The Three Body Problem

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1 Introduction

The Three-Body Problem is a classical problem in physics and celestial mechanics that involves studying the motion of three bodies interacting under gravitational forces. It is a complex problem due to its nonlinearity and lack of analytical solutions, which makes it an ideal candidate for numerical methods. In this presentation, we will delve into the Three-Body Problem and explore how numerical methods play a crucial role in understanding and solving this intricate problem.

1.1 Understanding the Three-Body Problem:

The Three-Body Problem refers to the motion of three celestial bodies, such as stars or planets, subject to their mutual gravitational attraction. It aims to determine the positions and velocities of the bodies as functions of time. The problem can be formulated using Newton's laws of motion and the law of universal gravitation. However, unlike the two-body problem, which has analytical solutions (e.g., elliptical orbits), the three-body problem lacks such simple solutions.

1.2 Challenges of Analytical Solutions:

The lack of analytical solutions arises from the complex nature of the problem. The gravitational forces between the three bodies create a system of coupled nonlinear differential equations. These equations cannot be solved analytically except for a few special cases. Consequently, numerical methods become indispensable for obtaining approximate solutions.

1.3 Numerical Method for Solving the Three-Body Problem:

Numerical methods provide a powerful toolset for approximating the motion of celestial bodies in the Three-Body Problem. One of the most commonly used methods is:

Numerical Integration Techniques:

- **a. Euler's Method:** The simplest numerical integration method, which approximates the solution by discretizing time and using forward differencing.

- **b. Runge-Kutta Methods:** More accurate than Euler's Method, these methods involve evaluating derivatives at multiple intermediate points to improve the approximation.

1.4 Benefits of Numerical Methods:

Numerical methods offer several key advantages in tackling the Three-Body Problem:

- **Flexibility:** Numerical methods allow us to study a wide range of scenarios, including arbitrary initial conditions and various configurations of the bodies. This flexibility enables us to explore the behavior of complex systems that cannot be easily analyzed analytically.
- **Accuracy and Precision:** While numerical methods provide approximations, they can yield highly accurate results with sufficient computational resources and appropriate algorithmic choices. The accuracy can be improved by using higher-order numerical integration schemes and reducing the time step size.
- **Visualization and Insight:** Numerical methods enable us to visualize the motion of the bodies over time. Through simulations and graphical representations, we can gain deeper insights into the dynamics, such as the emergence of chaotic behavior or the stability of orbital configurations.
- **Exploration of Unsolvable Cases:** Numerical methods allow us to investigate scenarios where analytical solutions are impossible to obtain. By employing numerical techniques, we can study systems with arbitrary masses, initial conditions, and external perturbations, thus expanding our understanding of the Three-Body Problem.

2 Effect of Initial Conditions on The Three Body Problem

When solving the Three-Body Problem using numerical methods, the initial conditions play a crucial role in determining the behavior and outcome of the system. The initial conditions refer to the positions and velocities of the three bodies at the starting time of the simulation. Here, we will discuss the effects and reasons behind the importance of initial conditions in solving the Three-Body Problem numerically.

2.1 1. Sensitivity to Initial Conditions:

The Three-Body Problem is known to exhibit sensitivity to initial conditions, commonly referred to as the butterfly effect. This means that even small variations in the initial conditions can lead to significant differences in the long-term behavior of the system. Due to the chaotic nature of the problem, even tiny changes in initial positions or velocities can result in drastically different trajectories and outcomes.

2.2 2. Stability of Orbital Configurations:

The initial conditions determine the stability of the orbital configurations in the Three-Body Problem. Stable configurations correspond to long-term, regular orbits where the bodies maintain relatively constant distances and velocities. Unstable configurations, on the other hand, lead to chaotic behavior or escape trajectories. Finding initial conditions that result in stable orbits is crucial for understanding the long-term dynamics of the system.

2.3 3. Conserving Energy and Angular Momentum:

The initial conditions must satisfy the conservation laws of energy and angular momentum. In an isolated system, the total energy (kinetic plus potential) and the total angular momentum of the bodies should remain constant throughout the simulation. Choosing appropriate initial conditions that conserve these quantities accurately ensures the numerical solution's fidelity to the underlying physical laws.

2.4 4. Special Cases and Resonances:

Certain special cases and resonant configurations in the Three-Body Problem have particular initial conditions that lead to stable or periodic behavior. For example, Lagrange points are points in space where the gravitational forces between the bodies and their orbital motions create equilibrium. Identifying and selecting these specific initial conditions can allow for studying interesting and unique dynamics.

2.5 5. Computational Considerations:

From a computational standpoint, the choice of initial conditions can affect the numerical stability and efficiency of the simulation. Certain initial conditions may lead to rapid changes or high velocities, requiring smaller time steps and increased computational resources. Carefully selecting initial conditions that balance accuracy and computational feasibility is crucial for obtaining reliable results within reasonable computational constraints.

In summary, the initial conditions have a significant impact on the behavior and outcomes of the Three-Body Problem when solved using numerical methods. Sensitivity to initial conditions, stability of orbital configurations, conservation of energy and angular momentum, special cases and resonances, and computational considerations all emphasize the importance of selecting appropriate initial conditions to accurately model and understand the complex dynamics of the Three-Body Problem.

3 Two Body Problem as a Special Case of Three Body Problem:

The Two-Body Problem can be considered as a condensed and simplified version of the Three-Body Problem, focusing solely on the gravitational interaction between two celestial bodies. This problem allows for analytical solutions and serves as a fundamental building block in the field of celestial mechanics.

In the Two-Body Problem, the motion of two bodies, such as planets, stars, or satellites, is governed by Newton's laws of motion and the law of universal gravitation. This problem aims to determine the positions and velocities of the bodies as they evolve over time.

One of the remarkable aspects of the Two-Body Problem is its ability to provide analytical solutions. Unlike the Three-Body Problem, which lacks general analytical solutions, the Two-Body Problem yields well-known and well-understood results. One of the key solutions derived from this problem is the description of elliptical orbits. According to Kepler's laws of planetary motion, if the two bodies have mass and the system is isolated, their orbits take the form of ellipses, with one focus occupied by the more massive body.

Kepler's laws, derived from the Two-Body Problem, have profound implications for understanding planetary motion. Johannes Kepler formulated three empirical laws based on observational data, which provide invaluable insights into the relationship between the periods, distances, and masses of celestial bodies. These laws serve as the foundation of modern celestial mechanics and contribute to our understanding of planetary systems.

Moreover, the Two-Body Problem acts as a useful approximation for specific scenarios in the more complex Three-Body Problem. In cases where the gravitational influence of a third body is negligible or when the mass of one body significantly exceeds that of the other two, it is reasonable to approximate the system as a Two-Body Problem. This simplification allows for easier analysis and provides practical insights into the behavior of the system under consideration.

To summarize, the Two-Body Problem represents a condensed version of the Three-Body Problem, focusing on the gravitational interaction between two bodies. With its analytical solutions, conservation laws, and Kepler's laws, the Two-Body Problem forms the cornerstone of celestial mechanics. It enables us to comprehend the fundamentals of orbital dynamics, contributes to astronomical calculations, and serves as a useful approximation for specific scenarios in more complex systems.

4 The Mathematical perspective of The Three Body Problem:

Consider three celestial bodies with masses m_1, m_2, and m_3 located at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ in three-dimensional space. The gravitational force between any two bodies can be described by Newton's law of universal gravitation:

$$\mathbf{F}_{ij} = G\frac{m_i m_j}{r_{ij}^3}\left(\mathbf{r}_j - \mathbf{r}_i\right),$$

where G is the gravitational constant, $r_{ij} = \|\mathbf{r}_j - \mathbf{r}_i\|$ is the distance between the bodies i and j, and $\mathbf{F}_{ij}$ is the force experienced by body i due to the gravitational pull of body j.

The equations of motion for the Three-Body Problem can be derived using Newton's second law of motion:

$$\mathbf{F}_i = m_i \mathbf{a}_i,$$

where $\mathbf{F}_i$ is the net force acting on body i and $\mathbf{a}_i$ is its acceleration. Summing up the forces acting on each body, we obtain the following equations:

$$m_1\mathbf{a}_1 = \mathbf{F}_{21} + \mathbf{F}_{31},$$

$$m_2\mathbf{a}_2 = \mathbf{F}_{12} + \mathbf{F}_{32},$$

$$m_3\mathbf{a}_3 = \mathbf{F}_{13} + \mathbf{F}_{23}.$$

We can express the acceleration $\mathbf{a}_i$ in terms of the second derivatives of the positions $\mathbf{r}_i$ with respect to time. Let's denote the derivatives as follows:

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt},$$

$$\mathbf{a}_i = \frac{d^2\mathbf{r}_i}{dt^2}.$$

Substituting these expressions into the equations of motion, we have:

$$m_1\frac{d^2\mathbf{r}_1}{dt^2} = \mathbf{F}_{21} + \mathbf{F}_{31},$$

$$m_2\frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{F}_{12} + \mathbf{F}_{32},$$

$$m_3\frac{d^2\mathbf{r}_3}{dt^2} = \mathbf{F}_{13} + \mathbf{F}_{23}.$$

To proceed further, we need to express the forces $\mathbf{F}_{ij}$ in terms of the positions $\mathbf{r}_i$. Substituting the expression for $\mathbf{F}_{ij}$ from earlier, we have:

$$\mathbf{F}_{21} = G\frac{m_1m_2}{r_{21}^3}\left(\mathbf{r}_2 - \mathbf{r}_1\right),$$

$$\mathbf{F}_{31} = G\frac{m_1m_3}{r_{31}^3}\left(\mathbf{r}_3 - \mathbf{r}_1\right),$$

$$\mathbf{F}_{12} = G\frac{m_2m_1}{r_{12}^3}\left(\mathbf{r}_1 - \mathbf{r}_2\right),$$

$$\mathbf{F}_{32} = G\frac{m_2m_3}{r_{32}^3}\left(\mathbf{r}_3 - \mathbf{r}_2\right),$$

$$\mathbf{F}_{13} = G\frac{m_3m_1}{r_{13}^3}\left(\mathbf{r}_1 - \mathbf{r}_3\right),$$

$$\mathbf{F}_{23} = G\frac{m_3m_2}{r_{23}^3}\left(\mathbf{r}_2 - \mathbf{r}_3\right).$$

Substituting these expressions back into the equations of motion, we arrive at the following set of second-order ordinary differential equations (ODEs) for the positions $\mathbf{r}_i$ of the three bodies:

$$m_1\frac{d^2\mathbf{r}_1}{dt^2} = G\frac{m_1m_2}{r_{21}^3}\left(\mathbf{r}_2 - \mathbf{r}_1\right) + G\frac{m_1m_3}{r_{31}^3}\left(\mathbf{r}_3 - \mathbf{r}_1\right),$$

$$m_2\frac{d^2\mathbf{r}_2}{dt^2} = G\frac{m_2m_1}{r_{12}^3}\left(\mathbf{r}_1 - \mathbf{r}_2\right) + G\frac{m_2m_3}{r_{32}^3}\left(\mathbf{r}_3 - \mathbf{r}_2\right),$$

$$m_3\frac{d^2\mathbf{r}_3}{dt^2} = G\frac{m_3m_1}{r_{13}^3}\left(\mathbf{r}_1 - \mathbf{r}_3\right) + G\frac{m_3m_2}{r_{23}^3}\left(\mathbf{r}_2 - \mathbf{r}_3\right).$$

These are the differential equations that govern the motion of the three bodies in the Three-Body Problem. Solving these equations numerically using appropriate numerical methods allows us to simulate and study the intricate dynamics of the system.

We will use the 4th order Runge Kutta method to solve the problem.

5 The explanation of the code:

In this code we simulated the motion of three bodies in three-dimensional space under the influence of gravitational forces. The bodies are represented by masses 'm1', 'm2', and 'm3', and their initial positions and velocities are specified. In the simulation we used the 4th Order Runge-Kutta method to numerically integrate the equations of motion. We first set up the necessary libraries and defines the constants, initial conditions, and time parameters.

It then performs an integration loop to calculate the updated positions and velocities at each time step. The integration process involves calculating constants (k1, k2, k3, and k4) that account for the gravitational forces between the bodies.

Finally, we updated the arrays with the new positions, velocities, and time values, and continued the loop until all time steps have been simulated.

We have also created an animation for the trajectories of the three bodies (will be submitted within the zip file).

Here is the output of our code:

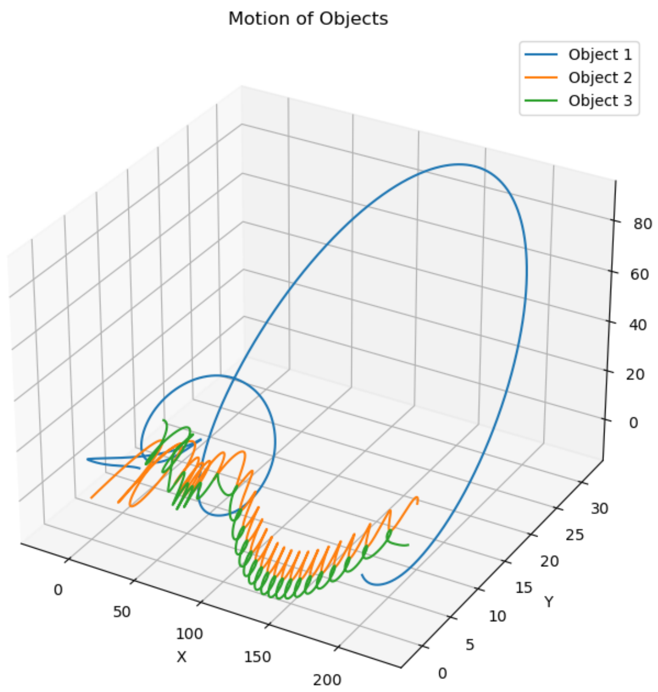

Figure 1: The motion of the three objects

6 Effect of change of the initial conditions in our code:

So, as we mentioned earlier in section 2, a little deviation may deviate the trajectory of the bodies hugely. Here, we changed the co-ordinate of the first object and velocity of the second object and correspondingly we can see the huge deviation in the trajectory:

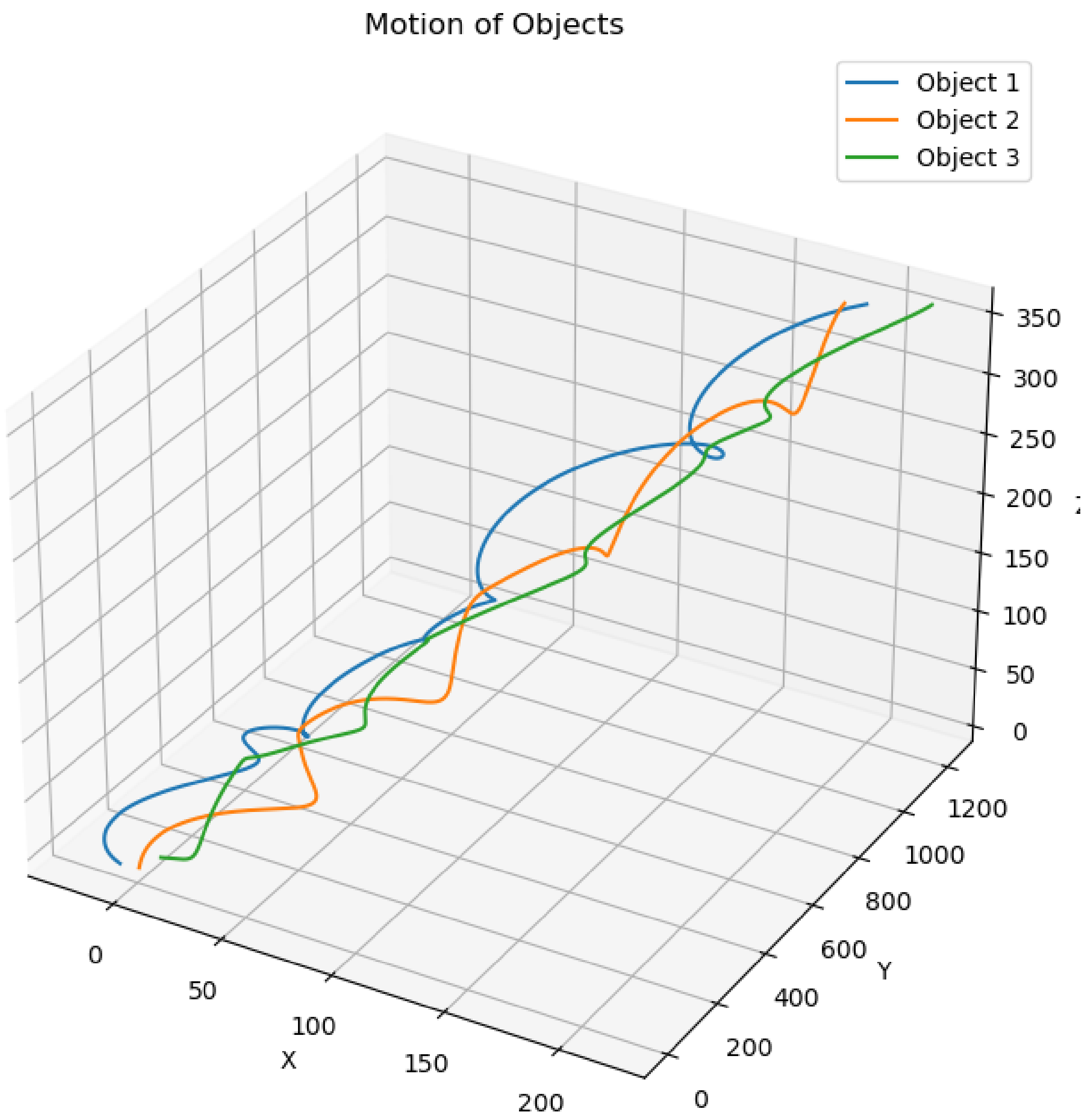

Figure 2: The motion of the three objects after changing the initial conditions of the first and second body

7 Effect of numerical precision:

The effect of numerical precision to the decimal places plays a significant role while solving The Three Body Problem. We noticed that, if we decrease the value of 'dt' from 0.001 to 0.0001 or increase the 'time step' value ('n_steps' in our code) from 200000 to 2000000, then we are not getting an expected result, instead we are getting a straight line instead. The reason lies in the fact that; while we are decreasing the value of 'dt' or increasing the value of 'n_steps', it affects the output values of the co-ordinate and velocities. The change takes place after 10th decimal places of our output and thus we face unexpected errors in the trajectories of the bodies.

8 Conclusion:

The Three-Body Problem poses a significant challenge in celestial mechanics, with limited analytical solutions available. Numerical methods provide indispensable tools for approximating the complex motion of three bodies under gravitational interactions. These methods offer flexibility, accuracy, and the ability to explore a wide range of scenarios. By leveraging numerical techniques, scientists can deepen their understanding of orbital dynamics and gain insights into the behavior of complex celestial systems.

Does it matter that we cannot solve the three body problem, given that we can just simulate the problem on a computer?

When considering solutions to the three-body problem, they are typically limited to a small subset of initial conditions, or they rely on numerical integration using computers. This leads to the notion that the three-body problem can be solved with the aid of high-speed computers by employing extremely precise numerical methods.

However, it is important to understand why computers alone can not completely solve this problem. Let's imagine a scenario where perfect observations are made. Would we then be able to use a program to accurately calculate the future trajectory of a planetary system? Even with flawless observations, it remains impossible. One obstacle lies in the nature of Newton's gravitational constant, which, like many other constants, is an irrational number. Even with a perfectly measured value of G, it would be infeasible to input it precisely into a computer due to the infinite time required for such exactness.

Nonetheless, computational methods excel in determining short-term trajectories. Additionally, when there is a significant disparity in mass between certain bodies (such as in the solar system, where the Sun outweighs all the planets combined), the ability to predict trajectories is greatly enhanced. However, similar to any aperiodic equation system, it remains impossible to determine trajectories for all time.