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Abstract

This proseminar paper provides a foundational overview of dual-/adjoint operators in a general Banach space setting.

Outline

Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as A^T over $\mathbb{R}$ or A^H over $\mathbb{C}$) to operators on normed $\mathbb{R}$ or $\mathbb{C}$ vector spaces.

Topics Covered

We will cover the following topics:

- The operator $\cdot^H$ is linear and isometric wrt. the spectral norm.
- The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem: $(\operatorname{Im} A)^{\perp} = \operatorname{Ker} A^H$.
- TODO Add more

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Terminology

The terms adjoint and dual are often used interchangeably. We will standardize to adjoint, to avoid unnecessary confusion. Following [**werner˙funktionalanalysis˙2018**], we write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ when the field is unspecified.

Definition: Linear Operator

Definition 1

We remind ourselves of the following concepts from the lecture [**krieg˙functional˙2025**]: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed $\mathbb{F}$-vector spaces

i) Let

$$T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call T a linear operator. We call T bounded, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation $T : X \to Y$ for linear operators rather than $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$.

Definition: Dual Space and Closed Unit Ball

Definition 2

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed $\mathbb{F}$-vector spaces

ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

iii) The closed unit ball in $(X, \|\cdot\|_X)$ is abbreviated with $\overline{B}_X$.

Revision: Foundational Statements (i)

Revision

The following statements are foundational for this topic:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces.

i) The set of continuous, linear operators $\mathcal{L}(X, Y)$ is a Banach space if and only if Y is a Banach space. In particular, the topological dual space $\mathcal{L}(X, \mathbb{F})$ is a Banach space.

ii) Let $T : X \to Y$ be a linear operator. Then T is bounded if and only if T is continous.

Revision

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces.

iii) Let $(Z, \|\cdot\|_Z)$ be a normed $\mathbb{F}$-vector space, let $T : X \to Y$ be a linear, bounded operator and let $S : Y \to Z$ be a linear, bounded operator. Then $S \circ T$ is a linear, bounded operator.

iv) Let $(Z, \|\cdot\|_Z)$ be a normed $\mathbb{F}$-vector space, let $T : X \to Y$ be a linear operator and let $S : Y \to Z$ be a linear operator. If T or S is compact, $S \circ T$ is compact.

Revision: Foundational Statements – Proof

Proof.

Please refer to the functional analysis lecture notes [**krieg˙functional˙2025**] from SS/2025. □

Definition: Adjoint Operator

Definition 3

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces and $T \in \mathcal{L}(X, Y)$.
Then $T' : Y' \to X', y' \mapsto y' \circ T$ is called the adjoint operator. From now on, we will implicitly refer to the normed $\mathbb{F}$ vector spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T'.

Remark: Adjoint Operator

Remark

In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from $\mathbb{F}$.
For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$.

Example: Dual Space of $\mathbb{R}^n$

Example 4

Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
(\mathbb{R}^n)' &= \{g : \mathbb{R}^n \to \mathbb{R} \text{ linear, continuous}\} && \text{(apply def. of dual space)} \\
&= \{g : \mathbb{R}^n \to \mathbb{R} \text{ linear}\} && \text{(result from linear algebra)} \\
&\cong \mathbb{R}^n && \text{(result from linear algebra see [\textbf{fischer˙bilinearformen}}
\end{aligned}$$

Example: Adjoint of Matrix Operator – Setup

Example 5

Let $m \in \mathbb{N}$ and let $A \in \mathbb{R}^{m \times n}$.
Consider the function $f : \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$. The signature of the adjoint is $f' : (\mathbb{R}^m)' \to (\mathbb{R}^n)'$. With e_i we denote the standard basis vectors and with e_i' we denote the dual standard basis vectors.

Example: Adjoint of Matrix Operator – Calculation

Example 6

Let $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We have

$$\begin{aligned} (f' e_i')(e_j) &= (e_i' \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\ &= f(e_j)_i = a_{ij} && \text{(apply def. of } e_i' \text{ and } f\text{)} \end{aligned}$$

So if we set $i = 1$ for instance, we get

$$(f' e_1')(e_j) = a_{1j}$$

The first "column" of f' (up to isomorphism) must be $(a_{1j})_{j=1,\ldots,n}$.

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Example 7

In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \to \mathbb{R}^n, x \mapsto A^T x$$

Introduction

The proofs will use the Hahn-Banach corollaries a couple of times. So first, we need to recall Hahn-Banach related theorems.

Revision: Hahn-Banach

Revision (Hahn-Banach)

Let $(X, \|\cdot\|)$ be a normed space and $0 \neq x \in X$.
Then we have

$$\exists f \in X' : \|f\| = 1 \wedge \ f(x) = \|x\|$$

Proof.

Please refer to the functional analysis lecture notes [**krieg˙functional˙2025**] from SS/2025. $\square$

Revision: Dual Unit Ball

Revision

The dual unit ball retains information about the norm of primal vectors:
Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space and $x \in X$.
Then we have

$$\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$$

Revision: Dual Unit Ball – Proof (Case 1)

Proof.

We will distinguist two cases:

Case 1 ($x = 0$): Since X' contains linear operators,

$$\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$$

$\square$

Revision: Dual Unit Ball – Proof (Case 2)

Proof.

Case 2 $x \neq 0$: We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

Using Hahn-Banach we get

$$\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|_X$$

So we get

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$$

□

Theorem: Properties of Adjoint Operator

Theorem 8

The adjoint operator has the following properties:

i) $T' \in \mathcal{L}(Y', X')$, *so* T' *is linear and bounded.*

  ▷ This implies $\forall y' Y' : T'y' \in X'$.

ii) $T \mapsto T'$ *is linear and isometric.*

Theorem: Properties of Adjoint Operator – Proof (i) Image

Proof.

"i)": Let $y' \in Y'$. Plugging it into the adjoint operator, we get $T'y' = y' \circ T$ with signature $X \to Y \to \mathbb{F}$. We can now see that $\operatorname{Im} T' \subset X'$.

$\square$

Theorem: Properties of Adjoint Operator – Proof (i) Linearity

Proof.

Let $y_1', y_2' \in Y', \alpha \in \mathbb{F}$. We then prove linearity:

$$
\begin{aligned}
T'(\alpha y_1' + y_2') &= (\alpha y_1' + y_2') \circ T && \text{(apply def. of adjoint operator)} \\
&= \alpha y_1' \circ T + y_2' \circ T && \text{(expand expression)} \\
&= \alpha T' y_1' + T' y_2' && \text{(apply def. of adjoint operator)}
\end{aligned}
$$

□

Theorem: Properties of Adjoint Operator – Proof (i) Boundedness

Proof.

Let $y' \in Y'$. We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\ &\leq \|y'\|_{Y'}\|T\|_{\mathcal{L}(X,Y)} && \text{(apply def. of op. norm)} \\ &:= C\|y'\|_{Y'} && \text{(def. the constant)} \end{aligned}$$

□

Theorem: Properties of Adjoint Operator – Proof (ii) Linearity

Proof.

Let $T_1, T_2 \in \mathcal{L}(X, Y), y' \in Y', x \in X, \alpha \in \mathbb{F}$. We first prove linearity:

$$
\begin{aligned}
(\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\
&= y'\left(\alpha T_1 x + T_2 x\right) && \text{(pull } x \text{ into the eq.)} \\
&= \alpha y'(T_1 x) + y'(T_2 x) && \text{(y' is linear)} \\
&= (\alpha T_1' y' + T_2' y')(x)
\end{aligned}
$$

$\square$

Theorem: Properties of Adjoint Operator – Proof (ii) Isometry

Proof.

We then prove isometry:

$$
\begin{aligned}
\|T\| &= \sup_{\|x\|_X \le 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\
&= \sup_{\|x\|_X \le 1} \sup_{\|y'\| \le 1} |y'(Tx)| && \text{(apply theorem of Hahn-Banach)} \\
&= \sup_{\|y'\| \le 1} \sup_{\|x\|_X \le 1} |y'(Tx)| && \text{(supremum order can be switched)} \\
&= \sup_{\|y'\| \le 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\
&= \|T'\| && \text{(apply def. of op. norm)}
\end{aligned}
$$

□

Example: Shift Operator – Setup

Example 9

Let $p \in (1, \infty)$ with $p \neq 2$. This makes l_p a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of l_p is isometrically isomorph to l_p^* where p^* is the Hölder conjugate with $1/p + 1/p^* = 1$.

Example 10

As a reminder, the general idea of the proof of $l_p' \cong l_p^*$ goes as follows:

Define the isometric isomorphism as

$$T : l_p^* \to l_p', s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$$

Verify $(Ts)x$ converges as

$$|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$$

and absolute convergence implies convergence.

Verify T is injective using the linearity.

Verify T is surjective and isometric (the long part). See **[werner˙funktionalanalysis˙2018]** for a full proof.

Example: Shift Operator – Left Shift

Example 11

To illustrate the adjoint operator, we now work through an example. Consider the left shift operator

$$T : l_p \to l_p, (x_k)_{k\in\mathbb{N}} \mapsto (x_{k+1})_{k\in\mathbb{N}}$$

It is well-defined since

$$\sum_{k\in\mathbb{N}} |x_{k+1}|^p \leq \sum_{k\in\mathbb{N}} |x_k|^p < \infty$$

Example: Shift Operator – Adjoint Signature

Example 12

We can now compute the adjoint operator T':

The adjoint T' must have the signature $l_p' \cong l_p^* \to l_p' \cong l_p^*$.

Example 13

Let $y' \in l_p' \cong l_p^*$. Then we can write $y' : l_p \to \mathbb{F}, x \mapsto \sum_{k\in\mathbb{N}} x_k s_k$ with $s \in l_p^*$.
Now for $x \in l_p$ we have

$$\begin{aligned}
(T'y')(x) &= (y' \circ T)(x) = y'(Tx) && \text{(apply def. of } T'\text{)}\\
&= y'((x_{k+1})_{k\in\mathbb{N}}) && \text{(apply def. of } T\text{)}\\
&= \sum_{k\in\mathbb{N}} x_{k+1} s_k && \text{(apply def. of } y'\text{)}\\
&= \sum_{k\in\mathbb{N}} x_k s_k' && \text{(with } s_1' = 0 \text{ and } s_k' = s_{k-1} \text{ for } k > 1\text{)}
\end{aligned}$$

Example: Shift Operator – Conclusion

Example 14

This tells us that the adjoint operator T' acts as a right shift (up to isomorphism):

$$T' : l_p^* \to l_p^*, (s_k)_{k\in\mathbb{N}} \mapsto (0, s_1, s_2, \dots)$$

Theorem: Adjoint Reverses Composition

Theorem 15

Let X, Y, Z be normed $\mathbb{F}$ vector spaces.
Then the adjoint operator reverses composition:

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

Theorem: Adjoint Reverses Composition – Proof

Proof.

Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$.
We know $ST = S \circ T$ is still a linear, bounded operator from X to Z. So $(ST)'$ is well-defined.
Let $z' \in Z' = \mathcal{L}(Z,\mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y,\mathbb{F})$. We have

$$\begin{aligned}
(ST)'(z') &= z' \circ (ST) && \text{(apply def. of adjoint operator)}\\
&= (z' \circ S) \circ T && \text{(write out chain explicitly)}\\
&= y' \circ T && \text{(subst. y')}\\
&= T'y' && \text{(apply def. of adjoint operator)}\\
&= T'(z' \circ S) && \text{(subst. y')}\\
&= T'S'z' && \text{(apply def. of adjoint operator)}
\end{aligned}$$

So in total $(ST)' = T'S'$. $\square$

Example: Composition Rule

Example 16

Referring back to the example with $A \in \mathbb{R}^{m \times n}$, the adjoint operator reverses composition rule is compatible with the way matrix transposition on $\mathbb{R}$ or the matrix adjoint on $\mathbb{C}$ behave.

Introduction

For starters, we need to recall some concepts from the lecture.

Definition 17

Let X be a normed $\mathbb{F}$ vector space.

i) X'' is called the bidual space.

ii) Let $J_X : X \to X'', p \mapsto (x' \mapsto x'(p))$. J_X is called the canonical embedding from X into X''.

So J_X returns a function that evaluates dual space elements at $p \in X$.

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Figure: Dual of Dual

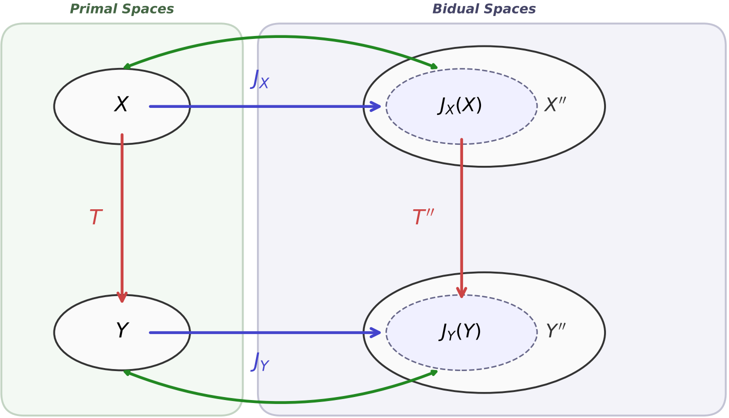

Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

Theorem: Bidual Adjoint Embedding

Theorem 18

Let $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed* $\mathbb{F}$ *vector spaces. Let* $T \in \mathcal{L}(X, Y)$ *be a bounded linear operator. Then we have:* $J_Y \circ T = T'' \circ J_X$.

Proof.

Before the proof, we can avoid confusion by typing out what T' and T'' evaluate to:

$T' : Y' \to X', y' \mapsto y' \circ T$ is the adjoint operator.

$T'' : X'' \to Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

$\square$

Theorem: Bidual Adjoint Embedding – Proof Signatures

Proof.

Now first of all, we need to check that the signatures of both sides of the equation match:

$$J_Y : Y \to Y'' \text{ and } T : X \to Y \text{ means } (J_Y \circ T) : X \to Y''.$$

$$T'' : X'' \to Y'' \text{ and } J_X : X \to X'' \text{ means } (T'' \circ J_X) : X \to Y''.$$

□

Theorem: Bidual Adjoint Embedding – Proof Calculation

Proof.

Finally, for $p \in X$ and $y' \in Y'$ we have

$$
\begin{aligned}
((J_Y \circ T)(p))\,(y') = J_Y(Tp)(y') = y'(Tp) \quad & \text{(subst. } p \text{ in and apply def. of } J_Y\text{)} \\
= (y'T)(p) \quad & \text{(use associativity)} \\
= (T'y')(p) \quad & \text{(apply def. of } T'\text{)} \\
= (x' \mapsto x'(p))(T'y') \quad & \text{(pull out subst. function)} \\
= J_X(p)(T'y') \quad & \text{(recognize this is just } J_X\text{)} \\
= T''(J_X(p))(y') \quad & \text{(apply def. use } T''\text{)} \\
= ((T'' \circ J_X)(p))\,(y') \quad & \text{(use } \circ \text{ notation)}
\end{aligned}
$$

$\square$

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Context

We will now characterize when a continuous operator between Y' and X' is an adjoint operator. We need to revise an important corollary from the lecture first.

Revision: Canonical Embedding Properties (i-ii)

Revision

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space. Then we have

i) The canonical embedding J_X is an isometric injective function.

ii) $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism.

Revision: Canonical Embedding Properties (iii-iv)

Revision

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space. Then we have

iii) J_X is a bounded, linear operator.

iv) $(J_X|_{X \to J_X(X)})^{-1}$ is a bounded, linear operator.

Revision: Canonical Embedding – Proof Idea (i-ii)

Proof Idea

"i)":
Please refer to the functional analysis lecture notes [**krieg˙functional˙2025**] from SS/2025.

"ii)": Follows by definition of J and from "i)".

Revision: Canonical Embedding – Proof Idea (iii)

Proof Idea

"iii)": Since $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

- J_X is linear since the $x' \in X'$ are linear.
- $\|J_X\| = 1$. This means J_X is bounded.

Revision: Canonical Embedding – Proof Idea (iv)

Proof Idea

"iv)": Since $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

$(J_X|_{X \to J_X(X)})^{-1}$ inherits linearity from J.

$\|J_X\| = 1$ and therefore $\|(J_X|_{X \to J_X(X)})^{-1}\| = 1$. This means $(J_X|_{X \to J_X(X)})^{-1}$ is bounded.

Corollary: Strengthened Bidual Adjoint Embedding

Corollary 19

We can strengthen the theorem using the isomorphism:
Let $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed* $\mathbb{F}$ *vector spaces. Let* $T \in \mathcal{L}(X, Y)$ *be a bounded linear operator. Then we have*

i) $J_Y \circ T = T'' \circ J_X$.

ii) $T''|_{X \to J_X(X)} = J_Y \circ T \circ (J_X|_{X \to J_X(X)})^{-1}$

iii) $T = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ T'' \circ J_X$.

Corollary: Strengthened Bidual Adjoint Embedding – Proof

Proof.

"i)": This is the statement of the theorem.

"ii)": The isomorphism from the revision gives us the result.

"iii)": The isomorphism from the revision gives us the result.

□

Theorem: Characterization of Adjoint Operators

Theorem 20

Let $S \in \mathcal{L}(Y', X')$ be a continuous, linear operator. Then we have

$$\exists T \in \mathcal{L}(X, Y) : T' = S \text{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

Theorem: Characterization of Adjoint Operators – Proof ($\Rightarrow$)

Proof.

"$\Rightarrow$": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T'\text{)} \\ &= J_Y(T(X)) && \text{(use the theorem)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y\text{)} \end{aligned}$$

$\square$

Proof.

"$\Leftarrow$": We have $S'(J_X(X)) \subset J_Y(Y)$ and the revision gives us $J_Y(Y) \cong Y$.
So for $x \in X$ and $y_x'' = S'(J_X(x))$ there is a (unique) $y_x \in Y$ with $y_x'' = J_Y(y_x)$.
Choose

$$T : X \to Y, x \mapsto y_x$$

We know T exists (and is unique) due to the previous argument. □

Theorem: Characterization of Adjoint Operators – Proof ($\Leftarrow$) Linearity

Proof.

We know T is linear and continuous, as

$$T = y. = \left(J_Y|_{Y \to J_Y(Y)}\right)^{-1} \circ S' \circ J_X$$

and all elements in the chain are bounded, linear operators. □

Theorem: Characterization of Adjoint Operators – Proof ($\Leftarrow$) $S = T'$

Proof.

Let $y' \in Y'$ and $x \in X$. Lastly, we need to prove that $S = T'$:

$$
\begin{aligned}
(Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X\text{)} \\
&= (S' J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of ajoint op.)} \\
&= J_Y(((J_Y|_{Y \to J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y \to J_Y(Y)})^{-1} = \mathrm{Id}\text{)} \\
&= y'(((J_Y|_{Y \to J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\
&= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X\text{)} \\
&= (T'y')(x) && \text{(apply def. of adjoint op.)}
\end{aligned}
$$

$\square$

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Introduction

For starters, we need to recall and review some concepts from the lecture.

Definition: Relatively Compact

Definition 21

The following definitions are revisions from the lecture:
Let X, Y be normed $\mathbb{F}$ vector spaces.

i) $M \subset X$ is relatively compact, if

$$\forall (x_n)_{n\in\mathbb{N}} \subset M : (x_n)_{n\in\mathbb{N}} \text{ has a converging subsequence in } X$$

Definition: Compact Operator and Rank

Definition 22

Let X, Y be normed $\mathbb{F}$ vector spaces.

ii) $T \in \mathcal{L}(X, Y)$ is compact, if $T\left(\overline{B}_X\right)$ is relatively compact.

iii) The rank of T is $\operatorname{rk} T = \dim T(X)$.

Context: Relatively Compact

As a reminder, in metric spaces a subset is compact if and only if all sequences contain a converging subsequence in that set. So relatively compact relaxes the requirement that the set must be closed. An important property of the closed unit ball in $\mathbb{R}^n$ or $\mathbb{C}$ is that it is compact. Therefore, a linear operator between $\mathbb{F}^n \to \mathbb{F}^m$ always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Context: Finite Rank Operators

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ have the same open sets. In finite dimensions, linear algebra tells us that $(X, \|\cdot\|_1)$ and $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$ have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space $\mathbb{R}^n$. In $\mathbb{R}^n$, the theorem of Heine-Borel tells us that all bounded subsets are relatively compact.

Since relatively compact is a topological property, the theorem transfers to $(X, \|\cdot\|)$. Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

Context: Infinite Rank

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general Banach spaces.

Definition: Uniformly Equicontinuous

Definition 23

The following definitions are critical for the theorem of Arzelà-Ascoli:
Let X, Y be metric spaces and $M \subset \{f : X \to Y\}$.

i) M is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

Definition: Pointwise Bounded

Definition 24

Let X, Y be metric spaces and $M \subset \{f : X \to Y\}$.

ii) M is pointwise bounded, if

$$\forall x \in X : \{f(x) \mid f \in M\} \text{ is bounded}$$

Context: Arzelà-Ascoli

The theorem of Arzelà-Ascoli from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions evaluations rather than the functions themselves. They can be easily verified to be true. Note that $C(D) = \{f : D \to \mathbb{F} \text{ continuous}\}$.

Revision: Arzelà-Ascoli

Revision (Arzelà-Ascoli)

Let D be a compact metric space and $M \subset C(D)$ with the supremum norm. Then we have M is uniformly equicontinuous and pointwise bounded implies M is relatively compact.

Revision: Arzelà-Ascoli – Proof Idea (1)

Proof Idea

D is separable, i.e. $\exists D_0 \subset D : \overline{D_0} = D$. Simply set $D_0 = \bigcup_{n\in\mathbb{N}} D_n$ where D_n is a finite $\frac{1}{n}$-covers of D.

For all $x \in D$ we can use the pointwise boundedness to invoke Bolzano-Weierstrass to find converging subsequences $(f_{n(x,k)}(x))_{k\in\mathbb{N}} \subset \mathbb{F}$.

Specifically, we have $\forall x \in D_0 : (f_{n(x,k)}(x))_{k\in\mathbb{N}}$ converges.

Proof Idea

Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on x: $\forall x \in D_0 : (f_{n(k)}(x))_{k\in\mathbb{N}}$ converges.

We already have $f_{n(k)}|_{D_0} \longrightarrow: f|_{D_0}$. And it seems sensible to assume that $(f_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence. But how can you extend this to the entire set D?

For each $x \in D$, we can choose an arbitrarily close $x_0 \in D_0$. Using two triangle inequalities, uniform equicontinuity allows us to extend the result to D.
As $C(D)$ is complete, the Cauchy sequence converges.

Context: Schauder

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

Theorem: Schauder

Theorem 25 (Schauder)

Let X, Y be Banach spaces and $T : X \to Y$ be a bounded, linear operator. Then we have T is compact if and only if T' is compact.

Theorem: Schauder – Proof ($\Rightarrow$) Setup

Proof.

"$\Rightarrow$": Let $(y_n')_{n\in\mathbb{N}} \subset Y' = \mathcal{L}(Y,\mathbb{F}) \subset C(Y)$ be bounded.
Our goal is to show that there is a convergent subsequence in $(T'y_n')_{n\in\mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm. For all $n \in \mathbb{N}$ set $f_n = y_n'|_{T(\overline{B}_X)}$. $\square$

Proof.

We have

$$
\begin{aligned}
\|T'y_n' - T'y_m'\| &= \|y_n' \circ T - y_m' \circ T\| && \text{(apply def. of adjoint op.)}\\
&= \sup_{\|x\|_X \le 1} |((y_n' \circ T) - (y_m' \circ T))(x)| && \text{(use supremum char. of the op. norm)}\\
&= \sup_{\|x\|_X \le 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m\text{)}\\
&= \sup_{d \in T\left(\overline{B}_X\right)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m\text{)}
\end{aligned}
$$

□

Proof.

We know $(T'y_n')_{n\in\mathbb{N}}$ converges if and only if it is a Cauchy sequence $(\mathcal{L}(X,\mathbb{F}), \|\cdot\|)$.
Therefore the convergence of $(T'y_n')_{n\in\mathbb{N}}$ in the operator norm is only dependent on the behaviour of $(f_n)_{n\in\mathbb{N}}$ on $\overline{T\left(\overline{B}_X\right)}$. $\square$

Theorem: Schauder – Proof ($\Rightarrow$) Setup for Arzelà-Ascoli

Proof.

We now set
$$D = \overline{T\left(\overline{B}_X\right)}$$
and pack the sequence into
$$M := \{f_n \mid n \in \mathbb{N}\}$$
and examine them for the conditions of Arzelà-Ascoli. $\square$

Theorem: Schauder – Proof ($\Rightarrow$) D is compact

Proof.

D is compact: We have

- $\overline{B}_X$ is bounded.
- T is a compact operator.

So $T\left(\overline{B}_X\right)$ is relatively compact and D is compact. $\square$

Theorem: Schauder – Proof ($\Rightarrow$) M is pointwise bounded

Proof.

M is pointwise bounded: For $x \in \overline{B}_X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} |f_n(Tx)| &= |y_n'(Tx)| := |y_n'(d)| && \text{(apply def. of } f_n \text{ and define } d\text{)} \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 && \text{(apply op. norm ineq. and D bounded means } \|d\|_Y \leq: C_2\text{)} \end{aligned}$$

For $d \in D$ we can choose $(x_k)_{k \in \mathbb{N}} \subset \overline{B}_X$ with $Tx_k \longrightarrow d$ and $f_n(Tx_k) \leq C_1 C_2$.
Using continuity we get $|f_n(d)| \leq C_1 C_2$.

□

Proof.

M is uniformly equicontinuous: For $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon / C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$ we have

$$\begin{aligned}
|f_n(d_1) - f_n(d_2)| &\leq \|y_n'\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)} \\
&\leq C_1 \|d_1 - d_2\|_Y && \text{(use the upper bound } \|y_n'\| \leq C_1\text{)} \\
&< C_1 \delta = \epsilon && \text{(substitute in } \delta\text{)}
\end{aligned}$$

$\square$

Theorem: Schauder – Proof ($\Rightarrow$) Conclusion

Proof.

The theorem of Arzelà-Ascoli now tells us that M is relatively compact. So every sequence in M has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ we have

$$\begin{aligned}
(f_{n(k)})_{k\in\mathbb{N}} &= (y'_n \circ T)_{k\in\mathbb{N}} && \text{(apply def. of } f_{n(k)}\text{)} \\
&= (T'y'_{n(k)})_{k\in\mathbb{N}} && \text{(apply def. of adjoint operator)}
\end{aligned}$$

So finally, $(T'y'_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence of $(T'y'_k)_{k\in\mathbb{N}}$. □

Theorem: Schauder – Proof ($\Leftarrow$)

Proof.

"$\Leftarrow$": The other proof direction tells us that

$$T \text{ compact } \Rightarrow T' \text{ compact}$$

So in extension this also yields

$$T' \text{ compact } \Rightarrow T'' \text{ compact}$$

The corollary tells us that

$$T = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ T'' \circ J_X$$

Lastly, the operator T'' is compact, the revision tells us J_X and $(J_Y|_{Y \to J_Y(Y)})^{-1}$ are bounded, linear operators and therefore T is a composition of bounded, linear operators.
Using the revision we can conclude that T is compact. $\square$

Example: Compact Integral Operator

Example 26

The lecture [**krieg˙functional˙2025**] states that the following integral operator is compact:

$$T : C[0,1] \to C[0,1], Tf(x) = \int_0^1 f(t)\,dx$$

So $T' : C[0,1]' \to C[0,1]'$ is compact too. Evaluating $C[0,1]'$ is beyond the scope of this paper.

Definition 27

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space, $U \subset X$ and $V \subset X'$. Then we define the annihilator of V in X as

$$V_\perp = \left\{x \in X \mid \forall x' \in V : x'(x) = 0\right\}$$

Context

To prove properties about this set, we need another corollary of the theorem of Hahn-Banach.

Revision: Hahn-Banach Corollary

Revision

Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space, $U \subset X$ a closed subspace and $x \in X \setminus U$. Then we have

$$\exists x' \in X' : x'|_U = 0 \wedge \; x'(x) \neq 0$$

Revision: Hahn-Banach Corollary – Proof

Proof.

Let $Y = X/U$ be the (canonical) quotient space. Then Y is a normed $\mathbb{F}$ vector space. (todo: why?) Set $y = x \in Y$. We can apply the theorem of Hahn-Banach to obtain $y' \in Y'$ with $y'(y) \neq 0$ and $y'|_U = 0$. $\square$

Theorem: Annihilator is Closed Linear Subspace

Theorem 28

Let $(X, \|\cdot\|)$ *be a normed* $\mathbb{F}$ *vector space,* $U \subset X$ *and* $V \subset X'$*. Then we have*

$$V_\perp \subset X \text{ is a closed linear subspace}$$

Theorem: Annihilator is Closed Linear Subspace – Proof

Proof.

We have

$$V_\perp = \bigcap_{x' \in V} (x')^{-1}(0)$$

As an intersection of closed sets, $V_\perp$ must be closed. $\square$

Theorem: Rank-Nullity Generalized

Theorem 29

Let $T \in \mathcal{L}(X, Y)$ be a bounded, linear operator. Then we have

$$\overline{\operatorname{Im} T} = (\operatorname{Ker} T')_{\perp}$$

▷ In linear algebra lectures this is proven for finite-dimensional vector spaces (see Satz 6.1.5 [**werner˙funktionalanalysis˙2018**])

Theorem: Rank-Nullity Generalized – Proof ($\subset$)

Proof.

"$\subset$": Let $Tx \in \operatorname{Im} T$ with $x \in X$ and $y' \in \operatorname{Ker} T'$.
We first prove $Tx \in (\operatorname{Ker} T')_\perp$:

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \operatorname{Ker} T'\text{)} \end{aligned}$$

Since this holds for all choices of Tx we get

$$\operatorname{Im} T \subset (\operatorname{Ker} T')_\perp$$

Since $(\operatorname{Ker} T')_\perp$ is closed, we also get

$$\overline{\operatorname{Im} T} \subset (\operatorname{Ker} T')_\perp$$

Theorem: Rank-Nullity Generalized – Proof ($\supset$)

Proof.

"$\supset$": We can prove the contraposition

$$\left(Y \setminus \overline{\operatorname{Im} T}\right) \subset \left(Y \setminus (\operatorname{Ker} T')_\perp\right)$$

Set $U = \overline{\operatorname{Im} T}$ and let $y \in Y \setminus U$. We know U that a closed linear subspace. The corollary of the theorem of Hahn-Banach tells us that

$$\exists y' \in Y' : y'|_U = 0 \wedge \; y'(y) \neq 0$$

Since $\operatorname{Ker} T' \subset Y'$ and

$$\forall y' \in \operatorname{Ker} T' : y'(y) \neq 0$$

we get

$$y \in Y \setminus (\operatorname{Ker} T')_\perp$$

$\square$

Corollary: Operator Solutions

Corollary 30

Let $T \in \mathcal{L}(X, Y)$ be a linear, continuous operator with $\operatorname{Im} T$ *closed. Then we have*

$$y \in \operatorname{Im} T \text{ if and only if } \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0$$

Corollary: Operator Solutions – Proof

Proof.

We have

$$
\begin{aligned}
y \in \operatorname{Im} T &= \overline{\operatorname{Im} T} && \text{(Im } T \text{ is closed)}\\
&= (\operatorname{Ker} T')_\perp && \text{(apply the theorem)}\\
\iff \forall y' \in \operatorname{Ker} T' &: y'(y) = 0 && \text{(apply def. of annihilator)}\\
\iff \forall y' \in Y' &: T'y' = 0 \Rightarrow y'(y) = 0 && \text{(write in equivalent way)}
\end{aligned}
$$

$\square$

Example: Shift Operator Revisited

Example 31

We can refer back to the example with the left shift operator.
The left shift operator T has $\operatorname{Im} T = l_p$, which is (trivially) closed.
Since we already know the adjoint operator we can verify that the last theorem works.
Everything is up to isometry: Let $(x_k)_{k\in\mathbb{N}} \in l_p$ such that $T\left((x_k)_{k\in\mathbb{N}}\right) = (x_2, \dots) \in l_p$. Then

$$T'y' = 0 \Rightarrow y'(y) = 0$$

amounts to "if the right shifted version of a sequence is 0 then

References

Thank you!

Questions?