(Co)Derived Equivalences in Algebra and Geometry

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Fix a field k, and let X and Ξ be dual k-vector spaces of dimension $n+1$ with dual bases (x_i) and (ξ_i) respectively. The goal of this exposition is to examine equivalences of various categories that arise naturally in this setting from algebro-geometric constructions. In particular, we look at chain complexes of

(i) modules over the symmetric algebra $A := \mathrm{Sym}^\bullet(X)$,

(ii) modules over the exterior algebra $A^! := \bigwedge^\bullet(\Xi)$,

(iii) coherent sheaves over $\mathbb{P}^n := \mathrm{Proj}(\mathrm{Sym}^\bullet(X))$, the projectivisation of Ξ.

The first hint at a correspondence between coherent sheaves on $\mathbb{P}^n$ and modules over the exterior algebra $A^! = \bigwedge^\bullet(\Xi)$ comes from the following observation in Bernstein, Gel'fand & Gel'fand (1978).

Example 0.1. Given a graded $A^!$-module $N_\bullet$ and a non-zero vector $\xi \in \Xi$, we have a complex of vector spaces

$$\mathcal{L}_\xi(N_\bullet): \qquad \cdots \to N_1 \xrightarrow{\xi} N_0 \xrightarrow{\xi} N_{-1} \to \cdots .$$

Writing $\mathcal{L}^j$ for the vector bundle $N_{-j} \otimes_k \mathcal{O}_{\mathbb{P}^n}(j)$, we can identify $\mathcal{L}^j_\xi = N_{-j}$ with the fiber of $\mathcal{L}^j$ at $[\xi] \in \mathbb{P}^n$ to obtain a complex

$$\mathcal{L}(N_\bullet): \qquad \cdots \to N_1 \otimes_k \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow N_0 \otimes_k \mathcal{O}_{\mathbb{P}^n} \longrightarrow N_{-1} \otimes_k \mathcal{O}_{\mathbb{P}^n}(1) \to \cdots$$

where the differential sends a section $f \in N_{-j} \otimes_k \mathcal{O}(j)$ (which is a degree j homogeneous function with values in N_{-j}) to $df = [\xi \mapsto \xi \cdot f(\xi)]$. We say the module $N_\bullet$ is *faithful* if the complex $\mathcal{L}(N_\bullet)$ is exact everywhere except in degree 0, in which case we write $\Phi(N_\bullet)$ for the vector bundle $H^0(\mathcal{L}(N_\bullet))$. In this case, the natural map $\mathcal{L}(N_\bullet) \to \Phi(N_\bullet)$ of omplexes (where we identify $\Phi(N_\bullet)$ with the corresponding complex concentrated in degree 0) gives a 'resolution' of $\Phi(N_\bullet)$ by powers of the Hopf bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

Example 0.2. In the simplest case when X, Ξ are one-dimensional, the data of a module over $A = k[x]$ involves a k-vector space M with a map $M \xrightarrow{x} M$ which can be seen as a cochain complex $F(M)$ of k-vector spaces with differential d of degree 1. The graded algebra $A^! = k[\xi]/(\xi^2)$ acts naturally on $F(M)$ via the degree 1 chain map $F(M) \xrightarrow{\xi} F(M)$ given by

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[dr] & M \arrow[r] \arrow[dr, "1"] & M \arrow[r] \arrow[dr] & \cdots & \\
& \cdots \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} .$$

Consider the cochain complex of A-modules

$$\cdots \to 0 \to M \otimes_k A \xrightarrow{d \otimes 1 + \xi \otimes x} M \otimes_k A \to 0 \to \cdots,$$

concentrated in degrees -1 and 0. This has the same underlying vector spaces as the complex $F(M) \otimes_k A$, but the differential has been 'twisted' to remember the $A^!$-action. This complex is exact everywhere except in degree 0, where it has cohomology M. Since the modules appearing in it are free, we have recovered a free resolution of M.

This is the first example of what may be called *Koszul duality*, a broad term encompassing various equivalences across algebra, geometry, and representation theory. The duality between symmetric and exterior algebras over finite dimensional vector spaces was first studied by Bernstein et al. (1978), who exhibit an adjunction between the categories of cochain complexes of graded modules over A and $A^!$.

Theorem. There are adjoint functors

$$\mathcal{C}(A\text{-grMod}) \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C}(A^!\text{-grMod})$$

such that any complex $M^\bullet$ of graded A-modules has free resolution $GF(M^\bullet)$, and any complex $N^\bullet$ of graded $A^!$-modules has injective resolution $FG(N^\bullet)$.

In Section 3, we look at Eisenbud, Floystad & Schreyer's (2003) treatment of the Bernstein-Gel'fand-Gel'fand (BGG) correspondence described above. To turn the adjunction into an equivalence of categories, we need to employ the machinery of Verdier's *derived categories*. Passing to the corresponding derived categories of modules has the effect that all *quasi-isomorphisms* (i.e. chain maps that induce isomorphisms on homology) become isomorphisms.

Bernstein et al. (1978) use the correspondence between coherent sheaves on $\mathbb{P}^n$ and graded A-modules (see for example Chapter II of Hartshorne (2008)) to describe the derived category $D^b(\text{coh-}\mathbb{P}^n)$ of projective n-space. In 4 we use

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1 Categories of complexes

1.1 Abelian categories

1.2 Bicomplexes

1.2.1 Spectral sequences

1.3 Homotopy and derived categories

1.3.1 Generators and exceptional sequences

1.3.2 Derived functors

2 Coherent sheaves on $\mathbb{P}^n$

The projectivisation of Ξ is the k-scheme $\mathbb{P}^n = \text{Proj}(\text{Sym}^\bullet(X))$, where $\text{Sym}^\bullet(X) = k[x_0, ..., x_n]$ is the standard symmetric algebra on X graded by degree. We strengthen the observation of Example 0.1 to the following result, the celebrated theorem of Beilinson.

Theorem (Beilinson (1978))**.** The derived category $\mathcal{D}^b(\mathbb{P}^n)$ is generated by the exceptional sequence

$$\langle \mathcal{O}(-n), \mathcal{O}(-n+1), ..., \mathcal{O}(-1), \mathcal{O} \rangle.$$

This section looks at Beilinson's (1978) original proof, following the treatment in Caldararu (2005) and Carbone (2016). There are two key ideas involved– the first is that the identity functor on $\mathcal{D}(\text{coh-}\mathbb{P}^n)$ admits a factorisation

$$\begin{tikzcd} \mathcal{D}^b(\mathbb{P}^n \times \mathbb{P}^n) \arrow[r, "- \otimes^{\mathbf{L}} \mathcal{O}_\Delta"] & \mathcal{D}^b(\mathbb{P}^n \times \mathbb{P}^n) \arrow[d, "\mathbf{R}\pi_{1*}"] \\ \mathcal{D}^b(\mathbb{P}^n) \arrow[u, "\pi_2^*"] \arrow[r, "\text{id}"] & \mathcal{D}^b(\mathbb{P}^n) \end{tikzcd}$$

where $\pi_1, \pi_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projection maps, and $\mathcal{O}_\Delta \in \text{coh-}(\mathbb{P}^n \times \mathbb{P}^n)$ is the structure sheaf of the diagonal subscheme. This follows from the geometric theory of a *Fourier-Mukai transform* associated to a pair of schemes $\mathcal{X}, \mathcal{Y}$, and we briefly sketch the construction in Section 2.1.

The second observation, called *Beilinson's resolution of the diagonal*, follows from the algebraic theory of *Koszul resolutions* and shows that that $\mathcal{O}_\Delta$ admits a resolution by locally free sheaves of the form $\pi_1^*(\Omega^i(i)) \otimes \pi_2^*(\mathcal{O}(-i))$, where Ω is the sheaf of differentials on $\mathbb{P}^n$. Combined with the factorisation of identity, this provides an algorithm to resolve any coherent sheaf on $\mathbb{P}^n$in terms of the $\mathcal{O}(i)$ thus proving Beilinson's result.

2.1 Fourier-Mukai transforms

The material in this section is from Huybrechts (2006), who covers the topic in great detail. Given two smooth projective k-schemes $\mathbf{X_1}$ and $\mathbf{X_2}$, we associate to each object $\mathcal{E} \in \mathcal{D}^b(\mathbf{X_1} \times \mathbf{X_2})$ an exact functor $\Phi_\mathcal{E} : \mathcal{D}^b(\mathbf{X_1}) \to \mathcal{D}^b(\mathbf{X_2})$ as follows.

Definition 2.1. The *Fourier-Mukai transform with kernel* $\mathcal{E}$ of a complex $\mathcal{A} \in \mathcal{D}^b(\mathbf{X_1})$ is defined as

$$\Phi_\mathcal{E}(\mathcal{A}) = \mathbf{R}\pi_{1*}(\pi_2^*\mathcal{A} \otimes^{\mathbf{L}} \mathcal{E}) \quad \in \mathcal{D}^b(\mathbf{X_2}).$$

Here $\pi_i : \mathbf{X_1} \times \mathbf{X_2} \to \mathbf{X_i}$ ($i = 1, 2$) be the projection maps, these are flat so the pullback functors π_i^* are exact and need no derivation. Being the composition of three exact functors, the Fourier-Mukai transform $\Phi_{\mathcal{E}}$ is an exact functor. Moreover, the dependence on the kernel is functorial– for a fixed $\mathcal{A} \in \mathcal{D}^b(\mathbf{X_1})$, the map

$$\begin{aligned} \Phi_{-}(\mathcal{A}) : \quad \mathcal{D}^b(\mathbf{X_1} \times \mathbf{X_2}) &\longrightarrow \mathcal{D}^b(\mathbf{X_2}) \\ \mathcal{E} &\longmapsto \Phi_{\mathcal{E}}(\mathcal{A}) \end{aligned}$$

is the composite $\pi_{1*}(\pi_2^* \mathcal{A} \otimes^{\mathbf{L}} -)$, hence is an exact functor.

The name comes from the following analogy with functional analysis– given a finite-dimensional vector space X and its dual Ξ, to any smooth function $E(x, \xi) : X \times \Xi \to \mathbb{C}$ we can associate a linear map $\phi_E : L^2(X) \to L^2(Y)$ between the spaces of square-integrable functions, given by $f \mapsto \int_X f(x)E(x, \xi)dx$. If $E(x, \xi) = e^{2\pi i \langle x, \xi \rangle}$, then ϕ_E is an isomorphism called the *Fourier transform*. Similarly, the Fourier-Mukai transform yields interesting functors based on choice of $\mathcal{E}$.

Example 2.2. If $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}$ and $\mathbf{X} \xhookrightarrow{\iota} \mathbf{X} \times \mathbf{X}$ is the diagonal inclusion, then we can consider the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta = \iota_* \mathcal{O}_X$, the structure sheaf of the diagonal subscheme. Since ι is a closed immersion, the pushforward ι_* is exact and $R\iota_* = \iota_*$ as derived functors. Hence $\mathcal{O}_\Delta = \mathbf{R}\iota_* \mathcal{O}_X$ in $\mathcal{D}^b(\mathbf{X})$, and we can use the projection formula to get

$$\begin{aligned} \Phi_{\mathcal{O}_\Delta}(\mathcal{A}) &= \mathbf{R}\pi_{1*}(\pi_2^* \mathcal{A} \otimes^{\mathbf{L}} \mathbf{R}\iota_* \mathcal{O}_X) \\ &= \mathbf{R}\pi_{1*} \circ \mathbf{R}\iota_* (\mathbf{L}\iota^* \pi_2^* \mathcal{A} \otimes^{\mathbf{L}} \mathcal{O}_X) \\ &= \mathbf{R}(\pi_1 \circ \iota)_* (\mathbf{L}(\pi_2 \circ \iota)^* \mathcal{A} \otimes^{\mathbf{L}} \mathcal{O}_X) \\ &= \mathcal{A} \otimes^{\mathbf{L}} \mathcal{O}_X. \end{aligned}$$

But $\mathcal{O}_X$ is a locally free sheaf so the functor $(- \otimes^{\mathbf{L}} \mathcal{O}_X)$ is the same as $(- \otimes \mathcal{O}_X)$, which is identity. In other words, the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$ is the identity functor.

Replacing the trivial bundle $\mathcal{O}_X$ in the above computation with some other line bundle $\mathcal{L}$ on $\mathbf{X}$, we see that the derived functor $(- \otimes \mathcal{L})$ is the Fourier-Mukai transform with kernel $\iota_* \mathcal{L}$. Similarly, one can show that the Fourier-Mukai kernel $\mathcal{O}_\Delta[1]$ yields the shift functor $\mathcal{A} \mapsto \mathcal{A}[1]$. Thus Fourier-Mukai transforms generalise many familiar constructions. It is in fact a theorem of Orlov that any fully faithful exact functor $\mathcal{D}^b(\mathbf{X}_1) \to \mathcal{D}^b(\mathbf{X}_2)$ that admits adjoints must arise as the Fourier-Mukai transform for some kernel determined uniquely up to isomorphism.

2.2 Koszul resolutions

Given a ring A and a sequence $(a_0, ..., a_n)$ of elements in A, the associated *Koszul complex* is a very useful construction which detects various homological properties of the ring, and often yields free resolutions of the A-module $A/(a_0, ..., a_n)$. The construction and theory of Koszul complexes is treated in its full generality in Eisenbud (1995); here we only study the behaviour in two special cases we use to resolve the diagonal as Beilinson did– the first is when $(a_0, ..., a_n)$ generate the unit ideal, and the second is when they form a regular sequence.

Definition 2.3. Given a ring A and a sequence $(a_0, ..., a_n)$ of elements in A, the associated *Koszul* *complex* is the complex of A-modules given by

$$K_A(a_0, ..., a_n): \qquad 0 \to \bigwedge_A^{n+1}(A^{n+1}) \to \bigwedge_A^{n}(A^{n+1}) \to \cdots \to \bigwedge_A^{2}(A^{n+1}) \to A^{n+1} \to A \to 0$$
$$d(e_{\alpha_1} \wedge ... \wedge e_{\alpha_i}) = \sum_j (-1)^{i+j+1} a_{\alpha_j} \cdot (e_{\alpha_1} \wedge ...\hat{e}_{\alpha_j} ... \wedge e_{\alpha_i})$$

where $e_0, ..., e_n$ are the standard generators of A^{n+1}, and $\hat{\cdot}$ denotes omission of a term. We put the term $\bigwedge_A^i(A^{n+1})$ in differential degree $-i$.

Observe that the modules appearing in $K_A(a_0, ..., a_n)$ are free, so acyclic Koszul complexes yield free A-resolutions. In the simplest case when the sequence contains a single element a_0, the Koszul complex is given by

$$K(a_0): \qquad 0 \to A \xrightarrow{a_0} A \to 0,$$

so it is exact if and only if a_0 is a unit in A. This result generalises to sequences $(a_0, ..., a_n)$ that generate the unit ideal.

Proposition 2.4. If A is a ring and $(a_0, ..., a_n) = A$, then the Koszul complex $K_A(a_0, ..., a_n)$ is exact everywhere.

Proof. We show that the the identity on $K_A(a_0, ..., a_n)$ is chain homotopic to the zero morphism. By assumption, there are elements $\lambda_0, ..., \lambda_n \in A$ such that $\sum_i \lambda_i a_i = -1$. Then consider the map given by

$$h : \bigwedge_A^i(A^{n+1}) \to \bigwedge_A^{i+1}(A^{n+1})$$
$$h(e) = \sum_j \lambda_j e \wedge e_j$$

A straightforward basis-wise check shows $d \circ h + h \circ d = \mathrm{id}$, showing h is the required chain homotopy. Since homotopic chain-maps induce the same map on homology, we must have that $K_A(a_0, ..., a_n)$ is exact. □

Looking again at the Koszul complex for a single element $a_0 \in A$, we have that $H^1(K(a_0)) = 0$ if and only if a_0 is not a zero-divisor in A– in this case the complex is a free resolution of $A/(a_0)$. Recall that $(a_0, ..., a_n)$ is an A-*regular sequence* if a_0 is not a zero-divisor in A, and for every $0 \leq i < n$, a_{i+1} is not a zero-divisor for the module $A/(a_0, ..., a_i)$. Then Eisenbud (1995) proves that whenever $(a_0, ..., a_n)$ is a regular sequence, the associated Koszul complex is exact everywhere except in degree 0 where it has cohomology $A/(a_0, ..., a_n)$. We prove a special case of the result, when A is a polynomial algebra and $a_0, ..., a_n$ are the indeterminates.

Proposition 2.5 (Loday (2012))**.** Suppose R contains a field of characteristic 0. Then for the polynomial ring $A = R[x_0, ..., x_n]$, we have

$$H^i(K_A(x_0, ..., x_n)) = \begin{cases} R, & i = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Write M for the free R-module generated by $x_0, ..., x_n$. Then we can identify $\bigwedge_A^\bullet(A^{n+1})$ with the algebra $\bigwedge_R^\bullet(M) \otimes_k A$, graded so that M lies in degree 1. Considering A as an R-algebra graded by degree, we see that $\mathbf{K} = K_A(x_0, \ldots, x_n)$ is a complex of graded R-algebras given by

$$\begin{aligned}\mathbf{K}: \qquad & 0 \to \textstyle\bigwedge_R^{n+1}(M) \otimes_R A\langle -n-1\rangle \to \bigwedge_R^n(M) \otimes_R A\langle -n\rangle \to \cdots \to A \to 0 \\ & d((x_{\alpha_1} \wedge \ldots \wedge x_{\alpha_i}) \otimes a) = \sum_j (-1)^{i+j+1} (x_{\alpha_1} \wedge \ldots \hat{x}_{\alpha_j} \ldots \wedge x_{\alpha_i}) \otimes a x_{\alpha_j}.\end{aligned}$$

Note the differential d preserves internal grading, so we can write the complex above as a direct sum $\mathbf{K} = \bigoplus_r \mathbf{K}_r$ where $\mathbf{K}_r$ is the complex of R-modules formed at Adam's degree r (called the rth *strand* of $\mathbf{K}$). Since cohomology is an additive functor, we have $H^i(\mathbf{K}) = \bigoplus_r H^i(\mathbf{K}_r)$.

Now the strands in negative degrees vanish everywhere, and the $\mathbf{K}_0$ has the module R concentrated in differential degree 0. Thus it suffices to prove every other strand is exact. We do this by showing that the identity map on $\mathbf{K}_r$ is nullhomotopic whenever $r > 0$. In this case, we know by assumption that $r \in R$ is a unit so consider the map

$$\begin{aligned} h : \textstyle\bigwedge^i(M) \otimes_k A_{r-i} &\to \textstyle\bigwedge^{i+1}(M) \otimes_k A_{r-i-1} \\ h\,(m \otimes (x_{\beta_1} \ldots x_{\beta_{r-i}})) &= -\frac{1}{r} \sum_j (m \wedge x_{\beta_j}) \otimes (x_{\beta_1} \ldots \hat{x}_{\beta_j} \ldots x_{\beta_{r-i}}).\end{aligned}$$

A straightforward basis-wise check shows $h \circ d + d \circ h = \mathrm{id}$, i.e. h is a chain homotopy between id and the zero map. □

Koszul complexes in geometry. We use Koszul complexes in the following geometric setting– on the affine scheme $\mathbf{X} = \operatorname{Spec} A$, an $(n+1)$-tuple $(a_0, \ldots, a_n)$ in A can be seen as a global section of the free sheaf $\mathcal{E} = \mathcal{O}_{\mathbf{X}}^{\oplus(n+1)}$. Then the Koszul complex associated to $(a_0, \ldots, a_n)$ yields a complex of coherent sheaves, given by

$$\mathcal{K}_{\mathbf{X}}(s): \qquad 0 \to \textstyle\bigwedge^{n+1}\mathcal{E}^\vee \to \bigwedge^n \mathcal{E}^\vee \to \cdots \to \bigwedge^2 \mathcal{E}^\vee \to \mathcal{O}_{\mathbf{X}} \to \mathcal{O}_{\mathbb{V}(s)} \to 0$$

where $\mathbb{V}(s)$ is the zero locus of s, i.e. the closed subscheme corresponding to the ideal $(a_0, \ldots, a_n)$. If $(a_0, \ldots, a_n)$ is a regular sequence then we have shown that the complex $\mathcal{K}_{\mathbf{X}}(s)$ is exact, giving a locally free resolution of $\mathcal{O}_{\mathbb{V}(s)}$.

The construction automatically extends to arbitrary schemes $\mathbf{X}$– given global section of a locally free sheaf $\mathcal{E} \in \text{coh-}\mathbf{X}$, we can cover $\mathbf{X}$ by affine open subschemes $\mathbf{X} = \bigcup_\alpha \mathbf{U}_\alpha$; then the associated Koszul complexes $\mathcal{K}_{\mathbf{U}_\alpha}(s|_{\mathbf{U}_\alpha})$ glue to give a Koszul complex $\mathcal{K}_{\mathbf{X}}(s)$ of coherent sheaves on $\mathbf{X}$. If s yields a regular sequence on each affine patch, then it is immediate that the Koszul complex associated to s is a locally free resolution of $\mathcal{O}_{\mathbb{V}(s)}$.

2.3 Beilinson's theorem

We briefly discuss the sheaves involved before proving the existence of Beilinson's resolution.

Sheaves on $\mathbb{P}^n$. Writing $x_{\alpha/0} = x_\alpha / x_0$ $(1 \leq \alpha \leq n)$, we see that the standard affine patch $\mathbf{U}_0 = \mathbb{P}^n \setminus \mathbb{V}(x_0)$ has coordinate ring $A^0 = k[x_{1/0}, \ldots, x_{n/0}]$. Serre's correspondence between graded A-modules and coherent sheaves on $\mathbb{P}^n$ (see Section II.5 of Hartshorne (2008)) asserts that any $\mathcal{M} \in \text{coh-}\mathbb{P}^n$ is the sheafification of some $M \in A\text{-grMod}$. Restricted to to the affine piece $\mathbf{U}_0$, such a sheaf is completely determined by the module of its global sections– we will view this

A^0-module M^0 as the degree 0 piece of the localisation $M[\frac{1}{x_0}]$. The transition functions for the sheaf come from the natural isomorphisms $M^0[\frac{x_0}{x_\alpha}] \cong M^\alpha[\frac{x_\alpha}{x_0}]$.

As usual, $\mathcal{O}_{\mathbb{P}^n}(j)$ denotes the jth twist of the structure sheaf on $\mathbb{P}^n$, which corresponds to the graded module $A\langle j\rangle$. On the patch $\mathbf{U}_0$, the corresponding A^0-module $A^0\langle j\rangle$ is free of rank 1, with distinguished generator x_0^j.

The cotangent sheaf Ω of $\mathbb{P}^n$ can be defined via the *Euler exact sequence* (see Section II.8 of Hartshorne (2008)) as the kernel of

$$\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n}\,; \quad (s_0, \dots, s_n) \longmapsto x_0 s_0 + \dots + x_n s_n.$$

Since the twisted sheaves $\mathcal{O}_{\mathbb{P}^n}(j)$ are flat, we have for each $j \in \mathbb{Z}$ an exact sequence

$$0 \to \Omega(j) \to \mathcal{O}_{\mathbb{P}^n}(j-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(j) \longrightarrow 0.$$

Writing $dx_{\alpha/0} = (x_0 e_\alpha - x_\alpha e_0)/x_0^2$ where $e_0, \dots, e_n$ are the standard generators of A^{n+1}, we see that the A^0-module corresponding to $\Omega(j)|_{\mathbf{U}_0}$ is freely generated by $x^j\, dx_{1/0}, \dots, x^j\, dx_{n/0}$. Write $D^0\langle j\rangle$ for this module. In particular, note that Ω and its twists are rank n locally free sheaves.

We write $\Omega^i(j)$ for $\bigwedge^i \Omega(j)$, the jth twist of the sheaf of i-forms on $\mathbb{P}^n$. On the affine patch $\mathbf{U}_0$, this corresponds to the module $\bigwedge^i_{A^0}(D^0\langle j\rangle)$.

Resolution of the diagonal. Let $\iota : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the inclusion of the diagonal subscheme Δ, and write $\pi_1, \pi_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ for the two coordinate projections. We now prove that $\mathcal{O}_\Delta = \iota_* \mathcal{O}_{\mathbb{P}^n}$ admits a locally free resolution by sheaves of the form $\pi_1^* \mathcal{O}(-j) \otimes \pi_2^* \Omega^j(j)$. We build the resolution locally in the form of a Koszul complex on each standard affine patch.

Theorem 2.6 (Beilinson (1978))**.** There is an exact sequence in coh-$(\mathbb{P}^n \times \mathbb{P}^n)$ of the form

$$0 \to \pi_1^*\mathcal{O}(-n) \otimes \pi_2^*\Omega^n(n) \to \cdots \to \pi_1^*\mathcal{O}(-1) \otimes \pi_2^*\Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

Proof. Write $(x_0 : \dots : x_n)$ and $(y_0 : \dots : y_n)$ for the homogeneous coordinates on the two copies of $\mathbb{P}^n$, and let $\mathbf{U}_\alpha = \mathbb{P}^n \setminus \mathbb{V}(x_\alpha)$ and $\mathbf{V}_\alpha = \mathbb{P}^n \setminus \mathbb{V}(y_\alpha)$ be the standard affine patches. We denote modules corresponding to sheaves on $\mathbf{U}_\alpha$ with a subscript x, and on $\mathbf{V}_\alpha$ with a subscript y. Thus $\mathbf{V}_\alpha$ has coordinate ring $A_y^0 \cong k[y_{1/0}, \dots, y_{n/0}]$.

We restrict to the affine open $\mathbf{U}_0 \times \mathbf{V}_0 \subset \mathbb{P}^n \times \mathbb{P}^n$, which has coordinate ring

$$A^{00} = A_x^0 \otimes_k A_y^0 \cong k[x_{1/0}, \dots, x_{n/0}, y_{1/0}, \dots, y_{n/0}].$$

Then the pullback $\pi_1^*\mathcal{O}(-j)$ corresponds to the A^{00}-module $A_x^0\langle -j\rangle \otimes_k A_y^0$, which is freely generated by x_0^{-j}. Likewise, the pullback $\pi_2^*\Omega^j(j)$ comes from the module $\bigwedge^j D_y^0(j) \otimes_k A_x^0$, which is freely generated over A^{00} by elements of the form $dy_{\alpha_1/n} \wedge \dots \wedge dy_{\alpha_i/n}$. □

3 The Bernstein-Gel'fand-Gel'fand correspondence

describe section

3.1 Data

3.1.1 Symmetric and exterior algebras

Given an $n+1$-dimensional k-vector space V, the *tensor algebra* is the k-vector space

$$T(V) = k \oplus \bigoplus_{i \geq 1} (\underbrace{V \otimes_k V \otimes_k \dots \otimes_k V}_{i \text{ times}})$$

with a product $\nabla : T(V) \otimes T(V) \to T(V)$ induced by the natural identifications $V^{\otimes i} \otimes V^{\otimes j} \xrightarrow{\sim} V^{\otimes (i+j)}$. This is an associative algebra with a natural $\mathbb{Z}_{\geq 0}$-grading. The *symmetric algebra* $\mathrm{Sym}^\bullet(V)$ and the *exterior algebra* $\bigwedge^\bullet(V)$ are then the graded algebras defined as quotients of $T(V)$ by certain two-sided ideals, namely

$$\mathrm{Sym}^\bullet(V) = \frac{T(V)}{(x \otimes y - y \otimes x \mid x, y \in V)}, \qquad \bigwedge^\bullet(V) = \frac{T(V)}{(x \otimes x \mid x \in V)}.$$

Since the ideals are generated by homogeneous elements, these algebras inherit gradings from $T(V)$.

We continue to use ∇ for the product morphism on either algebra, though the corresponding bilinear map on $\bigwedge^\bullet V$ is often written $\wedge$.

Remark 3.1. We can repeat the above constructions in the category of R-modules for any ring R. In this case, we write $T_R(M)$, $\mathrm{Sym}^\bullet_R(M)$, $\Lambda^\bullet_R(M)$ respectively for the tensor, symmetric, and exterior algebras over $M \in R$-Mod. In particular,

$$T(M) = R \oplus \bigoplus_{i \geq 1} (\underbrace{M \otimes_R M \otimes_R \dots \otimes_R M}_{i \text{ times}}).$$

Since we are primarily concerned with the algebras $A = \mathrm{Sym}^\bullet(X)$ and $A^! = \bigwedge^\bullet(\Xi)$, we redefine the grading on $A^!$ as $A^!_{-i} = \Lambda^i \Xi$. This amounts to a change of sign from the usual grading, but the convention ensures that the dual vector spaces X and Ξ lie in degrees 1 and -1 in their respective algebras.

3.1.2 The exterior coalgebra

The *exterior coalgebra* on Ξ is defined as the linear dual of $A^!$, written $A^{¡} := \mathrm{Hom}_k(A^!, k)$. $A^{¡}$ has the $\mathbb{Z}$-grading $A^{¡}_i = \mathrm{Hom}_k(A^!_{-i}, k)$ and is naturally an $A^!$-module via $a \cdot f(a') = (-1)^{\deg a} f(a \wedge a')$ for $a \in A^!$ homogeneous, $f \in \mathrm{Hom}(A^!, k)$. Moreover, for any k-vector space N we have the natural isomorphism of $A^!$-modules $\mathrm{Hom}_k(A^!, N) \cong A^{¡} \otimes_k N$.

Choosing a basis x_i for X fixes an isomorphism $X \cong \mathrm{Hom}_k(\Xi, k) = A^{¡}_1$, which can be extended to get the isomorphism of graded k-vector spaces

$$A^{¡} = \bigoplus_i \mathrm{Hom}_k(\Lambda^i \Xi, k) \cong \bigoplus_i \Lambda^i X = \bigwedge^\bullet(X).$$

Write $\tau : A^{¡} \to A$ for the k-linear map which identifies the subspaces of $A^{¡}$ and A corresponding to X, and is 0 elsewhere.

The coproduct on $A^{¡}$. Being the linear dual of a finite dimensional algebra, $A^{¡}$ has a natural (coassociative counital) coalgebra structure which comes from dualising the (associative unital) product $\nabla : A^! \otimes_k A^! \to A^!$. This is called the *shuffle coproduct*, and we give an explicit description of it as follows. Given a collection of indices $\underline{\alpha} = \{\alpha_1 < \ldots < \alpha_i\} \subseteq \{0, \ldots, n\}$, write $x_{\underline{\alpha}}$ for the standard basis element of $A^{¡}$ given by $x_{\alpha_1} \wedge x_{\alpha_2} \wedge \ldots \wedge x_{\alpha_i}$ (in particular, $x_\emptyset = 1$). The vector $\xi_{\underline{\alpha}}$ is defined similarly. We say a tuple $(\underline{\beta}, \underline{\beta'})$ of subsets is a *break* of $\underline{\alpha}$ if $(\beta_1 < \ldots < \beta_p, \beta'_1 < \ldots < \beta'_q)$ is a permutation of $(\alpha_1 < \ldots < \alpha_i)$ (in other words, $\underline{\alpha} = \underline{\beta} \sqcup \underline{\beta'}$). The *sign* of this break, written $\langle \underline{\beta}, \underline{\beta'} \rangle$, is defined to be the sign of the corresponding permutation. Thus we have have

$$\nabla(x_{\underline{\beta}} \otimes x_{\underline{\beta'}}) = x_{\underline{\beta}} \wedge x_{\underline{\beta'}} = \langle \underline{\beta}, \underline{\beta'} \rangle \, x_{\underline{\alpha}}.$$

This allows us to write the coproduct on $A^{¡}$ as

$$\Delta(x_{\underline{\alpha}}) = \sum_{(\underline{\beta}, \underline{\beta'}) \in \mathrm{br}(\underline{\alpha})} \langle \underline{\beta}, \underline{\beta'} \rangle \, x_{\underline{\beta}} \otimes x_{\underline{\beta'}}$$

where $\mathrm{br}(\underline{\alpha})$ is the set of all breaks of $\underline{\alpha}$. Recalling that $A^{¡} \otimes_k A^{¡}$ is $\mathbb{Z}$-graded with $\bigoplus_{p+q=i} A^{¡}_p \otimes A^{¡}_q$ in degree i, we observe that the map Δ respects grading hence $A^{¡}$ is a *graded coalgebra*.

3.1.3 Graded chain complexes

Objects of $\mathcal{C}(A\text{-grMod})$ are chain complexes of graded A-modules in which the differentials are morphisms in A-grMod (i.e. A-module homomorphisms which preserve degree). Such an object can be viewed as a $\mathbb{Z}^2$-graded k-vector space $\bigoplus_{i,j} M^i_j$ with an endomorphism d (the differential) such that

$d \circ d = 0$,

d has degree $(1, 0)$ i.e. $d(M^i_j) \subseteq M^{i+1}_j$, and

for each $i \in \mathbb{Z}$, $M^i_\bullet = \bigoplus_j M^i_j$ is a graded A-module.

Likewise, an object $N \in \mathcal{C}(A^!\text{-grMod})$ can be seen as a $\mathbb{Z}^2$-graded k-vector space $\oplus_{i,j} N^i_j$ with a differential ∂ of degree $(1, 0)$. We shall use the two viewpoints on interchangeably, switching between them whenever convenient to provide a clearer picture. In particular, the ability to view a complex as a single module with additional structure allows for cleaner definitions and proofs, see for instance Theorem 3.3.

For a chain complex $\mathbf{M} = \bigoplus_{i,j} M^i_j$, we say the lower indices denote the *internal* (or *Adam's*) grading, while the upper indices denote the *differential* (or *cohomological*) degree. We use '$\langle \cdot \rangle$' to denote shifts in Adam's gradings, continuing to use '$[\cdot]$' to denote shifts in differential gradings. Thus for example we have $M\langle q \rangle^i_j = M^i_{q+j}$.

3.2 Twisted functors

We now define additive functors

$$\mathcal{C}(A\text{-grMod}) \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{C}(A^!\text{-grMod})$$

on which the BGG correspondence is based. In the framework of $\mathbb{Z}^2$-graded vector spaces described in Section 3.1.3, we have

$$\bigoplus_{i,j} F(\mathbf{M})^i_j \cong \mathrm{Hom}_k\left(A^!, \bigoplus_{p,q} M^p_q\right) = A^{¡} \otimes_k \left(\bigoplus_{p,q} M^p_q\right), \qquad \bigoplus_{i,j} G(\mathbf{N})^i_j \cong A \otimes_k \left(\bigoplus_{p,q} N^p_q\right).$$

However, care is needed to define the gradings and differentials since, for example, naïvely applying the functor $\mathrm{Hom}_k(A^!, -)$ would lose all A-module structure. The key is to modify the naïve differential by adding a 'twist' as in Example 0.2.

3.2.1 Defining the functor F

We first define F on the category A-grMod, seen as the full subcategory of $\mathcal{C}(A\text{-grMod})$ whose objects are complexes concentrated in differential degree 0. If $M^0_\bullet$ is a graded A-module, we define $F(M^0_\bullet)$ to be the chain complex of $A^!$-modules given by a

$$\begin{aligned} \cdots \to A^!\langle -i\rangle \otimes_k M^0_i &\xrightarrow{\partial} A^!\langle -i-1\rangle \otimes_k M^0_{i+1} \to \cdots \\ a \otimes m &\longmapsto \sum_\alpha \xi_\alpha a \otimes x_\alpha m. \end{aligned}$$

The module $A^!\langle -i\rangle \otimes_k M^0_i$ is naturally isomorphic to $\mathrm{Hom}_k(A^!\langle i\rangle, M^0_i)$ and inherits an Adam's grading from $A^!$ with the vector space $A^!_{j-i} \otimes_k M^0_i$ forming the jth graded piece. These shifts in grading have been chosen precisely so that the differential ∂ has degree $(1, 0)$, while the graded commutativity of $A^!$ implies $\partial \circ \partial = 0$. Thus we indeed have a chain complex of $A^!$-modules.

Given a morphism $M^0_\bullet \to M^1_\bullet$ in A-grMod, the functoriality of tensor products induces $A^!$-module homomorphisms $A^!\langle -i\rangle \otimes_k M^0_i \to A^!\langle -i\rangle \otimes_k M^0_i$ which are compatible with the differentials (i.e. the natural squares commute). Thus we have an additive functor $F : A\text{-grMod} \to \mathcal{C}(A^!\text{-grMod})$.

To extend F to arbitrary chain complexes $\mathbf{M} = (\bigoplus_{i,j} M^i_j, d) \in \mathcal{C}(A\text{-grMod})$, we observe that the functoriality of F gives us a (commuting) bicomplex

$$\begin{tikzcd}
\vdots & & \vdots & \vdots & \\
F(M^{i+1}_\bullet) \arrow[u] & \cdots \arrow[r] & A^!\langle -j\rangle \otimes_k M^{i+1}_j \arrow[u] \arrow[r] & A^!\langle -j-1\rangle \otimes_k M^{i+1}_{j+1} \arrow[u] \arrow[r] & \cdots \\
F(M^i_\bullet) \arrow[u] & \cdots \arrow[r] & A^!\langle -j\rangle \otimes_k M^i_j \arrow[u] \arrow[r] & A^!\langle -j-1\rangle \otimes_k M^i_{j+1} \arrow[u] \arrow[r] & \cdots \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{1}$$

where the vertical maps are $1 \otimes d$. Define $F(\mathbf{M})$ to be the total complex of this bicomplex, i.e. $F(\mathbf{M})$ is given by

$$\begin{aligned} \cdots \to \bigoplus_{p+q=i} A^!\langle -q\rangle \otimes_k M^p_q &\xrightarrow{\partial} \bigoplus_{p+q=i+1} A^!\langle -q\rangle \otimes_k M^p_q \to \cdots, \\ \partial : a \otimes m &\longmapsto a \otimes dm + (-1)^{\#m} \sum_\alpha \xi_\alpha a \otimes x_\alpha m \end{aligned} \tag{2}$$

where $\#m$ is the differential degree of $m \in \mathbf{M}$. It is clear that each $F(\mathbf{M})^i_\bullet = \bigoplus_{p+q=i} A^!\langle -q\rangle \otimes_k M^p_q$ is a graded $A^!$ module, and the signs introduced in the total complex construction ensure $\partial \circ \partial = 0$. An explicit check confirms ∂ has degree $(1, 0)$, so we indeed have an object of $\mathcal{C}(A^!\text{-grMod})$.

The twist using comodules. Observe that the differential ∂ differs from the naïve differential $1 \otimes d$ on the tensor product by the horizontal maps, which are the 'twists' we have been alluding to. These have a nice description using the fact that a graded module $N_\bullet \in A^!$-grMod has the structure of a graded A^{i}-comodule via the map

$$\begin{aligned} \Delta : \quad N_\bullet &\longrightarrow N_\bullet \otimes_k A^{\mathsf{i}} \\ n &\longmapsto \sum_{\underline{\alpha} \subseteq \{0,\dots,n\}} \xi_{\underline{\alpha}} n \otimes x_{\underline{\alpha}}. \end{aligned}$$

Applying this idea to the $A^!$-modules $A^{\mathsf{i}}\langle -i \rangle$, we get a commuting square

$$\begin{tikzcd} \bigoplus_{u+v=j-q} A^{\mathsf{i}}_u \otimes_k A^{\mathsf{i}}_v \otimes_k M^{i-q}_q \arrow[r, "1\otimes\tau\otimes 1"] & A^{\mathsf{i}}_{j-q-1} \otimes_k A_1 \otimes_k M^{i-q}_q \arrow[d, "1\otimes\nabla"] \\ A^{\mathsf{i}}_{j-q} \otimes_k M^{i-q}_q \arrow[u, "\Delta\otimes 1"] \arrow[r, "(-1)^{i-q}(\partial - 1\otimes d)"] & A^{\mathsf{i}}_{j-q-1} \otimes_k M^{i-q}_{q+1} \end{tikzcd}$$

where $\nabla : A \otimes_k M^{i-q}_\bullet \to M^{i-q}_\bullet$ defines the A-module structure on M, and $\tau : A^{\mathsf{i}} \to A^!$ is the morphism defined in Section 3.1.2 which identifies A^{i}_1 with A_1, annihilating other graded pieces.

In summary, $F(\mathbf{M})$ as a $\mathbb{Z}^2$-graded vector space is simply $A^{\mathsf{i}} \otimes_k \mathbf{M}$ with (i,j)th piece

$$F(\mathbf{M})^i_j = \bigoplus_{p+q=i} A^{\mathsf{i}}_{j-q} \otimes_k M^p_q$$

and differential given on $A^{\mathsf{i}}_{j-q} \otimes_k M^p_q$ by

$$1 \otimes d + (-1)^p (1 \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1).$$

3.2.2 The left adjoint to F

The functor $G : \mathcal{C}(A^!\text{-grMod}) \to \mathcal{C}(A\text{-grMod})$ is analogously defined, and maps the chain complex $\mathbf{N} = (\bigoplus_{i,j} N, \partial)$ to $G(\mathbf{N})$ given by

$$\cdots \to \bigoplus_{p-q=i} N^p_q \otimes_k A\langle -q \rangle \xrightarrow{d} \bigoplus_{p-q=i+1} N^p_q \otimes_k A\langle -q \rangle \to \cdots \tag{3}$$
$$d : n \otimes a \longmapsto \partial n \otimes a + (-1)^{\#n} \sum_{\alpha} \xi_\alpha n \otimes x_\alpha a$$

where $\#n$ is the differential degree of $n \in \mathbf{N}$. The Adam's grading on each $G(\mathbf{N})^i_\bullet$ is inherited from A, and is given by

$$G(\mathbf{N})^i_j = \bigoplus_{p-q=i} N^p_q \otimes_k A_{j-q}.$$

Recalling that every $A^!$-module is a A^{i}-comodule (see Section 3.2.1), we can use the comodule structure-map $\Delta : N^i_\bullet \to N^i_\bullet \otimes A^{\mathsf{i}}$ to define the differential on $N^p_q \otimes_k A_{j-q}$ as

$$\partial \otimes 1 + (-1)^p (1 \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1).$$

The adjunction. Having defined the functors F and G, we show that G is left adjoint to F. Spelled out this means given $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$ and $\mathbf{N} \in \mathcal{C}(A^!\text{-grMod})$, there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathcal{C}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \cong \mathrm{Hom}_{\mathcal{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})).$$

At its heart this is a $\otimes$-Hom adjunction, as we shall illustrate in the special case of module categories below.

Lemma 3.2. Given modules $M \in A\text{-Mod}$ and $N \in A^!\text{-Mod}$, there are natural isomorphisms of abelian groups

$$\mathrm{Hom}_A(A \otimes_k N, M) \cong \mathrm{Hom}_k(N, M) \cong \mathrm{Hom}_{A^!}(N, \mathrm{Hom}_k(A^!, M)).$$

Proof. Consider the $(A, A^!)$-bimodule $T = A \otimes_k A^!$. Then the standard $\otimes$-Hom adjunction for bimodules (Bourbaki 1989) gives us a natural isomorphism

$$\mathrm{Hom}_A(T \otimes_{A^!} N, M) \cong \mathrm{Hom}_{A^!}(N, \mathrm{Hom}_A(T, M)).$$

Then observe that there are natural isomorphisms

$$T \otimes_{A^!} N \cong A \otimes_k A^! \otimes_{A^!} N \cong A \otimes_k N, \qquad \mathrm{Hom}_A(T, M) \cong \mathrm{Hom}_A(A \otimes_k A^!, M) \cong \mathrm{Hom}_k(A^!, M).$$

The isomorphism with $\mathrm{Hom}_k(N, M)$ comes similarly from treating A as an (A, k)-bimodule. □

We now exhibit the general adjunction for F and G, and it is here that the flexibility of interpreting a chain complex $\mathbf{M}$ of graded modules as a single $\mathbb{Z}^2$-graded module $\bigoplus_{i,j} M^i_j$ (see Section 3.1.3) really comes handy. Interpreting $\mathcal{C}(A\text{-grMod})$ as a subcategory of A-Mod (likewise for $A^!$), we use Lemma 3.2 to identify $\mathrm{Hom}_{\mathcal{C}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \subset \mathrm{Hom}_A(\mathbf{N} \otimes_k A, \mathbf{M})$ and $\mathrm{Hom}_{\mathcal{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})) \subset \mathrm{Hom}_{A^!}(\mathbf{N}, \mathrm{Hom}_k(A^!, \mathbf{M}))$ with the same subgroup of $\mathrm{Hom}_k(\mathbf{N}, \mathbf{M})$.

Theorem 3.3 (Bernstein et al. (1978))**.** The functor G, from the category of complexes of graded $A^!$-modules to the category of complexes of graded A-modules, is a left adjoint to the functor F.

Proof. Given $\bar{\varphi} \in \mathrm{Hom}_A(G(\mathbf{N}), \mathbf{M})$, the corresponding map $\varphi \in \mathrm{Hom}_k(\mathbf{N}, \mathbf{M})$ found in Lemma 3.2 is given by $\varphi(n) = \bar{\varphi}(n \otimes 1)$. Thus $\bar{\varphi}$ has degree $(0, 0)$ if and only if $\bar{\varphi}(N^i_j \otimes_k A_0) \subseteq M^{i-j}_j$, if and only if $\varphi(N^i_j) \subseteq M^{i-j}_j$. Moreover for $n \in N^i_j$, direct computation shows

$$(d_M \circ \bar{\varphi} - \bar{\varphi} \circ d_{G(\mathbf{N})})(n \otimes 1) \;=\; (d_M \circ \varphi - \varphi \circ \partial_N)(n) - (-1)^i \sum_{\alpha} x_\alpha \varphi(\xi_\alpha n),$$

thus $\bar{\varphi}$ is a morphism in $\mathcal{C}(A\text{-grMod})$ if and only if

$$\varphi(N^i_j) \subseteq M^{i-j}_j, \quad \text{and} \quad d_M \circ \varphi - \varphi \circ \partial_N = \sum_{\alpha} x_\alpha \varphi \xi_\alpha \tag{4}$$

where we write $\sum_\alpha x_\alpha \varphi \xi_\alpha$ for the map that takes $n \in N^i_j$ to $(-1)^i \sum_\alpha x_\alpha \varphi(\xi_\alpha n)$.

Likewise given $\varphi^! \in \mathrm{Hom}_{A^!}(\mathbf{N}, F(\mathbf{M}))$, repeating the above argument shows $\varphi^!$ is an element of $\mathrm{Hom}_{\mathcal{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})$ if and only if the corresponding map $\varphi \in \mathrm{Hom}_k(\mathbf{N}, \mathbf{M})$ satisfies (4). This shows that the isomorphisms given in Lemma 3.2 restrict to isomorphisms

$$\mathrm{Hom}_{\mathcal{C}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \cong \{\varphi \in \mathrm{Hom}_k(\mathbf{N}, \mathbf{M}) \text{ satisfying (4)}\} \cong \mathrm{Hom}_{\mathcal{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M}))$$

thereby showing G is left adjoint to F. □

3.3 Koszul resolutions

Given a complex $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$, the adjunction $F \vdash G$ takes the identity morphism

$$1_{F(\mathbf{M})} \in \mathrm{Hom}_{\mathcal{C}(A^!\text{-grMod})}(F(\mathbf{M}), F(\mathbf{M}))$$

to a map

$$\varepsilon_{\mathbf{M}} \in \mathrm{Hom}_{\mathcal{C}(A\text{-grMod})}(G(F(\mathbf{M})), \mathbf{M}).$$

The natural transformation $\varepsilon : G \circ F \to \mathbf{1}$ thus obtained is called the *counit* of the adjunction, and we say the morphism $\varepsilon_{\mathbf{M}}$ is the *component* of the transformation at $\mathbf{M}$. Likewise, there is the dual notion called the *unit* of the adjunction, which is a natural transformation $\eta : \mathbf{1} \to F \circ G$ giving, for any $\mathbf{N} \in \mathcal{C}(A^!\text{-grMod})$, a morphism $\eta_{\mathbf{N}} : \mathbf{N} \to F(G(\mathbf{N}))$.

Begin with the following observation.

Proposition 3.4. The functor F maps elements of $\mathcal{C}(A\text{-grMod})$ to complexes of injective $A^!$-modules, and the functor G maps elements of $\mathcal{C}(A^!\text{-grMod})$ to complexes of free A-modules.

Proof. The statement for G is immediate from definition, so we prove that for any $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$, the modules $F(\mathbf{M})^i_\bullet$ are injective over $A^!$.

Recall from Weibel (2003) (Proposition 2.3.10) that if $R : \mathcal{B} \to \mathcal{A}$ is an additive functor which is right adjoint to an exact functor $L : \mathcal{A} \to \mathcal{B}$, then for any injective object $I \in \mathcal{B}$ the object $R(I) \in \mathcal{A}$ is injective. Applying this to the pair of adjoint functors

$$R = \mathrm{Hom}_k(A^!, -) : k\text{-grMod} \to A^!\text{-grMod}, \quad L = (- \otimes_k A^!) : A^!\text{-grMod} \to k\text{-grMod}$$

and observing that L is exact since all k-vector spaces are flat over k, see that R preserves injectives. But every k-vector space is also injective, so the $A^!$-modules $A^i\langle -q\rangle \otimes_k M^p_q \cong R(M^p_q)$ appearing in (2) are all injective. To conclude, observe that the k-algebra $A^!$ is finite dimensional hence noetherian. By the theorem of Bass & Papp (see Lam (1999), Theorem 3.46) which asserts that a ring is (left) noetherian if and only if any direct sum of injective modules over it is injective, we are done. ☐

We show that the component $\varepsilon_{\mathbf{M}} : G(F(\mathbf{M})) \to \mathbf{M}$ is, in fact, a free resolution of the complex $\mathbf{M}$ and dually, the component $\eta_{\mathbf{N}}$ is an injective resolution of $\mathbf{N}$. A special but important case of this phenomenon is when the complex is

$$\cdots \to 0 \longrightarrow k \longrightarrow 0 \to \cdots ,$$

and this will be central in proving the result for general complexes.

3.3.1 The Koszul complex

The 1-dimensional vector space k can be considered a graded A-module concentrated degree 0, such that all $x_i \in A$ annihilate k. Then $F(k)$ is the complex $0 \to \Lambda^\bullet(X) \to 0$ concentrated in differential degree 0. We compute the complex $G(F(k))$ to be

$$0 \to A^i_{n+1} \otimes_k A\langle -n-1\rangle \to A^i_n \otimes_k A\langle -n\rangle \to \dots \to A^i_1 \otimes_k A\langle -1\rangle \to A^i_0 \otimes_k A \to 0 \qquad (5)$$
$$(x_{\alpha_1} \wedge \dots \wedge x_{\alpha_i}) \otimes 1 \longmapsto \sum_j (-1)^{i+j-1} (x_{\alpha_1} \wedge \dots \hat{x}_{\alpha_j} \dots \wedge x_{\alpha_i}) \otimes x_{\alpha_j}$$

where $\hat{\cdot}$ denotes omission of a term.

Observing the term in differential degree i is isomorphic to $\Lambda^i_A(A^{n+1})$, we can recognise the above complex as the *Koszul complex* associated to the regular sequence $(x_0, ..., x_n) \in A$. Then standard results on Koszul complexes found in Eisenbud (1995) (Chapter 17 and relevant sections of Appendix 2) show that this complex has cohomology 0 everywhere except in degree 0, where the cohomology is $A/(x_1, ..., x_n) \cong k$. We provide below a direct proof of the result in the special case when $\mathrm{char}(k) = 0$.

A similar argument shows that the complex $F(G(k))$ is exact everywhere except in degree 0, where it has cohomology k. Thus we have resolutions

$$G(F(k)) \to k \to 0, \qquad 0 \to k \to F(G(k))$$

of k by free A-modules and by injective $A^!$-modules, respectively. It is not hard to see that these maps are precisely the ones given by the counit and the unit of adjunction.

3.3.2 Resolutions in general

We show that the counit (resp. unit) gives a free (resp. injective) resolution, by first showing this is the case for graded modules (i.e. complexes concentrated in a single degree). When $\mathbf{M}$ is a graded A-module, we will show that the complex $G(F(\mathbf{M}))$ is 'built up' from the tensor product of $\mathbf{M}$ and the Koszul complex of k.

Lemma 3.5 (Eisenbud et al. (2003))**.** If $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$ is a chain complex concentrated in differential degree 0, then the natural map

$$\varepsilon_{\mathbf{M}} : G(F(\mathbf{M})) \to \mathbf{M}$$

is an epimorphism and induces an isomorphism on cohomology. Likewise, if $\mathbf{N} \in \mathcal{C}(A^!\text{-grMod})$ is a chain complex concentrated in differential degree 0, then the natural map

$$\eta_{\mathbf{N}} : \mathbf{N} \to F(G(\mathbf{N}))$$

is a monomorphism and induces an isomorphism on cohomology.

Proof. We first show that the complex $G(F(\mathbf{M}))$ has the same cohomology as the complex $\mathbf{M}$. Direct computation shows $G(F(\mathbf{M}))$ is given by

$$\cdots \to \bigoplus_p A^!_{-i} \otimes_k M^0_p \otimes_k A\langle i - p\rangle \longrightarrow \bigoplus_p A^!_{-i-1} \otimes_k M^0_p \otimes_k A\langle i + 1 - p\rangle \to \cdots,$$
$$a \otimes m \otimes b \longmapsto \sum_\alpha \xi_\alpha a \otimes x_\alpha m \otimes b \; + \; (-1)^{\deg m} \sum_\alpha \xi_\alpha a \otimes m \otimes x_\alpha b$$

so the strand of this in Adam's degree r is seen to be the total complex of the (commuting) bicomplex

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & A^!_{-i}\otimes_k M^0_p \otimes_k A_{i-p+r} \arrow[r]\arrow[d] & A^!_{-i-1}\otimes_k M^0_p\otimes_k A_{i-p+r+1} \arrow[r]\arrow[d] & \cdots \\ \cdots \arrow[r] & A^!_{-i-1}\otimes_k M^0_{p+1}\otimes_k A_{i-p+r} \arrow[r]\arrow[d] & A^!_{-i-2}\otimes_k M^0_{p+1}\otimes_k A_{i-p+r+1}\arrow[r]\arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd} . \qquad (6)$$

Here pth row is obtained by applying $(- \otimes_k M^0_p)$ to the complex $G(F(k))_{r-p}$, so is exact from Proposition 2.5 unless $p = r$. Moreover, the r th row is

$$\cdots 0 \to M^0_r \to 0 \to \cdots .$$

Thus first page of the spectral sequence (starting with horizontal cohomology) of (6) is

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots & M^0_r \arrow[d] & 0 \arrow[d] & \cdots \\ \cdots & 0 \arrow[d] & 0 \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd} .$$

By the theorem on spectral sequences, we conclude that the total complex $G(F(\mathbf{M}))_r$ has cohomology insert theorem

$$H^k(G(F(\mathbf{M}))_r) = \begin{cases} M^0_r, & k = 0 \\ 0, & \text{otherwise} \end{cases} .$$

Now it suffices to show that the map $\varepsilon_{\mathbf{M}} : G(F(\mathbf{M}))^0_r \to M^0_r$ is the cokernel of $G(F(\mathbf{M}))^{-1}_r \to G(F(\mathbf{M}))^0_r$. But this is immediate because the sequence

$$\begin{array}{ccccccc} \bigoplus_p X\otimes_k M^0_p \otimes_k \mathrm{Sym}^{\,r-p-1}(X) & \longrightarrow & \bigoplus_p M^0_p \otimes_k \mathrm{Sym}^{\,r-p}(X) & \longrightarrow & M^0_r & \longrightarrow & 0 \\ x_\alpha \otimes m \otimes a & \longmapsto & m\otimes x_\alpha a + (-1)^{\deg m} x_\alpha m \otimes a & & & & \\ & & m \otimes a & \longmapsto & (-1)^{\deg m} am & & \end{array}$$

is (split) exact.

The analogous statement about graded $A^!$-modules follows from a similar calculation. $\square$

The argument to extend this result to all chain complexes is purely formal.

Theorem 3.6 (Eisenbud et al. (2003))**.** For any complex $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$, the complex $G(F(\mathbf{M}))$ is a free resolution of $\mathbf{M}$ which surjects onto $\mathbf{M}$, and for any complex $\mathbf{N} \in \mathcal{C}(A^!\text{-grMod})$, the complex $F(G(\mathbf{N}))$ is an injective resolution of $\mathbf{N}$ which $\mathbf{N}$ injects into.

Proof. Given $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$, the surjectivity of $\varepsilon_{\mathbf{M}} : G(F(\mathbf{M})) \to \mathbf{M}$ can be checked on the level of underlying $\mathbb{Z}^2$-graded modules. The map is given on the (i,j)th component by

$$\bigoplus_{p,q} \operatorname{Hom}_k(A^!_{q-i}, M^q_{p-q}) \otimes_k A_{j-p+i} \longrightarrow M^i_j$$
$$f \otimes a \mapsto af(1)$$

hence any $m \in M^i_j$ can be written $\varepsilon_{\mathbf{M}}(f_m \otimes 1)$ where $f_m : A^!_0 \to M^i_j$ is the function $f_m(1) = m$. check sign

To prove that the induced map on cohomology is an isomorphism we first reduce the problem to bounded complexes using formal properties of the functors, and then induct on the length of the complex to reduce our problem to Lemma 3.5. The key properties we use are naturality of ε, and the fact that G, F and cohomology functors all preserve direct limits. prove

Any complex $\mathbf{M} \in \mathcal{C}(A\text{-grMod})$ can be written as the direct limit of bounded complexes $(\mathbf{M}^b)_{b\in B}$, giving us commuting diagrams

$$\begin{tikzcd} G(F(\mathbf{M}^b)) \arrow[r] \arrow[d, "\varepsilon_{\mathbf{M}^b}"] & G(F(\mathbf{M})) \arrow[r, equal] \arrow[d, "\varepsilon_{\mathbf{M}}"] & \varinjlim G(F(\mathbf{M}^b)) \\ \mathbf{M}^b \arrow[r] & \mathbf{M} \arrow[r, equal] & \varinjlim \mathbf{M}^b \end{tikzcd} . \tag{7}$$

Then applying the ith cohomology functor H^i to (7) then shows that the map $H^i(\varepsilon_{\mathbf{M}})$ is the limit of the maps $\varepsilon_{\mathbf{M}^b}$, so to show $H^i(\varepsilon_{\mathbf{M}})$ is an isomorphism it suffices to show all $H^i(\varepsilon_{\mathbf{M}^b})$ are. Thus without loss o generality the complex $\mathbf{M}$ is bounded. Since F and G respect translation in differential degree, say $\mathbf{M}$ has form

$$0 \to M^0_\bullet \to \ldots \to M^d_\bullet \to 0. \tag{8}$$

Let $\mathbf{M}^d$ be the chain complex with $M^d_\bullet$ in degree d, and 0 elsewhere. We have a short exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow \mathbf{M} \xrightarrow{\varphi} \mathbf{M}^d \longrightarrow 0$$

where φ is the obvious map. The complex $\ker(\varphi)$ is concentrated in degrees $0, \ldots, d-1$. Applying the exact functor H^i to the diagram formed by the naturality squares of ε on (8) gives us a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^i(G(F(\ker\varphi))) \arrow[r] \arrow[d, "H^i(\varepsilon_{\ker\varphi})"] & H^i(G(F(\mathbf{M}))) \arrow[r] \arrow[d, "H^i(\varepsilon_{\mathbf{M}})"] & H^i(G(F(\mathbf{M}^d))) \arrow[r] \arrow[d, "(H^i\varepsilon_{\mathbf{M}^d})"] & 0 \\ 0 \arrow[r] & H^i(\ker(\varphi)) \arrow[r] & H^i(\mathbf{M}) \arrow[r] & H^i(\mathbf{M}^d) \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact. By Lemma 3.5, $H^i(\varepsilon_{\mathbf{M}^d})$ is an isomorphism. By the Five lemma, $H^i(\varepsilon_{\mathbf{M}})$ is an isomorphism if and only if $H^i(\varepsilon_{\ker(\varphi)})$ is. Since $\ker(\varphi)$ is a strictly shorter complex than $\mathbf{M}$, we are done.

The analogous statement for $F \circ G$ follows from a similar calculation. $\square$

Thus we have a formulaic (albeit inefficient– the free A-module A is resolved to an n-term free complex) method to compute resolutions of complexes.

Syzygies and regularity of modules. We use the resolutions produced in Theorem 3.6 to prove a classical result of commutative algebra– Hilbert's syzygy theorem, and provide a way to compute the Castelnuovo-Mumford regularity of modules. We briefly discuss the notions involved, and refer to Eisenbud (1995) for details.

Writing a graded A-module M in terms of generators and relations produces a short exact sequence

$$0 \to S \to F \to M \to 0,$$

where F is a free module. The module S is unique up to direct sum with a free module (i.e. if $0 \to S' \to F' \to M \to 0$ is another such resolution then there are free modules L and L' such that $L \oplus S \cong L' \oplus S'$), and is called the *first syzygy* of M. Continuing the process, we can write S in terms of generators and relations and define the second syzygy of M to be the first syzygy of S. Thus the jth syzygy of M is the module S_j (up to direct sum with a free module) such that there is an exact sequence

$$0 \to S_j \to F_{j-1} \to ... \to F_0 \to M \to 0$$

where $F_0, ..., F_{j-1}$ are free modules. Note that if the jth syzygy of M is free then M has a free resolution of length $j+1$– thus syzygies form a measure of the 'complexity' of M. This is made precise using the notion of *projective dimension*, defined as

$$\mathrm{pd}(M) = \min\{j \mid \text{the } j\text{th syzygy module of } M \text{ is free or projective}\}.$$

Hilbert showed that the projective dimension of A-modules is bounded. The resolution produced using Theorem 3.6 provides a immediate constructive proof of this result.

Corollary 3.7 (Hilbert Syzygy Theorem)**.** If M is a graded module over $k[x_0, ..., x_n]$, then the $n+1$st syzygy module of M is free.

In fact, this bound is strict– for instance, the A-module k has projective dimension $n+1$. To see this, observe that (5) allows us to compute $\mathrm{Ext}_A^{n+1}(k, k) \cong k$ but by Lemma 4.1.6 of Weibel (2003), an A-module M has a projective resolution of length $\leq d$ if and only if $\mathrm{Ext}_A^d(M, N) = 0$ for all A-modules N. Defining the *graded global dimension* of a graded ring R to be

$$\mathrm{gr.gl.dim}(R) = \sup\{\mathrm{pd}(M) \mid M \in R\text{-grMod}\},$$

we have thus shown that $\mathrm{gr.gl.dim}(k[x_0, ..., x_n]) = n+1$.

The notion of *Castelnuovo-Mumford regularity* builds upon this, putting a bound on the degrees of generators and relations of a finitely generated graded A-module M. We say M is m-regular if the jth syzygy of m is generated in degrees $\leq m+j$. We state a homological characterisation of regularity, referring to Eisenbud & Goto (1984) for a proof.

Theorem 3.8 (Eisenbud & Goto (1984))**.** For a finitely generated graded A-module M, the following conditions are equivalent.

1. M is m-regular.

2. $M_{\geq m} = \bigoplus_{i \geq m} M_i$ is generated by M_m and has a *linear free resolution* (a free resolution in which the differentials are represented by matrices whose entries have degree ≤ 1.)

3. $M_{\geq m}$ is generated by M_m and $\mathrm{Tor}_A(M, k)_j^{j-i} = 0$ for all j and all $i > m$.

Using the Koszul complex (5), we extend the result above to the following.

Corollary 3.9 (Eisenbud et al. (2003)). A finitely generated graded A-module M is m-regular if and only if $M_{\geq m}$ is generated by M_m and the complex $F(M)$ is exact at degrees $> m$.

Proof. It suffices to show that the complex $F(M)$ has cohomology $H^i(F(M))_j = \mathrm{Tor}_A(M, k)_j^{j-i}$. To see this, note that the Koszul complex $G(F(k))$ given by (5) is a free resolution of k. Then the complex $M \otimes_A G(F(k))$ is given in differential degree $i - j$ by

$$M \otimes_A A^!_{j-i} \otimes_k A\langle i - j\rangle \cong A^!_{j-i} \otimes_k M\langle i - j\rangle.$$

The component in Adam's degree j is $A^!_{j-i} \otimes_k M_i$, which occurs as the degree (i, j) component of $F(M)$. Moreover, the differentials in both complexes coincide, hence we are done. $\square$

3.4 Descending to triangulated categories

Theorem 3.6 shows that the functors F and G preserve cohomology, so it is reasonable to ask whether they descend to the (triangulated) homotopy and derived categories which are the natural setting for formulating statements about cohomology. We show that the answer is positive for homotopy categories.

Lemma 3.10. The functors F and G take cones to cones– in other words, if $f : \mathbf{M} \to \mathbf{N}$ is a morphism in $\mathcal{C}(A\text{-grMod})$ then $F(\mathrm{cone}\, f) = \mathrm{cone}\, F(f)$, and likewise for G.

Proof. An easy explicit check from definition of F and G, ommited for brevity. $\square$

Theorem 3.11 ((Eisenbud et al. 2003)). The functors F and G descend to adjoint functors

$$\mathcal{K}(A\text{-grMod}) \underset{\bar{F}}{\overset{\bar{G}}{\leftrightarrows}} \mathcal{K}(A^!\text{-grMod})$$

between the triangulated homotopy categories of chain complexes.

Proof. In a category of chain complexes, a morphism $f : M \to N$ is nullhomotopic if and only the inclusion $0 \to N \to \mathrm{cone}\, f$ is split. Now we know F and G take cones to cones, and being additive functors they take split morphisms to split morphisms. Thus F and G factor through the homotopy categories.

To see that the induced functors are functors between triangulated categories, we show that F and G are exact functors hence preserve distinguished triangles. But this can be checked at the level of $\mathbb{Z}^2$-graded modules, and is immediate since the functors F and G are both given by tensor product with some k-vector space.

The adjunction between $\bar{F}$ and $\bar{G}$ follows immediately from the adjunction in Theorem 3.3 which identifies subgroups of nullhomotopic morphisms. $\square$

It is clear that this is not an equivalence, the natural maps $G(F(\mathbf{M})) \to \mathbf{M}$ and $\mathbf{N} \to F(G(\mathbf{N}))$ are not invertible in the homotopy categories. Theorem 3.6 however does show that they are quasi-isomorphisms, so one might expect F and G to induce isomorphisms of derived categories. There are multiple examples throughout the literature which show this fails– for instance (Keller 2003) shows that the complex $A \in \mathcal{D}(A\text{-grMod})$ is a compact object (i.e. the functor $\mathrm{Hom}_{\mathcal{D}(A\text{-grMod})}(A, -)$ commutes with infinite direct sums) but the object $F(A) \cong k\langle n+1\rangle[-n-1] \in \mathcal{D}(A^!\text{-grMod})$ is not compact. Below we exhibit explicitly the failure of our functors to descend to the derived category.

Example 3.12 (G does not preserve quasi-isomorphisms)**.** Let $n = 0$, so that $A = k[x]$ and $A^! = k[\xi]/(\xi^2)$. Consider the exact complex of graded $A^!$-modules

$$\cdots \to A^!\langle 2\rangle \xrightarrow{\xi} A^!\langle 1\rangle \xrightarrow{\xi} A^! \xrightarrow{\xi} A^!\langle -1\rangle \xrightarrow{\xi} A^!\langle -2\rangle \to \cdots$$

which is isomorphic to the zero complex in $\mathcal{D}(A^!\text{-grMod})$. The functor G maps this to

$$\cdots \to 0 \to \bigoplus_q A\langle -q\rangle \xrightarrow{1+x} \bigoplus_q A\langle -q\rangle \to 0 \to \cdots,$$

which is not acyclic (the only non-zero differential is not surjective), hence non-zero in $\mathcal{D}(A\text{-grMod})$.

Bernstein-Gel'fand-Gel'fand equivalence To work around this apparent problem, Bernstein et al. (1978) restricts to bounded complexes so that a simple spectral sequence argument shows F and G preserve acyclicity. This gives well-defined functors between the bounded derived categories which form an adjoint equivalence– the so-called 'BGG correspondence'.

Lemma 3.13. If $\mathbf{M}$ is a bounded acyclic complex of finitely generated A-modules, then the complex $F(\mathbf{M})$ is acyclic. Likewise, if $\mathbf{N}$ is a bounded acyclic complex of finitely generated $A^!$-modules, then the complex $G(\mathbf{M})$ is acyclic.

Proof. Given such an $\mathbf{M}$, the double complex (1) has exact columns. Then the first page of the spectral sequence (starting with vertical cohomology) vanishes everywhere. Since $M^p_\bullet = 0$ for large p, the double complex is bounded and the convergence theorem holds, indicating the total complex is acyclic.

The argument for G is similar. □

Then by Example 10.5.5 in (Weibel 2003), F and G descend to functors between derived categories prove

$$F_{\mathcal{D}} : \mathcal{D}^b(A\text{-grMod}) \to \mathcal{D}(A^!\text{-grMod}), \qquad G_{\mathcal{D}} : \mathcal{D}^b(A^!\text{-grMod}) \to \mathcal{D}(A\text{-grMod}).$$

To conclude, we show that $F_{\mathcal{D}}$ in fact has image $\mathcal{D}^b(A^!\text{-grMod})$ and likewise for $G_{\mathcal{D}}$.

Lemma 3.14. If $\mathbf{M}$ is a bounded complex of finitely generated A-modules, then the complex $F(\mathbf{M})$ has bounded cohomology and is quasi-isomorphic to a bounded complex of finitely generated $A^!$-modules.

Likewise, if $\mathbf{N}$ is a bounded complex of finitely generated $A^!$-modules, then the complex $G(\mathbf{N})$ has bounded cohomology and is quasi-isomorphic to a bounded complex of finitely generated $A^!$-modules.

Proof. If $\mathbf{N}$ is as given, then the complex $G(\mathbf{N})$ is bounded by definition– for any p, we have that the module $N^p_\bullet$ is finitely generated hence has only finitely many graded components. Then for sufficiently large i, we have $N^p_{p-i} = 0$ for all p.

For $\mathbf{M}$ as given the double complex (1) which computes M is bounded, and by Corollary 3.9 the first page of the corresponding spectral sequence (starting with horizontal cohomology) has finite support. Thus by the convergence theorem for spectral sequences, the cohomology of $F(\mathbf{M})$ is bounded. The existence of the quasi-isomorphic bounded complex of finitely generated modules follows from Hartshorne (2008), III Lemma 12.3. □

Theorem 3.15 (Bernstein et al. (1978))**.** The functors F and G induce an equivalence of derived categories

$$\mathcal{D}^b(A\text{-grMod}) \underset{F_{\mathcal{D}}}{\overset{G_{\mathcal{D}}}{\rightleftarrows}} \mathcal{D}^b(A^!\text{-grMod}).$$

Proof. From Lemma 3.14, the functors given are well-defined. Then Theorem 3.6 shows that $F_{\mathcal{D}} \circ G_{\mathcal{D}}$ and $G_{\mathcal{D}} \circ F_{\mathcal{D}}$ are naturally equivalent to the identity morphism, hence we have an equivalence of categories. □

4 Coherent sheaves on $\mathbb{P}^n$

4.1 The Tate resolution and Beilinson monads

5 Koszul duality after Keller (2003)

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