(Co)Derived Equivalences in Algebra and Geometry

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Fix a field k, and let X and Ξ be dual k-vector spaces of dimension $n+1$ with dual bases (x_i) and (ξ_i) respectively. The goal of this exposition is to examine equivalences of various categories that arise naturally in this setting from algebro-geometric constructions. In particular, we look at chain complexes of

(i) modules over the symmetric algebra $A := \mathrm{Sym}^\bullet(V)$,

(ii) modules over the exterior algebra $A^! := \Lambda^\bullet(V)$,

(iii) coherent sheaves over $\mathbb{P}^n := \mathrm{Proj}(\mathrm{Sym}^\bullet(V))$, the projectivisation of Ξ.

Example 0.1. In the simplest case when X, Ξ are one-dimensional, the data of a module over $A = k[x]$ involves a k-vector space M with a map $M \xrightarrow{x} M$ which can be seen as a cochain complex $F(M)$ of k-vector spaces with differential d of degree 1. The graded algebra $A^! = k[\xi]/(\xi^2)$ acts naturally on $F(M)$ via the degree 1 chain map $F(M) \xrightarrow{\xi} F(M)$ given by

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[dr] & M \arrow[r] \arrow[dr, "1"] & M \arrow[r] \arrow[dr] & \cdots & \\ & \cdots \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}.$$

Consider the cochain complex of A-modules

$$\cdots \to 0 \to M \otimes_k A \xrightarrow{d\otimes 1 + \xi \otimes x} M \otimes_k A \to 0 \to \cdots,$$

concentrated in degrees -1 and 0. This has the same underlying vector spaces as the complex $F(M) \otimes_k A$, but the differential has been 'twisted' to remember the $A^!$-action. This complex is exact everywhere except in degree 0, where it has cohomology M. Since the modules appearing in it are free, we have recovered a free resolution of M.

This is the first example of what may be called *Koszul duality*, a broad term encompassing various equivalences across algebra, geometry, and representation theory. The duality between symmetric and exterior algebras over finite dimensional vector spaces was first studied by Bernstein, Gel'fand & Gel'fand (1978), who exhibit an adjunction between the categories of cochain complexes of graded modules over A and $A^!$.

Theorem. There are adjoint functors

$$\mathrm{Ch}(A\text{-grMod}) \underset{F}{\overset{G}{\leftrightarrows}} \mathrm{Ch}(A^!\text{-grMod})$$

such that any complex $M^\bullet$ of graded A-modules has free resolution $GF(M^\bullet)$, and any complex $N^\bullet$ of graded $A^!$-modules has injective resolution $FG(N^\bullet)$.

In Section 2, we look at Eisenbud, Floystad & Schreyer's (2003) treatment of the Bernstein-Gel'fand-Gel'fand (BGG) correspondence described above. To turn the adjunction into an equivalence of categories, we need to employ the machinery of Verdier's *derived categories*. Passing to the corresponding derived categories of modules has the effect that all *quasi-isomorphisms* (i.e. chain maps that induce isomorphisms on homology) become isomorphisms.

Bernstein et al. (1978) use the correspondence between coherent sheaves on $\mathbb{P}^n$ and graded A-modules (see for example Chapter II of Hartshorne (2008)) to describe the derived category $D^b(\text{coh-}\mathbb{P}^n)$ of projective n-space. In 3 we use

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1 Categories of complexes

1.1 Chain complexes

Example 1.1 (Modules over an algebra).

Example 1.2 (Comodules over a coalgebra).

2 The Bernstein-Gel'fand-Gel'fand correspondence

describe section

2.1 Data

2.1.1 Symmetric and exterior algebras

Given an $n+1$-dimensional k-vector space V, the *tensor algebra* is the k-vector space

$$T(V) = \bigoplus_{i \geq 0} V^{\otimes i}$$

with a product $\nabla : T(V) \otimes T(V) \to T(V)$ induced by the natural identifications $V^{\otimes i} \otimes V^{\otimes j} \xrightarrow{\sim} V^{\otimes (i+j)}$. This is an associative algebra with a natural $\mathbb{Z}_{\geq 0}$-grading. The *symmetric algebra* $\mathrm{Sym}^\bullet(V)$ and the *exterior algebra* $\Lambda^\bullet(V)$ are then the graded algebras defined as quotients of $T(V)$ by certain two-sided ideals, namely

$$\mathrm{Sym}^\bullet(V) = \frac{T(V)}{(x \otimes y - y \otimes x \mid x, y \in V)}, \qquad \Lambda^\bullet(V) = \frac{T(V)}{(x \otimes x \mid x \in V)}.$$

Since the ideals are generated by homogeneous elements, these algebras inherit gradings from $T(V)$.

We continue to use ∇ for the product morphism on either algebra, though the corresponding bilinear map on $\Lambda^\bullet V$ is often written $\wedge$.

Since we are primarily concerned with the algebras $A = \mathrm{Sym}^\bullet(X)$ and $A^! = \Lambda^\bullet(\Xi)$, we redefine the grading on $A^!$ as $A^!_{-i} = \Lambda^i\Xi$. This amounts to a change of sign from the usual grading, but the convention ensures that the dual vector spaces X and Ξ lie in degrees 1 and -1 in their respective algebras.

2.1.2 The exterior coalgebra

The *exterior coalgebra* on Ξ is defined as the linear dual of $A^!$, written $A^¡ := \mathrm{Hom}_k(A^!, k)$. $A^¡$ has the $\mathbb{Z}$-grading $A^¡_i = \mathrm{Hom}_k(A^!_{-i}, k)$ and is naturally an $A^!$-module via $a \cdot f(a') = (-1)^{\deg a} f(a \wedge a')$ for $a \in A^!$ homogeneous, $f \in \mathrm{Hom}(A^!, k)$. Moreover, for any k-vector space N we have the natural isomorphism of $A^!$-modules $\mathrm{Hom}_k(A^!, N) \cong A^¡ \otimes_k N$.

Choosing a basis x_i for X fixes an isomorphism $X \cong \mathrm{Hom}_k(\Xi, k) = A^¡_1$, which can be extended to get the isomorphism of graded k-vector spaces

$$A^¡ = \bigoplus_i \mathrm{Hom}_k(\Lambda^i\Xi, k) \cong \bigoplus_i \Lambda^i X = \Lambda^\bullet(X).$$

Write $\tau : A^¡ \to A$ for the k-linear map which identifies the subspaces of $A^¡$ and A corresponding to X, and is 0 elsewhere.

The coproduct on $A^¡$. Being the linear dual of a finite dimensional algebra, $A^¡$ has a natural (coassociative counital) coalgebra structure which comes from dualising the (associative unital) product $\nabla : A^! \otimes_k A^! \to A^!$. This is called the *shuffle coproduct*, and we give an explicit description of it as follows. Given a collection of indices $\underline{\alpha} = \{\alpha_1 < \ldots < \alpha_i\} \subseteq \{0, \ldots, n\}$, write $x_{\underline{\alpha}}$ for the standard basis element of $A^¡$ given by $x_{\alpha_1} \wedge x_{\alpha_2} \wedge \ldots \wedge x_{\alpha_i}$ (in particular, $x_\emptyset = 1$). The vector $\xi_{\underline{\alpha}}$ is defined similarly. We say a tuple $(\underline{\beta}, \underline{\beta'})$ of subsets is a *break* of $\underline{\alpha}$ if $(\beta_1 < \ldots < \beta_p, \beta'_1 < \ldots < \beta'_q)$ is a permutation of $(\alpha_1 < \ldots < \alpha_i)$ (in other words, $\underline{\alpha} = \underline{\beta} \sqcup \underline{\beta'}$). The *sign* of this break, written $\langle \underline{\beta}, \underline{\beta'} \rangle$, is defined to be the sign of the corresponding permutation. Thus we have have

$$\nabla(x_{\underline{\beta}} \otimes x_{\underline{\beta'}}) = x_{\underline{\beta}} \wedge x_{\underline{\beta'}} = \langle \underline{\beta}, \underline{\beta'} \rangle\, x_{\underline{\alpha}}.$$

This allows us to write the coproduct on $A^¡$ as

$$\Delta(x_{\underline{\alpha}}) = \sum_{(\underline{\beta}, \underline{\beta'}) \in \mathrm{br}(\underline{\alpha})} \langle \underline{\beta}, \underline{\beta'} \rangle\, x_{\underline{\beta}} \otimes x_{\underline{\beta'}}$$

where $\mathrm{br}(\underline{\alpha})$ is the set of all breaks of $\underline{\alpha}$. Recalling that $A^¡ \otimes_k A^¡$ is $\mathbb{Z}$-graded with $\bigoplus_{p+q=i} A^¡_p \otimes A^¡_q$ in degree i, we observe that the map Δ respects grading hence $A^¡$ is a *graded coalgebra*.

2.1.3 Graded chain complexes

Objects of $\mathrm{Ch}(A\text{-grMod})$ are chain complexes of graded A-modules in which the differentials are morphisms in A-grMod (i.e. A-module homomorphisms which preserve degree). Such an object can be viewed as a $\mathbb{Z}^2$-graded k-vector space $\bigoplus_{i,j} M^i_j$ with an endomorphism d (the differential) such that

$d \circ d = 0$,

d has degree $(1, 0)$ i.e. $d(M^i_j) \subseteq M^{i+1}_j$, and

for each $i \in \mathbb{Z}$, $M^i_\bullet = \bigoplus_j M^i_j$ is a graded A-module.

Likewise, an object $N \in \mathrm{Ch}(A^!\text{-grMod})$ can be seen as a $\mathbb{Z}^2$-graded k-vector space $\oplus_{i,j} N^i_j$ with a differential ∂ of degree $(1, 0)$. We shall use the two viewpoints on interchangeably, switching between them whenever convenient to provide a clearer picture. In particular, the ability to view a complex as a single module with additional structure allows for cleaner definitions and proofs, see for instance Theorem 2.2.

For a chain complex $\mathbf{M} = \bigoplus_{i,j} M^i_j$, we say the lower indices denote the *internal* (or *Adam's*) grading, while the upper indices denote the *differential* (or *cohomological*) degree. We use '$\langle\cdot\rangle$' to denote shifts in Adam's gradings, continuing to use '$[\cdot]$' to denote shifts in differential gradings. Thus for example we have $M\langle q\rangle^i_j = M^i_{q+j}$.

2.2 Twisted Functors

We now define additive functors

$$\mathrm{Ch}(A\text{-grMod}) \underset{F}{\overset{G}{\leftrightarrows}} \mathrm{Ch}(A^!\text{-grMod})$$

on which the BGG correspondence is based. In the framework of $\mathbb{Z}^2$-graded vector spaces described in Section 2.1.3, we have

$$\bigoplus_{i,j} F(\mathbf{M})^i_j \cong \mathrm{Hom}_k\left(A^!, \bigoplus_{p,q} M^p_q\right) = A^i \otimes_k \left(\bigoplus_{p,q} M^p_q\right), \qquad \bigoplus_{i,j} G(\mathbf{N})^i_j \cong A \otimes_k \left(\bigoplus_{p,q} N^p_q\right).$$

However, care is needed to define the gradings and differentials since, for example, naïvely applying the functor $\mathrm{Hom}_k(A^!, -)$ would lose all A-module structure. The key is to modify the naïve differential by adding a 'twist' as in Example 0.1.

2.2.1 Defining the functor F

We first define F on the category A-grMod, seen as the full subcategory of $\mathrm{Ch}(A\text{-grMod})$ whose objects are complexes concentrated in differential degree 0. If $M^0_\bullet$ is a graded A-module, we define $F(M^0_\bullet)$ to be the chain complex of $A^!$-modules given by a

$$\cdots \to A^i\langle -i\rangle \otimes_k M^0_i \xrightarrow{\partial} A^i\langle -i-1\rangle \otimes_k M^0_{i+1} \to \cdots$$
$$a \otimes m \longmapsto \sum_\alpha \xi_\alpha a \otimes x_\alpha m.$$

The module $A^i\langle -i\rangle \otimes_k M^0_i$ is naturally isomorphic to $\mathrm{Hom}_k(A^!\langle i\rangle, M^0_i)$ and inherits an Adam's grading from A^i with the vector space $A^i_{j-i} \otimes_k M^0_i$ forming the jth graded piece. These shifts in grading have been chosen precisely so that the differential ∂ has degree $(1, 0)$, while the graded commutativity of $A^!$ implies $\partial \circ \partial = 0$. Thus we indeed have a chain complex of $A^!$-modules.

Given a morphism $M^0_\bullet \to M^1_\bullet$ in A-grMod, the functoriality of tensor products induces $A^!$-module homomorphisms $A^i\langle -i\rangle \otimes_k M^0_i \to A^i\langle -i\rangle \otimes_k M^0_i$ which are compatible with the differentials (i.e. the natural squares commute). Thus we have an additive functor $F : A\text{-grMod} \to \mathrm{Ch}(A^!\text{-grMod})$.

To extend F to arbitrary chain complexes $\mathbf{M} = (\bigoplus_{i,j} M^i_j, d) \in \mathrm{Ch}(A\text{-grMod})$, we observe that the functoriality of F gives us a (commuting) bicomplex

$$\begin{tikzcd}
\vdots & & \vdots & \vdots & \\
F(M^{i+1}_\bullet) \arrow[u] & \cdots \arrow[r] & A^{¡}\langle -j\rangle \otimes_k M^{i+1}_j \arrow[r] \arrow[u] & A^{¡}\langle -j-1\rangle \otimes_k M^{i+1}_{j+1} \arrow[r] \arrow[u] & \cdots \\
F(M^i_\bullet) \arrow[u] & \cdots \arrow[r] & A^{¡}\langle -j\rangle \otimes_k M^i_j \arrow[r] \arrow[u] & A^{¡}\langle -j-1\rangle \otimes_k M^i_{j+1} \arrow[r] \arrow[u] & \cdots \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

where the functorially induced vertical maps are $1 \otimes d$. Define $F(\mathbf{M})$ to be the total complex of this bicomplex, given by

$$\cdots \to \bigoplus_{p+q=i} A^{¡}\langle -q\rangle \otimes_k M^p_q \xrightarrow{\partial} \bigoplus_{p+q=i+1} A^{¡}\langle -q\rangle \otimes_k M^p_q \to \cdots,$$

$$\partial : a \otimes m \longmapsto a \otimes dm + (-1)^{\#m} \sum_\alpha \xi_\alpha a \otimes x_\alpha m$$

where $\#m$ is the differential degree of $m \in \mathbf{M}$. It is clear that each $F(\mathbf{M})^i_\bullet = \bigoplus_{p+q=i} A^{¡}\langle -q\rangle \otimes_k M^p_q$ is a graded $A^!$ module, and the signs introduced in the total complex construction ensure $\partial \circ \partial = 0$. An explicit check confirms ∂ has degree $(1, 0)$, so we indeed have an object of $\mathrm{Ch}(A^!\text{-grMod})$.

The twist using comodules. Observe that the differential ∂ differs from the naïve differential $1 \otimes d$ on the tensor product by the horizontal maps, which are the 'twists' we have been alluding to. These have a nice description using the fact that a graded module $N_\bullet \in A^!\text{-grMod}$ has the structure of a graded $A^{¡}$-comodule via the map

$$\begin{aligned} \Delta : \quad N_\bullet &\longrightarrow N_\bullet \otimes_k A^{¡} \\ n &\longmapsto \sum_{\underline{\alpha} \subseteq \{0,\dots,n\}} \xi_{\underline{\alpha}} n \otimes x_{\underline{\alpha}}. \end{aligned}$$

Applying this idea to the $A^!$-modules $A^{¡}\langle -i\rangle$, we get a commuting square

$$\begin{tikzcd}
\bigoplus_{u+v=j-q} A^{¡}_u \otimes_k A^{¡}_v \otimes_k M^{i-q}_q \arrow[r, "1\otimes\tau\otimes 1"] & A^{¡}_{j-q-1} \otimes_k A_1 \otimes_k M^{i-q}_q \arrow[d, "1\otimes\nabla"] \\
A^{¡}_{j-q} \otimes_k M^{i-q}_q \arrow[u, "\Delta\otimes 1"] \arrow[r, "(-1)^{i-q}(\partial - 1\otimes d)"] & A^{¡}_{j-q-1} \otimes_k M^{i-q}_{q+1}
\end{tikzcd}$$

where $\nabla : A \otimes_k M^{i-q}_\bullet \to M^{i-q}_\bullet$ defines the A-module structure on M, and $\tau : A^{¡} \to A^!$ is the morphism defined in Section 2.1.2 which identifies $A^{¡}_1$ with A_1, annihilating other graded pieces.

In summary, $F(\mathbf{M})$ as a $\mathbb{Z}^2$-graded vector space is simply $A^{i} \otimes_k \mathbf{M}$ with (i, j)th piece

$$F(\mathbf{M})^i_j = \bigoplus_{p+q=i} A^i_{j-q} \otimes_k M^p_q$$

and differential given on $A^i_{j-q} \otimes_k M^p_q$ by

$$1 \otimes d + (-1)^p (1 \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1).$$

2.2.2 The left adjoint to F

The functor $G : \mathrm{Ch}(A^!\text{-grMod}) \to \mathrm{Ch}(A\text{-grMod})$ is analogously defined, and maps the chain complex $\mathbf{N} = (\bigoplus_{i,j} N, \partial)$ to $G(\mathbf{N})$ given by

$$\cdots \to \bigoplus_{p-q=i} N^p_q \otimes_k A\langle -q\rangle \xrightarrow{d} \bigoplus_{p-q=i+1} N^p_q \otimes_k A\langle -q\rangle \to \cdots$$

$$d : n \otimes a \longmapsto \partial n \otimes a + (-1)^{\#n} \sum_{\alpha} \xi_\alpha n \otimes x_\alpha a$$

where $\#n$ is the differential degree of $n \in \mathbf{N}$. The Adam's grading on each $G(\mathbf{N})^i_\bullet$ is inherited from A, and is given by

$$G(\mathbf{N})^i_j = \bigoplus_{p-q=i} N^p_q \otimes_k A_{j-q}.$$

Recalling that every $A^!$-module is a A^i-comodule (see Section 2.2.1), we can use the comodule structure-map $\Delta : N^i_\bullet \to N^i_\bullet \otimes A^i$ to define the differential on $N^p_q \otimes_k A_{j-q}$ as

$$\partial \otimes 1 + (-1)^p (1 \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1).$$

The adjunction. Having defined the functors F and G, we show that G is left adjoint to F. Spelled out this means given $\mathbf{M} \in \mathrm{Ch}(A\text{-grMod})$ and $\mathbf{N} \in \mathrm{Ch}(A^!\text{-grMod})$, there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathrm{Ch}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \cong \mathrm{Hom}_{\mathrm{Ch}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})).$$

At its heart this is a $\otimes$-Hom adjunction, as we shall illustrate in the special case of module categories below.

Lemma 2.1 (Eisenbud et al. (2003)). Given modules $M \in A\text{-Mod}$ and $N \in A^!\text{-Mod}$, there are natural isomorphisms of abelian groups

$$\mathrm{Hom}_A(A \otimes_k N, M) \cong \mathrm{Hom}_k(N, M) \cong \mathrm{Hom}_{A^!}(N, \mathrm{Hom}_k(A^!, M)).$$

Proof. Choosing a basis n_α for N, the first isomorphism follows from observing that the A-module $A \otimes_k N$ is freely generated by $1 \otimes n_\alpha$.

The second isomorphism sends $\varphi \in \mathrm{Hom}_k(N, M)$ to the map $\varphi^! : N \to \mathrm{Hom}_k(A^!, M)$ such that for any $n \in N$ and $a \in A^!$ homogeneous we have

$$\varphi^!(n)(a) = (-1)^{\deg a} \varphi(an)$$

The inverse correspondence sends $\varphi^! \in \mathrm{Hom}_{A^!}(N, \mathrm{Hom}_k(A^!, M))$ to $\varphi \in \mathrm{Hom}_k(N, M)$ given by

$$\varphi(n) = \varphi^!(n)(1).$$

□

We now exhibit the general adjunction for F and G, and it is here that the flexibility of interpreting a chain complex $\mathbf{M}$ of graded modules as a single $\mathbb{Z}^2$-graded module $\bigoplus_{i,j} M^i_j$ (see Section 2.1.3) really comes handy. Interpreting $\mathrm{Ch}(A\text{-grMod})$ as a subcategory of A-Mod (likewise for $A^!$), we use Lemma 2.1 to identify $\mathrm{Hom}_{\mathrm{Ch}(A\text{-grMod})}(G(\mathbf{N}),\mathbf{M}) \subset \mathrm{Hom}_A(\mathbf{N}\otimes_k A,\mathbf{M})$ and $\mathrm{Hom}_{\mathrm{Ch}(A^!\text{-grMod})}(\mathbf{N},F(\mathbf{M})) \subset \mathrm{Hom}_{A^!}(\mathbf{N},\mathrm{Hom}_k(A^!,\mathbf{M}))$ with the same subgroup of $\mathrm{Hom}_k(\mathbf{N},\mathbf{M})$.

Theorem 2.2 (Bernstein et al. (1978))**.** The functor G, from the category of complexes of graded $A^!$-modules to the category of complexes of graded A-modules, is a left adjoint to the functor F.

Proof. Given $\bar{\varphi} \in \mathrm{Hom}_A(G(\mathbf{N}),\mathbf{M})$, the corresponding map $\varphi \in \mathrm{Hom}_k(\mathbf{N},\mathbf{M})$ found in Lemma 2.1 is given by $\varphi(n) = \bar{\varphi}(n \otimes 1)$. Thus $\bar{\varphi}$ has degree $(0,0)$ if and only if $\bar{\varphi}(N^i_j \otimes_k A_0) \subseteq M^{i-j}_j$, if and only if $\varphi(N^i_j) \subseteq M^{i-j}_j$. Moreover for $n \in N^i_j$, direct computation shows

$$(d_M \circ \bar{\varphi} - \bar{\varphi} \circ d_{G(\mathbf{N})})(n \otimes 1) \;=\; (d_M \circ \varphi - \varphi \circ \partial_N)(n) - (-1)^i \sum_\alpha x_\alpha \varphi(\xi_\alpha n),$$

thus $\bar{\varphi}$ is a morphism in $\mathrm{Ch}(A\text{-grMod})$ if and only if

$$\varphi(N^i_j) \subseteq M^{i-j}_j, \quad \text{and} \quad d_M \circ \varphi - \varphi \circ \partial_N = \sum_\alpha x_\alpha \varphi \xi_\alpha \tag{1}$$

where we write $\sum_\alpha x_\alpha \varphi \xi_\alpha$ for the map that takes $n \in N^i_j$ to $(-1)^i \sum_\alpha x_\alpha \varphi(\xi_\alpha n)$.

Likewise given $\varphi^! \in \mathrm{Hom}_{A^!}(\mathbf{N},F(\mathbf{M}))$, repeating the above argument shows $\varphi^!$ is an element of $\mathrm{Hom}_{\mathrm{Ch}(A^!\text{-grMod})}(\mathbf{N},F(\mathbf{M})$ if and only if the corresponding map $\varphi \in \mathrm{Hom}_k(\mathbf{N},\mathbf{M})$ satisfies (1). This shows that the isomorphisms given in Lemma 2.1 restrict to isomorphisms

$$\mathrm{Hom}_{\mathrm{Ch}(A\text{-grMod})}(G(\mathbf{N}),\mathbf{M}) \cong \{\varphi \in \mathrm{Hom}_k(\mathbf{N},\mathbf{M}) \text{ satisfying } (1)\} \cong \mathrm{Hom}_{\mathrm{Ch}(A^!\text{-grMod})}(\mathbf{N},F(\mathbf{M}))$$

thereby showing G is left adjoint to F. $\square$

Example 2.3.

2.2.3 The (co)unit of adjunction

3 Coherent sheaves on $\mathbb{P}^n$

3.1 The Tate resolution and Beilinson monads

4 Koszul duality after Keller (2003)

References

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