Lecture 4: Sequences with repetitions, distributing identical objects among distinct parties, the binomial theorem, and some properties of binomial coefficients

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References: Relevant parts of chapter 15 of the Math for CS book.

1 Sequences with repetitions revisted

Let us consider the following problem to warm up towards a more general statement.

Question . How many sequences are there of the form (A, B, C), where A, B and C are disjoint subsets of $S = \{1, \ldots, 1000\}$, and $|A| = 500$, $|B| = 325$, $|C| = 175$?

Proof. Note that $|A| + |B| + |C| = |S|$, and thus they form a partition of S. To distribute the 1000 elements of S between A, B, and C, it suffices to specify which elements of S are chosen to go into A, and which of them are chosen to go to B — the rest will then go to C.
Let's first pick which elements will go into A. We have to choose 500 elements from S, so there are $\binom{1000}{500}$ ways of doing that. Having picked elements for A, we would be left with $1000 - 500 = 500$ elements in S, and have to choose 325 out of them for B. For this, there are $\binom{500}{325}$ ways, and so the total is:

$$\binom{1000}{500} \cdot \binom{500}{325} = \frac{1000!}{500!500!} \cdot \frac{500!}{325!175!} = \frac{1000!}{500!325!175!}.$$

□

Let us try to generalize this to the following: how many sequences are there of the form $(A_1, \ldots, A_k)$ where $A_1, \ldots, A_k$ are disjoint subsets of $S = \{1, \ldots, n\}$ and $|A_1| = r_1, |A_2| = r_2, \ldots, |A_k| = r_k$ with $r_1 + r_2 + \ldots + r_k = n$?

Let us do exactly as we did above: first pick r_1 elements to put into A_1, then r_2 to put into A_2, $\ldots$, and finally r_{k-1} elements in A_{k-1}. As soon as have filled up the first $k-1$ elements, we will be left with $n - (r_1 + \ldots + r_k) = r_k$ elements, and thus all of them must go into A_k. Following the above calculations, we see that the number of ways of picking elements for $A_1, \ldots, A_{k-1}$ is

$$\binom{n}{r_1} \cdot \binom{n - r_1}{r_2} \cdot \binom{n - r_1 - r_2}{r_3} \cdot \ldots \cdot \binom{n - r_1 - \ldots - r_{k-2}}{r_{k-1}}$$

This can be written as

$$\frac{n!}{r_1!(n-r_1)!} \frac{(n-r_1)!}{r_2!(n-r_1-r_2)!} \frac{(n-r_1-r_2)!}{r_3!(n-r_1-r_2-r_3)!} \cdots \frac{(n-r_1-\ldots-r_{k-2})!}{r_{k-1}!(n-r_1-\ldots-r_{k-1})!}.$$

If you stare at this expression for a while you realize that all the numerators except $n!$ will get cancelled by some of the denominators, and the only denominators left will be $r_1!r_2!\dots r_k!$ (where did the $r_k!$ even come from?). Thus, the expression becomes: $\frac{n!}{r_1!r_2!\dots r_k!}$. Thus, we have

Lemma 1. *The number of sequences of the form* $(A_1, \dots, A_k)$ *where* $A_1, \dots, A_k$ *are disjoint subsets of* $S = \{1, \dots, n\}$ *and* $|A_1| = r_1, |A_2| = r_2, \dots, |A_k| = r_k$ *with* $r_1 + r_2 + \dots + r_k = n$ *is* $\frac{n!}{r_1!r_2!\dots r_k!}$. *In other words, the number of ways of distributing* n *distinct objects among* k *distinct parties such that party* i *gets exactly* r_i *many objects and* $r_1 + \dots + r_k = n$ *is* $\frac{n!}{r_1!\dots r_k!}$.

This also turns out to be the number of permutations when there are some repeated objects as we shall see now (the formula does look like the one we saw in the last lecture, doesn't it?)

Suppose we have n objects in total, r_1 of type 1, r_2 of type 2, ..., r_k of type k, such that $r_1+\dots+r_k = n$. How many distinct permutations are there of these n objects?
If they were all distinct, then the answer would have been $n!$, but as we have seen before, this will lead to an overcount when we have repetitions. In the last lecture, we suggested that one can use the division rule to get around this problem, and arrive at the number of distinct permutations being

$$\binom{n}{r_1, \dots, r_k} = \frac{n!}{r_1! \dots r_k!},$$

where the expression in the LHS is known as a multinomial coefficient.We will now give an alternate (and much cleaner) proof of this based on what we just did above with sequence of sets.

Let's think of coming up with one possible arrangement of these n objects. What needs to be specified in order to completely specify an arrangement? Well, I need to first tell you what positions the objects of type 1 go into, then I need to tell you what positions the objects of type 2 go into, ..., and finally what positions objects of type $k-1$ go into. As soon as I tell you all that, you'll automatically know where the objects of type k go into – they are forced to go to the remaining positions.
Now, how many ways are there to choose the positions for the objects of type 1. There are r_1 objects of type 1, so we need to choose r_1 out of n possible positions, and thus the number of ways is $\binom{n}{r_1}$. Having chosen the positions for the first type, we then choose positions for the objects of the second type. Now we need to choose r_2 positions from $n - r_1$ available ones, so that's $\binom{n-r_1}{r_2}$ ways...but, hold on? Isn't this turning out to be *exactly* what we did above?

Turns out you can do a simple trick (think bijection method): if you let A_i to be all the positions in the permutation where the r_i objects of type i will go into, then the problem becomes one of counting the number of sequences of the form $(A_1, \dots, A_k)$, with $A_1, \dots, A_k$ being disjoint subsets of $\{1, \dots, n\}$ (these are the n possible positions in the permutation) of sizes $r_1, \dots, r_k$ respectively, and $\sum_{i=1}^{k} r_i = n$. Now we can use Lemma 1 to finish the job, and conclude that the number of distinct permutations is $\frac{n!}{r_1!\dots r_k!}$.

2 Distributing identical objects among distinct parties

Many counting problems can be rephrased as the following problem:

Question . Suppose that there are n identical candies that need to be distributed among r children (obviously, no two children are identical, and so we have r distinct parties here). In how many ways can you do this if you are not allowed to break candies into smaller pieces?

To solve these problems, it's convenient to convert them into a linear equation with constraints (think bijection method) to be able to think clearly:

- Let $x_1, \ldots, x_r$ be r variables such that x_i denotes the number of candies child i receives.
- Since all n candies must get distributed, we have that $x_1 + x_2 + \ldots + x_r = n$.
- Furthermore, since you can't break the candies and must give whole candies to the children, we must enforce that the values taken by $x_1, \ldots, x_r$ be integer, i.e. we are interested in integer valued solutions to the above equation.
- Also, it doesn't make sense if a variable becomes negative (what would that even mean? instead of giving candies to the child, ask the child for candies?), so we will say that $x_1, \ldots, x_r \geq 0$.

So the number of ways of distributing candies among children become the same as the number of integer solutions to the above equation with the additional constraint that $x_1, \ldots, x_r \geq 0$.

While this way of phrasing the problem as finding solutions to a linear equation help us think clearly and make sure we got the problem right, it doesn't really do much in terms of helping us find the answer. Instead, we will not turn to our old friends: binary strings!

Let's work with $n = 10$ and $r = 4$, i.e. 10 candies among 4 children, or equivalently, solutions to $x_1 + x_2 + x_3 + x_4 = 10$ subject to $x_1, x_2, x_3, x_4 \geq 0$. Let's look at the binary string of length 10 consisting of all zeros, and think of those zeros as candies. We can now think of putting ones into the string as the process of dividing the zeros (candies) into two parts. Say we start out with 0000000000, and add a one somewhere in there: 00010000000. This can be interpreted as there being two parts, one to the left of the one and the other to the right of the one, and the first part has 3 zeros (candies), and the second part has 7 zeros (candies). If throw in another one, then we get three parts in the same manner: 000100010000 can be thought of as three parts with part one and two having three zeros each, and the third part having 4 zeros. Of course there could be parts that are empty, e.g., in 000110000000, the second part (between the two ones) is empty.
Now, if we throw in three ones in total, we get a distribution of the 10 zeros into 4 parts. Let's fix the covention that all the zeros (candies) to the left of the first one go to x_1 (or the first child), all the zeros between the 1st and 2nd one go to x_2 (the second child), and so on. Thus, the number of ways of distributing the 10 candies between 4 children, or the number of solutions to the above equations, is exactly the number of binary strings of length 13 that have 10 zeros and 3 ones, which we know using the subset rule is $\binom{13}{3}$.

We can generalize the above argument to conclude that the number of ways of distributing n identical objects among k distinct parties is equal to the number of binary strings that have exactly n zeros and $k-1$ ones (why $k-1$, and not k? Because a single one created two partitions, two ones creatd three, and so on, so $k-1$ ones will give us k partitions) which is $\binom{n+k-1}{k-1}$.

Question . In how many ways can you distribute 20 indivisible gold bars among 5 pirates so that every pirate gets at least one bar?

Proof. If we convert this an equation with constraints we get $x_1 + x_2 + \ldots + x_5 = 20$ with the constraints that $x_1 \geq 1, x_2 \geq 1, \ldots, x_5 \geq 1$. Unfortunately, the above method only works when we have constraints of the form $x_1 \geq 0, \ldots, x_5 \geq 0$. To get around this, let's first simply satisfy the bare minimum requirements of all the pirates (variables), i.e. let's given one bar to each of the pirates (variables) so that they all have at least one. This means we are left with $20 - 5 = 15$ bars.

We can now focus on distributing the remaining 15 bars among the 5 pirates without havin to worry about any additional constraints. Notice that now the equation would be simply $x_1 + \ldots + x_5 = 15$ with $x_1, \ldots, x_5 \geq 0$, and we know that the number of solutions to this equation is $\binom{19}{15}$. $\square$

Can you generalize this?

Question . In how many ways can you distribute n indivisible gold bars among k pirates so that every pirate gets at least r bars?

Sometimes the distribution problems disguise themselves in a very inconspicuous manner (the only way to call their bluff is to practice a lot of problems). Here is an example:

Question . Suppose we roll 10 identical dice together. How many possible outcomes are there?

Proof. Note that i) all the dice are identical ii) there is no order in which we rolled the dice (all of them were rolled simultaneously). Thus, the only thing that *distinguishes* an outcome from another, or the only thing that really *defines* an outcome is the following set of statistics: how many of the dice turned up 1, how many turned up 2, …, how many turned up 6?

Let x_i be the number of dice that turned up the number i (here $1 \leq i \leq 6$). Then we have $x_1 + \ldots + x_6 = 10$, and $x_1, \ldots, x_6 \geq 0$, and want to count the number of solutions to this set of equations. We are now on home turf, and can conclude that the answer must be $\binom{15}{10}$. $\square$

Note that had the dice been colored with distinct colors, or if they had been rolled one at a time, then the problem would be very different, and the answer would simply be 6^{10} using the product rule (convince yourself of this!)

Here is another one (this one is from your book) that can seem totally unrelated to distribution problems:

Question S. uppose there are 20 books arranged in a row on a book shelf. In how many ways can you choose 6 books so that no two of the chosen books are adjacent to each other on the rack?

At first glance, this problem might not seem to have anything to do with what we have been discussing so far. A little bit of rephrasing makes this illusion disappear.
Note that the most obvious way of specifying a selection/choice of 6 books is to literally specify the "index" of the book if one were to start counting from 1 starting at the left most book on the shelf. A more non-obvious way would be the following: specify how many *unchosen* books there are to the left of the first chosen book, then specify how many unchosen books there are between the first and second chosen book, then the number of unchosen books between the second and third chosen book, …, and finally the number of unchosen books to the right of the last (sixth) chosen

book. So, I can set this up in the following way: let x_1 denote the number of books to the left of the first chosen book, x_2 denote the number of books between the first and second chosen books, and so on, with x_7 representing the number of books to the right of the sixth chosen book.

What do conditions do we want to impose on these variables? Well, first of all there must be 14 unchosen books in all, and each of those 14 books must be either between two of the chosen books, or the left or right of all the chosen books, and thus: $x_1 + x_2 + \ldots + x_7 = 14$.
Furthermore, we want that no two books be adjacent, so there must be at least one unchosen book between the first and second chosen book, so we want $x_2 \geq 1$, and similarly we want $x_3, \ldots, x_6 \geq 1$. How about x_1 and x_7, there is no constraint on them: there could be as low as zero (it is possible to choose the first book from the left on the shelf, or the last book on the shelf), and so $x_1, x_7 \geq 0$. Again, we are back in familiar territory, so we can use the machinery we developed earlier in the section to finish the problem.

3 Binomial theorem

Recall that $(x+y)^2 = x^2 + y^2 + 2xy$. Actually, there is a lot going on when one tries to derive that:

- $(x+y)^2$ is nothing but $(x+y) \times (x+y)$. Since multiplication is *distributive* over addition, there will be four terms coming from the four pssible multiplications, and then all we be added up.
- Basically, each of the four terms will consist of exactly one entry each from the two copies of $(x+y)$. For the first copy, we have the choice of picking up either x or y, and we have the same choices for the second copy. This gives $2 \times 2 = 4$ terms in total.
- The four terms are $xx + xy + yx + yy$. Now recall that $xx = x^2$ and $yy = y^2$. Also, because of multiplication being *commutative* we have that $xy = yx$, and so we get $x^2 + y^2 + 2xy$.

One can now do the same for higher powers of $(x+y)$. How about $(x+y)^n$? This is nothing but n copies of $(x+y)$ multiplied together. Again, using distributivity of multiplication over addition, we will get 2^n terms: there are n copies of $(x+y)$, and we have two choices for the first copy, two for the second copy, and so on.
Obviously, as before, many terms will "collapse" to the same expression because of commutativity ($xy = yx$, and then combining consecutive xs and ys into powers). This means that, in the final simplified expression for $(x+y)^n$, different terms will have different coefficients in front of them depending on how many of the 2^n original terms collapse to them.

Let's ask the question, what's the coefficient of $x^k y^{n-k}$? In other words, how many of the 2^n initial terms collapse to $x^k y^{n-k}$. Clearly, only those terms that have exactly k xs in them will collapse to $x^k y^{n-k}$, so how many terms have exactly k xs?
Recall that we had to pick between x and y in each of the n copies of $(x+y)$. Thus, the terms that give us $x^k y^{n-k}$ must correspond to outcomes of the picking process where we decided to pick x in exactly k of the copies of $(x+y)$, and picked y in the rest. How many such outcomes are there? It's exactly the same as deciding in which k copies of $(x+y)$ (out of the n possible) will we pick x, and there are $\binom{n}{k}$ ways of doing that. So the coefficient of $x^k y^{n-k}$ must be $\binom{n}{k}$. Since there was nothing special about n and k in the above argument, we can conclude:

Theorem 2 (Binomial theorem)**.** *Let $n \geq 0$ be an integer, then we have that*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

4 Some properties of binomial coefficients

The binomial theorem lets us derive some interesting things about binomial coefficients. Let us first ask the following question what is the sum of all binomial coefficients for a fixed n, i.e.

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}.$$

We can answer this problem in two ways. The first one is combinatorial: recall that $\binom{n}{k}$ just denoted the number of size k subsets of the set $\{1, \ldots, n\}$. So, it seems like we are adding the number of subsets of size 0, the number of subsets of size 1, and so on up till the number of subsets of size n. But if you think about it this accounts for all possible subsets of $\{1, \ldots, n\}$, and thus the count must add up to the total number of different possible subsets of $\{1, \ldots, n\}$, which we know is 2^n.

A more algebraic way of seeing this is take the statement of the binomial theorem and set $x = y = 1$. What happens if we set $y = 1$ and $x = -1$? The LHS becomes zero, and we get

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

Now notice that $(-1)^k$ becomes 1 whenever k is even, and becomes -1 whenever k is odd, so we can write:

$$0 = \sum_{k \text{ is even and } \leq n} \binom{n}{k} - \sum_{k \text{ is odd and } \leq n} \binom{n}{k},$$

and so

$$\sum_{k \text{ is even and } \leq n} \binom{n}{k} = \sum_{k \text{ is odd and } \leq n} \binom{n}{k}.$$

This is another way of deriving something we proved using the bijection method before: $\sum_{k \text{ is even and } \leq n} \binom{n}{k}$ is nothing but the number of even size subsets of $\{1, \ldots, n\}$ and $\sum_{k \text{ is odd and } \leq n} \binom{n}{k}$ is the number of odd size subsets of $\{1, \ldots, n\}$, and the above expression tells us that there are as many odd size subsets as there are even size subsets (we gave a bijective proof for the case when n was odd, can you try for n even?).

We want to show the following:

$$\sum_{k < \frac{n}{2}} \binom{n}{k} = \sum_{k > \frac{n}{2}} \binom{n}{k},$$

i.e. the number of subsets of $\{1, \ldots, n\}$ of size less than $n/2$, is the same as the number of subsets of size more than $n/2$. I wouldn't give you the whole proof but it follows pretty simply by recalling that $\binom{n}{k} = \binom{n}{n-k}$ and so $\binom{n}{0} = \binom{n}{n}$, $\binom{n}{1} = \binom{n}{n-1}$, and so on.