Dual Operators

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Abstract

This seminar paper provides a foundational overview of Dual-/Adjunct-Operators in functional analysis.

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1 Revision

Definition 1. The following basic definitions:

i): Banach Space

ii): Hilbert Space

Definition 2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces on $\mathbb{R}$or $\mathbb{C}$.
Let $T : X \to Y$ be linear.
T is a linear operator. T is bounded, if $\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$.

Definition 3. The (topological) dual space is defined as $X^* := \mathcal{L}(X, \mathbb{R})$.

Theorem 1. *The topological dual space $\mathcal{L}(X, Y)$ is a Banach space.*

Proof Idea. □

Theorem 2. *For linear operators, continous and bounded are equivalent.*

Proof Idea. TOOD □

Definition 4. i):

ii):

Theorem 3. *i): Let $p \in [1, \infty)$. Then*

ii):

Proof Idea. TOOD □

Theorem 4.

Proof. See functional analysis lecture of SS/25. □

2 The Adjunct Operator

2.1 Definitions and Basic Properties

Remark 1. The terms adjunct and dual are often used interchangeably. We will standardize on adjunct, to avoid unnecessary confusion.
We will abstract the field as $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 5. Let X, Y be metric spaces and $T \in \mathcal{L}(X, Y)$.
Then $T' : Y' \to X', y' \mapsto y' \circ T$ is called the adjunct operator. From now on, we will implicitly refer to the metric spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T'.

Remark 2. In comparison to the operators we have previously worked with, the adjunct operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from $\mathbb{K}$.
For $y' \in Y' = \mathcal{L}(Y, \mathbb{K})$, the adjunct operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{K})$.

Theorem 5. *We can collect some basic properties of the dual operator:*

i): $T' \in \mathcal{L}(Y', X')$, *so* T' *is linear and bounded.*
This implies $\forall y' \in \mathcal{L}(Y', X') : T'y' \in X'$.

ii): $T \mapsto T'$ *is linear and isometric.*

iii): $T \mapsto T'$ *is not always surjective.*

Proof. "i)": We first prove linearity:

$$\begin{aligned}
T'(\alpha y_1' + y_2') &= (\alpha y_1' + y_2') \circ T && \text{(apply def. of dual operator)} \\
&= \alpha y_1' \circ T + y_2' \circ T && \text{(expand expression)} \\
&= \alpha T' y_1' + \beta T' y_2' && \text{(apply def. of dual operator)} \\
&\text{for all } y_1', y_2' \in Y' \text{ and } \alpha \in \mathbb{K}
\end{aligned}$$

We then prove boundedness of the operator norm:

$$\begin{aligned}
\|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of dual operator)} \\
&\le \|y'\|_{Y'} \|T\|_{\mathcal{L}(X,Y)} && \text{(Cauchy-Schwarz inequality)} \\
&:= C\|y'\|_{Y'} && \text{(def. the constant)} \\
&\text{for all } y' \in Y'
\end{aligned}$$

"ii)": We first prove linearity:

$$\begin{aligned}
(\alpha T_1 + T_2)'(y') &= y'(\alpha T_1 + T_2) && \text{(apply def. of dual operator)} \\
&= \alpha y'(T_1) + y'(T_2) && \text{(y' is linear)} \\
&\text{for all } T_1, T_2 \in \mathcal{L}(X, Y) \text{ and } y' \in Y' \text{ and } \alpha \in \mathbb{K}
\end{aligned}$$

We then prove isometry:

$$\|T\|$$

"iii)": We construct a counterexample:

□

Example 1.

Revision 1 (The Adjunct Operator reverses composition)**.** *Let* X, Y, Z *be normed metric spaces. Then we have* $\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$.

Proof. Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.
We know $ST = S \circ T$ is still a linear, bounded operator from X to Z. So $(ST)'$ is well-defined.
Let $z' \in Z' = \mathcal{L}(Z, \mathbb{K})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{K})$.
We can now evaluate the expression on z':

$$\begin{aligned}
(ST)'(z') &= z' \circ (ST) && \text{(apply def. of dual operator)}\\
&= (z' \circ S) \circ T && \text{(write out chain explicitly)}\\
&= y' \circ T && \text{(subst. y')}\\
&= T'y' && \text{(apply def. of dual operator)}\\
&= T'(z' \circ S) && \text{(subst. y')}\\
&= T'S'z' && \text{(apply def. of dual operator)}
\end{aligned}$$

So in total, $(ST)' = T'S'$. □

Example 2.

2.2 In Hilbert Spaces

Risz-Frechet for Comparison between Hilbetraum-Adjunct and Banachraum-Adjunct.

2.3 Examples for Adjunct Operators on Banach Spaces

2.4 Examples for Adjunct Operators on Hilbert Spaces

2.5 Examples for Adjunct Operators in Finite Dimensional Hilbert Spaces

2.6 Relation to Compactness

Definition 6. The following definitions are revisions from the lecture:
Let X, Y be normed metric spaces.

i): X is relatively compact, if $\forall (x_n)_{n\in\mathbb{N}} \subset X$ bounded : $(Tx_n)_{n\in\mathbb{N}}$ has a converging subsequence.

ii): $T \in \mathbb{N}$ is compact, if $TB(0, \leq 1, X)$ is relatively compact.

iii): The rank of T is $\operatorname{rk} T = \dim T(X)$.

Remark 3.

Definition 7. The following definitions are critical for the theorem of Arzelà-Ascoli:
Let X, Y be metric spaces and $M \subset \{f : X \to Y\}$.

i): M is uniformly equicontinous, if
$\forall \epsilon > 0, \exists \delta > 0, \forall f \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty)$.

ii): M is pointwise bounded, if $\forall x \in X : \{f(x) \mid f \in M\}$ is bounded.

Revision 2 (Arzelà-Ascoli)**.** *Let D be a compact metric space and $\subset C(D) = \{f : D \to \mathbb{K}$ continous$\}$. Then we have*

$$M \text{ relatively-compact} \iff M \text{ uniformly equicontinous} \wedge M \text{ pointwise bounded}$$

Proof Idea. TODO □

Definition 8. Let X be a normed vector space.

i): X'' is called the bidual space.

ii): $i : X' \to \mathbb{K}$

Theorem 6 (Schauder)**.** *Let X, Y be Banach spaces and $T : X \to Y$ a linear, continous operator. Then we have*

$$T \text{ compact} \iff T' \text{ compact}$$

Proof. „$\Rightarrow$“: TODO
„$\Leftarrow$“: We have T' is compact. Using the first part of the proof, we know T'' is compact.
Since $T'' \circ i_X = i_Y \circ T$ we know $i_Y \circ T$ is compact.
Since $Y \subset Y''$ is closed, we get T compact. □

2.7 Operator Equations