

Lecture 2: Sum rule, partition method, difference method, bijection method, product rules

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References: Relevant parts of chapter 15 of the Math for CS book.

1 Sum rule and the partition method

Recall the sum rule from the first lecture: if A and B are disjoint sets, then

$$|A \cup B| = |A| + |B|.$$

We can generalize this to any finite collection of disjoint sets: if A_1, \dots, A_n are disjoint, then

$$|A_1 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i|.$$

Suppose you are asked to compute the cardinality of a set A . One way to do so is to partition A into finitely many disjoint parts A_1, \dots, A_n such that together those parts cover A (that is what a partition means in the first place, so we are being redundant. I just wanted to make sure you understand what we mean by partitioning A), compute the cardinality of each of the parts separately, and then recover the cardinality of A using the sum rule. Let's see this in action using a problem:

Question . Suppose we roll two die, a black one and a white one, together. By a configuration or outcome we mean the numbers that show up on the die. We can specify an outcome using a tuple (b, w) where the first component is the number that shows up on the black dice, and the second component is the number that shows up on the white dice. If A is the set of all outcomes, what is $|A|$?

To compute the cardinality of the set of all possible outcomes, we break A into 6 parts: A_1, \dots, A_6 , where A_i is the set of all outcomes where the number on the black dice is equal to i . Notice that for each i we have that $|A_i| = 6$ (why?). Moreover, for every $i \neq j$, we have that $A_i \cap A_j = \emptyset$ (why?), and that $\bigcup_{i=1}^6 A_i = A$. Thus, using the sum rule, $|A| = \sum_{i=1}^6 |A_i| = 36$.

Use the same idea as above to solve the following problems:

Question . Suppose the setup is exactly as in the previous question. Compute the cardinality of each of the following sets:

- The set of all outcomes where the black and white dice end up with different numbers on them.
- The set of all outcomes where the number on the white dice is strictly greater than that on the black dice.

2 Difference method

Suppose we are dealing with a set A whose cardinality seems to be hard or cumbersome to compute using the partition method. A way to deal with such situations is the *difference method*. The idea is to find a larger set S such that $A \subset S$, such that the cardinality of S and $S \setminus A$ are easy to compute. Once you know $|S|$ and $|S \setminus A|$, using the sum rule, we know that

$$|S| = |A| + |S \setminus A| \implies |A| = |S| - |S \setminus A|.$$

Let us use this method to quickly compute the cardinality of the first set in the previous question (black and white dice have different numbers). Let A be the set of all outcomes where the two die have different numbers. Let S be the set of all possible outcomes. We know that $A \subset S$, and $|S| = 36$. Notice that $S \setminus A$ = all outcomes where the two die have the same number on them, and thus $|S \setminus A| = 6$. This then easily gives us $|A| = 30$! Cool, isn't it? Now use this method to solve the following problem:

Question . Find all 10-bit integers that have at least 2 ones in their binary representation.

(If you don't know what a 10-bit number is, or what a binary representation is, it might be time to ask your TA, or brush up some 205 concepts!)

3 Bijection method

Here is yet another method that's useful for computing $|A|$ when it's cumbersome to do so directly. Suppose you could find a set B such that i) it's easy to count and ii) there is a bijection $f : A \rightarrow B$, then we know from our review in the last lecture that $|A| = |B|$. This is called the *bijection method*. Let's use this method to solve find the cardinality of the set of all outcomes in the two-dice experiment where the white dice ends up with a strictly larger number than the black dice. Let's denote this set by $A_{w>b}$, and let's denote the set of outcomes where they end up with same numbers by $A_{w=b}$, and the set where the number on the black dice is strictly greater than the one on the white dice by $A_{b>w}$.

First notice that the three sets are disjoint, and account for all possible outcomes. Thus, by the sum rule, $|A| = |A_{w>b}| + |A_{w=b}| + |A_{b>w}|$, where A is the set of all possible outcomes. Recall that $|A| = 36$ and $|A_{w=b}| = 6$. Thus, we get $|A_{w>b}| + |A_{b>w}| = 30$.

Here the cool part now: we can set up a bijection between $A_{w>b}$ and $A_{b>w}$. Given a configuration/outcome in set $A_{w>b}$, we can map it to a unique configuration/outcome in the set $A_{b>w}$ by swapping the numbers on the white and black dice. For example, the outcome in the set $A_{w>b}$ where white has 5 and black has 1 can be mapped to the outcome in $A_{b>w}$ where we swap the two numbers, i.e. black has 5 and white has 1. Convince yourself that this is indeed a bijection between the two sets. This means that $|A_{w>b}| = |A_{b>w}|$, and thus, $|A_{w>b}| = 15$.

Question . How many subsets of $\{1, 2, \dots, 101\}$ are there that have an odd number of elements?

(We shall see very soon that the total number of possible subsets of a set S of size n is 2^n .) To solve this problem, one can set up a bijection between the set of all odd size subsets of $\{1, 2, \dots, 101\}$

and the set of all even subsets of $\{1, 2, \dots, 101\}$ in the following way: given an odd size subset S , we map it to $\{1, 2, \dots, 101\} \setminus S$, i.e. its complement inside $\{1, 2, \dots, 101\}$. Why is this map a bijection? What's the final answer then?

4 Product rule

In its simplest form, the product rule looks kind of obvious:

$$|A \times B| = |A| \cdot |B|,$$

where A and B could be two arbitrary sets (not necessarily disjoint). I claim that you can derive the product rule using the partition method. How? Assume that $|A| = k$, and $A = \{a_1, \dots, a_k\}$, and partition $A \times B$ into k parts A_i for $1 \leq i \leq k$, where A_i is all the elements of $A \times B$ which have a_i as their first component (in the tuple). Now complete the proof.

Of course, the product rule can be generalized to n sets A_1, \dots, A_n :

$$|A_1 \times A_2 \dots \times A_n| = \prod_{i=1}^n |A_i|.$$

(the symbol \prod is the equivalent of \sum for products.)

Whenever you see that you are dealing with a set that contains sequences, tuples, or outcomes from an experiment repeated multiples times, you might want to consider using the product rule for computing its cardinality (of course, this is not a hard and fast rule, just a rule-of-thumb. We will see cases where this will not work in a straightforward manner).

For example, going back to the black and white die experiment, we can estimate the total number of outcomes pretty easily now: if A is the set of outcomes for the white dice, and B the set of outcomes for the black dice, then the total number of outcomes is basically $|A \times B|$ (why?), and therefore, using the product rule, is equal to $|A| \cdot |B| = 36$.

Question . A valid code must satisfy the following conditions:

- It has total length four.
- The first two characters are digits between 0 and 9.
- The third character is an upper case or lower case letter (they are considered to be different possibilities, i.e. $a \neq A$).
- The last character must be one of the following: #, *, or @.

How many valid codes are there?

This can be viewed as finding $|A \times A \times B \times C|$, where A contains the digits 0 – 9, B contains the upper and lower case letters (52 in all), and C contains three special characters specified above. What's the answer?

Question . A valid password must satisfy the following conditions:

- It has total length six.
- Each character is an uppercase letter or a digit.
- The password must have at least one digit.

How many valid passwords are there?

Hint: Find all possible strings of length 6 that can be made from uppercase letters and digits, find all length six strings that follow the above rules except the last one, and use difference method to finish the problem.

5 Generalized product rule

Suppose you have 100 chairs arranged in a row. In how many ways can you seat 100 people? In other words, how many arrangements of people are there. For the first chair, any of the 100 people can be seated there. Once you fix the first choice, for the second chair there are 99 choices. Having fixed the first two choices, there are 98 choices for the third chair, and so on. It turns out that the answer is: $100 \times 99 \times 98 \times \dots \times 2 \times 1$. This is denoted by $100!$ (called “hundred factorial”). In general,

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1,$$

is the number of possible arrangements/permutations of n distinct objects.

The counting we did above falls under the *generalized product rule*: suppose we are trying to find the number of length n sequences such that there are P_1 choices for the first element in the sequence, P_2 choices for the second element in the sequence for every fixed choice of the first element, P_3 choices for the third element for every fixed choice of the first and second element, \dots , and P_n choices for the n^{th} element in the sequence for every fixed choice of the first $n - 1$ elements, then the total number of such sequences is $P_1 \cdot P_2 \cdot \dots \cdot P_n$.

Note that this is different from the product rule: in the product rule, the choices for each element in the sequence were in some sense “independent” of each other. For example, in the question on the number of valid codes, each character in the code could vary independently in its respective set (the first character could be any digit, independent from the second character, and so on), however, in the generalized product rule, we do not have such “independence of choice” for different elements in the sequence. Going back to the 100 chairs example, note that each chair could have any of the 100 people seated in it, however, these choices cannot vary independently of each other: if person 1 sits in chair 1, then no other chair can seat person 1. It’s important to be able to distinguish between the two versions of the product rule.

Question . In how many ways can we place two rooks, a black one and a white one, on an 8×8 chessboard so that they cannot attack each other?

Should we use the simple product rule or the generalized one? We should use the latter! Why? Suppose we were first placing the black rook and then placing the white rook, then the possible choices for the white rook depend on where the black rook is placed. As soon as you observe this kind of dependent choice in a problem, you know you have to go for the generalized product rule! There are 64 choices for the black rook. Once the black rook is placed, we cannot place the white rook in the same column or the same row as the black rook. That leaves us with 49 positions (why?), and so the answer is 64×49 .

6 Being careful while counting

When you attempt counting problems, especially when you are new to them, there is a chance that you might end up *overcounting*, i.e. you may count the same element in the set more than once, hence ending up with a larger number than the actual cardinality of the set. For example, consider the following problem: how many strings binary strings of length 8 are there that contain exactly 2 zeros? Here is a faulty way that ends up overcounting: there are 8 positions to choose from for placing the first zero, and after placing the first zero, there are 7 options for the placing the second zero (the rest of the string is just filled with ones), and thus the total number is 56.

What went wrong here? We ended up counting many of the strings twice! For example, consider 10111101. There are two ways to arrive at this string following the above procedure: you could place the first zero in position 2 and the second zero in position 7, or the other way around. This makes us count this string twice!

It turns out that in the above example, the proposed method ends up counting every string (with exactly two zeros) exactly twice, and thus we can arrive at the correct number by dividing the count obtained above by two (convince yourself of this!). My suggestion to you is to double or even triple check that you are not overcounting in any of the steps of your proof, and if you are overcounting, you are scaling down the count at some later point in the proof.