Quiz 2 (Solutions)

CS 206: Discrete Structures II
Summer 2018

July 26, 2018

Total points: 40 + 10 (extra credit)

Duration: ∞

Name:

NetID:

INSTRUCTIONS:

1. There are 4 problems in all. Each problem is worth 10 points. The exception is Problem 3: it has two parts, each worth 10 points, but the second part counts as extra credit.
2. Make sure you write your solutions ONLY in the space provided below each problem. There is plenty of space for each problem. You can ask for scrap paper for scratchwork.
3. You can leave your answers is terms of binomial coefficients, powers of numbers, and/or factorials.
4. You are allowed to bring a single sheet of paper as a cheatsheet with notes (possibly) written on both the sides.
5. Make sure you write your name and NetID in the space provided above.
6. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the quiz, and might even be reported to the authorities!

Problem 1. [10 pts]
Suppose A and B are independent events of a sample space Ω, and that $P(A) = 1/3$ and $P(B) = 2/5$. What is $P(A \cup B)$? Justify your answer and show the main steps of your solution.

Warning: Independent is not the same as disjoint or mutually exclusive!

Proof. Since we don't know if A and B are disjoint/mutually exclusive, all we can say is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

We do know that A and B are independent, so $P(A \cap B) = P(A)P(B) = 2/15$. This implies that

$$P(A \cup B) = 1/3 + 2/5 - 2/15 = 9/15.$$

$\square$

Problem 2. [10 pts]
Three fair dice colored red, blue, and green are rolled. We are interested in the probability that exactly two of the dice roll the same number given that the sum of the numbers that show up on the dice is equal to 10. What will you choose the outcomes and sample space to be? What is the size of the sample space? Which distribution will you use? What events do you need to define to solve this problem, and what are their sizes? Compute the desired probability. Show the main steps of your solution.

Proof. The sample space Ω is all possible sequences of length 3 where every entry is an integer between 1 and 6 inclusive, such that the first entry represents the number rolled by the red dice, the second by the blue dice, and the third by the green dice. So $|\Omega| = 6^3 = 216$. Since the dice are fair, we will use the uniform distribution.

Let A be the event of seeing that exactly two dice roll the same number, and let B be the event that the sum of the numbers rolled by the three dice is 10. We want to find:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{|A \cup B|}{|B|},$$

the last equation follows from the property of the uniform distribution.

A consists of all sequences where exactly two entries are same. There are $\binom{6}{1}$ of deciding what the two equal entries should be, and then for the third entry there are $\binom{5}{1}$ ways of choosing a number (note the third entry cannot be equal to the other two entries – we want exactly two entries to be the same). After we have chosen the number, there are $\frac{3!}{2!}$ ways of permuting them to get different possible sequences using the permutations with repetitions formula (we have one repetition here, hence we divide by 2!). Thus,

$$|A| = \binom{6}{1}\binom{5}{1}\frac{3!}{2!} = .90$$

We are intersted in finding out $A| \cap B|$, i.e. all sequences in A that sum to 10. We will do this by explicit enumeration:

$$|A \cap B| = |\{(2,2,6),(2,6,2),(6,2,2),(3,3,4),(3,4,3),(4,3,3),(4,4,2),(4,2,4),(2,4,4)\}| = 9.$$

To compute $|B|$, we basically want to count the number of solutions to $x_1 + x_2 + x_3 = 10$, where x_1 is the number rolled by the red dice, x_2 the one rolled by blue dice, and x_3 the one rolled by green dice. The constraints are $1 \le x_1, x_2, x_3 \le 6$. We first satisfy minimum requirements, and so we give 1 to each of x_1, x_2, x_3. So the new system of equation becomes, $y_1 + y_2 + y_3 = 7$, subject to $0 \le y_1, y_2, y_3 \le 5$ (It's 5 and not 6 because we have already given 1 to each variable and the original constraints said that no variable can be more than 6, so after getting one each, we don't want any variable to get more than 5 of the remaining 7.) The total number of solutions to this system if we ignore the upper bounds is $\binom{7+2}{2} = \binom{9}{2}$. Obviously, this count also includes solutions where some variable is 6 or 7 (note that at most one variable can be 6 or 7). The solutions where one of the variables is either 6 or 7 are

$\{(0,0,7),(0,7,0),(7,0,0),(1,0,6),(1,6,0),(0,1,6),(0,6,1),(6,1,0),(6,0,1)\}$, and the total number of thse solutions is thus 9. This means the solutions where none of the variables are 6 or 7 is $\binom{9}{2} - 9 = 36 - 9 = 27$.

Thus,

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{9}{27} = 1/3.$$

$\square$

More space for Problem 2:

Problem 3. [10 pts + 10 pts (extra credit)]
100 fair dice are rolled.

1. What is the probability that the product of the numbers that show up on the dice is divisible by 2?

2. **Extra credit (10 pts)** What is the probability that the product of the numbers that show up is divisible by 30?

Show the main steps of your solution.
Hint for part 1: What is the probability that the product is NOT divisible by 2?
Hint for part 2: Factorize 30. Use inclusion-exclusion.

Proof. 1. $P(\text{product is even}) = 1 - P(\text{product is odd})$. The product of the 100 numbers that show up on the dice is odd if and only if every dice rolls an odd number. The probability that a single dice rolls an odd number is $3/6 = 1/2$ (why?). Since all the dice are independent, the probability that all of the roll odd numbers is $(1/2)^{100}$. This implies that

$$P(\text{product is even}) = 1 - P(\text{product is odd}) = 1 - (1/2)^{100}.$$

2. Let E_2 be the event that at least one of the 100 dice rolls results in a multiple of 2, let E_3 be the event that at least one of the dice rolls results in multiple of 3, and E_5 be the event that at least one of the dice rolls results in a multiple of 5. We are interested in the event E that the product of all the numbers is divisble by 30. Observe that $E = E_2 \cap E_3 \cap E_5$. Sadly, E_2, E_3, E_5 are not independent (why?). However, we do know that

$$P(E) = P(E_2 \cap E_3 \cap E_5) = 1 - P((E_2 \cap E_3 \cap E_5)^c) = 1 - P(E_2^c \cup E_3^c \cup E_5^c).$$

So it suffices to compute $P(E_2^c \cup E_3^c \cup E_5^c)$. This is the probability of a union of three events, which are not disjoint (why?), and so we resort to inclusion-exclusion:

$$P(E_2^c \cup E_3^c \cup E_5^c) = P(E_2^c) + P(E_3^c) + P(E_5^c) - P(E_2^c \cap E_3^c) - P(E_2^c \cup E_5^c) - P(E_3^c \cap E_5^c) + P(E_2^c \cap E_3^c \cap E_5^c).$$

Let's see how we can compute each of these terms:

- $P(E_2^c)$ is the probabiltiy of that none of the dice rolls result in a mutiple of 2. From part 1 we know it's $(1/2)^{100}$. $P(E_3^c)$ is the probability that none of the dice rolls result in multiple of 3. Between 1 and 6, there are 2 multiplies of 3, and the probability that a single dice roll avoids them is $4/6$, and using independence, the probability that all 100 avoid them is $(4/6)^{100}$. Similarly, you can show that $P(E_5^c)$ is $(5/6)^{100}$.
- $P(E_2^c \cap E_3^c)$ is the probability of all the the dice rolls avoiding numbers that are either divisible by 2 or 3: these are $2, 3, 4, 6$, and thus the probability that a single dice roll avoids them is $2/6$, and again using independence, the probability that all dice rolls avoid them is $(2/6)^{100}$. Similarly, you can show that $P(E_2^c \cap E_5^c) = (3/6)^{100}$, and $P(E_3^c \cap E_5^c) = (3/6)^{100}$.

- Finally, $P(E_2^c \cap E_3^c \cap E_5^c)$ is the probability that all the dice rolls avoid that numbers that are divisible 2, 3, or 5. A single dice roll would have to roll 1 to avoid multiples of 2, 3, or 5, and thus the probability for a single dice roll is 1/6, and using independence, that of all 100 dice rolls avoiding these multiples is $(1/6)^{100}$.

Putting back all these values into inclusion-exclusion formula gives the probability of the product not being a multiple of 30. We then have subtract that from 1 to get the probability that the product is divisible by 30.

$\square$

More space for Problem 3:

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Problem 4. [10 pts]
An urn contains 10 identical red balls, 8 identical blue balls, and 6 identical green balls. You sample 6 balls from the urn, one at a time, and without replacement, so that each outcome is equally likely. What is the probability that all six balls are of the same color? Show the main steps of your solution.

Hint: What did I say about identical objects for probability problems?

Proof. First let's assume that all the balls are distinct (say they are numbered from 1 to 24 — this does not change the probability (why?)). Suppose E is the event that all six balls are of the same color. We can write this is the union of three events: E_R (all six red), E_B (all six blue), and E_G (all six green). Note that these three events are mutually exclusive/disjoint, and thus: $P(E) = P(E_R \cup E_B \cup E_G) = P(E_R) + P(E_B) + P(E_G)$. Thus, it suffices to compute each of the three probabilities separately and then adding them up.

Since we are picking 6 balls, one at a time, without replacement (i.e., once a ball is picked it is not put back into the urn), we can think of the outcomes as sequence of length 6, where every entry is one of the 24 balls, and no repetition is allowed. The total number of sequences of this kind is $\binom{24}{6}6!$ using choose-and-permute – first decide which six of the 24 will be there in the sequence and then permute them. So, $|\Omega| = \binom{24}{6}6!$.

In the case of the event E_R, we are looking at all sequences of length 6 that consist only of the red balls (and of course no repetition is allowed). There are $\binom{10}{6}$ ways of first choosing 6 red balls that will be present in the sequence, and then 6! ways of arranging those chosen 6 balls, and hence the total such sequences is $\binom{10}{6}6!$. Since every sequence is equally likely, we know that we are dealing with the uniform distribution, and so

$$P(E_R) = \frac{|E_R|}{|\Omega|} = \frac{\binom{10}{6}6!}{\binom{24}{6}6!} = \frac{\binom{10}{6}}{\binom{24}{6}}.$$

Using the exact same approach as the one we used for E_R, you can show that: $P(E_B) = \frac{\binom{8}{6}}{\binom{24}{6}}$ and $P(E_G) = \frac{\binom{6}{6}}{\binom{24}{6}}$.

$\square$

More space for Problem 4: