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FACULTY OF COMPUTER SCIENCE AND MATHEMATICS

Chair of Functional Analysis

Dual Operators

Seminar Functional Analysis

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Abstract: This proseminar paper provides a foundational overview of dual-/adjoint operators in a general Banach space setting.

1 Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as A^T over \mathbb{R} or A^H over \mathbb{C}) to operators on normed \mathbb{R} or \mathbb{C} vector spaces:

We will cover the following topics:

- The operator \cdot^H is linear and isometric wrt. the spectral norm.
- The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem:
 $(\text{Im } A)^\perp = \text{Ker } A^H$.
- TODO Add more

2 The adjoint operator

2.1 The basic definitions and conventions

The terms [adjoint](#) and [dual](#) are often used interchangeably. We will standardize to [adjoint](#), to avoid unnecessary confusion. Following [Wer18], we write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ when the field is unspecified.

Definition 1. We remind ourselves of the following concepts from the lecture [KP]: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed \mathbb{F} -vector spaces

i) Let

$$T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call T a [linear operator](#). We call T [bounded](#), if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation $T : X \rightarrow Y$ for linear operators rather than $T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$.

ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

iii) The closed unit ball in $(X, \|\cdot\|_X)$ is abbreviated with \overline{B}_X .

Revision 1. The following statements are foundational for this topic:

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces.

- i) The set of continuous, linear operators $\mathcal{L}(X, Y)$ is a Banach space if and only if Y is a Banach space. In particular, the topological dual space $\mathcal{L}(X, \mathbb{F})$ is a Banach space.
- ii) Let $T : X \rightarrow Y$ be a linear operator. Then T is bounded if and only if T is continuous.
- iii) Let $(Z, \|\cdot\|_Z)$ be a normed \mathbb{F} -vector space, let $T : X \rightarrow Y$ be a linear, bounded operator and let $S : Y \rightarrow Z$ be a linear, bounded operator. Then $S \circ T$ is a linear, bounded operator.
- iv) Let $(Z, \|\cdot\|_Z)$ be a normed \mathbb{F} -vector space, let $T : X \rightarrow Y$ be a linear operator and let $S : Y \rightarrow Z$ be a linear operator. If T or S is compact, $S \circ T$ is compact.

Proof. Please refer to the functional analysis lecture notes [KP] from SS/2025. □

Definition 2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces and $T \in \mathcal{L}(X, Y)$. Then $T' : Y' \rightarrow X', y' \mapsto y' \circ T$ is called the adjoint operator. From now on, we will implicitly refer to the normed \mathbb{F} vector spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T' .

Remark 1. In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from \mathbb{F} . For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$.

Example 1. Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned} (\mathbb{R}^n)' &= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear, continuous}\} && \text{(apply def. of dual space)} \\ &= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear}\} && \text{(result from linear algebra)} \\ &\cong \mathbb{R}^n && \text{(result from linear algebra see [FS25a])} \end{aligned}$$

Let $m \in \mathbb{N}$ and let $A \in \mathbb{R}^{m \times n}$.

Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m, x \mapsto Ax$. The signature of the adjoint is $f' : (\mathbb{R}^m)' \rightarrow (\mathbb{R}^n)'$. With e_i we denote the standard basis vectors and with e'_i we denote the dual standard basis vectors. Let $i = 1, \dots, m$ and $j = 1, \dots, n$. We have

$$\begin{aligned} (f'e'_i)(e_j) &= (e'_i \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\ &= f(e_j)_i = a_{ij} && \text{(apply def. of } e'_i \text{ and } f) \end{aligned}$$

So if we set $i = 1$ for instance, we get

$$(f'e'_1)(e_j) = a_{1j}$$

The first "column" of f' (up to isomorphism) must be $(a_{1j})_{j=1, \dots, n}$.

In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \rightarrow \mathbb{R}^n, x \mapsto A^T x$$

2.2 The basic properties

The proofs will use the Hahn-Banach corollaries a couple of times. So first, we need to recall Hahn-Banach related theorems:

Revision 2 (Hahn-Banach). *Let $(X, \|\cdot\|)$ be a normed space and $0 \neq x \in X$. Then we have*

$$\exists f \in X' : \|f\| = 1 \wedge f(x) = \|x\|$$

Proof. Please refer to the functional analysis lecture notes [KP] from SS/2025. □

Revision 3. *The dual unit ball retains information about the norm of primal vectors: Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space and $x \in X$. Then we have*

$$\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$$

Proof. We will distinguish two cases:

- Case 1 ($x = 0$): Since X' contains linear operators,

$$\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$$

- Case 2 $x \neq 0$: We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

Using Hahn-Banach from revision 2 we get

$$\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|_X$$

So we get

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$$

□

Theorem 1. *The adjoint operator has the following properties:*

- i) $T' \in \mathcal{L}(Y', X')$, so T' is linear and bounded.
▷ This implies $\forall y' Y' : T'y' \in X'$.
- ii) $T \mapsto T'$ is linear and isometric.

Proof. "i)": Let $y' \in Y'$. Plugging it into the adjoint operator, we get $T'y' = y' \circ T$ with signature $X \rightarrow Y \rightarrow \mathbb{F}$. We can now see that $\text{Im } T' \subset X'$.

Let $y'_1, y'_2 \in Y', \alpha \in \mathbb{F}$. We then prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of adjoint operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + T'y'_2 && \text{(apply def. of adjoint operator)} \end{aligned}$$

Let $y' \in Y'$. We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X, Y)} && \text{(apply def. of op. norm)} \\ &:= C \|y'\|_{Y'} && \text{(def. the constant)} \end{aligned}$$

"ii)": Let $T_1, T_2 \in \mathcal{L}(X, Y), y' \in Y', x \in X, \alpha \in \mathbb{F}$. We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\ &= y'(\alpha T_1 x + T_2 x) && \text{(pull } x \text{ into the eq.)} \\ &= \alpha y'(T_1 x) + y'(T_2 x) && \text{(y' is linear)} \\ &= (\alpha T'_1 y' + T'_2 y')(x) \end{aligned}$$

We then prove isometry:

$$\begin{aligned} \|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\ &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && \text{(apply theorem of Hahn-Banach 2)} \\ &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && \text{(supremum order can be switched)} \\ &= \sup_{\|y'\| \leq 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\ &= \|T'\| && \text{(apply def. of op. norm)} \end{aligned}$$

□

Example 2. Let $p \in (1, \infty)$ with $p \neq 2$. This makes l_p a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of l_p is isometrically isomorph to l_p^* where p^* is the [Hölder conjugate](#) with $1/p + 1/p^* = 1$. As a reminder, the general idea of the proof of $l_p^* \cong l_p^*$ goes as follows:

- Define the isometric isomorphism as

$$T : l_p^* \rightarrow l_p', s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$$

- Verify $(Ts)x$ converges as

$$|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$$

and absolute convergence implies convergence.

- Verify T is injective using the linearity.
- Verify T is surjective and isometric (the long part). See [Wer18] for a full proof.

To illustrate the adjoint operator, we now work through an example. Consider the [left shift](#) operator

$$T : l_p \rightarrow l_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$$

It is well-defined since

$$\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$$

We can now compute the adjoint operator T' :

- The adjoint T' must have the signature $l'_p \cong l_p^* \rightarrow l'_p \cong l_p^*$.
- Let $y' \in l'_p \cong l_p^*$. Then we can write $y' : l_p \rightarrow \mathbb{F}, x \mapsto \sum_{k \in \mathbb{N}} x_k s_k$ with $s \in l_p^*$.
Now for $x \in l_p$ we have

$$\begin{aligned}
(T'y')(x) &= (y' \circ T)(x) = y'(Tx) && \text{(apply def. of } T') \\
&= y'((x_{k+1})_{k \in \mathbb{N}}) && \text{(apply def. of } T) \\
&= \sum_{k \in \mathbb{N}} x_{k+1} s_k && \text{(apply def. of } y') \\
&= \sum_{k \in \mathbb{N}} x_k s'_k && \text{(with } s'_1 = 0 \text{ and } s'_k = s_{k-1} \text{ for } k > 1)
\end{aligned}$$

This tells us that the adjoint operator T' acts as a **right shift** (up to isomorphism):

$$T' : l_p^* \rightarrow l_p^*, (s_k)_{k \in \mathbb{N}} \mapsto (0, s_1, s_2, \dots)$$

Theorem 2. *Let X, Y, Z be normed \mathbb{F} vector spaces.
Then the adjoint operator reverses composition:*

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

Proof. Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.

We know $ST = S \circ T$ is still a linear, bounded operator from X to Z . So $(ST)'$ is well-defined. Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$. We have

$$\begin{aligned}
(ST)'(z') &= z' \circ (ST) && \text{(apply def. of adjoint operator)} \\
&= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\
&= y' \circ T && \text{(subst. } y') \\
&= T'y' && \text{(apply def. of adjoint operator)} \\
&= T'(z' \circ S) && \text{(subst. } y') \\
&= T'S'z' && \text{(apply def. of adjoint operator)}
\end{aligned}$$

So in total $(ST)' = T'S'$. □

Example 3. Referring back to the example with $A \in \mathbb{R}^{m \times n}$, the adjoint operator reverses composition rule is compatible with the way matrix transposition on \mathbb{R} or the matrix adjoint on \mathbb{C} behave.

2.3 The dual space of the dual space

For starters, we need to recall some concepts from the lecture.

Definition 3. Let X be a normed \mathbb{F} vector space.

- i) X'' is called the bidual space.
- ii) Let $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$. J_X is called the **canonical embedding** from X into X'' .

So J_X returns a function that evaluates dual space elements at $p \in X$.

Theorem 3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Then we have: $J_Y \circ T = T'' \circ J_X$.
This concept is illustrated in figure 1.*

Proof. Before the proof, we can avoid confusion by typing out what T' and T'' evaluate to:

- $T' : Y' \rightarrow X', y' \mapsto y' \circ T$ is the adjoint operator.
- $T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

- $J_Y : Y \rightarrow Y''$ and $T : X \rightarrow Y$ means $(J_Y \circ T) : X \rightarrow Y''$.

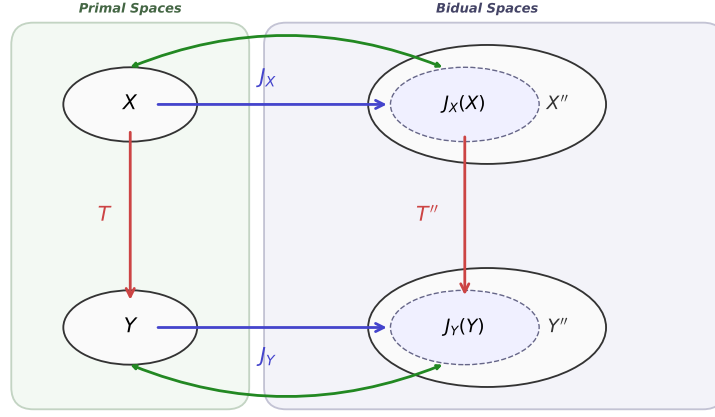


Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

- $T'' : X'' \rightarrow Y''$ and $J_X : X \rightarrow X''$ means $(T'' \circ J_X) : X \rightarrow Y''$.

Finally, for $p \in X$ and $y' \in Y'$ we have

$$\begin{aligned}
 ((J_Y \circ T)(p))(y') &= J_Y(Tp)(y') = y'(Tp) && \text{(subst. } p \text{ in and apply def. of } J_Y) \\
 &= (y'T)(p) && \text{(use associativity)} \\
 &= (T'y')(p) && \text{(apply def. of } T') \\
 &= (x' \mapsto x'(p))(T'y') && \text{(pull out subst. function)} \\
 &= J_X(p)(T'y') && \text{(recognize this is just } J_X) \\
 &= T''(J_X(p))(y') && \text{(apply def. use } T'') \\
 &= ((T'' \circ J_X)(p))(y') && \text{(use } \circ \text{ notation)}
 \end{aligned}$$

□

We will now characterize when a continuous operator between Y' and X' is an adjoint operator. We need to revise an important corollary from the lecture first.

Revision 4. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space. Then we have

- The canonical embedding J_X is an isometric injective function.
- $J : X \rightarrow J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism.
- J_X is a bounded, linear operator.
- $(J_X|_{X \rightarrow J_X(X)})^{-1}$ is a bounded, linear operator.

This concept is illustrated in figure 1.

Proof Idea.

"i)": Please refer to the functional analysis lecture notes [KP] from SS/2025.

"ii)": Follows by definition of J and from "i)".

"iii)": Since $J : X \rightarrow J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

- J_X is linear since the $x' \in X'$ are linear.
- $\|J_X\| = 1$. This means J_X is bounded.

"iv)": Since $J : X \rightarrow J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

- $(J_X|_{X \rightarrow J_X(X)})^{-1}$ inherits linearity from J .
- $\|J_X\| = 1$ and therefore $\|(J_X|_{X \rightarrow J_X(X)})^{-1}\| = 1$. This means $(J_X|_{X \rightarrow J_X(X)})^{-1}$ is bounded.

□

Corollary 1. *We can strengthen theorem 3 using the isomorphism from revision 4:
Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Then we have*

- i) $J_Y \circ T = T'' \circ J_X.$
- ii) $T''|_{X \rightarrow J_X(X)} = J_Y \circ T \circ (J_X|_{X \rightarrow J_X(X)})^{-1}$
- iii) $T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X.$

Proof. "i)": This is the statement of theorem 3.

"ii)": The isomorphism from revision 4 gives us the result.

"iii)": The isomorphism from revision 4 gives us the result. □

Theorem 4. *Let $S \in \mathcal{L}(Y', X')$ be a continuous, linear operator. Then we have*

$$\exists T \in \mathcal{L}(X, Y) : T' = S \text{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

Proof. " \Rightarrow ": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T') \\ &= J_Y(T(X)) && \text{(use theorem 3)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y) \end{aligned}$$

" \Leftarrow ": We have $S'(J_X(X)) \subset J_Y(Y)$ and revision 4 gives us $J_Y(Y) \cong Y$.

So for $x \in X$ and $y''_x = S'(J_X(x))$ there is a (unique) $y_x \in Y$ with $y''_x = J_Y(y_x)$.

Choose

$$T : X \rightarrow Y, x \mapsto y_x$$

We know T exists (and is unique) due to the previous argument.

We know T is linear and continuous, as

$$T = y. = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X$$

and all elements in the chain are bounded, linear operators.

Let $y' \in Y'$ and $x \in X$. Lastly, we need to prove that $S = T'$:

$$\begin{aligned} (Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X) \\ &= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of ajoint op.)} \\ &= J_Y(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} = \text{Id}) \\ &= y'(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\ &= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X) \\ &= (T'y')(x) && \text{(apply def. of adjoint op.)} \end{aligned}$$

□

2.4 Compact (adjoint) operators

For starters, we need to recall and review some concepts from the lecture.

Definition 4. The following definitions are revisions from the lecture:

Let X, Y be normed \mathbb{F} vector spaces.

- i) $M \subset X$ is relatively compact, if

$$\forall (x_n)_{n \in \mathbb{N}} \subset M : (x_n)_{n \in \mathbb{N}} \text{ has a converging subsequence in } X$$

- ii) $T \in \mathcal{L}(X, Y)$ is compact, if $T(\overline{B}_X)$ is relatively compact.

- iii) The rank of T is $\text{rk } T = \dim T(X)$.

As a reminder, in metric spaces a subset is compact if and only if all sequences contain a converging subsequence in that set. So relatively compact relaxes the requirement that the set must be closed. An important property of the closed unit ball in \mathbb{R}^n or \mathbb{C} is that it is compact. Therefore, a linear operator between $\mathbb{F}^n \rightarrow \mathbb{F}^m$ always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ have the same open sets. In finite dimensions, linear algebra tells us that $(X, \|\cdot\|_1)$ and $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$ have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space \mathbb{R}^n . In \mathbb{R}^n , the theorem of Heine-Borel tells us that all bounded subsets are relatively compact.

Since relatively compact is a topological property, the theorem transfers to $(X, \|\cdot\|)$. Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general Banach spaces.

Definition 5. The following definitions are critical for the theorem of [Arzelà-Ascoli](#): Let X, Y be metric spaces and $M \subset \{f : X \rightarrow Y\}$.

i) M is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

ii) M is pointwise bounded, if

$$\forall x \in X : \{f(x) \mid f \in M\} \text{ is bounded}$$

The theorem of [Arzelà-Ascoli](#) from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions [evaluations](#) rather than the functions themselves. They can be easily verified to be true. Note that $C(D) = \{f : D \rightarrow \mathbb{F} \text{ continuous}\}$.

Revision 5 (Arzelà-Ascoli). *Let D be a compact metric space and $M \subset C(D)$ with the supremum norm. Then we have M is uniformly equicontinuous and pointwise bounded implies M is relatively compact.*

Proof Idea.

- D is separable, i.e. $\exists D_0 \subset D : \overline{D_0} = D$. Simply set $D_0 = \bigcup_{n \in \mathbb{N}} D_n$ where D_n is a finite $\frac{1}{n}$ -covers of D .
- For all $x \in D$ we can use the pointwise boundedness to invoke Bolzano-Weierstrass to find converging subsequences $(f_{n(x,k)}(x))_{k \in \mathbb{N}} \subset \mathbb{F}$.
- Specifically, we have $\forall x \in D_0 : (f_{n(x,k)}(x))_{k \in \mathbb{N}}$ converges.
- Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on x : $\forall x \in D_0 : (f_{n(k)}(x))_{k \in \mathbb{N}}$ converges.
- We already have $f_{n(k)}|_{D_0} \rightarrow f|_{D_0}$. And it seems sensible to assume that $(f_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence. But how can you extend this to the entire set D ?
- For each $x \in D$, we can choose an arbitrarily close $x_0 \in D_0$. Using two triangle inequalities, uniform equicontinuity allows us to extend the result to D .
As $C(D)$ is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

Theorem 5 (Schauder). *Let X, Y be Banach spaces and $T : X \rightarrow Y$ be a bounded, linear operator. Then we have T is compact if and only if T' is compact.*

Proof. " \Rightarrow ": Let $(y'_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$ be bounded.

Our goal is to show that there is a convergent subsequence in $(T'y'_n)_{n \in \mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm. For all $n \in \mathbb{N}$ set $f_n = y'_n|_{T(\overline{B}_X)}$.

We have

$$\begin{aligned} \|T'y'_n - T'y'_m\| &= \|y'_n \circ T - y'_m \circ T\| && \text{(apply def. of adjoint op.)} \\ &= \sup_{\|x\|_X \leq 1} |((y'_n \circ T) - (y'_m \circ T))(x)| && \text{(use supremum char. of the op. norm)} \\ &= \sup_{\|x\|_X \leq 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m) \\ &= \sup_{d \in T(\overline{B}_X)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m) \end{aligned}$$

We know $(T'y'_n)_{n \in \mathbb{N}}$ converges if and only if it is a Cauchy sequence $(\mathcal{L}(X, \mathbb{F}), \|\cdot\|)$. Therefore the convergence of $(T'y'_n)_{n \in \mathbb{N}}$ in the operator norm is only dependent on the behaviour of $(f_n)_{n \in \mathbb{N}}$ on $T(\overline{B}_X)$.

We now set

$$D = \overline{T(\overline{B}_X)}$$

and pack the sequence into

$$M := \{f_n \mid n \in \mathbb{N}\}$$

and examine them for

- D is compact: We have
 - \overline{B}_X is bounded.
 - T is a compact operator.

So $T(\overline{B}_X)$ is relatively compact and D is compact.

- M is pointwise bounded: For $x \in \overline{B}_X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} |f_n(Tx)| &= |y'_n(Tx)| := |y'_n(d)| && \text{(apply def. of } f_n \text{ and define } d) \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 && \text{(apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq C_2) \end{aligned}$$

For $d \in D$ we can choose $(x_k)_{k \in \mathbb{N}} \subset \overline{B}_X$ with $Tx_k \rightarrow d$ and $f_n(Tx_k) \leq C_1 C_2$.

Using continuity we get $|f_n(d)| \leq C_1 C_2$.

- M is uniformly equicontinuous: For $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon/C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$ we have

$$\begin{aligned} |f_n(d_1) - f_n(d_2)| &\leq \|y'_n\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)} \\ &\leq C_1 \|d_1 - d_2\|_Y && \text{(use the upper bound } \|y'_n\| \leq C_1) \\ &< C_1 \delta = \epsilon && \text{(substitute in } \delta) \end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that M is relatively compact. So every sequence in M has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k \in \mathbb{N}}$ we have

$$\begin{aligned} (f_{n(k)})_{k \in \mathbb{N}} &= (y'_n \circ T)_{k \in \mathbb{N}} && \text{(apply def. of } f_{n(k)}) \\ &= (T'y'_{n(k)})_{k \in \mathbb{N}} && \text{(apply def. of adjoint operator)} \end{aligned}$$

So finally, $(T'y'_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence of $(T'y'_k)_{k \in \mathbb{N}}$.

" \Leftarrow ": The other proof direction tells us that

$$T \text{ compact} \Rightarrow T' \text{ compact}$$

So in extension this also yields

$$T' \text{ compact} \Rightarrow T'' \text{ compact}$$

Corollary 1 tells us that

$$T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$$

Lastly, the operator T'' is compact, theorem 4 tells us J_X and $(J_Y|_{Y \rightarrow J_Y(Y)})^{-1}$ are bounded, linear operators and therefore T is a composition of bounded, linear operators.

Using the revision theorem 1 we can conclude that T is compact. □

Example 4. The lecture [KP] states that the following integral operator is compact:

$$T : C[0, 1] \rightarrow C[0, 1], Tf(x) = \int_0^1 f(t) dx$$

So $T' : C[0, 1]' \rightarrow C[0, 1]'$ is compact too. Evaluating $C[0, 1]'$ is beyond the scope of this paper.

2.5 The rank-nullity theorem for operators

Definition 6. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ and $V \subset X'$. Then we define the annihilator of V in X as

$$V_\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$$

▷ In case of $X = \mathbb{R}^2$ and V 1-dimensional subspace (common in linear algebra), the annihilator is the set of straight lines through 0 in \mathbb{R}^2 .

To prove properties about this set, we need another corollary of the theorem of Hahn-Banach:

Revision 6. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ a closed subspace and $x \in X \setminus U$. Then we have

$$\exists x' \in X' : x'|_U = 0 \wedge x'(x) \neq 0$$

Proof. Let $Y = X/U$ be the (canonical) quotient space. Then Y is a normed \mathbb{F} vector space. (todo: why?) Set $y = x \in Y$. We can apply the theorem of Hahn-Banach 2 to obtain $y' \in Y'$ with $y'(y) \neq 0$ and $y'|_U = 0$. □

Theorem 6. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ and $V \subset X'$. Then we have

$$V_\perp \subset X \text{ is a closed linear subspace}$$

Proof. We have

$$V_\perp = \bigcap_{x' \in V} (x')^{-1}(0)$$

As an intersection of closed sets, V_\perp must be closed. □

Theorem 7. Let $T \in \mathcal{L}(X, Y)$ be a bounded, linear operator. Then we have

$$\overline{\text{Im } T} = (\text{Ker } T')_\perp$$

▷ In linear algebra lectures this is proven for finite-dimensional vector spaces (see Satz 6.1.5 [Wer18])

Proof. "⊂": Let $Tx \in \text{Im } T$ with $x \in X$ and $y' \in \text{Ker } T'$.

We first prove $Tx \in (\text{Ker } T')_\perp$:

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of Tx we get

$$\text{Im } T \subset (\text{Ker } T')_\perp$$

Since $(\text{Ker } T')_\perp$ is closed, we also get

$$\overline{\text{Im } T} \subset (\text{Ker } T')_\perp$$

" \supset ": We can prove the contraposition

$$(Y \setminus \overline{\text{Im } T}) \subset (Y \setminus (\text{Ker } T')^\perp)$$

Set $U = \overline{\text{Im } T}$ and let $y \in Y \setminus U$. We know U that a closed linear subspace. The corollary of the theorem of Hahn-Banach 6 tells us that

$$\exists y' \in Y' : y'|_U = 0 \wedge y'(y) \neq 0$$

Since $\text{Ker } T' \subset Y'$ and

$$\forall y' \in \text{Ker } T' : y'(y) \neq 0$$

we get

$$y \in Y \setminus (\text{Ker } T')^\perp$$

□

Corollary 2. *Let $T \in \mathcal{L}(X, Y)$ be a linear, continuous operator with $\text{Im } T$ closed. Then we have*

$$y \in \text{Im } T \text{ if and only if } \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0$$

Proof. We have

$$\begin{aligned} y \in \text{Im } T &= \overline{\text{Im } T} && (\text{Im } T \text{ is closed}) \\ &= (\text{Ker } T')^\perp && (\text{apply theorem 7}) \\ \iff \forall y' \in \text{Ker } T' : y'(y) = 0 &&& (\text{apply def. of annihilator}) \\ \iff \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0 &&& (\text{write in equivalent way}) \end{aligned}$$

□

Example 5. We can refer back to the example 2 with the left shift operator.

The left shift operator T has $\text{Im } T = l_p$, which is (trivially) closed.

Since we already know the adjoint operator we can verify that the last theorem 7 works.

Everything is up to isometry: Let $(x_k)_{k \in \mathbb{N}} \in l_p$ such that $T((x_k)_{k \in \mathbb{N}}) = (x_2, \dots) \in l_p$. Then

$$T'y' = 0 \Rightarrow y'(y) = 0$$

amounts to "if the right shifted version of a sequence is 0 then

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