Homework 5

CS 206: Discrete Structures II
Summer 2018

Due: 11:59 PM EDT, Sunday, August 5th, 2018

Total points: 60

Name:

NetID:

INSTRUCTIONS:

1. Print all the pages in this document and make sure you write the solutions in the space provided for each problem. This is very important! Even if you are using LaTeX, make sure your solutions fit into the given space.
2. Make sure you write your name and NetID in the space provided above.
3. After you are done writing the solutions, scan the sheets in the correct order into a PDF, and upload the PDF to Gradescope before the deadline mentioned above. No late submissions barring exceptional circumstances! **The submitted PDF should have all the 21 pages in the correct order even if you do not solve all the problems or use all the space provided for a problem.**
4. At the time of grading, the grader will randomly pick 3 problems out of the 8 problems to grade, and your final score (out of 60) will be based only on your solutions to the 3 selected problems.
5. As mentioned in the class, you may discuss with others but my suggestion would be that you try the problems on your own first. Even if you do end up discussing, make sure you understand the solution and write it in your own words. If we suspect that you have copied verbatim, you may be called to explain the solution.

Problem 1. [20 pts]
Let Ω be a sample space and A, B be independent events in the sample space. Prove that the following pairs of events are also independent:

1. A^c, B
2. A^c, B^c

More space for Problem 1:

Problem 2. [20 pts]
I have three urns, U_1, U_2 and U_3, such the first urn contains w_1 identical white balls and b_1 identical black balls, the second urn contains w_2 identical white balls and b_2 identical black balls, and the third urn contains w_3 identical white balls and b_3 identical black balls. I first choose a ball uniformly at random from U_1 (i.e., every ball is equally likely to be picked) and drop it into U_2. Next, I pick a ball uniformly at random from U_2 and drop it into U_3. Finally, I pick a ball uniformly at random from U_3 and it turns out to be white. What is the probability that the ball that was transferred from U_1 to U_2 was black? Show the main steps of your solution.

Hint: Note that this problem is asking for a conditional probability. Use the tree method.

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Problem 3. [20 pts]
You are playing a strange game of darts. It involves throwing 9 darts at an 18 inches $\times$ 18 inches board which has a 9×9 grid painted on it such that every cell in the grid has dimensions 2 inches $\times$ 2 inches (assume that the lines that define the grid are infinitesimally thin). You win the game if at least two darts land in the same cell. Since your aim is pretty bad, all we can say about your dart throws is that every dart is equally likely to land in any of the 81 cells (you can assume that every dart will always land in the interior of *some* cell and never on the boundary or outside the board.).
Now suppose that you end up winning the game. What is the probability that the first 7 darts you threw went into distinct cells but the last two ended up in the same cell? Show the main steps of your solution.

Hint: Again, this is a conditional probability problem.

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Problem 4. [20 pts]
Give examples of three events A, B and C such that A and B are independent, B and C are independent, but A and C are *not* independent. Explain your solution.

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Problem 5. [20 pts]
In this problem you will prove a general form of Bayes' theorem. Let Ω be a sample space. Let $A_1, A_2, \ldots, A_n$ be non-empty disjoint events in Ω such that

$$A_1 \cup A_2 \cup \ldots \cup A_n = \Omega.$$

Show that for any event $B \subseteq \Omega$ and $1 \leq i \leq n$,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}.$$

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Problem 6. [20 pts]
An urn contains 100 balls that have the numbers 1 to 100 painted on them (every ball has a distinct number). You keep sampling balls uniformly at random (i.e., every ball is equally likely to be picked), one at a time, and *without* replacement. For $1 \le i < j \le 100$, let $E_{\{i,j\}}$ denote the event that the ball with the number j was taken out of the urn *before* the ball with the number i. Prove that the events $E_{\{45,89\}}$ and $E_{\{23,60\}}$ are independent. Are $E_{\{13,72\}}$ and $E_{\{72,99\}}$ also independent? Why or why not?

Hint: You might want to think of the outcomes as permutations of 1 to 100, and Ω as the set of all possible permutations of 1 to 100 (why?).

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Problem 7. [20 pts]
Kirk, Spock, Uhura, and Scotty are playing a game of cards. They shuffle a complete deck of 52 cards and then distribute all the cards evenly among themselves. Given that each of them gets an ace, what is the probability that Kirk has the ace of spades?

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Problem 8. [20 pts] I have a special coin with "memory" that I will be tossing *exactly once* every day for the next 4 days (so tomorrow will be day 1). Let p_i denote the probability that the coin shows up heads when it's tossed on day i (this means it shows up tails with probability $1 - p_i$ when tossed on day i). Did I mention the coin has "memory"? By that I mean that:

1. On day 1, the coin is equally likely to show up as heads or tails, i.e. $p_1 = 1/2$.
2. For every subsequent day i, $i > 1$, $p_i = p_{i-1} - 0.1$ if the coin toss on day $i - 1$ resulted in heads, and $p_i = p_{i-1} + 0.1$ if the coin toss on day $i - 1$ resulted in tails.

What is the probability that the coin toss on day 4 results in heads?

Hint: Use the tree method and/or the law of total probability.

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