University of Passau

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# Dual Operators

Seminar Functional Analysis


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**Abstract:** This proseminar paper provides a foundational overview of dual-/adjoint operators in a general Banach space setting.

# 1 Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as $A^T$ over $\mathbb{R}$ or $A^H$ over $\mathbb{C}$) to operators on normed $\mathbb{R}$ or $\mathbb{C}$ vector spaces:
We will cover the following topics:

- The operator $\cdot^H$ is linear and isometric wrt. the spectral norm.
- The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem: $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^H$.
- TODO Add more

# 2 The adjoint operator

## 2.1 The basic definitions and conventions

The terms adjoint and dual are often used interchangeably. We will standardize to adjoint, to avoid unnecessary confusion. Following [Wer18], we write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ when the field is unspecified.

**Definition 1.** We remind ourselves of the following concepts from the lecture [KP]: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed $\mathbb{F}$-vector spaces

i) Let

$$T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call $T$ a linear operator. We call $T$ bounded, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation $T : X \to Y$ for linear operators rather than $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$.

ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

iii) The closed unit ball in $(X, \|\cdot\|_X)$ is abbreviated with $\overline{B}_X$.

**Revision 1.** *The following statements are foundational for this topic:*
*Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces.*

*i) The set of continuous, linear operators $\mathcal{L}(X, Y)$ is a Banach space if and only if $Y$ is a Banach space. In particular, the topological dual space $\mathcal{L}(X, \mathbb{F})$ is a Banach space.*

*ii) Let $T : X \to Y$ be a linear operator. Then $T$ is bounded if and only if $T$ is continous.*

*iii) Let $(Z, \|\cdot\|_Z)$ be a normed $\mathbb{F}$-vector space, let $T : X \to Y$ be a linear, bounded operator and let $S : Y \to Z$ be a linear, bounded operator. Then $S \circ T$ is a linear, bounded operator.*

*iv) Let $(Z, \|\cdot\|_Z)$ be a normed $\mathbb{F}$-vector space, let $T : X \to Y$ be a linear operator and let $S : Y \to Z$ be a linear operator. If $T$ or $S$ is compact, $S \circ T$ is compact.*

*Proof.* Please refer to the functional analysis lecture notes [KP] from SS/2025. □

**Definition 2.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$-vector spaces and $T \in \mathcal{L}(X, Y)$.
Then $T' : Y' \to X', y' \mapsto y' \circ T$ is called the adjoint operator. From now on, we will implicitly refer to the normed $\mathbb{F}$ vector spaces $X, Y$ and the topological dual spaces $X', Y'$ when talking about the dual operator $T'$.

**Remark 1.** In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from $\mathbb{F}$. For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$.

**Example 1.** Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
(\mathbb{R}^n)' &= \{g : \mathbb{R}^n \to \mathbb{R} \text{ linear, continuous}\} && \text{(apply def. of dual space)} \\
&= \{g : \mathbb{R}^n \to \mathbb{R} \text{ linear}\} && \text{(result from linear algebra)} \\
&\cong \mathbb{R}^n && \text{(result from linear algebra see [FS25a])}
\end{aligned}$$

Let $m \in \mathbb{N}$ and let $A \in \mathbb{R}^{m \times n}$.
Consider the function $f : \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$. The signature of the adjoint is $f' : (\mathbb{R}^m)' \to (\mathbb{R}^n)'$.
With $e_i$ we denote the standard basis vectors and with $e_i'$ we denote the dual standard basis vectors.
Let $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We have

$$\begin{aligned}
(f'e_i')(e_j) &= (e_i' \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\
&= f(e_j)_i = a_{ij} && \text{(apply def. of } e_i' \text{ and } f\text{)}
\end{aligned}$$

So if we set $i = 1$ for instance, we get

$$(f'e_1')(e_j) = a_{1j}$$

The first "column" of $f'$ (up to isomorphism) must be $(a_{1j})_{j=1,\ldots,n}$.
In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \to \mathbb{R}^n, x \mapsto A^T x$$

## 2.2 The basic properties

The proofs will use the Hahn-Banach corollaries a couple of times. So first, we need to recall Hahn-Banach related theorems:

**Revision 2** (Hahn-Banach)**.** *Let $(X, \|\cdot\|)$ be a normed space and $0 \neq x \in X$.*
*Then we have*

$$\exists f \in X' : \|f\| = 1 \wedge \ f(x) = \|x\|$$

*Proof.* Please refer to the functional analysis lecture notes [KP] from SS/2025. □

**Revision 3.** *The dual unit ball retains information about the norm of primal vectors:*
*Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space and $x \in X$.*
*Then we have*

$$\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$$

*Proof.* We will distinguist two cases:

- Case 1 ($x = 0$): Since $X'$ contains linear operators,

$$\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$$

- Case 2 $x \neq 0$: We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

  Using Hahn-Banach from revision 2 we get

$$\exists f \in X' : \|f\| = 1 \wedge \ |f(x)| = \|x\|_X$$

  So we get

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$$

□

**Theorem 1.** *The adjoint operator has the following properties:*

*i)* $T' \in \mathcal{L}(Y', X')$*, so* $T'$ *is linear and bounded.*
▷ This implies $\forall y'Y' : T'y' \in X'$.

*ii)* $T \mapsto T'$ *is linear and isometric.*

*Proof.* "i)": Let $y' \in Y'$. Plugging it into the adjoint operator, we get $T'y' = y' \circ T$ with signature $X \to Y \to \mathbb{F}$. We can now see that $\operatorname{Im} T' \subset X'$.
Let $y'_1, y'_2 \in Y', \alpha \in \mathbb{F}$. We then prove linearity:

$$\begin{aligned}
T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of adjoint operator)} \\
&= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\
&= \alpha T' y'_1 + T' y'_2 && \text{(apply def. of adjoint operator)}
\end{aligned}$$

Let $y' \in Y'$. We then prove boundedness of the operator norm:

$$\begin{aligned}
\|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\
&\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X,Y)} && \text{(apply def. of op. norm)} \\
&:= C\|y'\|_{Y'} && \text{(def. the constant)}
\end{aligned}$$

"ii)": Let $T_1, T_2 \in \mathcal{L}(X, Y), y' \in Y', x \in X, \alpha \in \mathbb{F}$. We first prove linearity:

$$\begin{aligned}
(\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\
&= y' \left(\alpha T_1 x + T_2 x\right) && \text{(pull $x$ into the eq.)} \\
&= \alpha y'(T_1 x) + y'(T_2 x) && \text{(y' is linear)} \\
&= (\alpha T'_1 y' + T'_2 y')(x)
\end{aligned}$$

We then prove isometry:

$$\begin{aligned}
\|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\
&= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && \text{(apply theorem of Hahn-Banach 2)} \\
&= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && \text{(supremum order can be switched)} \\
&= \sup_{\|y'\| \leq 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\
&= \|T'\| && \text{(apply def. of op. norm)}
\end{aligned}$$

□

**Example 2.** Let $p \in (1, \infty)$ with $p \neq 2$. This makes $l_p$ a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of $l_p$ is isometrically isomorph to $l_p^*$ where $p^*$ is the Hölder conjugate with $1/p + 1/p^* = 1$.
As a reminder, the general idea of the proof of $l'_p \cong l_p^*$ goes as follows:

- Define the isometric isomorphism as
$$T : l_p^* \to l'_p, s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$$
- Verify $(Ts)x$ converges as
$$|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$$
and absolute convergence implies convergence.
- Verify $T$ is injective using the linearity.
- Verify $T$ is surjective and isometric (the long part). See [Wer18] for a full proof.

To illustrate the adjoint operator, we now work through an example. Consider the left shift operator
$$T : l_p \to l_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$$
It is well-defined since
$$\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$$
We can now compute the adjoint operator $T'$:

- The adjoint $T'$ must have the signature $l_p' \cong l_p^* \to l_p' \cong l_p^*$.
- Let $y' \in l_p' \cong l_p^*$. Then we can write $y' : l_p \to \mathbb{F}, x \mapsto \sum_{k\in\mathbb{N}} x_k s_k$ with $s \in l_p^*$. Now for $x \in l_p$ we have

$$\begin{aligned}
(T'y')(x) &= (y' \circ T)(x) = y'(Tx) && \text{(apply def. of } T'\text{)} \\
&= y'((x_{k+1})_{k\in\mathbb{N}}) && \text{(apply def. of } T\text{)} \\
&= \sum_{k\in\mathbb{N}} x_{k+1} s_k && \text{(apply def. of } y'\text{)} \\
&= \sum_{k\in\mathbb{N}} x_k s_k' && \text{(with } s_1' = 0 \text{ and } s_k' = s_{k-1} \text{ for } k > 1\text{)}
\end{aligned}$$

This tells us that the adjoint operator $T'$ acts as a right shift (up to isomorphism):

$$T' : l_p^* \to l_p^*, (s_k)_{k\in\mathbb{N}} \mapsto (0, s_1, s_2, \dots)$$

**Theorem 2.** *Let $X, Y, Z$ be normed $\mathbb{F}$ vector spaces.*
*Then the adjoint operator reverses composition:*

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

*Proof.* Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.
We know $ST = S \circ T$ is still a linear, bounded operator from $X$ to $Z$. So $(ST)'$ is well-defined.
Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$. We have

$$\begin{aligned}
(ST)'(z') &= z' \circ (ST) && \text{(apply def. of adjoint operator)} \\
&= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\
&= y' \circ T && \text{(subst. y')} \\
&= T'y' && \text{(apply def. of adjoint operator)} \\
&= T'(z' \circ S) && \text{(subst. y')} \\
&= T'S'z' && \text{(apply def. of adjoint operator)}
\end{aligned}$$

So in total $(ST)' = T'S'$. □

**Example 3.** Referring back to the example with $A \in \mathbb{R}^{m\times n}$, the adjoint operator reverses composition rule is compatible with the way matrix transposition on $\mathbb{R}$ or the matrix adjoint on $\mathbb{C}$ behave.

## 2.3 The dual space of the dual space

For starters, we need to recall some concepts from the lecture.

**Definition 3.** Let $X$ be a normed $\mathbb{F}$ vector space.

i) $X''$ is called the bidual space.

ii) Let $J_X : X \to X'', p \mapsto (x' \mapsto x'(p))$. $J_X$ is called the canonical embedding from $X$ into $X''$.

So $J_X$ returns a function that evaluates dual space elements at $p \in X$.

**Theorem 3.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$ vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Then we have: $J_Y \circ T = T'' \circ J_X$.*
*This concept is illustrated in figure 1.*

*Proof.* Before the proof, we can avoid confusion by typing out what $T'$ and $T''$ evaluate to:

- $T' : Y' \to X', y' \mapsto y' \circ T$ is the adjoint operator.
- $T'' : X'' \to Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

- $J_Y : Y \to Y''$ and $T : X \to Y$ means $(J_Y \circ T) : X \to Y''$.

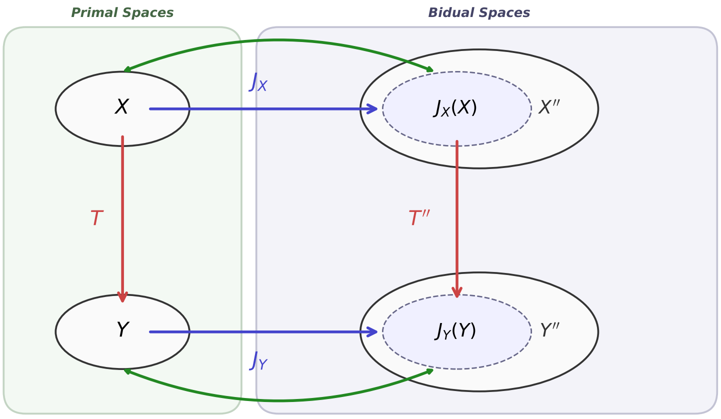


Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

- $T'' : X'' \to Y''$ and $J_X : X \to X''$ means $(T'' \circ J_X) : X \to Y''$.

Finally, for $p \in X$ and $y' \in Y'$ we have

$$\begin{aligned}
((J_Y \circ T)(p))\,(y') &= J_Y(Tp)(y') = y'(Tp) && \text{(subst. } p \text{ in and apply def. of } J_Y\text{)}\\
&= (y'T)(p) && \text{(use associativity)}\\
&= (T'y')(p) && \text{(apply def. of } T'\text{)}\\
&= (x' \mapsto x'(p))(T'y') && \text{(pull out subst. function)}\\
&= J_X(p)(T'y') && \text{(recognize this is just } J_X\text{)}\\
&= T''(J_X(p))(y') && \text{(apply def. use } T''\text{)}\\
&= ((T'' \circ J_X)(p))\,(y') && \text{(use } \circ \text{ notation)}
\end{aligned}$$

□

We will now characterize when a continuous operator between $Y'$ and $X'$ is an adjoint operator. We need to revise an important corollary from the lecture first.

**Revision 4.** *Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space. Then we have*

*i) The canonical embedding $J_X$ is an isometric injective function.*

*ii) $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism.*

*iii) $J_X$ is a bounded, linear operator.*

*iv) $(J_X|_{X \to J_X(X)})^{-1}$ is a bounded, linear operator.*

*This concept is illustrated in figure 1.*

*Proof Idea.*

"i)": Please refer to the functional analysis lecture notes [KP] from SS/2025.

"ii)": Follows by definition of $J$ and from "i)".

"iii)": Since $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

- $J_X$ is linear.
- $\|J_X\| = 1$. This means $J_X$ is bounded.

"iv)": Since $J : X \to J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

- $(J_X|_{X \to J_X(X)})^{-1}$ inherits linearity from $J$.
- $\|J_X\| = 1$ and therefore $\|(J_X|_{X \to J_X(X)})^{-1}\| = 1$. This means $(J_X|_{X \to J_X(X)})^{-1}$ is bounded.

□

**Corollary 1.** *We can strengthen theorem 3 using the isomorphism from revision 4:*
*Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{F}$ vector spaces. Let $T \in \mathcal{L}(X,Y)$ be a bounded linear operator. Then we have*

*i)* $J_Y \circ T = T'' \circ J_X$.

*ii)* $T''|_{X \to J_X(X)} = J_Y \circ T \circ (J_X|_{X \to J_X(X)})^{-1}$

*iii)* $T = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ T'' \circ J_X$.

*Proof.* "i)": This is the statement of theorem 3.

"ii)": The isomorphism from revision 4 gives us the result.

"iii)": The isomorphism from revision 4 gives us the result.

□

**Theorem 4.** *Let $S \in \mathcal{L}(Y', X')$ be a continuous, linear operator. Then we have*

$$\exists T \in \mathcal{L}(X,Y) : T' = S \text{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

*Proof.* "$\Rightarrow$": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T'\text{)} \\ &= J_Y(T(X)) && \text{(use theorem 3)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y\text{)} \end{aligned}$$

"$\Leftarrow$": We have $S'(J_X(X)) \subset J_Y(Y)$ and revision 4 gives us $J_Y(Y) \cong Y$.
So for $x \in X$ and $y''_x = S'(J_X(x))$ there is a (unique) $y_x \in Y$ with $y''_x = J_Y(y_x)$.
Choose

$$T : X \to Y, x \mapsto y_x$$

We know $T$ exists (and is unique) due to the previous argument.
We know $T$ is linear and continuous, as

$$T = y_\cdot = (J_Y|_{Y \to J_Y(Y)})^{-1} \circ S' \circ J_X$$

and all elements in the chain are bounded, linear operators.
Let $y' \in Y'$ and $x \in X$. Lastly, we need to prove that $S = T'$:

$$\begin{aligned} (Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X\text{)} \\ &= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of ajoint op.)} \\ &= J_Y(((J_Y|_{Y \to J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y \to J_Y(Y)})^{-1} = \mathrm{Id}\text{)} \\ &= y'(((J_Y|_{Y \to J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\ &= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X\text{)} \\ &= (T'y')(x) && \text{(apply def. of adjoint op.)} \end{aligned}$$

□

**Theorem 5.** *Lastly, we get one more property of the adjoint: $T \mapsto T'$ is not always surjective.*
▷ This is not the case with $\cdot^H$ e.g. between $\mathbb{R}^n \to \mathbb{R}^n$.

*Proof.* The operator and the adjoint operator have the following signatures:

- $T : X \to Y$
- $T' : Y' = \mathcal{L}(Y, \mathbb{F}) \to X' = \mathcal{L}(X, \mathbb{F})$

So the "not always" refers to a particular choice of $X$ and $Y$ we need to find.
Using the last theorem, we can find an example for an operator $S \in \mathcal{L}(Y', X')$ that is not an adjoint operator
(i.e. with $\nexists T \in \mathcal{L}(X,Y) : T' = S$).

□

## 2.4 Compact (adjoint) operators

For starters, we need to recall and review some concepts from the lecture.

**Definition 4.** The following definitions are revisions from the lecture:
Let $X, Y$ be normed $\mathbb{F}$ vector spaces.

i) $M \subset X$ is relatively compact, if

$$\forall (x_n)_{n\in\mathbb{N}} \subset M : (x_n)_{n\in\mathbb{N}} \text{ has a converging subsequence in } X$$

ii) $T \in \mathcal{L}(X, Y)$ is compact, if $T\left(\overline{B}_X\right)$ is relatively compact.

iii) The rank of T is $\operatorname{rk} T = \dim T(X)$.

As a reminder, in metric spaces a subset is compact if and only if all sequences contain a converging subsequence in that set. So relatively compact relaxes the requirement that the set must be closed. An important property of the closed unit ball in $\mathbb{R}^n$ or $\mathbb{C}$ is that it is compact. Therefore, a linear operator between $\mathbb{F}^n \to \mathbb{F}^m$ always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ have the same open sets. In finite dimensions, linear algebra tells us that $(X, \|\cdot\|_1)$ and $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$ have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space $\mathbb{R}^n$. In $\mathbb{R}^n$, the theorem of Heine-Borel tells us that all bounded subsets are relatively compact.

Since relatively compact is a topological property, the theorem transfers to $(X, \|\cdot\|)$. Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general Banach spaces.

**Definition 5.** The following definitions are critical for the theorem of Arzelà-Ascoli:
Let $X, Y$ be metric spaces and $M \subset \{f : X \to Y\}$.

i) $M$ is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

ii) $M$ is pointwise bounded, if

$$\forall x \in X : \{f(x) \mid f \in M\} \text{ is bounded}$$

The theorem of Arzelà-Ascoli from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions evaluations rather than the functions themselves. They can be easily verified to be true. Note that $C(D) = \{f : D \to \mathbb{F} \text{ continuous}\}$.

**Revision 5** (Arzelà-Ascoli)**.** *Let $D$ be a compact metric space and $M \subset C(D)$ with the supremum norm. Then we have $M$ is uniformly equicontinuous and pointwise bounded implies $M$ is relatively compact.*

*Proof Idea.*

- $D$ is separable, i.e. $\exists D_0 \subset D : \overline{D_0} = D$. Simply set $D_0 = \bigcup_{n\in\mathbb{N}} D_n$ where $D_n$ is a finite $\frac{1}{n}$-covers of $D$.
- For all $x \in D$ we can use the pointwise boundedness to invoke Bolzano-Weierstrass to find converging subsequences $(f_{n(x,k)}(x))_{k\in\mathbb{N}} \subset \mathbb{F}$.
- Specifically, we have $\forall x \in D_0 : (f_{n(x,k)}(x))_{k\in\mathbb{N}}$ converges.
- Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on $x$: $\forall x \in D_0 : (f_{n(k)}(x))_{k\in\mathbb{N}}$ converges.

- We already have $f_{n(k)}|_{D_0} \longrightarrow: f|_{D_0}$. And it seems sensible to assume that $(f_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence. But how can you extend this to the entire set $D$?
- For each $x \in D$, we can choose an arbitrarily close $x_0 \in D_0$. Using two triangle inequalities, uniform equicontinuity allows us to extend the result to $D$.
  As $C(D)$ is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

**Theorem 6** (Schauder)**.** *Let $X, Y$ be Banach spaces and $T : X \to Y$ be a bounded, linear operator. Then we have $T$ is compact if and only if $T'$ is compact.*

*Proof.* "$\Rightarrow$": Let $(y'_n)_{n\in\mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$ be bounded.
Our goal is to show that there is a convergent subsequence in $(T'y'_n)_{n\in\mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm. For all $n \in \mathbb{N}$ set $f_n = y'_n|_{T(\overline{B}_X)}$.
We have

$$\begin{aligned}
\|T'y'_n - T'y'_m\| &= \|y'_n \circ T - y'_m \circ T\| && \text{(apply def. of adjoint op.)}\\
&= \sup_{\|x\|_X \le 1} |((y'_n \circ T) - (y'_m \circ T))(x)| && \text{(use supremum char. of the op. norm)}\\
&= \sup_{\|x\|_X \le 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m)\\
&= \sup_{d \in T(\overline{B}_X)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m)
\end{aligned}$$

We know $(T'y'_n)_{n\in\mathbb{N}}$ converges if and only if it is a Cauchy sequence $(\mathcal{L}(X, \mathbb{F}), \|\cdot\|)$. Therefore the convergence of $(T'y'_n)_{n\in\mathbb{N}}$ in the operator norm is only dependent on the behaviour of $(f_n)_{n\in\mathbb{N}}$ on $\overline{T(\overline{B}_X)}$.
We now set

$$D = \overline{T(\overline{B}_X)}$$

and pack the sequence into

$$M := \{f_n \mid n \in \mathbb{N}\}$$

and examine them for

- $D$ is compact: We have
  - $\overline{B}_X$ is bounded.
  - $T$ is a compact operator.

  So $T(\overline{B}_X)$ is relatively compact and $D$ is compact.
- $M$ is pointwise bounded: For $x \in \overline{B}_X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
|f_n(Tx)| = |y'_n(Tx)| := |y'_n(d)| && \text{(apply def. of } f_n \text{ and define } d)\\
\le C_1\|d\|_Y \le C_1C_2 && \text{(apply op. norm ineq. and D bounded means } \|d\|_Y \le: C_2)
\end{aligned}$$

  For $d \in D$ we can choose $(x_k)_{k\in\mathbb{N}} \subset \overline{B}_X$ with $Tx_k \longrightarrow d$ and $f_n(Tx_k) \le C_1C_2$.
  Using continuity we get $|f_n(d)| \le C_1C_2$.
- $M$ is uniformly equicontinuous: For $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon/C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$ we have

$$\begin{aligned}
|f_n(d_1) - f_n(d_2)| &\le \|y'_n\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)}\\
&\le C_1\|d_1 - d_2\|_Y && \text{(use the upper bound } \|y'_n\| \le C_1)\\
&< C_1\delta = \epsilon && \text{(substitute in } \delta)
\end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that $M$ is relatively compact. So every sequence in $M$ has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ we have

$$\begin{aligned}(f_{n(k)})_{k\in\mathbb{N}} &= (y'_n \circ T)_{k\in\mathbb{N}} && \text{(apply def. of } f_{n(k)}\text{)}\\ &= (T'y'_{n(k)})_{k\in\mathbb{N}} && \text{(apply def. of adjoint operator)}\end{aligned}$$

So finally, $(T'y'_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence of $(T'y'_k)_{k\in\mathbb{N}}$.

"$\Leftarrow$": The other proof direction tells us that

$$T \text{ compact } \Rightarrow T' \text{ compact}$$

So in extension this also yields

$$T' \text{ compact } \Rightarrow T'' \text{ compact}$$

Corollary 1 tells us that

$$T = (J_Y|_{Y\to J_Y(Y)})^{-1} \circ T'' \circ J_X$$

Lastly, the operator $T''$ is compact, theorem 4 tells us $J_X$ and $(J_Y|_{Y\to J_Y(Y)})^{-1}$ are bounded, linear operators and therefore $T$ is a composition of bounded, linear operators.
Using the revision theorem 1 we can conclude that $T$ is compact.

□

**Example 4.** The lecture [KP] states that the following integral operator is compact:

$$T : C[0,1] \to C[0,1], Tf(x) = \int_0^1 f(t)\,dx$$

So $T' : C[0,1]' \to C[0,1]'$ is compact too. Evaluating $C[0,1]'$ is beyond the scope of this paper.

### 2.5 The rank-nullity theorem for operators

**Definition 6.** Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space, $U \subset X$ and $V \subset X'$. Then we define the annihilator of $V$ in $X$ as

$$V_\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$$

▷ In linear algebra, the annihilator is isomorphic to the orthogonal complement of a set. Similarly we generalize the idea of a dual structure isomorphic to the image or kernel to idk... todo

To prove properties about this set, we need another corollary of the theorem of Hahn-Banach:

**Revision 6.** *Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space, $U \subset X$ a closed subspace and $x \in X \setminus U$. Then we have*

$$\exists x' \in X' : x'|_U = 0 \wedge\ x'(x) \neq 0$$

*Proof.* Let $Y = X/U$ be the (canonical) quotient space. Then $Y$ is a normed $\mathbb{F}$ vector space. (todo: why?) Set $y = x \in Y$. We can apply the theorem of Hahn-Banach 2 to obtain $y' \in Y'$ with $y'(y) \neq 0$ and $y'|_U = 0$. □

**Theorem 7.** *Let $(X, \|\cdot\|)$ be a normed $\mathbb{F}$ vector space, $U \subset X$ and $V \subset X'$. Then we have*

$$V_\perp \subset X \textit{ is a closed linear subspace}$$

*Proof.* We will focus on $V_\perp \subset X'$. The other statement works analogously.
We first prove $V_\perp$ is not empty: TODO.
Let $x'_1, x'_2 \in X'$ and $\lambda \in \mathbb{F}$. We first prove $V_\perp \subset X'$ is a linear subspace:

$$\begin{aligned}(x'_1 + \lambda x'_2)(x) &= x'_1(x) + \lambda x'_2(x) && \text{(substitute } x \text{ in)}\\ &= 0 + \lambda 0 = 0 && \text{(use } x'_1, x'_2 \in X'\text{)}\end{aligned}$$

Let $(x'_k)_{k\in\mathbb{N}} \subset V_\perp$ be a converging sequence.
We now prove that $x'$ converges in $V_\perp$:

*todo*

□

In summary, TODO.

**Theorem 8.** *Let $T \in \mathcal{L}(X, Y)$ be a bounded, linear operator. Then we have*

$$\overline{\operatorname{Im} T} = (\operatorname{Ker} T')_{\perp}$$

▷ In linear algebra this is usually presented for finite-dimensional vector spaces (see Satz 6.1.5 [**Wer18**])

*Proof.* "$\subset$": Let $Tx \in \operatorname{Im} T$ with $x \in X$ and $y' \in \operatorname{Ker} T'$.
We first prove $Tx \in (\operatorname{Ker} T')_{\perp}$:

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \operatorname{Ker} T'\text{)} \end{aligned}$$

Since this holds for all choices of $Tx$ we get

$$\operatorname{Im} T \subset (\operatorname{Ker} T')_{\perp}$$

Since $(\operatorname{Ker} T')_{\perp}$ is closed, we also get

$$\overline{\operatorname{Im} T} \subset (\operatorname{Ker} T')_{\perp}$$

"$\supset$": We can prove the contraposition

$$\left(Y \setminus \overline{\operatorname{Im} T}\right) \subset \left(Y \setminus (\operatorname{Ker} T')_{\perp}\right)$$

Set $U = \overline{\operatorname{Im} T}$ and let $y \in Y \setminus U$. We know $U$ that a closed linear subspace. The corollary of the theorem of Hahn-Banach 6 tells us that

$$\exists y' \in Y' : y'|_U = 0 \land \ y'(y) \neq 0$$

Since $\operatorname{Ker} T' \subset Y'$ and

$$\forall y' \in \operatorname{Ker} T' : y'(y) \neq 0$$

we get

$$y \in Y \setminus (\operatorname{Ker} T')_{\perp}$$

□

**Corollary 2.** *Let $T \in \mathcal{L}(X, Y)$ be a linear, continuous operator with $\operatorname{Im} T$ closed. Then we have*

$$y \in \operatorname{Im} T \text{ if and only if } T'y' = 0 \Rightarrow y'(y) = 0$$

*Proof.* We have

$$\begin{aligned} y \in \operatorname{Im} T &= \overline{\operatorname{Im} T} && \text{(}\operatorname{Im} T \text{ is closed)} \\ &= (\operatorname{Ker} T')_{\perp} && \text{(apply theorem 8)} \\ \iff & \forall y' \in \operatorname{Ker} T' : y'(y) = 0 && \text{(apply def. of annihilator)} \\ \iff & \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0 && \text{(write in equivalent way)} \end{aligned}$$

□

**Example 5.** In nonlinear optimisation theory, the lemma of Farkas states [**harks˙lecture˙2026**]:
Let $A \in \mathbb{R}^{k \times n}$ and $d \in \mathbb{R}^k$. Then exactly one of the following statements is true:

- $\exists x \geq 0 : Ax = d$
- $\exists \lambda \in \mathbb{R}^k : \lambda A \geq 0 \land \ \lambda d < 0$

The lemma is equivalent to the ability to separate two convex sets using a hyperplane. It is used to prove the KKT-theorem which allows for nonlinear optimisation with constraints.

We can write down an equivalent version of Farkas lemma:
Let $A \in \mathbb{R}^{k \times n}$ and $d \in \mathbb{R}^k$. Then the following statements are equivalent:

- $\exists x \geq 0 : Ax = d$
- $\forall \lambda \in \mathbb{R}^k : A^T \lambda^T < 0 \lor \lambda d \geq 0$

Let $T = A$ and $T' = A^T$. This makes the lemma match the statement of corollary 2.

# References

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