Derivation of Output Secondary Beam

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1 Introduction

In this experiment, a high–energy proton beam impinges on a primary target to produce a secondary beam rich in various particles, including K^+ mesons. A charge–exchange target is then used to convert a fraction of these K^+ into neutral kaons via the reaction

$$K^+ + n \to K^0 + p.$$

Downstream, a magnet sweeps away the remaining charged particles, and a collimator defines a predominantly neutral beam. In the decay volume, the short–lived K_S decays rapidly, leaving an almost pure K_L beam for analysis.

2 Input Parameters

The derivation is based on the following system parameters:

- I_p: Primary proton beam intensity (particles/s).
- σ_{prod}: Production cross–section for K^+ mesons (cm^2).
- ϵ_{geom}: Geometric acceptance factor of the beamline.
- η_{K^+}: Fraction of produced particles that are K^+.
- P_{sel}: Momentum selection efficiency for K^+ mesons.
- f_{conv}: Conversion efficiency from K^+ to K^0 in the charge–exchange target.
- p_{min}, p_{max}: Lower and upper bounds of the selected momentum band (GeV/c).

- L_{decay}: Length of the decay volume (m).
- τ_S: Lifetime of the K_S (s).
- τ_L: Lifetime of the K_L (s).
- m_{K^+}, m_{K^0}: Masses of the charged and neutral kaons (GeV/c^2).
- m_n: Neutron mass (GeV/c^2).
- m_p: Proton mass (GeV/c^2).
- c: Speed of light ($\approx 3 \times 10^8$ m/s).
- t: Thickness of the hydrogen–rich target (in appropriate units).
- $\frac{\mathrm{d}E}{\mathrm{d}x}$: Stopping power in the target (MeV/(g/cm^2) or similar).

3 Derivation of the Neutral Kaon Flux

The total number of K^+ mesons produced per second is given by:

$$N_{K^+} = I_p \cdot \sigma_{\text{prod}} \cdot \epsilon_{\text{geom}} \cdot \eta_{K^+}.$$

After applying momentum selection and the charge–exchange conversion, the neutral kaon flux becomes:

$$N_{K^0} = N_{K^+} \cdot P_{\text{sel}} \cdot f_{\text{conv}}.$$

This expression represents the output secondary beam flux in terms of the primary beam and conversion efficiencies.

4 Momentum Distribution of the Output Beam

4.1 Kinematic Shift from Charge–Exchange Reaction

When the K^+ beam (with momentum p_i) passes through the thin hydrogen–rich target, a fraction undergoes the charge–exchange reaction:

$$K^+ + n \to K^0 + p.$$

Assuming the neutron in the target is at rest, energy and momentum conservation in the lab frame yields:

$$\sqrt{p_i^2 + m_{K^+}^2} + m_n = \sqrt{p_f^2 + m_{K^0}^2} + \sqrt{(p_i - p_f)^2 + m_p^2},$$

where p_f is the momentum of the outgoing K^0. For high–energy kaons, we set

$$p_f = p_i - \delta,$$

with δ a small shift. Expanding to first order in δ (and neglecting terms of order δ^2), one obtains

$$\delta \approx \frac{m_{K^0}^2 - m_{K^+}^2}{2\, p_i} + m_p - m_n.$$

Defining the kinematic momentum offset as

$$\Delta_{\text{kin}} \equiv -\delta,$$

we have

$$p_f \approx p_i + \Delta_{\text{kin}}.$$

For example, using typical masses (in MeV/c^2): $m_{K^+} \approx 493.7$, $m_{K^0} \approx 497.6$, $m_n \approx 939.6$, $m_p \approx 938.3$, and taking $p_i \sim 5000\,\text{MeV/c}$, one finds

$$\Delta_{\text{kin}} \approx 0.9\,\text{MeV/c}.$$

Thus, the charge–exchange reaction alone produces a very slight (sub–MeV) increase in the momentum of the outgoing K^0.

4.2 Momentum Loss Due to Ionization

In addition to the kinematic effect, as the beam passes through the thin target the kaons lose energy (and thus momentum) by ionizing the medium. If the target thickness is t and the stopping power is $\mathrm{d}E/\mathrm{d}x$, the energy loss is approximated by

$$\Delta E \approx \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right) t.$$

For relativistic particles where $E \approx p$, the momentum loss can be estimated as

$$\Delta p_{\text{loss}} \approx \Delta E.$$

4.3 Net Momentum Distribution

Combining the two effects, the net output momentum is given by

$$p_{\text{out}} = p_i + \Delta_{\text{kin}} - \Delta p_{\text{loss}}.$$

Since the initial K^+ momentum distribution, $f_{K^+}(p_i)$, is assumed to follow that of the high–energy proton beam, the transformation to the output K^0 momentum distribution is

$$f_{K^0}(p_{\text{out}}) = f_{K^+}\Big(p_{\text{out}} - \Delta_{\text{net}}(p_{\text{out}})\Big)$$

with the net momentum shift defined by

$$\Delta_{\text{net}}(p_i) = \Delta_{\text{kin}}(p_i) - \Delta p_{\text{loss}},$$

and an appropriate Jacobian factor arising from the change of variable from p_i to p_{out}.

In most cases the energy–loss term Δp_{loss} dominates over the tiny kinematic shift, so the net effect is a small reduction in momentum. These expressions allow one to calculate the momentum distribution of the output neutral beam from the known input distribution.

5 Decay Properties in the Decay Volume

The probability that a neutral kaon decays within the decay volume of length L_{decay} is given by:

$$P_{\text{decay}} = 1 - \exp\left(-\frac{L_{\text{decay}}}{\gamma c\tau}\right),$$

where γ is the Lorentz factor. For an approximate estimate, we take:

$$\gamma \approx \frac{\langle p\rangle}{m_K}.$$

Since the K_S decays very quickly (lifetime τ_S), after a few meters the surviving beam is predominantly K_L (with lifetime τ_L).

6 Beam Purity

If the conversion produces both K_S and K_L with initial probabilities P_S and P_L respectively, the fraction of K_L surviving after a distance L can be approximated by:

$$\text{Purity} \approx \frac{P_L}{P_S \exp\left(-\frac{L}{\gamma c\tau_S}\right) + P_L}.$$

This expression estimates the purity of the neutral beam in the detection region.

7 Summary and Conclusions

In summary, we have derived the following key expressions for the output secondary beam:

1. **Neutral Kaon Flux:**

$$N_{K^0} = I_p \cdot \sigma_{\text{prod}} \cdot \epsilon_{\text{geom}} \cdot \eta_{K^+} \cdot P_{\text{sel}} \cdot f_{\text{conv}}.$$

2. **Momentum Distribution (without energy loss):**

$$\langle p \rangle = \frac{p_{\min} + p_{\max}}{2}, \quad \Delta p = p_{\max} - p_{\min}.$$

3. **Kinematic Shift in the Charge–Exchange Reaction:**

$$\delta \approx \frac{m_{K^0}^2 - m_{K^+}^2}{2\,p_i} + m_p - m_n, \quad \Delta_{\text{kin}} = -\delta.$$

4. **Momentum Loss in the Target:**

$$\Delta p_{\text{loss}} \approx \left(\frac{\mathrm{d}E}{\mathrm{d}x}\right) t.$$

5. **Net Output Momentum:**

$$p_{\text{out}} = p_i + \Delta_{\text{kin}} - \Delta p_{\text{loss}},$$

with the transformed momentum distribution

$$f_{K^0}(p_{\text{out}}) = f_{K^+}\Big(p_{\text{out}} - \Delta_{\text{net}}(p_{\text{out}})\Big) \times \left|\frac{\mathrm{d}p_i}{\mathrm{d}p_{\text{out}}}\right|.$$

6. **Decay Probability in the Decay Volume:**

$$P_{\text{decay}} = 1 - \exp\left(-\frac{L_{\text{decay}}}{\gamma c \tau}\right), \quad \text{with } \gamma \approx \frac{\langle p \rangle}{m_K}.$$

7. **Beam Purity:**

$$\text{Purity} \approx \frac{P_L}{P_S \exp\left(-\frac{L}{\gamma c \tau_S}\right) + P_L}.$$