# Quiz 1

CS 206: Discrete Structures II
Summer 2018

July 10, 2018

**Total points:** 20

**Duration:** 30 minutes

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**Name:**

**NetID:**

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**INSTRUCTIONS:**

1. You have to solve 5 problems in 30 minutes. Each problem is worth 4 points, 2 points for the correct answer, and 2 points for providing a short explanation for the answer (doesn't need to be a full-fledged explanation — just explain the main steps). If you just write the correct answer without providing an explanation, you will be awarded 0 points for the problem.
2. Make sure you write your solutions ONLY in the space provided below each problem. There is plenty of space for each problem. You can use the back of the sheets for scratchwork.
3. You are allowed to bring a single sheet of paper as a cheatsheet with notes (possibly) written on both the sides.
4. Make sure you write your name and NetID in the space provided above.
5. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the quiz, and might even be reported to the authorities!

**Problem 1.** [4 pts]
Consider the following sentence:

MATTHEW SUCCESSFULLY DEFENDED HIS THESIS

Suppose you are allowed to permute the characters within each word (but not across different words) while letting the whitespaces remain in place. How many distinct permutations of the above the sentence are there? Justify your answer. You can leave your answer in terms of factorials and/or binomial coefficients.

Number of permutations of MATTHEW: $\frac{7!}{2!}$ (using the permutations with repetitions formula)
Number of permutations of SUCCESSFULLY: $\frac{12!}{3!2!2!2!}$ (same as above)
Number of permutations of DEFENDED: $\frac{8!}{3!3!}$ (same as above)
Number of permutations of HIS: 3!
Numbef of permutations of THESIS: $\frac{6!}{2!}$
Since we can permute the letters within every word independent of how the letters are permuted in the other words, the total possible distinct arrangements is the product of all the above counts:

$$\frac{7!12!8!6!}{(2!)^5(3!)^2}.$$

**Problem 2.** [4 pts]
Let $A_1, \ldots, A_k$ be disjoint nonempty subsets of a finite set $S$ such that $A_1 \cup A_2 \cup \ldots \cup A_k = S$. Let $B$ be a subset of $S$ such that for every $1 \leq i \leq k$, $|A_i \cap B| = 2$. What is the cardinality of $B$? Justify your answer.
**[Hint: Try to use the sum rule]**

Every element in $S$ must belong to one of $A_1, \ldots, A_k$. Since $B \subseteq S$, every element of $B$ either belongs to $A_1$, or $A_2$, ..., or $A_k$. Let $B_i$ be the elements of $B$ that belong to $A_i$. Note that $B_i = A_i \cap B$. Since $B_1, \ldots, B_k$ form a partition of $B$, using the sum rule, we have that

$$|B| = \sum_{i=1}^{k} |B_i| = \sum_{i=1}^{k} |A_i \cap B| = 2k$$

**Problem 3.** [4 pts]
Let $\mathcal{A}$ be an array of $n$ numbers, $a_1, a_2, \ldots, a_n$. A *segment* of an array is a block of consecutive elements of **size at least** 2 in $\mathcal{A}$. For example, if $n = 5$, $a_1, a_2, a_3$ is a segment, and so is $a_2, a_3, a_4, a_5$, but not $a_1, a_3, a_4$ because this is not a block of consecutive elements. What is the number of possible segments of an array when $n = 100$? Simplify your answer, and DO NOT leave it in terms of factorials and binomial coefficients.

To specify a segment, we need to specify its starting index and its ending index, and then the segment is basically all the elements between the starting and ending index (including the elements at the starting and ending index). Since the length is at least 2, the starting and ending index cannot be the same. Thus, the number of segments in an array of size 100 is equal to the number of ways in which you can choose two distinct indices from $\{1, \ldots, 100\}$ and so the answer is $\binom{100}{2} = 5050$.

**Problem 4.** [4 pts]
Consider the set of 7-digit numbers from 1000000 to 9999999. How many such numbers are there that satisfy ALL of the following conditions?

- Contain at least one 2 among their digits.
- Begin with an odd digit.
- End with an even digit.

**[Hint: Try to use the difference method]**

Let $A$ be the set of 7-digit numbers that satisfy all three conditions from above. Let $S$ be the set of all seven digit numbers that begin with an odd digit and end with an even digit.

First note that $|S| = 5 \times (10)^5 \times 5$ since there are 5 possibilities for the first digit (first digit must be odd), 10 each for the next 5 digits (middle digits can be anything), and finally 5 choices for the last digit (last digit must be even).

Next, note that $S \setminus A$ is the set of all seven digit numbers that begin with an odd digit and end with an even digit but contain **no** 2 among their digits. Also, there are $5 \times 9^5 \times 4$ such numbers (first digit is odd, middle digits can be anything except 2, and last digit is some even number other than 2), and so $|S \setminus A| = 5 \times 9^5 \times 4$.
Using difference method, $|A| = |S| - |S \setminus A| = 5 \cdot (10)^5 \cdot 5 - 5 \cdot (9)^5 \cdot 4$.

**Problem 5.** [4 pts]
Find the number of integer solutions to the following set of equations:

$$x_1 + x_2 + \ldots + x_{100} = 10000$$

$$x_1 \geq 1$$

$$x_2 \geq 2$$

$$.$$

$$.$$

$$.$$

$$x_{99} \geq 99$$

$$x_{100} \geq 100.$$

You can leave your answer in terms of factorials and/or binomial coefficients.

Let's first satisfy the minimum requirements for all 100 variables. That means we will give 1 to $x_1$, 2 to $x_2$, 3 to $x_3$, $\ldots$, 100 to $x_{100}$. In total we give away $1 + 2 + \ldots + 100 = 5050$. We are left with $10000 - 5050 = 4950$.

Now we are reduced to the problem of distributing the remaining 4950 among 100 parties. This is basically the same as the number of binary strings with 4950 zeros and 99 ones, and thus the answer is $\binom{4950+99}{99}$.