# Homework 1

CS 206: Discrete Structures II
Summer 2018

**Due:** 5:00 PM EDT, Thursday, July 5th 2018

**Total points:** 30 (10 for easy + 20 for hard)

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**INSTRUCTIONS:**

1. Print all the pages in this document and make sure you write the solutions in the space provided below each problem. This is very important!
2. Make sure you write your name and NetID in the space provided above.
3. After you are done writing, scan the sheets in the correct order into a PDF, and upload the PDF to Gradescope before the deadline mentioned above. No late submissions barring exceptional circumstances!
4. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.
5. As for grading, after the deadline, the grader will randomly pick one easy and one hard problem to grade. Needless to say the same problems will be graded in every student's submission. For e.g., the grader may randomly decide to grade the first easy problem and the second hard problem, and so all the submissions will be evaluated on the basis of those two problems.

## Part I: The easy stuff

**Problem 1.** [10 pts]
Let $S$ be defined as follows:

$$S = \{(A, B) \mid A \subseteq \{1, 2, \ldots, n\}, B \subseteq \{1, 2, \ldots, n\},\ |A \cap B| \geq 1\}.$$

What is the cardinality of S? Justify your answer.

**Problem 2.** [10 pts]
A binary string is called *balanced* if it has equal number of ones and zeros. Let $\mathcal{S}$ be the set of balanced binary strings of length 14, and let $\mathcal{P}$ be the set of paths from $A$ to $B$ in the grid below that use only downward and rightward steps (i.e., a path can never go up, left, or diagonal; only down or right). Give a bijection between $\mathcal{P}$ and $\mathcal{S}$. Justify your answer.

right = 1 (r)
down = 0 (d)
domain = r, d


= P

$$|P| = \frac{8}{8d/8r} \times \frac{8}{8d/8r} = 64 \text{ possible Paths from A To B}$$

$$\underset{1}{\frac{2}{d/r}}\ \underset{2}{\frac{2}{d/r}}\ \underset{3}{\frac{2}{d/r}}\ \underset{4}{\frac{2}{d/r}}\ \underset{5}{\frac{2}{d/r}}\ \underset{6}{\frac{2}{d/r}}\ \underset{7}{\frac{2}{d/r}}\ \underset{8}{\frac{1}{d/r}}\ \underset{9}{\frac{1}{d/r}}\ \underset{10}{\frac{1}{d/r}}\ \underset{11}{\frac{1}{d/r}}\ \underset{12}{\frac{1}{d/r}}\ \underset{13}{\frac{1}{d/r}}\ \underset{14}{\frac{1}{d/r}}$$

Can only choose one. Either all ones or all zeros.

$$\frac{2^7 \cdot 1^7 = 128}{2} = 64 \text{ possible arrengements for 14-bit balanced string}$$

eliminate repetition/over counting

1/0

right = 1
down = 0

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | B |

$= S$

$|S| = 64$

__All__ of the paths from A To B Can be modeled as __all__ of the possible Combinations of balanced 14-bit strings. thus there is a biyection between P and S.

$$|P| = |S|$$

$$64 = 64$$

## Part II: The hard stuff

**Problem 3.** [20 pts]
Let $a_1, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$. We say that this sequence is a *zig-zag* sequence if, for every $2 \le i \le n$, either:

- $a_i$ is greater than all the numbers occurring before it in the sequence, i.e. greater than each of $a_{i-1}, a_{i-2}, \ldots, a_1$, or
- $a_i$ is less than all the numbers occurring before it in the sequence, i.e. less than each of $a_{i-1}, a_{i-2}, \ldots, a_1$.

How many of the $n!$ permutations of $1, 2, \ldots, n$ are zig-zag sequences?

Zig-Zag Count for $n!$:
$f(n) \rightarrow$ all permutations of $n!$
$g(f) \rightarrow$ The zig-zag count for permutations of f

$g(2) = 2$ $g(13) = 4096$
$g(3) = 4$
$g(4) = 8$
$g(5) = 16$
$g(6) = 32$
$g(7) = 64$
$g(8) = 128$
$g(9) = 256$
$g(10) = 512$
$g(11) = 1024$
$g(12) = 2048$

Total Number of Zig-Zag permutations for n is $2^{n-1}$

$\boxed{2^{n-1}}$

**Problem 4.** [20 pts]
Let $\mathbb{Z}_m$ denote the set $\{0, 1, \ldots, m-1\}$. Then $\mathbb{Z}_m^n$ is the set of all sequences of length $n$ where every element in the sequence comes from the set $\mathbb{Z}_m$. For example, for $m = 3$ and $n = 2$, $\mathbb{Z}_3^2 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$. Let $f : \mathbb{Z}_m^n \to \mathbb{Z}_m$ be the function defined as:

$$f((b_1, b_2, \ldots, b_n)) = \left(\sum_{i=1}^{n} b_i\right) \bmod m.$$

What fraction of the points in the domain of $f$ map to 1? Write your answer in terms of $m$ and $n$. Justify it.

$\mathbb{Z}_m^n = \underbrace{g((n, m))}_{}$

The count for all possible sets of $(n, m)$.

$\underbrace{f((n, m))}_{}$

mod = 1 Count

$g(1,3) = 3$, $g(2,3) = 9$, $g(3,3) = 27$

$f(1,3) = 1$, $f(2,3) = 3$, $f(3,3) = 9$

$m^n$ is the total count of subsets generated by $\mathbb{Z}_m^n$. The fraction of that total wher $\left(\sum_{i=1}^{n}\right) \bmod m = 1$

is $\boxed{\dfrac{1}{m}}$

Thus $f(n,m) = \frac{1}{m} \cdot m^n$

$= m^{-1} \cdot m^n = m^{n-1}$

$$\boxed{m^{n-1}}$$