Lecture 5: Multinomial theorem, some more properties of binomial coefficients, combinatorial proofs, principle of inclusion-exclusion

Discrete Structures II (Summer 2018)
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References: Relevant parts of chapter 15 of the Math for CS book.

1 Multinomial theorem

The multinomial theorem is a generalization of the binomial theorem and let's us find the coefficients of terms in the expansion of $(x_1 + \ldots x_k)^n$. For the sake of simplicity and clarity, let's derive the formula for the case of three variables. The more general formula is easy to guess once we have the formula for three variables.

Let's consider $(x+y+z)^{10}$. If we didn't care about combining powers and commutativity, we would get 3^{10} terms in all, terms that look like $xxxxxxxxxx$, or $xxxxyyzzyx$, etc — we have 10 copies of $(x+y+z)$ multiplied together and we must pick x, y, or z from each of the 10 copies resulting in a sequence of length 10 consisting of xs, ys, and zs.
Of course, we combine powers and use commutativity to collapse these length 10 sequences into expressions of the form $x^{k_1}y^{k_2}z^{k_3}$, where $k_1 + k_2 + k_3 = 10$. The question is what is the coefficient of $x^{k_1}y^{k_2}z^{k_3}$, or in other words, how many of the sequences "collapse" to this term? Notice that any sequence with k_1 xs, k_2 ys, and k_3 zs will collapse to this term, and so we want to understand how many such sequences are there. This is easy. There must be exactly k_1 copies of $(x + y + z)$ from which we must choose x, and there are $\binom{10}{k_1}$ ways to decide which k_1 copies to go for. From the remaining $10 - k_1$ copies we must pick y from exactly k_2 of them, and there are $\binom{10-k_1}{k_2}$ ways of doing this. Finally, we are left with $10 - k_1 - k_2 = k_3$ copies of $(x + y + z)$ and we must choose z from those (why?).
Thus the total number of sequences that collapse to $x^{k_1}y^{k_2}z^{k_3}$ is

$$\binom{10}{k_1}\binom{10-k_1}{k_2} = \frac{10!}{k_1!k_2!(10-k_1-k_2)!} = \frac{10!}{k_1!k_2!k_3!}.$$

The above formula looks very similar to the formula for counting permutations of strings that have repeated letters in them...can you see why?

More generally, let's say we had $(x + y + z)^n$, then the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ for $k_1 + k_2 + k_3 = n$ in the expansion of $(x + y + z)^n$ is

$$\frac{n!}{k_1!k_2!k_3!}.$$

The above result is the **multinomial theorem**. We can consider a further generalization of the multinomial theorem where x,y, and z have a coefficient other than 1. We want to understand the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in $(ax + by + cz)^n$, where $k_1 + k_2 + k_3 = n$ and a, b, c are arbitrary real numbers. Notice that a, b, and c don't really affect the exponents of x, y, z, and so one can easily derive the formula (using the same ideas as above) for the coefficient of $x^{k_1}y^{k_2}z^{k_3}$ in this case as:

$$\frac{n!}{k_1!k_2!k_3!} \cdot a^{k_1}b^{k_2}c^{k_3}.$$

Question . Find the coefficient of $x^2y^3z^4$ in $(-3x+2y+z)^9$.

The answer follows pretty easily from what we just discussed: $\frac{9!}{2!3!4!} \cdot (-3)^2 2^3$.

2 Yet another property of binomial coefficients

Let's fix n to be some arbitrary positive integer. We want to understand how $\binom{n}{k}$ behaves as a function of k as we vary k from 0 to n.

Lemma 1. *When* $0 \le k < \frac{n-1}{2}$,

$$\binom{n}{k+1} > \binom{n}{k},$$

and when $\frac{n-1}{2} < k < n$,

$$\binom{n}{k+1} < \binom{n}{k}.$$

Also, $\binom{n}{k+1} = \binom{n}{k}$ *only when* $k = \frac{n-1}{2}$, *and this can only happen when* n *is odd.*

Proof. For $0 \le k \le n-1$, define r_k as follows:

$$r_k = \frac{\binom{n}{k+1}}{\binom{n}{k}},$$

i.e, the ratio of the $k+1^{th}$ binomial coefficient to the k^{th} binomial coefficient. We can simplify r_k as

$$r_k = \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{\frac{n!}{(k+1)!(n-k-1)!}}{\frac{n!}{k!(n-k)!}} = \frac{n-k}{k+1}.$$

(Can you explain why the last equality in the above series of equalities is true?) Note that whenever $r_k > 1$, then the $k+1^{th}$ binomial coefficient is larger than the k^{th} one, and it's the other way around when $r_k < 1$, so we just need to understand the behavior of r_k with k:

$$r_k = \frac{n-k}{k+1} < 1 \Leftrightarrow k > \frac{n-1}{2},$$

and

$$r_k = \frac{n-k}{k+1} > 1 \Leftrightarrow k < \frac{n-1}{2}.$$

Also, it's easy to see that the two binomial coefficients are equal when $r_k = 1$ which can only happen when $k = (n-1)/2$. This can only happen when n is odd. This completes the proof. $\square$

Thus, as a function of k, $\binom{n}{k}$ keeps increasing (strictly) till it becomes $\binom{n}{n/2}$ (when n is even) or $\binom{n}{(n-1)/2}$ (when n is odd). After that, in the even case, as k becomes more than $n/2$, $\binom{n}{k}$ decreases (strictly). In the odd case, $\binom{n}{(n-1)/2} = \binom{n}{(n+1)2}$, and after k goes beyond $(n+1)/2$ it strictly decreases. This means that the largest binomial coefficient in the n even case is $\binom{n}{n/2}$, and in the odd case there are two of them: $\binom{n}{(n-1)/2}$ and $\binom{n}{(n+1)/2}$.

3 Combinatorial proofs

Consider the equation:

$$\binom{n}{k} = \binom{n}{n-k}.$$

We know how to show that the two are same *algebraically*, i.e. by writing out their mathematical expressions and then showing that they both reduce to the same mathematical expression.

Yet another way to show that they are equal *without* appealing to algebra is to use combinatorics:

- Let S be the number of subsets of 1 to n of size exactly k.
- We know $|S| = \binom{n}{k}$ using the subset rule that we derived in lecture.
- On the other hand, if T is the set of all subsets of 1 to n of size exactly $n-k$ we can setup a bijection between S and T: simply map a size k subset in S to its complement. By the bijection method, we know that $|S| = |T|$, and the subset rule tells us that $|T| = \binom{n}{(n-k)}$.
- But this means $\binom{n}{k} = |S| = |T| = \binom{n}{(n-k)}$.

Notice that we didn't have to use any algebra in the above proof. All we did was define S appropriately so that the expression on the left hand side of the equation we want to prove is equal to $|S|$ by using some counting method, and then we showed that using another method $|S|$ is also equal to the expression on the right hand side, and thus the two expressions must be equal. This is called a *combinatorial proof* using *double counting* (since we counted the cardinality of the same set twice using different methods).

Let us consider another example: show that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ without using an algebra, i.e. give a combinatorial proof.

Let S be the powerset (i.e. the set of all subsets) of $\{1, \ldots, n\}$. One way we can count S by using the partition method: $|S|$ = number of subsets of $\{1, \ldots, n\}$ of size 1 + number of subsets of size 2 + … + number of subsets of size n. Since the number of subsets of $\{1, \ldots, n\}$ of size i is $\binom{n}{i}$, we get $|S| = \sum_{i=0}^{n} \binom{n}{i}$, and this show that $|S|$ is equal to the expression on the left hand side.

We now have to show that the expression of the right hand side is also equal to $|S|$. Recall that there is a bijection between the set of binary strings of length n and S: we can convert a binary string into a subset by including all those elements $1 \leq i \leq n$ for which the i^{th} position of x has a 1, and similarly, we can construct a binary string of length n given a subset of $\{1, \ldots, n\}$: given a subset A of $\{1, \ldots, n\}$, the string we construct for A has a 1 in the i^{th} position if i is in A otherwise there is a 0 in the i^{th} position. Since the number of binary strings of length n is 2^n, this means $|S| = 2^n$, which is the right hand side expression.

Consider another equation:

Question . Show that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. Of course you can give an algebraic proof by putting in the formula for $\binom{n}{k}$ and then showing that the left hand side and the right hand side reduce to the same mathematical expression, but we want to give a slick combinatorial proof which will be very clean and will not involve any nasty algebra.

Let S be the set of all k size subsets of $\{1, \ldots, n\}$. Using the subset rule we know that $|S| = \binom{n}{k}$. We can partition S into two parts: all the subsets of size k that contain the element "1" (say S_1), and all the ones that don't (call it S_2). To specify a subset of $\{1, \ldots, n\}$ that contains "1", we just need to specify what other $k-1$ elements are there in the subset. There are $\binom{n-1}{k-1}$ ways of choosing those $k-1$ elements from the remaining $n-1$ elements (elements other than "1"), thus $|S_1| = \binom{n-1}{k-1}$.

Now let us count the number of subsets that don't contain a "1". In this case, we have to choose all the k elements from $n-1$ elements (all elements other than "1"), and thus the number of such sets is $\binom{n-1}{k}$, and thus that's what $|S_2|$ is equal to. Using the partition rule we know that

$$|S| = |S_1| + |S_2| = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

This means that $\binom{n}{k} = |S| = \binom{n-1}{k-1} + \binom{n-1}{k}$. $\square$

So the general strategy is as follows: suppose you want to show that $E_1 = E_2$ where E_1, E_2 are mathematical expressions, then:

- Define a set S appropriately.
- Show that using some counting method, $|S| = E_1$.
- Show that using a different counting method, $|S| = E_2$.
- This implies that $E_1 = E_2$.

4 Principle of Inclusion-Exclusion

Suppose we have two sets $A, B \subseteq S$, and want to find $|A \cup B|$. We know that if $A \cap B = \emptyset$, then this is exactly the job for the sum rule:

$$|A \cup B| = |A| + |B|.$$

But this might not always be the case and A and B might have a nonzero intersection. What do we do then? You have seen in 205 that, more generally,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This takes care of the situation when we want to find the size of the union of two sets, how about three sets, i.e. find $|A \cup B \cup C|$? Let us try and derive a formula for the cardinality in this case.

Lemma 2. *Let A, B, C be subsets of S, then*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Proof. Let $X = B \cup C$. Then we have using the formula for union of two sets that

$$\begin{aligned} |A \cup B \cup C| &= |A \cup X| \\ &= |A| + |X| - |A \cap X| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \end{aligned} \tag{1}$$

We now again appeal to the formula for union of two sets to compute $|B \cup C|$ and $|(A \cap B) \cup (A \cap C)|$. We see that

$$|B \cup C| = |B| + |C| - |B \cap C|,$$

and

$$|(A \cap B) \cup (A \cap C)| = |(A \cap B)| + |(A \cap C)| - |(A \cap B) \cap (A \cap C)| = |(A \cap B)| + |(A \cap C)| - |A \cap B \cap C|.$$

If we substitute these back into Equation 1, we get that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

□

This is called the *principle of inclusion-exclusion*: we first overestimate $|A \cup B \cup C|$ by including too much (this is the $|A| + |B| + |C|$ part), then we underestimate it by excluding too much (this is the $-|A \cap B| - |A \cap C| - |B \cap C|$ part), and then we finally include again to get the right count for $|A \cup B \cup C|$ (this is the $|A \cap B \cap C|$ part).
In the next lecture we will see how to generalize this to the union of an arbitrary number of sets, but for now let's see some applications of the formula for union of three sets.

Question . Let $S = \{1, \ldots, 100\}$. How many numbers are there in S that are either multiples of 2 or 3 or 5?

Proof. Inclusion-exclusion comes in handy when you are dealing with a set of objects and want to know how many of those objects satisfy at least one out of two or more given conditions (in some cases, you have to count the number of objects that satisfy *all* the given conditions. We will see how to deal with those in the next lecture). In this case, the "objects" are the numbers from 1 to 100, and there are three given conditions:

1. divisible by 2
2. divisible by 3
3. divisible by 5

and we want to find the number of "objects", i.e. numbers, that satisfy at least one of the three above conditions.

The next step in solving such problems is to define a set for every condition: the set of the objects that satisfy that condition. So in this case we will define three sets for the three conditions:

1. Let A be the set of numbers in S that are divisible by 2
2. Let B be the set of numbers in S that are divisible by 3
3. Let C be the set of numbers in S that are divisible by 5

Notice that we want to find all the numbers that are either divisible by 2 or by 3 or by 5, and this translates to $|A \cup B \cup C|$ in the language of set theory ("or" is $\cup$, i.e. union, and "and" is $\cap$, i.e., intersection). Now we can apply the inclusion-exclusion formula for the union of three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

To find the cardinality of the union of the three sets, we need to find the cardinality of all the sets that occur on the right hand side of the above formula:

- $|A|$ is just the number of numbers in S that are divisble by 2, and this is just 50.
- $|B|$ is the number of numbers divisible by 3, and this is just 33.
- $|C|$ is the number of numbers divisible by 5, and this is just 20.
- $|A \cap B|$ is the number of numbers divisible by both 2 and 3 (i.e. by 6), and this is just 16.
- $|A \cap C|$ is the number of numbers divisible by both 2 and 5 (i.e. by 10), and this is just 10.
- $|B \cap C|$ is the number of numbers divisible by both 3 and 5 (i.e. by 15), and this is just 6.
- $|A \cap B \cap C|$ is the number of numbers divisible by 2, 3, and 5 (i.e. by 30), and this is just 3.

We can now substitute in all these values in the inclusion-exclusion formula we stated above, and we see that

$$|A \cup B \cup C| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74.$$

$\square$

In the next lecture we will see many more examples of inclusion-exclusion and will state a general formula for the union of an arbitrary number of sets.