Lecture 6: General inclusion-exclusion

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Rutgers University
Instructor: Abhishek Bhrushundi

References: Relevant parts of chapter 15 of the Math for CS book.

1 General inclusion-exclusion

Last time we saw the inclusion-exclusion based formula for the cardinality of the union of three sets. What happens when there are four or more sets? We will derive the formula for union of four sets using the formula for three sets, and then I will state (without proof) the general inclusion-exclusion formula.

1.1 Inclusion-exclusion for union of four sets

Suppose we want to find $|A \cup B \cup C \cup D|$. Let's set $X = C \cup D$. Then, using the inclusion-exclusion formula for three sets we have

$$\begin{aligned}
|A \cup B \cup X| &= |A| + |B| + |X| - |A \cap B| - |B \cap X| - |A \cap X| - |A \cap B \cap X| \\
&= |A| + |B| + |C \cup D| - |A \cap B| - |B \cap (C \cup D)| - |A \cap (C \cup D)| - |A \cap B \cap (C \cup D)| \\
&= |A| + |B| + |C \cup D| - |A \cap B| - |(B \cap C) \cup (B \cap D)| \\
&- |(A \cap C) \cup (A \cap D)| - |(A \cap B \cap C) \cup (A \cap B \cap D)| \qquad (1)
\end{aligned}$$

The last three terms involve union of two sets and we know the formula for that. Let's compute each of them individually and put them back in Equation 1.

- $|(B \cap C) \cup (B \cap D)| = |B \cap C| + |B \cap D| - |B \cap C \cap B \cap D| = |B \cap C| + |B \cap D| - |B \cap C \cap D|$.
- Similarly,
$$|(A \cap C) \cup (A \cap D)| = |A \cap C| + |A \cap D| - |A \cap C \cap D|.$$
- $|(A \cap B \cap C) \cup (A \cap B \cap D)| = |A \cap B \cap C| + |A \cap B \cap D| - |A \cap B \cap C \cap A \cap B \cap D|$. This is the same as:
$$|(A \cap B \cap C) \cup (A \cap B \cap D)| = |A \cap B \cap C| + |A \cap B \cap D| - |A \cap B \cap C \cap D|.$$

Substituting these back into Equation 1 we get,

$$\begin{aligned}
|A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \text{ (Level I terms, } \binom{4}{1} \text{ of them)} \\
&- (|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|) \text{ (Level II terms, } \binom{4}{2} \text{ of them)} \\
&+ (|A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D| + |A \cap C \cap D|) \text{ (Level III terms, } \binom{4}{3} \text{ of them)} \\
&- |A \cap B \cap C \cap D| \text{ (Level IV terms, } \binom{4}{4} = 1 \text{ of them).}
\end{aligned}$$

Let's see a problem that uses inclusion-exclusion formula for four sets:

Question . Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ be two finite sets of size 4 each. A function $f : A \to B$ is said to have a fixed point if for some $i \in \{1, 2, 3, 4\}$, $f(a_i) = b_i$. How many bijective functions mapping A to B are there that have no fixed points?

Proof. As discussed in the previous lecture, we first want to identify the objects we are dealing with (numbers, subsets, functions, etc.). In this case we are asked "how many bijective functions mapping to A to B are there..", and this means the objects we are dealing with is the set of bijective functions from A to B. Let S be the set of all bijective functions from A to B. Recall that $|S| = 4!$.

Typically, in problems where we can use inclusion-exclusion, we are asked to find the number of objects (among the set of all objects we identified in first step above) that satisfy some given conditions. What are the conditions in this case? We want to find the number of bijective functions that do not have fixed points. It's always better to break down the conditions into further smaller conditions. In this case, note that not having a fixed point can be broken down into four conditions:

- $f(a_1) \neq b_1$
- $f(a_2) \neq b_2$
- $f(a_3) \neq b_3$
- $f(a_4) \neq b_4$

Once we have identified the conditions, we define a set for every condition. For $1 \leq i \leq 4$, define A_i be the set of all bijective functions for which $f(a_i) \neq b_i$. Having defined the sets, we can now translate the question being asked into the language of set theory: it is easy to see that the number of bijective functions that have no fixed points is $|A_1 \cap A_2 \cap A_3 \cap A_4|$.

Notice that we are dealing with the intersection of four sets whereas inclusion-exclusion helps us deal with the union of sets. The next step is to use the difference method to go from the intersection to union. Note that

$$|A_1 \cap \ldots \cap A_4| = |S| - |(A_1 \cap \ldots \cap A_4)^c|,$$

where $(A_1 \cap \ldots \cap A_4)^c$ is the complement of $(A_1 \cap \ldots \cap A_4)$ inside S. Using De Morgan's law, we know that $(A_1 \cap \ldots \cap A_4)^c = A_1^c \cup A_2^c \cup A_3^c \cup A_4^c$. We are now in good position: we have to compute the cardinality of the union of four sets and can use inclusion-exclusion to do so.

Notice that A_i^c (for $1 \le i \le 4$) is the set of all bijective functions for which $f(a_i) = b_i$. You can convince yourself that $|A_i| = 3!$ (that's how many ways there are to map the remaining elements in A with the remaining elements in B while making sure the function is bijective). For $i \neq j$, $A_i^c \cap A_j^c$ is the number of bijective functions for which $f(a_i) = b_i$ and $f(a_j) = b_j$. Again, it's easy to count that $|A_i^c \cap A_j^c| = 2!$. Similarly, one can show that for three distinct indices i, j, k, $|A_i^c \cap A_j^c \cap A_k^c| = 1!$, and $|A_1^a \cap A_2^c \cap A_3^c \cap A_4^c| = 1$.

We can now apply the inclusion-exclusion formula for four sets to compute $|A_1^c \cup \ldots \cup A_4^c|$. All the level I terms in the formula are equal to 4! and there are $\binom{4}{1}$ level I terms, all the level II terms are equal to 3! and there are $\binom{4}{2}$ of them, all the level III terms are equal to 1! and there are $\binom{4}{3}$ of them, and finally there is only one level IV term and it's equal to 1. If we combine all these terms with the correct signs (remember the alternation between + and − as we go from level to level), we get:

$$|A_1^c \cup \ldots \cup A_4^c| = 3! \cdot \binom{4}{1} - 2! \cdot \binom{4}{2} + 1!\binom{4}{3} - \binom{4}{4}.$$

To finish the problem, we use this to find $|A_1 \cap A_2 \cap A_3 \cap A_4|$:

$$|A_1 \cap \ldots \cap A_4| = 4! \text{ (that's } |S|) \; - \left(3! \cdot \binom{4}{1} - 2! \cdot \binom{4}{2} + 1!\binom{4}{3} - \binom{4}{4}\right).$$

$\square$

1.2 General inclusion-exclusion formula

Suppose now we want to find $|A_1 \cup \ldots \cup A_n|$. Using induction and following the same ideas we used for the union of three and four sets, one can show that (observe the alternation of signs as we go from one level to the next!)

$$\begin{aligned}
|A_1 \cup A_2 \ldots \cup A_n| = {} & |A_1| + \ldots + |A_n| \text{ (Level 1 terms, } \binom{n}{1} \text{ of them)} \\
& - (|A_1 \cap A_2| + \ldots + (\text{all pairs } A_i \cap A_j) + \ldots + |A_{n-1} \cap A_n|) \text{ (Level 2 terms, } \binom{n}{2} \text{ of them)} \\
& + (|A_1 \cap A_2 \cap A_3| + (\text{all three-wise intersections } |A_i \cap A_j \cap A_k|)\ ) \text{ (Level 3 terms, } \binom{n}{3} \text{ of them} \\
& \ldots \\
& (-1)^{\ell-1} ((\text{all } \ell\text{-wise intersections } |A_{i_1} \cap \ldots \cap A_{i_\ell}|)\ ) \text{ (Level } \ell \text{ terms, } \binom{n}{\ell} \text{ of them)} \\
& \ldots \\
& (-1)^{n-1}|A_1 \cap A_2 \ldots \cap A_n| \text{ (Level } n \text{ term, } \binom{n}{n} = 1 \text{ of them).}
\end{aligned}$$

Note that the number of terms at level ℓ is $\binom{n}{\ell}$ — it's all the ℓ-wise intersections of the n sets, and the sign at level ℓ is $(-1)^{\ell-1}$.

In the in-class exams, you will typically never need more than 4 or 5 levels of inclusion-exclusion, however your third HW has a problem that will need you to do 21 levels. The trick is to try to get a general expression for level ℓ and then sum over all the expressions for the n different levels with the appropriate sign. For example, in the problem we discussed in the previous subsection, the expression for level ℓ (for $1 \leq \ell \leq 4$) is $(4-\ell)! \cdot \binom{4}{\ell}$ (we assume that $0! = 1$), and thus we can just sum over all these expression with the right sign (the sign of level ℓ is $(-1)^{\ell-1}$). Thus, we get:

$$|A_1^c \cup A_2^c \cup A_3^c \cup A_4^c| = \sum_{\ell=1}^{4} (-1)^{\ell-1} (4-\ell)! \cdot \binom{4}{\ell}.$$