere in this sequence (since we are considering permutations of $a_1, a_2, \ldots, a_n$), and they both will occur before a in the sequence since a appears last. But this means that neither of the above two conditions can be satisfied since there is the element $min_i\ a_i$ that occurs before a and is less than a, and there is also the element $max_i\ a_i$ that occurs before a in the sequence but is more than a in value. This means that the sequence cannot be zig-zag, and this is a contradiction to our assumption.

Let $T(n)$ be the number of zig-zag sequences that can be formed using n distinct numbers $\{a_1, \ldots, a_n\}$. Note that $T(2)$ is 2. Based on the above observations, the set of zig-zag sequences made up of n distinct numbers $\{a_1, \ldots, a_n\}$ can be partitioned into two types: ones that end in $min_i\ a_i$ and the ones that end in $max_i\ a_i$. The number of zig-zag sequences of the former type is exactly equal to the number of zig-zag sequences of the set $\{a_1, \ldots, a_n\} \setminus \{min_i\ a_i\}$, and the number of the latter type is exactly equal to the number of zig-zag sequences of the set $\{a_1, \ldots, a_n\} \setminus \{max_i\ a_i\}$. Thus, using induction, $T(n) = 2 \cdot T(n-1)$. This together with the fact that $T(2) = 2$ implies that $T(n) = 2^{n-1}$. This completes the proof. $\square$

Problem 4. [20 pts]
Let $\mathbb{Z}_m$ denote the set $\{0, 1, \ldots, m-1\}$. Then $\mathbb{Z}_m^n$ is the set of all sequences of length n where every element in the sequence comes from the set $\mathbb{Z}_m$. For example, for $m = 3$ and $n = 2$, $\mathbb{Z}_3^2 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$. Let $f : \mathbb{Z}_m^n \to \mathbb{Z}_m$ be the function defined as:

$$f((b_1, b_2, \ldots, b_n)) = \left(\sum_{i=1}^{n} b_i\right) \mod m.$$

What fraction of the points in the domain of f map to 1? Write your answer in terms of m and n. Justify it.

To solve this problem, we need to find the number of sequences $(b_1, \ldots, b_n)$ in $\mathbb{Z}_m^n$ such that $\sum_{i=1}^n b_i \mod m = 1$. Notice that if we were simply counting the total number of sequences in $\mathbb{Z}_m^n$ without this additional constraint, there would be m^n such sequences: there are m choices (we can use one of the following values: 0, 1, …, $m-1$) for the first element in the sequence (i.e. b_1), m choices for the second element, and so on, and all these choices are independent, and thus by the product rule, we would get m^n. Unfortunately, with the additional constraint that $\sum_{i=1}^n b_i \mod m = 1$, the choices are no longer independent!

Notice that no matter what $b_1, b_2, \ldots, b_{n-1}$ are chosen to be, there is always a unique choice for b_n so that $\sum_{i=1}^n b_i \mod m = 1$, i.e. b_n must be equal to $\left(1 - \left(\sum_{i=1}^{n-1} b_i\right)\right) \mod m$. This means the first $n-1$ choices can be made independently, and there are m^{n-1} ways to do so (using the product rule). As soon as we fix our choices for the first $n-1$ elements in the sequence, there is EXACTLY one choice left for b_n. This means that the total number of sequences in $\mathbb{Z}_m^n$ that satisfy $\sum_{i=1}^n b_i \mod m = 1$ is m^{n-1}. Since $|\mathbb{Z}_m^n| = m^n$, the fraction of points that satisfy the desired condition is $m^{n-1}/m^n = 1/m$. This finishes the proof.