

Dual Operators

Seminar Functional Analysis

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Motivation

The goal of dual- or adjoint operators is to generalize the notion of an adjoint matrix (often denoted as A^T over \mathbb{R} or A^H over \mathbb{C}) to operators on normed \mathbb{R} or \mathbb{C} vector spaces.
We will cover the following topics:

The operator \cdot^H is linear and isometric wrt. the spectral norm.

The fundamental theorem of linear algebra (Gilbert Strang) or rank-nullity theorem:
 $(\text{Im } A)^\perp = \text{Ker } A^H$.

TODO Add more

Terminology

The terms **adjoint** and **dual** are often used interchangeably. We will standardize to **adjoint**, to avoid unnecessary confusion. Following [Wer18], we write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ when the field is unspecified.

Definition: Linear Operator

Definition 1

We remind ourselves of the following concepts from the lecture [KP25]: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed \mathbb{F} -vector spaces

i) Let

$$T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$$

be a linear mapping. Then we call T a **linear operator**. We call T **bounded**, if

$$\exists C > 0, \forall x \in X : \|Tx\|_Y \leq C\|x\|_X$$

For brevity, we will use the notation $T : X \rightarrow Y$ for linear operators rather than $T : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y)$.

ii) The (topological) dual space is defined as

$$X' := \mathcal{L}(X, \mathbb{F})$$

T

Revision

The following statements are foundational for this topic:

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces.

- i) The set of continuous, linear operators $\mathcal{L}(X, Y)$ is a Banach space if and only if Y is a Banach space. In particular, the topological dual space $\mathcal{L}(X, \mathbb{F})$ is a Banach space.
- ii) Let $T : X \rightarrow Y$ be a linear operator. Then T is bounded if and only if T is continuous.
- iii) Let $(Z, \|\cdot\|_Z)$ be a normed \mathbb{F} -vector space, let $T : X \rightarrow Y$ be a linear, bounded operator and let $S : Y \rightarrow Z$ be a linear, bounded operator. Then $S \circ T$ is a linear, bounded operator.
- iv) Let $(Z, \|\cdot\|_Z)$ be a normed \mathbb{F} -vector space, let $T : X \rightarrow Y$ be a linear operator and let $S : Y \rightarrow Z$ be a linear operator. If T or S is compact, $S \circ T$ is compact.

Definition: Adjoint Operator

Definition 2

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} -vector spaces and $T \in \mathcal{L}(X, Y)$. Then $T' : Y' \rightarrow X'$, $y' \mapsto y' \circ T$ is called the adjoint operator. From now on, we will implicitly refer to the normed \mathbb{F} vector spaces X, Y and the topological dual spaces X', Y' when talking about the dual operator T' .

Remark

In comparison to the operators we have previously worked with, the adjoint operator takes and outputs linear, bounded operators. It's one more level of abstraction removed from \mathbb{F} .

For $y' \in Y' = \mathcal{L}(Y, \mathbb{F})$, the adjoint operator evaluates to $T'y' \in \mathcal{L}(X, \mathbb{F}) = X'$.

Example: Dual Space of \mathbb{R}^n

Example 3

Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned}(\mathbb{R}^n)' &= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear, continuous}\} && (\text{apply def. of dual space}) \\&= \{g : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear}\} && (\text{result from linear algebra}) \\&\cong \mathbb{R}^n && (\text{result from linear algebra see [FS25]})\end{aligned}$$

Example: Adjoint of Matrix Operator

Example 4

Let $m \in \mathbb{N}$ and let $A \in \mathbb{R}^{m \times n}$.

Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $x \mapsto Ax$. The signature of the adjoint is $f' : (\mathbb{R}^m)' \rightarrow (\mathbb{R}^n)'$. With e_i we denote the standard basis vectors and with e'_i we denote the dual standard basis vectors.

Let $i = 1, \dots, m$ and $j = 1, \dots, n$. We have

$$\begin{aligned}(f'e'_i)(e_j) &= (e'_i \circ f)(e_j) && \text{(apply def. of adjoint op.)} \\ &= f(e_j)_i = a_{ij} && \text{(apply def. of } e'_i \text{ and } f\text{)}\end{aligned}$$

So if we set $i = 1$ for instance, we get $(f'e'_1)(e_j) = a_{1j}$. The first "column" of f' (up to isomorphism) must be $(a_{1j})_{j=1,\dots,n}$. In conclusion, we have (up to isomorphism):

$$f' : \mathbb{R}^m \rightarrow \mathbb{R}^n, x \mapsto A^T x$$

Revision: Hahn-Banach and Dual Unit Ball

Revision (Hahn-Banach)

Let $(X, \|\cdot\|)$ be a normed space and $0 \neq x \in X$.

Then we have $\exists f \in X' : \|f\| = 1 \wedge f(x) = \|x\|$

Revision

The dual unit ball retains information about the norm of primal vectors:

Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space and $x \in X$.

Then we have $\|x\|_X = \sup_{f \in X', \|f\| \leq 1} |f(x)|$

Theorem: Properties of Adjoint Operator

Theorem 5

The adjoint operator has the following properties:

- i) $T' \in \mathcal{L}(Y', X')$, so T' is linear and bounded.
 - ▷ This implies $\forall y' \in Y' : T'y' \in X'$.
- ii) $T \mapsto T'$ is linear and isometric.

Theorem: Properties of Adjoint Operator – Proof (i)

Proof.

"i)": Let $y' \in Y'$. Plugging it into the adjoint operator, we get $T'y' = y' \circ T$ with signature $X \rightarrow Y \rightarrow \mathbb{F}$. We can now see that $\text{Im } T' \subset X'$.

Let $y'_1, y'_2 \in Y', \alpha \in \mathbb{F}$. We then prove linearity:

$$\begin{aligned} T'(\alpha y'_1 + y'_2) &= (\alpha y'_1 + y'_2) \circ T && \text{(apply def. of adjoint operator)} \\ &= \alpha y'_1 \circ T + y'_2 \circ T && \text{(expand expression)} \\ &= \alpha T'y'_1 + T'y'_2 && \text{(apply def. of adjoint operator)} \end{aligned}$$

Let $y' \in Y'$. We then prove boundedness of the operator norm:

$$\begin{aligned} \|T'y'\|_{X'} &= \|y' \circ T\|_{X'} && \text{(apply def. of adjoint operator)} \\ &\leq \|y'\|_{Y'} \|T\|_{\mathcal{L}(X, Y)} && \text{(apply def. of op. norm)} \\ &:= C \|y'\|_{Y'} && \text{(def. the constant)} \end{aligned}$$

T

Theorem: Properties of Adjoint Operator – Proof (ii) Linearity

Proof.

"ii)": Let $T_1, T_2 \in \mathcal{L}(X, Y)$, $y' \in Y'$, $x \in X$, $\alpha \in \mathbb{F}$. We first prove linearity:

$$\begin{aligned} (\alpha T_1 + T_2)'(y')(x) &= y'(\alpha T_1 x + T_2 x) && \text{(apply def. of adjoint operator)} \\ &= y'(\alpha T_1 x + T_2 x) && \text{(pull } x \text{ into the eq.)} \\ &= \alpha y'(T_1 x) + y'(T_2 x) && \text{(} y' \text{ is linear)} \\ &= (\alpha T'_1 y' + T'_2 y')(x) \end{aligned}$$



Theorem: Properties of Adjoint Operator – Proof (ii) Isometry

Proof.

We then prove isometry:

$$\begin{aligned}\|T\| &= \sup_{\|x\|_X \leq 1} \|Tx\|_Y && (\text{use supremum char. of op. norm}) \\ &= \sup_{\|x\|_X \leq 1} \sup_{\|y'\| \leq 1} |y'(Tx)| && (\text{apply theorem of Hahn-Banach}) \\ &= \sup_{\|y'\| \leq 1} \sup_{\|x\|_X \leq 1} |y'(Tx)| && (\text{supremum order can be switched}) \\ &= \sup_{\|y'\| \leq 1} \|T'y'\| && (\text{apply def. of adjoint operator and op. norm}) \\ &= \|T'\| && (\text{apply def. of op. norm})\end{aligned}$$



Example: Shift Operator (1/3)

Example 6

Let $p \in (1, \infty)$ with $p \neq 2$. This makes l_p a Banach space according to the lecture, but not a Hilbert space as the parallelogram rule is not satisfied. We know that the dual space of l_p is isometrically isomorphic to l_p^* where p^* is the Hölder conjugate with $1/p + 1/p^* = 1$.

As a reminder, the general idea of the proof of $l_p' \cong l_p^*$ goes as follows:

Define the isometric isomorphism as $T : l_p^* \rightarrow l_p', s \mapsto (x \mapsto \sum_{k \in \mathbb{N}} x_k s_k)$

Verify $(Ts)x$ converges as $|(Ts)x| \leq \|x\|_p \|s\|_{p^*} < \infty$ and absolute convergence implies convergence.

Verify T is injective using the linearity.

Verify T is surjective and isometric (the long part). See [Wer18] for a full proof.

Example: Shift Operator (2/3)

Example 7

To illustrate the adjoint operator, we now work through an example. Consider the **left shift** operator $T : I_p \rightarrow I_p, (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}$. It is well-defined since $\sum_{k \in \mathbb{N}} |x_{k+1}|^p \leq \sum_{k \in \mathbb{N}} |x_k|^p < \infty$.

We can now compute the adjoint operator T' : The adjoint T' must have the signature $I'_p \cong I_p^* \rightarrow I'_p \cong I_p^*$.

Example: Shift Operator (3/3)

Example 8

Let $y' \in I_p' \cong I_p^*$. Then we can write $y' : I_p \rightarrow \mathbb{F}, x \mapsto \sum_{k \in \mathbb{N}} x_k s_k$ with $s \in I_p^*$. Now for $x \in I_p$ we have

$$\begin{aligned}(T'y')(x) &= (y' \circ T)(x) = y'(Tx) && (\text{apply def. of } T') \\&= y'((x_{k+1})_{k \in \mathbb{N}}) && (\text{apply def. of } T) \\&= \sum_{k \in \mathbb{N}} x_{k+1} s_k && (\text{apply def. of } y') \\&= \sum_{k \in \mathbb{N}} x_k s'_k && (\text{with } s'_1 = 0 \text{ and } s'_k = s_{k-1} \text{ for } k > 1)\end{aligned}$$

This tells us that the adjoint operator T' acts as a **right shift** (up to isomorphism):
 $T' : I_p^* \rightarrow I_p^*, (s_k)_{k \in \mathbb{N}} \mapsto (0, s_1, s_2, \dots)$



Theorem: Adjoint Reverses Composition

Theorem 9

Let X, Y, Z be normed \mathbb{F} vector spaces.

Then the adjoint operator reverses composition:

$$\forall T \in \mathcal{L}(X, Y), \forall S \in \mathcal{L}(Y, Z) : (S \circ T)' = T' \circ S'$$

Theorem: Adjoint Reverses Composition – Proof

Proof.

Let $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$.

We know $ST = S \circ T$ is still a linear, bounded operator from X to Z . So $(ST)'$ is well-defined.

Let $z' \in Z' = \mathcal{L}(Z, \mathbb{F})$ and set $y' = z' \circ S \in \mathcal{L}(Y, \mathbb{F})$. We have

$$\begin{aligned} (ST)'(z') &= z' \circ (ST) && \text{(apply def. of adjoint operator)} \\ &= (z' \circ S) \circ T && \text{(write out chain explicitly)} \\ &= y' \circ T && \text{(subst. } y') \\ &= T'y' && \text{(apply def. of adjoint operator)} \\ &= T'(z' \circ S) && \text{(subst. } y') \\ &= T'S'z' && \text{(apply def. of adjoint operator)} \end{aligned}$$

So in total $(ST)' = T'S'$.



Example: Composition Rule

Example 10

Referring back to the example with $A \in \mathbb{R}^{m \times n}$, the adjoint operator reverses composition rule is compatible with the way matrix transposition on \mathbb{R} or the matrix adjoint on \mathbb{C} behave.

Definition: Bidual Space

Definition 11

Let X be a normed \mathbb{F} vector space.

- i) X'' is called the bidual space.
- ii) Let $J_X : X \rightarrow X'', p \mapsto (x' \mapsto x'(p))$. J_X is called the **canonical embedding** from X into X'' .

So J_X returns a function that evaluates dual space elements at $p \in X$.

Figure: Dual of Dual

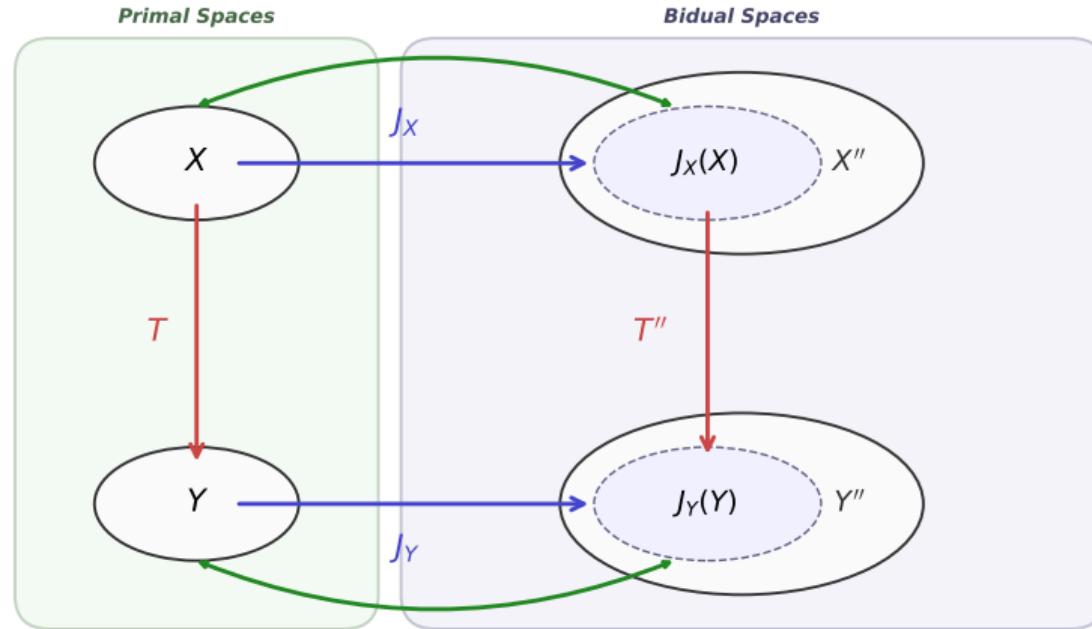


Figure 1: An illustration of the main concepts in the dual of duals chapter. The operator and bidual operator are in **red**. The canonical embeddings are in **blue**. Isomorphisms are in **green**.

Theorem: Bidual Adjoint Embedding

Theorem 12

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Then we have: $J_Y \circ T = T'' \circ J_X$.

Theorem: Bidual Adjoint Embedding – Proof Setup

Proof.

Before the proof, we can avoid confusion by typing out what T' and T'' evaluate to:

$T' : Y' \rightarrow X', y' \mapsto y' \circ T$ is the adjoint operator.

$T'' : X'' \rightarrow Y'', x'' \mapsto x'' \circ T'$ is the biadjoint operator.

Now first of all, we need to check that the signatures of both sides of the equation match:

$J_Y : Y \rightarrow Y''$ and $T : X \rightarrow Y$ means $(J_Y \circ T) : X \rightarrow Y''$.

$T'' : X'' \rightarrow Y''$ and $J_X : X \rightarrow X''$ means $(T'' \circ J_X) : X \rightarrow Y''$.



Theorem: Bidual Adjoint Embedding – Proof Calculation

Proof.

Finally, for $p \in X$ and $y' \in Y'$ we have

$$\begin{aligned} ((J_Y \circ T)(p))(y') &= J_Y(Tp)(y') = y'(Tp) && (\text{subst. } p \text{ in and apply def. of } J_Y) \\ &= (y'T)(p) && (\text{use associativity}) \\ &= (T'y')(p) && (\text{apply def. of } T') \\ &= (x' \mapsto x'(p))(T'y') && (\text{pull out subst. function}) \\ &= J_X(p)(T'y') && (\text{recognize this is just } J_X) \\ &= T''(J_X(p))(y') && (\text{apply def. use } T'') \\ &= ((T'' \circ J_X)(p))(y') && (\text{use } \circ \text{ notation}) \end{aligned}$$



Revision: Canonical Embedding Properties

Revision

Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space. Then we have

- i) The canonical embedding J_X is an isometric injective function.
- ii) $J : X \rightarrow J_X(X), x \mapsto J_X(x)$ is an isometric isomorphism.
- iii) J_X is a bounded, linear operator.
- iv) $(J_X|_{x \rightarrow J_X(x)})^{-1}$ is a bounded, linear operator.

Corollary: Strengthened Bidual Adjoint Embedding

Corollary 13

We can strengthen the theorem using the isomorphism:

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed \mathbb{F} vector spaces. Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator. Then we have

- i) $J_Y \circ T = T'' \circ J_X$.
- ii) $T''|_{X \rightarrow J_X(X)} = J_Y \circ T \circ (J_X|_{X \rightarrow J_X(X)})^{-1}$
- iii) $T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$.

Proof.

"i)": This is the statement of the theorem.

"ii)": The isomorphism from the revision gives us the result.

"iii)": The isomorphism from the revision gives us the result.



Theorem: Characterization of Adjoint Operators

Theorem 14

Let $S \in \mathcal{L}(Y', X')$ be a continuous, linear operator. Then we have

$$\exists T \in \mathcal{L}(X, Y) : T' = S \text{ if and only if } S'(J_X(X)) \subset J_Y(Y)$$

Theorem: Characterization of Adjoint Operators – Proof (\Rightarrow)

Proof.

" \Rightarrow ": We have

$$\begin{aligned} S'(J_X(X)) &= T''(J_X(X)) && \text{(substitute in } S = T') \\ &= J_Y(T(X)) && \text{(use the theorem)} \\ &\subset J_Y(Y) && \text{(use } T(X) \subset Y) \end{aligned}$$



Theorem: Characterization of Adjoint Operators – Proof (\Leftarrow) Setup

Proof.

" \Leftarrow ": We have $S'(J_X(X)) \subset J_Y(Y)$ and the revision gives us $J_Y(Y) \cong Y$.

So for $x \in X$ and $y''_x = S'(J_X(x))$ there is a (unique) $y_x \in Y$ with $y''_x = J_Y(y_x)$.

Choose $T : X \rightarrow Y, x \mapsto y_x$. We know T exists (and is unique) due to the previous argument.

We know T is linear and continuous, as $T = y \cdot = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X$ and all elements in the chain are bounded, linear operators. □

Theorem: Characterization of Adjoint Operators – Proof (\Leftarrow) $S = T'$

Proof.

Let $y' \in Y'$ and $x \in X$. Lastly, we need to prove that $S = T'$:

$$\begin{aligned}(Sy')(x) &= J_X(x)(Sy') && \text{(express using } J_X\text{)} \\&= (S'J_X(x))(y') = (S' \circ J_X)(x)(y') && \text{(apply def. of adjoint op.)} \\&= J_Y(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x))(y') && \text{(use } J_Y \circ (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} = \text{Id}\text{)} \\&= y'(((J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ S' \circ J_X)(x)) && \text{(evaluate } J_Y \text{ at the vector)} \\&= y'(Tx) = (y' \circ T)(x) && \text{(subst. in } T = J_Y^{-1} \circ S' \circ J_X\text{)} \\&= (T'y')(x) && \text{(apply def. of adjoint op.)}\end{aligned}$$



Definition: Relatively Compact, Compact Operator, Rank

Definition 15

The following definitions are revisions from the lecture:

Let X, Y be normed \mathbb{F} vector spaces.

- i) $M \subset X$ is relatively compact, if $\forall (x_n)_{n \in \mathbb{N}} \subset M : (x_n)_{n \in \mathbb{N}}$ has a converging subsequence in X
- ii) $T \in \mathcal{L}(X, Y)$ is compact, if $T(\overline{B}_X)$ is relatively compact.
- iii) The rank of T is $\text{rk } T = \dim T(X)$.

Definition: Uniformly Equicontinuous and Pointwise Bounded

Definition 16

The following definitions are critical for the theorem of [Arzelà-Ascoli](#):

Let X, Y be metric spaces and $M \subset \{f : X \rightarrow Y\}$.

i) M is uniformly equicontinuous, if

$$\forall \epsilon > 0, \exists \delta > 0, \forall T \in M, \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(Tx, Ty) < \epsilon$$

ii) M is pointwise bounded, if $\forall x \in X : \{f(x) \mid f \in M\}$ is bounded

Revision (Arzelà-Ascoli)

Let D be a compact metric space and $M \subset C(D)$ with the supremum norm. Then we have M is uniformly equicontinuous and pointwise bounded implies M is relatively compact.

Theorem: Schauder

Theorem 17 (Schauder)

Let X, Y be Banach spaces and $T : X \rightarrow Y$ be a bounded, linear operator. Then we have T is compact if and only if T' is compact.

Theorem: Schauder – Proof (\Rightarrow) Setup

Proof.

" \Rightarrow ": Let $(y'_n)_{n \in \mathbb{N}} \subset Y' = \mathcal{L}(Y, \mathbb{F}) \subset C(Y)$ be bounded.

Our goal is to show that there is a convergent subsequence in $(T'y'_n)_{n \in \mathbb{N}}$ with respect to $(X', \|\cdot\|)$ where $\|\cdot\|$ is the operator norm. For all $n \in \mathbb{N}$ set $f_n = y'_n|_{T(\overline{B}_X)}$.

We have

$$\begin{aligned} \|T'y'_n - T'y'_m\| &= \|y'_n \circ T - y'_m \circ T\| && \text{(apply def. of adjoint op.)} \\ &= \sup_{\|x\|_X \leq 1} |((y'_n \circ T) - (y'_m \circ T))(x)| && \text{(use supremum char. of the op. norm)} \\ &= \sup_{\|x\|_X \leq 1} |((f_n \circ T) - (f_m \circ T))(x)| && \text{(subst. in } f_n \text{ and } f_m) \\ &= \sup_{d \in T(\overline{B}_X)} |f_n(d) - f_m(d)| && \text{(subst. in } f_n \text{ and } f_m) \end{aligned}$$

T

Theorem: Schauder – Proof (\Rightarrow) Setup for Arzelà-Ascoli

Proof.

We know $(T'y'_n)_{n \in \mathbb{N}}$ converges if and only if it is a Cauchy sequence $(\mathcal{L}(X, \mathbb{F}), \|\cdot\|)$. Therefore the convergence of $(T'y'_n)_{n \in \mathbb{N}}$ in the operator norm is only dependent on the behaviour of $(f_n)_{n \in \mathbb{N}}$ on $\overline{T(\bar{B}_X)}$.

We now set $D = \overline{T(\bar{B}_X)}$ and pack the sequence into $M := \{f_n \mid n \in \mathbb{N}\}$ and examine them for the conditions of Arzelà-Ascoli. □

Theorem: Schauder – Proof (\Rightarrow) D is compact, M is pointwise bounded

Proof.

D is compact: We have \overline{B}_X is bounded and T is a compact operator. So $T(\overline{B}_X)$ is relatively compact and D is compact.

M is pointwise bounded: For $x \in \overline{B}_X$ and $n \in \mathbb{N}$ we have

$$|f_n(Tx)| = |y'_n(Tx)| := |y'_n(d)| \quad (\text{apply def. of } f_n \text{ and define } d)$$

$$\leq C_1 \|d\|_Y \leq C_1 C_2 \quad (\text{apply op. norm ineq. and } D \text{ bounded means } \|d\|_Y \leq C_2)$$

For $d \in D$ we can choose $(x_k)_{k \in \mathbb{N}} \subset \overline{B}_X$ with $Tx_k \rightarrow d$ and $f_n(Tx_k) \leq C_1 C_2$. Using continuity we get $|f_n(d)| \leq C_1 C_2$.



Theorem: Schauder – Proof (\Rightarrow) M is uniformly equicontinuous

Proof.

M is uniformly equicontinuous: For $n \in \mathbb{N}, \epsilon > 0, \delta = \epsilon/C_1, \forall d_1, d_2 \in D, \|d_1 - d_2\|_Y < \delta$ we have

$$\begin{aligned} |f_n(d_1) - f_n(d_2)| &\leq \|y'_n\| \cdot \|d_1 - d_2\|_Y && \text{(factor out and apply op. norm ineq.)} \\ &\leq C_1 \|d_1 - d_2\|_Y && \text{(use the upper bound } \|y'_n\| \leq C_1) \\ &< C_1 \delta = \epsilon && \text{(substitute in } \delta) \end{aligned}$$



Theorem: Schauder – Proof (\Rightarrow) Conclusion

Proof.

The theorem of Arzelà-Ascoli now tells us that M is relatively compact. So every sequence in M has a convergent subsequence. In particular for the convergent subsequence $(f_{n(k)})_{k \in \mathbb{N}}$ we have

$$\begin{aligned}(f_{n(k)})_{k \in \mathbb{N}} &= (y'_n \circ T)_{k \in \mathbb{N}} && (\text{apply def. of } f_{n(k)}) \\ &= (T'y'_{n(k)})_{k \in \mathbb{N}} && (\text{apply def. of adjoint operator})\end{aligned}$$

So finally, $(T'y'_{n(k)})_{k \in \mathbb{N}}$ is a convergent subsequence of $(T'y'_k)_{k \in \mathbb{N}}$. □

Theorem: Schauder – Proof (\Leftarrow)

Proof.

" \Leftarrow ": The other proof direction tells us that T compact $\Rightarrow T'$ compact So in extension this also yields T' compact $\Rightarrow T''$ compact The corollary tells us that

$T = (J_Y|_{Y \rightarrow J_Y(Y)})^{-1} \circ T'' \circ J_X$ Lastly, the operator T'' is compact, the revision tells us J_X and $(J_Y|_{Y \rightarrow J_Y(Y)})^{-1}$ are bounded, linear operators and therefore T is a composition of bounded, linear operators.

Using the revision we can conclude that T is compact. □

Example: Compact Integral Operator

Example 18

The lecture [KP25] states that the following integral operator is compact:

$$T : C[0, 1] \rightarrow C[0, 1], T f(x) = \int_0^1 f(t) dx$$

So $T' : C[0, 1]' \rightarrow C[0, 1]'$ is compact too. Evaluating $C[0, 1]'$ is beyond the scope of this paper.

Definition: Annihilator

Definition 19

Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ and $V \subset X'$. Then we define the annihilator of V in X as

$$V_{\perp} = \{x \in X \mid \forall x' \in V : x'(x) = 0\}$$

Revision

Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ a closed subspace and $x \in X \setminus U$. Then we have

$$\exists x' \in X' : x'|_U = 0 \wedge x'(x) \neq 0$$

Theorem: Annihilator is Closed Linear Subspace

Theorem 20

Let $(X, \|\cdot\|)$ be a normed \mathbb{F} vector space, $U \subset X$ and $V \subset X'$. Then we have

$V_\perp \subset X$ is a closed linear subspace

Proof.

We have $V_\perp = \bigcap_{x' \in V} (x')^{-1}(0)$ As an intersection of closed sets, V_\perp must be closed. □

Theorem: Rank-Nullity Generalized

Theorem 21

Let $T \in \mathcal{L}(X, Y)$ be a bounded, linear operator. Then we have

$$\overline{\text{Im } T} = (\text{Ker } T')_{\perp}$$

- ▷ In linear algebra lectures this is proven for finite-dimensional vector spaces (see Satz 6.1.5 [Wer18])

Theorem: Rank-Nullity Generalized – Proof (\subset)

Proof.

" \subset ": Let $Tx \in \text{Im } T$ with $x \in X$ and $y' \in \text{Ker } T'$.

We first prove $Tx \in (\text{Ker } T')^\perp$:

$$\begin{aligned} y'(Tx) &= (y' \circ T)(x) && \text{(associativity and use } \circ \text{ notation)} \\ &= (T'y')(x) && \text{(apply adjoint op. definition)} \\ &= 0(x) = 0 && \text{(apply } y' \in \text{Ker } T') \end{aligned}$$

Since this holds for all choices of Tx we get $\text{Im } T \subset (\text{Ker } T')^\perp$. Since $(\text{Ker } T')^\perp$ is closed, we also get $\overline{\text{Im } T} \subset (\text{Ker } T')^\perp$



Theorem: Rank-Nullity Generalized – Proof (\supset)

Proof.

" \supset ": We can prove the contraposition $(Y \setminus \overline{\text{Im } T}) \subset (Y \setminus (\text{Ker } T')^\perp)$. Set $U = \overline{\text{Im } T}$ and let $y \in Y \setminus U$. We know U that a closed linear subspace. The corollary of the theorem of Hahn-Banach tells us that $\exists y' \in Y' : y'|_U = 0 \wedge y'(y) \neq 0$. Since $\text{Ker } T' \subset Y'$ and $\forall y' \in \text{Ker } T' : y'(y) = 0$ we get $y \in Y \setminus (\text{Ker } T')^\perp$ □

Corollary: Operator Solutions

Corollary 22

Let $T \in \mathcal{L}(X, Y)$ be a linear, continuous operator with $\text{Im } T$ closed. Then we have

$$y \in \text{Im } T \text{ if and only if } \forall y' \in Y' : T'y' = 0 \Rightarrow y'(y) = 0$$

Proof.

We have

$$\begin{aligned} y \in \text{Im } T &= \overline{\text{Im } T} && (\text{Im } T \text{ is closed}) \\ &= (\text{Ker } T')_{\perp} && (\text{apply the theorem}) \\ \iff \forall y' \in \text{Ker } T' &: y'(y) = 0 && (\text{apply def. of annihilator}) \\ \iff \forall y' \in Y' &: T'y' = 0 \Rightarrow y'(y) = 0 && (\text{write in equivalent way}) \end{aligned}$$



Example: Shift Operator Revisited

Example 23

We can refer back to the example with the left shift operator.

The left shift operator T has $\text{Im } T = I_p$, which is (trivially) closed.

Since we already know the adjoint operator we can verify that the last theorem works.

Everything is up to isometry: Let $(x_k)_{k \in \mathbb{N}} \in I_p$ such that $T((x_k)_{k \in \mathbb{N}}) = (x_2, \dots) \in I_p$. Then $T'y' = 0 \Rightarrow y'(y) = 0$ amounts to "if the right shifted version of a sequence is 0 then

References

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Summary

Defined the adjoint operator $T' : Y' \rightarrow X'$ for bounded linear operators

Showed $T \mapsto T'$ is linear and isometric

Proved the adjoint reverses composition: $(ST)' = T'S'$

Characterized when an operator $S : Y' \rightarrow X'$ is an adjoint via the canonical embedding

Proved Schauder's theorem: T compact $\iff T'$ compact

Generalized the rank-nullity theorem: $\overline{\text{Im } T} = (\text{Ker } T')^\perp$