Introduction to Entropy

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Abstract

Your abstract.

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1 Entropy and Mutual information

1.1 Definitions and Conventions

Let $(\mathcal{X}, \mathcal{A}, \mathbb{P}_{\mathcal{A}})$ be a probability space. Let $X : \mathcal{X} \to \mathbb{R}$ be a discrete random variable on the space with probability density function $f_X : \mathcal{X} \to \mathbb{R}_+$. We use the shorthand notation $p(x) = \mathbb{P}_{\mathcal{A}}[X = x]$. Let $(\mathcal{Y}, \mathcal{B}, \mathbb{P}_{\mathcal{B}})$ be a probability space. Let $Y : \mathcal{Y} \to \mathbb{R}$ be a discrete random variable on the space with probability density function $f_Y : \mathcal{Y} \to \mathbb{R}_+$. We use the shorthand notation $p(y) = \mathbb{P}_{\mathcal{B}}[X = x]$.

This is a test.

Definition 1. TODO: This is all incorrectly defined. Let X be a discrete random variable with distribution $p(x)$.
We define entropy as $H_q(X) = \mathbb{E}\left(-\log_q p(X)\right)$. Since entropy was originally defined in the context of compression by Shannon in TODO, We usually use $q = 2$ and $H(X) = \mathbb{E}\left(-\log_2 p(X)\right)$. TODO: Add $log = log_2$ convention.

Theorem 1. *Existence of Entropy: If $\mathcal{X}$ is finite, $H_q(X)$ exists.*

Example, when Entropy does not exist: Let $X =$,

Proof. □

Convention: $0 \log(0) = 0$.

Using definition of expected value: $H_p(X) = \mathbb{E}\left(-\log_q(p(X)) = \mathbb{E}\left(\frac{1}{\log_q(p(X))}\right)\right)$.

Definition 2. Let X, Y be discrete random variables with marginal distributions $p(x), p(y)$ and with joint distribution $p(x, y)$. We define the following:

$$
\begin{aligned}
\text{Conditional Entropy:} \quad & H(X \mid Y) = -\mathbb{E}\left(\log p(X \mid Y)\right) \\
\text{Joint Entropy:} \quad & H(X, Y) = -\mathbb{E}\left(\log p(X, Y)\right) \\
\text{Relative Entropy:} \quad & D(p(x) \| q(x)) = \mathbb{E}_p\left(\log \frac{p(X)}{q(X)}\right) \text{ with the conventions...} \\
\text{Mutual information:} \quad & I(X; Y) := D(p(x, y) \| p(x)p(y)) \\
\text{Conditional Mutual information:} \quad & I(X; Y \mid Z) := H(X \mid Z) - H(X \mid Y, Z)
\end{aligned}
$$

Remark 1. From now on, we always define the two discrete random variables X, Y from Definition 2.

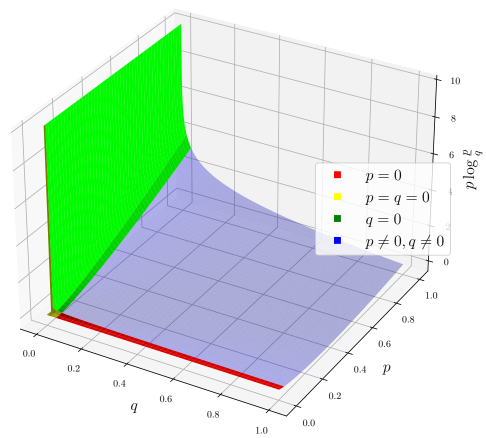
Remark 2. Under the assumptions of Definition 2, we have

$$\begin{aligned} D(p(x)\|q(x)) &= \mathbb{E}_p\left(\log \frac{p(X)}{q(X)}\right) && \text{(def. of relative entropy)} \\ &= \sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)} && \text{(def. of expected value)} \end{aligned}$$

To understand the conventions, we can look at the limit cases:

1. **Case** $p \in (0,1], q = 0$**:** $\lim_{q\to 0^+} p\log\frac{p}{q} = \lim_{q\to 0^+}(p\log p - p\log q) = \infty$.
2. **Case** $p = 0, q \in (0,1]$**:** $0\log\frac{0}{q} = 0$.
3. **Case** $p = q = 0$**:** Case 1 logic yields $\lim_{q\to 0^+} p\log\frac{p}{q} = \infty$ and Case 2 logic yields $0\log\frac{0}{0} = 0$.

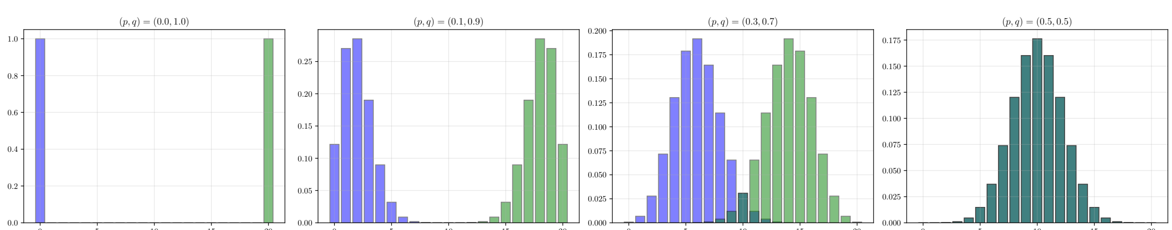
As we want $\sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)}$ to sum over $x \in \mathcal{X}, p(x) > 0$, we choose the convention $0\log\frac{0}{0} = 0$. We can visualize the the pointwise relative entropy function $(p,q) \mapsto \log\frac{p}{q}$:



We can calculate the relative entropies for an example. The more alike the distributions are, the closer to zero the relative entropy is. Let $X \sim B(20,\alpha)$ and $Y \sim B(20,\beta)$ with $(\alpha,\beta) \in [0,1]^2$.

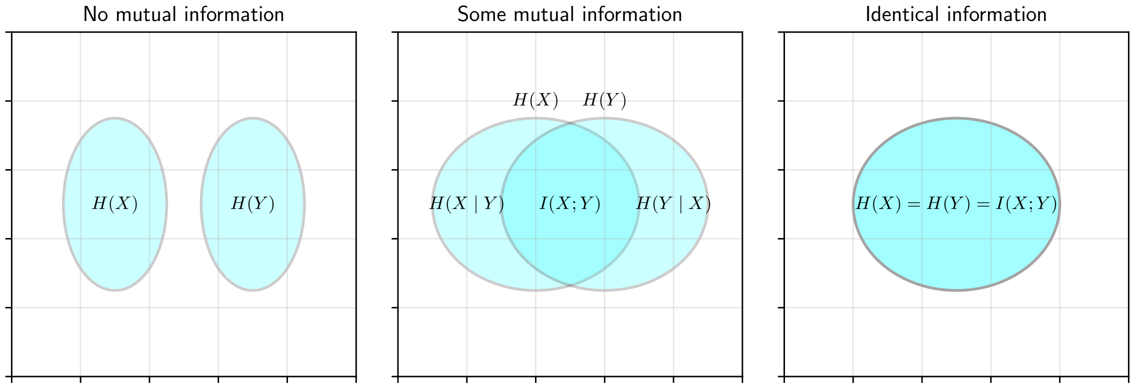
$$D(p(x)\|q(x)) = \sum_{x=0}^{20} p(x)\log\frac{p(x)}{q(x)}$$

$$\begin{aligned} \alpha = 0, \beta = 1: &\quad D(p(x)\|q(x)) = 0\log\frac{0}{1} + 1\log\frac{1}{0} = 0 + \infty = \infty \\ \alpha = 0.1, \beta = 0.9: &\quad D(p(x)\|q(x)) \approx 50.7 \\ \alpha = 0.3, \beta = 0.7: &\quad D(p(x)\|q(x)) \approx 9.8 \\ \alpha = 0.5, \beta = 0.5: &\quad D(p(x)\|q(x)) = \sum_{x=0}^{20} p(x)\log 1 = 0 \end{aligned}$$



1.2 Chain Rules and Mutual Information

Remark 3. The relationship between entropy, conditional entropy and mutual information can be visualized:



Theorem 2. *Let $n \in \mathbb{N}, n \geq 2$, $(X_1, \cdots, X_n) \sim p(x_1, \cdots, x_n)$. The following statements about Entropy and Mutual Information are called Chain Rules:*

1. $H(X_1, \cdots, X_n) = \sum_{i=1}^{n} H(X_i \mid X_{i-1}, \cdots, X_1)$
2.
3.

Proof. 1. Prove this result using induction by n.
Base case $n = 2$:

$$\begin{aligned} H(X_1, X_2) &= -\mathbb{E}\left(\log p(X_1, X_2)\right) \\ &= -\mathbb{E}\left(\log p(X_2 \mid X_1) p(X_1)\right) \\ &= -\mathbb{E}\left(\log p(X_2 \mid X_1)\right) + -\mathbb{E}\left(\log p(X_1)\right) \\ &= H(X_1) + H(X_2 \mid X_1) \end{aligned}$$

Assume the theorem holds for $n-1$. Induction case $n-1$ to n:

$$\begin{aligned} H(X_1, \cdots, H_n) &= H(X_n | X_1, \cdots, H_{n-1}) + H(X_1, \cdots, H_{n-1}) && \text{(apply base case)} \\ &= H(X_n | X_1, \cdots, H_{n-1}) + \sum_{i=1}^{n-1} H(X_i \mid X_{i-1}, \cdots, X_1) && \text{(induction hypothesis)} \end{aligned}$$

□

Corollary 1. *We can specialize the Chain Rule for Entropy and Mutual Information to two variables:*

1. $H(X, Y) = H(X) + H(Y \mid X)$
2. $I(X; Y) =$

Proof. Follows from Theorem 2. □

Theorem 3. *There are multiple equivalent ways to express Mutual Information:*

1. $I(X;Y) = H(Y) - H(Y|X)$
2. $I(X;Y) = I(Y;X)$
3. $I(Y;X) = H(X) - H(X|Y)$
4. $I(X;Y) = H(X,Y)$
5. $I(X;X) = H(X)$

Proof. 1. We can use the definition of mutual information and relative entropy to obtain:

$$\begin{aligned}
I(X;Y) &= D(p(x,y)\|p(x)p(y)) && \text{(by def. of mutual info.)}\\
&= \mathbb{E}_{p(x,y)}\left(\log\frac{p(x,y)}{p(x)p(y)}\right) && \text{(by def. relative entropy)}\\
&= \mathbb{E}_{p(x,y)}\left(\log\frac{p(x)p(y|x)}{p(x)p(y)}\right) && \text{(using cond. probability)}\\
&= \mathbb{E}_{p(x,y)}\left(\log\frac{p(y|x)}{p(y)}\right) && \text{(simplify fraction)}\\
&= \mathbb{E}_{p(x,y)}\left(\log p(y|x)\right) - \mathbb{E}_p(x,y)\left(\log p(y)\right) && \text{(simplify logarithm)}\\
&= H(Y|X) - H(Y) && \text{(by def. of entropy)}
\end{aligned}$$

2. The definition of mutual information yields:

$$\begin{aligned}
I(X;Y) &= D(p(x,y)\|p(x)p(y)) && \text{(by def. of mutual info.)}\\
&= D(p(y,x)\|p(y)p(x)) && \text{(TODO: Idk)}\\
&= I(X;Y)
\end{aligned}$$

3. Follows directly from 2 and 3.

4.

$$\begin{aligned}
I(X;Y) &= H(Y) - H(X|Y) && \text{(by 1)}\\
&= H(Y) - (H(X,Y) - H(X)) && \text{(chain rule)}\\
&= H(X) + H(Y) - H(X,Y)
\end{aligned}$$

5. Using 1 we get $I(X;X) = H(X) - H(X|X) = H(X)$.

□

2 Inequalities for Entropy and Mutual Information

2.1 Convexity and Jensen Inequality

Remark 4. We use the common definition of convex functions, concave functions from analysis.

Theorem 4. *Let X be a random variable (not necessarily discrete) and f a function.*

1. *If f is convex, we have $\mathbb{E}f(X) \geq f(\mathbb{E}X)$.*
2. *If f is concave, we have $\mathbb{E}f(X) \leq f(\mathbb{E}X)$.*

Proof.

1. Distinguish two cases. If $\mathcal{X}$ is finite:
   We show $f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)$ by induction.
   The definition of convexity yields the base case $i = 2$: $f(p_1x_1 + p_2x_2) \leq p_1 f(x_1) + p_2 f(x_2)$.
   We assume the claim holds for $n-1$ and the induction case goes as follows:

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &= f\left(p_1 x_1 + (1-p_1)\sum_{i=2}^{n} \frac{p_i}{1-p_1} x_i\right) \\ &\leq p_1 f(x_1) + (1-p_1) f\left(\sum_{i=2}^{n} \frac{p_i}{1-p_1} x_i\right) && \text{(def. of convexity)} \\ &\leq p_1 f(x_1) + (1-p_1)\sum_{i=2}^{n} \frac{p_i}{1-p_1} f(x_i) && \text{(induct. hypo. applies bc. of } \sum_{i=2}^{n} \frac{p_i}{1-p_1} = 1) \\ &= \sum_{i=1}^{n} p_i f(x_i) \end{aligned}$$

   Else:

2. Follows from part 1 applied to $-f$.

□

Corollary 2. *Entropy and Mutual Information are non-negative:*

1. $0 \leq H(X)$
2. $0 \leq D(p(x)\|q(x))$
3. $0 \leq I(X;Y)$

Proof.

1. Note that $\log(\frac{1}{[0,1]}) = \log([1,\infty]) = [0,\infty]$ and $p(X)(\mathcal{X}) \in [0,1]$.
   Using the monotonicity of the expected value, we obtain

$$0 \leq \mathbb{E}\left(\log\left(\frac{1}{p(X)}\right)\right) = -\mathbb{E}\left(\log\left(p(X)\right)\right) = H(X)$$

2. We can prove this using Jensens inequality on a *concave* function:

$$\begin{aligned} -D(p(x)\|q(x)) &= -\mathbb{E}_p\left(\log\frac{p(X)}{q(X)}\right) && \text{(def. of relative entropy)} \\ &\leq -\log\left(\mathbb{E}_p\frac{p(X)}{q(X)}\right) = \log\left(\mathbb{E}_p\frac{q(X)}{p(X)}\right) && \text{(-log is convex, property of log)} \\ &= \log\left(\sum_{x\in\mathcal{X}} p(x)\frac{q(x)}{p(x)}\right) = \log\left(\sum_{x\in\mathcal{X}} q(x)\right) && \text{(def. of exp. value, simplify expr.)} \\ &= \log(1) = 0 && \text{(q is a prob. function)} \end{aligned}$$

   So equivalently, we have $D(p(x)\|q(x)) \geq 0$.

3. Follows from part 1: $I(X;Y) = D(p(x,y)\|p(x)p(y)) \geq 0$.

$\square$

Remark 5. There are multiple natural questions we can ask about Entropy. We will look at an example for each of them and then prove the results in the follow-up theorem.

1. What distribution maximises the value of Entropy? I simulated

   From the data, it seems plausible that the uniform distribution maximises entropy. We

2. Can joint entropy increase if we add redundant information?
   Let $\Omega = \{1,2\}$, p uniform TODO, $X, Y : \Omega \to \mathbb{R}$, $X(\omega) = \omega$ and $Y(\omega) = 2\omega$. Y is redundant to X, as $Y = 2X$.

$$H(X) = \sum_{i=1}^{2} -0.5 \log 0.5 = -\log 0.5 = \log 2 = 1$$
$$H(X,Y) = \sum_{(i,j)\in\{(1,2),(2,4)\}} -0.25 \log 0.25 = 2 * 0.25 * 2 = 1$$

   Pairs like $(1,4)$ have probability zero and do not contribute to the sum in $H(X,Y)$. This example suggest entropy never increases. We will prove that in the follow-up theorem.

3. What happens to the Entropy if we add independent noise to our measurements?
   Let $X \sim U(\{1,2,3,4\})$ be the original signal, $N \sim B(3, 0.5)$ the noise and let X, N be independent. Then $S := X + N$ is a noisy signal.

   We can compute the cririal variance a.

Theorem 5. *We can formalise the previous observations including some more:*

1. *More information can only decrease entropy: $H(X \mid Y) \leq H(X)$.*
2. *The uniform distribution maximizes entropy:*
   $H(X) \leq \log|\mathcal{X}|$ *and* $H(X) = \log|\mathcal{X}| \iff X \sim U(\mathcal{X})$.
3. *If information from Y does not add anything to X, then Y must be derived from X:*
   $H(Y \mid X) = 0 \implies \exists f : Y = f(X)$ *almost surely*
4. *If independent noise is added to a random variable, entropy can only increase:*
   Set $Z = X + Y$. Then we have X, Y independent $\implies H(X) \leq H(Z) \wedge H(Y) \leq H(Z)$

Proof. 1. $0 \leq I(X;Y) = H(X) - H(X \mid Y) \iff H(X \mid Y) \leq H(X)$

2. Let $Y \sim U(\mathcal{X})$ st. $\forall x \in \mathcal{X} : q(x) = \frac{1}{|\mathcal{X}|}$.

$$\begin{aligned} 0 \leq D(p(x)\|q(x)) &= \mathbb{E}_p\left(\log \frac{p(X)}{q(X)}\right) = \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \\ &= \sum_{x\in\mathcal{X}} p(x) \log\left(p(x)|\mathcal{X}|\right) = \sum_{x\in\mathcal{X}} p(x)\log p(x) + |\mathcal{X}| \sum_{x\in\mathcal{X}} p(x) \\ &= -H(X) + |\mathcal{X}| = |\mathcal{X}| - H(X) \end{aligned}$$

   This is equivalent to $H(X) \leq \log|\mathcal{X}|$. TODO: Iff part.

3. We have $H(Y \mid X) = -\mathbb{E}\left(\log p(Y \mid X)\right) = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} -p(x,y)\log p(y \mid x)$.
   Additionally, we have $\forall (x,y) \in \mathcal{X}\times\mathcal{Y} : -p(x,y)\log p(y \mid x) \geq 0$.
   Combining those facts, we get

$$\begin{aligned} & H(Y \mid X) = 0 \\ \iff & \forall (x,y) \in \mathcal{X}\times\mathcal{Y} : p(x,y)\log p(y \mid x) = 0 \\ \iff & \forall (x,y) \in \mathcal{X}\times\mathcal{Y} : p(x,y) = 0 \oplus \log p(y \mid x) = 0 \\ \iff & \forall (x,y) \in \mathcal{X}\times\mathcal{Y} : p(x,y) = 0 \oplus p(y \mid x) = 1 \end{aligned}$$

This tells us, that either $p(x,y) = 0$ or $p(x,y) = p(y \mid x)p(x) = p(x) = 1$.
Now we can finish up the argument. Set $A = \{x \in \mathcal{X} : p(x) > 0\}$.
Define $(y_x)_{x\in\mathcal{X}}$ such that $\forall x \in \mathcal{X} : p(x, y_x) > 0$.
Set $f : A \to \mathcal{Y}, x \mapsto y_x$. This gets us $Imf = y_x : x \in \mathcal{X} = y \in \mathcal{Y} : p(x,y) > 0$.
So $Y = f(X)$ almost surely.

4. We have

$$\begin{aligned}
H(Z \mid X) = -\mathbb{E}\left(\log p(Z \mid X)\right) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} -p(x,y)\log P[Z = x+y \mid X = x] \\
&= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} -p(x,y)\log P[X+Y = x+y \mid X = x] \\
&= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} -p(x,y)\log P[Y = y \mid X = x] \\
&= -\mathbb{E}\left(\log p(Y \mid X)\right) = H(Y \mid X)
\end{aligned}$$

$H(Z,X) = H(X) + H(Z \mid X)$ $H(Y,X) = H(X) + H(Y \mid X)$
$H(Y,X) = H(X) + H(Y \mid X)$ Using the Chain rule, we get $H(X,Y) = H(Y \mid X) + H(X)$

$\square$

2.2 The Information Inequality

Hello

2.3 Application to Optimisation

Hello