

# Dual Operators

## Seminar Functional Analysis

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# Outline

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## 2 The adjoint operator

- The basic definitions and conventions
- The basic properties
- The dual space of the dual space
- Compact (adjoint) operators
- The rank-nullity theorem for operators

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# Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as  $A^T$  over  $\mathbb{R}$  or  $A^H$  over  $\mathbb{C}$ ) to operators on normed  $\mathbb{R}$  or  $\mathbb{C}$  vector spaces.

We will cover the following topics:

The general version of the operator  $\cdot^H$  is linear and isometric wrt. the spectral norm.

The general version of the fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem:  $(\text{Im } A)^\perp = \text{Ker } A^H$ .

How the adjoint operation behaves on compact operators.

A characterisation of  $\exists x \in X : Tx = y$ .

# Terminology

The terms **adjoint** and **dual** are often used interchangeably. We will standardize to **adjoint**, to avoid unnecessary confusion. Following [Wer18], we write  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  when the field is unspecified.

# Definition: Linear Operator

## Definition 1

We remind ourselves of the following concepts from the lecture [KP25]: Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  and  $(Z, \|\cdot\|_Z)$  be normed  $\mathbb{F}$ -vector spaces

i) Let

$$T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call  $T$  a **linear operator**. We call  $T$  **bounded**, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation  $T : X \rightarrow Y$  for linear operators rather than  $T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$ .

ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

# Revision: Foundational Statements

## Revision

The following statements are foundational for this topic:

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$ -vector spaces.

- i) The set of continuous, linear operators  $\mathcal{L}(X, Y)$  is a Banach space if and only if  $Y$  is a Banach space. In particular, the topological dual space  $\mathcal{L}(X, \mathbb{F})$  is a Banach space.
- ii) Let  $T : X \rightarrow Y$  be a linear operator. Then  $T$  is bounded if and only if  $T$  is continuous.
- iii) Let  $(Z, \|\cdot\|_Z)$  be a normed  $\mathbb{F}$ -vector space, let  $T : X \rightarrow Y$  be a linear, bounded operator and let  $S : Y \rightarrow Z$  be a linear, bounded operator. Then  $S \circ T$  is a linear, bounded operator.
- iv) Let  $(Z, \|\cdot\|_Z)$  be a normed  $\mathbb{F}$ -vector space, let  $T : X \rightarrow Y$  be a linear operator and let  $S : Y \rightarrow Z$  be a linear operator. If  $T$  or  $S$  is compact,  $S \circ T$  is compact.

# Definition: Adjoint Operator

## Definition 2

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$ -vector spaces and  $T \in \mathcal{L}(X, Y)$ . Then  $T' : Y' \rightarrow X', y' \mapsto y' \circ T$  is called the adjoint operator.

## Remark

In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from  $\mathbb{F}$ . For  $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$ , the adjoint operator evaluates to  $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$ .

# Example: Dual Space of $\mathbb{R}^n$

## Example 3

Let  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned}(\mathbb{R}^n)' &= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear, continuous}\} && (\text{apply def. of dual space}) \\&= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear}\} && (\text{result from linear algebra}) \\&\cong \mathbb{R}^n && (\text{result from linear algebra see [FS25]})\end{aligned}$$

# Example: Adjoint of Matrix Operator

## Example 4

Let  $m \in \mathbb{N}$  and let  $A \in \mathbb{R}^{m \times n}$ .

Consider the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $x \mapsto Ax$ . The signature of the adjoint is  $f' : (\mathbb{R}^m)' \rightarrow (\mathbb{R}^n)'$ . With  $e_i$  we denote the standard basis vectors and with  $e'_i$  we denote the dual standard basis vectors.

Let  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . We have

$$\begin{aligned}(f'e'_i)(e_j) &= (e'_i \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\ &= f(e_j)_i = a_{ij} && \text{(apply def. of } e'_i \text{ and } f\text{)}\end{aligned}$$

So if we set  $i = 1$  for instance, we get  $(f'e'_1)(e_j) = a_{1j}$ . The first "column" of  $f'$  (up to isomorphism) must be  $(a_{1j})_{j=1,\dots,n}$ . In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \rightarrow \mathbb{R}^n, x \mapsto A^T x$$

# Revision: Hahn-Banach and Dual Unit Ball

## Revision (Hahn-Banach)

Let  $(X, \|\cdot\|_X)$  be a normed space and  $0 \neq x \in X$ .  
Then we have  $\exists f \in X' : \|f\|_{X'} = 1 \wedge f(x) = \|x\|_X$

## Revision

The dual unit ball retains information about the norm of primal vectors:  
Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space and  $x \in X$ .  
Then we have  $\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$

# Theorem: Properties of Adjoint Operator

## Theorem 5

*The adjoint operator has the following properties:*

- i)  $T' \in \mathcal{L}(Y', X')$ , so  $T'$  is linear and bounded.
  - ▷ This implies  $\forall y' \in Y' : T'y' \in X'$ .
- ii)  $T \mapsto T'$  is linear and isometric.

# Theorem: Properties of Adjoint Operator – Proof (i)

## Proof

"i)": Let  $y' \in Y'$ . Plugging it into the adjoint operator, we get  $T'y' = y' \circ T$  with signature  $X \rightarrow Y \rightarrow \mathbb{F}$ . We can now see that  $\text{Im } T' \subset X'$ .

Let  $y'_1, y'_2 \in Y'$ ,  $\alpha \in \mathbb{F}$ . We then prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of adjoint operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + T'y'_2 && \text{(apply def. of adjoint operator)} \end{aligned}$$

Let  $y' \in Y'$ . We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X, Y)} && \text{(apply def. of op. norm)} \\ &:= C \|y'\|_{Y'} && \text{(def. the constant)} \end{aligned}$$

## Theorem: Properties of Adjoint Operator – Proof (ii) Linearity

### Proof

"ii)": Let  $T_1, T_2 \in \mathcal{L}(X, Y)$ ,  $y' \in Y'$ ,  $x \in X$ ,  $\alpha \in \mathbb{F}$ . We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\ &= \alpha y'(T_1 x) + y'(T_2 x) && \text{(\(y'\) is linear)} \\ &= (\alpha T'_1 y' + T'_2 y')(x) \end{aligned}$$

# Theorem: Properties of Adjoint Operator – Proof (ii) Isometry

Proof.

We then prove isometry:

$$\begin{aligned}\|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && (\text{use supremum char. of op. norm}) \\ &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && (\text{apply Hahn-Banach corollary}) \\ &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && (\text{supremum order can be switched}) \\ &= \sup_{\|y'\| \leq 1} \|T'y'\| && (\text{apply def. of adjoint operator and op. norm}) \\ &= \|T'\|\end{aligned}$$



## Example: Shift Operator (1/3)

### Example 6

Let  $p \in (1, \infty)$  with  $p \neq 2$ . This makes  $l_p$  a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of  $l_p$  is isometrically isomorphic to  $l_{p^*}$  where  $p^*$  is the Hölder conjugate with  $1/p + 1/p^* = 1$ .

As a reminder, the general idea of the proof of  $l'_p \cong l_{p^*}$  goes as follows:

Define the isometric isomorphism as  $T : l_{p^*} \rightarrow l'_p, s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$

Verify  $(Ts)x$  converges as  $|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$  and absolute convergence implies convergence.

Verify  $T$  is injective using the linearity.

Verify  $T$  is surjective and isometric (the long part). See [Wer18] for a full proof.

## Example: Shift Operator (2/3)

### Example (cont.)

To illustrate the adjoint operator, we now work through an example. Consider the [left shift](#) operator  $T : I_p \rightarrow I_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$ . It is well-defined since  $\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$ .

We can now compute the adjoint operator  $T'$ : The adjoint  $T'$  must have the signature  $I'_p \cong I_{p^*} \rightarrow I'_p \cong I_{p^*}$ .

## Example: Shift Operator (3/3)

### Example (cont.)

Let  $y' \in I_p' \cong I_{p^*}$ . Then we can write  $y' : I_p \rightarrow \mathbb{F}, x \mapsto \sum_{k \in \mathbb{N}} x_k s_k$  with  $s \in I_{p^*}$ . Now for  $x \in I_p$  we have

$$\begin{aligned} (T'y')(x) &= (y' \circ T)(x) = y'(Tx) && (\text{apply def. of } T') \\ &= y'((x_{k+1})_{k \in \mathbb{N}}) && (\text{apply def. of } T) \\ &= \sum_{k \in \mathbb{N}} x_{k+1} s_k && (\text{apply def. of } y') \\ &= \sum_{k \in \mathbb{N}} x_k s'_k && (\text{with } s'_1 = 0 \text{ and } s'_k = s_{k-1} \text{ for } k > 1) \end{aligned}$$

This tells us that the adjoint operator  $T'$  acts as a **right shift** (up to isomorphism):  
 $T' : I_{p^*} \rightarrow I_{p^*}, (s_k)_{k \in \mathbb{N}} \mapsto (0, s_1, s_2, \dots)$

# Theorem: Adjoint Reverses Composition

## Theorem 7

Let  $X, Y, Z$  be normed  $\mathbb{F}$  vector spaces.

Then the adjoint operator reverses composition:

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

# Theorem: Adjoint Reverses Composition – Proof

Proof.

Let  $T \in \mathcal{L}(X, Y)$  and  $S \in \mathcal{L}(Y, Z)$ .

We know  $ST = S \circ T$  is still a linear, bounded operator from  $X$  to  $Z$ . So  $(ST)'$  is well-defined.

Let  $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$  and set  $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$ . We have

$$\begin{aligned} (ST)'(z') &= z' \circ (ST) && \text{(apply def. of adjoint operator)} \\ &= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\ &= y' \circ T && \text{(subst. } y'\text{)} \\ &= T'y' && \text{(apply def. of adjoint operator)} \\ &= T'(z' \circ S) && \text{(subst. } y'\text{)} \\ &= T'S'z' && \text{(apply def. of adjoint operator)} \end{aligned}$$

So in total  $(ST)' = T'S'$ .

□

## Example: Composition Rule

### Example 8

Referring back to the example with  $A \in \mathbb{R}^{m \times n}$ , the adjoint operator reverses composition rule is compatible with the way matrix transposition on  $\mathbb{R}$  or the matrix adjoint on  $\mathbb{C}$  behave.

# Definition: Bidual Space

## Definition 9

Let  $X$  be a normed  $\mathbb{F}$  vector space.

- i)  $X''$  is called the bidual space.
- ii) Let  $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$ .  $J_X$  is called the **canonical embedding** from  $X$  into  $X''$ .

So  $J_X$  returns a function that evaluates dual space elements at  $p \in X$ .

# Figure: Dual of Dual

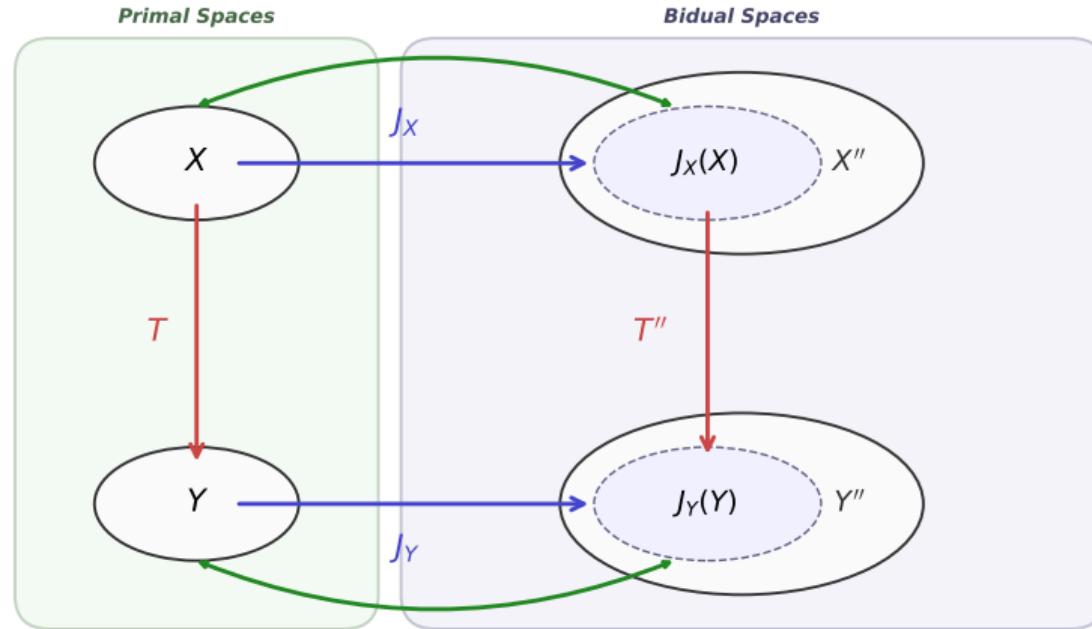


Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

# Theorem: Bidual Adjoint Embedding

## Theorem 10

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$  vector spaces. Let  $T \in \mathcal{L}(X, Y)$  be a bounded linear operator. Then we have:  $J_Y \circ T = T'' \circ J_X$ .

# Theorem: Bidual Adjoint Embedding – Proof Setup

## Proof

Before the proof, we can avoid confusion by typing out what  $T'$  and  $T''$  evaluate to:

$T' : Y' \rightarrow X', y' \mapsto y' \circ T$  is the adjoint operator.

$T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$  is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

$J_Y : Y \rightarrow Y''$  and  $T : X \rightarrow Y$  means  $(J_Y \circ T) : X \rightarrow Y''$ .

$T'' : X'' \rightarrow Y''$  and  $J_X : X \rightarrow X''$  means  $(T'' \circ J_X) : X \rightarrow Y''$ .

# Theorem: Bidual Adjoint Embedding – Proof Calculation

Proof.

Finally, for  $p \in X$  and  $y' \in Y'$  we have

$$\begin{aligned} ((J_Y \circ T)(p))(y') &= J_Y(Tp)(y') = y'(Tp) && (\text{subst. } p \text{ in and apply def. of } J_Y) \\ &= (y'T)(p) && (\text{use associativity}) \\ &= (T'y')(p) && (\text{apply def. of } T') \\ &= (x' \mapsto x'(p))(T'y') && (\text{pull out subst. function}) \\ &= J_X(p)(T'y') && (\text{recognize this is just } J_X) \\ &= T''(J_X(p))(y') && (\text{apply def. use } T'') \\ &= ((T'' \circ J_X)(p))(y') && (\text{use } \circ \text{ notation}) \end{aligned}$$



# Revision: Canonical Embedding Properties

## Revision

Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space. Then we have

- i) The canonical embedding  $J_X$  is an isometric injective function.
- ii)  $J : X \rightarrow J_X(X), x \mapsto J_X(x)$  is an isometric isomorphism.
- iii)  $J_X$  is a bounded, linear operator.
- iv)  $(J_X|_{x \rightarrow J_X(x)})^{-1}$  is a bounded, linear operator.

# Revision: Canonical Embedding Properties – Proof

## Proof

- i) Please refer to the functional analysis lecture notes [KP25] from SS/2025.
- ii) Follows by definition of  $J$  and from "i)".
- iii) Since  $J : X \rightarrow J_X(X)$ ,  $x \mapsto J_X(x)$  is an isometric isomorphism, we have
  - $J_X$  is linear since the  $x' \in X'$  are linear.
  - $\|J_X\| = 1$ . This means  $J_X$  is bounded.
- iv) Since  $J : X \rightarrow J_X(X)$ ,  $x \mapsto J_X(x)$  is an isometric isomorphism, we have
  - $(J_X|_{x \rightarrow J_X(x)})^{-1}$  inherits linearity from  $J$ .
  - $\|J_X\| = 1$  and therefore  $\|(J_X|_{x \rightarrow J_X(x)})^{-1}\| = 1$ . This means  $(J_X|_{x \rightarrow J_X(x)})^{-1}$  is bounded.

# Corollary: Strengthened Bidual Adjoint Embedding

## Corollary 11

We can strengthen the theorem using the isomorphism:

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed  $\mathbb{F}$  vector spaces. Let  $T \in \mathcal{L}(X, Y)$  be a bounded linear operator. Then we have

- i)  $J_Y \circ T = T'' \circ J_X$ .
- ii)  $T''|_{J_X(X)} = J_Y \circ T \circ (J_X|_{X \rightarrow J_X(X)})^{-1}$
- iii)  $T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$ .

## Proof.

"i)": This is the statement of the theorem.

"ii)": The isomorphism from the revision gives us the result.

"iii)": The isomorphism from the revision gives us the result.



# Figure: Strengthened Bidual Adjoint Embedding

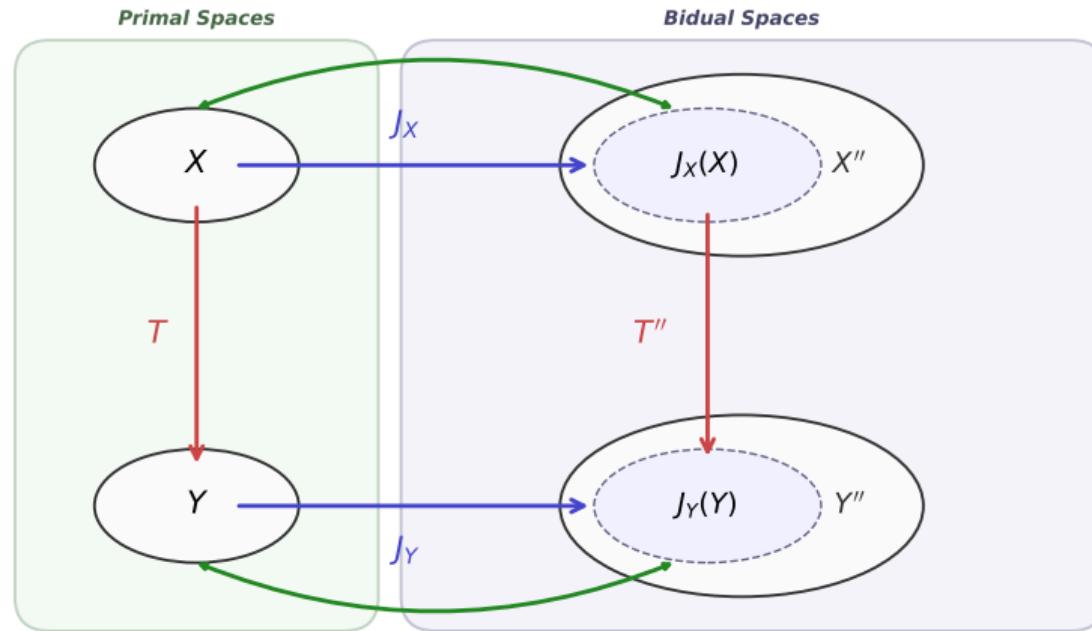


Figure 2: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

# Theorem: Characterization of Adjoint Operators

## Theorem 12

Let  $S \in \mathcal{L}(Y', X')$  be a continuous, linear operator. Then we have

$$\exists T \in \mathcal{L}(X, Y) : T' = S \text{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

# Theorem: Characterization of Adjoint Operators – Proof ( $\Rightarrow$ )

## Proof

" $\Rightarrow$ ": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T') \\ &= J_Y(T(X)) && \text{(use the theorem)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y) \end{aligned}$$

# Theorem: Characterization of Adjoint Operators – Proof ( $\Leftarrow$ ) Setup

## Proof (cont.)

" $\Leftarrow$ ": We have  $S'(J_X(X)) \subset J_Y(Y)$  and the revision gives us  $J_Y(Y) \cong Y$ . So for  $x \in X$  and  $y''_x = S'(J_X(x))$  there is a (unique)  $y_x \in Y$  with  $y''_x = J_Y(y_x)$ . Choose  $T : X \rightarrow Y, x \mapsto y_x$ . We know  $T$  exists (and is unique) due to the previous argument. We know  $T$  is linear and continuous, as  $T = y. = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X$  and all elements in the chain are bounded, linear operators.

# Theorem: Characterization of Adjoint Operators – Proof ( $\Leftarrow$ ) $S = T'$

Proof.

Let  $y' \in Y'$  and  $x \in X$ . Lastly, we need to prove that  $S = T'$ :

$$\begin{aligned}(Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X\text{)} \\&= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of adjoint op.)} \\&= J_Y(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} = \text{Id}) \\&= y'(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\&= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X\text{)} \\&= (T'y')(x) && \text{(apply def. of adjoint op.)}\end{aligned}$$



# Definition: Relatively Compact, Compact Operator, Rank

## Definition 13

The following definitions are revisions from the lecture:

Let  $X, Y$  be normed  $\mathbb{F}$  vector spaces.

- i)  $M \subset X$  is relatively compact, if  $\forall (x_n)_{n \in \mathbb{N}} \subset M : (x_n)_{n \in \mathbb{N}}$  has a converging subsequence in  $X$
- ii)  $T \in \mathcal{L}(X, Y)$  is compact, if  $T(\overline{B}_X)$  is relatively compact.
- iii) The rank of  $T$  is  $\text{rk } T = \dim T(X)$ .

# Definition: Uniformly Equicontinuous and Pointwise Bounded

## Definition 14

The following definitions are critical for the theorem of [Arzelà-Ascoli](#):

Let  $X, Y$  be metric spaces and  $M \subset \{f : X \rightarrow Y\}$ .

i)  $M$  is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

ii)  $M$  is pointwise bounded, if  $\forall x \in X : \{f(x) \mid f \in M\}$  is bounded

## Revision (Arzelà-Ascoli)

Let  $D$  be a compact metric space and  $M \subset C(D)$  with the supremum norm. Then we have  $M$  is uniformly equicontinuous and pointwise bounded implies  $M$  is relatively compact.

# Theorem: Schauder

## Theorem 15 (Schauder)

*Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  be a bounded, linear operator. Then we have  $T$  is compact if and only if  $T'$  is compact.*

# Theorem: Schauder – Proof ( $\Rightarrow$ ) Setup

## Proof

" $\Rightarrow$ ": Let  $(y'_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$  be bounded.

Our goal is to show that there is a convergent subsequence in  $(T'y'_n)_{n \in \mathbb{N}}$  with respect to  $(X', \|\cdot\|)$  where  $\|\cdot\|$  is the operator norm. For all  $n \in \mathbb{N}$  set  $f_n = y'_n|_{T(\overline{B}_X)}$ .

We have

$$\begin{aligned} \|T'y'_n - T'y'_m\| &= \|y'_n \circ T - y'_m \circ T\| && \text{(apply def. of adjoint op.)} \\ &= \sup_{\|x\|_X \leq 1} |((y'_n \circ T) - (y'_m \circ T))(x)| && \text{(use supremum char. of the op. norm)} \\ &= \sup_{\|x\|_X \leq 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m) \\ &= \sup_{d \in T(\overline{B}_X)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m) \end{aligned}$$

# Theorem: Schauder – Proof ( $\Rightarrow$ ) Setup for Arzelà-Ascoli

## Proof (cont.)

We know  $(T'y'_n)_{n \in \mathbb{N}}$  converges if and only if it is a Cauchy sequence  $(\mathcal{L}(X, \mathbb{F}), \|\cdot\|)$ . Therefore the convergence of  $(T'y'_n)_{n \in \mathbb{N}}$  in the operator norm is only dependent on the behaviour of  $(f_n)_{n \in \mathbb{N}}$  on  $\overline{T(\bar{B}_X)}$ .

We now set  $D = \overline{T(\bar{B}_X)}$  and pack the sequence into  $M := \{f_n \mid n \in \mathbb{N}\}$  and examine them for the conditions of Arzelà-Ascoli.

Theorem: Schauder – Proof ( $\Rightarrow$ )  $D$  is compact,  $M$  is pointwise bounded

### Proof (cont.)

$D$  is compact: We have  $\overline{B}_X$  is bounded and  $T$  is a compact operator. So  $T(\overline{B}_X)$  is relatively compact and  $D$  is compact.

$M$  is pointwise bounded: For  $x \in \overline{B}_X$  and  $n \in \mathbb{N}$  we have

$$|f_n(Tx)| = |y'_n(Tx)| := |y'_n(d)| \quad (\text{apply def. of } f_n \text{ and define } d)$$

$$\leq C_1 \|d\|_Y \leq C_1 C_2 \quad (\text{apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq C_2)$$

For  $d \in D$  we can choose  $(x_k)_{k \in \mathbb{N}} \subset \overline{B}_X$  with  $Tx_k \rightarrow d$  and  $|f_n(Tx_k)| \leq C_1 C_2$ .

Using continuity we get  $|f_n(d)| \leq C_1 C_2$ .

# Theorem: Schauder – Proof ( $\Rightarrow$ ) $M$ is uniformly equicontinuous

## Proof (cont.)

$M$  is uniformly equicontinuous: For  $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon/C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$  we have

$$\begin{aligned} |f_n(d_1) - f_n(d_2)| &\leq \|y'_n\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)} \\ &\leq C_1 \|d_1 - d_2\|_Y && \text{(use the upper bound } \|y'_n\| \leq C_1) \\ &< C_1 \delta = \epsilon && \text{(substitute in } \delta) \end{aligned}$$

# Theorem: Schauder – Proof ( $\Rightarrow$ ) Conclusion

## Proof (cont.)

The theorem of Arzelà-Ascoli now tells us that  $M$  is relatively compact. So every sequence in  $M$  has a convergent subsequence. In particular for the convergent subsequence  $(f_{n(k)})_{k \in \mathbb{N}}$  we have

$$\begin{aligned}(f_{n(k)} \circ T)_{k \in \mathbb{N}} &= (y'_{n(k)} \circ T)_{k \in \mathbb{N}} && (\text{apply def. of } f_{n(k)}) \\ &= (T'y'_{n(k)})_{k \in \mathbb{N}} && (\text{apply def. of adjoint operator})\end{aligned}$$

So finally,  $(T'y'_{n(k)})_{k \in \mathbb{N}}$  is a convergent subsequence of  $(T'y'_k)_{k \in \mathbb{N}}$ .

# Theorem: Schauder – Proof ( $\Leftarrow$ )

## Proof.

" $\Leftarrow$ ": The other proof direction tells us that  $T$  compact  $\Rightarrow T'$  compact So in extension this also yields  $T'$  compact  $\Rightarrow T''$  compact The corollary tells us that

$T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$  Lastly, the operator  $T''$  is compact, the revision tells us  $J_X$  and  $(J_Y|_{Y \rightarrow J_Y(Y)})^{-1}$  are bounded, linear operators and therefore  $T$  is a composition of bounded, linear operators.

Using the revision we can conclude that  $T$  is compact. □

# Example: Compact Integral Operator

## Example 16

The lecture [KP25] states that the following integral operator is compact:

$$T : C[0, 1] \rightarrow C[0, 1], T f(x) = \int_0^1 f(t) dx$$

So  $T' : C[0, 1]' \rightarrow C[0, 1]'$  is compact too. Evaluating  $C[0, 1]'$  is beyond the scope of this paper.

# Definition: Annihilator

## Definition 17

Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space and  $V \subset X'$ . Then we define the annihilator of  $V$  in  $X$  as

$$V^\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$$

- ▷ The annihilator is the set of linear, bounded functionals that "see" exactly the opposite of  $V$  and are "blind" to  $V$ .

# Revision: Hahn-Banach Corollary

## Revision

Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space,  $U \subset X$  a closed subspace and  $x \in X \setminus U$ . Then we have

$$\exists x' \in X' : x'|_U = 0 \wedge x'(x) \neq 0$$

# Theorem: Annihilator is Closed Linear Subspace

## Theorem 18

Let  $(X, \|\cdot\|)$  be a normed  $\mathbb{F}$  vector space and  $V \subset X'$ . Then we have

$V^\perp \subset X$  is a closed linear subspace

## Proof.

We have  $V^\perp = \bigcap_{x' \in V} (x')^{-1}(0)$  As an intersection of closed sets,  $V^\perp$  must be closed. □

# Theorem: Rank-Nullity Generalized

## Theorem 19

Let  $T \in \mathcal{L}(X, Y)$  be a bounded, linear operator. Then we have

$$\overline{\text{Im } T} = (\text{Ker } T')_{\perp}$$

- ▷ In linear algebra lectures this is proven for finite-dimensional vector spaces (see Satz 6.1.5 [Wer18])

# Theorem: Rank-Nullity Generalized – Proof ( $\subset$ )

## Proof

" $\subset$ ": Let  $Tx \in \text{Im } T$  with  $x \in X$  and  $y' \in \text{Ker } T'$ .

We first prove  $Tx \in (\text{Ker } T')^\perp$ :

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of  $Tx$  we get  $\text{Im } T \subset (\text{Ker } T')^\perp$ . Since  $(\text{Ker } T')^\perp$  is closed, we also get  $\overline{\text{Im } T} \subset (\text{Ker } T')^\perp$

# Theorem: Rank-Nullity Generalized – Proof ( $\supset$ )

Proof.

" $\supset$ ": We can prove the contraposition  $(Y \setminus \overline{\text{Im } T}) \subset (Y \setminus (\text{Ker } T')^\perp)$ . Set  $U = \overline{\text{Im } T}$  and let  $y \in Y \setminus U$ . We know  $U$  that a closed linear subspace. The corollary of the theorem of Hahn-Banach tells us that  $\exists y' \in Y' : y'|_U = 0 \wedge y'(y) \neq 0$ . Since  $\text{Ker } T' \subset Y'$  and  $\forall y' \in \text{Ker } T' : y'(y) = 0$  we get  $y \in Y \setminus (\text{Ker } T')^\perp$  □

# Corollary: Operator Solutions

## Corollary 20

Let  $T \in \mathcal{L}(X, Y)$  be a linear, continuous operator with  $\text{Im } T$  closed. Then we have

$$y \in \text{Im } T \text{ if and only if } \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0$$

## Proof.

We have

$$\begin{aligned} y \in \text{Im } T &= \overline{\text{Im } T} && (\text{Im } T \text{ is closed}) \\ &= (\text{Ker } T')_{\perp} && (\text{apply the theorem}) \\ \iff \forall y' \in \text{Ker } T' &: y'(y) = 0 && (\text{apply def. of annihilator}) \\ \iff \forall y' \in Y' &: T'y' = 0 \Rightarrow y'(y) = 0 && (\text{write in equivalent way}) \end{aligned}$$



## Example: Left Shift

### Example 21

Consider the left shift operator  $T : I_p \rightarrow I_p$  from earlier. Its adjoint  $T' : I_{p^*} \rightarrow I_{p^*}$  is the right shift operator:  $T'(s_1, s_2, \dots) = (0, s_1, s_2, \dots)$

To apply the theorem, we first find  $\text{Ker } T'$ . If  $T's = 0$ , then  $(0, s_1, s_2, \dots) = (0, 0, 0, \dots)$ , which implies  $s = 0$ . Thus,  $\text{Ker } T' = \{0\}$ .

The theorem states that  $\overline{\text{Im } T} = (\text{Ker } T')^\perp = \{0\}^\perp = I_p$ . This is consistent with the fact that the left shift is surjective ( $\text{Im } T = I_p$ ).

## Example: Right Shift

### Example 22

Consider the right shift operator  $T : I_p \rightarrow I_p$ ,  $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ . Its image is the closed subspace  $\text{Im } T = \{y \in I_p \mid y_1 = 0\}$ .

The adjoint  $T' : I_{p^*} \rightarrow I_{p^*}$  is the left shift,  $T'(s_1, s_2, s_3, \dots) = (s_2, s_3, \dots)$ . Thus  $\text{Ker } T' = \{(s_1, 0, 0, \dots) \mid s_1 \in \mathbb{F}\} = \text{span}\{e'_1\}$ .

By the corollary,  $y \in \text{Im } T$  iff it is annihilated by every functional in  $\text{Ker } T'$ . For any  $s \in \text{Ker } T'$ , the condition  $s(y) = 0$  becomes:  $\sum_{k=1}^{\infty} y_k s_k = y_1 s_1 = 0$

Since this must hold for all  $s_1 \in \mathbb{F}$ , we must have  $y_1 = 0$ . This matches  $\text{Im } T$ , verifying the theorem.

## References

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- [KP25] David Krieg and Joscha Prochno. *Functional Analysis – Lecture Notes UoP*. 2025. (Visited on 07/22/2025).
- [Wer18] Dirk Werner. *Funktionalanalysis*. Springer-Lehrbuch. Berlin, Heidelberg: Springer Berlin Heidelberg, 2018. ISBN: 9783662554067 9783662554074. DOI: 10.1007/978-3-662-55407-4. URL: <http://link.springer.com/10.1007/978-3-662-55407-4> (visited on 01/20/2026).

# Summary

Defined the adjoint operator  $T' : Y' \rightarrow X'$  for bounded linear operators

Showed  $T \mapsto T'$  is linear and isometric

Proved the adjoint reverses composition:  $(ST)' = T'S'$

Characterized when an operator  $S : Y' \rightarrow X'$  is an adjoint via the canonical embedding

Proved Schauder's theorem:  $T$  compact  $\iff T'$  compact

Generalized the rank-nullity theorem:  $\overline{\text{Im } T} = (\text{Ker } T')^\perp$