

CS 206: Practice problems - Set I
Summer 2018
(Based on lectures 1-4)

Problem 1

Answer the following questions. Explain in one line how you got the answer.

1. Consider the word “OHMYGODIMONFIRE”. How many distinguishable ways are there to rearrange the letters?
2. How many sequences of 1s and -1s of length 10 are there that sum up to 2?
3. How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition of letters is allowed?
4. How many 4-letter words can be obtained using any of the 26 letters of the alphabet, if repetition is not allowed?
5. How many 4-letter words contain at least one repeated letter? **Hint: Difference method**
6. How many 4-letter words contain the letter X? **Hint: How many have no Xs and difference method**
7. How many 4-letter words consist of only the letters X and/or Y? (The words XXXX and YYYY are included in this count.)
8. How many ways are there of forming a committee of 10 members that has at least 4 women if you have to choose from 30 men and 20 women? **Hint: How many have at most 3 women? Use partition method (case analysis) for that and then use difference method to find what you really need.**
9. A group of n friends are all about to go separate ways in life and so they all hug each other. If one hug involves exactly 2 people and every pair of friends hugged exactly once, how many hugs were there in total?
10. A group of n friends are all about to go separate ways in life and so they all give each other gifts. If every pair of friends exchanged gifts exactly once, how many gifts were there in total?

Problem 2

Given a binary string S , let $\Delta(S)$ = number of 1's in S - number of 0's in S . S is balanced if $-2 \leq \Delta(S) \leq 2$. How many balanced binary strings of length n are there. Provide a formal proof for your answer. **Hint: Think about n being even/ n being odd. For each case, count for each of the possible values of Δ and add up.**

Problem 3

In each of the following cases, how many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 17$ where x_1, x_2, x_3, x_4, x_5 are non-negative integers.

1. $x_1 \geq 1$.
2. $x_i \geq 2$ for $i = 1, 2, \dots, 5$.
3. $0 \leq x_1 \leq 10$. **Hint: Find number of solutions when there is no constraint on x_1 , find solutions when $x_1 > 10$, and use difference method.**
4. $0 \leq x_1 \leq 3, 1 \leq x_2 \leq 4, x_3 \geq 15$. **Hint: if x_3 is given a 15 to satisfy to its requirements, then we are left with 2, and thus automatically in all possible solutions x_1, x_2 will be within their specified ranges.**
5. $x_1 \geq x_2$. **Hint: Do a case analysis: what is the number of solutions when $x_1 = 1, 2, \dots$, and get an expression for the number of solutions for each case, and then add them up. You will get a sum of binomial coefficients.**

Justify your answers.

Problem 4

What is the number of binary strings of length 100 that contain exactly three occurrences of the strings 01? **Hint: a binary string can be thought as having many zones — a zone is just a contiguous block of ones or zeros. For example, 000110001 has four zones, a zone of 0s, then a zone of 1s, a zone of 0s, and finally a zone of 1s (consisting of a single 1). If a string has exactly three occurrences of 01, how many zones can it have, and how should the zones be arranged?**

Problem 5

Suppose you have 10 bins, all of different color (they are identical except for their distinct colors), and 30 balls, each ball having a unique number painted on it (they are identical except for the distinct numbers on them).

1. In how many ways can you distribute the balls into the bins?

2. Suppose you only want to put exactly one ball in every bin (and discard the rest). How many ways of doing that are there?
3. Now suppose that all the bins are painted with the same color so that they all look the same. What will your answer to the second question be? **Hint: This becomes exactly the problem of choosing 10 balls from 30 balls since it doesn't matter which bin a ball goes into, and all that matters is which 10 balls were chosen from among the 30.**
4. Now suppose the numbers painted on the balls get erased and the balls look identical (but the bins still have their colors): what will your answer to the first question be? **Hint: This is the pirates and gold bars setup. What are the goldbars and pirates in this case though?**
5. Now suppose that the numbers on the balls are erased, and someone paints a third of the balls white, a third of the balls black, and the remaining third blue, while the bins still retain their original color: what's your answer to the first question? What about the second question? **Hint: For the first part of this question, find the number of ways of distributing only the white balls among the bins first, then find the number of ways of distributing only black balls, and also find the number of ways of distributing only the blue balls. Now notice that how you distribute white balls has no bearing whatsoever on how you distribute the other color balls, and so on. This suggests using product rule!**
For second part, notice that for every bin you have three choices now: use a blue, black, or a white ball. (Which specific ball of a particular color you use doesn't matter – the balls are all identical. All that matters is the color).
6. Assuming the setting of the previous part, how many ways are there of distributing all the balls so that every bin contains at least one ball of every color? **Hint: This problem is a bit tricky. In combinatorics, whenever we say "how many ways" we are basically interested in the number of visually distinct outcomes, and not "how many ways" the way you would mean it in colloquial English. What's the answer then?**
7. Now suppose that the distinct numbers are painted back onto the balls while still retaining their color (as in the previous two parts), and the bins still have their color. How many ways of distributing the balls are there so that every bin gets at least one ball of every color? **Hint: Can a bin get more than one ball of any color?**
8. Assume that the balls still have their colors and the distinct numbers on them from the previous part, however, now, the bins are all painted with the same color so that they become identical. What is your answer to the second question if you want to ensure that at least 8 white balls and at least 7 blue balls are discarded while distributing? **Hint: See hint for the third part.**