Dual Operators

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Abstract

This seminar paper provides a foundational overview of Dual-/Adjunct-Operators in functional analysis.

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1 Motivation

2 Revision

Definition 1. The following basic definitions:

i): Banach Space

ii): Hilbert Space

Definition 2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces on $\mathbb{R}$or $\mathbb{C}$.
Let $T : X \to Y$ be linear.
T is a linear operator. T is bounded, if $\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$.

Definition 3. The (topological) dual space is defined as $X^* := \mathcal{L}(X, \mathbb{R})$.

Theorem 1. *The topological dual space $\mathcal{L}(X, Y)$ is a Banach space.*

Proof Idea. □

Theorem 2. *For linear operators, continous and bounded are equivalent.*

Proof Idea. TOOD □

Definition 4. i):

ii):

Theorem 3. *i): Let $p \in [1, \infty)$. Then*

ii):

Proof Idea. TOOD □

Theorem 4.

Proof. See functional analysis lecture of SS/25. □

3 The Adjunct Operator

3.1 Definitions and Basic Properties

Remark 1. The terms adjunct and dual are often used interchangeably. We will standardize on adjunct, to avoid unnecessary confusion.
We will abstract the field as $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 5. Let X, Y be metric spaces and $T \in \mathcal{L}(X, Y)$.
Then $T' : Y' \to X', y' \mapsto y' \circ T$ is called the adjunct operator. From now on, we will implicitly refer to the metric spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T'.

Remark 2. In comparison to the operators we have previously worked with, the adjunct operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from $\mathbb{F}$.
For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjunct operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F})$.

Theorem 5. *We can collect some basic properties of the dual operator:*

i): $T' \in \mathcal{L}(Y', X')$, *so* T' *is linear and bounded.*
This implies $\forall y' \in \mathcal{L}(Y', X') : T'y' \in X'$.

ii): $T \mapsto T'$ *is linear and isometric.*

iii): $T \mapsto T'$ *is not always surjective.*

Proof. "i)": We first prove linearity:

$$\begin{aligned} T'(\alpha y_1' + y_2') &= (\alpha y_1' + y_2') \circ T && \text{(apply def. of dual operator)} \\ &= \alpha y_1' \circ T + y_2' \circ T && \text{(expand expression)} \\ &= \alpha T' y_1' + \beta T' y_2' && \text{(apply def. of dual operator)} \\ &\text{for all } y_1', y_2' \in Y' \text{ and } \alpha \in \mathbb{F} \end{aligned}$$

We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of dual operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X,Y)} && \text{(Cauchy-Schwarz inequality)} \\ &:= C\|y'\|_{Y'} && \text{(def. the constant)} \\ &\text{for all } y' \in Y' \end{aligned}$$

"ii)": We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y') &= y'(\alpha T_1 + T_2) && \text{(apply def. of dual operator)} \\ &= \alpha y'(T_1) + y'(T_2) && \text{(y' is linear)} \\ &\text{for all } T_1, T_2 \in \mathcal{L}(X, Y) \text{ and } y' \in Y' \text{ and } \alpha \in \mathbb{F} \end{aligned}$$

We then prove isometry:

$$\|T\|$$

"iii)": We construct a counterexample:

□

Example 1.

Revision 1 (The Adjunct Operator reverses composition)**.** *Let* X, Y, Z *be normed metric spaces. Then we have* $\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$.

Proof. Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.
We know $ST = S \circ T$ is still a linear, bounded operator from X to Z. So $(ST)'$ is well-defined.
Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$.
We can now evaluate the expression on z':

$$\begin{aligned}
(ST)'(z') &= z' \circ (ST) && \text{(apply def. of dual operator)} \\
&= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\
&= y' \circ T && \text{(subst. y')} \\
&= T'y' && \text{(apply def. of dual operator)} \\
&= T'(z' \circ S) && \text{(subst. y')} \\
&= T'S'z' && \text{(apply def. of dual operator)}
\end{aligned}$$

So in total, $(ST)' = T'S'$. □

Example 2.

3.2 The Dual Space of the Dual Space

For starters, we need to recall some concepts from the lecture.

Definition 6. Let X be a normed vector space.

i): X'' is called the bidual space.

ii): Let $J_x : X' \to \mathbb{F}, x' \mapsto x'(x)$ and $J : X \to X'', x \mapsto J_x$. J is called the canonical embedding from X into X''.

Revision 2.

Proof. □

Theorem 6 (The Operator, Adjunct Operator and the Canonical Embeddings)**.**

Proof. □

3.3 Relation to Compactness

For starters, we need to recall and revise some concepts from the lecture.

Definition 7. The following definitions are revisions from the lecture:
Let X, Y be normed metric spaces.

i): X is relatively compact, if $\forall (x_n)_{n\in\mathbb{N}} \subset X$ bounded : $(Tx_n)_{n\in\mathbb{N}}$ has a converging subsequence.

ii): $T \in \mathbb{N}$ is compact, if $TB(0, \leq 1, X)$ is relatively compact.

iii): The rank of T is $\operatorname{rk} T = \dim T(X)$.

As a reminder, in metric spaces a subset is compact iff. all sequences contain a converging subsequence in that set. So relatively compact is a relaxed version of compactness: it disposes of the closed requirement. An important property of the closed unit ball in $\mathbb{R}^n$ or $\mathbb{C}$ is that it is compact. Therefore, a linear operator between $\mathbb{F}^n \to \mathbb{F}^m$ always maps the closed unit ball to a compact set. But the domain might not be a finite-dimensional normed space and the unit ball of the domain might not be compact either.

Even when we only know that a bounded operator has finite rank, we can see that it is always compact: In finite dimensions, all norms are equivalent. So $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ have the same open sets. In finite dimensions, linear algebra tells us that $(X, \|\cdot\|_1)$ and $(\mathbb{R}^{\dim X}, \|\cdot\|_1)$ have the same open sets. So all finite-dimensional normed spaces are topologically equivalent to some space $\mathbb{R}^n$. In $\mathbb{R}^n$, the theorem of Heine-Borel tells us that all bounded subsets are relatively-compact. Since relatively-compact is a topological property, the theorem transfers to $(X, \|\cdot\|)$. Finally, a finite rank causes the bounded image of the operator to be finite-dimensional, which then causes it to be relatively-compact. This is a very strong statement, considering that the domain unit ball might not be compact at all.

This finally breaks down when the rank is infinite. So a compact operator simply ensures that we still can enjoy this finite-dimensional behavior between general banach spaces.

Definition 8. The following definitions are critical for the theorem of Arzelà-Ascoli!
Let X, Y be metric spaces and $M \subset \{f : X \to Y\}$.

i): M is uniformly equicontinous, if
$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon.$

ii): M is pointwise bounded, if $\forall x \in X : \{f(x) \mid f \in M\}$ is bounded.

The theorem of Arzelà-Ascoli from the lecture gives us a characterisation of relative compactness for the set of continous functions on a compact metric space using the two previous concepts. This is interesting, because uniform equicontinouity and pointwise boundedness are (relatively) simple properties of the functions evaluations rather than the functions themselves. They can be easily verified to be true.

Revision 3 (Arzelà-Ascoli)**.** *Let D be a compact metric space and $M \subset C(D) = \{f : D \to \mathbb{F}$ continous$\}$ with the supremum norm.*
Then we have M is uniformly equicontinous and pointwise bounded implies M is relatively compact.

Proof Idea.

1. D is seperable, i.e. $\exists D_0 \subset D : \overline{D_0} = D$. Simply set $D_0 = \bigcup_{n\in\mathbb{N}} D_n$ where D_n is a finite $\frac{1}{n}$-covers of D.
2. For all $x \in M$ we can use the pointwise boundedness to invoke Bolzano-Weierstraß to find converging subsequences $(f_{n(x,k)}(x))_{k\in\mathbb{N}} \subset \mathbb{F}$.
3. Specifically, we have $\forall x \in D_0 \cap M =: M_0 : (f_{n(x,k)}(x))_{k\in\mathbb{N}}$ converges.
4. Using the commonly used diagonal argument from the lecture, we can find a unified subsequence with no dependence on x: $\forall x \in M_0 : (f_{n(k)}(x))_{k\in\mathbb{N}}$ converges.
5. We already have $f_{n(k)}|_{M_0} \longrightarrow: f|_{M_0}$. And it seems sensible to assume that $(f_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence. But how can you extend this to the entire set M?
6. For each $x \in M$, we can choose an arbitrarily close $x_0 \in M_0$. Using two triangle inequalities, uniform equicontinouity allows us to extend the result to M.
   As $C(D)$ is complete, the Cauchy sequence converges.

□

Using the revision, we can now prove the theorem of Schauder and then trace the arguments through both proofs to get a better understanding of the ideas.

Theorem 7 (Schauder)**.** *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded, linear operator. Then we have T is compact iff. T' is compact.*

Proof. „$\Rightarrow$“: Let $(y'_n)_{n\in\mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$ be bounded.
Our goal is to show that there is a convergent subsequence in $(T'y'_n)_{n\in\mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm.
Let $K = B(0, \leq 1, X)$ and for all $n \in \mathbb{N}$ set $f_n = (y'_n \circ T)|_B \in \mathcal{L}(X, \mathbb{F})$. Then we have

$$\begin{aligned}\|f - f_m\|_\infty &= \sup_{\|x\|_X = 1} ((y' \circ T) - (y'_m \circ T))x && \text{(apply def. of } f_n \text{ and } f_m) \\ &= \|y' \circ T - y'_m \circ T\| && \text{(use supremum char. of operator norm)}\end{aligned}$$

This tells us that convergence in the operator norm only cares about the behavior on the closed unit ball! Let $D = \overline{TB(0, \leq 1, X)}$. We have

1. D is an (inherited) metric space.
2. D is closed by definition.
3. $T(B(0, \leq 1, X))$ is a bounded set under a bounded operator, so D is bounded.

Since D is in a metric space, it must be compact.

We can now pack our sequence into $M := \{f_n \mid n \in \mathbb{N}\}$ and examine it for

1. Pointwise boundedness: For $x \in B(0, \leq 1, X)$ and $n \in \mathbb{N}$

$$\begin{aligned}|f_n(x)| = |y'_n(Tx)| := |y'_n(d)| && \text{(apply def. of } f_n \text{ and define } d) \\ \leq C_1 \|d\|_Y \leq C_1 C_2 && \text{(apply op. norm ineq. and D bounded means } \|d\|_Y \leq: C_2)\end{aligned}$$

2. Uniform equicontinuity: For $n \in \mathbb{N}, \epsilon > 0, \delta = \frac{\epsilon}{C_1 C_2}, \forall x, y \in K, \|x - y\|_X < \delta$

$$\begin{aligned}\|f_n(x) - f_n(y)\|_Y &\leq \|y'_n \circ Tx - y'_n \circ Ty\|_Y && \text{(apply def. of } f_n) \\ &\leq \|y'_n\| \|T(x - y)\|_Y && \text{(factor out and apply op. norm ineq.)} \\ &\leq \|y'_n\| \|T\| \|x - y\|_X && \text{(apply op. norm ineq.)} \\ &\leq C_1 C_2 \|x - y\|_X < C_1 C_2 \delta = \epsilon && \text{(apply bounded and substitute in } \delta)\end{aligned}$$

The theorem of Arzelà-Ascoli now tells us that M is relatively compact. So every sequence in M has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ we have

$$\begin{aligned}(f_{n(k)})_{k\in\mathbb{N}} &= (y'_n \circ T)_{k\in\mathbb{N}} && \text{(apply def. of } f_{n(k)}) \\ &= (T'y'_{n(k)})_{k\in\mathbb{N}} && \text{(apply def. of adjunct operator)}\end{aligned}$$

So finally, $(T'y'_{n(k)})_{k\in\mathbb{N}}$ is a convergent subsequence of $(T'y'_k)_{k\in\mathbb{N}}$.

„$\Leftarrow$“: $\square$

Example 3.

3.4 Operator Equations