

# Midterm exam

CS 206: Discrete Structures II  
Summer 2018

(Sample - Solutions)

**Total points:** 130

**Duration:** 2 hours

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**Name:**

**NetID:**

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## INSTRUCTIONS:

1. You have to solve 13 problems in 2 hours. Each problem is worth 10 points. To get full points for a problem, you must give details for all the steps involved in solving the problem AND arrive at the correct answer. Giving partial details or arriving at the wrong answer will result in a partial score.
2. You may leave your answer in terms of factorials, binomial coefficients, and/or power of numbers.
3. Make sure you write your solutions **ONLY** in the space provided below each problem. There is plenty of space for each problem. You can use the back of the sheets for scratchwork.
4. You may refer to physical copies of any books or lecture notes you want to during the exam. However, the use of any electronic devices will lead to cancellation of your exam and a zero score, with the possibility of the authorities getting involved.
5. Make sure you write your name and NetID in the space provided above.
6. If we catch you cheating, or later suspect that your answers were copied from someone else, you will be given a zero on the exam, and might even be reported to the authorities!

**Problem 1.** [10 pts]

How many 11 letter words contain only vowels?

*Proof.* There are 5 vowels, and so for each of the 11 locations in the word we have 5 choices. Also, the choices at the 11 locations in the word are all independent (they don't affect each other) and so we can use the product rule. Thus, the answer is  $5^{11}$ .

□

**Problem 2.** [10 pts]

How many 11 letter words contain only consonants?

*Proof.* There are 21 consonants, and we are looking at strings/words of length 11. This means that for every position in the string/word, we have 21 choices of consonants, and these choices are all independent. Thus, using the product rule, the answer is  $(21)^{11}$ .  $\square$

**Problem 3.** [10 pts]

How many 11 letter words contain BOTH vowels and consonants?

*Proof.* Let  $A$  be the set of 11 letter words that contain both vowels and consonants. Let  $S$  be all possible strings that can be made using a-z. Thus,  $|S|$  is simply  $(26)^{11}$  using the product rule since we have 26 choices for every position, and the choices at the different positions are all independent.

We will now find  $|S - A|$  or  $|A^c|$  and then use that to find  $|A|$ . Note that  $S - A$  is the set of all length 11 strings that either do not contain any vowels or do not contain any letters. The number of strings of length 11 that do not contain vowels, or in other words, contain only consonants is  $(21)^{11}$  (using Problem 2), and similarly the strings that do not contain any consonants is  $(5)^{26}$  (Using Problem 3). Thus, the total number of strings that either don't contain vowels or don't contain consonants is  $(5)^{26} + (21)^{11}$ . This follows from the sum rule since the set of words that don't contain vowels is disjoint from those that don't contain consonants.

Appealing to the difference method, we see that  $|A| = |S| - |S - A| = (26)^{11} - (21)^{11} - (5)^{11}$ .  $\square$

**Problem 4.** [10 pts]

In how many ways can you distribute 1000 identical M&M's among 200 kids?

*Proof.* The M&M's are identical and the kids are distinct (identical kids is absurd) so we are trying to distribute 1000 identical objects among 200 distinct parties, or 1000 identical balls into 200 distinct bins. The number of ways of distributing is the same as binary strings with 1000 zeros and 199 ones since we can setup a bijection between the two: the 1000 zeros represent the M&M's and the 199 ones serve as “dividers” partitioning the zeros into 200 parts. Since the number of strings of this form is  $\binom{1199}{199}$ , this is also the number of ways of distributing among the kids.

□

**Problem 5.** [10 pts]

In how many ways can you distribute 1000 identical M&M's among 200 kids, if you are allowed to keep some of the M&M's for yourself?

*Proof.* Let  $x_i$  denote the number of M&M's received by the  $i^{th}$  kid ( $1 \leq i \leq 200$ ). Let  $y$  be the number of M&M's you keep for yourself. Then the number of ways to distribute is same as the number of non-negative integer solutions to

$$x_1 + \dots + x_{200} + y = 1000.$$

The number of integers solutions to the above equation is the same as the number of binary strings with 1000 zeros and 200 ones, and is  $\binom{1200}{200}$ .

Thus, the number of ways of distributing is  $\binom{1200}{200}$ . □

**Problem 6.** [10 pts]

What is the number of integer solutions to the equation  $x_1 + x_2 + x_3 + x_4 = 30$  with  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $x_3 \geq 3$ , and  $x_4 \leq 4$ ?

*Proof.* We will use the difference method.

Let  $A$  denote the set of integer solutions to the given equation with the given constraints. Let  $S$  be the set of solutions to the given equation with the constraints  $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$ . To find the number of solutions in  $S$ , we first give 3 to  $x_3$  to satisfy its minimum requirement, that leaves us with 27, and then we know that the number of solutions is just the number of binary strings with 27 zeros and 3 ones, and is equal to  $\binom{30}{3}$ , i.e.  $|S| = \binom{30}{3}$ .

Notice that  $S - A$  is just the set of all integer solutions to the above equation with constraints  $x_1, x_2 \geq 0, x_3 \geq 3$ , and  $x_4 > 4$  (or equivalently,  $x_4 \geq 5$ ). We can easily count the number of solutions under these constraints by first satisfying the minimum requirements of  $x_3$  and  $x_4$  by giving them 3 and 5 respectively, and then distributing the remaining 22 among the 4 variables, which can be done in  $\binom{25}{3}$  ways (this is the number of binary strings with 22 zeros and 3 ones). Thus,  $|S - A| = \binom{25}{3}$ .

This means  $|A| = |S| - |S - A| = \binom{30}{3} - \binom{25}{3}$ . □

**Problem 7.** [10 pts]

What is the coefficient of  $x^8y^6z^1$  in  $(-12x + 3y + 4z)^{15}$ ?

*Proof.* We know that, using the multinomial theorem, the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  ( $k_1 + k_2 + k_3 = n$ ) in  $(ax + by + cz)^n$  is

$$a^{k_1}b^{k_2}c^{k_3} \cdot \frac{n!}{k_1!k_2!k_3!}.$$

In our case,  $a = -12$ ,  $b = 3$ ,  $c = 4$ , and  $k_1 = 8$ ,  $k_2 = 6$ ,  $k_3 = 1$ , and thus the coefficient of  $x^8y^6z^1$  in  $(-12x + 3y + 4z)^{15}$  is

$$(-12)^8 \cdot 3^6 \cdot 4 \cdot \frac{15!}{8!6!1!}.$$

□



**Problem 8.** [10 pts]

How many different strings of length 12 can you make using the letters of the word “**vicissitudes**”?

*Proof.* The given string has length 12 and contains 1 “v”, 3 “i”, 1 “c”, 3 “s”, 1 “t”, 1 “u”, 1 “d”, and 1 “e”. Thus, using the permutations with repetitions formula or Bookkeeper’s rule, we have that the number of distinct permutations of the given string is

$$\frac{12!}{1!3!1!3!1!1!1!} = \frac{12!}{(3!)^2}.$$

□

**Problem 9.** [10 pts]

Consider  $S = \{1, \dots, 100\}$ . You want to pick a subset of  $S$  that contains at least one number from each of the following sets:

- Even numbers.
- Multiples of three.
- Multiples of five.

In how many ways can you do so?

*Proof.* We will precompute some counts that will be useful later:

1. The number of numbers between 1 and 100 inclusive that are divisible by 2 is 50.
2. The number of numbers between 1 and 100 inclusive that are divisible by 5 is 20.
3. The number of numbers between 1 and 100 inclusive that are divisible by 3 is 33.
4. The number of numbers between 1 and 100 inclusive that are divisible by both 2 and 5 is 10.
5. The number of numbers between 1 and 100 inclusive that are divisible by both 2 and 3 is 16.
6. The number of numbers between 1 and 100 inclusive that are divisible by both 3 and 5 is 6.
7. The number of numbers between 1 and 100 inclusive that are divisible by 2, 3 and 5 is 3.

We can use the values computed above along with inclusion-exclusion to compute that the number of numbers between 1 and 100 inclusive that are:

1. divisible by 2 OR 3 is  $50 + 33 - 16 = 67$ .
2. divisible by 2 OR 5 is  $50 + 20 - 10 = 60$ .
3. divisible by 3 OR 5 is  $20 + 33 - 6 = 47$ .
4. divisible by 2 OR 3 OR 5 is  $50 + 33 + 20 - 10 - 16 - 6 + 3 = 74$ .

Let  $A$  be the set of all subsets of  $S$  that contain at least one multiple of 2, let  $B$  be the set of all subsets of  $S$  that contain at least one multiple of 3, and let  $C$  be the set of all subsets of  $S$  that contain at least one multiple of 5. We are interested in finding  $|A \cap B \cap C|$ .

Let  $T$  be the set of all possible subsets of  $S$ . We will first find  $|(A \cap B \cap C)^c| = |A^c \cup B^c \cup C^c|$ , where the complements are defined with respect to  $T$  being the universe. Using the principle of inclusion-exclusion we can write:

$$\begin{aligned} |A^c \cup B^c \cup C^c| &= |A^c| + |B^c| + |C^c| \\ &\quad - (|A^c \cap B^c| + |A^c \cap C^c| + |B^c \cap C^c|) \\ &\quad + |A^c \cap B^c \cap C^c| \end{aligned}$$

We now need to compute the cardinality of each of the sets that occurs in the above expression:

- $A^c$  is the set of all subsets of  $S$  that do not contain multiples of 2. Using the counts computed above, we know there are 50 numbers not divisible by 2 inside  $S$  and so  $|A^c| = 2^{50}$ .
- Similarly,  $|B^c| = 2^{80}$ .
- Using the same arguments as above, we can compute  $|C^c| = 2^{67}$ .
- $A^c \cap B^c$  is all subset of  $S$  that contain neither multiples of 2 nor 3. Using the counts computed above, we know that the number of numbers in  $S$  that are neither divisible by 2 nor 3 is  $100 - 67 = 33$ , and thus  $|A^c \cap B^c| = 2^{33}$ .
- Using similar arguments as above, we can conclude that  $|A^c \cap C^c| = 2^{40}$ , and  $|B^c \cap C^c| = 2^{53}$ .
- $A^c \cap B^c \cap C^c$  is the set of all subsets of  $S$  that are free of multiples of 2, multiples of 3, and multiples of 5. Using the counts above, we know that the number of numbers in  $S$  that are not divisible by either of 2, 3 or 5 is  $100 - 74 = 26$ , and thus  $|A^c \cap B^c \cap C^c| = 2^{26}$ .

We can conclude that

$$|(A \cap B \cap C)^c| = 2^{50} + 2^{67} + 2^{80} - 2^{33} - 2^{40} - 2^{53} + 2^{26}.$$

Since  $|T| = 2^{100}$ , we have that  $|(A \cap B \cap C)| = |T| - |(A \cap B \cap C)^c|$  which is equal to

$$2^{100} - (2^{50} + 2^{67} + 2^{80} - 2^{33} - 2^{40} - 2^{53} + 2^{26}).$$

□

More space for Problem 9

**Problem 10.** [10 pts]

How many integer solutions are there to the equation  $x_1 \times x_2 \times \dots \times x_n = -1$  if for every  $1 \leq i \leq n$  we have that  $-1 \leq x_i \leq 1$ ?

**Hint:** Surely none of the  $x_i$ s can be equal to zero since otherwise their product will become 0. This means each  $x_i$  is either 1 or  $-1$ . Think about the scenarios in which the product of all the  $x_i$ s can be  $-1$ ? Can it be  $-1$  if exactly 4 of the  $x_i$ s are  $-1$  and the rest are 1?

*Proof.* No variables can be 0 otherwise the whole product is zero. Observe that the product  $x_1 \times \dots \times x_n$  is  $-1$  if and only if the number of variables that are equal to  $-1$  is odd. This is because an odd number of  $-1$ 's will always multiply to give  $-1$ , and all the other variables (the ones that are not set to  $-1$ ) are all 1 and don't affect the product. So we are interested in the number of ways we can set an odd number of variables to be  $-1$  (and the rest are set to 1). For every odd number  $k$  between 0 and  $n$ , there are exactly  $\binom{n}{k}$  ways of choosing  $k$  variables that are to be set to  $-1$ , so the total number of ways in which you can set an odd number of variables to  $-1$  is, using the partition rule, equal to

$$\sum_{0 \leq k \leq n, k \text{ is odd}} \binom{n}{k}.$$

We know that the sum of all odd binomial coefficients is equal the sum of all even binomial coefficients, while the former and latter sum up together to  $2^n$ . This means that

$$\sum_{0 \leq k \leq n, k \text{ is odd}} \binom{n}{k} = 2^{n-1}.$$

□

More space for Problem 10

**Problem 11.** [10 pts]

Prove using induction that  $n! > 2^n$  for  $n \geq 4$ .

*Proof.* For the base case, let  $n = 4$ , then  $n! = 24$  and  $2^n = 16$ , and so the base case is true since  $24 > 16$ . Let  $n \geq 4$  be arbitrary and suppose  $(n-1)! > 2^{n-1}$ . We want to show that  $n! > 2^n$ . Since we know that  $(n-1)! > 2^{n-1}$  we can multiply both sides by  $n$  and we get  $n \cdot (n-1)! > n \cdot 2^{n-1} > 2 \cdot 2^{n-1}$ , where the last inequality follows from the fact that  $n \geq 4 > 2$ . Since  $n \cdot (n-1)! = n!$  and  $2 \cdot 2^{n-1} = 2^n$ , this finishes the proof.  $\square$

More space for Problem 11



**Problem 12.** [10 pts]

A group consists of the following people: a boy and a girl from NJ, a boy and a girl from Delaware, a boy and a girl from Texas, and a boy and a girl from California. We want to match every girl to a boy so that the following conditions are satisfied:

- No two girls are matched to the same boy.
- A girl is not matched to the boy from her state.

In how many ways can you do the matching?

*Proof.* Let  $A_1$  be the set of all matchings where the girl from NJ is not matched to the boy from NJ, let  $A_2$  be the set of all matching where the girl from Delaware is not matched to the boy from Delaware, let  $A_3$  be all the matching where the girl from Texas is not matching to the boy from Texas, and let  $A_4$  be the set of matchings where the girl from California is not matched to the boy from California. Then we want to find  $|(A_1 \cap \dots \cap A_4)^c|$

Let  $S$  be the set of all possible matchings where no two girls are matched to the same boy, and every girl is matched to some boy. We will compute the cardinality of the complement of  $A_1 \cap \dots \cap A_4$  inside  $S$ , i.e.  $|(A_1 \cap \dots \cap A_4)^c| = |A_1^c \cup \dots \cup A_4^c|$ .

We will use the principle of inclusion-exclusion:

$$\begin{aligned} |A_1^c \cup \dots \cup A_4^c| &= |A_1^c| + \dots + |A_4^c| \text{ (Level 1 terms)} \\ &\quad - \left( \sum_{1 \leq i < j \leq 4} |A_i^c \cap A_j^c| \right) \text{ (Level 2 terms)} \\ &\quad + \left( \sum_{1 \leq i < j < k \leq 4} |A_i^c \cap A_j^c \cap A_k^c| \right) \text{ (Level 3 terms)} \\ &\quad - |A_1^c \cap \dots \cap A_4^c| \text{ (Level 4 terms)}. \end{aligned}$$

Note that  $|A_1^c|$  is the number of ways of matching where the girl from NJ is matched with the boy from NJ, and thus there are  $3!$  ways to match the other people. In fact, by symmetry,  $|A_2^c| = |A_3^c| = |A_4^c| = |A_1^c| = 3!$ . There are 4 such terms so we get a total contribution of  $4 \cdot 3!$  from Level 1 terms.

As for  $|A_1^c \cap A_2^c|$ , this is the number of ways of matching where the girls from NJ and Delaware matched with the boys from their own state, and thus there are  $2!$  ways of watching the remaining people. By symmetry, we have that for every  $1 \leq i < j \leq 4$ ,  $|A_i^c \cap A_j^c| = 2!$ . Since there are  $\binom{4}{2}$  terms in all and each is equal to  $2!$ , we get a contribution of  $-\binom{4}{2} \cdot 2!$  from the Level 2 terms (note the  $-$  sign — this is level 2!)

At Level 3, we have terms of the form  $|A_i^c \cap A_j^c \cap A_k^c|$ , i.e. three girls are matched to boys from their own states, thus the number of ways to match the remaining one girl and boy is just 1. This is true for every term at Level 3, and since there are  $\binom{4}{3}$  terms, we have a total contribution of  $\binom{4}{3} \cdot 1$ .

At the last level, we have only one term which captures the number of ways in which every girl is matched to the boy from her state. There is only way to do this, and there is only  $\binom{4}{4} = 1$  term, so the total contribution is just  $-1$  (the sign is because its the fourth level, and hence the sign is  $(-1)^{4-1} = -1$ ).

Combining all the terms we get that

$$|A_1^c \cup \dots \cup A_4^c| = \binom{4}{1} 3! - \binom{4}{2} 2! + \binom{4}{3} 1! - \binom{4}{4}.$$

Since we are interested in finding  $|S - (A_1^c \cup \dots \cup A_4^c)| = |A_1 \cap \dots \cap A_4|$ , and  $|S| = 4!$ , we get that

$$|A_1 \cap \dots \cap A_4| = 4! - \left( \binom{4}{1} 3! - \binom{4}{2} 2! + \binom{4}{3} 1! - \binom{4}{4} \right).$$

□

More space for Problem 12

**Problem 13.** [10 pts]

Show that:

$$100 \cdot 2^{99} = \sum_{k=1}^{100} k \cdot \binom{100}{k}.$$

**Hint:** Use the binomial theorem with  $y$  set to 1, differentiate both sides with respect to  $x$ , and then set the value of  $x$  appropriately.

*Proof.* Using the binomial theorem with  $y$  set to 1, we know that  $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$ . Taking derivative with respect to  $x$  on both sides, we get

$$n(1+x)^{n-1} = \frac{d}{dx} \left( \binom{n}{0} + \binom{n}{1}x + \dots + \binom{n}{n}x^n \right) = \binom{n}{1} + 2\binom{n}{2}x + \dots + n\binom{n}{n-1}x^{n-1}.$$

Setting  $x = 1$  on both the sides, we get

$$n2^{n-1} = \sum_{k=1}^n k \binom{n}{k}.$$

Substituting  $n = 100$  gives us the desired equation. □

More space for Problem 13