

Dual Operators

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Abstract

This proseminar paper provides a foundational overview of dual-/adjoint operators in functional analysis.

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1 Introduction

1.1 Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as A^T over \mathbb{R} or A^H over \mathbb{C}) to general operators on banach spaces.

We will cover and generalize the following topics:

1. The operator \cdot^H is linear and isometric wrt. the spectral norm.
2. The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem:
 $(\text{Im } A)^\perp = \text{Ker } A^H$.
3. The lagrange duality from nonlinear optimisation theory.
4. TODO Add more

1.2 Revision

Definition 1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces on \mathbb{R} or \mathbb{C} .
Let $T : X \rightarrow Y$ be linear.

T is a **linear operator**. T is **bounded**, if $\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$.

Definition 2. The (topological) dual space is defined as $X' := \mathcal{L}(X, \mathbb{R})$.

Theorem 1. *The topological dual space $\mathcal{L}(X, Y)$ is a Banach space.*

Proof Idea.

□

Theorem 2. *For linear operators, continuous and bounded are equivalent.*

Proof Idea. TOOD

□

Definition 3. i):

ii):

Theorem 3. i): *Let $p \in [1, \infty)$. Then*

ii):

Proof Idea. TOOD

□

Theorem 4.

Proof. See functional analysis lecture of SS/25.

□

2 The adjoint operator

2.1 Definitions and basic properties

Remark 1. The terms **adjoint** and **dual** are often used interchangeably. We will standardize on **adjoint**, to avoid unnecessary confusion. We will abstract the field as $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 4. Let X, Y be metric spaces and $T \in \mathcal{L}(X, Y)$.

Then $T' : Y' \rightarrow X', y' \mapsto y' \circ T$ is called the adjoint operator. From now on, we will implicitly refer to the metric spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T' .

Remark 2. In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from \mathbb{F} . For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F})$.

First, we need to recall a corollary of the Hahn-Banach theorems:

Revision 1. Let $(X, \|\cdot\|)$ be a normed vector space and $x \in X$.

Then we have $\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$.

Proof. Case 1 $x = 0$: Since X' contains linear operators, $\|x\|_X = 0 = \sup_{f \in X', \|f\| \leq 1} |f(0)|$.
Case 2 $x \neq 0$: We have

$$\sup_{f \in X', \|f\| \leq 1} |f(x)| \leq \sup_{f \in X', \|f\| \leq 1} \|f\| \|x\|_X \leq 1 \|x\|_X = \|x\|_X$$

By Hahn-Banach, we have $\exists f \in X' : \|f\| = 1 \wedge |f(x)| = \|x\|_X$.

So we get $\sup_{f \in X', \|f\| \leq 1} |f(x)| = \|x\|_X$.

□

Theorem 5. We can collect some basic properties of the adjoint operator:

i): $T' \in \mathcal{L}(Y', X')$, so T' is linear and bounded.

This implies $\forall y' \in \mathcal{L}(Y', X') : T'y' \in X'$.

ii): $T \mapsto T'$ is linear and isometric.

iii): $T \mapsto T'$ is not always surjective.

Proof. "i)": We first prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of dual operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + \beta T'y'_2 && \text{(apply def. of dual operator)} \\ &\quad \text{for all } y'_1, y'_2 \in Y' \text{ and } \alpha \in \mathbb{F} \end{aligned}$$

We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of dual operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X, Y)} && \text{(Cauchy-Schwarz inequality)} \\ &:= C \|y'\|_{Y'} && \text{(def. the constant)} \\ &\quad \text{for all } y' \in Y' \end{aligned}$$

"ii)": We first prove linearity:

$$\begin{aligned}
 (\alpha T_1 + T_2)'(y') &= y'(\alpha T_1 + T_2) && \text{(apply def. of dual operator)} \\
 &= \alpha y'(T_1) + y'(T_2) && \text{(y' is linear)} \\
 &\quad \text{for all } T_1, T_2 \in \mathcal{L}(X, Y) \text{ and } y' \in Y' \text{ and } \alpha \in \mathbb{F}
 \end{aligned}$$

We then prove isometry:

$$\begin{aligned}
 \|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && \text{(use supremum char. of op. norm)} \\
 &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && \text{(todo consult script)} \\
 &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && \text{(supremum order can be switched)} \\
 &= \sup_{\|y'\| \leq 1} \|T'y'\| && \text{(apply def. of adjoint operator and op. norm)} \\
 &= \|T'\| && \text{(apply def. of op. norm)}
 \end{aligned}$$

"iii)": We construct a counterexample:

□

Example 1.

Revision 2. Let X, Y, Z be normed metric spaces.

Then the adjoint operator reverses composition:

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'.$$

Proof. Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.

We know $ST = S \circ T$ is still a linear, bounded operator from X to Z . So $(ST)'$ is well-defined.

Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$.

We can now evaluate the expression on $z' \in Z'$:

$$\begin{aligned}
 (ST)'(z') &= z' \circ (ST) && \text{(apply def. of dual operator)} \\
 &= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\
 &= y' \circ T && \text{(subst. y')} \\
 &= T'y' && \text{(apply def. of dual operator)} \\
 &= T'(z' \circ S) && \text{(subst. y')} \\
 &= T'S'z' && \text{(apply def. of dual operator)}
 \end{aligned}$$

So in total, $(ST)' = T'S'$.

□

Example 2.

2.2 The dual space of the dual space

For starters, we need to recall some concepts from the lecture.

Definition 5. Let X be a normed vector space.

- i): X'' is called the bidual space.
- ii): Let $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$. J_X is called the [canonical embedding](#) from X into X'' .

So J_X returns a function that evaluates dual space elements at $p \in X$.

Theorem 6. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator and $T' \in \mathcal{L}(Y', X')$ the adjoint operator.

Then we have: $J_Y \circ T = T'' \circ J_X$.

Proof. Before the proof, we can avoid confusion by typing out what T' and T'' evaluate to:

- $T' : Y' \rightarrow X', y' \mapsto y' \circ T$ is the adjoint operator.
- $T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

- $J_Y : Y \rightarrow Y''$ and $T : X \rightarrow Y$ means $(J_Y \circ T) : X \rightarrow Y''$.
- $T'' : X'' \rightarrow Y''$ and $J_X : X \rightarrow X''$ means $(T'' \circ J_X) : X \rightarrow Y''$.

Finally, for $p \in X$ and $y' \in Y'$ we have

$$\begin{aligned}
 [(J_Y \circ T)(p)](y') &= J_Y(Tp)(y') = y'(Tp) && (\text{subst. } p \text{ in and apply def. of } J_Y) \\
 &= (y'T)(p) && (\text{use associativity}) \\
 &= (T'y')(p) && (\text{apply def. of } T') \\
 &= (x' \mapsto x'(p))(T'y') && (\text{pull out subst. function}) \\
 &= J_X(p)(T'y') && (\text{recognize this is just } J_X) \\
 &= T''(J_X(p))(y') && (\text{apply def. use } T'') \\
 &= [(T'' \circ J_X)(p)](y') && (\text{use } \circ \text{ notation})
 \end{aligned}$$

□

Example 3.

2.3 Compact (adjoint) operators

For starters, we need to recall and revise some concepts from the lecture.

Definition 6. The following definitions are revisions from the lecture:
Let X, Y be normed metric spaces.

- i): X is relatively compact, if $\forall(x_n)_{n \in \mathbb{N}} \subset X$ bounded : $(Tx_n)_{n \in \mathbb{N}}$ has a converging subsequence.
- ii): $T \in \mathbb{N}$ is compact, if $TB(0, \leq 1, X)$ is relatively compact.
- iii): The rank of T is $\text{rk } T = \dim T(X)$.

As a reminder, in metric spaces a subset is compact iff. all sequences contain a converging subsequence in that set. So relatively compact is a relaxed version of compactness: it dispenses of the closed requirement. An important property of the closed unit ball in \mathbb{R}^n or \mathbb{C} is that it is compact. Therefore, a linear operator between $\mathbb{F}^n \rightarrow \mathbb{F}^m$ always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ have the same open sets. In finite dimensions, linear algebra tells us that $(X, \|\cdot\|_1)$ and $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$ have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space \mathbb{R}^n . In \mathbb{R}^n , the theorem of Heine-Borel tells us that all bounded subsets are relatively-compact. Since relatively-compact is a topological property, the theorem transfers to $(X, \|\cdot\|)$. Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively-compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general banach spaces.

Definition 7. The following definitions are critical for the theorem of [Arzelà-Ascoli](#)!
Let X, Y be metric spaces and $M \subset \{f : X \rightarrow Y\}$.

- i): M is uniformly equicontinuous, if
 $\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$.
- ii): M is pointwise bounded, if $\forall x \in X : \{f(x) \mid f \in M\}$ is bounded.

The theorem of [Arzelà-Ascoli](#) from the lecture gives us a characterisation of relative compactness for the set of continuous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinuity and pointwise boundedness are (relatively) simple properties of the functions [evaluations](#) rather than the functions themselves. They can be easily verified to be true.

Revision 3 (Arzelà-Ascoli). Let D be a compact metric space and $M \subset C(D) = \{f : D \rightarrow \mathbb{F} \text{ continuous}\}$ with the supremum norm.

Then we have M is uniformly equicontinuous and pointwise bounded implies M is relatively compact.

Proof Idea. 1. D is separable, i.e. $\exists D_0 \subset D : \overline{D_0} = D$. Simply set $D_0 = \bigcup_{n \in \mathbb{N}} D_n$ where D_n is a finite $\frac{1}{n}$ -covers of D .

2. For all $x \in M$ we can use the pointwise boundedness to invoke Bolzano-Weierstraß to find converging subsequences $(f_{n(x,k)}(x))_{k \in \mathbb{N}} \subset \mathbb{F}$.

3. Specifically, we have $\forall x \in D_0 \cap M =: M_0 : (f_{n(x,k)}(x))_{k \in \mathbb{N}}$ converges.

4. Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on x : $\forall x \in M_0 : (f_{n(k)}(x))_{k \in \mathbb{N}}$ converges.

5. We already have $f_{n(k)}|_{M_0} \longrightarrow f|_{M_0}$. And it seems sensible to assume that $(f_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence. But how can you extend this to the entire set M ?

6. For each $x \in M$, we can choose an arbitrarily close $x_0 \in M_0$. Using two triangle inequalities, uniform equicontinuity allows us to extend the result to M .

As $C(D)$ is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

Theorem 7 (Schauder). *Let X, Y be Banach spaces and $T : X \rightarrow Y$ be a bounded, linear operator. Then we have T is compact iff. T' is compact.*

Proof. „ \Rightarrow “: Let $(y_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$ be bounded.

Our goal is to show that there is a convergent subsequence in $(T'y'_n)_{n \in \mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm.

Let $K = B(0, \leq 1, X)$ and for all $n \in \mathbb{N}$ set $f_n = (y'_n \circ T)|_B \in \mathcal{L}(X, \mathbb{F})$. Then we have

$$\begin{aligned} \|f - f_m\|_\infty &= \sup_{\|x\|_X=1} ((y' \circ T) - (y'_m \circ T))x && (\text{apply def. of } f_n \text{ and } f_m) \\ &= \|y' \circ T - y'_m \circ T\| && (\text{use supremum char. of operator norm}) \end{aligned}$$

This tells us that convergence in the operator norm only cares about the behavior on the closed unit ball! Let $D = \overline{T B(0, \leq 1, X)}$. We have

1. D is an (inherited) metric space.
2. D is closed by definition.
3. $T(B(0, \leq 1, X))$ is a bounded set under a bounded operator, so D is bounded.

Since D is in a metric space, it must be compact.

We can now pack our sequence into $M := \{f_n \mid n \in \mathbb{N}\}$ and examine it for

1. Pointwise boundedness: For $x \in B(0, \leq 1, X)$ and $n \in \mathbb{N}$

$$\begin{aligned} |f_n(x)| &= |y'_n(Tx)| := |y'_n(d)| && (\text{apply def. of } f_n \text{ and define } d) \\ &\leq C_1 \|d\|_Y \leq C_1 C_2 && (\text{apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq: C_2) \end{aligned}$$

2. Uniform equicontinuity: For $n \in \mathbb{N}, \epsilon > 0, \delta = \frac{\epsilon}{C_1 C_2}, \forall x, y \in K, \|x - y\|_X < \delta$

$$\begin{aligned} \|f_n(x) - f_n(y)\|_Y &\leq \|y'_n \circ Tx - y'_n \circ Ty\|_Y && (\text{apply def. of } f_n) \\ &\leq \|y'_n\| \|T(x - y)\|_Y && (\text{factor out and apply op. norm ineq.}) \\ &\leq \|y'_n\| \|T\| \|x - y\|_X && (\text{apply op. norm ineq.}) \\ &\leq C_1 C_2 \|x - y\|_X < C_1 C_2 \delta = \epsilon && (\text{apply bounded and substitute in } \delta) \end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that M is relatively compact. So every sequence in M has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k \in \mathbb{N}}$ we have

$$\begin{aligned} (f_{n(k)})_{k \in \mathbb{N}} &= (y'_n \circ T)_{k \in \mathbb{N}} && (\text{apply def. of } f_{n(k)}) \\ &= (T'y'_{n(k)})_{k \in \mathbb{N}} && (\text{apply def. of adjoint operator}) \end{aligned}$$

So finally, $(T'y'_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence of $(T'y'_k)_{k \in \mathbb{N}}$.

„ \Leftarrow “:

□

Example 4.

2.4 The rank-nullity theorem for operators

Definition 8. Let $(X, \|\cdot\|)$ be a normed vector space, $U \subset X$ and $V \subset X'$. Then define

- i): $U^\perp = \{x' \in X' \mid \forall x \in U : x'(x) = 0\}$ as the annihilator of U in X' .
- ii): $V_\perp = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$ as the annihilator of V in X .

To prove properties about these sets, we need a corollary of the theorem of Hahn-Banach.

Revision 4. TODO: Hahn-Banach corollary.

Proof Idea. TODO

□

Theorem 8. Let $(X, \|\cdot\|)$ be a normed vector space, $U \subset X$ and $V \subset X'$. Then we have

- i): U^\perp and V_\perp are not empty.
- ii): $U^\perp \subset X'$ and $V_\perp \subset X$ are both respectively closed linear subspaces.
- iii): Let U be closed. Then we have $(X/U)' \cong U^\perp$ (they are isomorphic).
- iv): Let U be closed. Then we have $U' \cong X'/U^\perp$ (they are isomorphic).

Proof. "i)":

"ii)": We will focus on $U^\perp \subset X'$. The other statement works analogously.

Let $x'_1, x'_2 \in X'$ and $\lambda \in \mathbb{F}$. We first prove $\emptyset \neq U^\perp \subset X'$ is a linear subspace:

$$\begin{aligned} (x'_1 + \lambda x'_2)(x) &= x'_1(x) + \lambda x'_2(x) && \text{(substitute } x \text{ in)} \\ &= 0 + \lambda 0 = 0 && \text{(use } x'_1, x'_2 \in X') \end{aligned}$$

Let $(x'_k)_{k \in \mathbb{N}} \subset U^\perp$ be a converging sequence.

We now prove that x' converges in U^\perp :

todo

"iii)":

"iv)":

□

In summary, TODO.

Theorem 9. Let $T \in \mathcal{L}(X, Y)$ be a bounded, linear operator. Then we have $\overline{\text{Im } T} = (\text{Ker } T')^\perp$.

Proof. " \subset ": Let $Tx \in \text{Im } T$ with $x \in X$ and $y' \in \text{Ker } T'$.

We first prove $Tx \in (\text{Ker } T')^\perp$:

$$\begin{aligned} y'Tx &= (y' \circ T)(x) && \text{(use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply dual op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of Tx we get $\text{Im } T \subset (\text{Ker } T')^\perp$.

Since $(\text{Ker } T')^\perp$ is closed, we also get $\overline{\text{Im } T} \subset (\text{Ker } T')^\perp$.

" \supset ": TODO

□

Theorem 10. TODO: The stuff with the $\text{Id} - T$ operator.

Proof. TODO

□

Corollary 1. Let $T \in \mathcal{L}(X, Y)$ be a linear, continuous operator with $\text{Im } T$ closed. Then we have $Tx = y$ has a solution iff. $T'y' = 0 \Rightarrow y'(y) = 0$.

Proof.

□

Example 5.

References

- [Noa] Dual operators between Hilbert spaces: with or without Riesz representation. MathOverflow. URL: <https://mathoverflow.net/questions/71040/dual-operators-between-hilbert-spaces-with-or-without-riesz-representation> (visited on 01/20/2026).