

Homework 5 (Solutions)

CS 206: Discrete Structures II
Summer 2018

Due: 11:59 PM EDT, Sunday, August 5th, 2018

Total points: 60

Name:

NetID:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided for each problem. This is very important! Even if you are using LaTeX, make sure your solutions fit into the given space.
2. Make sure you write your name and NetID in the space provided above.
3. After you are done writing the solutions, scan the sheets in the correct order into a PDF, and upload the PDF to Gradescope before the deadline mentioned above. No late submissions barring exceptional circumstances! **The submitted PDF should have all the 21 pages in the correct order even if you do not solve all the problems or use all the space provided for a problem.**
4. At the time of grading, the grader will randomly pick 3 problems out of the 8 problems to grade, and your final score (out of 60) will be based only on your solutions to the 3 selected problems.
5. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [20 pts]

Let Ω be a sample space and A, B be independent events in the sample space. Prove that the following pairs of events are also independent:

1. A^c, B
2. A^c, B^c

Proof. 1. Recall the basic fact that $B = (B \cap A) \cup (B \cap A^c)$ where $B \cap A$ and $B \cap A^c$ are disjoint. Using the sum rule:

$$P(B) = P(A \cap B) + P(A^c \cap B) \implies P(A^c \cap B) = P(B) - P(A \cap B).$$

Since A and B are independent, $P(A \cap B) = P(A)P(B)$. Plugging this into the above equation, we get

$$P(A^c \cap B) = P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(B)P(A^c),$$

where the last equation follows from the fact that $P(A^c) = 1 - P(A)$.

2. Using De Morgan's law,

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B).$$

Using inclusion-exclusion for two sets, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and using independence $P(A \cap B) = P(A)P(B)$, thus, putting these into the above equation, we get

$$\begin{aligned} P(A^c \cap B^c) &= 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - (P(A) + P(B) - P(A)P(B)) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c). \end{aligned}$$

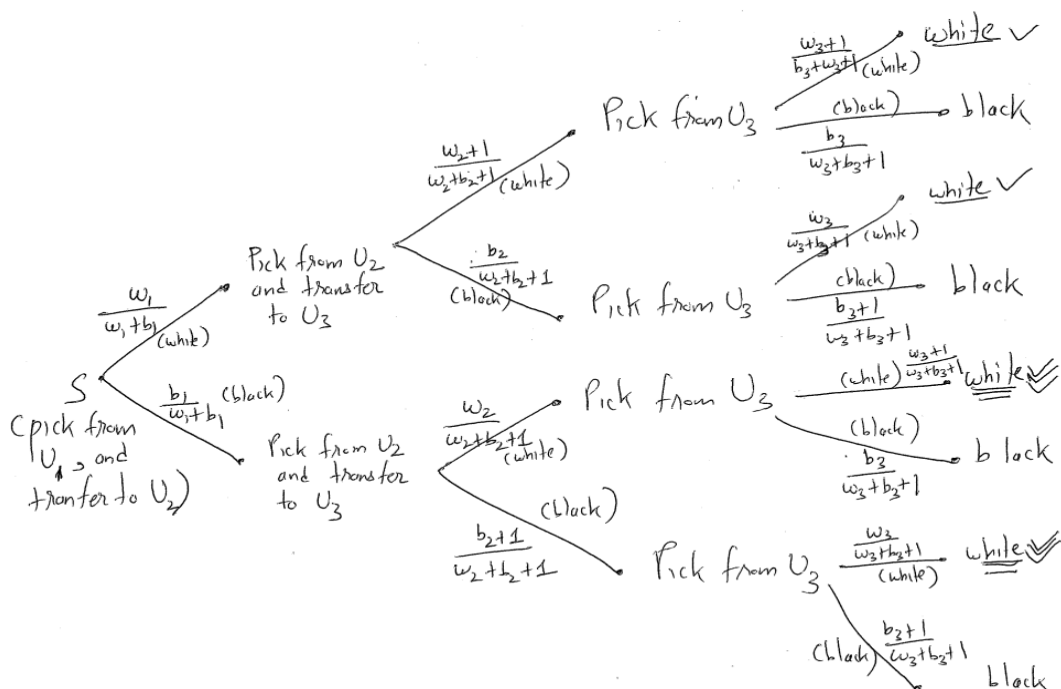
□

Problem 2. [20 pts]

I have three urns, U_1, U_2 and U_3 , such the first urn contains w_1 identical white balls and b_1 identical black balls, the second urn contains w_2 identical white balls and b_2 identical black balls, and the third urn contains w_3 identical white balls and b_3 identical black balls. I first choose a ball uniformly at random from U_1 (i.e., every ball is equally likely to be picked) and drop it into U_2 . Next, I pick a ball uniformly at random from U_2 and drop it into U_3 . Finally, I pick a ball uniformly at random from U_3 and it turns out to be white. What is the probability that the ball that was transferred from U_1 to U_2 was black? Show the main steps of your solution.

Hint: Note that this problem is asking for a conditional probability. Use the tree method.

Proof. The tree for this problem is as follows:



Let A be the event of transferring a black ball from U_1 to U_2 , and let B be the event that the ball that was picked from U_3 is white. Then the outcomes (leaves) that correspond to B are the ones with at least one check mark next to them. To find $P(B)$, we multiply probabilities along root-to-leaf for every checked leaf, and then all of them. We get

$$P(B) = \frac{w_3}{b_3 + w_3 + 1} \frac{b_2 + 1}{w_2 + b_2 + 1} \frac{b_1}{b_1 + w_1} + \frac{w_3 + 1}{b_3 + w_3 + 1} \frac{w_2}{w_2 + b_2 + 1} \frac{b_1}{b_1 + w_1} +$$

$$\frac{w_3}{b_3 + w_3 + 1} \frac{b_2}{w_2 + b_2 + 1} \frac{w_1}{b_1 + w_1} + \frac{w_3 + 1}{b_3 + w_3 + 1} \frac{w_2 + 1}{w_2 + b_2 + 1} \frac{w_1}{b_1 + w_1}.$$

The outcomes corresponding to $A \cap B$ are the ones with three check marks next to them (our underlined thrice) (Do you see why?). We get

$$P(A \cap B) = \frac{w_3}{b_3 + w_3 + 1} \frac{b_2 + 1}{w_2 + b_2 + 1} \frac{b_1}{b_1 + w_1} + \frac{w_3 + 1}{b_3 + w_3 + 1} \frac{w_2}{w_2 + b_2 + 1} \frac{b_1}{b_1 + w_1}.$$

Then, $P(A|B)$ is just $\frac{P(A \cap B)}{P(B)}$.

□

Problem 3. [20 pts]

You are playing a strange game of darts. It involves throwing 9 darts at an 18 inches \times 18 inches board which has a 9×9 grid painted on it such that every cell in the grid has dimensions 2 inches \times 2 inches (assume that the lines that define the grid are infinitesimally thin). You win the game if at least two darts land in the same cell. Since your aim is pretty bad, all we can say about your dart throws is that every dart is equally likely to land in any of the 81 cells (you can assume that every dart will always land in the interior of *some* cell and never on the boundary or outside the board.).

Now suppose that you end up winning the game. What is the probability that the first 7 darts you threw went into distinct cells but the last two ended up in the same cell? Show the main steps of your solution.

Hint: Again, this is a conditional probability problem.

Proof. Let's assume that you continue to throw the remaining darts even if you get two darts to land in the same cell (Does it matter?). Let B be the event of winning the game, and let A be the event of the first seven darts going into distinct cells and the last two going into the same cell (I will assume that the last two darts go into a cell *distinct* from the cells that the first seven went into).

The total number of outcomes is $(81)^9$ (81 possibilities for each dart, and 9 independent dart throws in all). The outcomes in B^c (i.e., losing the game) are those where all darts land in different cells. There are $\binom{81}{9}$ ways of deciding which distinct 9 cells the 9 darts will land in, and then $9!$ ways to arrange the 9 darts in those chosen cells, and so

$$P(B^c) = \frac{\binom{81}{9}9!}{(81)^9} \implies P(B) = 1 - \frac{\binom{81}{9}9!}{(81)^9}.$$

Note that $A \subseteq B$ (why?), and so $P(A \cap B) = P(A)$. To find the number of outcomes in A , note that there are $\binom{81}{7}$ ways of choosing cells for the first seven darts to land in, and $7!$ ways of arranging the seven darts in the chosen cells. After that there are $\binom{74}{1}$ ways of choosing a cell for the last two darts to land in (why?), and so

$$P(A) = \frac{\binom{81}{7}7!\binom{74}{1}}{(81)^9}.$$

Now, $P(A|B)$ can be found using the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)},$$

since in our case $A \cap B = A$. □

Problem 4. [20 pts]

Give examples of three events A, B and C such that A and B are independent, B and C are independent, but A and C are *not* independent. Explain your solution.

Proof. Consider the experiment of tossing a fair coin twice. Let A be the event of seeing a heads in the first toss, B the event of seeing a heads in the second toss, and C the event of seeing a tails in the first toss. Then clearly A, B and B, C are independent (why?), but A and C are not independent because $A \cap C = \emptyset$, i.e. A and C are mutually exclusive and hence cannot be independent. \square

Problem 5. [20 pts]

In this problem you will prove a general form of Bayes' theorem. Let Ω be a sample space. Let A_1, A_2, \dots, A_n be non-empty disjoint events in Ω such that

$$A_1 \cup A_2 \cup \dots \cup A_n = \Omega.$$

Show that for any event $B \subseteq \Omega$ and $1 \leq i \leq n$,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^n P(A_j)P(B|A_j)}.$$

Proof. Observe that the A_1, \dots, A_n form a partition of the probability space. Thus, using the law of total probability,

$$\Pr(B) = \sum_{j=1}^n \Pr(A_j) \Pr(B|A_j)$$

Using Bayes rule, for every $1 \leq i \leq n$,

$$\Pr(A_i|B) = \frac{\Pr(B|A_i)P(A_i)}{\Pr(B)}$$

Combining the two equations, we get

$$\Pr(A_i|B) = \frac{\Pr(A_i) \Pr(B|A_i)}{\sum_{j=1}^n \Pr(A_j) \Pr(B|A_j)}.$$

□

Problem 6. [20 pts]

An urn contains 100 balls that have the numbers 1 to 100 painted on them (every ball has a distinct number). You keep sampling balls uniformly at random (i.e., every ball is equally likely to be picked), one at a time, and *without* replacement. For $1 \leq i < j \leq 100$, let $E_{\{i,j\}}$ denote the event that the ball with the number j was taken out of the urn *before* the ball with the number i . Prove that the events $E_{\{45,89\}}$ and $E_{\{23,60\}}$ are independent. Are $E_{\{13,72\}}$ and $E_{\{72,99\}}$ also independent? Why or why not?

Hint: You might want to think of the outcomes as permutations of 1 to 100, and Ω as the set of all possible permutations of 1 to 100 (why?).

Proof. The outcomes are just permutations of 1 to 100, where the number j appears in position i if the ball with number j was picked in the i^{th} round (think of the balls being picked one at a time, and let's call the picking of one ball as one round).

What are the outcomes in $E_{\{i,j\}}$ for $i < j$? These are all the permutations where j appears before i in the permutation. To count the number of such permutations, first let's decide the positions into which we want to place i and j . There are $\binom{100}{2}$ ways of picking two spots for i and j . Once we pick the two spots, it is clear that j goes into the spot that appears first between the two spots. For example, if the spots we pick are 5, 40, then j must go into position 5 and i into 40 in order to have j appear before i . Now, we must arrange the remaining 98 elements in remaining 98 spots, and this can be done in $98!$ ways. So, the size of $E_{\{i,j\}}$ is $\binom{100}{2}98!$, and

$$P(E_{\{i,j\}}) = \frac{\binom{100}{2}98!}{100!} = \frac{1}{2}.$$

This is true regardless of what i and j are.

What are the outcomes in $E_{\{45,89\}} \cap E_{\{23,60\}}$? These are all the permutations where 89 appears before 45 and 60 appears before 23. Let's count the outcomes in the same way as before. Let's first pick 2 spots for 89 and 45. This can be done in $\binom{100}{2}$ ways. There is only one way to arrange 89 and 45 in those slots. Next, having placed 89 and 45, let's pick 2 out of the remaining 98 spots for 23 and 60. There are $\binom{98}{2}$ ways of doing this. Again, there is only one way to place 23, 60 in the two chosen spots. Now we can arrange the remaining 96 numbers in the remaining 96 spots, and there are $96!$ ways of doing so. Hence,

$$P(E_{\{45,89\}} \cap E_{\{23,60\}}) = \frac{\binom{100}{2}\binom{98}{2}96!}{100!} = \frac{1}{4}.$$

This proves that $E_{\{45,89\}}$ and $E_{\{23,60\}}$ are independent since

$$P(E_{\{45,89\}} \cap E_{\{23,60\}}) = P(E_{\{45,89\}})P(E_{\{23,60\}}).$$

We will now prove that $E_{\{13,72\}}$ and $E_{\{72,99\}}$ are not independent. To see this first recall that we have already proven that

$$P(E_{\{13,72\}})P(E_{\{72,99\}}) = \frac{1}{4},$$

since each of the terms is $1/2$ (see above). Let's try to show that $P(E_{\{13,72\}} \cap E_{\{72,99\}}) \neq \frac{1}{4}$. What are the outcomes in $E_{\{13,72\}} \cap E_{\{72,99\}}$? It's all permutations where 99 occurs before 72, which occurs before 13. Again, to count such permutations, let's first pick three spots for these numbers to be placed. This can be done in $\binom{100}{3}$ ways. After choosing the three spots, there is exactly one way to place 99, 72, 13 in them (why?). The remaining 97 elements can be arranged in $97!$ ways, and so

$$P(E_{\{13,72\}} \cap E_{\{72,99\}}) = \frac{\binom{100}{3} 97!}{100!} = \frac{1}{6} \neq \frac{1}{4}.$$

□

Problem 7. [20 pts]

Kirk, Spock, Uhura, and Scotty are playing a game of cards. They shuffle a complete deck of 52 cards and then distribute all the cards evenly among themselves. Given that each of them gets an ace, what is the probability that Kirk has the ace of spades?

Proof. The sample space is all possible ways to distribute the cards among the four people. How to count the number of possibilities:

Pick 4 sets of 13 cards in order. In other words, pick 13 cards from 52, give them to Kirk, next pick 13 from the rest, give them to Spock, and so on. Number of ways to do it is

$$\binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13}.$$

The size of the sample space is, thus, $\binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13}$.

The probability of interest is $P(\text{Kirk has an ace of spades} \mid \text{everyone gets an ace}) =$

$$\frac{P(\text{Kirk has an ace of spades} \cap \text{everyone gets an ace})}{P(\text{everyone gets an ace})}.$$

We need to find both the probabilities.

To find $P(\text{everyone gets an ace})$:

1. First distribute all the aces among the four people. There are $4!$ ways to do it, since there are 4 different aces.
2. Next, pick 4 sets of 12 cards from the remaining sequentially, and distribute. The number of ways to do it is

$$\binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12}$$

3. Hence, the number of ways so that everyone gets an ace is

$$4! \binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12}$$

4. Hence, $P(\text{everyone gets an ace}) = \frac{4! \binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12}}{\binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13}}$

Now, we need to find the probability that $P(\text{Kirk has an ace of spades} \cap \text{everyone gets an ace})$. How do we count this? Give Kirk the ace of spades, and distribute the rest of the aces among everyone else: $3!$ ways of doing that. Next, distribute the remaining cards (the non-ace cards, 48 of them) among the four players. Thus, the total number of ways to do the whole process is $3! \binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12}$, and so

$P(\text{Kirk has an ace of spades} \cap \text{everyone gets an ace}) =$

$$\frac{3! \binom{48}{12} \binom{36}{12} \binom{24}{12} \binom{12}{12}}{\binom{52}{13} \binom{39}{13} \binom{26}{13} \binom{13}{13}}.$$

Hence, the required conditional probability is

$\Pr(\text{Kirk has an ace of spades} \mid \text{everyone gets an ace})$

$$\begin{aligned}
 & \frac{3! \times \binom{48}{12} \times \binom{36}{12} \times \binom{24}{12} \times \binom{12}{12}}{\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13}} \\
 = & \frac{4! \times \binom{48}{12} \times \binom{36}{12} \times \binom{24}{12} \times \binom{12}{12}}{\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13}} \\
 = & \frac{3!}{4!} = \frac{1}{4}
 \end{aligned}$$

This shouldn't be surprising because of the nature of these events. Is there an easier way to do this? □

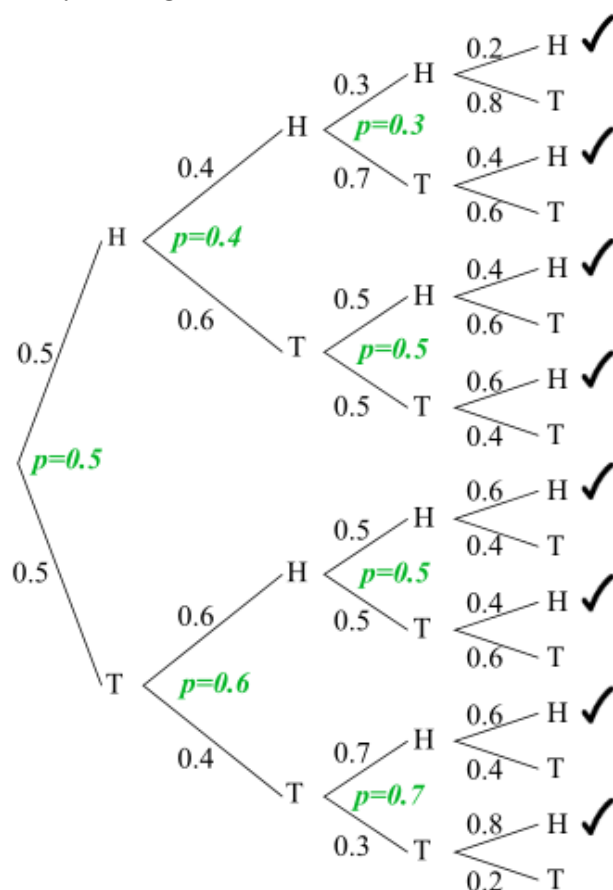
Problem 8. [20 pts] I have a special coin with “memory” that I will be tossing *exactly once* every day for the next 4 days (so tomorrow will be day 1). Let p_i denote the probability that the coin shows up heads when it’s tossed on day i (this means it shows up tails with probability $1 - p_i$ when tossed on day i). Did I mention the coin has “memory”? By that I mean that:

1. On day 1, the coin is equally likely to show up as heads or tails, i.e. $p_1 = 1/2$.
2. For every subsequent day i , $i > 1$, $p_i = p_{i-1} - 0.1$ if the coin toss on day $i - 1$ resulted in heads, and $p_i = p_{i-1} + 0.1$ if the coin toss on day $i - 1$ resulted in tails.

What is the probability that the coin toss on day 4 results in heads?

Hint: Use the tree method and/or the law of total probability.

Proof. Using the tree method we can create the tree and mark the desired outcomes.



To calculate the probability a marked outcome we will multiply the probabilities along the root-to-leaf path for that outcome. To get the total probability, we add the probabilities we computed for each marked outcome. We get:

$$\begin{aligned}
P(\text{Last flip is Head}) &= 0.5 \times 0.4 \times 0.3 \times 0.2 + 0.5 \times 0.4 \times 0.7 \times 0.4 \\
&\quad + 0.5 \times 0.6 \times 0.5 \times 0.4 + 0.5 \times 0.6 \times 0.5 \times 0.6 \\
&\quad + 0.5 \times 0.6 \times 0.5 \times 0.6 + 0.5 \times 0.6 \times 0.5 \times 0.4 \\
&\quad + 0.5 \times 0.4 \times 0.7 \times 0.6 + 0.5 \times 0.4 \times 0.3 \times 0.8 \\
&= 0.5
\end{aligned}$$

□