

Quiz 3 (Solutions)

CS 206: Discrete Structures II
Summer 2018

August 09, 2018

Total points: 40

Duration: ∞

Name:

NetID:

INSTRUCTIONS:

1. There are 3 problems in all. Problem 1 and 2 are worth 10 points each. Problem 3 has two parts, each part being worth 10 points.
2. Make sure you write your solutions **ONLY** in the space provided below each problem. There is plenty of space for each problem. You can ask for scrap paper for scratchwork.
3. You are allowed to refer to physical copies of any books or notes during the quiz. However, the use of electronic devices is not permitted during the quiz, and violating this will lead to the cancellation of your exam.
4. Make sure you write your name and NetID in the space provided above.
5. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the quiz, and might even be reported to the authorities!

Problem 1. [10 pts]

Let X be a discrete random variable defined on a finite probability space (Ω, P) . Suppose that $\text{Range}(X) = \{-2, -1, 0, 1, 2\}$, and $\mathbb{E}[X] = -\frac{1}{4}$. Furthermore,

$P(X = -2) = p$
$P(X = -1) = \frac{1}{16}$
$P(X = 0) = q$
$P(X = 1) = \frac{1}{16}$
$P(X = 2) = \frac{1}{8}$

Find the values of p and q . Express your answers as fractions reduced to their lowest terms. Show all the steps of your solution.

Hint: What is $\sum_{a \in \text{Range}(X)} P(X = a)$?

Proof. Since $\mathbb{E}[X] = \sum_{a \in \text{Range}(X)} aP(X = a)$, and also since it's given that $\mathbb{E}[X] = -1/4$, we write

$$\begin{aligned} (-2 \times p) + \left(\frac{1}{16} \times (-1)\right) + (0 \times q) + \left(1 \times \frac{1}{16}\right) + \left(2 \times \frac{1}{8}\right) &= -\frac{1}{4} \\ \implies -2p - \frac{1}{16} + \frac{1}{16} + \frac{1}{4} &= -\frac{1}{4} \implies p = \frac{1}{4}. \end{aligned}$$

Moreover, we also know that for any random variable X ,

$$\sum_{a \in \text{Range}(X)} P(X = a) = 1,$$

and so

$$p + \frac{1}{16} + q + \frac{1}{16} + \frac{1}{8} = 1 \implies q = \frac{3}{4} - p = \frac{1}{2}.$$

□

Problem 2. [10 pts]

Consider the experiment of rolling a fair dice 90 times. Let X be the number of times an even number is rolled and let Y be the number of times a '6' is rolled. Define the random variable Z as $Z = X - Y$. Find the expected value of Z . Simplify the final answer as much as possible (Do NOT leave your answer in terms of factorials, binomial coefficients, or powers!). Show all the steps of your solution.

Proof. Since X is the total number of times a 2, a 4 or a 6 is rolled (i.e., total number of times an even number is rolled), and Y is the number of times a 6 is rolled, clearly, $Z = X - Y$ is the total number of times a 2 or a 4 is rolled.

Let Z_i , for $1 \leq i \leq 90$, be the indicator random variable that takes value 1 if a 2 or 4 is rolled in the i^{th} roll and takes value 0 otherwise. Then clearly,

$$Z = \sum_{i=1}^{90} Z_i.$$

Thus, using linearity of expectation,

$$\mathbb{E}[Z] = \sum_{i=1}^{90} \mathbb{E}[Z_i].$$

For any i , since Z_i is an indicator random variable,

$$\mathbb{E}[Z_i] = P(Z_i = 1) = P(\text{a 2 or a 4 is rolled in the } i^{\text{th}} \text{ roll}).$$

Since all 90 dice roll are independent and fair, the probability of rolling a 2 or 4 in roll i is $\frac{2}{6} = \frac{1}{3}$. Thus, $\mathbb{E}[Z_i] = \frac{1}{3}$, and thus

$$\mathbb{E}[Z] = 90 \times \frac{1}{3} = 30.$$

□

Problem 3. [10 + 10 = 20 pts]

Suppose we roll a fair dice n times. Let X_1, \dots, X_n be random variables such that X_i is equal to the number observed in the i^{th} roll.

1. Let $2 \leq i \leq n$. What is $P(X_i > X_1)$, i.e. what is the probability that the number observed in the i^{th} roll is larger than the number observed in the first roll?
2. Define X to be the total number of rolls (other than the first one) in which the number observed is larger than the number observed in the first roll. What is $\mathbb{E}[X]$?

Show all the steps of your solution.

Proof. 1. All dice rolls are fair and independent. Fix any $1 \leq i \leq n$. We want to calculate $P(X_i > X_1)$. Note that X_1 takes a value in $\{1, \dots, 6\}$, and these are the only possible values X_1 can take, i.e. the events $[X_1 = 1], [X_1 = 2], \dots, [X_1 = 6]$ form a partition of Ω , and thus, using the law of total probability,

$$P(X_i > X_1) = \sum_{k=1}^6 P(X_1 = k)P(X_i > X_1 | X_1 = k) = \frac{\sum_{k=1}^6 P(X_i > X_1 | X_1 = k)}{6}.$$

Here the last equality uses the fact that for every $1 \leq k \leq 6$, $P(X_1 = k) = 1/6$.

Next, note that $P(X_i > X_1 | X_1 = k)$ is basically the same $P(X_i > k | X_1 = k)$. Also, since X_i and X_1 are independent (why?),

$$P(X_i > k | X_1 = k) = P(X_i > k) = \frac{6 - k}{6}.$$

(Do you see why $P(X_i > k) = (6 - k)/6$?). Thus,

$$P(X_i > X_1) = \frac{\sum_{k=1}^6 \frac{6-k}{6}}{6} = \frac{\sum_{k=1}^6 (6 - k)}{36} = \frac{36 - \sum_{k=1}^6 k}{36} = \frac{36 - 21}{36} = \frac{15}{36} = \frac{5}{12}.$$

2. For $2 \leq i \leq n$, let Z_i be the indicator random variable for the event that the number observed in the i^{th} roll is larger than that observed in the first roll, i.e. $X_i > X_1$. Then,

$$X = Z_2 + Z_3 + \dots + Z_n.$$

Using linearity of expectation,

$$\mathbb{E}[X] = \sum_{i=2}^n \mathbb{E}[Z_i].$$

Since each Z_i is an indicator random variable, we know that $\mathbb{E}[Z_i] = P(Z_i = 1) = P(X_i > X_1) = \frac{5}{12}$, where the last equality follows from the first part of this problem. Thus,

$$\mathbb{E}[X] = \sum_{i=2}^n \mathbb{E}[Z_i] = \sum_{i=2}^n \frac{5}{12} = \frac{5(n-1)}{12}.$$

□