Introduction to Entropy

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December 8, 2025

Abstract

This seminar paper provides a foundational overview of Information Theory, focusing on the mathematical properties of Shannon Entropy and Mutual Information. We establish key definitions and rigorously prove fundamental inequalities, including the non-negativity of entropy and the Data Processing Inequality, using Jensen's Inequality. To build intuition, theoretical results are complemented by numerical simulations of Bernoulli and Geometric random variables. Finally, we introduce Relative Entropy (KL-Divergence) and discuss its geometric properties and applications in optimization.

Contents

1 Entropy and Mutual information

1.1 Definitions and Conventions

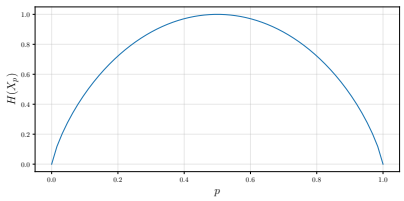
Definition 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{X}$ be countable sets and $X : \Omega \to \mathcal{X}$ be a discrete random variable on $(\Omega, \mathcal{A}, \mathbb{P})$. We can then define

$$\textbf{Entropy wrt. base:} \quad H_q(X) = \mathbb{E}\left(-\log_q p(X)\right) = -\sum_{x \in \operatorname{supp}(X)} p(x) \log_q p(x)$$

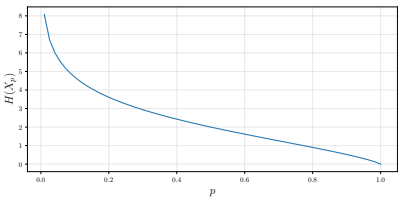
$$\textbf{Entropy conventionally:} \quad H(X) = H_2(X)$$

Remark 1.
1. Let $\mathcal{X}, \mathcal{Y}$ be countable sets and $X : \Omega \to \mathcal{X}$, $Y : \Omega \to \mathcal{Y}$ be discrete random variables on $(\Omega, \mathcal{A}, \mathbb{P})$. From now on, the random variables X, Y are always available for use.
2. We do **not** use the shorthand notations $p(x) = \mathbb{P}[X = x]$ and $p(y) = \mathbb{P}[Y = y]$ from [CT05], to keep the notation easily understandable.
3. Currently, $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}$ is not required. Later theorems like Jensens Inequality do require real-valued random variables.
4. We use the convention $\log = \log_2$, as the entropy H is defined wrt. base 2.
5. We also use the following convention and justify it through a continuity argument:
$$0 \log 0 = \lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{\ln(2) x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{-x}{\ln(2)} = 0$$
This choice is sensible, as $\log x$ is not defined for negative x.
6. The conventions, definitions and theorems are from Definitions, Theorems, Remarks and Exercises in *Elements of Information Theory, second edition* (see [CT05]).

Remark 2. Note that if $|\mathcal{X}|$ is finite, $(\forall p \in \mathbb{R}_+ : H_p(X)$ finite) and $H(X) \leq |\mathcal{X}|$ (see Theorem 4). For $|\mathcal{X}|$ countably infinite, there are counterexamples where $H_p(X) = \infty$ (see [Mat]). From now on, we will assume that entropy is finite.



(a) Bernoulli Rand. Variable Entropy $H(X_p)$



(b) Geometric Rand. Variable Entropy $H(X_p)$

Example 1. Let $p \in (0,1)$ and $X_p \sim B(1,p)$ be a weighted coin flip.
We can calculate the Entropy of X_p: $H(X_p) = -p\log p - (1-p)\log(1-p)$.
A visual inspection (see Figure 1a) reveals that $H(X_p)$ seems to be maximised for p and minimised for $p \in \{0,1\}$. An increase in uncertainty about the result of the coin flip seems to correspond with an increase in entropy.

Example 2. Let $p \in (0,1)$ and $X_p \sim G(p)$ be the number of times a weighted coin is flipped, until the first head occurs. We will calculate the Entropy of X_p. It will require the two well-known series:

$$\forall r \in (0,1) : \sum_{n\in\mathbb{N}_0} r^n = \frac{1}{1-r} \tag{1}$$

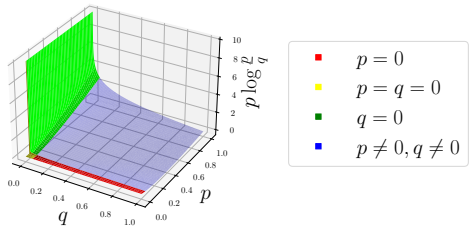
$$\forall r \in (0,1) : \sum_{n\in\mathbb{N}_0} nr^n = \frac{r}{(1-r)^2} \tag{2}$$

We can now directly calculate the Entropy of X_p:

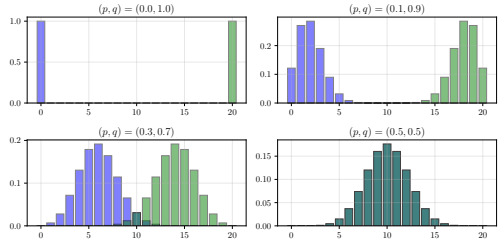
$$\begin{aligned}
H(X_p) &= \sum_{x\in\mathbb{N}} -p(x)\log p(x) && \text{(Def. of Entropy)}\\
&= -\sum_{x\in\mathbb{N}} (1-p)^{x-1}p\log\left((1-p)^{x-1}p\right) && \text{(Subst. in geometric density function)}\\
&= -\sum_{x\in\mathbb{N}} (1-p)^{x-1}p\left((x-1)\log(1-p)+\log p\right) && \text{(Log rules)}\\
&= -p\log(1-p)\sum_{x\in\mathbb{N}_0}\left((1-p)^x x\right) - p\log p\sum_{x\in\mathbb{N}_0}\left((1-p)^x\right) && \text{(Factor out constants)}\\
&= -p\log(1-p)\frac{1-p}{(1-(1-p))^2} - p\log p\frac{1}{1-(1-p)} && \text{(Use series 1 and 2)}\\
&= -p\log(1-p)\frac{1-p}{p^2} - p\log p\frac{1}{p} && \text{(Simplify expr.)}\\
&= \frac{-(1-p)\log(1-p) - p\log p}{p} && \text{(Simplify expr.)}
\end{aligned}$$

For $p = 0.5$ we get $H(X_{0.5}) = \frac{-0.5\log 0.5 - 0.5\log 0.5}{0.5} = -2\log 0.5 = 2$.
We can visually inspect $(0,1) \to \mathbb{R}, p \mapsto H(X_p)$ (see Figure 1b) to get a feeling for the entropy of X_p. An increase in p is linked to lower variance and more concentration of the distribution towards zero. Based on the plot, that increase looks to be linked to a lower entropy and vice-versa.

TODO: Complete the q



(a) Pointwise Relative Entropy



(b) Relative Entropies of Binomial Distributions

Definition 2.

We define the following:

$$\textbf{Conditional Entropy:} \quad H(X \mid Y) = -\mathbb{E}\left(\log p_{(X,Y)}(X \mid Y)\right)$$
$$\textbf{Joint Entropy:} \quad H(X, Y) = -\mathbb{E}\left(\log p_{(X,Y)}(X, Y)\right)$$

Let p and q be two probability mass functions on the same set $\mathcal{Z}$. We define the following:

$$\textbf{Relative Entropy:} \quad D(p\|q) = \mathbb{E}_{X\sim p}\left(\log \frac{p(X)}{q(X)}\right) \quad \text{with conventions from Remark 3}$$

(Sometimes also called KL-Divergence)

Using Relative Entropy, we can define the Mutual Information between the variables X, Y:

$$\textbf{Mutual Information:} \quad I(X;Y) = D(p_{(X,Y)}\|p_X\, p_Y)$$

Remark 3. We defined have

$$\begin{aligned} D(p(x)\|q(x)) &= \mathbb{E}_p\left(\log \frac{p(X)}{q(X)}\right) && \text{(def. of relative entropy)} \\ &= \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} && \text{(def. of expected value)} \end{aligned}$$

To understand the conventions, we can look at the limit cases:

1. Case $p \in (0, 1], q = 0$: $\lim_{q\to 0^+} p \log \frac{p}{q} = \lim_{q\to 0^+} (p \log p - p \log q) = \infty$.
2. Case $p = 0, q \in (0, 1]$: $0 \log \frac{0}{q} = 0$.
3. Case $p = q = 0$: Case 1 logic yields $\lim_{q\to 0^+} p \log \frac{p}{q} = \infty$ and Case 2 logic yields $0 \log \frac{0}{0} = 0$.

As we want $\sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$ to sum over $x \in \mathcal{X}, p(x) > 0$, we choose the convention $0 \log \frac{0}{0} = 0$. Figure 2a visualizes the the pointwise relative entropy function $(p, q) \mapsto \log \frac{p}{q}$.

Example 3. To understand the concept, we can now calculate the relative entropies for an example. Let $X \sim B(20, \alpha)$ and $Y \sim B(20, \beta)$ with $(\alpha, \beta) \in [0, 1]^2$.

$$\begin{aligned} & D(p(x)\|q(x)) = \sum_{x=0}^{20} p(x) \log \frac{p(x)}{q(x)} \\ \alpha = 0, \beta = 1: \quad & D(p(x)\|q(x)) = 0 \log \frac{0}{1} + 1 \log \frac{1}{0} = 0 + \infty = \infty \\ \alpha = 0.1, \beta = 0.9: \quad & D(p(x)\|q(x)) \approx 50.7 \\ \alpha = 0.3, \beta = 0.7: \quad & D(p(x)\|q(x)) \approx 9.8 \\ \alpha = 0.5, \beta = 0.5: \quad & D(p(x)\|q(x)) = \sum_{x=0}^{20} p(x) \log 1 = 0 \end{aligned}$$

Figure 2b visualizes the two discrete distribution functions in the cases above. Intuitively, the more overlap the distributions have, the closer to zero the relative entropy is.

1.2 Mutual Information and Chain Rules

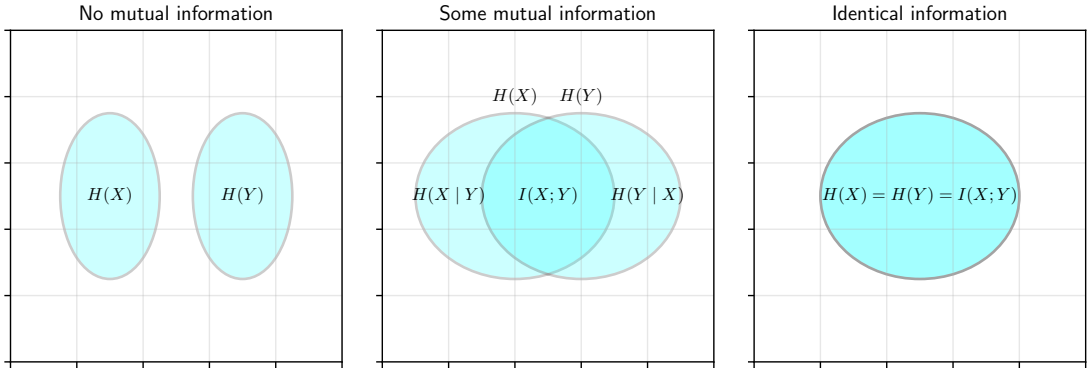


Figure 3: Relationship between Entropy, Conditional Entropy and Mutual Information

Theorem 1. *There are multiple equivalent ways to express Mutual Information (see Figure 3):*

1. $I(X;Y) = H(Y) - H(Y|X)$
2. $I(X;Y) = I(Y;X)$
3. $I(Y;X) = H(X) - H(X|Y)$
4. $I(X;Y) = H(X,Y)$
5. $I(X;X) = H(X)$

Proof. 1. We can use the definition of mutual information and relative entropy to obtain:

$$\begin{aligned}
I(X;Y) &= D(p(x,y)\|p(x)p(y)) && \text{(by def. of mutual info.)}\\
&= \mathbb{E}_{p(x,y)}\left(\log\frac{p(x,y)}{p(x)p(y)}\right) && \text{(by def. relative entropy)}\\
&= \mathbb{E}_{p(x,y)}\left(\log\frac{p(x)p(y|x)}{p(x)p(y)}\right) && \text{(using cond. probability)}\\
&= \mathbb{E}_{p(x,y)}\left(\log\frac{p(y|x)}{p(y)}\right) && \text{(simplify fraction)}\\
&= \mathbb{E}_{p(x,y)}\left(\log p(y|x)\right) - \mathbb{E}_p(x,y)\left(\log p(y)\right) && \text{(simplify logarithm)}\\
&= H(Y|X) - H(Y) && \text{(by def. of entropy)}
\end{aligned}$$

2. The definition of mutual information yields:

$$\begin{aligned}
I(X;Y) &= D(p(x,y)\|p(x)p(y)) && \text{(by def. of mutual info.)}\\
&= D(P[X=x,Y=y]\|P[X=x]P[Y=y]) && \text{(use clear notation)}\\
&= D(P[Y=y,X=x]\|P[Y=y]P[X=x]) && \text{(use commutativity)}\\
&= D(p(y,x)\|p(y)p(x)) && \text{(back to conventional notation)}\\
&= I(X;Y)
\end{aligned}$$

3. Follows directly from 2 and 3.

4.

$$\begin{aligned}
I(X;Y) &= H(Y) - H(X|Y) && \text{(by 1)}\\
&= H(Y) - (H(X,Y) - H(X)) && \text{(chain rule)}\\
&= H(X) + H(Y) - H(X,Y)
\end{aligned}$$

5. Using 1 we get $I(X;X) = H(X) - H(X|X) = H(X)$.

□

Definition 3.

Let p and q be a joint probability mass function on $\mathcal{X} \times \mathcal{Y}$.

$$\textbf{Conditional Relative Entropy:} \quad D(p(x \mid y) \| q(x \mid y)) = \mathbb{E}_{(X,Y)\sim p(x,y)} \left(\log \frac{p(X \mid Y)}{q(X \mid Y)} \right)$$

$$\textbf{Conditional Mutual Information:} \quad I(X;Y \mid Z) = H(X \mid Z) - H(X \mid Y, Z)$$

Example 4. To understand the concept, we can now calculate the conditional relative entropies for an example.

Example 5. To understand the concept, we can now calculate the conditional relative entropies for an example.

Theorem 2. *Let $n \in \mathbb{N}, n \geq 2$, $(X_1, \cdots, X_n) \sim p(x_1, \cdots, x_n)$. The following statements about Entropy and Mutual Information are called Chain Rules:*

1. $H(X_1, \cdots, X_n) = \sum_{i=1}^{n} H(X_i \mid X_{i-1}, \cdots, X_1)$
2. $D(p(x)\|q(x)) = D(p(x \mid y)\|q(x \mid y)) + D(p(y)\|q(y))$
3. $I()$

Proof. 1. Prove this result using induction by n.
Base case $n = 2$:

$$\begin{aligned} H(X_1, X_2) &= -\mathbb{E}\left(\log p(X_1, X_2)\right) \\ &= -\mathbb{E}\left(\log p(X_2 \mid X_1)p(X_1)\right) \\ &= -\mathbb{E}\left(\log p(X_2 \mid X_1)\right) + -\mathbb{E}\left(\log p(X_1)\right) \\ &= H(X_1) + H(X_2 \mid X_1) \end{aligned}$$

Assume the theorem holds for $n-1$. Induction case $n-1$ to n:

$$\begin{aligned} H(X_1, \cdots, H_n) &= H(X_n|X_1, \cdots, H_{n-1}) + H(X_1, \cdots, H_{n-1}) && \text{(apply base case)} \\ &= H(X_n|X_1, \cdots, H_{n-1}) + \sum_{i=1}^{n-1} H(X_i \mid X_{i-1}, \cdots, X_1) && \text{(induction hypothesis)} \end{aligned}$$

2.

$$\begin{aligned} D(p(x)\|q(x)) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\frac{p(x)}{q(x)} = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\frac{p(x)}{q(x)} = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\frac{p(x \mid y)p(y)}{q(x \mid y)p(y)} \\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\frac{p(x \mid y)}{q(x \mid y)} + \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\frac{p(y)}{p(y)} \\ &= D(p(x \mid y)\|q(x \mid y)) + D(p(y)\|q(y)) \end{aligned}$$

3. TODO

□

Corollary 1. *The following specialization of the Chain Rules for Entropy and Mutual Information is commonly used:*

1. $H(X, Y) = H(X) + H(Y \mid X)$
2. $D(p\|q) = D()$
3. $I(X; Y) =$

Proof. Follows from Theorem 2. □

2 Inequalities for Entropy and Mutual Information

2.1 Convexity and Jensens Inequality

Remark 4. We use the common definition of convex functions, concave functions from analysis. From now on, X, Y have to be real-valued random variables.

Theorem 3. *Let $f : \Omega \to \mathbb{R}$ a function.*

1. *If f is convex, we have $\mathbb{E}f(X) \geq f(\mathbb{E}X)$.*
2. *If f is concave, we have $\mathbb{E}f(X) \leq f(\mathbb{E}X)$.*

Proof. 1. Let $n \in \mathbb{N}$
$\{1\}$ and $p_1, \cdots, p_n \in (0,1)$ such that $\sum_{i=1}^n p_i = 1$ and $x_1, \cdots, x_n \in \mathbb{R}$. Distinguish two cases.
If $\mathcal{X}$ is finite: We show $f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i)$ by induction.
The definition of convexity yields the base case $i = 2$: $f(p_1 x_1 + p_2 x_2) \leq p_1 f(x_1) + p_2 f(x_2)$.
We assume the claim holds for $n-1$ and the induction case goes as follows:

$$\begin{aligned}
f\left(\sum_{i=1}^n p_i x_i\right) &= f\left(p_1 x_1 + (1-p_1)\sum_{i=2}^n \frac{p_i}{1-p_1} x_i\right) \\
&\leq p_1 f(x_1) + (1-p_1) f\left(\sum_{i=2}^n \frac{p_i}{1-p_1} x_i\right) \quad \text{(def. of convexity)} \\
&\leq p_1 f(x_1) + (1-p_1)\sum_{i=2}^n \frac{p_i}{1-p_1} f(x_i) \quad \text{(induct. hypo. applies bc. of } \sum_{i=2}^n \frac{p_i}{1-p_1} = 1) \\
&= \sum_{i=1}^n p_i f(x_i)
\end{aligned}$$

Else:

2. Follows from part 1 applied to $-f$.

□

Corollary 2. *Entropy and Mutual Information are non-negative:*

1. $0 \leq H(X)$
2. $0 \leq D(p(x)\|q(x))$
3. $0 \leq I(X;Y)$

Proof. 1. Note that $\log(\frac{1}{[0,1]}) = \log([1,\infty]) = [0,\infty]$ and $p(X)(\mathcal{X}) \in [0,1]$.
Using the monotonicity of the expected value, we obtain

$$0 \leq \mathbb{E}\left(\log\left(\frac{1}{p(X)}\right)\right) = -\mathbb{E}\left(\log\left(p(X)\right)\right) = H(X)$$

2. We can prove this using Jensens Inequality on a *concave* function:

$$\begin{aligned}
-D(p(x)\|q(x)) &= -\mathbb{E}_p\left(\log\frac{p(X)}{q(X)}\right) && \text{(def. of relative entropy)} \\
&\leq -\log\left(\mathbb{E}_p\frac{p(X)}{q(X)}\right) = \log\left(\mathbb{E}_p\frac{q(X)}{p(X)}\right) && \text{(-log is convex, property of log)} \\
&= \log\left(\sum_{x\in\mathcal{X}} p(x)\frac{q(x)}{p(x)}\right) = \log\left(\sum_{x\in\mathcal{X}} q(x)\right) && \text{(def. of exp. value, simplify expr.)} \\
&= \log(1) = 0 && \text{(q is a prob. function)}
\end{aligned}$$

So equivalently, we have $D(p(x)\|q(x)) \geq 0$.

3. Follows from part 1: $I(X;Y) = D(p(x,y) \| p(x)p(y)) \geq 0$.

□

Remark 5. There are multiple natural questions we can ask about Entropy. We will look at an example for each of them and then prove the results in the follow-up theorem.

1. What distribution maximises the value of Entropy? I simulated

   From the data, it seems plausible that the uniform distribution maximises entropy. We

2. Can joint entropy increase if we add redundant information?
   Let $\Omega = \{1, 2\}$, p uniform TODO, $X, Y : \Omega \to \mathbb{R}$, $X(\omega) = \omega$ and $Y(\omega) = 2\omega$. Y is redundant to X, as $Y = 2X$.

$$H(X) = \sum_{i=1}^{2} -0.5 \log 0.5 = -\log 0.5 = \log 2 = 1$$
$$H(X,Y) = \sum_{(i,j) \in \{(1,2),(2,4)\}} -0.25 \log 0.25 = 2 * 0.25 * 2 = 1$$

   Pairs like $(1, 4)$ have probability zero and do not contribute to the sum in $H(X, Y)$. This example suggest entropy never increases. We will prove that in the follow-up theorem.

3. What happens to the Entropy if we add independent noise to our measurements?
   Let $X \sim U(\{1, 2, 3, 4\})$ be the original signal, $N \sim B(3, 0.5)$ the noise and let X, N be independent. Then $S := X + N$ is a noisy signal.

   We can compute the cririca variance a.

Theorem 4. *We can formalise the previous observations including some more:*

1. *More information can only decrease entropy: $H(X \mid Y) \leq H(X)$.*
2. *The uniform distribution maximizes entropy:*
   $H(X) \leq \log |\mathcal{X}|$ and $H(X) = \log |\mathcal{X}| \iff X \sim U(\mathcal{X})$.
3. *If information from Y does not add anything to X, then Y must be derived from X:*
   $H(Y \mid X) = 0 \implies \exists f : Y = f(X)$ almost surely
4. *If independent noise is added to a random variable, entropy can only increase:*
   Set $Z = X + Y$. Then we have X, Y independent $\implies H(X) \leq H(Z) \wedge H(Y) \leq H(Z)$
5. *TODO: Maybe add independence bound?*

Proof. 1. $0 \leq I(X;Y) = H(X) - H(X \mid Y) \iff H(X \mid Y) \leq H(X)$

2. Let $Y \sim U(\mathcal{X})$ st. $\forall x \in \mathcal{X} : q(x) = \frac{1}{|\mathcal{X}|}$.

$$\begin{aligned} 0 \leq D(p(x) \| q(x)) &= \mathbb{E}_p \left(\log \frac{p(X)}{q(X)} \right) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \\ &= \sum_{x \in \mathcal{X}} p(x) \log \left(p(x) |\mathcal{X}| \right) = \sum_{x \in \mathcal{X}} p(x) \log p(x) + |\mathcal{X}| \sum_{x \in \mathcal{X}} p(x) \\ &= -H(X) + |\mathcal{X}| = |\mathcal{X}| - H(X) \end{aligned}$$

   This is equivalent to $H(X) \leq \log |\mathcal{X}|$. TODO: Iff part.

3. We have $H(Y \mid X) = -\mathbb{E} \left(\log p(Y \mid X) \right) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} -p(x,y) \log p(y \mid x)$.
   Additionally, we have $\forall (x, y) \in \mathcal{X} \times \mathcal{Y} : -p(x, y) \log p(y \mid x) \geq 0$.
   Combining those facts, we get

$$\begin{aligned} & H(Y \mid X) = 0 \\ \iff & \forall (x,y) \in \mathcal{X} \times \mathcal{Y} : p(x,y) \log p(y \mid x) = 0 \\ \iff & \forall (x,y) \in \mathcal{X} \times \mathcal{Y} : p(x,y) = 0 \oplus \log p(y \mid x) = 0 \\ \iff & \forall (x,y) \in \mathcal{X} \times \mathcal{Y} : p(x,y) = 0 \oplus p(y \mid x) = 1 \end{aligned}$$

This tells us, that either $p(x,y) = 0$ or $p(x,y) = p(y \mid x)p(x) = p(x) = 1$.
Now we can finish up the argument. Set $A = \{x \in \mathcal{X} : p(x) > 0\}$.
Define $(y_x)_{x \in \mathcal{X}}$ such that $\forall x \in \mathcal{X} : p(x, y_x) > 0$.
Set $f : A \to \mathcal{Y}, x \mapsto y_x$. This gets us $Imf = \{y_x : x \in \mathcal{X}\} = \{y \in \mathcal{Y} : p(x,y) > 0\}$.
So $Y = f(X)$ almost surely.

4. We have

$$\begin{aligned} H(Z \mid X) &= -\mathbb{E}\left(\log p(Z \mid X)\right) = \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} -p(x,y) \log P[Z = x + y \mid X = x] \\ &= \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} -p(x,y) \log P[X + Y = x + y \mid X = x] \\ &= \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}} -p(x,y) \log P[Y = y \mid X = x] \\ &= -\mathbb{E}\left(\log p(Y \mid X)\right) = H(Y \mid X) \end{aligned}$$

$H(Z,X) = H(X) + H(Z \mid X)$ $H(Y,X) = H(X) + H(Y \mid X)$
$H(Y,X) = H(X) + H(Y \mid X)$ Using the Chain rule, we get $H(X,Y) = H(Y \mid X) + H(X)$
□

2.2 The Log-Sum Inequality

Hello

2.3 Application to Optimisation

Hello

References

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[Mat] *Can the entropy of a random variable with countably many outcomes be infinite?* URL: https://math.stackexchange.com/questions/279304/can-the-entropy-of-a-random-variable-with-countably-many-outcomes-be-infinite (visited on 08/12/2025).