

# Dual Operators

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## Abstract

This proseminar paper provides a foundational overview of dual-/adjoint operators in a general Banach space setting.

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## 1 Introduction

### 1.1 Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as  $A^T$  over  $\mathbb{R}$  or  $A^H$  over  $\mathbb{C}$ ) to general operators on Banach spaces.

We will cover and generalize the following topics:

1. The operator  $\cdot^H$  is linear and isometric wrt. the spectral norm.
2. The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem:  
 $(\text{Im } A)^\perp = \text{Ker } A^H$ .
3. The Lagrange duality from nonlinear optimisation theory.
4. TODO Add more

### 1.2 General revision

**Definition 1.** Remember the following concepts:

- i): Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed vector spaces on  $\mathbb{R}$  or  $\mathbb{C}$ .  
Let  $T : X \rightarrow Y$  be linear.  
 $T$  is a **linear operator**.  $T$  is **bounded**, if  $\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$ .
- ii): The (topological) dual space is defined as  $X' := \mathcal{L}(X, \mathbb{R})$ .

**Revision 1.** The following statements are foundational for this topic:

- i): The topological dual space  $\mathcal{L}(X, Y)$  is a Banach space.
- ii): For linear operators, continuous and bounded are equivalent.
- iii): A linear, bounded operator between  $X$  and  $Y$  and one between  $Y$  and  $Z$  can be chained to produce a linear, bounded operator between  $X$  and  $Z$ .
- iv): In a chain of linear, bounded operators with at least one compact operator the resulting operator is compact.

*Proof.* Please refer to the functional analysis lecture notes from the summer of 2025. □

## 2 The adjoint operator

### 2.1 The basic definitions and conventions

**Remark 1.** The terms [adjoint](#) and [dual](#) are often used interchangeably. We will standardize on [adjoint](#), to avoid unnecessary confusion. We will abstract the field as  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ .

**Definition 2.** Let  $X, Y$  be metric spaces and  $T \in \mathcal{L}(X, Y)$ .

Then  $T' : Y' \rightarrow X', y' \mapsto y' \circ T$  is called the adjoint operator. From now on, we will implicitly refer to the metric spaces  $X, Y$  and the topological dual spaces  $X', Y'$  when talking about the dual operator  $T'$ .

**Remark 2.** In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from  $\mathbb{F}$ . For  $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$ , the adjoint operator evaluates to  $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$ .

**Example 1.**

### 2.2 The dual space of the dual space

For starters, we need to recall some concepts from the lecture.

**Definition 3.** Let  $X$  be a normed vector space.

- i):  $X''$  is called the bidual space.
- ii): Let  $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$ .  $J_X$  is called the [canonical embedding](#) from  $X$  into  $X''$ .

So  $J_X$  returns a function that evaluates dual space elements at  $p \in X$ .

Figure [todo] illustrates why the bidual space in conjunction with the adjoint operator is interesting. TODO

**Theorem 1.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed vector spaces. Let  $T \in \mathcal{L}(X, Y)$  be a bounded linear operator. Then we have:  $J_Y \circ T = T'' \circ J_X$ .

Or equivalently:  $T|_{J_X(X)} = J_Y \circ T'' \circ J_X|_{J_X(X)}^{-1}$ .

Or equivalently:  $T'' = J_Y^{-1} \circ T \circ J_X$ .

*Proof.* Before the proof, we can avoid confusion by typing out what  $T'$  and  $T''$  evaluate to:

- $T' : Y' \rightarrow X', y' \mapsto y' \circ T$  is the adjoint operator.
- $T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$  is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

- $J_Y : Y \rightarrow Y''$  and  $T : X \rightarrow Y$  means  $(J_Y \circ T) : X \rightarrow Y''$ .
- $T'' : X'' \rightarrow Y''$  and  $J_X : X \rightarrow X''$  means  $(T'' \circ J_X) : X \rightarrow Y''$ .

Finally, for  $p \in X$  and  $y' \in Y'$  we have

$$\begin{aligned}
 [(J_Y \circ T)(p)](y') &= J_Y(Tp)(y') = y'(Tp) && \text{(subst. } p \text{ in and apply def. of } J_Y) \\
 &= (y'T)(p) && \text{(use associativity)} \\
 &= (T'y')(p) && \text{(apply def. of } T') \\
 &= (x' \mapsto x'(p))(T'y') && \text{(pull out subst. function)} \\
 &= J_X(p)(T'y') && \text{(recognize this is just } J_X) \\
 &= T''(J_X(p))(y') && \text{(apply def. use } T'') \\
 &= [(T'' \circ J_X)(p)](y') && \text{(use } \circ \text{ notation)}
 \end{aligned}$$

□

Secondly, we can answer when a continuous operator between  $Y'$  and  $X'$  is an adjoint operator. We need to revise an important corollary from the lecture first.

**Revision 2.** Let  $(X, \|\cdot\|)$  be a normed vector space. Then we have

i): The canonical embedding  $J_X$  is an isometric injective function.

ii):  $J : X \rightarrow J_X(X), x \mapsto J_X(x)$  is an isometric isomorphism.

iii): TODO  $J_X^{\text{inv}}$  is basically a bounded, linear operator. I just hope that's true m'fucka!

*Proof Idea.* "i)": Please refer to the functional analysis lecture notes from the summer of 2025.

"ii)": Follows by definition of  $J$  and from "i)".

"iii)": TODO

□

**Theorem 2.** Let  $S \in \mathcal{L}(Y', X')$  be a continuous, linear operator. Then we have  $\exists T \in \mathcal{L}(X, Y) : T' = S \iff S'(J_X(X)) \subset J_Y(Y)$ .

*Proof.* „ $\Rightarrow$ “: We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T') \\ &= J_Y(T(X)) && \text{(use theorem 1)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y) \end{aligned}$$

„ $\Leftarrow$ “: We have  $S'(J_X(X)) \subset J_Y(Y)$  and revision 2 gives us  $J_Y(Y) \cong Y$ .

So for  $x \in X$  and  $y_x'' = S'(J_X(x))$  there is a (unique)  $y_x \in Y$  with  $y_x'' = J_Y(y_x)$ .

Define  $T : X \rightarrow Y, x \mapsto y_x$ . We know  $T$  exists (and is unique) due to the previous argument.

We know  $T$  is linear and continuous, as  $T = y. = J_Y^{-1} \circ S' \circ J_X$  and all elements in the chain are bounded, linear operators.

Let  $y' \in Y'$  and  $x \in X$ . Lastly, we need to prove that  $S = T'$ :

$$\begin{aligned} (Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X) \\ &= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of adjoint op.)} \\ &= J_Y((J_Y^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ J_Y^{-1} = \text{Id}) \\ &= y'((J_Y^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\ &= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X) \\ &= (T'y')(x) && \text{(apply def. of adjoint op.)} \end{aligned}$$

□

**Example 2.** We have already seen an adjoint operator. Using the last theorem, we can find an example for an operator  $S \in \mathcal{L}(Y', X')$  that is not an adjoint operator (i.e. with  $\nexists T \in \mathcal{L}(X, Y) : T' = S$ ).

TODO.

## 2.3 The basic properties

The proofs will use the Hahn-Banach corollaries a couple of times. So first, we need to recall Hahn-Banach related theorems:

**Revision 3** (Hahn-Banach). Let  $(X, \|\cdot\|)$  be a normed space and  $0 \neq x \in X$ . Then we have  $\exists f \in X' : \|f\| = 1 \wedge |f(x)| = 1$ .

*Proof.* Please refer to the functional analysis lecture notes from the summer of 2025.

□

**Revision 4.** Let  $(X, \|\cdot\|)$  be a normed vector space and  $x \in X$ .

Then we have  $\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$ .

*Proof.* Case 1  $x = 0$ : Since  $X'$  contains linear operators,  $\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$ .

Case 2  $x \neq 0$ : We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

Using Hahn-Banach from revision 3 we get  $\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|_X$ .

So we get  $\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$ .

□

**Theorem 3.** We can collect some basic properties of the adjoint operator:

i):  $T' \in \mathcal{L}(Y', X')$ , so  $T'$  is linear and bounded.  
This implies  $\forall y' \in \mathcal{L}(Y', X') : T'y' \in X'$ .

ii):  $T \mapsto T'$  is linear and isometric.

iii):  $T \mapsto T'$  is not always surjective.

*This is not the case with  $\cdot^H$  e.g. between  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ .*

*Proof.* "i)": Let  $y' \in Y'$ . Plugging it into the adjoint operator, we get  $T'y' = y' \circ T$  with signature  $X \rightarrow Y \rightarrow \mathbb{F}$ . We can now see that  $\text{Im } T' \subset X'$ .

We then prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of dual operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + T'y'_2 && \text{(apply def. of dual operator)} \\ &\text{for all } y'_1, y'_2 \in Y' \text{ and } \alpha \in \mathbb{F} \end{aligned}$$

We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of dual operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X, Y)} && \text{(apply def. of op. norm)} \\ &:= C \|y'\|_{Y'} && \text{(def. the constant)} \\ &\text{for all } y' \in Y' \end{aligned}$$

"ii)": We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y') &= y'(\alpha T_1 + T_2) && \text{(apply def. of dual operator)} \\ &= \alpha y'(T_1) + y'(T_2) && \text{(y' is linear)} \\ &\text{for all } T_1, T_2 \in \mathcal{L}(X, Y) \text{ and } y' \in Y' \text{ and } \alpha \in \mathbb{F} \end{aligned}$$

We then prove isometry:

$$\begin{aligned} \|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\ &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && \text{(apply theorem of Hahn-Banach 3)} \\ &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && \text{(supremum order can be switched)} \\ &= \sup_{\|y'\| \leq 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\ &= \|T'\| && \text{(apply def. of op. norm)} \end{aligned}$$

"iii)": The operator and the adjoint operator have the following signatures:

- $T : X \rightarrow Y$
- $T' : Y' = \mathcal{L}(Y, \mathbb{F}) \rightarrow X' = \mathcal{L}(X, \mathbb{F})$

So the "not always" refers to a particular choice of  $X$  and  $Y$  we need to find. Fortunately in example 2 we already found a counterexample!

□

**Example 3.**

**Example 4.** Let  $p \in (1, \infty)$  with  $p \neq 2$ . This makes  $l_p$  a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of  $l_p$  is isometrically isomorph to  $l_p^*$  where  $p^*$  is the Hölder conjugate with  $\frac{1}{p} + \frac{1}{p^*} = 1$ . As a reminder, the general idea of the proof of  $l_p' \cong l_p^*$  goes as follows:

1. Define the isometric isomorphism as  $T : l_p^* \rightarrow l_p', s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$ .

2. Verify  $(Ts)x$  converges as  $\|(Ts)x\| \leq \|x\|_p \|s\|_{p^*} < \infty$  and absolute convergence implies convergence.
3. Verify  $T$  is injective using the linearity.
4. Verify  $T$  is surjective and isometric through todo (the annoying part).

Now, let's work through an example!

Consider the **left shift** operator  $T : l_p \rightarrow l_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$ .

It is well-defined since  $\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$ .

What should the adjoint operator  $T'$  intuitively be?

The adjoint  $T'$  must have the signature  $l'_p \cong l_p^* \rightarrow l'_p \cong l_p^*$ .

Let  $y' \in l'_p \cong l_p^*$ . Then we can write  $y' : l_p \rightarrow \mathbb{F}, x \mapsto \sum_{k \in \mathbb{N}} x_k s_k$  with  $s \in l_p^*$ .

Now for  $x \in l_p$  we have

$$\begin{aligned}
 (T'y')(x) &= (y' \circ T)(x) = y'(Tx) && \text{(apply def. of } T') \\
 &= y'((x_{k+1})_{k \in \mathbb{N}}) && \text{(apply def. of } T) \\
 &= \sum_{k \in \mathbb{N}} x_{k+1} s_k && \text{(apply def. of } y') \\
 &= \sum_{k \in \mathbb{N}} x_k s'_k && \text{(with } s'_1 = 0 \text{ and } s'_k = s_{k-1} \text{ for } k > 1)
 \end{aligned}$$

This tells us that the adjoint operator  $T'$  acts as a **right shift** (up to isomorphism):

$$T' : l_p^* \rightarrow l_p^*, (s_k)_{k \in \mathbb{N}} \mapsto (0, s_1, s_2, \dots)_{k \in \mathbb{N}}$$

**Revision 5.** Let  $X, Y, Z$  be normed metric spaces.

Then the adjoint operator reverses composition:

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'.$$

*Proof.* Let  $T \in \mathcal{L}(X, Y)$  and  $S \in \mathcal{L}(Y, Z)$ .

We know  $ST = S \circ T$  is still a linear, bounded operator from  $X$  to  $Z$ . So  $(ST)'$  is well-defined.

Let  $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$  and set  $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$ .

We can now evaluate the expression on  $z' \in Z'$ :

$$\begin{aligned}
 (ST)'(z') &= z' \circ (ST) && \text{(apply def. of dual operator)} \\
 &= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\
 &= y' \circ T && \text{(subst. } y') \\
 &= T'y' && \text{(apply def. of dual operator)} \\
 &= T'(z' \circ S) && \text{(subst. } y') \\
 &= T'S'z' && \text{(apply def. of dual operator)}
 \end{aligned}$$

So in total,  $(ST)' = T'S'$ . □

**Example 5.**

## 2.4 Compact (adjoint) operators

For starters, we need to recall and revise some concepts from the lecture.

**Definition 4.** The following definitions are revisions from the lecture:

Let  $X, Y$  be normed metric spaces.

- i):  $X$  is relatively compact, if  $\forall (x_n)_{n \in \mathbb{N}} \subset X$  bounded :  $(Tx_n)_{n \in \mathbb{N}}$  has a converging subsequence.
- ii):  $T \in \mathbb{N}$  is compact, if  $TB(0, \leq 1, X)$  is relatively compact.
- iii): The rank of  $T$  is  $\text{rk } T = \dim T(X)$ .

As a reminder, in metric spaces a subset is compact iff. all sequences contain a converging subsequence in that set. So relatively compact is a relaxed version of compactness: it disposes of the closed requirement. An important property of the closed unit ball in  $\mathbb{R}^n$  or  $\mathbb{C}$  is that it is compact. Therefore, a linear operator between  $\mathbb{F}^n \rightarrow \mathbb{F}^m$  always maps the closed unit ball to a compact set.

But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So  $(X, \|\cdot\|)$  and  $(X, \|\cdot\|_1)$  have the same open sets. In finite dimensions, linear algebra tells us that  $(X, \|\cdot\|_1)$  and  $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$  have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space  $\mathbb{R}^n$ . In  $\mathbb{R}^n$ , the theorem of Heine-Borel tells us that all bounded subsets are relatively-compact. Since relatively-compact is a topological property, the theorem transfers to  $(X, \|\cdot\|)$ . Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively-compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general Banach spaces.

**Definition 5.** The following definitions are critical for the theorem of [Arzelà-Ascoli](#)!

Let  $X, Y$  be metric spaces and  $M \subset \{f : X \rightarrow Y\}$ .

- i):  $M$  is uniformly equicontinuous, if  
 $\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$ .
- ii):  $M$  is pointwise bounded, if  $\forall x \in X : \{f(x) \mid f \in M\}$  is bounded.

The theorem of [Arzelà-Ascoli](#) from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions [evaluations](#) rather than the functions themselves. They can be easily verified to be true. Note that  $C(D) = \{f : D \rightarrow \mathbb{F} \text{ continuous}\}$ .

**Revision 6** (Arzelà-Ascoli). *Let  $D$  be a compact metric space and  $M \subset C(D)$  with the supremum norm. Then we have  $M$  is uniformly equicontinuous and pointwise bounded implies  $M$  is relatively compact.*

*Proof Idea.* 1.  $D$  is separable, i.e.  $\exists D_0 \subset D : \overline{D_0} = D$ . Simply set  $D_0 = \bigcup_{n \in \mathbb{N}} D_n$  where  $D_n$  is a finite  $\frac{1}{n}$ -covers of  $D$ .

2. For all  $x \in M$  we can use the pointwise boundedness to invoke Bolzano-Weierstraß to find converging subsequences  $(f_{n(x,k)}(x))_{k \in \mathbb{N}} \subset \mathbb{F}$ .
3. Specifically, we have  $\forall x \in D_0 \cap M =: M_0 : (f_{n(x,k)}(x))_{k \in \mathbb{N}}$  converges.
4. Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on  $x$ :  $\forall x \in M_0 : (f_{n(k)}(x))_{k \in \mathbb{N}}$  converges.
5. We already have  $f_{n(k)}|_{M_0} \rightarrow f|_{M_0}$ . And it seems sensible to assume that  $(f_{n(k)})_{k \in \mathbb{N}}$  is a convergent subsequence. But how can you extend this to the entire set  $M$ ?
6. For each  $x \in M$ , we can choose an arbitrarily close  $x_0 \in M_0$ . Using two triangle inequalities, uniform equicontinuity allows us to extend the result to  $M$ .  
 As  $C(D)$  is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

**Theorem 4** (Schauder). *Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  be a bounded, linear operator. Then we have  $T$  is compact iff.  $T'$  is compact.*

*Proof.* „ $\Rightarrow$ “: Let  $(y'_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$  be bounded.

Our goal is to show that there is a convergent subsequence in  $(T'y'_n)_{n \in \mathbb{N}}$  with respect to  $(X', \|\cdot\|)$  where  $\|\cdot\|$  is the operator norm.

Let  $K = B(0, \leq 1, X)$  and for all  $n \in \mathbb{N}$  set  $f_n = (y'_n \circ T)|_K \in \mathcal{L}(X, \mathbb{F})$ . Then we have

$$\begin{aligned} \|f - f_m\|_\infty &= \sup_{\|x\|_X=1} ((y' \circ T) - (y'_m \circ T))x && \text{(apply def. of } f_n \text{ and } f_m) \\ &= \|y' \circ T - y'_m \circ T\| && \text{(use supremum char. of operator norm)} \end{aligned}$$

This tells us that convergence in the operator norm only cares about the behavior on the closed unit ball! Let  $D = \overline{TB(0, \leq 1, X)}$ . We have

1.  $D$  is an (inherited) metric space.
2.  $D$  is closed by definition.
3.  $T(B(0, \leq 1, X))$  is a bounded set under a bounded operator, so  $D$  is bounded.

Since  $D$  is in a metric space, it must be compact.

We can now pack our sequence into  $M := \{f_n \mid n \in \mathbb{N}\}$  and examine it for

1. Pointwise boundedness: For  $x \in B(0, \leq 1, X)$  and  $n \in \mathbb{N}$

$$\begin{aligned} |f_n(x)| &= |y'_n(Tx)| := |y'_n(d)| \quad (\text{apply def. of } f_n \text{ and define } d) \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 \quad (\text{apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq C_2) \end{aligned}$$

2. Uniform equicontinuity: For  $n \in \mathbb{N}, \epsilon > 0, \delta = \frac{\epsilon}{C_1 C_2}, \forall x, y \in K, \|x - y\|_X < \delta$

$$\begin{aligned} \|f_n(x) - f_n(y)\|_Y &\leq \|y'_n \circ Tx - y'_n \circ Ty\|_Y && (\text{apply def. of } f_n) \\ &\leq \|y'_n\| \|T(x - y)\|_Y && (\text{factor out and apply op. norm ineq.}) \\ &\leq \|y'_n\| \|T\| \|x - y\|_X && (\text{apply op. norm ineq.}) \\ &\leq C_1 C_2 \|x - y\|_X < C_1 C_2 \delta = \epsilon && (\text{apply bounded and substitute in } \delta) \end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that  $M$  is relatively compact. So every sequence in  $M$  has a convergent subsequence. In particular for the convergent subsequence  $(f_{n(k)})_{k \in \mathbb{N}}$  we have

$$\begin{aligned} (f_{n(k)})_{k \in \mathbb{N}} &= (y'_n \circ T)_{k \in \mathbb{N}} && (\text{apply def. of } f_{n(k)}) \\ &= (T' y'_{n(k)})_{k \in \mathbb{N}} && (\text{apply def. of adjoint operator}) \end{aligned}$$

So finally,  $(T' y'_{n(k)})_{k \in \mathbb{N}}$  is a convergent subsequence of  $(T' y'_k)_{k \in \mathbb{N}}$ .

„ $\Leftarrow$ “: The other proof direction tells us that  $T$  compact  $\Rightarrow T'$  compact.

So in extension this also yields  $T'$  compact  $\Rightarrow T''$  compact.

Theorem 1 tells us that  $T = J_Y^{-1} \circ T'' \circ J_X$ .

And revision 1 tells us that  $T$  is also compact, as it is a composition of at least one compact operator- in this case  $T''$ . □

**Example 6.**

## 2.5 The rank-nullity theorem for operators

**Definition 6.** Let  $(X, \|\cdot\|)$  be a normed vector space,  $U \subset X$  and  $V \subset X'$ . Then define

- i):  $U^\perp = \{x' \in X' \mid \forall x \in U : x'(x) = 0\}$  as the annihilator of  $U$  in  $X'$ .
- ii):  $V_\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$  as the annihilator of  $V$  in  $X$ .

To prove properties about these sets, we need a corollary of the theorem of Hahn-Banach.

**Revision 7.** *TODO: Hahn-Banach corollary.*

*Proof Idea.* *TODO* □

**Theorem 5.** Let  $(X, \|\cdot\|)$  be a normed vector space,  $U \subset X$  and  $V \subset X'$ . Then we have

- i):  $U^\perp$  and  $V_\perp$  are not empty.
- ii):  $U^\perp \subset X'$  and  $V_\perp \subset X$  are both respectively closed linear subspaces.
- iii): Let  $U$  be closed. Then we have  $(X/U)' \cong U^\perp$  (they are isomorphic).
- iv): Let  $U$  be closed. Then we have  $U' \cong X'/U^\perp$  (they are isomorphic).

*Proof.* "i)":

"ii)": We will focus on  $U^\perp \subset X'$ . The other statement works analogously.  
 Let  $x'_1, x'_2 \in X'$  and  $\lambda \in \mathbb{F}$ . We first prove  $\emptyset \neq U^\perp \subset X'$  is a linear subspace:

$$\begin{aligned} (x'_1 + \lambda x'_2)(x) &= x'_1(x) + \lambda x'_2(x) && \text{(substitute } x \text{ in)} \\ &= 0 + \lambda 0 = 0 && \text{(use } x'_1, x'_2 \in X') \end{aligned}$$

Let  $(x'_k)_{k \in \mathbb{N}} \subset U^\perp$  be a converging sequence.  
 We now prove that  $x'$  converges in  $U^\perp$ :

*todo*

"iii)":

"iv)":

□

In summary, TODO.

**Theorem 6.** *Let  $T \in \mathcal{L}(X, Y)$  be a bounded, linear operator. Then we have  $\overline{\text{Im } T} = (\text{Ker } T')^\perp$ .*

*Proof.* „ $\subset$ “: Let  $Tx \in \text{Im } T$  with  $x \in X$  and  $y' \in \text{Ker } T'$ .

We first prove  $Tx \in (\text{Ker } T')^\perp$ :

$$\begin{aligned} y'Tx &= (y' \circ T)(x) && \text{(use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply dual op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of  $Tx$  we get  $\text{Im } T \subset (\text{Ker } T')^\perp$ .

Since  $(\text{Ker } T')^\perp$  is closed, we also get  $\overline{\text{Im } T} \subset (\text{Ker } T')^\perp$ .

„ $\supset$ “: TODO

□

**Theorem 7.** *TODO: The stuff with the  $\text{Id} - T$  operator.*

*Proof.* TODO

□

**Corollary 1.** *Let  $T \in \mathcal{L}(X, Y)$  be a linear, continuous operator with  $\text{Im } T$  closed. Then we have  $Tx = y$  has a solution iff.  $T'y' = 0 \Rightarrow y'(y) = 0$ .*

*Proof.*

□

**Example 7.**

## References

- [Noa] *Dual operators between Hilbert spaces: with or without Riesz representation.* MathOverflow. URL: <https://mathoverflow.net/questions/71040/dual-operators-between-hilbert-spaces-with-or-without-riesz-representation> (visited on 01/20/2026).