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Chair of Functional Analysis

Dual Operators

Seminar Functional Analysis

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Abstract: This proseminar paper provides a foundational overview of dual-/adjoint operators in a general Banach space setting.

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1 Introduction

1.1 Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as A^T over \mathbb{R} or A^H over \mathbb{C}) to operators on normed \mathbb{R} or \mathbb{C} vector spaces:

We will cover the following topics:

1. The operator \cdot^H is linear and isometric wrt. the spectral norm.
2. The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem: $(\text{Im } A)^\perp = \text{Ker } A^H$.
3. The Lagrange duality from nonlinear optimisation theory.
4. TODO Add more

1.2 General revision

The terms **adjoint** and **dual** are often used interchangeably. We will standardize on **adjoint**, to avoid unnecessary confusion. Following [Wer18], we write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ when the field is unspecified.

Definition 1. We remind ourselves of the following concepts from the lecture [KP25]:

- i) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces and let

$$T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call T a **linear operator**. We call T **bounded**, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation

$$T : X \rightarrow Y$$

for linear operators rather than

$$T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$$

- ii) Let $(X, \|\cdot\|_X)$ be a normed \mathbb{F} -vector. The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

Revision 1. *The following statements are foundational for this topic:*

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces.

- i) *The set of continuous, linear operators $\mathcal{L}(X, Y)$ is a Banach space if and only if Y is a Banach space. In particular, the topological dual space $\mathcal{L}(X, \mathbb{F})$ is a Banach space.*
- ii) *Let $T : X \rightarrow Y$ be a linear operator. Then T is bounded if and only if T is continuous.*
- iii) *Let $(Z, \|\cdot\|_Z)$ be a normed \mathbb{F} -vector space, let $T : X \rightarrow Y$ be a linear, bounded operator and let $S : Y \rightarrow Z$ be a linear, bounded operator. Then $S \circ T$ is a linear, bounded operator.*
- iv) *Let $(Z, \|\cdot\|_Z)$ be a normed \mathbb{F} -vector space, let $T : X \rightarrow Y$ be a linear operator and let $S : Y \rightarrow Z$ be a linear operator. If T or S is compact, $S \circ T$ is compact.*

Proof. Please refer to the functional analysis lecture notes [KP25] from SS/2025. □

2 The adjoint operator

2.1 The basic definitions and conventions

Definition 2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces and $T \in \mathcal{L}(X, Y)$. Then $T' : Y' \rightarrow X'$, $y' \mapsto y' \circ T$ is called the adjoint operator. From now on, we will implicitly refer to the normed \mathbb{F} vector spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T' .

Remark 1. In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from \mathbb{F} . For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$.

Example 1.

2.2 The basic properties

The proofs will use the Hahn-Banach corollaries a couple of times. So first, we need to recall Hahn-Banach related theorems:

Revision 2 (Hahn-Banach). *Let $(X, \|\cdot\|)$ be a normed space and $0 \neq x \in X$.*

Then we have

$$\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|$$

Proof. Please refer to the functional analysis lecture notes [KP25] from SS/2025. \square

Revision 3. *TODO (Explain):*

Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space and $x \in X$.

Then we have

$$\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$$

Proof. We will distinguish two cases:

- Case 1 ($x = 0$): Since X' contains linear operators,

$$\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$$

- Case 2 $x \neq 0$: We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

Using Hahn-Banach from revision 2 we get

$$\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|_X$$

So we get

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$$

\square

Theorem 1. *The adjoint operator has the following properties:*

- i) $T' \in \mathcal{L}(Y', X')$, so T' is linear and bounded.
This implies $\forall y' \in \mathcal{L}(Y', X') : T'y' \in X'$.

- ii) $T \mapsto T'$ is linear and isometric.

Proof. "i)": Let $y' \in Y'$. Plugging it into the adjoint operator, we get $T'y' = y' \circ T$ with signature $X \rightarrow Y \rightarrow \mathbb{F}$. We can now see that $\text{Im } T' \subset X'$.

Let $y'_1, y'_2 \in Y'$, $\alpha \in \mathbb{F}$. We then prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of adjoint operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + T'y'_2 && \text{(apply def. of adjoint operator)} \end{aligned}$$

Let $y' \in Y'$. We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X,Y)} && \text{(apply def. of op. norm)} \\ &:= C\|y'\|_{Y'} && \text{(def. the constant)} \end{aligned}$$

"ii)": Let $T_1, T_2 \in \mathcal{L}(X, Y)$, $y' \in Y'$, $x \in X$, $\alpha \in \mathbb{F}$. We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\ &= y'(\alpha T_1 x + T_2 x) && \text{(pull } x \text{ into the eq.)} \\ &= \alpha y'(T_1 x) + y'(T_2 x) && \text{(y' is linear)} \end{aligned}$$

We then prove isometry:

$$\begin{aligned} \|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\ &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && \text{(apply theorem of Hahn-Banach 2)} \\ &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && \text{(supremum order can be switched)} \\ &= \sup_{\|y'\| \leq 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\ &= \|T'\| && \text{(apply def. of op. norm)} \end{aligned}$$

□

Example 2.

Example 3. Let $p \in (1, \infty)$ with $p \neq 2$. This makes l_p a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of l_p is isometrically isomorphic to l_p^* where p^* is the Hölder conjugate with $1/p + 1/p^* = 1$.

As a reminder, the general idea of the proof of $l'_p \cong l_p^*$ goes as follows:

1. Define the isometric isomorphism as

$$T : l_p^* \rightarrow l'_p, s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$$

2. Verify $(Ts)x$ converges as

$$|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$$

and absolute convergence implies convergence.

3. Verify T is injective using the linearity.

4. Verify T is surjective and isometric through todo (the annoying part).

To illustrate the adjoint operator, we now work through an example. Consider the left shift operator

$$T : l_p \rightarrow l_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$$

It is well-defined since

$$\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$$

We can now compute the adjoint operator T' :

- The adjoint T' must have the signature $l'_p \cong l_p^* \rightarrow l'_p \cong l_p^*$.
- Let $y' \in l'_p \cong l_p^*$. Then we can write $y' : l_p \rightarrow \mathbb{F}, x \mapsto \sum_{k \in \mathbb{N}} x_k s_k$ with $s \in l_p^*$. Now for $x \in l_p$ we have

$$\begin{aligned} (T'y')(x) &= (y' \circ T)(x) = y'(Tx) && \text{(apply def. of } T') \\ &= y'((x_{k+1})_{k \in \mathbb{N}}) && \text{(apply def. of } T) \\ &= \sum_{k \in \mathbb{N}} x_{k+1} s_k && \text{(apply def. of } y') \\ &= \sum_{k \in \mathbb{N}} x_k s'_k && \text{(with } s'_1 = 0 \text{ and } s'_k = s_{k-1} \text{ for } k > 1) \end{aligned}$$

This tells us that the adjoint operator T' acts as a **right shift** (up to isomorphism):

$$T' : l_p^* \rightarrow l_p^*, (s_k)_{k \in \mathbb{N}} \mapsto (0, s_1, s_2, \dots)$$

Theorem 2. Let X, Y, Z be normed \mathbb{F} vector spaces.

Then the adjoint operator reverses composition:

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

Proof. Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.

We know $ST = S \circ T$ is still a linear, bounded operator from X to Z . So $(ST)'$ is well-defined. Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$. We have

$$\begin{aligned} (ST)'(z') &= z' \circ (ST) && (\text{apply def. of adjoint operator}) \\ &= (z' \circ S) \circ T && (\text{write out chain explicitly}) \\ &= y' \circ T && (\text{subst. } y') \\ &= T'y' && (\text{apply def. of adjoint operator}) \\ &= T'(z' \circ S) && (\text{subst. } y') \\ &= T'S'z' && (\text{apply def. of adjoint operator}) \end{aligned}$$

So in total $(ST)' = T'S'$. □

Example 4.

2.3 The dual space of the dual space

For starters, we need to recall some concepts from the lecture.

Definition 3. Let X be a normed \mathbb{F} vector space.

- i) X'' is called the bidual space.
- ii) Let $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$. J_X is called the **canonical embedding** from X into X'' . So J_X returns a function that evaluates dual space elements at $p \in X$.

Figure [todo] illustrates why the bidual space in conjunction with the adjoint operator is interesting.
TODO

Theorem 3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Then we have: $J_Y \circ T = T'' \circ J_X$.

Or equivalently: $T''|_{J_X(X)} = J_Y \circ T \circ J_X|_{J_X(X)}^{-1}$.

Or equivalently: $T = J_Y|_{J_Y(Y)}^{-1} \circ T'' \circ J_X$.

Proof. Before the proof, we can avoid confusion by typing out what T' and T'' evaluate to:

- $T' : Y' \rightarrow X', y' \mapsto y' \circ T$ is the adjoint operator.
- $T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

- $J_Y : Y \rightarrow Y''$ and $T : X \rightarrow Y$ means $(J_Y \circ T) : X \rightarrow Y''$.
- $T'' : X'' \rightarrow Y''$ and $J_X : X \rightarrow X''$ means $(T'' \circ J_X) : X \rightarrow Y''$.

Finally, for $p \in X$ and $y' \in Y'$ we have

$$\begin{aligned} ((J_Y \circ T)(p))(y') &= J_Y(Tp)(y') = y'(Tp) && (\text{subst. } p \text{ in and apply def. of } J_Y) \\ &= (y'T)(p) && (\text{use associativity}) \\ &= (T'y')(p) && (\text{apply def. of } T') \\ &= (x' \mapsto x'(p))(T'y') && (\text{pull out subst. function}) \\ &= J_X(p)(T'y') && (\text{recognize this is just } J_X) \\ &= T''(J_X(p))(y') && (\text{apply def. use } T'') \\ &= ((T'' \circ J_X)(p))(y') && (\text{use } \circ \text{ notation}) \end{aligned}$$

□

Secondly, we can answer when a continuous operator between Y' and X' is an adjoint operator. We need to revise an important corollary from the lecture first.

Revision 4. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space. Then we have

- i) The canonical embedding J_X is an isometric injective function.
- ii) $J : X \rightarrow J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism.
- iii) $J_X|_{J_X(X)}^{-1}$ is a bounded, linear operator.

Proof Idea. "i)": Please refer to the functional analysis lecture notes [KP25] from SS/2025.

"ii)": Follows by definition of J and from "i)".

"iii)": Since $J : X \rightarrow J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism, we have

- $J_X|_{J_X(X)}^{-1}$ inherits linearity from J .
- $\|J\| = 1$ and therefore $\|J_X|_{J_X(X)}^{-1}\| = 1$. This means $J_X|_{J_X(X)}^{-1}$ is bounded.

□

Theorem 4. Let $S \in \mathcal{L}(Y', X')$ be a continuous, linear operator.

Then we have $\exists T \in \mathcal{L}(X, Y) : T' = S \iff S'(J_X(X)) \subset J_Y(Y)$.

Proof. " \Rightarrow ": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && (\text{substitute in } S = T') \\ &= J_Y(T(X)) && (\text{use theorem 3}) \\ &\subset J_Y(Y) && (\text{use } T(X) \subset Y) \end{aligned}$$

" \Leftarrow ": We have $S'(J_X(X)) \subset J_Y(Y)$ and revision 4 gives us $J_Y(Y) \cong Y$.

So for $x \in X$ and $y''_x = S'(J_X(x))$ there is a (unique) $y_x \in Y$ with $y''_x = J_Y(y_x)$.

Define $T : X \rightarrow Y, x \mapsto y_x$. We know T exists (and is unique) due to the previous argument.

We know T is linear and continuous, as $T = y. = J_Y|_{J_Y(Y)}^{-1} \circ S' \circ J_X$ and all elements in the chain are bounded, linear operators. Let $y' \in Y'$ and $x \in X$. Lastly, we need to prove that $S = T'$:

$$\begin{aligned} (Sy')(x) &= J_X(x)(Sy') && (\text{express using } J_X) \\ &= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && (\text{apply def. of adjoint op.}) \\ &= J_Y((J_Y|_{J_Y(Y)}^{-1} \circ S' \circ J_X)(x))(y') && (\text{use } J_Y \circ J_Y|_{J_Y(Y)}^{-1} = \text{Id}) \\ &= y'((J_Y|_{J_Y(Y)}^{-1} \circ S' \circ J_X)(x)) && (\text{evaluate } J_Y \text{ at the vector}) \\ &= y'(Tx) = (y' \circ T)(x) && (\text{subst. in } T = J_Y|_{J_Y(Y)}^{-1} \circ S' \circ J_X) \\ &= (T'y')(x) && (\text{apply def. of adjoint op.}) \end{aligned}$$

□

Example 5. We have already seen an adjunct operator. Using the last theorem, we can find an example for an operator $S \in \mathcal{L}(Y', X')$ that is not an adjoint operator
(i.e. with $\nexists T \in \mathcal{L}(X, Y) : T' = S$).

TODO.

Theorem 5. Lastly, we get one more property of the adjoint: $T \mapsto T'$ is not always surjective.
This is not the case with \mathbb{R}^H e.g. between $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Proof. The operator and the adjoint operator have the following signatures:

- $T : X \rightarrow Y$
- $T' : Y' = \mathcal{L}(Y, \mathbb{F}) \rightarrow X' = \mathcal{L}(X, \mathbb{F})$

So the "not always" refers to a particular choice of X and Y we need to find. Fortunately in example 5 we already found a counterexample!

□

2.4 Compact (adjoint) operators

For starters, we need to recall and revise some concepts from the lecture.

Definition 4. The following definitions are revisions from the lecture:
Let X, Y be normed \mathbb{F} vector spaces.

- i) X is relatively compact, if

$$\forall (x_n)_{n \in \mathbb{N}} \subset X \text{ bounded} : (x_n)_{n \in \mathbb{N}} \text{ has a converging subsequence in } Y$$

- ii) $T \in \mathcal{L}(X, Y)$ is compact, if

$$T(B_X) \text{ is relatively compact}$$

- iii) The rank of T is $\text{rk } T = \dim T(X)$.

As a reminder, in metric spaces a subset is compact if and only if all sequences contain a converging subsequence in that set. So relatively compact is a relaxed version of compactness: it dispenses of the closed requirement. An important property of the closed unit ball in \mathbb{R}^n or \mathbb{C} is that it is compact. Therefore, a linear operator between $\mathbb{F}^n \rightarrow \mathbb{F}^m$ always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ have the same open sets. In finite dimensions, linear algebra tells us that $(X, \|\cdot\|_1)$ and $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$ have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space \mathbb{R}^n . In \mathbb{R}^n , the theorem of Heine-Borel tells us that all bounded subsets are relatively compact. Since relatively compact is a topological property, the theorem transfers to $(X, \|\cdot\|)$. Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general Banach spaces.

Definition 5. The following definitions are critical for the theorem of [Arzelà-Ascoli](#):
Let X, Y be metric spaces and $M \subset \{f : X \rightarrow Y\}$.

- i) M is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

- ii) M is pointwise bounded, if

$$\forall x \in X : \{f(x) \mid f \in M\} \text{ is bounded}$$

The theorem of [Arzelà-Ascoli](#) from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions [evaluations](#) rather than the functions themselves. They can be easily verified to be true. Note that $C(D) = \{f : D \rightarrow \mathbb{F} \text{ continuous}\}$.

Revision 5 (Arzelà-Ascoli). Let D be a compact metric space and $M \subset C(D)$ with the supremum norm. Then we have M is uniformly equicontinuous and pointwise bounded implies M is relatively compact.

Proof Idea. 1. D is separable, i.e. $\exists D_0 \subset D : \overline{D_0} = D$. Simply set $D_0 = \bigcup_{n \in \mathbb{N}} D_n$ where D_n is a finite $\frac{1}{n}$ -covers of D .

2. For all $x \in D$ we can use the pointwise boundedness to invoke Bolzano-Weierstraß to find converging subsequences $(f_{n(x,k)}(x))_{k \in \mathbb{N}} \subset \mathbb{F}$.
3. Specifically, we have $\forall x \in D_0 : (f_{n(x,k)}(x))_{k \in \mathbb{N}}$ converges.
4. Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on x : $\forall x \in D_0 : (f_{n(k)}(x))_{k \in \mathbb{N}}$ converges.

5. We already have $f_{n(k)}|_{D_0} \rightarrow f|_{D_0}$. And it seems sensible to assume that $(f_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence. But how can you extend this to the entire set D ?
6. For each $x \in D$, we can choose an arbitrarily close $x_0 \in D_0$. Using two triangle inequalities, uniform equicontinuity allows us to extend the result to D .
As $C(D)$ is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

Theorem 6 (Schauder). *Let X, Y be Banach spaces and $T : X \rightarrow Y$ be a bounded, linear operator. Then we have T is compact if and only if T' is compact.*

Proof. TODO the proof here is (slightly) still wrong bc of the way the f 's are defined! But conceptually, should be good! „⇒“: Let $(y'_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$ be bounded.

Our goal is to show that there is a convergent subsequence in $(T'y'_n)_{n \in \mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm.

Let $K = B_X$ and for all $n \in \mathbb{N}$ set $f_n = (y'_n \circ T)|_B \in \mathcal{L}(X, \mathbb{F})$. Then we have

$$\begin{aligned} \|f - f_m\|_\infty &= \sup_{\|x\|_X=1} ((y' \circ T) - (y'_m \circ T))x && (\text{apply def. of } f_n \text{ and } f_m) \\ &= \|y' \circ T - y'_m \circ T\| && (\text{use supremum char. of operator norm}) \end{aligned}$$

This tells us that convergence in the operator norm only cares about the behavior on the closed unit ball! Let $D = \overline{T(B_X)}$. We know

1. B_X is bounded.
2. T is a compact operator.

So TB_X is relatively compact and D is compact.

We can now pack our sequence into $M := \{f_n \mid n \in \mathbb{N}\}$ and examine it for

1. Pointwise boundedness: For $x \in B_X$ and $n \in \mathbb{N}$

$$\begin{aligned} |f_n(x)| &= |y'_n(Tx)| := |y'_n(d)| && (\text{apply def. of } f_n \text{ and define } d) \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 && (\text{apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq C_2) \end{aligned}$$

2. Uniform equicontinuity: For $n \in \mathbb{N}, \epsilon > 0, \delta = \frac{\epsilon}{C_1 C_2}, \forall x, y \in K, \|x - y\|_X < \delta$

$$\begin{aligned} \|f_n(x) - f_n(y)\|_Y &\leq \|y'_n \circ Tx - y'_n \circ Ty\|_Y && (\text{apply def. of } f_n) \\ &\leq \|y'_n\| \cdot \|T(x - y)\|_Y && (\text{factor out and apply op. norm ineq.}) \\ &\leq \|y'_n\| \cdot \|T\| \cdot \|x - y\|_X && (\text{apply op. norm ineq.}) \\ &\leq C_1 C_2 \|x - y\|_X < C_1 C_2 \delta = \epsilon && (\text{apply bounded and substitute in } \delta) \end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that M is relatively compact. So every sequence in M has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k \in \mathbb{N}}$ we have

$$\begin{aligned} (f_{n(k)})_{k \in \mathbb{N}} &= (y'_{n(k)} \circ T)_{k \in \mathbb{N}} && (\text{apply def. of } f_{n(k)}) \\ &= (T' y'_{n(k)})_{k \in \mathbb{N}} && (\text{apply def. of adjoint operator}) \end{aligned}$$

So finally, $(T' y'_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence of $(T' y'_k)_{k \in \mathbb{N}}$.

„⇐“: The other proof direction tells us that T compact $\Rightarrow T'$ compact.
So in extension this also yields T' compact $\Rightarrow T''$ compact.

Theorem 3 tells us that $T = J_Y^{-1} \circ T'' \circ J_X$.

And revision 1 tells us that T is also compact, as it is a composition of at least one compact operator - in this case T'' .

□

Example 6.

2.5 The rank-nullity theorem for operators

Definition 6. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ and $V \subset X'$. Then define

- i) $U^\perp = \{x' \in X' \mid \forall x \in U : x'(x) = 0\}$ as the annihilator of U in X' .
- ii) $V_\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$ as the annihilator of V in X .

In linear algebra, the annihilator is isomorphic to the orthogonal complement of a set. Similarly we generalize the idea of a dual structure isomorphic to the image or kernel to idk... todo

To prove properties about these sets, we need another corollary of the theorem of Hahn-Banach:

Revision 6. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ a closed subspace and $x \in X \setminus U$. Then we have $\exists x' \in X' : x'|_U = 0 \wedge x'(x) \neq 0$.

Proof. TODO □

Theorem 7. Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ and $V \subset X'$. Then we have

- i) $U^\perp \subset X'$ and $V_\perp \subset X$ are both closed linear subspaces of their respective supsets.
- ii) Let U be closed. Then we have $(X/U)' \cong U^\perp$ (they are isomorphic).
- iii) Let U be closed. Then we have $U' \cong X'/U^\perp$ (they are isomorphic).

Proof. "i)": We will focus on $U^\perp \subset X'$. The other statement works analogously. We first prove U^\perp is not empty: TODO.

Let $x'_1, x'_2 \in X'$ and $\lambda \in \mathbb{F}$. We first prove $U^\perp \subset X'$ is a linear subspace:

$$\begin{aligned} (x'_1 + \lambda x'_2)(x) &= x'_1(x) + \lambda x'_2(x) && \text{(substitute } x \text{ in)} \\ &= 0 + \lambda 0 = 0 && \text{(use } x'_1, x'_2 \in X') \end{aligned}$$

Let $(x'_k)_{k \in \mathbb{N}} \subset U^\perp$ be a converging sequence.

We now prove that x' converges in U^\perp :

todo

"ii)":

"iii)":

□

In summary, TODO.

Theorem 8. Let $T \in \mathcal{L}(X, Y)$ be a bounded, linear operator. Then we have $\overline{\text{Im } T} = (\text{Ker } T')^\perp$.

Proof. „ \subset “: Let $Tx \in \text{Im } T$ with $x \in X$ and $y' \in \text{Ker } T'$.

We first prove $Tx \in (\text{Ker } T')^\perp$:

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of Tx we get $\text{Im } T \subset (\text{Ker } T')^\perp$.

Since $(\text{Ker } T')^\perp$ is closed, we also get $\overline{\text{Im } T} \subset (\text{Ker } T')^\perp$.

„ \supset “: TODO □

Corollary 1. Let $T \in \mathcal{L}(X, Y)$ be a linear, continuous operator with $\text{Im } T$ closed. Then we have $Tx = y$ has a solution if and only if $T'y' = 0 \Rightarrow y'(y) = 0$.

Proof. □

Example 7.

References

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