# Dual Operators

## Seminar Functional Analysis

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January 2026

# Outline

# Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as $A^T$ over $\mathbb{R}$ or $A^H$ over $\mathbb{C}$) to operators on normed $\mathbb{R}$ or $\mathbb{C}$ vector spaces.
We will cover the following topics:

- The operator $\cdot^H$ is linear and isometric wrt. the spectral norm.
- The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem: $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^H$.
- TODO Add more

# Terminology

The terms adjoint and dual are often used interchangeably. We will standardize to adjoint, to avoid unnecessary confusion. Following [Wer18], we write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ when the field is unspecified.

# Definition: Linear Operator

## Definition 1

We remind ourselves of the following concepts from the lecture [KP25]: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed $\mathbb{F}$-vector spaces

i) Let

$$T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call $T$ a linear operator. We call $T$ bounded, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation $T : X \to Y$ for linear operators rather than $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$.

ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

# Revision: Foundational Statements

## Revision

The following statements are foundational for this topic:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces.

i) The set of continuous, linear operators $\mathcal{L}(X, Y)$ is a Banach space if and only if $Y$ is a Banach space. In particular, the topological dual space $\mathcal{L}(X, \mathbb{F})$ is a Banach space.

ii) Let $T : X \to Y$ be a linear operator. Then $T$ is bounded if and only if $T$ is continous.

iii) Let $(Z, \|\cdot\|_Z)$ be a normed $\mathbb{F}$-vector space, let $T : X \to Y$ be a linear, bounded operator and let $S : Y \to Z$ be a linear, bounded operator. Then $S \circ T$ is a linear, bounded operator.

iv) Let $(Z, \|\cdot\|_Z)$ be a normed $\mathbb{F}$-vector space, let $T : X \to Y$ be a linear operator and let $S : Y \to Z$ be a linear operator. If $T$ or $S$ is compact, $S \circ T$ is compact.

# Definition: Adjoint Operator

## Definition 2

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces and $T \in \mathcal{L}(X, Y)$.
Then $T' : Y' \to X', y' \mapsto y' \circ T$ is called the adjoint operator.

## Remark

In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from $\mathbb{F}$.
For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$.

# Example: Dual Space of $\mathbb{R}^n$

## Example 3

Let $n \in \mathbb{N}$. Then we have

$$
\begin{aligned}
(\mathbb{R}^n)' &= \{g : \mathbb{R}^n \to \mathbb{R} \text{ linear, continuous}\} && \text{(apply def. of dual space)} \\
&= \{g : \mathbb{R}^n \to \mathbb{R} \text{ linear}\} && \text{(result from linear algebra)} \\
&\cong \mathbb{R}^n && \text{(result from linear algebra see [FS25])}
\end{aligned}
$$

# Example: Adjoint of Matrix Operator

## Example 4

Let $m \in \mathbb{N}$ and let $A \in \mathbb{R}^{m \times n}$.
Consider the function $f : \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$. The signature of the adjoint is $f' : (\mathbb{R}^m)' \to (\mathbb{R}^n)'$. With $e_i$ we denote the standard basis vectors and with $e_i'$ we denote the dual standard basis vectors.
Let $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We have

$$\begin{aligned}(f'e_i')(e_j) &= (e_i' \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\ &= f(e_j)_i = a_{ij} && \text{(apply def. of } e_i' \text{ and } f\text{)}\end{aligned}$$

So if we set $i = 1$ for instance, we get $(f'e_1')(e_j) = a_{1j}$. The first "column" of $f'$ (up to isomorphism) must be $(a_{1j})_{j=1,\ldots,n}$. In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \to \mathbb{R}^n, x \mapsto A^T x$$

# Revision: Hahn-Banach and Dual Unit Ball

## Revision (Hahn-Banach)

Let $(X, \|\cdot\|_X)$ be a normed space and $0 \neq x \in X$.
Then we have $\exists f \in X' : \|f\|_{X'} = 1 \wedge \ f(x) = \|x\|_X$

## Revision

The dual unit ball retains information about the norm of primal vectors:
Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space and $x \in X$.
Then we have $\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$

# Theorem: Properties of Adjoint Operator

## Theorem 5

*The adjoint operator has the following properties:*

*i)* $T' \in \mathcal{L}(Y', X')$, *so* $T'$ *is linear and bounded.*

▷ This implies $\forall y' Y' : T'y' \in X'$.

*ii)* $T \mapsto T'$ *is linear and isometric.*

# Theorem: Properties of Adjoint Operator – Proof (i)

## Proof

"i)": Let $y' \in Y'$. Plugging it into the adjoint operator, we get $T'y' = y' \circ T$ with signature $X \to Y \to \mathbb{F}$. We can now see that $\operatorname{Im} T' \subset X'$.
Let $y_1', y_2' \in Y', \alpha \in \mathbb{F}$. We then prove linearity:

$$\begin{aligned}
T'(\alpha y_1' + y_2') &= (\alpha y_1' + y_2') \circ T && \text{(apply def. of adjoint operator)} \\
&= \alpha y_1' \circ T + y_2' \circ T && \text{(expand expression)} \\
&= \alpha T' y_1' + T' y_2' && \text{(apply def. of adjoint operator)}
\end{aligned}$$

Let $y' \in Y'$. We then prove boundedness of the operator norm:

$$\begin{aligned}
\|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\
&\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X,Y)} && \text{(apply def. of op. norm)} \\
&:= C\|y'\|_{Y'} && \text{(def. the constant)}
\end{aligned}$$

# Theorem: Properties of Adjoint Operator – Proof (ii) Linearity

## Proof

"ii)": Let $T_1, T_2 \in \mathcal{L}(X, Y), y' \in Y', x \in X, \alpha \in \mathbb{F}$. We first prove linearity:

$$\begin{aligned}
(\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\
&= \alpha y'(T_1 x) + y'(T_2 x) && \text{(y' is linear)} \\
&= (\alpha T_1' y' + T_2' y')(x)
\end{aligned}$$

# Theorem: Properties of Adjoint Operator – Proof (ii) Isometry

## Proof.

We then prove isometry:

$$
\begin{aligned}
\|T\| &= \sup_{\|x\|_X \le 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\
&= \sup_{\|x\|_X \le 1} \sup_{\|y'\| \le 1} |y'(Tx)| && \text{(apply Hahn-Banach corollary)} \\
&= \sup_{\|y'\| \le 1} \sup_{\|x\|_X \le 1} |y'(Tx)| && \text{(supremum order can be switched)} \\
&= \sup_{\|y'\| \le 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\
&= \|T'\| && \text{(apply def. of op. norm)}
\end{aligned}
$$

$\square$

# Example: Shift Operator (1/3)

## Example 6

Let $p \in (1, \infty)$ with $p \neq 2$. This makes $l_p$ a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of $l_p$ is isometrically isomorph to $l_{p^*}$ where $p^*$ is the Hölder conjugate with $1/p + 1/p^* = 1$.

As a reminder, the general idea of the proof of $l_p' \cong l_{p^*}$ goes as follows:

- Define the isometric isomorphism as $T : l_{p^*} \to l_p', s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$
- Verify $(Ts)x$ converges as $|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$ and absolute convergence implies convergence.
- Verify $T$ is injective using the linearity.
- Verify $T$ is surjective and isometric (the long part). See [Wer18] for a full proof.

# Example: Shift Operator (2/3)

## Example (cont.)

To illustrate the adjoint operator, we now work through an example. Consider the left shift operator $T : l_p \to l_p, (x_k)_{k\in\mathbb{N}} \mapsto (x_{k+1})_{k\in\mathbb{N}}$ It is well-defined since $\sum_{k\in\mathbb{N}} |x_{k+1}|^p \leq \sum_{k\in\mathbb{N}} |x_k|^p < \infty$
We can now compute the adjoint operator $T'$: The adjoint $T'$ must have the signature $l_p' \cong l_{p^*} \to l_p' \cong l_{p^*}$.

# Example: Shift Operator (3/3)

## Example (cont.)

Let $y' \in l_p' \cong l_{p^*}$. Then we can write $y' : l_p \to \mathbb{F}, x \mapsto \sum_{k\in\mathbb{N}} x_k s_k$ with $s \in l_{p^*}$.
Now for $x \in l_p$ we have

$$
\begin{aligned}
(T'y')(x) &= (y' \circ T)(x) = y'(Tx) && \text{(apply def. of } T'\text{)} \\
&= y'((x_{k+1})_{k\in\mathbb{N}}) && \text{(apply def. of } T\text{)} \\
&= \sum_{k\in\mathbb{N}} x_{k+1} s_k && \text{(apply def. of } y'\text{)} \\
&= \sum_{k\in\mathbb{N}} x_k s_k' && \text{(with } s_1' = 0 \text{ and } s_k' = s_{k-1} \text{ for } k > 1\text{)}
\end{aligned}
$$

This tells us that the adjoint operator $T'$ acts as a right shift (up to isomorphism):
$T' : l_{p^*} \to l_{p^*}, (s_k)_{k\in\mathbb{N}} \mapsto (0, s_1, s_2, \dots)$

# Theorem: Adjoint Reverses Composition

**Theorem 7**

*Let $X, Y, Z$ be normed $\mathbb{F}$ vector spaces.*
*Then the adjoint operator reverses composition:*
$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$

# Theorem: Adjoint Reverses Composition – Proof

## Proof.

Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.
We know $ST = S \circ T$ is still a linear, bounded operator from $X$ to $Z$. So $(ST)'$ is well-defined.
Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$. We have

$$
\begin{aligned}
(ST)'(z') &= z' \circ (ST) && \text{(apply def. of adjoint operator)} \\
&= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\
&= y' \circ T && \text{(subst. y')} \\
&= T'y' && \text{(apply def. of adjoint operator)} \\
&= T'(z' \circ S) && \text{(subst. y')} \\
&= T'S'z' && \text{(apply def. of adjoint operator)}
\end{aligned}
$$

So in total $(ST)' = T'S'$. □

# Example: Composition Rule

## Example 8

Referring back to the example with $A \in \mathbb{R}^{m \times n}$, the adjoint operator reverses composition rule is compatible with the way matrix transposition on $\mathbb{R}$ or the matrix adjoint on $\mathbb{C}$ behave.

# Definition: Bidual Space

## Definition 9

Let $X$ be a normed $\mathbb{F}$ vector space.

i) $X''$ is called the bidual space.

ii) Let $J_X : X \to X'', p \mapsto (x' \mapsto x'(p))$. $J_X$ is called the canonical embedding from $X$ into $X''$.

So $J_X$ returns a function that evaluates dual space elements at $p \in X$.

# Figure: Dual of Dual

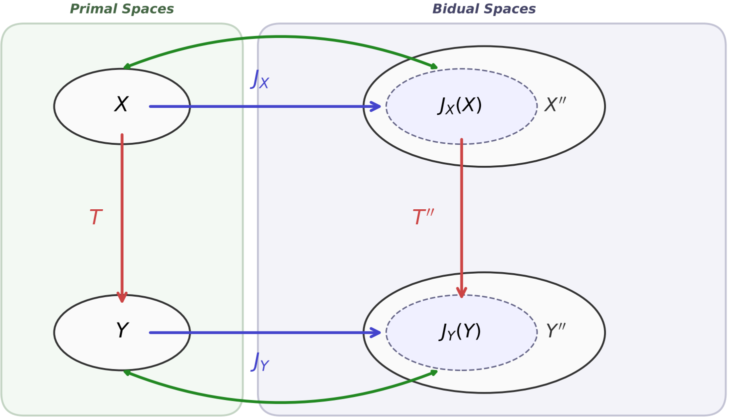


Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

# Theorem: Bidual Adjoint Embedding

**Theorem 10**

*Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed* $\mathbb{F}$ *vector spaces. Let* $T \in \mathcal{L}(X, Y)$ *be a bounded linear operator. Then we have:* $J_Y \circ T = T'' \circ J_X$.

# Theorem: Bidual Adjoint Embedding – Proof Setup

## Proof

Before the proof, we can avoid confusion by typing out what $T'$ and $T''$ evaluate to:

$T' : Y' \to X', y' \mapsto y' \circ T$ is the adjoint operator.

$T'' : X'' \to Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

$J_Y : Y \to Y''$ and $T : X \to Y$ means $(J_Y \circ T) : X \to Y''$.

$T'' : X'' \to Y''$ and $J_X : X \to X''$ means $(T'' \circ J_X) : X \to Y''$.

# Theorem: Bidual Adjoint Embedding – Proof Calculation

**Proof.**

Finally, for $p \in X$ and $y' \in Y'$ we have

$$
\begin{aligned}
((J_Y \circ T)(p))\,(y') = J_Y(Tp)(y') = y'(Tp) \quad && \text{(subst. } p \text{ in and apply def. of } J_Y) \\
= (y'T)(p) \quad && \text{(use associativity)} \\
= (T'y')(p) \quad && \text{(apply def. of } T') \\
= (x' \mapsto x'(p))(T'y') \quad && \text{(pull out subst. function)} \\
= J_X(p)(T'y') \quad && \text{(recognize this is just } J_X) \\
= T''(J_X(p))(y') \quad && \text{(apply def. use } T'') \\
= ((T'' \circ J_X)(p))\,(y') \quad && \text{(use } \circ \text{ notation)}
\end{aligned}
$$

□

# Revision: Canonical Embedding Properties

## Revision

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space. Then we have

i) The canonical embedding $J_X$ is an isometric injective function.

ii) $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism.

iii) $J_X$ is a bounded, linear operator.

iv) $(J_X|_{X \to J_X(X)})^{-1}$ is a bounded, linear operator.

# Revision: Canonical Embedding Properties – Proof

## Proof

i) Please refer to the functional analysis lecture notes [KP25] from SS/2025.

ii) Follows by definition of $J$ and from "i)".

iii) Since $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

  $J_X$ is linear since the $x' \in X'$ are linear.
  $\|J_X\| = 1$. This means $J_X$ is bounded.

iv) Since $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

  $(J_X|_{X \to J_X(X)})^{-1}$ inherits linearity from $J$.
  $\|J_X\| = 1$ and therefore $\|(J_X|_{X \to J_X(X)})^{-1}\| = 1$. This means $(J_X|_{X \to J_X(X)})^{-1}$ is bounded.

# Corollary: Strengthened Bidual Adjoint Embedding

## Corollary 11

*We can strengthen the theorem using the isomorphism:*
*Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed* $\mathbb{F}$ *vector spaces. Let* $T \in \mathcal{L}(X, Y)$ *be a bounded linear operator. Then we have*

*i)* $J_Y \circ T = T'' \circ J_X$.

*ii)* $T''|_{J_X(X)} = J_Y \circ T \circ (J_X|_{X \to J_X(X)})^{-1}$

*iii)* $T = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ T'' \circ J_X$.

## Proof.

"i)": This is the statement of the theorem.

"ii)": The isomorphism from the revision gives us the result.

"iii)": The isomorphism from the revision gives us the result.

□

# Figure: Strengthened Bidual Adjoint Embedding

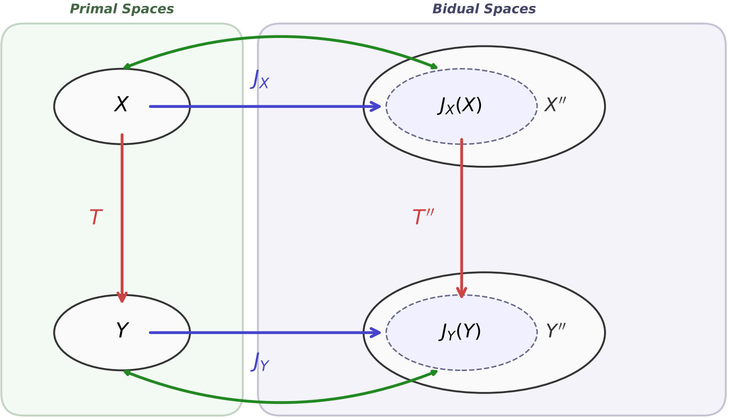


Figure 2: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

# Theorem: Characterization of Adjoint Operators

**Theorem 12**

*Let $S \in \mathcal{L}(Y', X')$ be a continuous, linear operator. Then we have*

$$\exists T \in \mathcal{L}(X, Y) : T' = S \textit{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

# Theorem: Characterization of Adjoint Operators – Proof ($\Rightarrow$)

**Proof**

"$\Rightarrow$": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T'\text{)} \\ &= J_Y(T(X)) && \text{(use the theorem)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y\text{)} \end{aligned}$$

# Theorem: Characterization of Adjoint Operators – Proof ($\Leftarrow$) Setup

## Proof (cont.)

"$\Leftarrow$": We have $S'(J_X(X)) \subset J_Y(Y)$ and the revision gives us $J_Y(Y) \cong Y$.
So for $x \in X$ and $y''_x = S'(J_X(x))$ there is a (unique) $y_x \in Y$ with $y''_x = J_Y(y_x)$.
Choose $T : X \to Y, x \mapsto y_x$ We know $T$ exists (and is unique) due to the previous argument.
We know $T$ is linear and continuous, as $T = y_{\cdot} = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ S' \circ J_X$ and all elements in the chain are bounded, linear operators.

# Theorem: Characterization of Adjoint Operators – Proof ($\Leftarrow$) $S = T'$

**Proof.**

Let $y' \in Y'$ and $x \in X$. Lastly, we need to prove that $S = T'$:

$$
\begin{aligned}
(Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X\text{)}\\
&= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of ajoint op.)}\\
&= J_Y(((J_Y|_{Y\to J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y\to J_Y(Y)})^{-1} = \mathrm{Id}\text{)}\\
&= y'(((J_Y|_{Y\to J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)}\\
&= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X\text{)}\\
&= (T'y')(x) && \text{(apply def. of adjoint op.)}
\end{aligned}
$$

$\square$

# Definition: Relatively Compact, Compact Operator, Rank

## Definition 13

The following definitions are revisions from the lecture:

Let $X, Y$ be normed $\mathbb{F}$ vector spaces.

i) $M \subset X$ is relatively compact, if $\forall (x_n)_{n\in\mathbb{N}} \subset M : (x_n)_{n\in\mathbb{N}}$ has a converging subsequence in $X$

ii) $T \in \mathcal{L}(X, Y)$ is compact, if $T\left(\overline{B}_X\right)$ is relatively compact.

iii) The rank of T is $\operatorname{rk} T = \dim T(X)$.

# Definition: Uniformly Equicontinuous and Pointwise Bounded

## Definition 14

The following definitions are critical for the theorem of Arzelà-Ascoli:
Let $X, Y$ be metric spaces and $M \subset \{f : X \to Y\}$.

i) $M$ is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

ii) $M$ is pointwise bounded, if $\forall x \in X : \{f(x) \mid f \in M\}$ is bounded

# Revision: Arzelà-Ascoli

**Revision (Arzelà-Ascoli)**

Let $D$ be a compact metric space and $M \subset C(D)$ with the supremum norm. Then we have $M$ is uniformly equicontinuous and pointwise bounded implies $M$ is relatively compact.

**Theorem 15 (Schauder)**

*Let $X, Y$ be Banach spaces and $T : X \to Y$ be a bounded, linear operator. Then we have $T$ is compact if and only if $T'$ is compact.*

# Theorem: Schauder – Proof ($\Rightarrow$) Setup

## Proof

"$\Rightarrow$": Let $(y_n')_{n\in\mathbb{N}} \subset Y' = \mathcal{L}(Y,\mathbb{F}) \subset C(Y)$ be bounded.
Our goal is to show that there is a convergent subsequence in $(T'y_n')_{n\in\mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm. For all $n \in \mathbb{N}$ set $f_n = y_n'|_{T(\overline{B}_X)}$.
We have

$$
\begin{aligned}
\|T'y_n' - T'y_m'\| &= \|y_n' \circ T - y_m' \circ T\| && \text{(apply def. of adjoint op.)}\\
&= \sup_{\|x\|_X \le 1} |((y_n' \circ T) - (y_m' \circ T))(x)| && \text{(use supremum char. of the op. norm)}\\
&= \sup_{\|x\|_X \le 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m)\\
&= \sup_{d \in T(\overline{B}_X)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m)
\end{aligned}
$$

# Theorem: Schauder – Proof ($\Rightarrow$) Setup for Arzelà-Ascoli

## Proof (cont.)

We know $(T'y_n')_{n\in\mathbb{N}}$ converges if and only if it is a Cauchy sequence $(\mathcal{L}(X,\mathbb{F}), \|\cdot\|)$.
Therefore the convergence of $(T'y_n')_{n\in\mathbb{N}}$ in the operator norm is only dependent on the behaviour of $(f_n)_{n\in\mathbb{N}}$ on $\overline{T\left(\overline{B}_X\right)}$.
We now set $D = \overline{T\left(\overline{B}_X\right)}$ and pack the sequence into $M := \{f_n \mid n \in \mathbb{N}\}$ and examine them for the conditions of Arzelà-Ascoli.

# Theorem: Schauder – Proof ($\Rightarrow$) $D$ is compact, $M$ is pointwise bounded

## Proof (cont.)

$D$ is compact: We have $\overline{B}_X$ is bounded and $T$ is a compact operator. So $T\left(\overline{B}_X\right)$ is relatively compact and $D$ is compact.

$M$ is pointwise bounded: For $x \in \overline{B}_X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} |f_n(Tx)| = |y_n'(Tx)| :&= |y_n'(d)| && \text{(apply def. of } f_n \text{ and define } d\text{)} \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 && \text{(apply op. norm ineq. and D bounded means } \|d\|_Y \leq: C_2\text{)} \end{aligned}$$

For $d \in D$ we can choose $(x_k)_{k \in \mathbb{N}} \subset \overline{B}_X$ with $Tx_k \longrightarrow d$ and $|f_n(Tx_k)| \leq C_1 C_2$. Using continuity we get $|f_n(d)| \leq C_1 C_2$.

# Theorem: Schauder – Proof ($\Rightarrow$) $M$ is uniformly equicontinuous

## Proof (cont.)

$M$ is uniformly equicontinuous: For $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon / C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$ we have

$$\begin{aligned}
|f_n(d_1) - f_n(d_2)| &\leq \|y_n'\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)} \\
&\leq C_1 \|d_1 - d_2\|_Y && \text{(use the upper bound } \|y_n'\| \leq C_1\text{)} \\
&< C_1 \delta = \epsilon && \text{(substitute in } \delta\text{)}
\end{aligned}$$

# Theorem: Schauder – Proof ($\Rightarrow$) Conclusion

## Proof (cont.)

The theorem of Arzelà-Ascoli now tells us that $M$ is relatively compact. So every sequence in $M$ has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ we have

$$\begin{aligned}(f_{n(k)} \circ T)_{k\in\mathbb{N}} &= (y'_{n(k)} \circ T)_{k\in\mathbb{N}} && \text{(apply def. of } f_{n(k)}\text{)}\\ &= (T' y'_{n(k)})_{k\in\mathbb{N}} && \text{(apply def. of adjoint operator)}\end{aligned}$$

So finally, $(T' y'_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence of $(T' y'_k)_{k\in\mathbb{N}}$.

# Theorem: Schauder – Proof ($\Leftarrow$)

**Proof.**

"$\Leftarrow$": The other proof direction tells us that $T$ compact $\Rightarrow T'$ compact So in extension this also yields $T'$ compact $\Rightarrow T''$ compact The corollary tells us that $T = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ T'' \circ J_X$ Lastly, the operator $T''$ is compact, the revision tells us $J_X$ and $(J_Y|_{Y \to J_Y(Y)})^{-1}$ are bounded, linear operators and therefore $T$ is a composition of bounded, linear operators.
Using the revision we can conclude that $T$ is compact. $\square$

# Example: Compact Integral Operator

## Example 16

The lecture [KP25] states that the following integral operator is compact:

$$T : C[0,1] \to C[0,1], Tf(x) = \int_0^1 f(t)\, dx$$

So $T' : C[0,1]' \to C[0,1]'$ is compact too. Evaluating $C[0,1]'$ is beyond the scope of this paper.

# Definition: Annihilator

## Definition 17

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space and $V \subset X'$. Then we define the annihilator of $V$ in $X$ as

$$V_\perp = \left\{x \in X \mid \forall x' \in V : x'(x) = 0\right\}$$

▷ The annihilator is the set of linear, bounded functionals that "see" exactly the opposite of $V$ and are "blind" to $V$.

# Revision: Hahn-Banach Corollary

**Revision**

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space, $U \subset X$ a closed subspace and $x \in X \setminus U$. Then we have

$$\exists x' \in X' : x'|_U = 0 \wedge \; x'(x) \neq 0$$

# Theorem: Annihilator is Closed Linear Subspace

## Theorem 18

*Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space and $V \subset X'$. Then we have*

$$V_\perp \subset X \text{ is a closed linear subspace}$$

## Proof.

We have $V_\perp = \bigcap_{x' \in V} (x')^{-1}(0)$ As an intersection of closed sets, $V_\perp$ must be closed. $\square$

# Theorem: Rank-Nullity Generalized

## Theorem 19

*Let* $T \in \mathcal{L}(X, Y)$ *be a bounded, linear operator. Then we have*

$$\overline{\operatorname{Im} T} = (\operatorname{Ker} T')_\perp$$

▷ In linear algebra lectures this is proven for finite-dimensional vector spaces (see Satz 6.1.5 [Wer18])

# Theorem: Rank-Nullity Generalized – Proof ($\subset$)

## Proof

"$\subset$": Let $Tx \in \operatorname{Im} T$ with $x \in X$ and $y' \in \operatorname{Ker} T'$.
We first prove $Tx \in (\operatorname{Ker} T')_\perp$:

$$
\begin{aligned}
y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use $\circ$ notation)} \\
&= (T'y')(x) && \text{(apply adjoint op. definition)} \\
&= 0(x) = 0 && \text{(apply $y' \in \operatorname{Ker} T'$)}
\end{aligned}
$$

Since this holds for all choices of $Tx$ we get $\operatorname{Im} T \subset (\operatorname{Ker} T')_\perp$ Since $(\operatorname{Ker} T')_\perp$ is closed, we also get $\overline{\operatorname{Im} T} \subset (\operatorname{Ker} T')_\perp$

# Theorem: Rank-Nullity Generalized – Proof ($\supset$)

**Proof.**

"$\supset$": We can prove the contraposition $\left(Y \setminus \overline{\operatorname{Im} T}\right) \subset (Y \setminus (\operatorname{Ker} T')_{\perp})$ Set $U = \overline{\operatorname{Im} T}$ and let $y \in Y \setminus U$. We know $U$ that a closed linear subspace. The corollary of the theorem of Hahn-Banach tells us that $\exists y' \in Y' : y'|_U = 0 \wedge \ y'(y) \neq 0$ Since $\operatorname{Ker} T' \subset Y'$ and $\forall y' \in \operatorname{Ker} T' : y'(y) \neq 0$ we get $y \in Y \setminus (\operatorname{Ker} T')_{\perp}$ □

# Corollary: Operator Solutions

## Corollary 20

*Let* $T \in \mathcal{L}(X, Y)$ *be a linear, continuous operator with* $\operatorname{Im} T$ *closed. Then we have*

$$y \in \operatorname{Im} T \text{ if and only if } \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0$$

## Proof.

We have

$$\begin{aligned} y \in \operatorname{Im} T &= \overline{\operatorname{Im} T} && \text{(Im } T \text{ is closed)} \\ &= (\operatorname{Ker} T')_\perp && \text{(apply the theorem)} \\ \iff & \forall y' \in \operatorname{Ker} T' : y'(y) = 0 && \text{(apply def. of annihilator)} \\ \iff & \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0 && \text{(write in equivalent way)} \end{aligned}$$

□

# Example: Left Shift

## Example 21

Consider the left shift operator $T : l_p \to l_p$ from earlier. Its adjoint $T' : l_{p^*} \to l_{p^*}$ is the right shift operator: $T'(s_1, s_2, \dots) = (0, s_1, s_2, \dots)$

To apply the theorem, we first find Ker $T'$. If $T's = 0$, then $(0, s_1, s_2, \dots) = (0, 0, 0, \dots)$, which implies $s = 0$. Thus, Ker $T' = \{0\}$.

The theorem states that $\overline{\text{Im } T} = (\text{Ker } T')_\perp = \{0\}_\perp = l_p$. This is consistent with the fact that the left shift is surjective (Im $T = l_p$).

# Example: Right Shift

## Example 22

Consider the right shift operator $T : l_p \to l_p$, $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. Its image is the closed subspace $\operatorname{Im} T = \{y \in l_p \mid y_1 = 0\}$.

The adjoint $T' : l_{p^*} \to l_{p^*}$ is the left shift, $T'(s_1, s_2, s_3, \dots) = (s_2, s_3, \dots)$. Thus $\operatorname{Ker} T' = \{(s_1, 0, 0, \dots) \mid s_1 \in \mathbb{F}\} = \operatorname{span}\{e_1'\}$.

By the corollary, $y \in \operatorname{Im} T$ iff it is annihilated by every functional in $\operatorname{Ker} T'$. For any $s \in \operatorname{Ker} T'$, the condition $s(y) = 0$ becomes: $\sum_{k=1}^{\infty} y_k s_k = y_1 s_1 = 0$

Since this must hold for all $s_1 \in \mathbb{F}$, we must have $y_1 = 0$. This matches $\operatorname{Im} T$, verifying the theorem.

# References


[FS25] Gerd Fischer and Boris Springborn. "Bilinearformen und Skalarprodukte". In: *Lineare Algebra*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2025, pp. 309–372. ISBN: 9783662712603 9783662712610. DOI: 10.1007/978-3-662-71261-0_7. URL: https://link.springer.com/10.1007/978-3-662-71261-0_7 (visited on 01/26/2026).

[KP25] David Krieg and Joscha Prochno. *Functional Analysis – Lecture Notes UoP*. 2025. (Visited on 07/22/2025).

[Wer18] Dirk Werner. *Funktionalanalysis*. Springer-Lehrbuch. Berlin, Heidelberg: Springer Berlin Heidelberg, 2018. ISBN: 9783662554067 9783662554074. DOI: 10.1007/978-3-662-55407-4. URL: http://link.springer.com/10.1007/978-3-662-55407-4 (visited on 01/20/2026).

# Summary

Defined the adjoint operator $T' : Y' \to X'$ for bounded linear operators

Showed $T \mapsto T'$ is linear and isometric

Proved the adjoint reverses composition: $(ST)' = T'S'$

Characterized when an operator $S : Y' \to X'$ is an adjoint via the canonical embedding

Proved Schauder's theorem: $T$ compact $\iff$ $T'$ compact

Generalized the rank-nullity theorem: $\overline{\operatorname{Im} T} = (\operatorname{Ker} T')_\perp$