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Chair of Functional Analysis

# Dual Operators

Seminar Functional Analysis

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**Abstract:** This proseminar paper provides a foundational overview of dual-/adjoint operators in a general Banach space setting.

# 1 Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as  $A^T$  over  $\mathbb{R}$  or  $A^H$  over  $\mathbb{C}$ ) to operators on normed  $\mathbb{R}$  or  $\mathbb{C}$  vector spaces:

We will cover the following topics:

- The operator  $\cdot^H$  is linear and isometric wrt. the spectral norm.
- The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem:  $(\text{Im } A)^\perp = \text{Ker } A^H$ .
- TODO Add more

## 2 The adjoint operator

### 2.1 The basic definitions and conventions

The terms **adjoint** and **dual** are often used interchangeably. We will standardize to **adjoint**, to avoid unnecessary confusion. Following [Wer18], we write  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  when the field is unspecified.

**Definition 1.** We remind ourselves of the following concepts from the lecture [KP]: Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  and  $(Z, \|\cdot\|_Z)$  be normed  $\mathbb{F}$ -vector spaces

- i) Let

$$T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call  $T$  a **linear operator**. We call  $T$  **bounded**, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation  $T : X \rightarrow Y$  for linear operators rather than  $T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$ .

- ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

- iii) The closed unit ball in  $(X, \|\cdot\|_X)$  is abbreviated with  $\overline{B}_X$ .

**Revision 1.** *The following statements are foundational for this topic:*

*Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$ -vector spaces.*

- The set of continuous, linear operators  $\mathcal{L}(X, Y)$  is a Banach space if and only if  $Y$  is a Banach space. In particular, the topological dual space  $\mathcal{L}(X, \mathbb{F})$  is a Banach space.*
- Let  $T : X \rightarrow Y$  be a linear operator. Then  $T$  is bounded if and only if  $T$  is continuous.*
- Let  $(Z, \|\cdot\|_Z)$  be a normed  $\mathbb{F}$ -vector space, let  $T : X \rightarrow Y$  be a linear, bounded operator and let  $S : Y \rightarrow Z$  be a linear, bounded operator. Then  $S \circ T$  is a linear, bounded operator.*
- Let  $(Z, \|\cdot\|_Z)$  be a normed  $\mathbb{F}$ -vector space, let  $T : X \rightarrow Y$  be a linear operator and let  $S : Y \rightarrow Z$  be a linear operator. If  $T$  or  $S$  is compact,  $S \circ T$  is compact.*

*Proof.* Please refer to the functional analysis lecture notes [KP] from SS/2025. □

**Definition 2.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$ -vector spaces and  $T \in \mathcal{L}(X, Y)$ .

Then  $T' : Y' \rightarrow X'$ ,  $y' \mapsto y' \circ T$  is called the adjoint operator. From now on, we will implicitly refer to the normed  $\mathbb{F}$  vector spaces  $X, Y$  and the topological dual spaces  $X', Y'$  when talking about the dual operator  $T'$ .

**Remark 1.** In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from  $\mathbb{F}$ . For  $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$ , the adjoint operator evaluates to  $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$ .

**Example 1.** Let  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} (\mathbb{R}^n)' &= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear, continuous}\} && \text{(apply def. of dual space)} \\ &= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear}\} && \text{(result from linear algebra)} \\ &\cong \mathbb{R}^n && \text{(result from linear algebra see [FS25a])} \end{aligned}$$

Let  $m \in \mathbb{N}$  and let  $A \in \mathbb{R}^{m \times n}$ .

Consider the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m, x \mapsto Ax$ . The signature of the adjoint is  $f' : (\mathbb{R}^m)' \rightarrow (\mathbb{R}^n)'$ . With  $e_i$  we denote the standard basis vectors and with  $e'_i$  we denote the dual standard basis vectors. Let  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . We have

$$\begin{aligned} (f'e'_i)(e_j) &= (e'_i \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\ &= f(e_j)_i = a_{ij} && \text{(apply def. of } e'_i \text{ and } f) \end{aligned}$$

So if we set  $i = 1$  for instance, we get

$$(f'e'_1)(e_j) = a_{1j}$$

The first "column" of  $f'$  (up to isomorphism) must be  $(a_{1j})_{j=1, \dots, n}$ . In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \rightarrow \mathbb{R}^n, x \mapsto A^T x$$

## 2.2 The basic properties

The proofs will use the Hahn-Banach corollaries a couple of times. So first, we need to recall Hahn-Banach related theorems:

**Revision 2** (Hahn-Banach). *Let  $(X, \|\cdot\|)$  be a normed space and  $0 \neq x \in X$ . Then we have*

$$\exists f \in X' : \|f\| = 1 \wedge f(x) = \|x\|$$

*Proof.* Please refer to the functional analysis lecture notes [KP] from SS/2025.  $\square$

**Revision 3.** *The dual unit ball retains information about the norm of primal vectors: Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space and  $x \in X$ .*

*Then we have*

$$\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$$

*Proof.* We will distinguish two cases:

- Case 1 ( $x = 0$ ): Since  $X'$  contains linear operators,

$$\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$$

- Case 2  $x \neq 0$ : We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

Using Hahn-Banach from revision 2 we get

$$\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|_X$$

So we get

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$$

$\square$

**Theorem 1.** *The adjoint operator has the following properties:*

- i)  $T' \in \mathcal{L}(Y', X')$ , so  $T'$  is linear and bounded.  
▷ This implies  $\forall y' \in Y' : T'y' \in X'$ .
- ii)  $T \mapsto T'$  is linear and isometric.

*Proof.* "i)": Let  $y' \in Y'$ . Plugging it into the adjoint operator, we get  $T'y' = y' \circ T$  with signature  $X \rightarrow Y \rightarrow \mathbb{F}$ . We can now see that  $\text{Im } T' \subset X'$ .

Let  $y'_1, y'_2 \in Y'$ ,  $\alpha \in \mathbb{F}$ . We then prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of adjoint operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + T'y'_2 && \text{(apply def. of adjoint operator)} \end{aligned}$$

Let  $y' \in Y'$ . We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X, Y)} && \text{(apply def. of op. norm)} \\ &:= C \|y'\|_{Y'} && \text{(def. the constant)} \end{aligned}$$

"ii)": Let  $T_1, T_2 \in \mathcal{L}(X, Y)$ ,  $y' \in Y'$ ,  $x \in X$ ,  $\alpha \in \mathbb{F}$ . We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\ &= y'(\alpha T_1 x + T_2 x) && \text{(pull } x \text{ into the eq.)} \\ &= \alpha y'(T_1 x) + y'(T_2 x) && \text{(y' is linear)} \\ &= (\alpha T'_1 y' + T'_2 y')(x) \end{aligned}$$

We then prove isometry:

$$\begin{aligned} \|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\ &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && \text{(apply theorem of Hahn-Banach 2)} \\ &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && \text{(supremum order can be switched)} \\ &= \sup_{\|y'\| \leq 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\ &= \|T'\| && \text{(apply def. of op. norm)} \end{aligned}$$

□

**Example 2.** Let  $p \in (1, \infty)$  with  $p \neq 2$ . This makes  $l_p$  a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of  $l_p$  is isometrically isomorphic to  $l_p^*$  where  $p^*$  is the Hölder conjugate with  $1/p + 1/p^* = 1$ . As a reminder, the general idea of the proof of  $l_p' \cong l_p^*$  goes as follows:

- Define the isometric isomorphism as

$$T : l_p^* \rightarrow l_p', s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$$

- Verify  $(Ts)x$  converges as

$$|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$$

and absolute convergence implies convergence.

- Verify  $T$  is injective using the linearity.

- Verify  $T$  is surjective and isometric (the long part). See [Wer18] for a full proof.

To illustrate the adjoint operator, we now work through an example. Consider the left shift operator

$$T : l_p \rightarrow l_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$$

It is well-defined since

$$\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$$

We can now compute the adjoint operator  $T'$ :

- The adjoint  $T'$  must have the signature  $l'_p \cong l_p^* \rightarrow l'_p \cong l_p^*$ .
- Let  $y' \in l'_p \cong l_p^*$ . Then we can write  $y' : l_p \rightarrow \mathbb{F}, x \mapsto \sum_{k \in \mathbb{N}} x_k s_k$  with  $s \in l_p^*$ . Now for  $x \in l_p$  we have

$$\begin{aligned}
(T'y')(x) &= (y' \circ T)(x) = y'(Tx) && (\text{apply def. of } T') \\
&= y'((x_{k+1})_{k \in \mathbb{N}}) && (\text{apply def. of } T) \\
&= \sum_{k \in \mathbb{N}} x_{k+1} s_k && (\text{apply def. of } y') \\
&= \sum_{k \in \mathbb{N}} x_k s'_k && (\text{with } s'_1 = 0 \text{ and } s'_k = s_{k-1} \text{ for } k > 1)
\end{aligned}$$

This tells us that the adjoint operator  $T'$  acts as a **right shift** (up to isomorphism):

$$T' : l_p^* \rightarrow l_p^*, (s_k)_{k \in \mathbb{N}} \mapsto (0, s_1, s_2, \dots)$$

**Theorem 2.** *Let  $X, Y, Z$  be normed  $\mathbb{F}$  vector spaces.  
Then the adjoint operator reverses composition:*

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

*Proof.* Let  $T \in \mathcal{L}(X, Y)$  and  $S \in \mathcal{L}(Y, Z)$ .

We know  $ST = S \circ T$  is still a linear, bounded operator from  $X$  to  $Z$ . So  $(ST)'$  is well-defined. Let  $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$  and set  $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$ . We have

$$\begin{aligned}
(ST)'(z') &= z' \circ (ST) && (\text{apply def. of adjoint operator}) \\
&= (z' \circ S) \circ T && (\text{write out chain explicitly}) \\
&= y' \circ T && (\text{subst. } y') \\
&= T'y' && (\text{apply def. of adjoint operator}) \\
&= T'(z' \circ S) && (\text{subst. } y') \\
&= T'S'z' && (\text{apply def. of adjoint operator})
\end{aligned}$$

So in total  $(ST)' = T'S'$ . □

**Example 3.** Referring back to the example with  $A \in \mathbb{R}^{m \times n}$ , the adjoint operator reverses composition rule is compatible with the way matrix transposition on  $\mathbb{R}$  or the matrix adjoint on  $\mathbb{C}$  behave.

## 2.3 The dual space of the dual space

For starters, we need to recall some concepts from the lecture.

**Definition 3.** Let  $X$  be a normed  $\mathbb{F}$  vector space.

- $X''$  is called the bidual space.
- Let  $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$ .  $J_X$  is called the **canonical embedding** from  $X$  into  $X''$ .

So  $J_X$  returns a function that evaluates dual space elements at  $p \in X$ .

**Theorem 3.** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$  vector spaces. Let  $T \in \mathcal{L}(X, Y)$  be a bounded linear operator. Then we have:  $J_Y \circ T = T'' \circ J_X$ .*

*This concept is illustrated in figure 1.*

*Proof.* Before the proof, we can avoid confusion by typing out what  $T'$  and  $T''$  evaluate to:

- $T' : Y' \rightarrow X', y' \mapsto y' \circ T$  is the adjoint operator.
- $T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$  is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

- $J_Y : Y \rightarrow Y''$  and  $T : X \rightarrow Y$  means  $(J_Y \circ T) : X \rightarrow Y''$ .

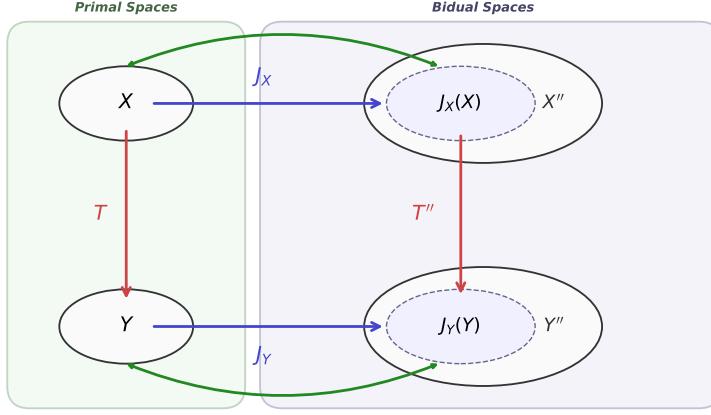


Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

- $T'' : X'' \rightarrow Y''$  and  $J_X : X \rightarrow X''$  means  $(T'' \circ J_X) : X \rightarrow Y''$ .

Finally, for  $p \in X$  and  $y' \in Y'$  we have

$$\begin{aligned}
 ((J_Y \circ T)(p))(y') &= J_Y(Tp)(y') = y'(Tp) && \text{(subst. } p \text{ in and apply def. of } J_Y) \\
 &= (y'T)(p) && \text{(use associativity)} \\
 &= (T'y')(p) && \text{(apply def. of } T') \\
 &= (x' \mapsto x'(p))(T'y') && \text{(pull out subst. function)} \\
 &= J_X(p)(T'y') && \text{(recognize this is just } J_X) \\
 &= T''(J_X(p))(y') && \text{(apply def. use } T'') \\
 &= ((T'' \circ J_X)(p))(y') && \text{(use } \circ \text{ notation)}
 \end{aligned}$$

□

We will now characterize when a continuous operator between  $Y'$  and  $X'$  is an adjoint operator. We need to revise an important corollary from the lecture first.

**Revision 4.** Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space. Then we have

- i) The canonical embedding  $J_X$  is an isometric injective function.
- ii)  $J : X \rightarrow J_X(X), x \mapsto J_X(x)$  is an isometric isomorphism.
- iii)  $J_X$  is a bounded, linear operator.
- iv)  $(J_X|_{X \rightarrow J_X(X)})^{-1}$  is a bounded, linear operator.

This concept is illustrated in figure 1.

*Proof Idea.*

"i)": Please refer to the functional analysis lecture notes [KP] from SS/2025.

"ii)": Follows by definition of  $J$  and from "i)".

"iii)": Since  $J : X \rightarrow J_X(X), x \mapsto J_X(x)$  is an isometric isomorphism, we have

- $J_X$  is linear since the  $x' \in X'$  are linear.
- $\|J_X\| = 1$ . This means  $J_X$  is bounded.

"iv)": Since  $J : X \rightarrow J_X(X), x \mapsto J_X(x)$  is an isometric isomorphism, we have

- $(J_X|_{X \rightarrow J_X(X)})^{-1}$  inherits linearity from  $J$ .
- $\|J_X\| = 1$  and therefore  $\|(J_X|_{X \rightarrow J_X(X)})^{-1}\| = 1$ . This means  $(J_X|_{X \rightarrow J_X(X)})^{-1}$  is bounded.

□

**Corollary 1.** We can strengthen theorem 3 using the isomorphism from revision 4:  
Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$  vector spaces. Let  $T \in \mathcal{L}(X, Y)$  be a bounded linear operator. Then we have

- i)  $J_Y \circ T = T'' \circ J_X$ .
- ii)  $T''|_{X \rightarrow J_X(X)} = J_Y \circ T \circ (J_X|_{X \rightarrow J_X(X)})^{-1}$
- iii)  $T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$ .

*Proof.* "i)": This is the statement of theorem 3.

"ii)": The isomorphism from revision 4 gives us the result.

"iii)": The isomorphism from revision 4 gives us the result. □

**Theorem 4.** Let  $S \in \mathcal{L}(Y', X')$  be a continuous, linear operator. Then we have

$$\exists T \in \mathcal{L}(X, Y) : T' = S \text{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

*Proof.* " $\Rightarrow$ ": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T') \\ &= J_Y(T(X)) && \text{(use theorem 3)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y) \end{aligned}$$

" $\LeftarrowS'(J_X(X)) \subset J_Y(Y)$  and revision 4 gives us  $J_Y(Y) \cong Y$ .

So for  $x \in X$  and  $y''_x = S'(J_X(x))$  there is a (unique)  $y_x \in Y$  with  $y''_x = J_Y(y_x)$ .

Choose

$$T : X \rightarrow Y, x \mapsto y_x$$

We know  $T$  exists (and is unique) due to the previous argument.

We know  $T$  is linear and continuous, as

$$T = y. = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X$$

and all elements in the chain are bounded, linear operators.

Let  $y' \in Y'$  and  $x \in X$ . Lastly, we need to prove that  $S = T'$ :

$$\begin{aligned} (Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X) \\ &= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of adjoint op.)} \\ &= J_Y(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} = \text{Id}) \\ &= y'(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\ &= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X) \\ &= (T'y')(x) && \text{(apply def. of adjoint op.)} \end{aligned}$$

□

**Theorem 5.** Lastly, we get one more property of the adjoint:  $T \mapsto T'$  is not always surjective.

▷ This is not the case with the transpose between  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ .

*Proof.* The operator and the adjoint operator have the following signatures:

- $T : X \rightarrow Y$
- $T' : Y' = \mathcal{L}(Y, \mathbb{F}) \rightarrow X' = \mathcal{L}(X, \mathbb{F})$

So the "not always" refers to a particular choice of  $X$  and  $Y$  we need to find.

Using the last theorem, we can find an example for an operator  $S \in \mathcal{L}(Y', X')$  that is not an adjoint operator  
(i.e. with  $\nexists T \in \mathcal{L}(X, Y) : T' = S$ ). □

## 2.4 Compact (adjoint) operators

For starters, we need to recall and review some concepts from the lecture.

**Definition 4.** The following definitions are revisions from the lecture:  
Let  $X, Y$  be normed  $\mathbb{F}$  vector spaces.

- i)  $M \subset X$  is relatively compact, if

$$\forall (x_n)_{n \in \mathbb{N}} \subset M : (x_n)_{n \in \mathbb{N}} \text{ has a converging subsequence in } X$$

- ii)  $T \in \mathcal{L}(X, Y)$  is compact, if  $T(\overline{B}_X)$  is relatively compact.
- iii) The rank of  $T$  is  $\text{rk } T = \dim T(X)$ .

As a reminder, in metric spaces a subset is compact if and only if all sequences contain a converging subsequence in that set. So relatively compact relaxes the requirement that the set must be closed. An important property of the closed unit ball in  $\mathbb{R}^n$  or  $\mathbb{C}$  is that it is compact. Therefore, a linear operator between  $\mathbb{F}^n \rightarrow \mathbb{F}^m$  always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So  $(X, \|\cdot\|)$  and  $(X, \|\cdot\|_1)$  have the same open sets. In finite dimensions, linear algebra tells us that  $(X, \|\cdot\|_1)$  and  $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$  have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space  $\mathbb{R}^n$ . In  $\mathbb{R}^n$ , the theorem of Heine-Borel tells us that all bounded subsets are relatively compact.

Since relatively compact is a topological property, the theorem transfers to  $(X, \|\cdot\|)$ . Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general Banach spaces.

**Definition 5.** The following definitions are critical for the theorem of [Arzelà-Ascoli](#):  
Let  $X, Y$  be metric spaces and  $M \subset \{f : X \rightarrow Y\}$ .

- i)  $M$  is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

- ii)  $M$  is pointwise bounded, if

$$\forall x \in X : \{f(x) \mid f \in M\} \text{ is bounded}$$

The theorem of [Arzelà-Ascoli](#) from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions [evaluations](#) rather than the functions themselves. They can be easily verified to be true. Note that  $C(D) = \{f : D \rightarrow \mathbb{F} \text{ continuous}\}$ .

**Revision 5 (Arzelà-Ascoli).** Let  $D$  be a compact metric space and  $M \subset C(D)$  with the supremum norm. Then we have  $M$  is uniformly equicontinuous and pointwise bounded implies  $M$  is relatively compact.

*Proof Idea.*

- $D$  is separable, i.e.  $\exists D_0 \subset D : \overline{D_0} = D$ . Simply set  $D_0 = \bigcup_{n \in \mathbb{N}} D_n$  where  $D_n$  is a finite  $\frac{1}{n}$ -covers of  $D$ .
- For all  $x \in D$  we can use the pointwise boundedness to invoke Bolzano-Weierstrass to find converging subsequences  $(f_{n(x,k)}(x))_{k \in \mathbb{N}} \subset \mathbb{F}$ .
- Specifically, we have  $\forall x \in D_0 : (f_{n(x,k)}(x))_{k \in \mathbb{N}}$  converges.
- Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on  $x$ :  $\forall x \in D_0 : (f_{n(k)}(x))_{k \in \mathbb{N}}$  converges.

- We already have  $f_{n(k)}|_{D_0} \rightarrow f|_{D_0}$ . And it seems sensible to assume that  $(f_{n(k)})_{k \in \mathbb{N}}$  is a convergent subsequence. But how can you extend this to the entire set  $D$ ?
- For each  $x \in D$ , we can choose an arbitrarily close  $x_0 \in D_0$ . Using two triangle inequalities, uniform equicontinuity allows us to extend the result to  $D$ . As  $C(D)$  is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

**Theorem 6** (Schauder). *Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  be a bounded, linear operator. Then we have  $T$  is compact if and only if  $T'$  is compact.*

*Proof.* " $\Rightarrow$ ": Let  $(y'_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$  be bounded.

Our goal is to show that there is a convergent subsequence in  $(T'y'_n)_{n \in \mathbb{N}}$  with respect to  $(X', \|\cdot\|)$  where  $\|\cdot\|$  is the operator norm. For all  $n \in \mathbb{N}$  set  $f_n = y'_n|_{T(\overline{B}_X)}$ .

We have

$$\begin{aligned} \|T'y'_n - T'y'_m\| &= \|y'_n \circ T - y'_m \circ T\| && \text{(apply def. of adjoint op.)} \\ &= \sup_{\|x\|_X \leq 1} |((y'_n \circ T) - (y'_m \circ T))(x)| && \text{(use supremum char. of the op. norm)} \\ &= \sup_{\|x\|_X \leq 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m) \\ &= \sup_{d \in T(\overline{B}_X)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m) \end{aligned}$$

We know  $(T'y'_n)_{n \in \mathbb{N}}$  converges if and only if it is a Cauchy sequence  $(\mathcal{L}(X, \mathbb{F}), \|\cdot\|)$ . Therefore the convergence of  $(T'y'_n)_{n \in \mathbb{N}}$  in the operator norm is only dependent on the behaviour of  $(f_n)_{n \in \mathbb{N}}$  on  $T(\overline{B}_X)$ .

We now set

$$D = \overline{T(\overline{B}_X)}$$

and pack the sequence into

$$M := \{f_n \mid n \in \mathbb{N}\}$$

and examine them for

- $D$  is compact: We have

- $\overline{B}_X$  is bounded.
- $T$  is a compact operator.

So  $T(\overline{B}_X)$  is relatively compact and  $D$  is compact.

- $M$  is pointwise bounded: For  $x \in \overline{B}_X$  and  $n \in \mathbb{N}$  we have

$$\begin{aligned} |f_n(Tx)| &= |y'_n(Tx)| := |y'_n(d)| && \text{(apply def. of } f_n \text{ and define } d) \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 && \text{(apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq C_2) \end{aligned}$$

For  $d \in D$  we can choose  $(x_k)_{k \in \mathbb{N}} \subset \overline{B}_X$  with  $Tx_k \rightarrow d$  and  $f_n(Tx_k) \leq C_1 C_2$ . Using continuity we get  $|f_n(d)| \leq C_1 C_2$ .

- $M$  is uniformly equicontinuous: For  $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon/C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$  we have

$$\begin{aligned} |f_n(d_1) - f_n(d_2)| &\leq \|y'_n\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)} \\ &\leq C_1 \|d_1 - d_2\|_Y && \text{(use the upper bound } \|y'_n\| \leq C_1) \\ &< C_1 \delta = \epsilon && \text{(substitute in } \delta) \end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that  $M$  is relatively compact. So every sequence in  $M$  has a convergent subsequence. In particular for the convergent subsequence  $(f_{n(k)})_{k \in \mathbb{N}}$  we have

$$\begin{aligned} (f_{n(k)})_{k \in \mathbb{N}} &= (y'_n \circ T)_{k \in \mathbb{N}} && (\text{apply def. of } f_{n(k)}) \\ &= (T'y'_{n(k)})_{k \in \mathbb{N}} && (\text{apply def. of adjoint operator}) \end{aligned}$$

So finally,  $(T'y'_{n(k)})_{k \in \mathbb{N}}$  is a convergent subsequence of  $(T'y'_k)_{k \in \mathbb{N}}$ .

" $\Leftarrow$ ": The other proof direction tells us that

$$T \text{ compact} \Rightarrow T' \text{ compact}$$

So in extension this also yields

$$T' \text{ compact} \Rightarrow T'' \text{ compact}$$

Corollary 1 tells us that

$$T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$$

Lastly, the operator  $T''$  is compact, theorem 4 tells us  $J_X$  and  $(J_Y|_{Y \rightarrow J_Y(Y)})^{-1}$  are bounded, linear operators and therefore  $T$  is a composition of bounded, linear operators.

Using the revision theorem 1 we can conclude that  $T$  is compact.

□

**Example 4.** The lecture [KP] states that the following integral operator is compact:

$$T : C[0, 1] \rightarrow C[0, 1], Tf(x) = \int_0^1 f(t) dx$$

So  $T' : C[0, 1]' \rightarrow C[0, 1]'$  is compact too. Evaluating  $C[0, 1]'$  is beyond the scope of this paper.

## 2.5 The rank-nullity theorem for operators

**Definition 6.** Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space,  $U \subset X$  and  $V \subset X'$ . Then we define the annihilator of  $V$  in  $X$  as

$$V_\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$$

▷ In linear algebra, the annihilator is isomorphic to the orthogonal complement of a set. Similarly we generalize the idea of a dual structure isomorphic to the image or kernel to idk... todo

To prove properties about this set, we need another corollary of the theorem of Hahn-Banach:

**Revision 6.** Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space,  $U \subset X$  a closed subspace and  $x \in X \setminus U$ . Then we have

$$\exists x' \in X' : x'|_U = 0 \wedge x'(x) \neq 0$$

*Proof.* Let  $Y = X/U$  be the (canonical) quotient space. Then  $Y$  is a normed  $\mathbb{F}$  vector space. (todo: why?) Set  $y = x \in Y$ . We can apply the theorem of Hahn-Banach 2 to obtain  $y' \in Y'$  with  $y'(y) \neq 0$  and  $y'|_U = 0$ . □

**Theorem 7.** Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space,  $U \subset X$  and  $V \subset X'$ . Then we have

$$V_\perp \subset X \text{ is a closed linear subspace}$$

*Proof.* We will focus on  $V_\perp \subset X'$ . The other statement works analogously.

We first prove  $V_\perp$  is not empty: TODO.

Let  $x'_1, x'_2 \in X'$  and  $\lambda \in \mathbb{F}$ . We first prove  $V_\perp \subset X'$  is a linear subspace:

$$\begin{aligned} (x'_1 + \lambda x'_2)(x) &= x'_1(x) + \lambda x'_2(x) && (\text{substitute } x \text{ in}) \\ &= 0 + \lambda 0 = 0 && (\text{use } x'_1, x'_2 \in X') \end{aligned}$$

Let  $(x'_k)_{k \in \mathbb{N}} \subset V_\perp$  be a converging sequence.

We now prove that  $x'$  converges in  $V_\perp$ :

todo

□

In summary, TODO.

**Theorem 8.** Let  $T \in \mathcal{L}(X, Y)$  be a bounded, linear operator. Then we have

$$\overline{\text{Im } T} = (\text{Ker } T')_{\perp}$$

▷ In linear algebra this is usually presented for finite-dimensional vector spaces (see Satz 6.1.5 [Wer18])

*Proof.* " $\subset$ ": Let  $Tx \in \text{Im } T$  with  $x \in X$  and  $y' \in \text{Ker } T'$ .

We first prove  $Tx \in (\text{Ker } T')_{\perp}$ :

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of  $Tx$  we get

$$\text{Im } T \subset (\text{Ker } T')_{\perp}$$

Since  $(\text{Ker } T')_{\perp}$  is closed, we also get

$$\overline{\text{Im } T} \subset (\text{Ker } T')_{\perp}$$

" $\supset$ ": We can prove the contraposition

$$(Y \setminus \overline{\text{Im } T}) \subset (Y \setminus (\text{Ker } T')_{\perp})$$

Set  $U = \overline{\text{Im } T}$  and let  $y \in Y \setminus U$ . We know  $U$  that a closed linear subspace. The corollary of the theorem of Hahn-Banach 6 tells us that

$$\exists y' \in Y' : y'|_U = 0 \wedge y'(y) \neq 0$$

Since  $\text{Ker } T' \subset Y'$  and

$$\forall y' \in \text{Ker } T' : y'(y) = 0$$

we get

$$y \in Y \setminus (\text{Ker } T')_{\perp}$$

□

**Corollary 2.** Let  $T \in \mathcal{L}(X, Y)$  be a linear, continuous operator with  $\text{Im } T$  closed. Then we have

$$y \in \text{Im } T \text{ if and only if } T'y' = 0 \Rightarrow y'(y) = 0$$

*Proof.* We have

$$\begin{aligned} y \in \text{Im } T &= \overline{\text{Im } T} && \text{(Im } T \text{ is closed)} \\ &= (\text{Ker } T')_{\perp} && \text{(apply theorem 8)} \\ \iff \forall y' \in \text{Ker } T' &: y'(y) = 0 && \text{(apply def. of annihilator)} \\ \iff \forall y' \in Y' &: T'y' = 0 \Rightarrow y'(y) = 0 && \text{(write in equivalent way)} \end{aligned}$$

□

**Example 5.** In nonlinear optimisation theory, the lemma of Farkas states [harks lecture 2026]: Let  $A \in \mathbb{R}^{k \times n}$  and  $d \in \mathbb{R}^k$ . Then exactly one of the following statements is true:

- $\exists x \geq 0 : Ax = d$
- $\exists \lambda \in \mathbb{R}^k : \lambda A \geq 0 \wedge \lambda d < 0$

The lemma is equivalent to the ability to separate two convex sets using a hyperplane. It is used to prove the KKT-theorem which allows for nonlinear optimisation with constraints.

We can write down an equivalent version of Farkas lemma:

Let  $A \in \mathbb{R}^{k \times n}$  and  $d \in \mathbb{R}^k$ . Then the following statements are equivalent:

- $\exists x \geq 0 : Ax = d$
- $\forall \lambda \in \mathbb{R}^k : A^T \lambda^T < 0 \vee \lambda d \geq 0$

Let  $T = A$  and  $T' = A^T$ . This makes the lemma match the statement of corollary 2.

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