(Co)Derived Equivalences in Algebra and Geometry

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Fix a field k, and let X and Ξ be dual k-vector spaces of dimension $n+1$ with dual bases (x_i) and (ξ_i) respectively. The goal of this exposition is to examine equivalences of various categories that arise naturally in this setting from algebro-geometric constructions. In particular, we look at chain complexes of

(i) modules over the symmetric algebra $A := \mathrm{Sym}^\bullet(X)$,

(ii) modules over the exterior algebra $A^! := \bigwedge^\bullet(\Xi)$,

(iii) coherent sheaves over $\mathbb{P}^n := \mathrm{Proj}(\mathrm{Sym}^\bullet(X))$, the projectivisation of Ξ.

The first hint at a correspondence between coherent sheaves on $\mathbb{P}^n$ and modules over the exterior algebra $A^! = \bigwedge^\bullet(\Xi)$ comes from the following observation in Bernstein, Gel'fand & Gel'fand (1978).

Example 0.1. Given a graded $A^!$-module $N_\bullet$ and a non-zero vector $\xi \in \Xi$, we have a complex of vector spaces

$$\mathcal{L}_\xi(N_\bullet): \qquad \cdots \to N_1 \xrightarrow{\xi} N_0 \xrightarrow{\xi} N_{-1} \to \cdots .$$

Writing $\mathcal{L}^j$ for the vector bundle $N_{-j} \otimes_k \mathcal{O}_{\mathbb{P}^n}(j)$, we can identify $\mathcal{L}^j_\xi = N_{-j}$ with the fiber of $\mathcal{L}^j$ at

$[\xi] \in \mathbb{P}^n$ to obtain a complex

$$\mathcal{L}(N_\bullet): \qquad \cdots \to N_1 \otimes_k \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow N_0 \otimes_k \mathcal{O}_{\mathbb{P}^n} \longrightarrow N_{-1} \otimes_k \mathcal{O}_{\mathbb{P}^n}(1) \to \cdots$$

where the differential sends a section $f \in N_{-j} \otimes_k \mathcal{O}(j)$ (which is a degree j homogeneous function with values in N_{-j}) to $df = [\xi \mapsto \xi \cdot f(\xi)]$. We say the module $N_\bullet$ is *faithful* if the complex $\mathscr{L}(N_\bullet)$ is exact everywhere except in degree 0, in which case we write $\Phi(N_\bullet)$ for the vector bundle $H^0(\mathscr{L}(N_\bullet))$. In this case, the natural map $\mathscr{L}(N_\bullet) \to \Phi(N_\bullet)$ of omplexes (where we identify $\Phi(N_\bullet)$ with the corresponding complex concentrated in degree 0) gives a 'resolution' of $\Phi(N_\bullet)$ by powers of the Hopf bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

Example 0.2. In the simplest case when X, Ξ are one-dimensional, the data of a module over $A = k[x]$ involves a k-vector space M with a map $M \xrightarrow{x} M$ which can be seen as a cochain complex $F(M)$ of k-vector spaces with differential d of degree 1. The graded algebra $A^! = k[\xi]/(\xi^2)$ acts naturally on $F(M)$ via the degree 1 chain map $F(M) \xrightarrow{\xi} F(M)$ given by

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[dr] & M \arrow[r] \arrow[dr, "1"] & M \arrow[r] \arrow[dr] & \cdots & \\
& \cdots \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd} .$$

Consider the cochain complex of A-modules

$$\cdots \to 0 \to M \otimes_k A \xrightarrow{d \otimes 1 + \xi \otimes x} M \otimes_k A \to 0 \to \cdots ,$$

concentrated in degrees -1 and 0. This has the same underlying vector spaces as the complex $F(M) \otimes_k A$, but the differential has been 'twisted' to remember the $A^!$-action. This complex is exact everywhere except in degree 0, where it has cohomology M. Since the modules appearing in it are free, we have recovered a free resolution of M.

This is the first example of what may be called *Koszul duality*, a broad term encompassing various equivalences across algebra, geometry, and representation theory. The duality between symmetric and exterior algebras over finite dimensional vector spaces was first studied by Bernstein et al. (1978), who exhibit an adjunction between the categories of cochain complexes of graded modules over A and $A^!$.

Theorem. There are adjoint functors

$$\mathbf{C}(A\text{-grMod}) \underset{F}{\overset{G}{\leftrightarrows}} \mathbf{C}(A^!\text{-grMod})$$

such that any complex $M^\bullet$ of graded A-modules has free resolution $GF(M^\bullet)$, and any complex $N^\bullet$ of graded $A^!$-modules has injective resolution $FG(N^\bullet)$.

In Section 3, we look at Eisenbud, Floystad & Schreyer's (2003) treatment of the Bernstein-Gel'fand-Gel'fand (BGG) correspondence described above. To turn the adjunction into an equivalence of categories, we need to employ the machinery of Verdier's *derived categories*. Passing to the corresponding derived categories of modules has the effect that all *quasi-isomorphisms* (i.e. chain maps that induce isomorphisms on homology) become isomorphisms.

Bernstein et al. (1978) use the correspondence between coherent sheaves on $\mathbb{P}^n$ and graded A-modules (see for example Chapter II of Hartshorne (2008)) to describe the derived category $D^b(\mathscr{C}oh\, \mathbb{P}^n)$ of projective n-space. In 2 we use

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1 Categories of complexes

We set up the basic framework of homological algebra and derived categories necessary to formulate results in later sections. The material is largely taken from Weibel (2003) and the initial chapters of Huybrechts (2006), with heuristics taken from Thomas (2001).

Definition 1.1. A category $\mathfrak{A}$ is *additive* if it satisfies the following.

1. Each hom-set $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ has the structure of an abelian group such that composition distributes over addition, i.e. given a diagram in $\mathfrak{A}$ of the form

$$A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

   we have $f \circ (g + g') \circ h = f \circ g \circ h + f \circ g' \circ h$ in $\mathrm{Hom}_{\mathfrak{A}}(A, D)$.

2. $\mathfrak{A}$ has finite products, i.e. for any $A, B \in \mathfrak{A}$ there is an object $A \times B$ satisfying the usual universal property. In this case, it can be shown that finite products are the same as finite coproducts.

3. There is an object $0 \in \mathfrak{A}$ such that $\mathrm{Hom}_{\mathfrak{A}}(0, 0)$ is the trivial (zero) group.

If in addition $\mathfrak{A}$ also satisfies the axioms below, then it is called an *abelian* category.

4. For every morphism $f : A \to B$, there are morphisms $\ker(f) : K \to A$ and $\mathrm{coker}(f) : B \to C$ (called the *kernel* and *cokernel* respectively), such that every morphism g with $f \circ g = 0$ uniquely factors through $\ker(f)$, and every morphism h with $h \circ f = 0$ uniquely factors through $\mathrm{coker}(f)$.

5. Every monomorphism in $\mathfrak{A}$ is the kernel of its cokernel.

6. Every epimorphism in $\mathfrak{A}$ is the cokernel of its kernel.

Remark 1.2. It is an easy consequence of the definitions that kernels are always monic, and cokernels are always epic. For a morphism $f : A \to B$ with kernel $K \to A$, we often abuse notation to call K the kernel, leaving the morphism to A implicit. Likewise, we call the codomain of $\mathrm{coker}(f)$ the cokernel.

Abelian categories provide the right framework to perform operations with kernels and cokernels, which is the essence of homological algebra. The prototypical example of an abelian category is the category R-Mod of (left) modules on a ring R. In this case the notions of kernel and cokernel coincide with the usual ones.

Example 1.3 (Endomorphism rings)**.** For any object A in an additive category $\mathfrak{A}$, the group $\mathrm{Hom}_{\mathfrak{A}}(A, A)$ naturally has the structure of a unital ring, where multiplication is given by composition. Then any unital ring R can be seen as an additive category with a single non-zero object $*$, such that $\mathrm{Hom}_{\mathfrak{A}}(*, *) \cong R$. Thus rings are two-object abelian categories!

Example 1.4 (Chain complexes)**.** A *chain complex* $\mathbf{A}$ in an additive category $\mathfrak{A}$ is a sequence of objects $(A^i)_{i \in \mathbb{Z}}$ and morphisms $d^i : A^i \to A^{i+1}$ (called the *differentials*) such that the compositions $d^i \circ d^{i-1}$ are all 0. We say the object A^i sits in differential degree i. If all but finitely many A^is are the zero object, then the chain complex is said to be *bounded*.

A *chain-map* $f : \mathbf{A} \to \mathbf{B}$ is a sequence $(f^i : A^i \to B^i)_{i \in \mathbb{Z}}$ of morphisms in $\mathfrak{A}$ such that $f^{i+1} \circ d^i = d^i \circ f^i$ for all i. Then associated to $\mathfrak{A}$ is a category $\mathbf{C}(\mathfrak{A})$ whose objects are chain complexes in $\mathfrak{A}$, and morphisms are chain-maps. This is naturally an abelian category, where the zero object is the complex which has 0 in every differential degree. Write $\mathbf{C}^b(\mathfrak{A})$, $\mathbf{C}^+(\mathfrak{A})$, and $\mathbf{C}^-(\mathfrak{A})$ for the full subcategories whose objects are bounded complexes, complexes bounded below, and complexes bounded above respectively. These are again examples of abelian categories.

Short exact sequences. Cokernels and kernels can be defined in any additive category $\mathfrak{A}$. We define a diagram in $\mathfrak{A}$ of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

to be a *short exact sequence* if $f = \ker(g)$ and $g = \ker(f)$. Then in an abelian category, any morphism thats monic or epic forms a part of some exact sequence.

Cohomology. In an abelian category $\mathfrak{A}$, a morphism always has a kernel, a cokernel and an image. Then given a chain complex $\mathbf{A} \in \mathbf{C}(\mathfrak{A})$, we define the ith *cohomology* to be the object

$$H^i(\mathbf{A}) = \mathrm{im}(\ker d^{i-1} \to A^i \to \mathrm{coker}\ d^i) \quad \in \mathfrak{A}.$$

For each i, this defines an additive functor $H^i : \mathbf{C}(\mathfrak{A}) \to \mathfrak{A}$. In case $\mathfrak{A}$ is the category of R-modules (which will be the case whenever we consider cohomology in this exposition), this coincides with the usual definition

$$H^i(\mathbf{A}) = \frac{\ker(d^n)}{\mathrm{im}(d^{n-1})}.$$

We say $\mathbf{A}$ is *exact* at A^i if $H^i(A^\bullet) = 0$, and we say $\mathbf{A}$ is *exact* (or *acyclic*) if it is exact at every A^i. Short exact sequences can then be seen as instances of exact complexes.

The notions of additive and abelian categories come naturally with notions of functors which preserve the additional structures– these are called *additive* and *exact* functors.

Definition 1.5. A functor $F : \mathfrak{A} \to \mathfrak{B}$ between additive categories is called an *additive functor* if the maps $F : \mathrm{Hom}_{\mathfrak{A}}(A, B) \to \mathrm{Hom}_{\mathfrak{B}}(FA, FB)$ are group homomorphisms. We say F is *exact* if, in addition, it sends short exact sequences to short exact sequences.

Example 1.6 (Exact functors on chain complexes)**.** There are two exact functors which are always defined for any additive category $\mathfrak{A}$.

1. There is a natural inclusion $\mathfrak{A} \to \mathbf{C}(\mathfrak{A})$ which sends an object A to the chain complex $A^\bullet$ where $A^0 = A$, and $A^i = 0$ for $i \neq 0$. This is an exact functor, identifying $\mathfrak{A}$ as a full abelian subcategory of $\mathbf{C}(\mathfrak{A})$.

2. Given $A^\bullet \in \mathbf{C}(\mathfrak{A})$, we define the *translate by* 1 of $A^\bullet$ to be the chain complex $A^\bullet[1]$, which has $A^{i+1}, -d^{i+1}$ in degree i. This defines an exact functor $[1] : \mathfrak{A} \to \mathfrak{A}$ which is an equivalence of categories. Write $[i]$ for the i-fold composition of $[1]$ with itself, and $[-i]$ for the functor inverse to $[i]$.

1.1 Bicomplexes

If $\mathfrak{A}$ is an abelian category, then so is $\mathbf{C}(\mathfrak{A})$ so we can construct a category $\mathbf{C}(\mathbf{C}(\mathfrak{A}))$ whose objects are *bicomplexes* of the form

$$\begin{tikzcd}
\vdots & & \vdots & \vdots & \\
\mathbf{A}^{p+1}: \arrow[u] & \cdots \arrow[r] & A^{p+1,q} \arrow[r] \arrow[u] & A^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\mathbf{A}^{p}: \arrow[u] & \cdots \arrow[r] & A^{p,q} \arrow[r] \arrow[u] & A^{p,q+1} \arrow[r] \arrow[u] & \cdots \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \quad . \tag{1}$$

where all the squares commute. Here the horizontal differentials (which we write $d_{>}$ for) come from the internal differentials of the chain complexes $\mathbf{A}^i \in \mathbf{C}(\mathfrak{A})$, while the vertical differentials (which we write $d_{\wedge}$ for) come from the differentials of the complex in $\mathbf{C}(\mathbf{C}(\mathfrak{A}))$.

Such bicomplexes have an associated *total (direct sum) complex* in $\mathbf{C}(\mathfrak{A})$, given by

$$\cdots \to \bigoplus_{p+q=i-1} A^{p,q} \longrightarrow \bigoplus_{p+q=i} A^{p,q} \to \cdots$$

where the differential sends $a \in A^{p,q}$ to $d_{\wedge}(a) + (-1)^i d_{>}(a)$. If direct products are taken instead of direct sums, we have the total direct product complex. In the case of bounded bicomplexes the two notions coincide.

Example 1.7. Given two chain complexes $\mathbf{A}, \mathbf{B} \in R\text{-Mod}$ for some ring R, there is an associated bicomplex $\mathbf{A} \otimes_R \mathbf{B}$ given in degree (p, q) by $A^p \otimes_R B^q$. Then the *total tensor product complex* $\mathrm{Tot}(\mathbf{A} \otimes_R \mathbf{B})$ is the total direct sum complex of this bicomplex.

Likewise, defining the bicomplex $\mathrm{Hom}_R(\mathbf{A}, \mathbf{B})$ in degree (p, q) by $\mathrm{Hom}_R(A^p, B^q)$. The *total Hom complex* $\mathrm{Tot}(\mathrm{Hom}_R(\mathbf{A}, \mathbf{B}))$ is the total direct product complex of this bicomplex. The usual $\otimes - \mathrm{Hom}$ adjunction extends to these total complexes, the details can be found in Weibel (2003).

1.1.1 Spectral sequences.

Often, one is interested in the cohomology of the total complex associated to a bicomplex. *Spectral sequences* provide a bookkeeping tool for this purpose, and often allow us to extract the cohomology of the bicomplex from the cohomologies of the rows or columns. We will only deal with spectral sequences associated to bicomplexes of modules, so the category $\mathfrak{A}$ is a module category for the purposes of this section.

The *spectral sequence (starting with horizontal cohomology)* $\mathbf{E}$ of a bicomplex (1) is a sequence of *pages* $\mathbf{E}_i$, where each page has objects of $\mathfrak{A}$ arranged in a grid $E_i^{p,q}$ with specified morphisms which we now describe. The zeroth page is given by forgetting the vertical differentials in (1), as

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ \cdots & \longrightarrow & A^{p+1,q} & \longrightarrow & A^{p+1,q+1} & \longrightarrow & \cdots \\ \cdots & \longrightarrow & A^{p,q} & \longrightarrow & A^{p,q+1} & \longrightarrow & \cdots \\ & & \vdots & & \vdots & & \end{array}.$$

The objects in the first page will be the cohomologies of the sequences in $\mathbf{E}_0$, with maps between them induced by the vertical differentials $d_\wedge$.

$$\begin{array}{cccc} & \vdots & \vdots & \\ & \uparrow & \uparrow & \\ \cdots & H^q(\mathbf{A}^{p+1}) & H^{q+1}(\mathbf{A}^{p+1}) & \cdots \\ & \uparrow & \uparrow & \\ \cdots & H^q(\mathbf{A}^{p}) & H^{q+1}(\mathbf{A}^{p}) & \cdots \\ & \uparrow & \uparrow & \\ & \vdots & \vdots & \end{array}.$$

The subsequent pages are defined likewise– in particular, the morphisms on the ith page go from $E_i^{p,q}$ to $E_i^{p+i,q-i-1}$ and the compositions of the morphisms, whenever defined, are zero (so that the ith page contains a sequence of complexes.) The objects on $\mathbf{E}_i$ are then the cohomologies of the complexes on $\mathbf{E}_{i-1}$, and the morphisms described above are induced from the previous pages.

All the spectral sequences we consider in this exposition are *regular*, i.e. there is an $r \geq 2$ such that the morphisms on every page after $\mathbf{E}_r$ are zero. In this case, we have $E_r^{p,q} = E_{r+1}^{p,q} = ... = E_\infty^{p,q}$ for some object $E_\infty^{p,q} \in \mathfrak{A}$.

Definition 1.8. Given a sequence $H^\bullet = (..., H^{-1}, H^0, H^1, ...)$ of objects in $\mathfrak{A}$, we say a (regular) spectral sequence $\mathbf{E}$ *converges weakly* to $H^\bullet$ if for every i there exists a filtration

$$\cdots \subseteq F^2H^i \subseteq F^1H^i \subseteq F^0H^i = H^i$$

such that for every $p \geq 0$ and for every q, we have

$$\frac{F^pH^{p+q}}{F^{p+1}H^{p+q}} \cong E_\infty^{p,q}.$$

If in addition for all i we have $\bigcap_p F^pH^i = 0$ and $H^i = \lim_p H^i/F^pH^i$, then we say the spectral sequence *converges* to $H^\bullet$.

For a general bicomplex as in (1), not much can be said about convergence of the spectral sequence. However, if the bicomplex is *bounded* i.e. for every i there are only finitely many non-zero $A^{p,q}$ with $p+q=i$ then we have the following result.

Proposition 1.9. If the bicomplex (1) is bounded, then the spectral sequence $\mathbf{E}$ converges to the sequence $H^\bullet$ where H^i is the ith homology object of the total (direct sum) complex of the bicomplex.

Proof. See Lemma 12.25.3 of The Stacks project authors (2022). □

1.2 Three abelian categories

Before proceeding with more constructions from homological algebra, we describe the three abelian categories central to this exposition– these are the module categories of the symmetric and exterior algebra, and the category of coherent sheaves on $\mathbb{P}^n$. These are defined in the usual way, but we record the constructions involved for completeness of exposition, and to set conventions.

Let k, X, and Ξ be as in the introduction.

1.2.1 Symmetric and exterior algebras.

Definition 1.10. Given an $n+1$-dimensional k-vector space V, the *tensor algebra* is the k-vector space

$$T(V) = k \oplus \bigoplus_{i \geq 1} (\underbrace{V \otimes_k V \otimes_k \ldots \otimes_k V}_{i \text{ times}})$$

with a product $\nabla : T(V) \otimes T(V) \to T(V)$ induced by the natural identifications $V^{\otimes i} \otimes V^{\otimes j} \xrightarrow{\sim} V^{\otimes(i+j)}$. This is an associative algebra with a natural $\mathbb{Z}_{\geq 0}$-grading.

The *symmetric algebra* $\mathrm{Sym}^\bullet(V)$ and the *exterior algebra* $\bigwedge^\bullet(V)$ are then the graded algebras defined as quotients of $T(V)$ by certain two-sided ideals, namely

$$\mathrm{Sym}^\bullet(V) = \frac{T(V)}{(x \otimes y - y \otimes x \mid x, y \in V)}, \qquad \bigwedge\nolimits^\bullet(V) = \frac{T(V)}{(x \otimes x \mid x \in V)}.$$

Since the ideals are generated by homogeneous elements, these algebras inherit gradings from $T(V)$.

We continue to use ∇ for the product morphism on either algebra, though the corresponding bilinear map on $\bigwedge^\bullet V$ is often written $\wedge$.

Remark 1.11. We can repeat the above constructions in the category of R-modules for any ring R. In this case, we write $T_R(M)$, $\mathrm{Sym}^\bullet_R(M)$, $\Lambda^\bullet_R(M)$ respectively for the tensor, symmetric, and exterior algebras over $M \in R$-Mod. In particular,

$$T(M) = R \oplus \bigoplus_{i \geq 1} (\underbrace{M \otimes_R M \otimes_R \ldots \otimes_R M}_{i \text{ times}}).$$

Since we are primarily concerned with the algebras $A = \mathrm{Sym}^\bullet(X)$ and $A^! = \bigwedge^\bullet(\Xi)$, we redefine the grading on $A^!$ as $A^!_{-i} = \Lambda^i \Xi$. This amounts to a change of sign from the usual grading, but the convention ensures that the dual vector spaces X and Ξ lie in degrees 1 and -1 in their respective algebras.

Graded modules. A graded A-module is a $\mathbb{Z}$-graded k-vector space $\mathbf{M} = \oplus_i M_i$ with an A-module structure such that $A_i M_j \subseteq M_{i+j}$. For any i, we say the elements of M_i are homogeneous of degree i. A morphism of graded modules then is an A-module homomorphism that preserves the degree of homogeneous elements. Define the category A-grMod to have finitely genered graded A-modules as its objects, and graded module homomorphisms as the morphisms. This is an abelian category.

The abelian category $A^!$-grMod is defined likewise, with objects being finitely generated graded $A^!$-modules.

1.2.2 The exterior coalgebra

The *exterior coalgebra* on Ξ is defined as the linear dual of $A^!$, written $A^i := \mathrm{Hom}_k(A^!, k)$. A^i has the $\mathbb{Z}$-grading $A^i_i = \mathrm{Hom}_k(A^!_{-i}, k)$ and is naturally an $A^!$-module via

$$a \cdot f(a') = (-1)^{\deg a} f(a \wedge a')$$

whenever $a \in A^!$ is homogeneous, and $f \in \mathrm{Hom}(A^!, k)$.

For any vector space N, there is the natural isomorphism of $A^!$-modules $\mathrm{Hom}_k(A^!, N) \cong A^i \otimes_k N$.

Choosing a basis x_i for X fixes an isomorphism $X \cong \mathrm{Hom}_k(\Xi, k) = A^i_1$, which can be extended to get the isomorphism of graded k-vector spaces

$$A^i = \bigoplus_i \mathrm{Hom}_k(\Lambda^i \Xi, k) \cong \bigoplus_i \Lambda^i X = \bigwedge^\bullet(X).$$

In particular, X is a subspace of both A^i and A. This observation is essential in defining the Koszul duality functors, so we write $\tau : A^i \to A$ for the k-linear map which identifies the subspaces of A^i and A corresponding to X, and is 0 elsewhere.

The coproduct on A^i. Being the linear dual of a finite dimensional algebra, A^i has a natural (coassociative counital) coalgebra structure which comes from dualising the (associative unital) product $\nabla : A^! \otimes_k A^! \to A^!$. This is called the *shuffle coproduct*, and it is helpful to have an explicit description of it which we now describe.

Given a collection of indices $\underline{\alpha} = \{\alpha_1 < \ldots < \alpha_i\} \subseteq \{0, \ldots, n\}$, write $x_{\underline{\alpha}}$ for the standard basis element of A^i given by $x_{\alpha_1} \wedge x_{\alpha_2} \wedge \ldots \wedge x_{\alpha_i}$ (in particular, $x_\emptyset = 1$). The vector $\xi_{\underline{\alpha}}$ is defined similarly. We say a tuple $(\underline{\beta}, \underline{\beta'})$ of subsets is a *break* of $\underline{\alpha}$ if $(\beta_1 < \ldots < \beta_p, \beta'_1 < \ldots < \beta'_q)$ is a permutation of $(\alpha_1 < \ldots < \alpha_i)$ (in other words, $\underline{\alpha} = \underline{\beta} \sqcup \underline{\beta'}$). The *sign* of this break, written $\langle \underline{\beta}, \underline{\beta'} \rangle$, is defined to be the sign of the corresponding permutation. Thus we have have

$$\nabla(x_{\underline{\beta}} \otimes x_{\underline{\beta'}}) = x_{\underline{\beta}} \wedge x_{\underline{\beta'}} = \langle \underline{\beta}, \underline{\beta'} \rangle \, x_{\underline{\alpha}}.$$

This allows us to write the coproduct on A^i as

$$\Delta(x_{\underline{\alpha}}) = \sum_{(\underline{\beta}, \underline{\beta'}) \in \mathrm{br}(\underline{\alpha})} \langle \underline{\beta}, \underline{\beta'} \rangle \, x_{\underline{\beta}} \otimes x_{\underline{\beta'}}$$

where $\mathrm{br}(\underline{\alpha})$ is the set of all breaks of $\underline{\alpha}$. Recalling that $A^i \otimes_k A^i$ is $\mathbb{Z}$-graded with $\bigoplus_{p+q=i} A^i_p \otimes A^i_q$ in degree i, we observe that the map Δ respects grading hence A^i is a *graded coalgebra*.

1.2.3 Graded chain complexes

Objects of $\mathbf{C}(A\text{-grMod})$ are chain complexes of graded A-modules in which the differentials are morphisms in A-grMod (i.e. A-module homomorphisms which preserve degree). Such an object can be viewed as a $\mathbb{Z}^2$-graded k-vector space $\mathbf{M} = \bigoplus_{i,j} M^i_j$ with an endomorphism d (the differential) such that

1. $d \circ d = 0$,
2. d has degree $(1, 0)$ i.e. $d(M^i_j) \subseteq M^{i+1}_j$, and
3. for each $i \in \mathbb{Z}$, $M^i_\bullet = \bigoplus_j M^i_j$ is a graded A-module.

Likewise, an object $\mathbf{N} \in \mathbf{C}(A^!\text{-grMod})$ can be seen as a $\mathbb{Z}^2$-graded k-vector space $\oplus_{i,j} N^i_j$ with a differential ∂ of degree $(1, 0)$. We shall use the two viewpoints on interchangeably, switching between them whenever convenient to provide a clearer picture. In particular, the ability to view a complex as a single module with additional structure allows for cleaner definitions and proofs, see for instance Theorem 3.2.

For a chain complex $\mathbf{M} = \bigoplus_{i,j} M^i_j$, we say the lower indices denote the *internal* (or *Adam's*) grading, while the upper indices denote the *differential* (or *cohomological*) degree. We use '$\langle\cdot\rangle$' to denote shifts in Adam's gradings, continuing to use '$[\cdot]$' to denote shifts in differential gradings. Thus for example we have $\mathbf{M}\langle q\rangle^i_j = M^i_{q+j}$.

1.2.4 Coherent sheaves

The details of all the constructions described in this section can be found in Sections II.5 and II.8 of Hartshorne (2008).

For any scheme $\mathbf{X}$, the category $\mathscr{Sh}\mathbf{X}$ of sheaves on $\mathbf{X}$ is abelian with the usual notions of kernel and cokernel. If the scheme $\mathbf{X}$ is noetherian, then the (co)kernel of a morphism of coherent sheaves is coherent. In this case, the category $\mathscr{Coh}\,\mathbf{X}$ whose objects are coherent sheaves of $\mathcal{O}_\mathbf{X}$-modules is abelian.

The category $\mathscr{Coh}\,\mathbf{X}$ supports an additional operation, the tensor product $\otimes$. The tensor product of coherent sheaves will always be over the structure sheaf, unless otherwise specified. Given a coherent sheaf $\mathcal{E}$, we can then define the sheaves of algebras $T\mathcal{E}$, $\mathrm{Sym}\,\mathcal{E}$, and $\bigwedge^\bullet \mathcal{E}$ as coming from the presheaves which assign to an open set $U \subset \mathbf{X}$ the corresponding tensor operation applied to $\mathcal{E}(U)$ as an $\mathcal{O}_\mathbf{X}(U)$-module.

If $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ is a morphism of noetherian schemes, then the *pullback* of a coherent sheaf on $\mathbf{Y}$ is again coherent. This gives an additive functor

$$f^* : \mathscr{Coh}\,\mathbf{Y} \longrightarrow \mathscr{Coh}\,\mathbf{X}.$$

The pullback functor commutes with the various tensor operations described above.

Sheaves on $\mathbb{P}^n$. The *projectivisation* of Ξ is the k-scheme defined as $\mathbb{P}^n = \mathrm{Proj}(\mathrm{Sym}(\Xi^\vee))$. Since $\mathrm{Sym}(\Xi^\vee) = A$ is the polynomial algebra on $n+1$ variables, we see that $\mathbb{P}^n$ is the usual projective n-space over $\mathrm{Spec}(k)$. This is a noetherian scheme, and so $\mathscr{Coh}\,\mathbb{P}^n$ is an abelian category.

Serre gives a correspondence between A-grMod and $\mathscr{Coh}\,\mathbb{P}^n$ using a construction similar to Proj, called the 'sheafification' of graded modules. This correspondence is functorial, but *not* an equivalence of categories. Indeed, much of Section 2 is dedicated to obtaining a precise formulation of the equivalence between coherent sheaves on $\mathbb{P}^n$ and finitely generated graded A-modules.

In particular, the module $A \in A$-grMod corresponds to the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$. Since $\mathbb{P}^n$ is central to the exposition, we omit the subscript $\mathbb{P}^n$ unless there is a possibility of confusion– thus writing $\mathcal{O}$ for the structure sheaf.

The invertible sheaf $\mathcal{O}(i)$ is defined as the sheafification of the graded module $A\langle i\rangle$. In particular, the global sections of $\mathcal{O}(i)$ correspond to degree i polynomials. The various sheaves $\mathcal{O}(i)$ form an abelian group under $\otimes$, there being natural isomorphisms $\mathcal{O}(i)\otimes\mathcal{O}(j)\cong\mathcal{O}(i+j)$. Note that tensor products of sheaves are always taken over the structure sheaf.

For any sheaf $\mathcal{E}\in\mathscr{Coh}\,\mathbb{P}^n$, its ith *twist* is defined by $\mathcal{E}(i)=\mathcal{E}\otimes\mathcal{O}(i)$. Since the sheaves $\mathcal{O}(i)$ are flat, taking the ith twist is an exact functor.

The *cotangent sheaf* Ω and the *tangent sheaf* $\mathcal{T}=\mathscr{Hom}(\Omega,\mathcal{O})$ are locally free sheaves of rank n which fit in dual exact sequences given by

$$0\to\Omega\to\mathcal{O}(-1)^{\oplus(n+1)}\to\mathcal{O}\to 0,\qquad 0\to\mathcal{O}\to\mathcal{O}(1)^{\oplus(n+1)}\to\mathcal{T}\to 0. \tag{2}$$

These are called the *Euler exact sequences*, and form our primary means of getting hold of the sheaves. In particular, Ω corresponds to the graded module given by the kernel of

$$A^{n+1}\langle -1\rangle\longrightarrow A;\qquad (a_0,\dots,a_n)\longmapsto a_0x_0+\dots+a_nx_n.$$

1.3 Homotopy and Derived categories

We return to our discussion of homological algebra, to build the framework of *triangulated categories* and the derived category. Before describing these, we provide some motivation as to why we construct them by first reviewing how homological algebra is typically used to find invariants in mathematics.

Given a category $\mathfrak{C}$, we choose for each $X\in\mathfrak{C}$ a complex $\Phi(X)\in\mathbf{C}(\mathfrak{A})$ for some abelian category $\mathfrak{A}$. Then the compositions $H^i\circ\Phi:\mathfrak{C}\to\mathfrak{A}$ allow us to assign a sequence of $\mathfrak{A}$-objects associated to X, and this assignment (in the good cases) is functorial. The following examples illustrate this.

1. Consider the category $\mathscr{Simp}$ of simplicial complexes (in the sense of algebraic topology), and the functor $F:\mathscr{Simp}\to\mathbb{Z}\text{-Mod}$ which associates to a simplicial complex $\mathbf{X}\in\mathscr{Simp}$ the chain complex $F(\mathbf{X})$ of abelian groups where the group in degree $-i$ is generated by i-simplices, and the differentials are boundary maps. If $\mathscr{Tr}$ is the category of triangulable topological spaces, then we can choose a map $\Phi:\mathscr{Tr}\to\mathscr{Simp}$ which associates to each $\mathbf{X}\in\mathscr{Tr}$ a simplicial complex which is its triangulation. The ith *simplicial homology* of $\mathbf{X}$ is defined as $H^{-i}(F(\Phi(\mathbf{X})))$.

2. For a ring R, let $\mathbf{C}^-_{\text{free}}(R\text{-Mod})$ be the full subcategory of $\mathbf{C}^-(R\text{-Mod})$ whose objects are complexes of free R-modules. Choose for each $M\in R\text{-Mod}$, choose a complex $\Phi(M)\in\mathbf{C}^-_{\text{free}}(R\text{-Mod})$ such that $\Phi(M)$ is exact in non-zero degrees and has $H^0(\Phi(M))=M$. Then for a fixed R-module M, we define the ith Ext functor by $\mathrm{Ext}^i_R(M,-)=H^{-i}\circ\mathrm{Hom}_R(M,-)\circ\Phi$.

3. For a scheme $\mathbf{X}$, let $\mathbf{C}^+_{\text{inj}}(\mathscr{Qcoh}\,\mathbf{X})$ be the full subcategory of $\mathbf{C}^+(\mathscr{Qcoh}\,\mathbf{X})$ whose objects are complexes given in each degree by an injective sheaf. If $\mathbf{X}$ is noetherian, then the category $\mathscr{Qcoh}\,\mathbf{X}$ has enough injectives (see Section III.3 of Hartshorne (2008)) and for any coherent sheaf $\mathcal{E}$ it is possible to choose a complex $\Phi(\mathcal{E})\in\mathbf{C}^+_{\text{inj}}(\mathscr{Qcoh}\,\mathbf{X})$ such that $\Phi(\mathcal{E})$ is exact in non-zero degrees and has $H^0(\Phi(\mathcal{E}))=\mathcal{E}$. Then the ith *sheaf cohomology* of $\mathcal{E}$ is defined as $H^i(\Gamma(\Phi(\mathcal{E})))$, where Γ is the global sections functor.

All of the constructions above have two 'problems'– the first is that the choice of Φ is often neither unique nor functorial. However, it turns out in all of the examples that the actual homology computed is functorial and independent of the choice of Φ. For triangulable spaces, proving the independence of homology from choice of triangulation amounts to showing that whenever $\Phi_1(\mathbf{X})$ and $\Phi_2(\mathbf{X})$ are two triangulations of $\mathbf{X}$, there is a third simplicial complex $\Phi_3(\mathbf{X})$ (a 'common refinement' of the two triangulations) such that there is a diagram

$$\begin{tikzcd} & \Phi^3(\mathbf{X}) \arrow[dl] \arrow[dr] & \\ \Phi^1(\mathbf{X}) & & \Phi^2(\mathbf{X}) \end{tikzcd}$$

in which the two morphisms induce isomorphisms on cohomology.

Definition 1.12. A morphism $f : \mathbf{A} \to \mathbf{B}$ in $\mathbf{C}(\mathfrak{A})$ is a *quasi-isomorphism* if for all $i \in \mathbb{Z}$, the induced maps $H^i(f) : H^i(\mathbf{A}) \to H^i(\mathbf{B})$ are isomorphisms.

Cohomology is hard to compute directly in the original category $\mathfrak{C}$, so we choose to pass through an intermediate category $\mathbf{C}(\mathfrak{A})$. In an ideal situation, we would have, in place of $\mathbf{C}(\mathfrak{A})$, another category $\mathbf{D}(\mathfrak{A})$ such that the homology functors $H^i : \mathbf{D}(\mathfrak{A}) \to \mathfrak{A}$ are just as easy to compute, but the assignment $\Phi : \mathfrak{C} \to \mathbf{D}(\mathfrak{A})$ would actually be functorial. The derived category will play this role.

The second problem with cohomology is that it is too crude an invariant– the triangulable spaces $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have the same simplicial homology, but are clearly not homotopy equivalent. In fact, by Whitehead's theorem, two simply connected triangulable spaces $\mathbf{X}, \mathbf{Y}$ are homotopy equivalent if and only if there are maps

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl] \arrow[dr] & \\ \mathbf{X} & & \mathbf{Y} \end{tikzcd}$$

which induce isomorphisms on homology. Thus instead of simply stating if two objects in $\mathfrak{C}$ have isomorphic cohomology, we also wish to specify whether these isomorphisms are induced by morphisms in $\mathfrak{C}$. Thus a better invariant than cohomology is the associated complex itself, identified with other complexes it maps to via quasi-isomorphisms. In the words of Thomas (2001),

Complexes good, (co)homology bad.

Accordingly, the *derived category* of an abelian category $\mathfrak{A}$ is defined via a universal property.

Theorem 1.13. There is a category $\mathbf{D}(\mathfrak{A})$ with an additive functor $\Theta : \mathbf{C}(\mathfrak{A}) \to \mathbf{D}(\mathfrak{A})$ universal among additive functors that send quasi-isomorphisms to isomorphisms, i.e. whenever $F : \mathbf{C}(\mathfrak{A}) \to \mathfrak{D}$ is an additive functor such that every quasi-isomorphism f in $\mathbf{C}(\mathfrak{A})$ is sent to an isomorphism $F(f)$ in $\mathfrak{D}$, then F factors uniquely through Θ.

$$\begin{tikzcd} \mathbf{C}(\mathfrak{A}) \arrow[r, "F"] \arrow[d, "\Theta"] & \mathfrak{D} \\ \mathbf{D}(\mathfrak{A}) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

We don't prove this, directing the reader to Weibel (2003) for details. We will, however, describe the construction and state some useful properties. The derived category is constructed in two steps– we first pass to the homotopy category where homotopic morphisms are identified, and then invert quasi-isomorphisms by the operation of localisation.

1.3.1 The homotopy category

To any abelian category $\mathfrak{A}$, we associate a category $\mathbf{K}(\mathfrak{A})$ (the *homotopy category*) such that there is an additive functor $\mathbf{C}(\mathfrak{A}) \to \mathbf{K}(\mathfrak{A})$ which H^i factors through.

Definition 1.14. A chain map $\mathbf{A} \xrightarrow{f} \mathbf{B}$ in $\mathbf{C}(\mathfrak{A})$ is said to be *nullhomotopic* if there are morphisms $s^i : A^i \to B^{i-1}$ such that $f^i = d^i \circ s^{i+1} + s^i \circ d^i$ for all i. Two chain maps $f, g : \mathbf{A} \rightrightarrows \mathbf{B}$ are *homotopic* if the difference $f - g$ is nullhomotopic. We call s the *chain homotopy*.

We define the *homotopy category* $\mathbf{K}(\mathfrak{A})$ to have the same objects as $\mathbf{C}(\mathfrak{A})$ and morphisms given by equivalence classes of chain homotopic maps. For any $\mathbf{A}, \mathbf{B} \in \mathbf{C}(\mathfrak{A})$, the nullhomotopic chain maps form a subgroup $N(\mathfrak{A}, \mathfrak{B})$ of $\mathrm{Hom}_{\mathbf{C}(\mathfrak{A})}(\mathbf{A}, \mathbf{B})$. Then we have

$$\mathrm{Hom}_{\mathbf{K}(\mathfrak{A})}(\mathbf{A}, \mathbf{B}) = \frac{\mathrm{Hom}_{\mathbf{C}(\mathfrak{A})}(\mathbf{A}, \mathbf{B})}{N(\mathbf{A}, \mathbf{B})}.$$

The categories $\mathbf{K}^+(\mathfrak{A}), \mathbf{K}^-(\mathfrak{A})$, and $\mathbf{K}^b(\mathfrak{A})$ are analogously defined from $\mathbf{C}^+(\mathfrak{A}), \mathbf{C}^-(\mathfrak{A})$, and $\mathbf{C}^b(\mathfrak{A})$ respectively.

Isomorphisms in $\mathbf{K}(\mathfrak{A})$ are called *homotopy equivalences*.

The motivation to consider the homotopy category comes from algebraic topology, where homotopy equivalent spaces are guaranteed to have the same homology– so techniques in algebraic topology can only give us information *up to homotopy*, a weaker equivalence than homeomorphism. Likewise, techniques of homological algebra fail to distinguish between homotopic morphisms.

Proposition 1.15 (Weibel (2003))**.** If f and g are homotopic chain maps, then the induced morphisms $H^i(f)$ and $H^i(g)$ on homology are the same.

We often use the result above to show a complex is exact, by showing first that the identity map on the complex is nullhomotopic. Such complexes are called *contractible*.

1.3.2 Localisation and the derived category

Let Q be the class of all quasi-isomorphisms in $\mathbf{K}(\mathfrak{A})$. We define the derived category $\mathbf{D}(\mathfrak{A}) = Q^{-1}\,\mathbf{K}(\mathfrak{A})$ by 'adding inverses' to quasi-isomorphisms, in the sense of localisations defined below.

Definition 1.16. Let S be a collection of morphisms in a category $\mathfrak{C}$. A localisation of $\mathfrak{C}$ with respect to S is a category $S^{-1}\mathfrak{C}$ with a functor $q : \mathfrak{C} \to S^{-1}\mathfrak{C}$ such that the following hold.

1. For every $s \in S$, $q(s)$ is an isomorphism in $S^{-1}\mathfrak{C}$.
2. Any functor $F : \mathfrak{C} \to \mathfrak{D}$ such that F takes elements of S to isomorphisms factors uniquely through q.

Example 1.17 (Localisation of rings)**.** Considering a commutative ring R to be an additive category $\mathfrak{A}$ with a single non-zero object $*$ as in Example 1.3, we can choose S to be a multiplicative subset of $R = \mathrm{Hom}_{\mathfrak{A}}(*, *)$. Then $S^{-1}\mathfrak{A}$ is again additive with a single non-zero object $*$. The corresponding ring is precisely the ring of fractions $S^{-1}R$.

We describe the objects and morphisms of $\mathbf{D}(\mathfrak{A})$ explicitly, directing the reader to Weibel (2003) for proofs and further motivations.

Proposition 1.18. 1. Under the localisation map $q : \mathbf{K}(\mathfrak{A}) \to \mathbf{D}(\mathfrak{A})$, the objects of the two categories are identified.

2. The cohomology functors $H^i : \mathbf{K}(\mathfrak{A}) \to \mathfrak{A}$ factor through $q : \mathbf{K}(\mathfrak{A}) \to \mathbf{D}(\mathfrak{A})$, so the cohomology objects $H^i(\mathbf{A})$ of any $\mathbf{A} \in \mathbf{D}(\mathfrak{A})$ are well-defined.

3. Viewing an object $A \in \mathfrak{A}$ as a complex concentrated in degree 0 yields an equivalence between $\mathfrak{A}$ and the full subcategory of $\mathbf{D}(\mathfrak{A})$ whose objects are all complexes $\mathbf{A}$ with $H^i(\mathbf{A}) = 0$ for all $i \neq 0$.

To describe the morphisms, we go back to Example 1.17. Recall that elements of the ring of fractions $S^{-1}R$ can be given by a tuple (s, r) (conventionally written r/s) for $s \in S$, $r \in R$. Likewise, morphisms $\mathbf{A} \to \mathbf{B}$ in the localised category $\mathbf{D}(\mathfrak{A})$ are given by diagrams in $\mathbf{K}(\mathfrak{A})$ of the form

$$\begin{tikzcd} & \mathbf{C} \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ \mathbf{A} & & \mathbf{B} \end{tikzcd}$$

where the first map is a quasi-isomorphism.

The derived categories $\mathbf{D}^+(\mathfrak{A})$, $\mathbf{D}^-(\mathfrak{A})$, and $\mathbf{D}^b(\mathfrak{A})$ are defined analogously as localisations of $\mathbf{K}^+(\mathfrak{A})$, $\mathbf{K}^-(\mathfrak{A})$, and $\mathbf{K}^b(\mathfrak{A})$ respectively.

1.3.3 Triangulated categories

Both the homotopy category and the derived category of $\mathfrak{A}$ are additive, but neither is usually abelian. Recall that the structure of an abelian category on $\mathbf{C}(\mathfrak{A})$ involves, associating to each morphism f in $\mathbf{C}(\mathfrak{A})$ its kernel and cokernel, which fit into an exact sequence

$$0 \to \mathbf{C} \xrightarrow{\ker f} \mathbf{A} \xrightarrow{f} \mathbf{B} \xrightarrow{\operatorname{coker} f} \mathbf{D} \to 0.$$

Although (co)kernels are no longer guaranteed to be well-defined, we describe a structure on the homotopy and derived categories that 'remembers' this information. Define the *mapping cone* of f to be the complex $\mathscr{C}\!one f$ given in degree i by $A^{i+1} \oplus B^i$, with differential

$$\begin{aligned} \cdots \to A^{i+1} \oplus B^i &\longrightarrow A^{i+2} \oplus B^{i+1} \to \cdots \\ (a, b) &\longmapsto (-da, db + fa). \end{aligned}$$

The complex $\mathscr{C}\!one f$ has natural injections from and projections to $\mathbf{B}$ and $\mathbf{A}[1]$ respectively. This gives us a triple of morphisms

$$(\mathbf{A} \to \mathbf{B}, \mathbf{B} \to \mathscr{C}\!one f, \mathscr{C}\!one f \to \mathbf{A}[1])$$

in $\mathbf{K}(\mathfrak{A})$, which we call a *strict triangle*.

Definition 1.19. We say a triple of morphisms (u, v, w) in $\mathbf{K}(\mathfrak{A})$ is an *exact triangle* if there is a strict triangle (f, g, h) and a commuting diagram in $\mathbf{K}(\mathfrak{A})$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "f"] \arrow[d, "\alpha"] & \mathbf{B} \arrow[r, "g"] \arrow[d] & \mathscr{C}\!one f \arrow[r, "h"] \arrow[d] & \mathbf{A}[1] \arrow[d, "\alpha[1]"] \\ \mathbf{E} \arrow[r, "u"] & \mathbf{F} \arrow[r, "v"] & \mathbf{G} \arrow[r, "w"] & \mathbf{E}[1] \end{tikzcd}$$

where the vertical maps are isomorphisms.

Strict triangles and distinguished triangles in $\mathbf{D}(\mathfrak{A})$ are analogously defined. Then whenever

$$0 \to \mathbf{A} \xrightarrow{f} \mathbf{B} \longrightarrow \mathbf{C} \to 0$$

is a short exact sequence in $\mathbf{C}(\mathfrak{A})$, the category $\mathbf{K}(\mathfrak{A})$ (and hence also $\mathbf{D}(\mathfrak{A})$) has an exact triangle

$$(\mathbf{A} \to \mathscr{Cyl}\, f,\ \mathscr{Cyl}\, f \to \mathscr{Cone}\, f,\ \mathscr{Cone}\, f \to \mathbf{A}[1])$$

where the complex $\mathscr{Cyl}\, f$ (called the *mapping cylinder* of f), is defined as the mapping cone of $\mathscr{Cone}\, f[-1] \to \mathbf{A}$. It can be shown that there are quasi-isomorphisms $\mathscr{Cyl}\, f \to \mathbf{B}$ and $\mathscr{Cone}\, f \to \mathbf{C}$ so that the derived category 'remembers' the short exact sequence.

Every exact triangle in $\mathbf{D}(\mathfrak{A})$ (and hence in particular, every short exact sequence in $\mathbf{C}(\mathfrak{A})$ has an associated *long exact sequence of cohomology* which is a very useful computational tool.

Proposition 1.20. If $(\mathbf{A} \to \mathbf{B},\ \mathbf{B} \to \mathbf{C},\ \mathbf{C} \to \mathbf{A}[1])$ is an exact triangle in $\mathbf{D}(\mathfrak{A})$, then there is an associated exact complex in $\mathfrak{A}$ given by

$$\cdots \to H^i(\mathbf{A}) \to H^i(\mathbf{B}) \to H^i(\mathbf{C}) \to H^{i+1}(\mathbf{A}) \to \cdots$$

where the maps are functorially induced. Here we identify $H^i(\mathbf{A}[1])$ with $H^{i+1}(\mathbf{A})$.

Proof. See Corollary 10.1.4 of Weibel (2003). □

The structure given to $\mathbf{K}(\mathfrak{A})$ by exact triangles is abstracted by the notion of a *triangulated category*, which is a category $\mathfrak{C}$ with an automorphism $T : \mathfrak{C} \to \mathfrak{C}$ (called the *translation*) and a collection of triples of morphisms (u, v, w) (called *distinguished triangles*) subject to four axioms, the details of which are in Definition 10.2.1 of Weibel (2003). To us, the fact that is the most important is that the derived category of an abelian category satisfies these axioms, with the role of T played by [1], and the distinguished triangles being the exact triangles.

1.3.4 Derived functors

A *morphism of triangulated categories* is an additive functor that commutes with translation and sends distinguished triangles to distinguished triangles.

Remark 1.21. The localisation functor q defined in Definition 1.16 is not necessarily additive, however Weibel (2003) describes conditions on the class S which ensure $q : \mathfrak{C} \to S^{-1}\mathfrak{C}$ is well-behaved. For the purposes of this exposition, the following fact suffices: when $\mathfrak{A}$ is the abelian category of sheaves or modules, the class Q of quasi-isomorphisms in a (triangulated subcategory of) $\mathbf{K}(\mathfrak{A})$ is sufficiently nice that the localisation functor q and the functors induced by the universal property are morphisms of triangulated categories.

Morphisms between derived categories will typically come from additive functors $F : \mathbf{C}(\mathfrak{A}) \to \mathbf{C}(\mathfrak{B})$ on the corresponding chain complex categories.

Lemma 1.22. An additive functor $F : \mathbf{C}(\mathfrak{A}) \to \mathbf{C}(\mathfrak{B})$ descends to a morphism of triangulated categories $F : \mathbf{K}(\mathfrak{A}) \to \mathbf{K}(\mathfrak{B})$ it takes cones to cones, i.e. for any morphism f in $\mathbf{C}(\mathfrak{A})$ we have $F(\mathscr{Cone}\, f) = \mathscr{Cone}\, F(f)$.

Proof. It is immediate from the splitting lemma that a chain map $f : \mathbf{A} \to \mathbf{B}$ is nullhomotopic if and only if the short exact sequence

$$0 \to \mathbf{B} \longrightarrow \mathscr{C}\!one f \longrightarrow \mathbf{A}[1] \to 0$$

is split. Since F maps cones to cones, the sequence above is mapped by F to

$$0 \to F(\mathbf{B}) \longrightarrow \mathscr{C}\!one F(f) \longrightarrow F(\mathbf{A})[1] \to 0.$$

But additive functors send split exact sequences to split exact sequences, so $F(f)$ is nullhomotopic and F descends to an additive functor $\mathbf{K}(\mathfrak{A}) \to \mathbf{K}(\mathfrak{B})$. Moreover, the image of a strict triangle is a strict triangle so we have a morphism of triangulated categories. □

In order to induce a functor $\mathbf{D}(\mathfrak{A}) \to \mathbf{D}(\mathfrak{B})$, the functor F must send quasi-isomorphisms to quasi-isomorphisms. Often, however, this is not the case and we resort to one of two options– restrict to a triangulated subcategory (for example the bounded derived category) so that F does map quasi-isomorphisms to isomorphisms, or define functors on the derived category that preserve *some* of the properties of F. We describe both the approaches.

Descent to smaller triangulated subcategories. Given a morphism $F : \mathbf{K}(\mathfrak{A}) \to \mathbf{K}(\mathfrak{B})$ of triangulated categories, we say a complex $\mathbf{A} \in \mathbf{K}(\mathfrak{A})$ is F-*acyclic* if the complex $F(\mathbf{A})$ is acyclic.

Proposition 1.23. Suppose $\mathbf{K}$ is a triangulated subcategory of $\mathbf{K}(\mathfrak{A})$ such that every acyclic complex in $\mathbf{K}$ is also F-acyclic. Then the restricted functor $F : \mathbf{K} \to \mathbf{K}(\mathfrak{B})$ descends to a morphism of triangulated categories $F : Q^{-1}\mathbf{K} \to \mathbf{D}(\mathfrak{B})$ where Q is the class of quasi-isomorphisms in $\mathbf{K}$.

Proof. If $f : \mathbf{A} \to \mathbf{B}$ is a quasi-isomorphism in $\mathbf{K}$, then examining the long exact sequence in cohomology associated to $0 \to \mathbf{B} \to \mathscr{C}\!one f \to \mathbf{A}[1] \to 0$ shows that $\mathscr{C}\!one f$ is acyclic. Hence by assumption $F(\mathscr{C}\!one f)$ is acyclic. Now F preserves exact triangles, so the triangle

$$(F(\mathbf{A}) \to F(\mathbf{B}),\ F(\mathbf{B}) \to F(\mathscr{C}\!one f),\ F(\mathscr{C}\!one f) \to F(\mathbf{A})[1])$$

is exact. Examining the associated long exact sequence on homology shows that $F(f) : F(\mathbf{A}) \to F(\mathbf{B})$ is a quasi-isomorphism. Thus the composite $\mathbf{K} \to \mathbf{K}(\mathfrak{B}) \to \mathbf{D}(\mathfrak{B})$ sends quasi-isomorphisms to isomorphisms, hence by the universal property of localisation we have an induced functor $Q^{-1}\mathbf{K} \to \mathbf{D}(\mathfrak{A})$. By Remark 1.21, we are done. □

This approach is used, for instance, to establish the Bernstein-Gel'fand-Gel'fand correspondence of Section 3 which gives an equivalence of bounded derived categories but fails to extend to the unbounded situation directly.

Derived functors. Let $\mathbf{K}$ be a triangulated subcategory of $\mathbf{K}(\mathfrak{A})$, and $q : \mathbf{K} \to \mathbf{D}$ the localisation map with respect to the class of quasi-isomorphisms. For $F : \mathbf{K} \to \mathbf{K}(\mathfrak{A})$ a morphism of triangulated categories, we define associated *derived functors* by their universal properties.

Definition 1.24. A *(total) right derived functor* of F on $\mathbf{K}$ is a morphism of triangulated categories $\mathbf{R}F : \mathbf{D} \to \mathbf{D}(\mathfrak{B})$ with a natural transformation

$$\begin{tikzcd}
\mathbf{K} \arrow[r, "F"] & \mathbf{K}(\mathfrak{B}) \arrow[r, "q"] \arrow[d, phantom, "\Downarrow\zeta"] & \mathbf{D}(\mathfrak{B}) \\
\mathbf{K} \arrow[r, "q"] & \mathbf{D} \arrow[r, "\mathbf{R}F"] & \mathbf{D}(\mathfrak{B})
\end{tikzcd}$$

such that if $G : \mathbf{D} \to \mathbf{D}(\mathfrak{B})$ is a morphism equipped with a natural transformation $\zeta' : qF \Rightarrow Gq$, then there is a unique natural transformation $\eta : \mathbf{R}F \Rightarrow G$ such that $\zeta'_{\mathbf{A}} = \eta_{q\mathbf{A}} \circ \zeta_{\mathbf{A}}$ for every $\mathbf{A} \in \mathbf{D}$.

The *(total) left derived functor* of F on $\mathbf{K}$ is a morphism $\mathbf{L}F : \mathbf{D} \to \mathbf{D}(\mathfrak{B})$ with a natural transformation $\zeta : (\mathbf{L}F)q \Rightarrow qF$ satisfying a universal property similar to the right derived functor.

We provide a brief account of various right and left derived functors that come up in this exposition.

Example 1.25. If $F : \mathfrak{A} \to \mathfrak{B}$ is an exact functor, then the induced functor $F : \mathbf{C}(\mathfrak{A}) \to \mathbf{C}(\mathfrak{B})$ preserves quasi-isomorphisms (since f is a quasi-isomorphism if and only if $\mathscr{C}one\, f$ is an acyclic complex) and we have a functor $F : \mathbf{D}(\mathfrak{A}) \to \mathbf{D}(\mathfrak{B})$. In effect, F is its own right and left derived functor on $\mathbf{K}(\mathfrak{A})$.

Proposition 1.23 can be seen as a generalisation of above, saying that if a triangulated subcategory $\mathbf{K}$ is such that every F-acyclic complex is F-acyclic then F is its own right and left derived functor on $\mathbf{K}$. Often, we can choose the subcategory $\mathbf{K}$ (with localisation $\mathbf{D}$) such that $\mathbf{D} \cong \mathbf{K}(\mathfrak{A})$. This allows us to show existence of derived functors in certain special cases.

Example 1.26. For a noetherian scheme $\mathbf{X}$, let $\mathbf{K}^+_{\mathrm{inj}}(\mathfrak{Qcoh}\,\mathbf{X})$ be the full subcategory of $\mathbf{K}(\mathfrak{Qcoh}\,\mathbf{X})$ containing complexes of injective sheaves that are bounded below. Since injective sheaves are flasque, every acyclic complex in $\mathbf{K}^+_{\mathrm{inj}}(\mathfrak{Qcoh}\,\mathbf{X})$ is Γ-acyclic where Γ is the global sections functor.

Now every quasi-isomorphism in $\mathbf{K}^+_{\mathrm{inj}}$ is an isomorphism and moreover, every complex in $\mathbf{K}^+(\mathfrak{Qcoh}\,\mathbf{X})$ is quasi-isomorphic to a bounded below complex of injectives. This can be used to show that the derived category $\mathbf{D}^+(\mathfrak{Qcoh}\,\mathbf{X})$ is equivalent to $\mathbf{K}^+_{\mathrm{inj}}(\mathfrak{Qcoh}\,\mathbf{X})$. Thus Γ has left and right derived functors $\mathbf{D}^+(\mathfrak{Qcoh}\,\mathbf{X}) \to \mathbf{D}(\mathbb{Z}\text{-Mod})$. In practice, computing these on a complex $\mathcal{E}$ involves first finding a quasi-isomorphic complex of injectives and then applying Γ. In particular, if $\mathcal{E}$ is a complex concentrated in degree 0, then the ith sheaf cohomology is $H^i(\mathbf{R}\Gamma(\mathcal{E}))$.

In fact, Weibel (2003) shows that the right derived functor extends to the whole derived category, giving a morphism of triangulated categories $\mathbf{R}\Gamma : \mathbf{D}(\mathfrak{Qcoh}\,\mathbf{X}) \to \mathbf{D}(\mathbb{Z}\text{-Mod})$.

Example 1.27. If $\mathfrak{A}$ has enough injectives (i.e. any object of $\mathfrak{A}$ can be resolved by injectives), then for each complex $\mathbf{A} \in \mathbf{K}(\mathfrak{A})$ the functor $\mathrm{Tot}(\mathrm{Hom}(\mathbf{A}, -))$ has left and right derived functors $\mathbf{D}^+(\mathfrak{A}) \to \mathbf{D}(\mathbb{Z}\text{-Mod})$. Then Weibel (2003) shows that the right derived functor is a bifunctor

$$\mathbf{R}\,\mathrm{Hom} : \mathbf{D}(\mathfrak{A})^{\mathrm{op}} \times \mathbf{D}^+(\mathfrak{A}) \to \mathbf{D}(\mathbb{Z}\text{-Mod}).$$

Moreover, if $\mathbf{A}$ and $\mathbf{B}$ are both complexes bounded below, then we have the *hyperext* groups

$$\mathrm{Ext}^i(\mathbf{A}, \mathbf{B}) \cong H^i(\mathbf{R}\,\mathrm{Hom}(\mathbf{A}, \mathbf{B})) = \mathrm{Hom}_{\mathbf{D}(\mathbf{A})}(\mathbf{A}, \mathbf{B}[i]).$$

Likewise, if $\mathfrak{A} = R\text{-Mod}$ has enough projectives, then the total tensor product functor has a left derived functor

$$\otimes^{\mathbf{L}}_R : \mathbf{D}^-(\mathfrak{A}) \times \mathbf{D}^-(\mathfrak{A}) \to \mathbf{D}(\mathbb{Z}\text{-Mod}).$$

The cohomologies of this are the *hypertor* functors

$$\mathrm{Tor}^R_i(\mathbf{A}, \mathbf{B}) = H^{-i}(\mathbf{A} \otimes^{\mathbf{L}}_R \mathbf{B}).$$

The derived tensor product construction extends to the bounded derived category of quasicoherent sheaves on a noetherian scheme $\mathbf{X}$, if one uses locally free sheaves instead of projectives. If

$f : \mathbf{X} \to \mathbf{Y}$ is a proper morphism of projective schemes, then the derived functors $\mathbf{R}f^*$ and $\mathbf{L}f_*$ exist on $\mathbf{K}^b(\mathfrak{Qcoh}\,\mathbf{X})$. This allows us to state a useful result, which generalises the classical projection formula which states

$$f_*(\mathcal{F} \otimes f^*(\mathcal{E})) \cong f_*(\mathcal{F}) \otimes \mathcal{E}$$

whenever $\mathcal{E}, \mathcal{F} \in \mathfrak{Qcoh}\,\mathbf{X}$, and $\mathcal{E}$ is locally free. When extended to the derived categories, Weibel (2003) shows that whenever $\mathcal{E}, \mathcal{F}$ are complexes in $\mathbf{D}^b(\mathfrak{Qcoh}\,\mathbf{X})$, we have

$$\mathbf{R}f_*(\mathcal{F} \otimes^{\mathbf{L}} \mathbf{L}f^*(\mathcal{E})) \cong \mathbf{R}f_*(\mathcal{F}) \otimes^{\mathbf{L}} \mathcal{E}.$$

1.3.5 Generators of a triangulated category

Triangulated categories have two ‘fundamental operations’ built in– translations and taking mapping cones. Then we say a collection of objects S in a triangulated category $\mathfrak{C}$ *generates it* if, up to isomorphism, every object of $\mathfrak{C}$ can be reached by taking these objects, shifting them, taking arbitrary morphisms between them, taking mapping cones of these morphisms, and repeating these operations finitely many times.

Definition 1.28. If $\mathfrak{C}$ is a triangulated category, we say a collection S of objects generates $\mathfrak{C}$ if there is no proper triangulated subcategory of $\mathfrak{C}$ containing S.

Generators give us an explicit handle on the triangulated category, as is illustrated by the following lemma from Beilinson (1978).

Lemma 1.29. Let $F : \mathfrak{C} \to \mathfrak{D}$ be a morphism of triangulated categories, and $\{X_i\}$ a collection of generators of $\mathfrak{C}$ such that the collection $\{FX_i\}$ generates $\mathfrak{D}$. If we have that for any pair X_i, X_j and any integer m the map

$$F : \mathrm{Hom}_{\mathfrak{C}}(X_i[m], X_j) \to \mathrm{Hom}_{\mathfrak{D}}(FX_i[m], FX_j)$$

is an isomorphism, then F is an equivalence of categories.

Proof. Note the image of F is a triangulated subcategory containing all the FX_i, i.e. the whole of $\mathfrak{D}$. It suffices to show that the functor F is fully faithful, i.e. that $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(FX, FY)$ for all $X, Y \in \mathfrak{C}$.

Let $\mathfrak{C}'$ be the full subcategory of those $X \in \mathfrak{C}$ satisfying $\mathrm{Hom}_{\mathfrak{C}}(X[m], X_i) \cong \mathrm{Hom}_{\mathfrak{D}}(FX[m], FX_i)$ for all X_i and all m. We show that this is a triangulated subcategory of $\mathfrak{C}$ and contains all the X_i by assumption, so $\mathfrak{C}' = \mathfrak{C}$. Note $\mathfrak{C}'$ is clearly closed under taking translations. If $(X \to Y,\ Y \to Z,\ Z \to X)$ is an exact triangle in $\mathfrak{C}$ such that X and Y are in $\mathfrak{C}'$, then for any generator X_i we have a commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}(X, X_i) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(Y, X_i) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(Z, X_i) \arrow[r] \arrow[d] & \mathrm{Hom}(X[1], X_i) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(Y[1], X_i) \arrow[d, "\cong"] \\
\mathrm{Hom}(FX, FX_i) \arrow[r] & \mathrm{Hom}(FY, FX_i) \arrow[r] & \mathrm{Hom}(FZ, FX_i) \arrow[r] & \mathrm{Hom}(FX[1], FX_i) \arrow[r] & \mathrm{Hom}(FY[1], FX_i)
\end{tikzcd}$$

in $\mathbf{D}(\mathbb{Z}\text{-Mod})$. By the derived version of the five lemma, the morphism in the middle is also an isomorphism. This shows $\mathfrak{C}'$ is closed under taking mapping cones, hence is a triangulated subcategory as required.

A similar argument shows that if $\mathfrak{C}''$ is the full subcategory of those $Y \in \mathfrak{C}$ satisfying $\mathrm{Hom}_{\mathfrak{C}}(X, Y) \cong \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ for all $X \in \mathfrak{C}$, then $\mathfrak{C}''$ is a triangulated subcategory and contains all the X_i. Thus $\mathfrak{C}'' = \mathfrak{C}$, and $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(FX, FY)$ for all $X, Y \in \mathfrak{C}$ as required. $\square$

2 Coherent sheaves on $\mathbb{P}^n$

The projectivisation of Ξ is the k-scheme $\mathbb{P}^n = \mathrm{Proj}(\mathrm{Sym}^\bullet(X))$, where $\mathrm{Sym}^\bullet(X) = k[x_0, ..., x_n]$ is the standard symmetric algebra on X graded by degree. We strengthen the observation of Example 0.1 to the following result, the celebrated theorem of Beilinson.

Theorem (Beilinson (1978)). The derived category $\mathbf{D}^b(\mathbb{P}^n)$ is generated by the exceptional sequence

$$\langle \mathcal{O}(-n), \mathcal{O}(-n+1), ..., \mathcal{O}(-1), \mathcal{O} \rangle.$$

This section looks at Beilinson's (1978) original proof, following the treatment in Caldararu (2005) and Carbone (2016). There are two key ideas involved– the first is that the identity functor on $\mathbf{D}(\mathscr{Coh}\,\mathbb{P}^n)$ admits a factorisation

$$\begin{tikzcd}
\mathbf{D}^b(\mathbb{P}^n \times \mathbb{P}^n) \arrow[r, "-\otimes^{\mathbf{L}}\mathcal{O}_\Delta"] & \mathbf{D}^b(\mathbb{P}^n \times \mathbb{P}^n) \arrow[d, "\mathbf{R}\pi_{1*}"] \\
\mathbf{D}^b(\mathbb{P}^n) \arrow[u, "\pi_2^*"] \arrow[r, "\mathrm{id}"] & \mathbf{D}^b(\mathbb{P}^n)
\end{tikzcd}$$

where $\pi_1, \pi_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projection maps, and $\mathcal{O}_\Delta \in \mathscr{Coh}(\mathbb{P}^n \times \mathbb{P}^n)$ is the structure sheaf of the diagonal subscheme. This follows from the geometric theory of a *Fourier-Mukai transform* associated to a pair of schemes $\mathcal{X}, \mathcal{Y}$, and we briefly sketch the construction in Section 2.1.

The second observation, called *Beilinson's resolution of the diagonal*, follows from the algebraic theory of *Koszul resolutions* and shows that that $\mathcal{O}_\Delta$ admits a resolution by locally free sheaves of the form $\pi_1^*(\Omega^i(i)) \otimes \pi_2^*(\mathcal{O}(-i))$, where Ω is the sheaf of differentials on $\mathbb{P}^n$. Combined with the factorisation of identity, this provides an algorithm to resolve any coherent sheaf on $\mathbb{P}^n$in terms of the $\mathcal{O}(i)$ thus proving Beilinson's result.

2.1 Fourier-Mukai transforms

The material in this section is from Huybrechts (2006), who covers the topic in great detail. Given two smooth projective k-schemes $\mathbf{X_1}$ and $\mathbf{X_2}$, we associate to each object $\mathcal{E} \in \mathbf{D}^b(\mathbf{X_1} \times \mathbf{X_2})$ an exact functor $\Phi_\mathcal{E} : \mathbf{D}^b(\mathbf{X_1}) \to \mathbf{D}^b(\mathbf{X_2})$ as follows.

Definition 2.1. The *Fourier-Mukai transform with kernel* $\mathcal{E}$ of a complex $\mathcal{A} \in \mathbf{D}^b(\mathbf{X_1})$ is defined as

$$\Phi_\mathcal{E}(\mathcal{A}) = \mathbf{R}\pi_{1*}(\pi_2^*\mathcal{A} \otimes^{\mathbf{L}} \mathcal{E}) \quad \in \mathbf{D}^b(\mathbf{X_2}).$$

Here $\pi_i : \mathbf{X_1} \times \mathbf{X_2} \to \mathbf{X_i}$ $(i = 1, 2)$ be the projection maps, these are flat so the pullback functors π_i^* are exact and need no derivation. Being the composition of three exact functors, the Fourier-Mukai transform $\Phi_\mathcal{E}$ is an exact functor. Moreover, the dependence on the kernel is functorial– for a fixed $\mathcal{A} \in \mathbf{D}^b(\mathbf{X_1})$, the map

$$\begin{aligned} \Phi_{-}(\mathcal{A}) : \quad \mathbf{D}^b(\mathbf{X_1} \times \mathbf{X_2}) &\longrightarrow \mathbf{D}^b(\mathbf{X_2}) \\ \mathcal{E} &\longmapsto \Phi_\mathcal{E}(\mathcal{A}) \end{aligned}$$

is the composite $\pi_{1*}(\pi_2^*\mathcal{A} \otimes^{\mathbf{L}} -)$, hence is an exact functor.

The name comes from the following analogy with functional analysis– given a finite-dimensional vector space X and its dual Ξ, to any smooth function $E(x, \xi) : X \times \Xi \to \mathbb{C}$ we can associate a linear map $\phi_E : L^2(X) \to L^2(Y)$ between the spaces of square-integrable functions, given by $f \mapsto \int_X f(x)E(x,\xi)dx$. If $E(x,\xi) = e^{2\pi i\langle x,\xi\rangle}$, then ϕ_E is an isomorphism called the *Fourier transform*. Similarly, the Fourier-Mukai transform yields interesting functors based on choice of $\mathcal{E}$.

Example 2.2. If $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}$ and $\mathbf{X} \overset{\iota}{\hookrightarrow} \mathbf{X} \times \mathbf{X}$ is the diagonal inclusion, then we can consider the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta = \iota_*\mathcal{O}_X$, the structure sheaf of the diagonal subscheme. Since ι is a closed immersion, the pushforward ι_* is exact and $R\iota_* = \iota_*$ as derived functors. Hence $\mathcal{O}_\Delta = \mathbf{R}\iota_*\mathcal{O}_X$ in $\mathbf{D}^b(\mathbf{X})$, and we can use the projection formula to get

$$\begin{aligned}\Phi_{\mathcal{O}_\Delta}(\mathcal{A}) &= \mathbf{R}\pi_{1*}(\pi_2^*\mathcal{A} \otimes^{\mathbf{L}} \mathbf{R}\iota_*\mathcal{O}_X)\\ &= \mathbf{R}\pi_{1*} \circ \mathbf{R}\iota_*(\mathbf{L}\iota^*\,\pi_2^*\mathcal{A} \otimes^{\mathbf{L}} \mathcal{O}_X)\\ &= \mathbf{R}(\pi_1 \circ \iota)_*(\mathbf{L}(\pi_2 \circ \iota)^*\mathcal{A} \otimes^{\mathbf{L}} \mathcal{O}_X)\\ &= \mathcal{A} \otimes^{\mathbf{L}} \mathcal{O}_X.\end{aligned}$$

But $\mathcal{O}_X$ is a locally free sheaf so the functor $(- \otimes^{\mathbf{L}} \mathcal{O}_X)$ is the same as $(- \otimes \mathcal{O}_X)$, which is identity. In other words, the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$ is the identity functor.

Replacing the trivial bundle $\mathcal{O}_X$ in the above computation with some other line bundle $\mathcal{L}$ on $\mathbf{X}$, we see that the derived functor $(- \otimes \mathcal{L})$ is the Fourier-Mukai transform with kernel $\iota_*\mathcal{L}$. Similarly, one can show that the Fourier-Mukai kernel $\mathcal{O}_\Delta[1]$ yields the shift functor $\mathcal{A} \mapsto \mathcal{A}[1]$. Thus Fourier-Mukai transforms generalise many familiar constructions. It is in fact a theorem of Orlov that any fully faithful exact functor $\mathbf{D}^b(\mathbf{X}_1) \to \mathbf{D}^b(\mathbf{X}_2)$ that admits adjoints must arise as the Fourier-Mukai transform for some kernel determined uniquely up to isomorphism.

2.2 Koszul resolutions

Given a ring A and a sequence $(a_0, ..., a_n)$ of elements in A, the associated *Koszul complex* is a very useful construction which detects various homological properties of the ring, and often yields free resolutions of the A-module $A/(a_0, ..., a_n)$. The construction and theory of Koszul complexes is treated in its full generality in Eisenbud (1995); here we only study the behaviour in two special cases we use to resolve the diagonal as Beilinson did– the first is when $(a_0, ..., a_n)$ generate the unit ideal, and the second is when they form a regular sequence.

Definition 2.3. Given a ring A and a sequence $(a_0, ..., a_n)$ of elements in A, the associated *Koszul complex* is the complex of A-modules given by

$$K_A(a_0, ..., a_n): \qquad 0 \to \textstyle\bigwedge_A^{n+1}(A^{n+1}) \to \bigwedge_A^n(A^{n+1}) \to \cdots \to \bigwedge_A^2(A^{n+1}) \to A^{n+1} \to A \to 0$$
$$d(e_{\alpha_1} \wedge ... \wedge e_{\alpha_i}) = \sum_j (-1)^{i+j+1} a_{\alpha_j} \cdot (e_{\alpha_1} \wedge ... \hat{e}_{\alpha_j} ... \wedge e_{\alpha_i})$$

where $e_0, ..., e_n$ are the standard generators of A^{n+1}, and $\hat{}$ denotes omission of a term. We put the term $\bigwedge_A^i(A^{n+1})$ in differential degree $-i$.

Observe that the modules appearing in $K_A(a_0, ..., a_n)$ are free, so acyclic Koszul complexes yield free A-resolutions. In the simplest case when the sequence contains a single element a_0, the Koszul complex is given by

$$K(a_0): \qquad 0 \to A \xrightarrow{a_0} A \to 0,$$

so it is exact if and only if a_0 is a unit in A. This result generalises to sequences $(a_0, ..., a_n)$ that generate the unit ideal.

Proposition 2.4. If A is a ring and $(a_0, ..., a_n) = A$, then the Koszul complex $K_A(a_0, ..., a_n)$ is exact everywhere.

Proof. We show that the the identity on $K_A(a_0, ..., a_n)$ is chain homotopic to the zero morphism. By assumption, there are elements $\lambda_0, ..., \lambda_n \in A$ such that $\sum_i \lambda_i a_i = -1$. Then consider the map given by

$$h : \textstyle\bigwedge_A^i (A^{n+1}) \to \bigwedge_A^{i+1}(A^{n+1})$$
$$h(e) = \sum_j \lambda_j e \wedge e_j$$

A straightforward basis-wise check shows $d \circ h + h \circ d = \mathrm{id}$, showing h is the required chain homotopy. Since homotopic chain-maps induce the same map on homology, we must have that $K_A(a_0, ..., a_n)$ is exact. $\square$

Looking again at the Koszul complex for a single element $a_0 \in A$, we have that $H^1(K(a_0)) = 0$ if and only if a_0 is not a zero-divisor in A– in this case the complex is a free resolution of $A/(a_0)$. Recall that $(a_0, ..., a_n)$ is an A-*regular sequence* if a_0 is not a zero-divisor in A, and for every $0 \leq i < n$, a_{i+1} is not a zero-divisor for the module $A/(a_0, ..., a_i)$. Then Eisenbud (1995) proves that whenever $(a_0, ..., a_n)$ is a regular sequence, the associated Koszul complex is exact everywhere except in degree 0 where it has cohomology $A/(a_0, ..., a_n)$. We prove a special case of the result, when A is a polynomial algebra and $a_0, ..., a_n$ are the indeterminates.

Proposition 2.5 (Loday (2012))**.** Suppose R contains a field of characteristic 0. Then for the polynomial ring $A = R[x_0, ..., x_n]$, we have

$$H^i(K_A(x_0, ..., x_n)) = \begin{cases} R, & i = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Proof. Write M for the rank $n+1$ free R-module with generators $m_0, ..., m_n$. Then we can identify $\bigwedge_A^\bullet (A^{n+1})$ with the algebra $\bigwedge_R^\bullet (M) \otimes_k A$, these are graded so that M lies in degree 1. Considering A as an R-algebra graded by degree, we see that $\mathbf{K} = K_A(x_0, ..., x_n)$ is a complex of graded R-algebras given by

$$\mathbf{K}: \qquad 0 \to \textstyle\bigwedge_R^{n+1}(M) \otimes_R A\langle -n-1 \rangle \to \bigwedge_R^n(M) \otimes_R A\langle -n \rangle \to \cdots \to A \to 0$$
$$d((m_{\alpha_1} \wedge ... \wedge m_{\alpha_i}) \otimes a) = \sum_j (-1)^{i+j+1} (m_{\alpha_1} \wedge ... \hat{m}_{\alpha_j} ... \wedge m_{\alpha_i}) \otimes a x_{\alpha_j}.$$

Note the differential d preserves internal grading, so we can write the complex above as a direct sum $\mathbf{K} = \bigoplus_r \mathbf{K}_r$ where $\mathbf{K}_r$ is the complex of R-modules formed at Adam's degree r (called the rth *strand* of $\mathbf{K}$). Since cohomology is an additive functor, we have $H^i(\mathbf{K}) = \bigoplus_r H^i(\mathbf{K}_r)$.

Now the strands in negative degrees vanish everywhere, and the $\mathbf{K}_0$ has the module R concentrated in differential degree 0. Thus it suffices to prove every other strand is exact. We do this by showing that the identity map on $\mathbf{K}_r$ is nullhomotopic whenever $r > 0$. In this case, we know by assumption that $r \in R$ is a unit so consider the map

$$h : \textstyle\bigwedge^i (M) \otimes_k A_{r-i} \to \bigwedge^{i+1}(M) \otimes_k A_{r-i-1}$$
$$h(m \otimes (x_{\beta_1} ... x_{\beta_{r-i}})) = -\frac{1}{r} \sum_j (m \wedge x_{\beta_j}) \otimes (x_{\beta_1} ... \hat{x}_{\beta_j} ... x_{\beta_{r-i}}).$$

A straightforward basis-wise check shows $h \circ d + d \circ h = \mathrm{id}$, i.e. h is a chain homotopy between id and the zero map. $\square$

Koszul complexes in geometry. We use Koszul complexes in the following geometric setting– on the affine scheme $\mathbf{X} = \mathrm{Spec}\, A$, an $(n+1)$-tuple $(a_0, ..., a_n)$ in A can be seen as a global section of the free sheaf $\mathcal{E} = \mathcal{O}_{\mathbf{X}}^{\oplus(n+1)}$. Then the Koszul complex associated to $(a_0, ..., a_n)$ yields a complex of coherent sheaves, given by

$$\mathcal{K}_{\mathbf{X}}(s): \qquad 0 \to \bigwedge^{n+1}\mathcal{E}^\vee \to \bigwedge^n \mathcal{E}^\vee \to \cdots \to \mathcal{E}^\vee \to \mathcal{O}_{\mathbf{X}} \to \mathcal{O}_{\mathbb{V}(s)} \to 0$$

where $\mathbb{V}(s)$ is the zero locus of s, i.e. the closed subscheme corresponding to the ideal $(a_0, ..., a_n)$. The differential then can be interpreted as 'contracting' an i-form in $\bigwedge^i \mathcal{E}^\vee$ with the section s. If $(a_0, ..., a_n)$ is a regular sequence then we have shown that the complex $\mathcal{K}_{\mathbf{X}}(s)$ is exact, giving a locally free resolution of $\mathcal{O}_{\mathbb{V}(s)}$.

The construction automatically extends to arbitrary schemes **X**– given global section of a locally free sheaf $\mathcal{E} \in \mathcal{C}oh\, \mathbf{X}$, we can cover **X** by affine open subschemes $\mathbf{X} = \bigcup_\alpha \mathbf{U}_\alpha$; then the associated Koszul complexes $\mathcal{K}_{\mathbf{U}_\alpha}(s|_{\mathbf{U}_\alpha})$ glue to give a Koszul complex $\mathcal{K}_{\mathbf{X}}(s)$ of coherent sheaves on **X**.

Remark 2.6. The exactness of this complex can be checked stalk-locally– given $p \in \mathbf{X}$, the section s gives a tuple $(s_0, ..., s_n)$ in the local ring $\mathcal{O}_{\mathbf{X},p}$. Localising the complex $\mathcal{K}_{\mathbf{X}}(s)$ at p then yields the Koszul complex $K_{\mathcal{O}_{\mathbf{X},p}}(s_0, ..., s_n)$. In particular, if $p \notin \mathbb{V}(s)$ then $(s_0, ..., s_n)$ generate the unit ideal so the localised complex is exact by Proposition 2.4. Thus it suffices to check that the complex $\mathcal{K}_{\mathbf{X}}(s)$ is exact at points of $\mathbb{V}(s)$.

2.3 Beilinson's theorem

Let $\iota : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the inclusion of the diagonal subscheme Δ, and write $\pi_1, \pi_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ for the two coordinate projections. To prove Beilinson's theorem, we will show that the Koszul complex associated to a particular locally free sheaf $\mathcal{E} \in \mathcal{C}oh(\mathbb{P}^n \times \mathbb{P}^n)$ gives a resolution of $\mathcal{O}_\Delta = \iota_* \mathcal{O}_{\mathbb{P}^n}$. The sheaf $\mathcal{E}$ is chosen so that the sheaves arising in the resolution are of the form $\pi_1^* \mathcal{O}_{\mathbb{P}^n}(-i) \otimes \pi_2^* \Omega^i_{\mathbb{P}^n}(i)$, where $\Omega_{\mathbb{P}^n}$ is the cotangent sheaf of $\mathbb{P}^n$ and $\Omega^i_{\mathbb{P}^n}$ denotes its ith exterior power. Given such a resolution, the functoriality (in kernel) of the Fourier-Mukai transform can be used to obtain a resolution of any $\mathcal{A} \in \mathbf{D}^b(\mathbb{P}^n)$.

In particular, the (-1)th twist of the second Euler exact sequence yields an associated long exact sequence of cohomology

$$0 \to \Gamma(\mathcal{O}(-1)) \to \Gamma(\mathcal{O}^{\oplus(n+1)}) \to \Gamma(\mathcal{T}(-1)) \to H^1(\mathcal{O}(-1)) \to \cdots .$$

But all cohomologies of $\mathcal{O}(-1)$ vanish, so we have an isomorphism $\Gamma(\mathcal{T}(-1)) \cong \Gamma(\mathcal{O}^{\oplus(n+1)})$. Thus the global sections of $\mathcal{T}(-1)$ form an $n+1$-dimensional k-vector space. Write $\frac{\partial}{\partial x_0}, ..., \frac{\partial}{\partial x_n}$ for the standard basis.

Resolution of the diagonal. Given two coherent sheaves $\mathcal{A}, \mathcal{B}$ on $\mathbb{P}^n$, write $\mathcal{A} \boxtimes \mathcal{B}$ for the coherent sheaf $\pi_1^* \mathcal{A} \otimes \pi_2^* \mathcal{B}$ on $\mathbb{P}^n \times \mathbb{P}^n$. Using this notation, the locally free sheaf on $\mathbb{P}^n \times \mathbb{P}^n$ and global section which we use to resolve the diagonal is given by

$$\mathcal{E} = \mathcal{O}(1) \boxtimes \mathcal{T}(-1), \qquad s = \sum_{\alpha=0}^{n} x_\alpha \boxtimes \frac{\partial}{\partial y_\alpha} \quad \in \Gamma(\mathcal{E}),$$

where $(x_0 : ... : x_n)$ and $(y_0 : ... : y_n)$ are homogeneous coordinates on the first and second copy of $\mathbb{P}^n$ respectively. To understand how the section s is defined, note that global sections of a sheaf $\mathcal{A}$ are same as morphisms from the structure sheaf to $\mathcal{A}$. Then by functoriality of pullbacks (and since pullback of the structure sheaf is again the structure sheaf), we lift $x_\alpha \in \Gamma(\mathcal{O}(1))$ to a global section $\pi_1^*(x_\alpha) \in \Gamma(\pi_1^*\mathcal{O}(1))$. The sections $\pi_2^*(\frac{\partial}{\partial y_\alpha}) \in \Gamma(\pi_2^*\mathcal{T}(-1))$ are likewise defined. Thus we have a global section $\pi_1^*(x_\alpha) \otimes \pi_2^*(\frac{\partial}{\partial y_\alpha})$ of the tensor product *presheaf*, which gets mapped to a global section $x_\alpha \boxtimes \frac{\partial}{\partial y_\alpha}$ of the tensor product sheaf by the canonical map

$$\Gamma(\pi_1^*\mathcal{O}(1)) \otimes_k \Gamma(\pi_2^*\mathcal{T}(-1)) \to \Gamma(\mathcal{O}(1) \boxtimes \mathcal{T}(-1)).$$

The following results justify our choice of $\mathcal{E}$ and s.

Lemma 2.7. For $\mathcal{E}$ and s as given, the section s vanishes precisely on the diagonal subscheme.

Proof. We check this on affine patches– writing $\mathbf{U}_\alpha = \mathbb{P}^n \setminus \mathbb{V}(x_\alpha)$ and $\mathbf{V}_\beta = \mathbb{P}^n \setminus \mathbb{V}(y_\beta)$, we see that the various open sets of the form $\mathbf{U}_\alpha \times \mathbf{V}_\beta$ are isomorphic to $\mathbb{A}^2$ and form an open cover of $\mathbb{P}^n \times \mathbb{P}^n$. We focus on the patch $\mathbf{U}_0 \times \mathbf{V}_0$, though the calculation on other patches is similar.

Write $x_{\alpha/0} = x_\alpha/x_0$ $(1 \le \alpha \le n)$ for the standard coordinates on $\mathbf{U}_0$, the coordinates on $\mathbf{V}_0$ are written similarly. Then writing $S = k[y_{1/0}, ..., y_{n/0}]$ for the coordinate ring of $\mathbf{V}_0$, we can read off the module of sections of $\mathcal{T}(-1)$ on $\mathbf{V}_0$ as the quotient

$$T = \frac{S^{n+1}}{S \cdot (1, y_{1/0}, ..., y_{n/0})}.$$

Since the quotient map commutes with restriction of sections, the restriction of $\frac{\partial}{\partial y_\alpha} \in \Gamma(\mathcal{T}(-1))$ is the image in T of the αth standard generator of S^{n+1}. Then we must have

$$\left.\frac{\partial}{\partial y_0}\right|_{\mathbf{V}_0} + y_{1/0}\left.\frac{\partial}{\partial y_1}\right|_{\mathbf{V}_0} + \cdots + y_{n/0}\left.\frac{\partial}{\partial y_n}\right|_{\mathbf{V}_0} = 0.$$

Writing $S' = k[x_{1/0}, ..., k_{n/0}]$ for the coordinate ring on $\mathbf{U}_0$, it is clear that $\mathcal{O}(1)$ restricts to the module S'. The restriction of the section $x_\alpha \in \Gamma(\mathcal{O}(1))$ is given by $\frac{x_\alpha}{x_0}$.

Write $R = k[x_{1/0}, ..., y_{n/0}] \cong S' \otimes_k S$ for the structure sheaf of $\mathbf{U}_0 \times \mathbf{V}_0$. Since pulling back a sheaf on an affine schemes corresponds to taking tensor product with the coordinate ring, the restriction of $\mathcal{O}(1) \boxtimes \mathcal{T}(-1)$ to $\mathbf{U}_0 \times \mathbf{V}_0$ corresponds to the R-module

$$(S' \otimes_{S'} R) \otimes_R (R \otimes_S T) \cong R \otimes_S T.$$

Then we have

$$\begin{aligned} s\big|_{\mathbf{U}_0 \times \mathbf{V}_0} &= \sum_{\alpha=0}^{n} x_\alpha\Big|_{\mathbf{U}_0} \otimes \left.\frac{\partial}{\partial y_\alpha}\right|_{\mathbf{V}_0} \\ &= \sum_{\alpha=1}^{n} \left(-1 \otimes y_{\alpha/0}\frac{\partial}{\partial y_\alpha} + x_{\alpha/0} \otimes \left.\frac{\partial}{\partial y_\alpha}\right|_{\mathbf{V}_0} \right) \\ &= \sum_{\alpha=1}^{n} (x_{\alpha/0} - y_{\alpha/0}) \otimes \left.\frac{\partial}{\partial y_\alpha}\right|_{\mathbf{V}_0}. \end{aligned}$$

Thus the restriction of $\mathbb{V}(s)$ to $\mathbf{U}_0 \times \mathbf{V}_0$ is defined by the ideal $(x_{1/0} - y_{1/0}, ..., x_{n/0} - y_{n/0})$, which gives the closed subscheme $\Delta \cap (\mathbf{U}_0 \times \mathbf{V}_0)$. □

Lemma 2.8. For $\mathcal{E}$ as given and $i \geq 1$, there is a natural isomorphism $\bigwedge^i(\mathcal{E}^\vee) \cong \mathcal{O}(-i) \boxtimes \Omega^i(i)$.

Proof. Observe we have natural isomorphisms

$$\begin{aligned}
\mathcal{E}^\vee &= \mathcal{H}om(\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{T}(-1), \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}) \\
&\cong \mathcal{H}om(\pi_1^*\mathcal{O}(1), \mathcal{H}om(\pi_2^*\mathcal{T}(-1), \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n})) \\
&\cong \mathcal{H}om(\pi_1^*\mathcal{O}(1), \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}) \otimes \mathcal{H}om(\pi_2^*\mathcal{T}(-1), \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}) \\
&= (\pi_1^*\mathcal{O}(1))^\vee \otimes (\pi_2^*\mathcal{T}(-1))^\vee
\end{aligned}$$

where the first isomorphism comes from the $\otimes - \mathcal{H}om$ adjunction, and the second uses the fact that $\mathcal{O}(1)$ (and hence its pullback) is locally free; see Section II.5 of Hartshorne (2008). Moreover, since pullback commutes with dualising for locally free sheaves of finite rank, we have the isomorphisms

$$(\pi_1^*\mathcal{O}(1))^\vee \cong \pi_1^*\mathcal{O}(-1), \qquad \begin{aligned}(\pi_2^*\mathcal{T}(-1))^\vee &\cong \pi_2^*(\mathcal{T}(-1))^\vee \\ &\cong \pi_2^*(\mathcal{T}^\vee \otimes \mathcal{O}(1)) \\ &\cong \pi_2^*\Omega(1)\end{aligned}$$

hence $\mathcal{E}^\vee = \mathcal{O}(-1) \boxtimes \Omega(1)$, proving the result for $i = 1$. To prove the result for $i > 1$, we record the following observation.

Claim For locally free coherent sheaves $\mathcal{L}, \mathcal{M}$ such that $\mathcal{L}$ has rank 1, there is a natural isomorphism

$$\textstyle\bigwedge^i(\mathcal{L} \otimes \mathcal{M}) \cong \mathcal{L}^{\otimes i} \otimes \bigwedge^i \mathcal{M}.$$

Given this, it is immediate that we have the isomorphisms

$$\begin{aligned}
\textstyle\bigwedge^i(\mathcal{E}^\vee) &= \textstyle\bigwedge^i(\pi_1^*\mathcal{O}(-1) \otimes \pi_2^*\Omega(1)) \\
&\cong (\pi_1^*\mathcal{O}(-1))^{\otimes i} \otimes \textstyle\bigwedge^i(\pi_2^*\Omega(1)) \\
&\cong \pi_1^*(\mathcal{O}(-1))^{\otimes i} \otimes \pi_2^*\left(\textstyle\bigwedge^i\Omega(1)\right) \\
&\cong \pi_1^*\mathcal{O}(-i) \otimes \pi_2^*\left(\textstyle\bigwedge^i\Omega \otimes (\mathcal{O}(1))^{\otimes i}\right) \\
&\cong \mathcal{O}(-i) \boxtimes \Omega^i(i).
\end{aligned}$$

To prove the claim, observe that there is a natural map

$$(\mathcal{M} \otimes \mathcal{L})^{\otimes i} \to \mathcal{L}^{\otimes i} \otimes \textstyle\bigwedge^i \mathcal{M}.$$

We show that this induces the required isomorphism $\bigwedge^i(\mathcal{L} \otimes \mathcal{M}) \cong \mathcal{L}^{\otimes i} \otimes \bigwedge^i \mathcal{M}$. This can be checked locally– on an affine open Spec R, say $\mathcal{L}$ and $\mathcal{M}$ are given by the free R-modules L and M respectively. Then the natural map

$$T_R(L \otimes_R M) \to \bigoplus_j (L^{\otimes j} \otimes_R \textstyle\bigwedge_R^j M)$$

on the tensor algebra sends $(\ell \otimes m) \otimes (\ell \otimes m) \mapsto 0$, thus descending to a map on ${\bigwedge^\bullet}_R(L \otimes_R M)$. This induced map preserves grading. Thus we have a map $\bigwedge^i(\mathcal{L} \otimes \mathcal{M}) \to (\mathcal{L})^{\otimes i} \otimes \bigwedge^i \mathcal{M}$. Comparing ranks of the sheaves, this must be an isomorphism. $\square$

We now show that the Koszul complex of s gives the required resolution of the diagonal.

Theorem 2.9 (Beilinson (1978)). There is an exact sequence in $\mathcal{Coh}(\mathbb{P}^n \times \mathbb{P}^n)$ of the form

$$0 \to \mathcal{O}(-n) \boxtimes \Omega^n(n) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega(1) \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

Proof. From Lemma 2.7 and Lemma 2.8, it is clear that the Koszul complex $\mathcal{K}_{\mathbb{P}^n \times \mathbb{P}^n}(s)$ has the required form. It remains to show the complex is exact. From the discussion in Remark 2.6, we can check this on the affine open sets $\mathbf{U}_\alpha \times \mathbf{V}_\alpha$ which cover $\mathbf{V}(s)$.

Using the notation of Lemma 2.7, the coordinate ring of $\mathbf{U}_0 \times \mathbf{V}_0$ can be written as $R = S[z_1, ..., z_n]$ where $z_\alpha = x_{\alpha/0} - y_{\alpha/0}$. Then the section $s|_{\mathbf{U}_0 \times \mathbf{V}_0}$ corresponds to the n-tuple $(z_1, ..., z_n)$. Hence the restriction of $\mathcal{K}_{\mathbb{P}^n \times \mathbb{P}^n}(s)$ is given by $K_R(z_1, ..., z_n)$, the Koszul complex associated to the regular sequence $(z_1, ..., z_n)$ in R. By Proposition 2.5, this is exact. □

3 The Bernstein-Gel'fand-Gel'fand correspondence

describe section

3.1 Data

3.2 Twisted functors

We now define additive functors

$$\mathbf{C}(A\text{-grMod}) \underset{F}{\overset{G}{\leftrightarrows}} \mathbf{C}(A^!\text{-grMod})$$

on which the BGG correspondence is based. In the framework of $\mathbb{Z}^2$-graded vector spaces described in Section 1.2.3, we have

$$\bigoplus_{i,j} F(\mathbf{M})^i_j \cong \mathrm{Hom}_k\left(A^!, \bigoplus_{p,q} M^p_q\right) = A^i \otimes_k \left(\bigoplus_{p,q} M^p_q\right), \qquad \bigoplus_{i,j} G(\mathbf{N})^i_j \cong A \otimes_k \left(\bigoplus_{p,q} N^p_q\right).$$

However, care is needed to define the gradings and differentials since, for example, naïvely applying the functor $\mathrm{Hom}_k(A^!, -)$ would lose all A-module structure. The key is to modify the naïve differential by adding a 'twist' as in Example 0.2.

3.2.1 Defining the functor F

We first define F on the category A-grMod, seen as the full subcategory of $\mathbf{C}(A\text{-grMod})$ whose objects are complexes concentrated in differential degree 0. If $M^0_\bullet$ is a graded A-module, we define $F(M^0_\bullet)$ to be the chain complex of $A^!$-modules given by a

$$\begin{aligned} \cdots \to A^i\langle -i\rangle \otimes_k M^0_i &\xrightarrow{\partial} A^i\langle -i-1\rangle \otimes_k M^0_{i+1} \to \cdots \\ a \otimes m &\longmapsto \sum_\alpha \xi_\alpha a \otimes x_\alpha m. \end{aligned}$$

The module $A^i\langle -i\rangle \otimes_k M^0_i$ is naturally isomorphic to $\mathrm{Hom}_k(A^!\langle i\rangle, M^0_i)$ and inherits an Adam's grading from A^i with the vector space $A^i_{j-i} \otimes_k M^0_i$ forming the jth graded piece. These shifts in grading have been chosen precisely so that the differential ∂ has degree $(1, 0)$, while the graded commutativity of $A^!$ implies $\partial \circ \partial = 0$. Thus we indeed have a chain complex of $A^!$-modules.

Given a morphism $M^0_\bullet \to M^1_\bullet$ in A-grMod, the functoriality of tensor products induces $A^!$-module homomorphisms $A^{\mathrm{i}}\langle -i\rangle \otimes_k M^0_i \to A^{\mathrm{i}}\langle -i\rangle \otimes_k M^0_i$ which are compatible with the differentials (i.e. the natural squares commute). Thus we have an additive functor $F : A\text{-grMod} \to \mathbf{C}(A^!\text{-grMod})$.

To extend F to arbitrary chain complexes $\mathbf{M} = (\bigoplus_{i,j} M^i_j, d) \in \mathbf{C}(A\text{-grMod})$, we observe that the functoriality of F gives us a (commuting) bicomplex

$$\begin{tikzcd}
\vdots & & \vdots & \vdots & \\
F(M^{i+1}_\bullet) \arrow[u] & \cdots \arrow[r] & A^{\mathrm{i}}\langle -j\rangle \otimes_k M^{i+1}_j \arrow[r] \arrow[u] & A^{\mathrm{i}}\langle -j-1\rangle \otimes_k M^{i+1}_{j+1} \arrow[r] \arrow[u] & \cdots \\
F(M^i_\bullet) \arrow[u] & \cdots \arrow[r] & A^{\mathrm{i}}\langle -j\rangle \otimes_k M^i_j \arrow[r] \arrow[u] & A^{\mathrm{i}}\langle -j-1\rangle \otimes_k M^i_{j+1} \arrow[r] \arrow[u] & \cdots \\
\vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{3}$$

where the vertical maps are $1 \otimes d$. Define $F(\mathbf{M})$ to be the total complex of this bicomplex, i.e. $F(\mathbf{M})$ is given by

$$\cdots \to \bigoplus_{p+q=i} A^{\mathrm{i}}\langle -q\rangle \otimes_k M^p_q \xrightarrow{\partial} \bigoplus_{p+q=i+1} A^{\mathrm{i}}\langle -q\rangle \otimes_k M^p_q \to \cdots, \tag{4}$$
$$\partial : a \otimes m \longmapsto a \otimes dm + (-1)^{\#m} \sum_\alpha \xi_\alpha a \otimes x_\alpha m$$

where $\#m$ is the differential degree of $m \in \mathbf{M}$. It is clear that each $F(\mathbf{M})^i_\bullet = \bigoplus_{p+q=i} A^{\mathrm{i}}\langle -q\rangle \otimes_k M^p_q$ is a graded $A^!$ module, and the signs introduced in the total complex construction ensure $\partial \circ \partial = 0$. An explicit check confirms ∂ has degree $(1, 0)$, so we indeed have an object of $\mathbf{C}(A^!\text{-grMod})$.

The twist using comodules. Observe that the differential ∂ differs from the naïve differential $1 \otimes d$ on the tensor product by the horizontal maps, which are the 'twists' we have been alluding to. These have a nice description using the fact that a graded module $N_\bullet \in A^!$-grMod has the structure of a graded A^{i}-comodule via the map

$$\begin{aligned} \Delta : \quad N_\bullet &\longrightarrow N_\bullet \otimes_k A^{\mathrm{i}} \\ n &\longmapsto \sum_{\underline{\alpha} \subseteq \{0,\dots,n\}} \xi_{\underline{\alpha}} n \otimes x_{\underline{\alpha}}. \end{aligned}$$

Applying this idea to the $A^!$-modules $A^{\mathrm{i}}\langle -i\rangle$, we get a commuting square

$$\begin{tikzcd}
\bigoplus_{u+v=j-q} A^{\mathrm{i}}_u \otimes_k A^{\mathrm{i}}_v \otimes_k M^{i-q}_q \arrow[r, "1\otimes\tau\otimes 1"] & A^{\mathrm{i}}_{j-q-1} \otimes_k A_1 \otimes_k M^{i-q}_q \arrow[d, "1\otimes\nabla"] \\
A^{\mathrm{i}}_{j-q} \otimes_k M^{i-q}_q \arrow[u, "\Delta\otimes 1"] \arrow[r, "(-1)^{i-q}(\partial - 1\otimes d)"] & A^{\mathrm{i}}_{j-q-1} \otimes_k M^{i-q}_{q+1}
\end{tikzcd}$$

where $\nabla : A \otimes_k M_\bullet^{i-q} \to M_\bullet^{i-q}$ defines the A-module structure on M, and $\tau : A^{i} \to A^{!}$ is the morphism defined in Section 1.2.2 which identifies A^{i}_1 with A_1, annihilating other graded pieces.

In summary, $F(\mathbf{M})$ as a $\mathbb{Z}^2$-graded vector space is simply $A^{i} \otimes_k \mathbf{M}$ with (i,j)th piece

$$F(\mathbf{M})^i_j = \bigoplus_{p+q=i} A^{i}_{j-q} \otimes_k M^p_q$$

and differential given on $A^{i}_{j-q} \otimes_k M^p_q$ by

$$1 \otimes d + (-1)^p (1 \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1).$$

3.2.2 The left adjoint to F

The functor $G : \mathbf{C}(A^!\text{-grMod}) \to \mathbf{C}(A\text{-grMod})$ is analogously defined, and maps the chain complex $\mathbf{N} = (\bigoplus_{i,j} N, \partial)$ to $G(\mathbf{N})$ given by

$$\cdots \to \bigoplus_{p-q=i} N^p_q \otimes_k A\langle -q\rangle \xrightarrow{d} \bigoplus_{p-q=i+1} N^p_q \otimes_k A\langle -q\rangle \to \cdots \tag{5}$$
$$d : n \otimes a \longmapsto \partial n \otimes a + (-1)^{\#n} \sum_\alpha \xi_\alpha n \otimes x_\alpha a$$

where $\#n$ is the differential degree of $n \in \mathbf{N}$. The Adam's grading on each $G(\mathbf{N})^i_\bullet$ is inherited from A, and is given by

$$G(\mathbf{N})^i_j = \bigoplus_{p-q=i} N^p_q \otimes_k A_{j-q}.$$

Recalling that every $A^!$-module is a A^{i}-comodule (see Section 3.2.1), we can use the comodule structure-map $\Delta : N^i_\bullet \to N^i_\bullet \otimes A^{i}$ to define the differential on $N^p_q \otimes_k A_{j-q}$ as

$$\partial \otimes 1 + (-1)^p (1 \otimes \nabla) \circ (1 \otimes \tau \otimes 1) \circ (\Delta \otimes 1).$$

The adjunction. Having defined the functors F and G, we show that G is left adjoint to F. Spelled out this means given $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$ and $\mathbf{N} \in \mathbf{C}(A^!\text{-grMod})$, there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathbf{C}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \cong \mathrm{Hom}_{\mathbf{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})).$$

At its heart this is a $\otimes$-Hom adjunction, as we shall illustrate in the special case of module categories below.

Lemma 3.1. Given modules $M \in A$-Mod and $N \in A^!$-Mod, there are natural isomorphisms of abelian groups

$$\mathrm{Hom}_A(A \otimes_k N, M) \cong \mathrm{Hom}_k(N, M) \cong \mathrm{Hom}_{A^!}(N, \mathrm{Hom}_k(A^!, M)).$$

Proof. Consider the $(A, A^!)$-bimodule $T = A \otimes_k A^!$. Then the standard $\otimes$-Hom adjunction for bimodules (Bourbaki 1989) gives us a natural isomorphism

$$\mathrm{Hom}_A(T \otimes_{A^!} N, M) \cong \mathrm{Hom}_{A^!}(N, \mathrm{Hom}_A(T, M)).$$

Then observe that there are natural isomorphisms

$$T \otimes_{A^!} N \cong A \otimes_k A^! \otimes_{A^!} N \cong A \otimes_k N, \qquad \mathrm{Hom}_A(T, M) \cong \mathrm{Hom}_A(A \otimes_k A^!, M) \cong \mathrm{Hom}_k(A^!, M).$$

The isomorphism with $\mathrm{Hom}_k(N, M)$ comes similarly from treating A as an (A, k)-bimodule. $\square$

We now exhibit the general adjunction for F and G, and it is here that the flexibility of interpreting a chain complex $\mathbf{M}$ of graded modules as a single $\mathbb{Z}^2$-graded module $\bigoplus_{i,j} M^i_j$ (see Section 1.2.3) really comes handy. Interpreting $\mathbf{C}(A\text{-grMod})$ as a subcategory of A-Mod (likewise for $A^!$), we use Lemma 3.1 to identify $\mathrm{Hom}_{\mathbf{C}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \subset \mathrm{Hom}_A(\mathbf{N} \otimes_k A, \mathbf{M})$ and $\mathrm{Hom}_{\mathbf{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})) \subset \mathrm{Hom}_{A^!}(\mathbf{N}, \mathrm{Hom}_k(A^!, \mathbf{M}))$ with the same subgroup of $\mathrm{Hom}_k(\mathbf{N}, \mathbf{M})$.

Theorem 3.2 (Bernstein et al. (1978))**.** The functor G, from the category of complexes of graded $A^!$-modules to the category of complexes of graded A-modules, is a left adjoint to the functor F.

Proof. Given $\bar{\varphi} \in \mathrm{Hom}_A(G(\mathbf{N}), \mathbf{M})$, the corresponding map $\varphi \in \mathrm{Hom}_k(\mathbf{N}, \mathbf{M})$ found in Lemma 3.1 is given by $\varphi(n) = \bar{\varphi}(n \otimes 1)$. Thus $\bar{\varphi}$ has degree $(0, 0)$ if and only if $\bar{\varphi}(N^i_j \otimes_k A_0) \subseteq M^{i-j}_j$, if and only if $\varphi(N^i_j) \subseteq M^{i-j}_j$. Moreover for $n \in N^i_j$, direct computation shows

$$(d_M \circ \bar{\varphi} - \bar{\varphi} \circ d_{G(\mathbf{N})})(n \otimes 1) \;=\; (d_M \circ \varphi - \varphi \circ \partial_N)(n) - (-1)^i \sum_\alpha x_\alpha \varphi(\xi_\alpha n),$$

thus $\bar{\varphi}$ is a morphism in $\mathbf{C}(A\text{-grMod})$ if and only if

$$\varphi(N^i_j) \subseteq M^{i-j}_j, \quad \text{and} \quad d_M \circ \varphi - \varphi \circ \partial_N = \sum_\alpha x_\alpha \varphi \xi_\alpha \tag{6}$$

where we write $\sum_\alpha x_\alpha \varphi \xi_\alpha$ for the map that takes $n \in N^i_j$ to $(-1)^i \sum_\alpha x_\alpha \varphi(\xi_\alpha n)$.

Likewise given $\varphi^! \in \mathrm{Hom}_{A^!}(\mathbf{N}, F(\mathbf{M}))$, repeating the above argument shows $\varphi^!$ is an element of $\mathrm{Hom}_{\mathbf{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M})$ if and only if the corresponding map $\varphi \in \mathrm{Hom}_k(\mathbf{N}, \mathbf{M})$ satisfies (6). This shows that the isomorphisms given in Lemma 3.1 restrict to isomorphisms

$$\mathrm{Hom}_{\mathbf{C}(A\text{-grMod})}(G(\mathbf{N}), \mathbf{M}) \cong \{\varphi \in \mathrm{Hom}_k(\mathbf{N}, \mathbf{M}) \text{ satisfying (6)}\} \cong \mathrm{Hom}_{\mathbf{C}(A^!\text{-grMod})}(\mathbf{N}, F(\mathbf{M}))$$

thereby showing G is left adjoint to F. □

3.3 Koszul resolutions

Given a complex $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$, the adjunction $F \vdash G$ takes the identity morphism

$$1_{F(\mathbf{M})} \in \mathrm{Hom}_{\mathbf{C}(A^!\text{-grMod})}(F(\mathbf{M}), F(\mathbf{M}))$$

to a map

$$\varepsilon_{\mathbf{M}} \in \mathrm{Hom}_{\mathbf{C}(A\text{-grMod})}(G(F(\mathbf{M})), \mathbf{M}).$$

The natural transformation $\varepsilon : G \circ F \to \mathbf{1}$ thus obtained is called the *counit* of the adjunction, and we say the morphism $\varepsilon_{\mathbf{M}}$ is the *component* of the transformation at $\mathbf{M}$. Likewise, there is the dual notion called the *unit* of the adjunction, which is a natural transformation $\eta : \mathbf{1} \to F \circ G$ giving, for any $\mathbf{N} \in \mathbf{C}(A^!\text{-grMod})$, a morphism $\eta_{\mathbf{N}} : \mathbf{N} \to F(G(\mathbf{N}))$.

Begin with the following observation.

Proposition 3.3. The functor F maps elements of $\mathbf{C}(A\text{-grMod})$ to complexes of injective $A^!$-modules, and the functor G maps elements of $\mathbf{C}(A^!\text{-grMod})$ to complexes of free A-modules.

Proof. The statement for G is immediate from definition, so we prove that for any $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$, the modules $F(\mathbf{M})^i_\bullet$ are injective over $A^!$.

Recall from Weibel (2003) (Proposition 2.3.10) that if $R : \mathscr{B} \to \mathscr{A}$ is an additive functor which is right adjoint to an exact functor $L : \mathscr{A} \to \mathscr{B}$, then for any injective object $I \in \mathscr{B}$ the object $R(I) \in \mathscr{A}$ is injective. Applying this to the pair of adjoint functors

$$R = \mathrm{Hom}_k(A^!, -) : k\text{-grMod} \to A^!\text{-grMod}, \quad L = (- \otimes_k A^!) : A^!\text{-grMod} \to k\text{-grMod}$$

and observing that L is exact since all k-vector spaces are flat over k, see that R preserves injectives. But every k-vector space is also injective, so the $A^!$-modules $A^i\langle -q\rangle \otimes_k M^p_q \cong R(M^p_q)$ appearing in (4) are all injective. To conclude, observe that the k-algebra $A^!$ is finite dimensional hence noetherian. By the theorem of Bass & Papp (see Lam (1999), Theorem 3.46) which asserts that a ring is (left) noetherian if and only if any direct sum of injective modules over it is injective, we are done. $\square$

We show that the component $\varepsilon_{\mathbf{M}} : G(F(\mathbf{M})) \to \mathbf{M}$ is, in fact, a free resolution of the complex $\mathbf{M}$ and dually, the component $\eta_{\mathbf{N}}$ is an injective resolution of $\mathbf{N}$. A special but important case of this phenomenon is when the complex is

$$\cdots \to 0 \longrightarrow k \longrightarrow 0 \to \cdots,$$

and this will be central in proving the result for general complexes.

3.3.1 The Koszul complex

The 1-dimensional vector space k can be considered a graded A-module concentrated degree 0, such that all $x_i \in A$ annihilate k. Then $F(k)$ is the complex $0 \to \Lambda^\bullet(X) \to 0$ concentrated in differential degree 0. We compute the complex $G(F(k))$ to be

$$0 \to A^i_{n+1} \otimes_k A\langle -n-1\rangle \to A^i_n \otimes_k A\langle -n\rangle \to \dots \to A^i_1 \otimes_k A\langle -1\rangle \to A^i_0 \otimes_k A \to 0 \tag{7}$$
$$(x_{\alpha_1} \wedge \dots \wedge x_{\alpha_i}) \otimes 1 \longmapsto \sum_j (-1)^{i+j-1}(x_{\alpha_1} \wedge \dots \hat{x}_{\alpha_j} \dots \wedge x_{\alpha_i}) \otimes x_{\alpha_j}$$

where $\hat{\cdot}$ denotes omission of a term.

Observing the term in differential degree i is isomorphic to $\Lambda^i_A(A^{n+1})$, we can recognise the above complex as the *Koszul complex* associated to the regular sequence $(x_0, \dots, x_n) \in A$. Then standard results on Koszul complexes found in Eisenbud (1995) (Chapter 17 and relevant sections of Appendix 2) show that this complex has cohomology 0 everywhere except in degree 0, where the cohomology is $A/(x_1, \dots, x_n) \cong k$. We provide below a direct proof of the result in the special case when $\mathrm{char}(k) = 0$.

A similar argument shows that the complex $F(G(k))$ is exact everywhere except in degree 0, where it has cohomology k. Thus we have resolutions

$$G(F(k)) \to k \to 0, \qquad 0 \to k \to F(G(k))$$

of k by free A-modules and by injective $A^!$-modules, respectively. It is not hard to see that these maps are precisely the ones given by the counit and the unit of adjunction.

3.3.2 Resolutions in general

We show that the counit (resp. unit) gives a free (resp. injective) resolution, by first showing this is the case for graded modules (i.e. complexes concentrated in a single degree). When $\mathbf{M}$ is a graded A-module, we will show that the complex $G(F(\mathbf{M}))$ is 'built up' from the tensor product of $\mathbf{M}$ and the Koszul complex of k.

Lemma 3.4 (Eisenbud et al. (2003))**.** If $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$ is a chain complex concentrated in differential degree 0, then the natural map

$$\varepsilon_{\mathbf{M}} : G(F(\mathbf{M})) \to \mathbf{M}$$

is an epimorphism and induces an isomorphism on cohomology. Likewise, if $\mathbf{N} \in \mathbf{C}(A^!\text{-grMod})$ is a chain complex concentrated in differential degree 0, then the natural map

$$\eta_{\mathbf{N}} : \mathbf{N} \to F(G(\mathbf{N}))$$

is a monomorphism and induces an isomorphism on cohomology.

Proof. We first show that the complex $G(F(\mathbf{M}))$ has the same cohomology as the complex $\mathbf{M}$. Direct computation shows $G(F(\mathbf{M}))$ is given by

$$\begin{gathered} \cdots \to \bigoplus_p A^{!}_{-i} \otimes_k M^0_p \otimes_k A\langle i-p\rangle \longrightarrow \bigoplus_p A^{!}_{-i-1} \otimes_k M^0_p \otimes_k A\langle i+1-p\rangle \to \cdots, \\ a \otimes m \otimes b \longmapsto \sum_\alpha \xi_\alpha a \otimes x_\alpha m \otimes b \;+\; (-1)^{\deg m} \sum_\alpha \xi_\alpha a \otimes m \otimes x_\alpha b \end{gathered}$$

so the strand of this in Adam's degree r is seen to be the total complex of the (commuting) bicomplex

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & A^{!}_{-i} \otimes_k M^0_p \otimes_k A_{i-p+r} \arrow[r] \arrow[d] & A^{!}_{-i-1} \otimes_k M^0_p \otimes_k A_{i-p+r+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & A^{!}_{-i-1} \otimes_k M^0_{p+1} \otimes_k A_{i-p+r} \arrow[r] \arrow[d] & A^{!}_{-i-2} \otimes_k M^0_{p+1} \otimes_k A_{i-p+r+1} \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd} \quad . \tag{8}$$

Here pth row is obtained by applying $(- \otimes_k M^0_p)$ to the complex $G(F(k))_{r-p}$, so is exact from Proposition 2.5 unless $p = r$. Moreover, the rth row is

$$\cdots 0 \to M^0_r \to 0 \to \cdots .$$

Thus first page of the spectral sequence (starting with horizontal cohomology) of (8) is

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & M_r^0 \arrow[d] & 0 \arrow[d] & \cdots \\
\cdots & 0 \arrow[d] & 0 \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}.$$

By the theorem on spectral sequences, we conclude that the total complex $G(F(\mathbf{M}))_r$ has cohomology insert theorem

$$H^k(G(F(\mathbf{M}))_r) = \begin{cases} M_r^0, & k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Now it suffices to show that the map $\varepsilon_{\mathbf{M}} : G(F(\mathbf{M}))_r^0 \to M_r^0$ is the cokernel of $G(F(\mathbf{M}))_r^{-1} \to G(F(\mathbf{M}))_r^0$. But this is immediate because the sequence

$$\begin{array}{ccccccc}
\bigoplus_p X \otimes_k M_p^0 \otimes_k \mathrm{Sym}^{r-p-1}(X) & \longrightarrow & \bigoplus_p M_p^0 \otimes_k \mathrm{Sym}^{r-p}(X) & \longrightarrow & M_r^0 & \longrightarrow & 0 \\
x_\alpha \otimes m \otimes a & \longmapsto & m \otimes x_\alpha a + (-1)^{\deg m} x_\alpha m \otimes a & & & & \\
 & & m \otimes a & \longmapsto & (-1)^{\deg m} am & &
\end{array}$$

is (split) exact.

The analogous statement about graded $A^!$-modules follows from a similar calculation. □

The argument to extend this result to all chain complexes is purely formal.

Theorem 3.5 (Eisenbud et al. (2003))**.** For any complex $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$, the complex $G(F(\mathbf{M}))$ is a free resolution of $\mathbf{M}$ which surjects onto $\mathbf{M}$, and for any complex $\mathbf{N} \in \mathbf{C}(A^!\text{-grMod})$, the complex $F(G(\mathbf{N}))$ is an injective resolution of $\mathbf{N}$ which $\mathbf{N}$ injects into.

Proof. Given $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$, the surjectivity of $\varepsilon_{\mathbf{M}} : G(F(\mathbf{M})) \to \mathbf{M}$ can be checked on the level of underlying $\mathbb{Z}^2$-graded modules. The map is given on the (i,j)th component by

$$\bigoplus_{p,q} \mathrm{Hom}_k(A^!_{q-i}, M^q_{p-q}) \otimes_k A_{j-p+i} \longrightarrow M^i_j$$

$$f \otimes a \mapsto af(1)$$

hence any $m \in M^i_j$ can be written $\varepsilon_{\mathbf{M}}(f_m \otimes 1)$ where $f_m : A^!_0 \to M^i_j$ is the function $f_m(1) = m$. check sign

To prove that the induced map on cohomology is an isomorphism we first reduce the problem to bounded complexes using formal properties of the functors, and then induct on the length of the complex to reduce our problem to Lemma 3.4. The key properties we use are naturality of ε, and the fact that G, F and cohomology functors all preserve direct limits. prove

Any complex $\mathbf{M} \in \mathbf{C}(A\text{-grMod})$ can be written as the direct limit of bounded complexes $(\mathbf{M}^b)_{b \in B}$, giving us commuting diagrams

$$\begin{tikzcd} G(F(\mathbf{M}^b)) \arrow[r] \arrow[d, "\varepsilon_{\mathbf{M}^b}"] & G(F(\mathbf{M})) \arrow[r, equal] \arrow[d, "\varepsilon_{\mathbf{M}}"] & \overset{\lim}{\to} G(F(\mathbf{M}^b)) \\ \mathbf{M}^b \arrow[r] & \mathbf{M} \arrow[r, equal] & \overset{\lim}{\to} \mathbf{M}^b \end{tikzcd} . \tag{9}$$

Then applying the ith cohomology functor H^i to (9) then shows that the map $H^i(\varepsilon_{\mathbf{M}})$ is the limit of the maps $\varepsilon_{\mathbf{M}^b}$, so to show $H^i(\varepsilon_{\mathbf{M}})$ is an isomorphism it suffices to show all $H^i(\varepsilon_{\mathbf{M}^b})$ are. Thus without loss o generality the complex $\mathbf{M}$ is bounded. Since F and G respect translation in differential degree, say $\mathbf{M}$ has form

$$0 \to M^0_\bullet \to \ldots \to M^d_\bullet \to 0. \tag{10}$$

Let $\mathbf{M}^d$ be the chain complex with $M^d_\bullet$ in degree d, and 0 elsewhere. We have a short exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow \mathbf{M} \xrightarrow{\varphi} \mathbf{M}^d \longrightarrow 0$$

where φ is the obvious map. The complex $\ker(\varphi)$ is concentrated in degrees $0,\ldots, d-1$. Applying the exact functor H^i to the diagram formed by the naturality squares of ε on (10) gives us a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^i(G(F(\ker\varphi))) \arrow[r] \arrow[d, "H^i(\varepsilon_{\ker\varphi})"] & H^i(G(F(\mathbf{M}))) \arrow[r] \arrow[d, "H^i(\varepsilon_{\mathbf{M}})"] & H^i(G(F(\mathbf{M}^d))) \arrow[r] \arrow[d, "(H^i\varepsilon_{\mathbf{M}^d})"] & 0 \\ 0 \arrow[r] & H^i(\ker(\varphi)) \arrow[r] & H^i(\mathbf{M}) \arrow[r] & H^i(\mathbf{M}^d) \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact. By Lemma 3.4, $H^i(\varepsilon_{\mathbf{M}^d})$ is an isomorphism. By the Five lemma, $H^i(\varepsilon_{\mathbf{M}})$ is an isomorphism if and only if $H^i(\varepsilon_{\ker(\varphi)})$ is. Since $\ker(\varphi)$ is a strictly shorter complex than $\mathbf{M}$, we are done.

The analogous statement for $F \circ G$ follows from a similar calculation. $\square$

Thus we have a formulaic (albeit inefficient– the free A-module A is resolved to an n-term free complex) method to compute resolutions of complexes.

Syzygies and regularity of modules. We use the resolutions produced in Theorem 3.5 to prove a classical result of commutative algebra– Hilbert's syzygy theorem, and provide a way to compute the Castelnuovo-Mumford regularity of modules. We briefly discuss the notions involved, and refer to Eisenbud (1995) for details.

Writing a graded A-module M in terms of generators and relations produces a short exact sequence

$$0 \to S \to F \to M \to 0,$$

where F is a free module. The module S is unique up to direct sum with a free module (i.e. if $0 \to S' \to F' \to M \to 0$ is another such resolution then there are free modules L and L' such that $L \oplus S \cong L' \oplus S'$), and is called the *first syzygy* of M. Continuing the process, we can write S in terms of generators and relations and define the second syzygy of M to be the first syzygy of S.

Thus the jth syzygy of M is the module S_j (up to direct sum with a free module) such that there is an exact sequence

$$0 \to S_j \to F_{j-1} \to \dots \to F_0 \to M \to 0$$

where $F_0, \dots, F_{j-1}$ are free modules. Note that if the jth syzygy of M is free then M has a free resolution of length $j+1$– thus syzygies form a measure of the 'complexity' of M. This is made precise using the notion of *projective dimension*, defined as

$$\mathrm{pd}(M) = \min\{j \mid \text{the jth syzygy module of } M \text{ is free or projective}\}.$$

Hilbert showed that the projective dimension of A-modules is bounded. The resolution produced using Theorem 3.5 provides a immediate constructive proof of this result.

Corollary 3.6 (Hilbert Syzygy Theorem)**.** If M is a graded module over $k[x_0, \dots, x_n]$, then the $n+1$st syzygy module of M is free.

In fact, this bound is strict– for instance, the A-module k has projective dimension $n+1$. To see this, observe that (7) allows us to compute $\mathrm{Ext}_A^{n+1}(k,k) \cong k$ but by Lemma 4.1.6 of Weibel (2003), an A-module M has a projective resolution of length $\leq d$ if and only if $\mathrm{Ext}_A^d(M,N) = 0$ for all A-modules N. Defining the *graded global dimension* of a graded ring R to be

$$\mathrm{gr.gl.dim}(R) = \sup\{\mathrm{pd}(M) \mid M \in R\text{-grMod}\},$$

we have thus shown that $\mathrm{gr.gl.dim}(k[x_0, \dots, x_n]) = n+1$.

The notion of *Castelnuovo-Mumford regularity* builds upon this, putting a bound on the degrees of generators and relations of a finitely generated graded A-module M. We say M is m-regular if the jth syzygy of m is generated in degrees $\leq m+j$. We state a homological characterisation of regularity, referring to Eisenbud & Goto (1984) for a proof.

Theorem 3.7 (Eisenbud & Goto (1984))**.** For a finitely generated graded A-module M, the following conditions are equivalent.

1. M is m-regular.
2. $M_{\geq m} = \bigoplus_{i \geq m} M_i$ is generated by M_m and has a *linear free resolution* (a free resolution in which the differentials are represented by matrices whose entries have degree ≤ 1.)
3. $M_{\geq m}$ is generated by M_m and $\mathrm{Tor}_A(M,k)_j^{j-i} = 0$ for all j and all $i > m$.

Using the Koszul complex (7), we extend the result above to the following.

Corollary 3.8 (Eisenbud et al. (2003))**.** A finitely generated graded A-module M is m-regular if and only if $M_{\geq m}$ is generated by M_m and the complex $F(M)$ is exact at degrees $> m$.

Proof. It suffices to show that the complex $F(M)$ has cohomology $H^i(F(M))_j = \mathrm{Tor}_A(M,k)_j^{j-i}$. To see this, note that the Koszul complex $G(F(k))$ given by (7) is a free resolution of k. Then the complex $M \otimes_A G(F(k))$ is given in differential degree $i-j$ by

$$M \otimes_A A^{!}_{j-i} \otimes_k A\langle i-j \rangle \cong A^{!}_{j-i} \otimes_k M\langle i-j \rangle.$$

The component in Adam's degree j is $A^{!}_{j-i} \otimes_k M_i$, which occurs as the degree (i,j) component of $F(M)$. Moreover, the differentials in both complexes coincide, hence we are done. □

3.4 Descending to triangulated categories

Theorem 3.5 shows that the functors F and G preserve cohomology, so it is reasonable to ask whether they descend to the (triangulated) homotopy and derived categories which are the natural setting for formulating statements about cohomology. We show that the answer is positive for homotopy categories.

Lemma 3.9. The functors F and G take cones to cones– in other words, if $f : \mathbf{M} \to \mathbf{N}$ is a morphism in $\mathbf{C}(A\text{-grMod})$ then $F(\operatorname{cone} f) = \operatorname{cone} F(f)$, and likewise for G.

Proof. An easy explicit check from definition of F and G, ommited for brevity. □

Theorem 3.10 ((Eisenbud et al. 2003))**.** The functors F and G descend to adjoint functors

$$\mathbf{K}(A\text{-grMod}) \underset{\bar{F}}{\overset{\bar{G}}{\leftrightarrows}} \mathbf{K}(A^!\text{-grMod})$$

between the triangulated homotopy categories of chain complexes.

Proof. In a category of chain complexes, a morphism $f : M \to N$ is nullhomotopic if and only the inclusion $0 \to N \to \operatorname{cone} f$ is split. Now we know F and G take cones to cones, and being additive functors they take split morphisms to split morphisms. Thus F and G factor through the homotopy categories.

To see that the induced functors are functors between triangulated categories, we show that F and G are exact functors hence preserve distinguished triangles. But this can be checked at the level of $\mathbb{Z}^2$-graded modules, and is immediate since the functors F and G are both given by tensor product with some k-vector space.

The adjunction between $\bar{F}$ and $\bar{G}$ follows immediately from the adjunction in Theorem 3.2 which identifies subgroups of nullhomotopic morphisms. □

It is clear that this is not an equivalence, the natural maps $G(F(\mathbf{M})) \to \mathbf{M}$ and $\mathbf{N} \to F(G(\mathbf{N}))$ are not invertible in the homotopy categories. Theorem 3.5 however does show that they are quasi-isomorphisms, so one might expect F and G to induce isomorphisms of derived categories. There are multiple examples throughout the literature which show this fails– for instance (Keller 2003) shows that the complex $A \in \mathbf{D}(A\text{-grMod})$ is a compact object (i.e. the functor $\operatorname{Hom}_{\mathbf{D}(A\text{-grMod})}(A, -)$ commutes with infinite direct sums) but the object $F(A) \cong k\langle n+1\rangle[-n-1] \in \mathbf{D}(A^!\text{-grMod})$ is not compact. Below we exhibit explicitly the failure of our functors to descend to the derived category.

Example 3.11 (G does not preserve quasi-isomorphisms)**.** Let $n = 0$, so that $A = k[x]$ and $A^! = k[\xi]/(\xi^2)$. Consider the exact complex of graded $A^!$-modules

$$\cdots \to A^!\langle 2\rangle \xrightarrow{\xi} A^!\langle 1\rangle \xrightarrow{\xi} A^! \xrightarrow{\xi} A^!\langle -1\rangle \xrightarrow{\xi} A^!\langle -2\rangle \to \cdots$$

which is isomorphic to the zero complex in $\mathbf{D}(A^!\text{-grMod})$. The functor G maps this to

$$\cdots \to 0 \to \bigoplus_q A\langle -q\rangle \xrightarrow{1+x} \bigoplus_q A\langle -q\rangle \to 0 \to \cdots,$$

which is not acyclic (the only non-zero differential is not surjective), hence non-zero in $\mathbf{D}(A\text{-grMod})$.

Bernstein-Gel'fand-Gel'fand equivalence To work around this apparent problem, Bernstein et al. (1978) restricts to bounded complexes so that a simple spectral sequence argument shows F and G preserve acyclicity. This gives well-defined functors between the bounded derived categories which form an adjoint equivalence– the so-called 'BGG correspondence'.

Lemma 3.12. If $\mathbf{M}$ is a bounded acyclic complex of finitely generated A-modules, then the complex $F(\mathbf{M})$ is acyclic. Likewise, if $\mathbf{N}$ is a bounded acyclic complex of finitely generated $A^!$-modules, then the complex $G(\mathbf{M})$ is acyclic.

Proof. Given such an $\mathbf{M}$, the double complex (3) has exact columns. Then the first page of the spectral sequence (starting with vertical cohomology) vanishes everywhere. Since $M^p_\bullet = 0$ for large p, the double complex is bounded and the convergence theorem holds, indicating the total complex is acyclic.

The argument for G is similar. $\square$

Then by Example 10.5.5 in (Weibel 2003), F and G descend to functors between derived categories prove

$$F_\mathbf{D} : \mathbf{D}^b(A\text{-grMod}) \to \mathbf{D}(A^!\text{-grMod}), \qquad G_\mathbf{D} : \mathbf{D}^b(A^!\text{-grMod}) \to \mathbf{D}(A\text{-grMod}).$$

To conclude, we show that $F_\mathbf{D}$ in fact has image $\mathbf{D}^b(A^!\text{-grMod})$ and likewise for $G_\mathbf{D}$.

Lemma 3.13. If $\mathbf{M}$ is a bounded complex of finitely generated A-modules, then the complex $F(\mathbf{M})$ has bounded cohomology and is quasi-isomorphic to a bounded complex of finitely generated $A^!$-modules.

Likewise, if $\mathbf{N}$ is a bounded complex of finitely generated $A^!$-modules, then the complex $G(\mathbf{N})$ has bounded cohomology and is quasi-isomorphic to a bounded complex of finitely generated $A^!$-modules.

Proof. If $\mathbf{N}$ is as given, then the complex $G(\mathbf{N})$ is bounded by definition– for any p, we have that the module $N^p_\bullet$ is finitely generated hence has only finitely many graded components. Then for sufficiently large i, we have $N^p_{p-i} = 0$ for all p.

For $\mathbf{M}$ as given the double complex (3) which computes M is bounded, and by Corollary 3.8 the first page of the corresponding spectral sequence (starting with horizontal cohomology) has finite support. Thus by the convergence theorem for spectral sequences, the cohomology of $F(\mathbf{M})$ is bounded. The existence of the quasi-isomorphic bounded complex of finitely generated modules follows from Hartshorne (2008), III Lemma 12.3. $\square$

Theorem 3.14 (Bernstein et al. (1978))**.** The functors F and G induce an equivalence of derived categories

$$\mathbf{D}^b(A\text{-grMod}) \underset{F_\mathbf{D}}{\overset{G_\mathbf{D}}{\leftrightarrows}} \mathbf{D}^b(A^!\text{-grMod}).$$

Proof. From Lemma 3.13, the functors given are well-defined. Then Theorem 3.5 shows that $F_\mathbf{D} \circ G_\mathbf{D}$ and $G_\mathbf{D} \circ F_\mathbf{D}$ are naturally equivalent to the identity morphism, hence we have an equivalence of categories. $\square$

3.5 The Tate resolution and Beilinson monads

4 Koszul duality after Keller (2003)

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