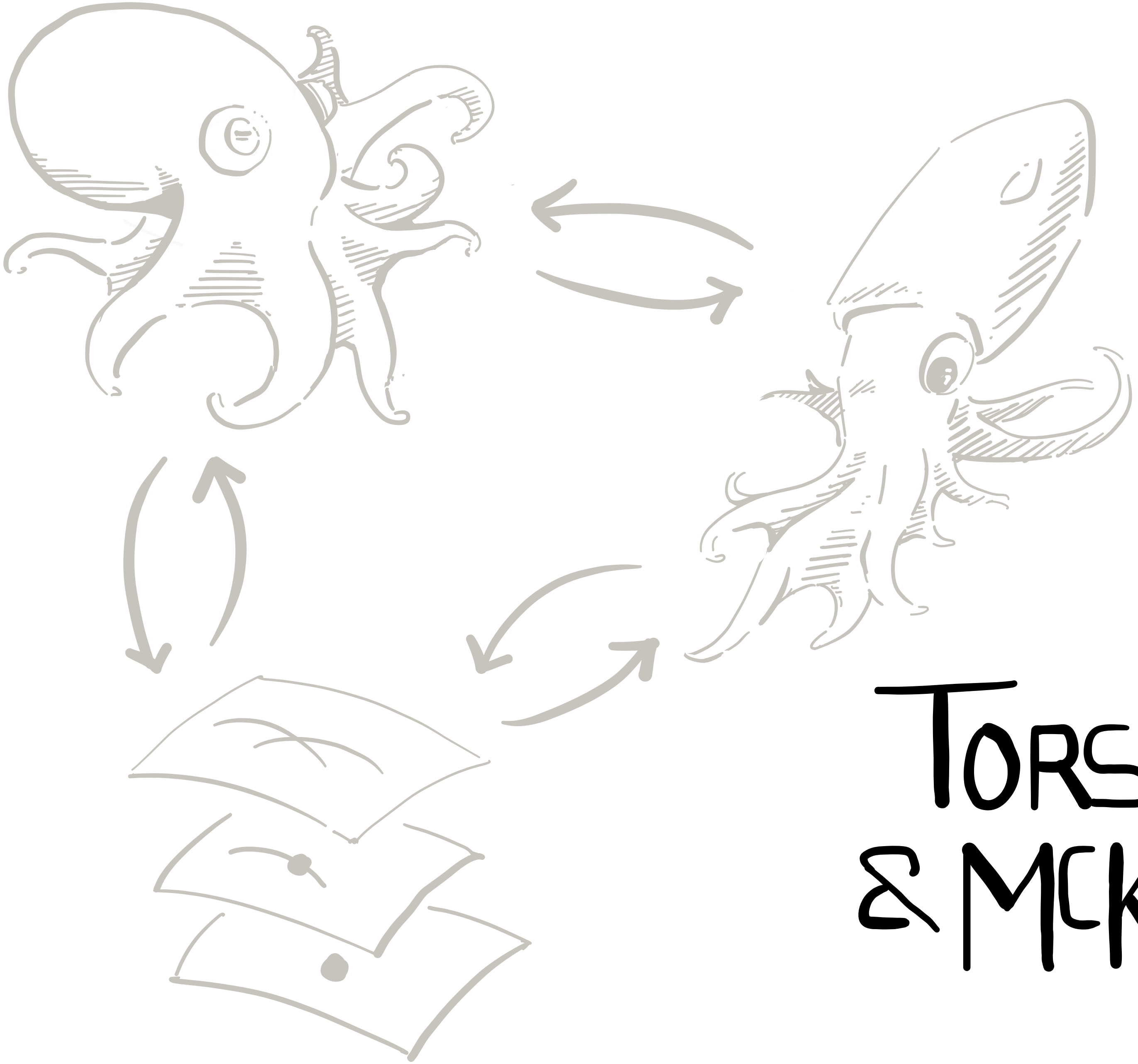

Torsion Pairs & McKay Quivers

Parth Shimpi
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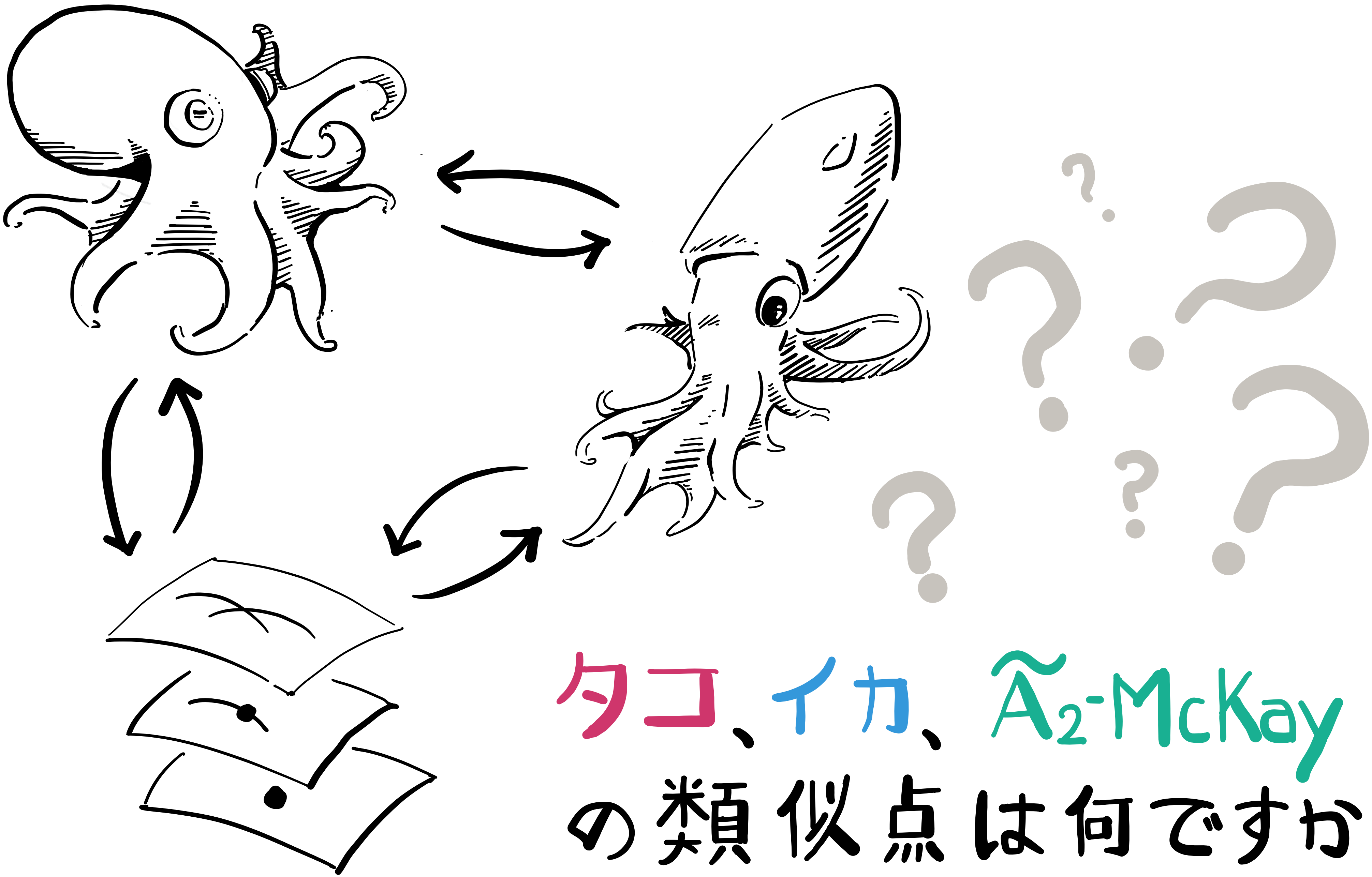

タコ、イカ、$\widetilde{A}_2$-McKayの類似点は何ですか

Let Λ be a (contracted) affine preprojective algebra.

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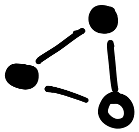
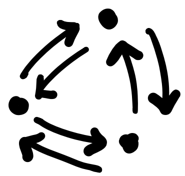
$\Delta =$  (ADE)

extend

$\underline{\Delta} =$ 

double

$Q =$ 

path algebra

$$\mathbb{C}Q = \mathbb{C}\langle \text{paths in } Q \rangle$$

add relations

$$\Lambda = \mathbb{C}Q \big/ \left([a, a^*] \mid a \in Q_1 \right)$$

Let Λ be a (contracted) affine preprojective algebra.

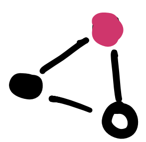
$$\Lambda = \Big(\sum_{j\notin J} e_j\Big)\cdot\Lambda'\cdot\Big(\sum_{j\notin J} e_j\Big)$$

contract

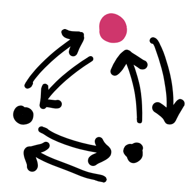
$$\Lambda' = \mathbb{C}Q \Big/ \big([a,a^*] \mid a\in Q_1 \big)$$

$J \subset \Delta =$  (ADE)

extend

$\underline{\Delta} =$ 

double

$Q =$ 

path algebra

$$\mathbb{C}Q = \mathbb{C}\langle \text{paths in } Q \rangle$$

add relations

Let Λ be a (contracted) affine preprojective algebra.

! Many have studied this

Derived autoequivalences. [Crawley-Boevey '00]
[Ishii–Uehara '05]
[Hirano–Wemyss '23]

Tilting theory. [Buan-Iyama-Reiten-Scott '09]
[Amiot–Iyama–Reiten '15]

Stability. [Bridgeland '09]
[Ishii–Ueda–Uehara '10]
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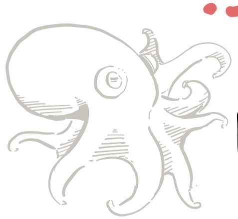
... so I also tried.



Hearts of t-structures. [S. '25]

Let Λ be a (contracted) affine preprojective algebra.

Question. Is it possible to classify hearts in $D^b\operatorname{mod}\Lambda$?

If $\mathcal{T}$ is a triangulated category, we say $\mathcal{K} \subseteq \mathcal{T}$ is a (bounded) heart if $\quad \operatorname{Hom}(\mathcal{K}, \mathcal{K}[<0]) = 0$ and $\mathcal{T} = \langle \mathcal{K}[n] \mid n \in \mathbb{Z} \rangle$

Hearts of t-structures. [S. '25]

Let Λ be a (contracted) affine preprojective algebra.

Question. Is it possible to classify hearts in $D^b \operatorname{mod} \Lambda$?

Theorem [S]. If $\mathcal{K}$ is a bounded heart in $D^b \operatorname{mod} \Lambda$ and $\mathcal{K} \subseteq \operatorname{fl} \operatorname{mod} \Lambda [-1,0]$ then $\mathcal{K}$ is a member of a short, explicit list.

Let Λ be a (contracted) affine preprojective algebra.

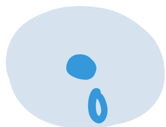
Question. Is it possible to classify hearts in $D^0\mathrm{mod}\,\Lambda$?

Theorem [S]. If K is a bounded heart in $D^0\mathrm{mod}\,\Lambda$ and $K \subseteq \mathrm{fl}\,\mathrm{mod}\,\Lambda[-1,0]$ then K is a member of a short, explicit list.

Full subcategory of objects with finite-length cohomology.

Equivalently, viewing Λ as a sheaf of nc-algebras over $Z = \mathrm{Spec}(Z\Lambda)$, we have

$$D^0\Lambda := D^0\mathrm{mod}\,\Lambda = \{ M \in \mathrm{Coh}(Z,\Lambda) \mid \mathrm{Supp}\, M \subseteq \{0\} \text{ in } Z \}.$$



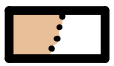
$Z = \mathrm{Spec}\, \dfrac{\mathbb{C}[x,y]}{(xy=z^3)}$

Let Λ be a (contracted) affine preprojective algebra.

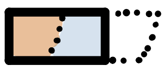
Question. Is it possible to classify hearts in $D^b \mathrm{mod}\, \Lambda$?

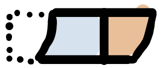
Theorem [S]. If K is a bounded heart in $D^b \mathrm{mod}\, \Lambda$ and $K \subseteq \mathrm{fl}\,\mathrm{mod}\, \Lambda[-1,0]$ then K is a member of a short, explicit list.

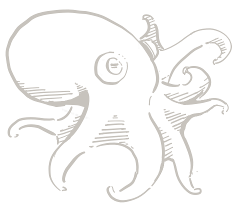
K is a Happel-Reiten-Smalø tilt of $H = \mathrm{fl}\,\mathrm{mod}\, \Lambda$

$\therefore$  $T = H \cap K[1]$ is a torsion class in H

 $F = H \cap K$ is a torsion-free class in H

 $H = T * F$

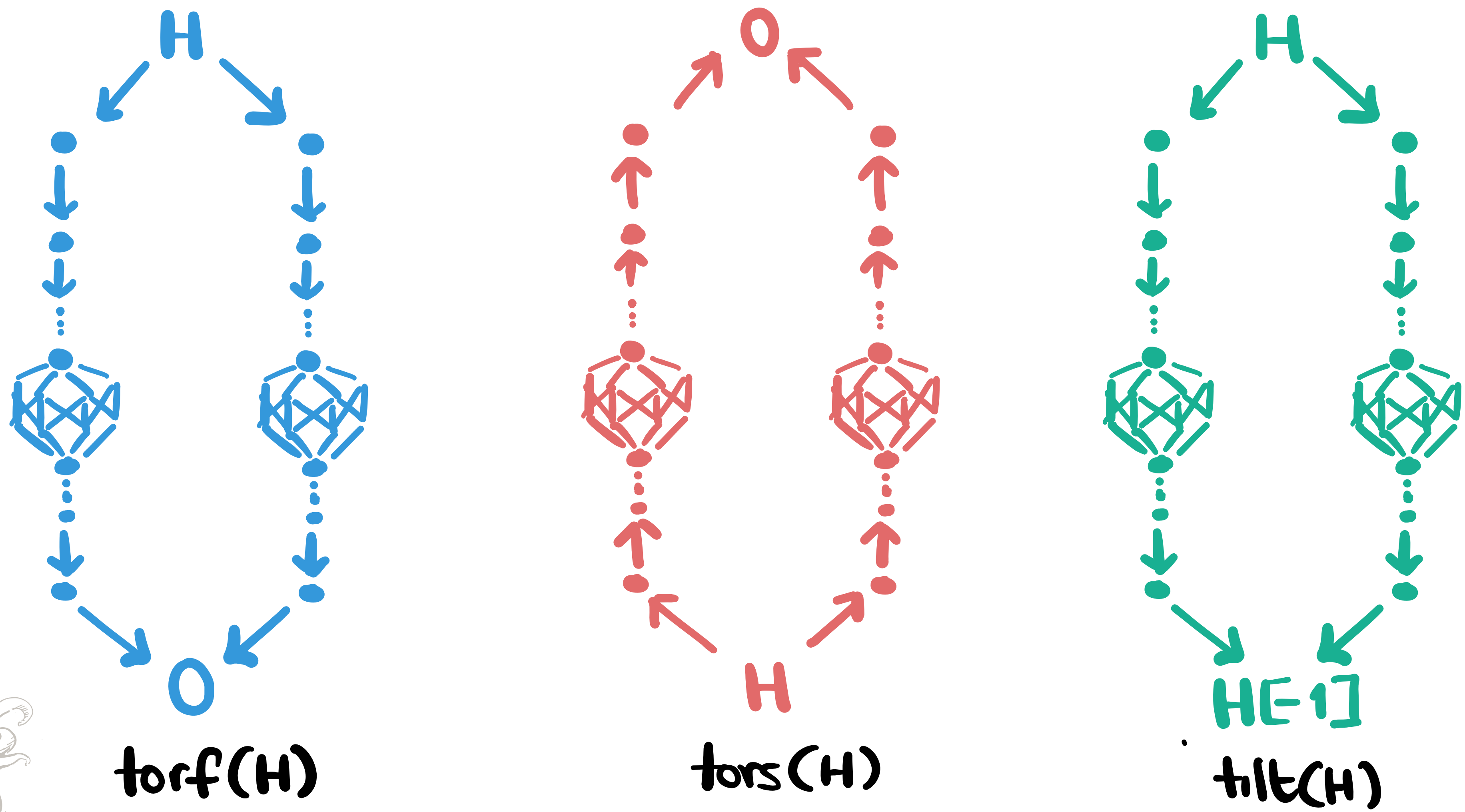
 $K = F * T[-1]$



 $Z = \mathrm{Spec}\, \dfrac{\mathbb{C}[x,y]}{(xy = z^3)}$

Further, $\mathrm{torf}(H) \cong \mathrm{tors}(H)^{op} \cong \mathrm{tilt}(H)$ is a complete lattice

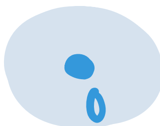
poset that admits arbitrary
suprema (least upper bounds)
and infima (gr. lower bounds)



What hearts are already known?

① $\mathrm{coh}(Z,\Lambda) \cong \mathrm{fl}\ \mathrm{mod}\,\Lambda$ ... "natural" heart

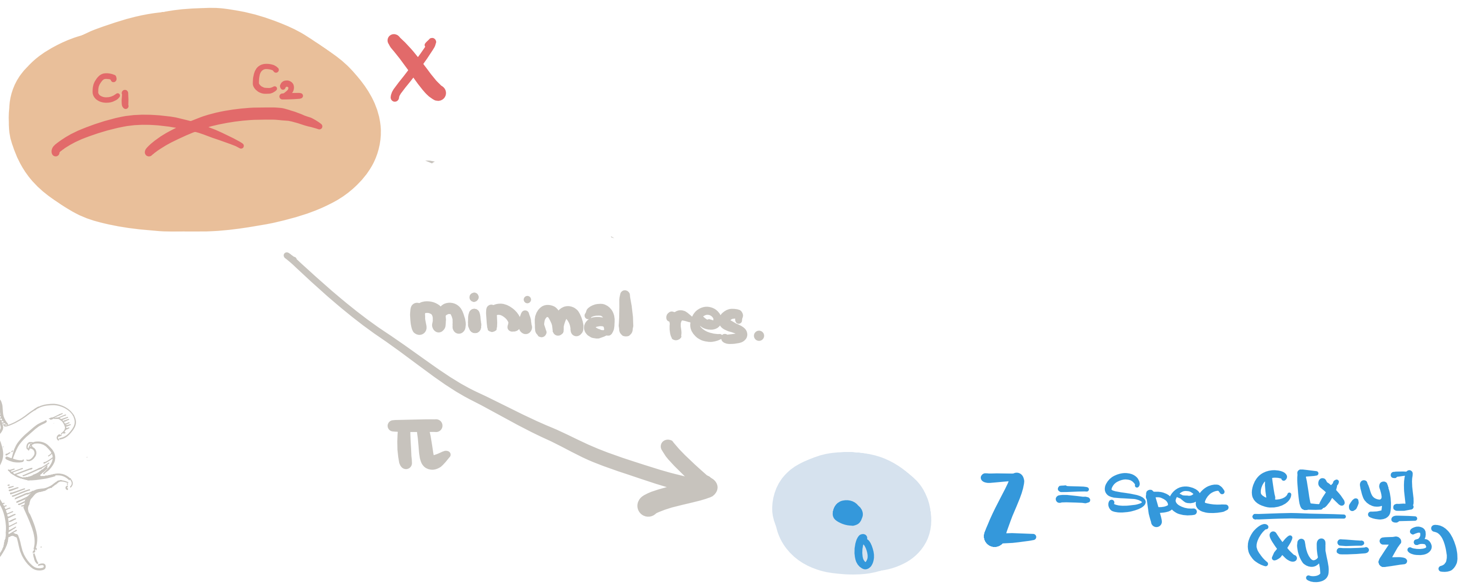
$$= \{M \in \mathrm{Coh}\,\Lambda \mid \mathrm{Supp}\, M \subseteq \{0\}\}$$



$Z = \mathrm{Spec}\,\dfrac{\mathbb{C}[x,y]}{(xy = z^3)}$

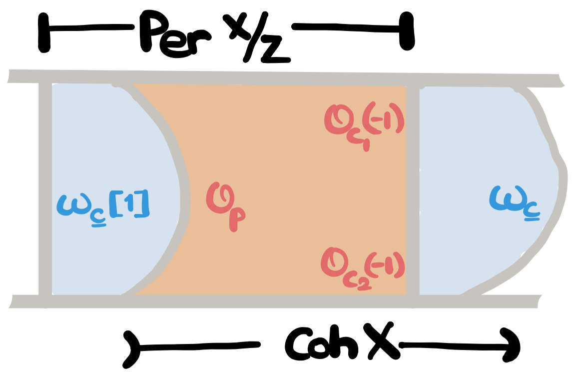
Theorem [Kapranov-Vasserot, Van den Bergh]. There is a derived equivalence between $(X, \mathcal{O}_X)$ and (Z, Λ) that identifies $\operatorname{mod}\Lambda$ with the category of π-perverse sheaves on X.

$$\begin{tikzcd}
D^b\operatorname{mod}\Lambda \arrow[r, "\sim"] & D^b\operatorname{Coh}X \\
\operatorname{mod}\Lambda \arrow[u, hook] \arrow[r, "\sim"] & \operatorname{Per}(X/Z) \arrow[u, hook]
\end{tikzcd}$$



Theorem [Kapranov–Vasserot, Van den Bergh]. There is a derived equivalence between $(X, \mathcal{O}_X)$ and (Z, Λ) that identifies $\mathrm{mod}\,\Lambda$ with the category of π-perverse sheaves on X.

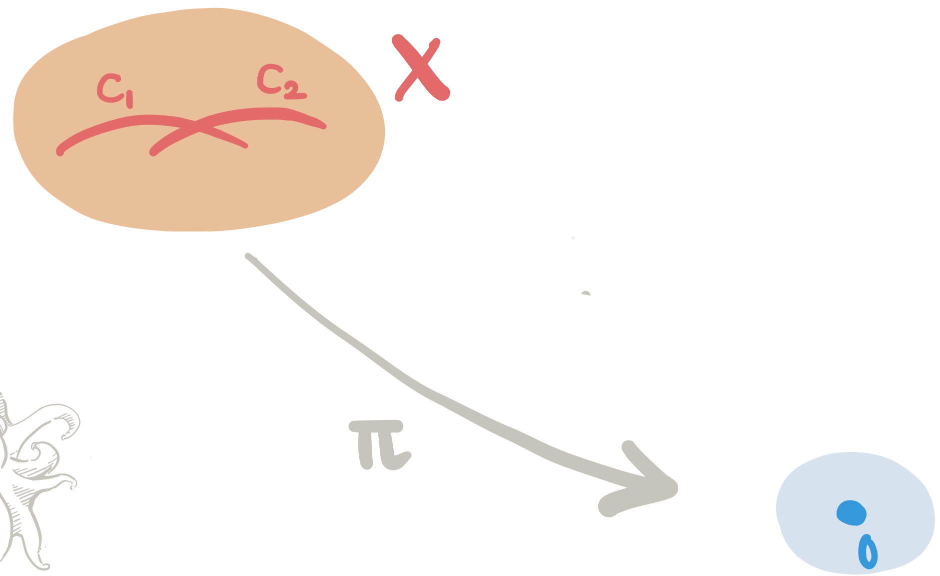
$\mathrm{Per}(X/Z)$ is the $(+\mathrm{ve})$ tilt of $\mathrm{Coh}\,X$ in the torsion class $\ker(R^1\pi_*) \subseteq \mathrm{Coh}\,X$



$$\mathrm{mod}\,\Lambda \xrightarrow{\sim} \mathrm{Per}(X/Z)$$

$$S_0 \longmapsto \omega_C[1]$$

$$S_i \longmapsto \mathcal{O}_{C_i}(-1)$$

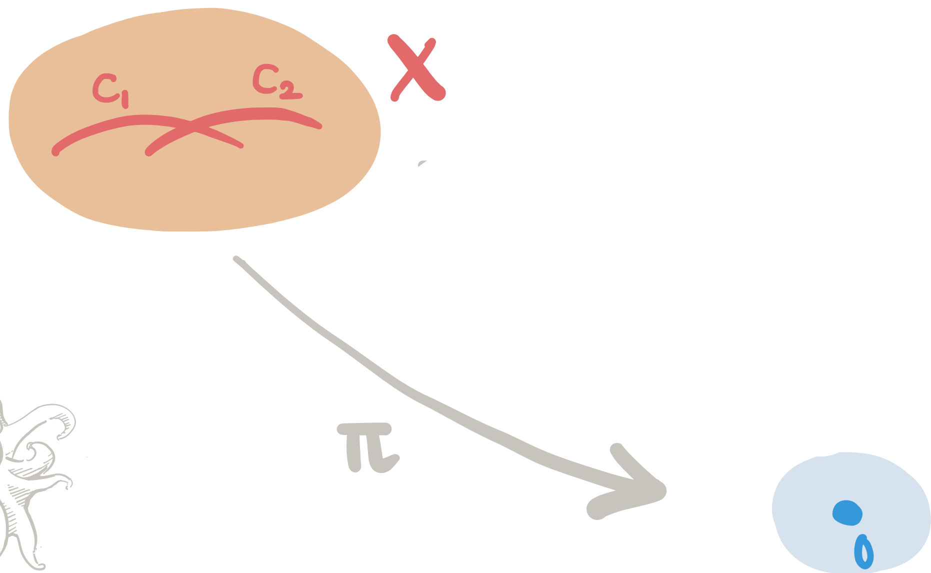


$$Z = \mathrm{Spec}\,\frac{\mathbb{C}[x,y]}{(xy = z^3)}$$

Theorem [Kapranov-Vasserot, Van den Bergh]. There is a derived equivalence between $(X, \mathcal{O}_X)$ and (Z, Λ) that identifies $\mathrm{mod}\,\Lambda$ with the category of π-perverse sheaves on X.

These are compatible with support-restriction.

$$\begin{tikzcd}
D^b \mathrm{mod}\,\Lambda \arrow[r, "\sim"] & D^b \mathrm{Coh}\, X \\
D^0 \mathrm{mod}\,\Lambda \arrow[r, "\sim"] \arrow[u] & D^0 \mathrm{Coh}\, X \arrow[u] \\
\mathrm{fl}\,\mathrm{mod}\,\Lambda \arrow[r, "\sim"] \arrow[u, hook] & \mathrm{per}(X/Z) \arrow[u, hook]
\end{tikzcd}$$

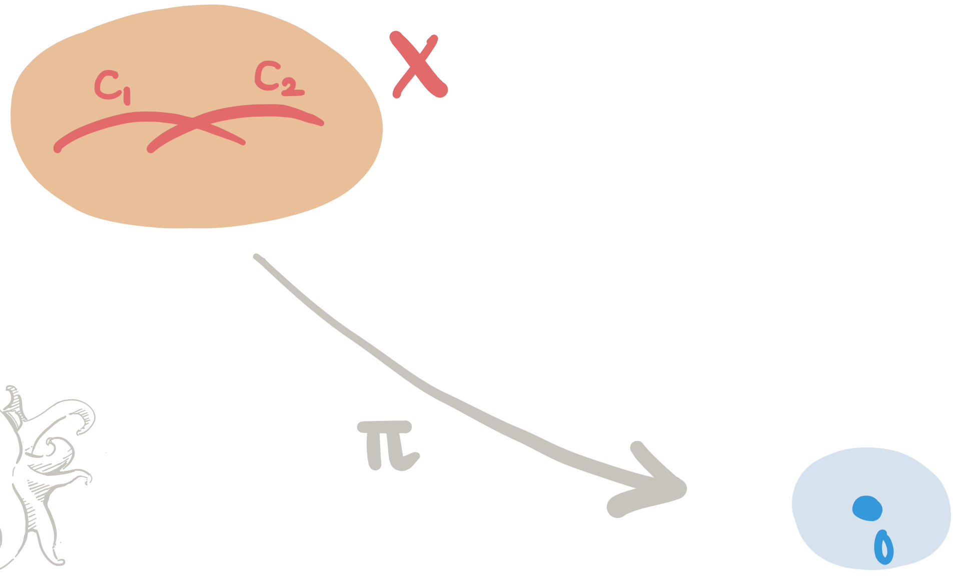


$$Z = \mathrm{Spec}\, \frac{\mathbb{C}[x,y]}{(xy = z^3)}$$

What hearts are already known?

① $\mathrm{coh}(Z, \Lambda) \cong \mathrm{fl}\ \mathrm{mod}\ \Lambda \cong \mathrm{per}\ X/Z$

② $\mathrm{coh}(X, \mathcal{O}_X) = \mathrm{coh}\ X \cong \mathrm{per}\ X/X$



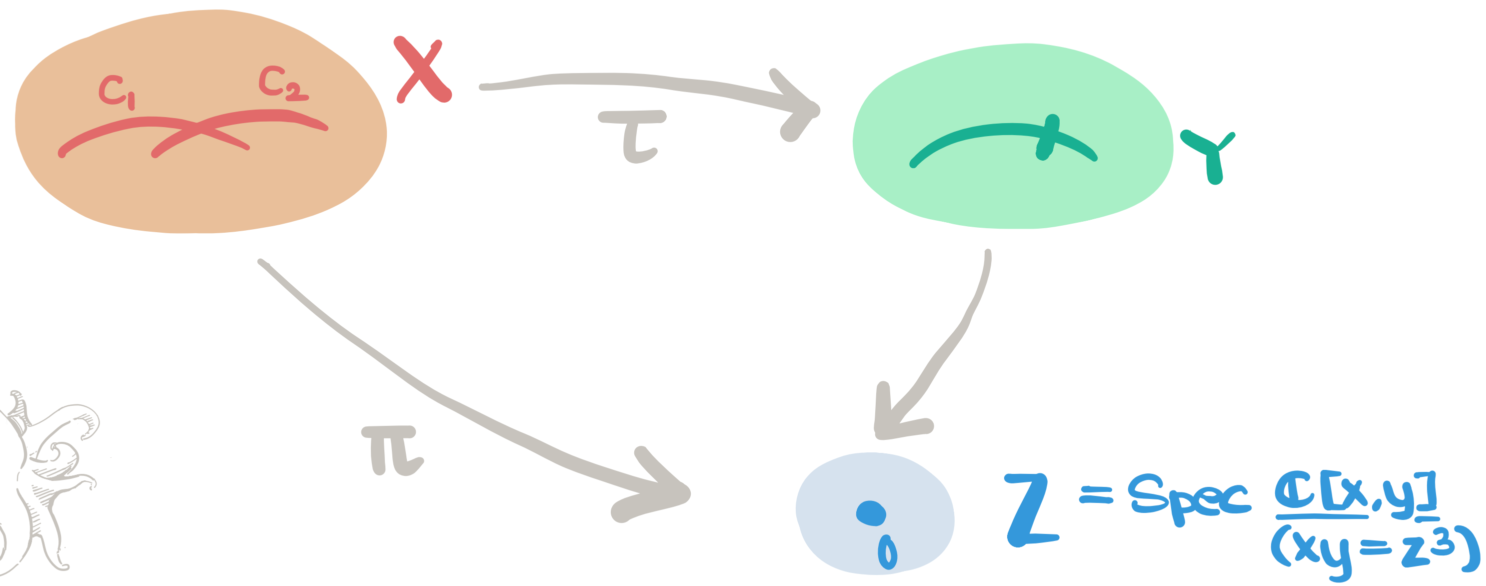
$Z = \mathrm{Spec}\ \dfrac{\mathbb{C}[x,y]}{(xy = z^3)}$

What hearts are already known?

① $\mathrm{coh}(Z, \Lambda) \cong \mathrm{fl}\ \mathrm{mod}\ \Lambda \cong \mathrm{per}(X/Z)$

② $\mathrm{coh}(X, \mathcal{O}_X) = \mathrm{coh}\ X \cong \mathrm{per}(X/X)$

③ $\mathrm{coh}(Y, \mathcal{A}) \cong \mathrm{per}(X/Y)$ for each partial contraction.



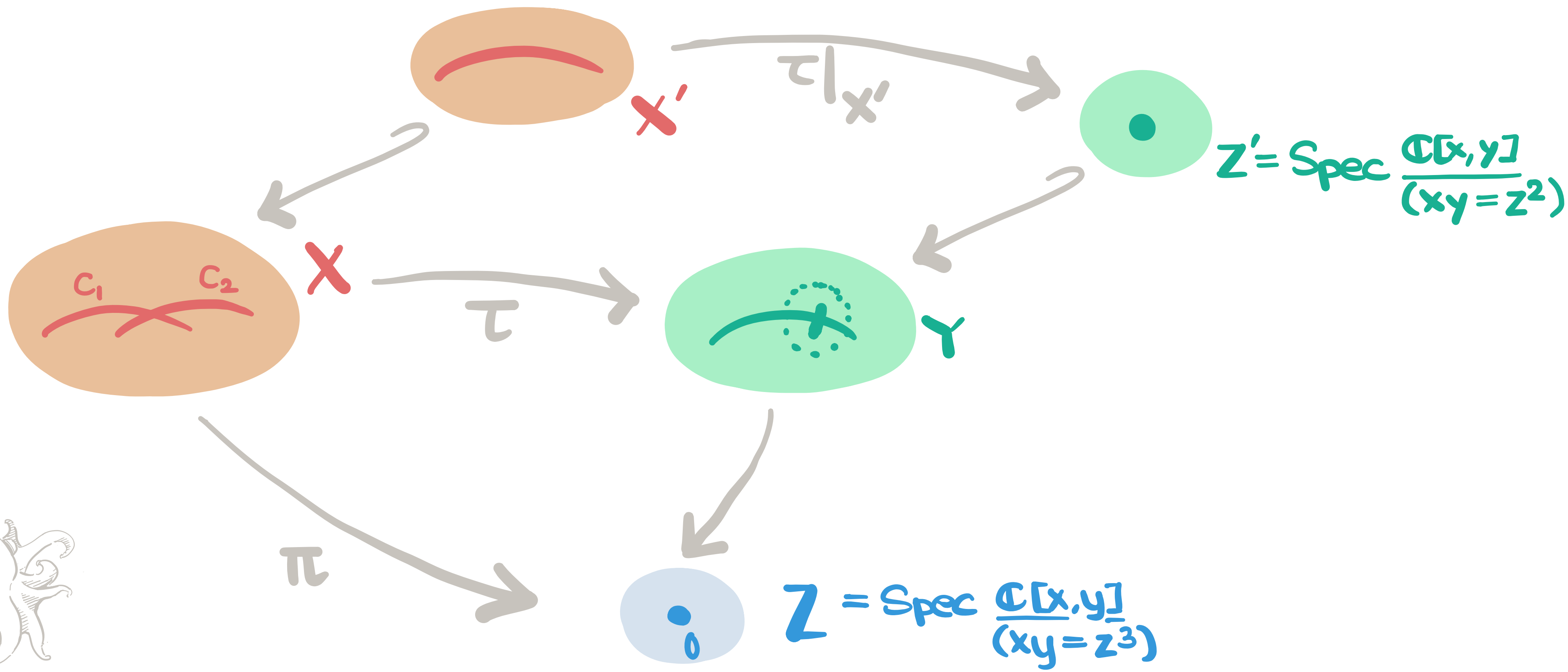
The construction of per from coh works flat-locally.

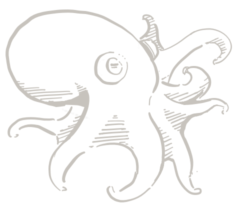
$\Rightarrow Y$ has a sheaf of nc-algebras Λ such that

$\Lambda \cong \mathcal{O}_Y$ away from contracted locus,

Λ is an NCCR near the contracted locus,

and $(X, \mathcal{O}_X)$ is derived equivalent to (Y, Λ).

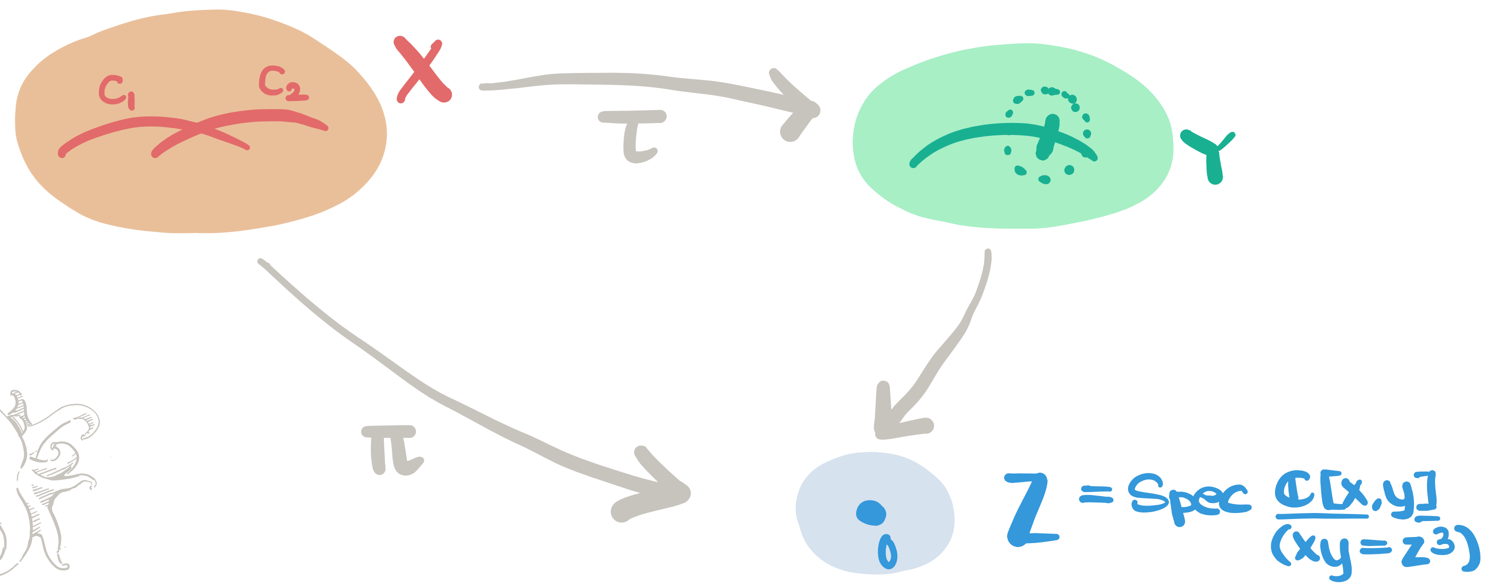




Lemma.
$$\mathrm{per}(X/Y) = \Big\langle \underbrace{\{\mathcal{F} \in \mathrm{per}(X/Z) \mid \mathrm{Supp}\,\mathcal{F} \text{ contracted by } \tau\}}_{=\,\tau\text{-perverse supported on contacted}},\ \underbrace{\{\mathcal{F} \in \mathrm{per}(X/X) \mid \tau \text{ is an iso. on } \mathrm{Supp}\,\mathcal{F}\}}_{=\,\tau\text{-perverse supported on uncontracted.}} \Big\rangle$$

"="

So semi-geometric hearts are "made up of" cohX and $H = \mathrm{flmod}\,\Lambda$.



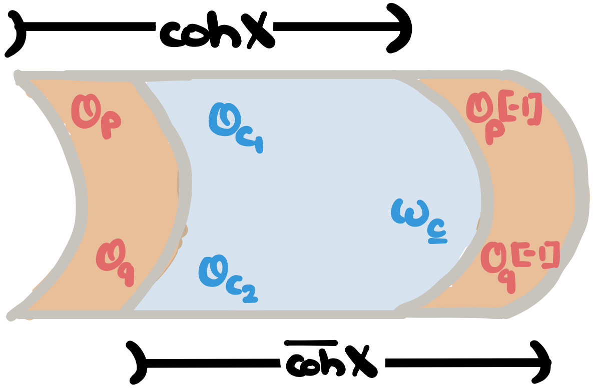
What hearts are already known?

① $\mathrm{coh}(Z,\Lambda) \cong \mathrm{fl}\ \mathrm{mod}\,\Lambda \cong \mathrm{per}(X/Z)$

② $\mathrm{coh}(X,\mathcal{O}_X) = \mathrm{coh}\,X \cong \mathrm{per}(X/X)$

③ $\mathrm{coh}(Y,\mathcal{A}) \cong \mathrm{per}(X/Y)$ for each partial contraction.

More hearts come from modifying the above, eg by tilting $\mathrm{coh}\,X$ in (some or all) skyscrapers.



To get a whole interval $[\overline{\mathrm{coh}}\,X, \mathrm{coh}\,X] \cong \mathrm{Bool}(C)$ in $\mathrm{tilt}(H)$.

(Likewise tilt $\mathrm{per}(X/Y)$ in skyscrapers in "geometric locus").

More interestingly, have derived equivalences from tilting modules.

$T \in \mathrm{mod}\,\Lambda$ is tilting if

$\mathrm{Ext}^i(T,T)=0 \quad \forall i \neq 0$,

and there exist sequences

$0 \to P_2 \to P_1 \to T \to 0$ w/ $P_1, P_2 \in \mathrm{add}\,\Lambda$

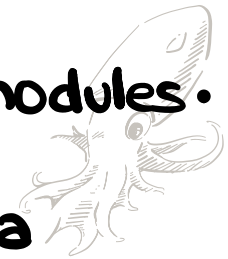
$0 \to \Lambda \to T_2 \to T_1 \to 0$ w/ $T_1, T_2 \in \mathrm{add}\,T$

Get equivalence $\Phi_T : D^b(\mathrm{End}\,T) \xrightarrow{-\otimes T} D^b\Lambda$ such that

$\Phi_T(\mathrm{mod}(\mathrm{End}\,T)[-1]) \in \mathrm{tilt}(\mathrm{mod}\,\Lambda)$, equivalently

$\Psi_T(\mathrm{mod}\,\Lambda) \in \mathrm{tilt}(\mathrm{mod}(\mathrm{End}\,T))$.

where $\Psi_T := \Phi_T^{-1}$



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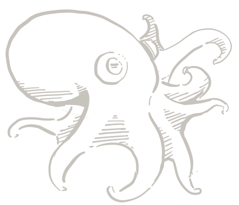
[Buan-Iyama-Reiten-Scott, Iyama-Wemyss] provide a complete understanding of tilting Λ-modules.

① in the uncontracted setting, for all T tilting, $\mathrm{End}(T)$ is canonically isomorphic to Λ.

$$\Phi_T(\mathrm{mod}(\mathrm{End}\,T)[-1]) \in \mathrm{tilt}(\mathrm{mod}\,\Lambda)$$

$$\Psi_T(\mathrm{mod}\,\Lambda) \in \mathrm{tilt}(\mathrm{mod}(\mathrm{End}\,T)).$$

$\Rightarrow$ For each tilting module T, get two tilts $\Psi_T(\mathcal{H})$ and $\Phi_T(\mathcal{H}[-1])$.



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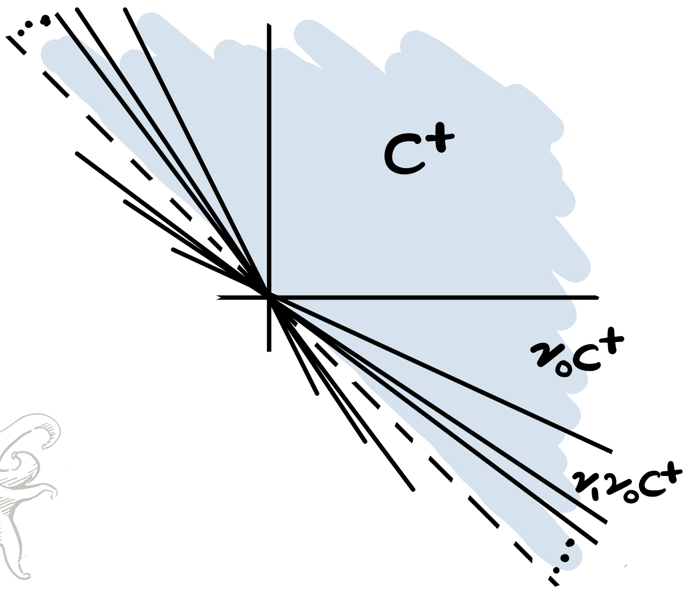
① in the uncontracted setting, for all T tilting, $\mathrm{End}(T)$ is canonically isomorphic to Λ.

② Iso-classes of tilting Λ-modules are in bijection with chambers of the $\hat{\Delta}$-Tits cone.



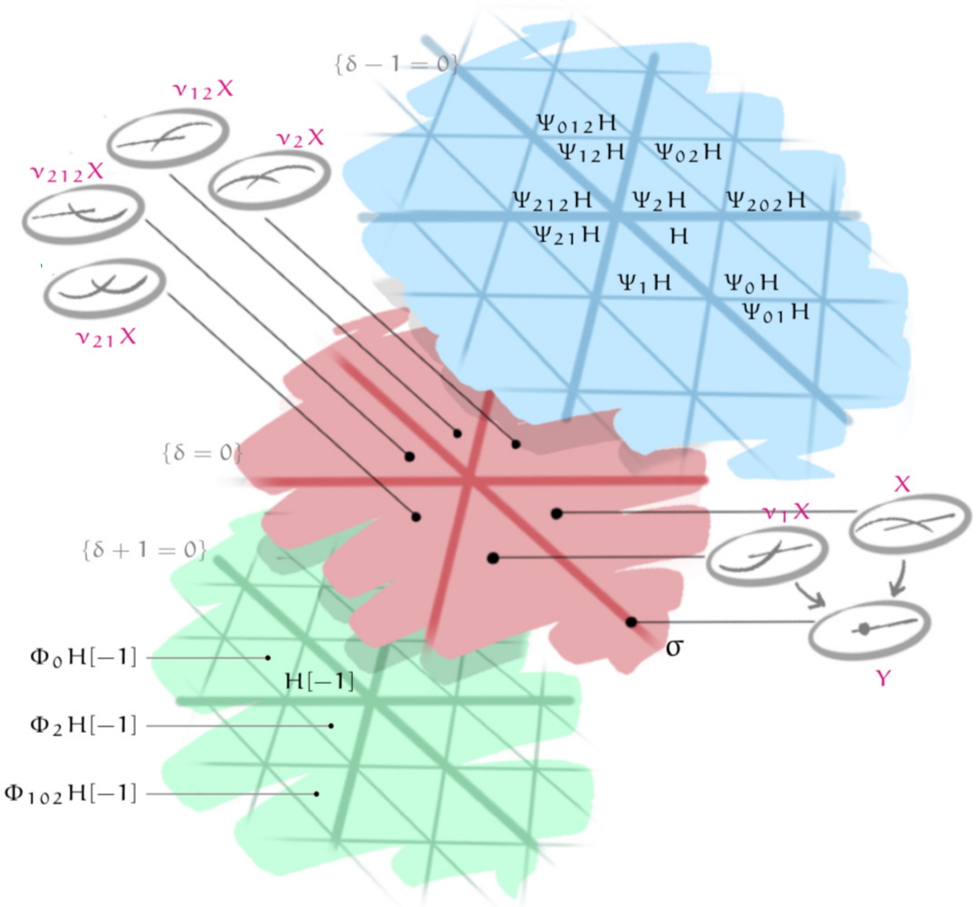
$\Rightarrow$ For each sequence of wall-crossings γ from starting chamber C^+, get a tilting module T_γ with corresponding tilted hearts $\Psi_\gamma(\mathcal{H})$, $\Phi_\gamma(\mathcal{H}[-1])$.

What hearts are already known?

① $H = \mathrm{coh}(Z, \Lambda)$, $\Psi_\nu H$, and $\Phi_\nu H[-1]$ for any ν.

② $K \in [\overline{\mathrm{coh}}\, X, \mathrm{coh}\, X]$, and $\Psi_\nu K$ for ν not containing 0.

③ Appropriate combinations of above, eg $\mathrm{per}(X/Y)$ and $\overline{\mathrm{per}}\, X/Y$.



Let Λ be a (contracted) affine preprojective algebra.

Question. Is it possible to classify hearts in $D^b \operatorname{mod} \Lambda$?

Theorem [S]. If K is a bounded heart in $D^b \operatorname{mod} \Lambda$ and $K \subseteq \operatorname{fl} \operatorname{mod} \Lambda [-1,0]$ then one of the following holds.

① $K = H = \operatorname{coh}(Z, \Lambda)$, $\Psi_\nu H$, or $\Phi_\nu H[-1]$ for any ν.

② $K \in \Psi_\nu^{-1} [\overline{\operatorname{coh}} X, \operatorname{coh} X]$ for ν not containing 0.

③ K is an appropriate combination of above.

Let Λ be a (contracted) affine preprojective algebra.

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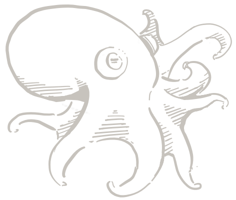
③ K is an appropriate combination of above.

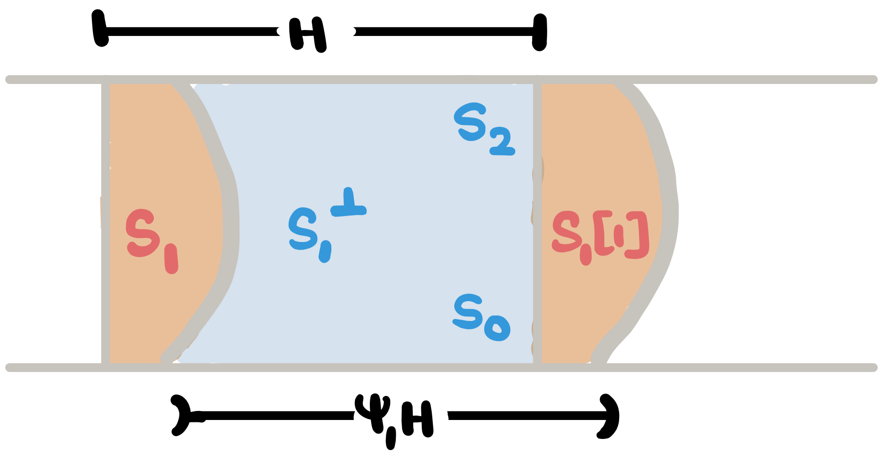
Corollary. If $M \in \mathrm{mod}\,\Lambda$ is a finite length brick, then M is either a simple Λ-module or a skyscraper up to mutation and shifts.



Local understanding of the poset.

$\psi_i H$ is a simple tilt of H, i.e. the corresponding torsion class is minimal containing S_i.

$\Rightarrow$ the relation $H > \psi_i H$ is covering, and all covering relations $H > K$ arise in this way.

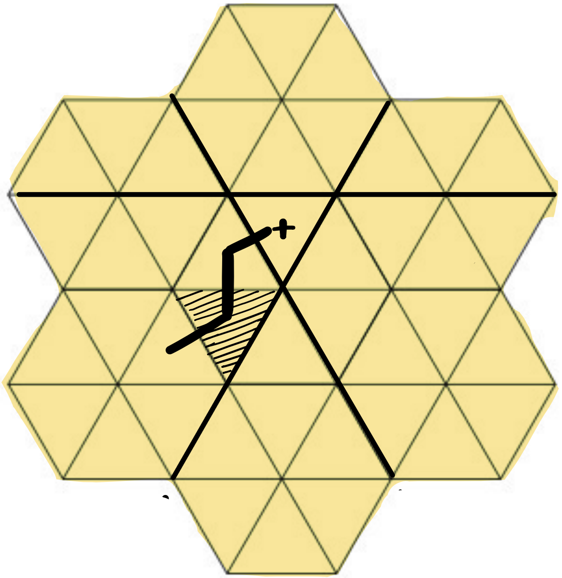


Local understanding of the poset.

$\Psi_i H$ is a simple tilt of H, i.e. the corresponding torsion class is minimal containing S_i.

$\Rightarrow$ the relation $H > \Psi_i H$ is covering, and all covering relations $H > ?$ arise in this way.

Likewise if $\gamma = \gamma_i \mu$ is a minimal path, then $\Psi_\gamma H$ is a simple tilt of $\Psi_\mu H$ and $\Phi_\mu H$ is a simple tilt of $\Phi_\gamma H[-1]$.



Global, numerical understanding of the poset.

Recap. The $\tilde{\Delta}$-Cartan algebra $\mathfrak{h}$ can be identified with $K_0\Lambda$ in a way that

- $[S_i] \in K_0\Lambda$ correspond to simple roots $\alpha_i \in \mathfrak{h}$
- $[\mathcal{O}_p] \in K_0 X$ correspond to imaginary root $\delta \in \mathfrak{h}$
- $\Theta = \mathrm{Hom}(K_0\Lambda, \mathbb{R})$ is identified with $K^{\mathrm{split}}_{\mathrm{proj}}\Lambda$
- Action of mutation coincides with Weyl group i.e. Ψ_i acts by simple reflection in α_i.

Then [BIRS] obtain the bijection

$$\{\text{Tilting } \Lambda\text{-modules}\} \longrightarrow \left\{\begin{array}{l}\text{chambers in Tits cone} \\ \text{in } \Theta\end{array}\right\}$$

by assigning

$$T \longmapsto \text{g-vector cone}\,(T).$$

cone in $\Theta \cong K^{\text{split}}\,\mathrm{proj}\,\Lambda$ recording 2-term projective resolutions of the summands of T.

Turns out, it can be obtained differently.

Then [BIRS] obtain the bijection

$$\{\text{Tilting } \Lambda\text{-modules}\} \longrightarrow \{\text{chambers in Tits cone in } \Theta\}$$

by assigning

$$T \longmapsto \{\theta \in \Theta \mid \theta[k] \geqslant 0 \ \forall k \in \Psi_T H\}$$

Forget tilting altogether?

$$\{\Psi_\gamma H \mid \gamma \text{ a path}\} \longrightarrow \{\text{chambers in Tits cone in } \Theta\}$$

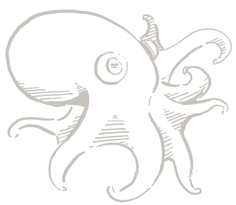
$$K \longmapsto \{\theta \in \Theta \mid \theta[k] \geq 0 \ \forall k \in \Psi_T H\}$$

Theorem [Broomhead–Pauksztello–Ploog–Woolf].
If H is an algebraic abelian category, then assigning each $K \in \mathrm{tilt}(H)$ to its **heart cone**

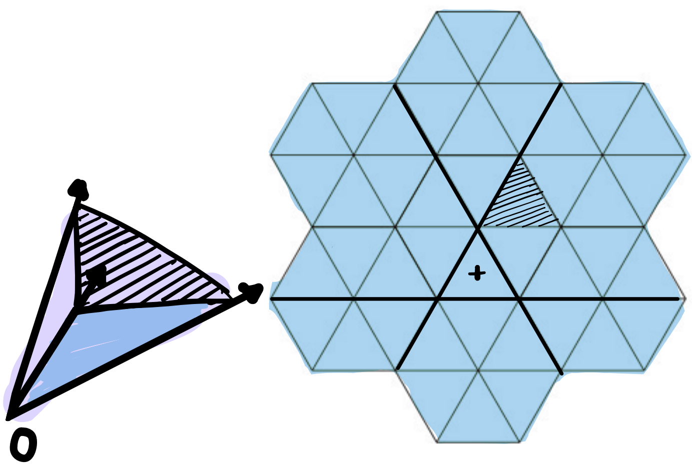
$$C_K = \{\theta \in \mathrm{Hom}(K_0 H, \mathbb{R}) \mid \theta(k) \geq 0 \ \forall k \in K\}$$

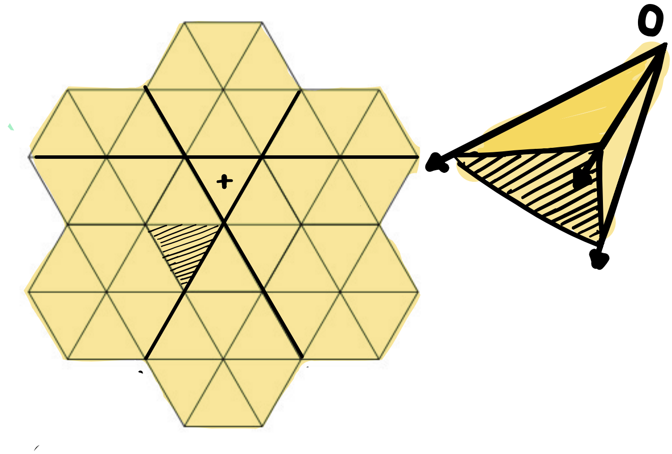
gives a complete simplicial fan, **the heart fan of H**

$$\mathrm{HFan}(H) = \{C_K \mid K \in \mathrm{tilt}(H)\}.$$



For $H = \mathrm{fl}\,\mathrm{mod}\,\Lambda$, thus fill out "most of" Θ.

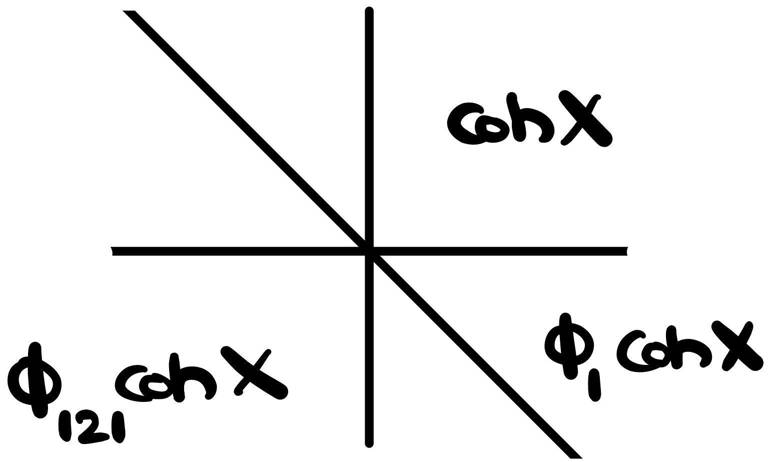




$$C(\Psi_\gamma H) = \gamma \cdot C^+ \qquad C(\Phi_\gamma H[-1]) = -(\gamma \cdot C^+).$$

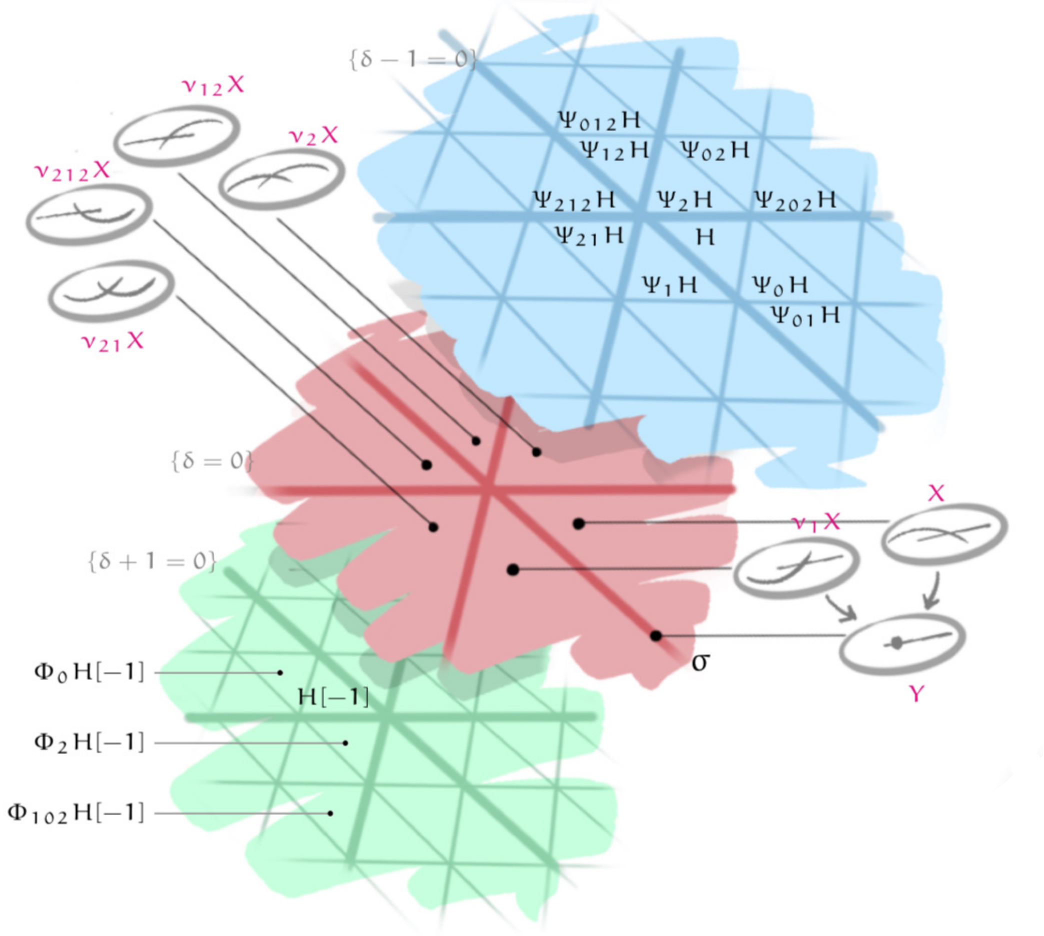
(Draw slices wrt $\delta = \sum_i [S_i]$ normalised at ± 1)

Identifying $\{\delta=0\}\subset\Theta$ with $\mathbb{R}\otimes\operatorname{Pic}X$ shows the chamber $C^0=\{\theta \mid \theta(\delta)=0,\ \theta(\alpha_i)\geq 0\ \forall i\neq 0\}$ is the heart cone of $\operatorname{coh}X$,



Likewise for semi-geometric hearts.

Thus HFan(flmod Λ) is induced by root hyperplanes in Θ.



Theorem [Asai-Pfeifer]. If $\sigma \in \mathrm{HFan}(H)$ is a heart cone, then the set

$$\{K \in \mathrm{tilt}(H) \mid CK \supseteq \sigma\}$$

is an <u>interval</u> of the form

$$\{K \mid SS(\theta) * A(\theta) \geqslant K \geqslant A(\theta) * SS(\theta)[-1].\}$$

Abelian subcategory of θ-semistables in H

$$\{h \in H \mid \theta(s) > 0 \ \forall\ 0 \neq s \hookrightarrow h\}$$
$$*$$
$$\{h \in H \mid \theta(f) < 0 \ \forall\ h \twoheadrightarrow f \neq 0\}[-1]$$

for $\theta \in \sigma$ generic.

Theorem [Asai-Pfeifer]. If $\sigma \in \mathrm{HFan}(\mathcal{H})$ is a heart cone, then the set

$$\{ K \in \mathrm{tilt}(\mathcal{H}) \mid C_K \supseteq \sigma \}$$

is an <u>interval</u> of the form

$$\{ K \mid SS(\theta) * A(\theta) \geq K \geq A(\theta) * SS(\theta)[-1]. \}.$$

Thus full dimensional cones (where $SS(\theta) = \emptyset$) correspond to a unique tilt of $\mathcal{H}$.

Theorem [Asai–Pfeifer]. If $\sigma \in \mathrm{HFan}(H)$ is a heart cone, then the set

$$\{K \in \mathrm{tilt}(H) \mid C_K \supseteq \sigma\}$$

is an <u>interval</u> of the form

$$\{K \mid SS(\theta) * A(\theta) \geq K \geq A(\theta) * SS(\theta)[-1]\},$$

further in bijection with $\mathrm{tilt}(SS(\theta))$.

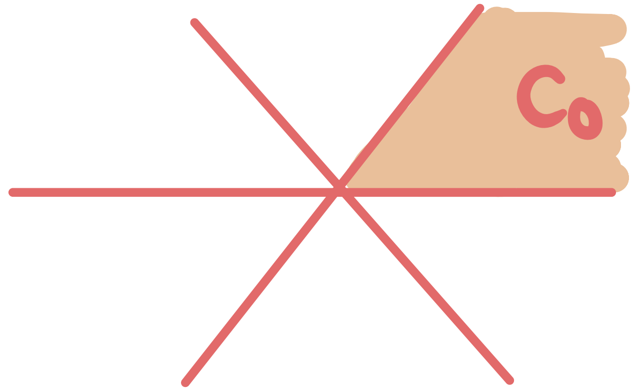
If σ is not a full dimensional cone, then get inductive behaviour.

(If $SS(\theta) = U * V$ is a torsion pair then consider

$K = V * A(\theta) * U[-1]$)

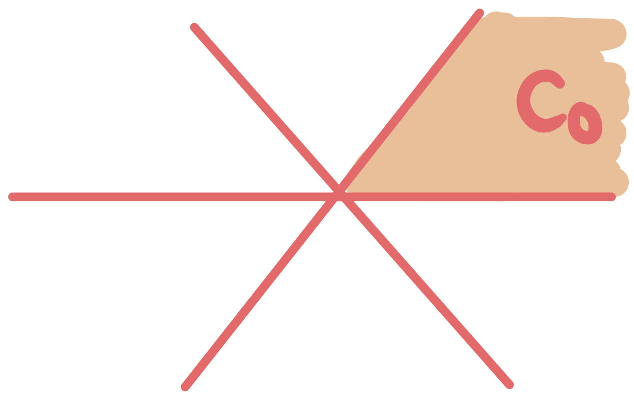
Eg for $\sigma = C_0$, compute

$$SS(\Theta) = \langle \Theta_p \mid p \in C \rangle$$



(essentially because $X = \mathcal{M}_\Theta$).

Eg for $\sigma = C_0$, compute

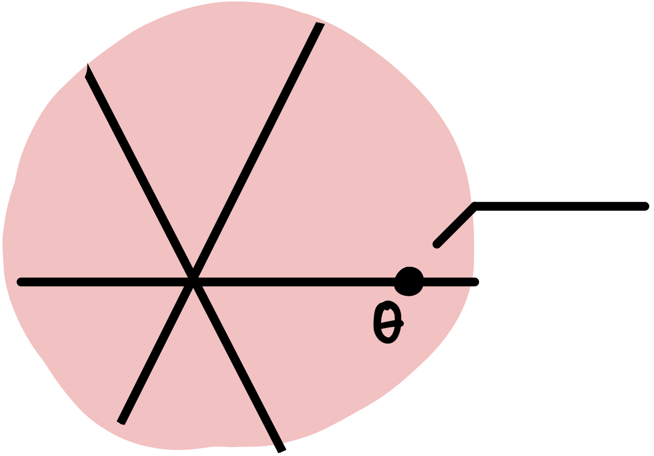


$$ss(\Theta) = \langle \mathcal{O}_p \mid p \in C \rangle$$

So must have $\mathrm{coh}\, X = ss(\Theta) * A(\Theta)$

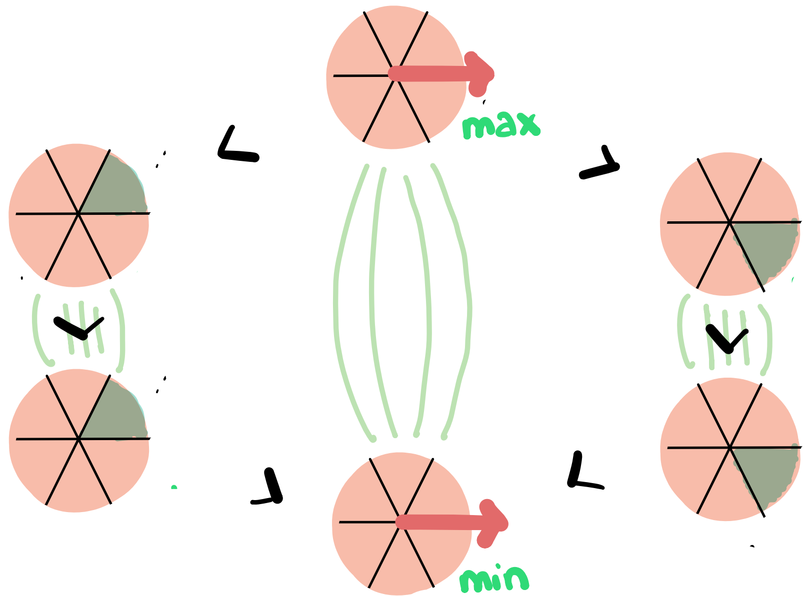
$$\Rightarrow \quad A(\Theta) * ss(\Theta)[-1] = \overline{\mathrm{coh}}\, X$$

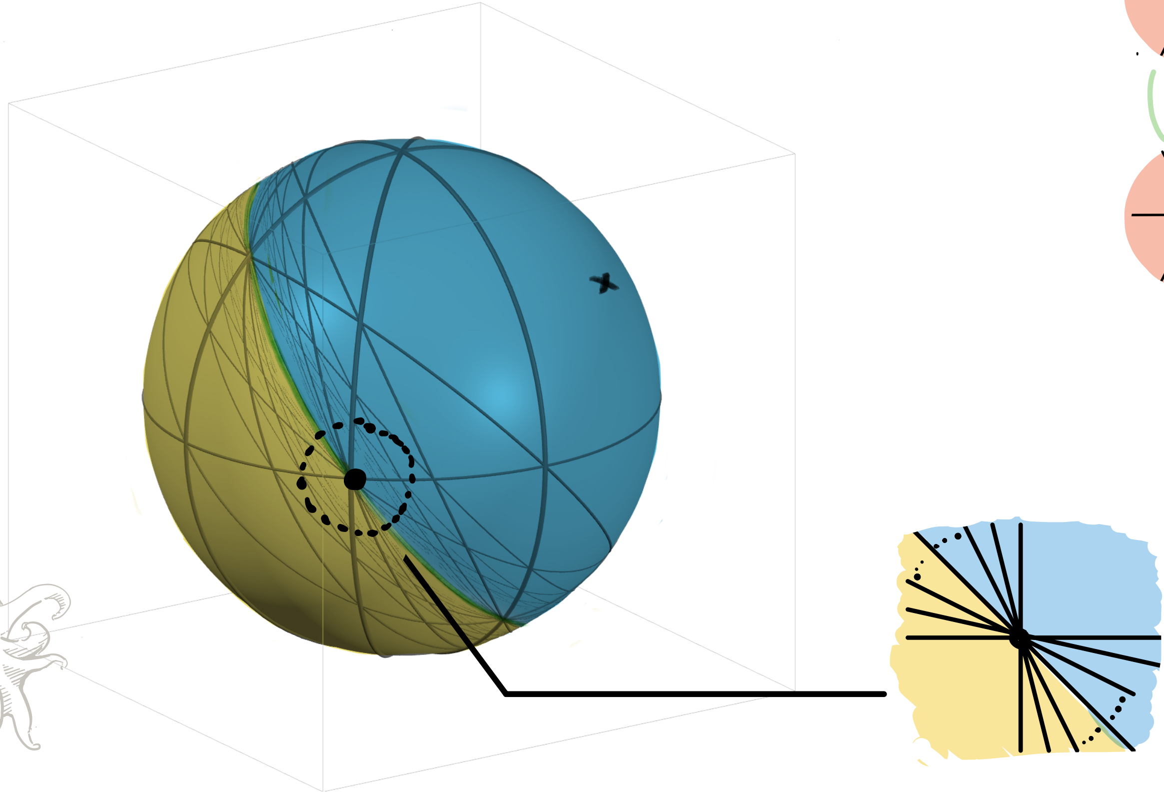
i.e. $K \in \mathrm{tilt}(H)$ has heart cone C_0 iff it is a tilt of $\mathrm{coh}\, X$ in skyscrapers.



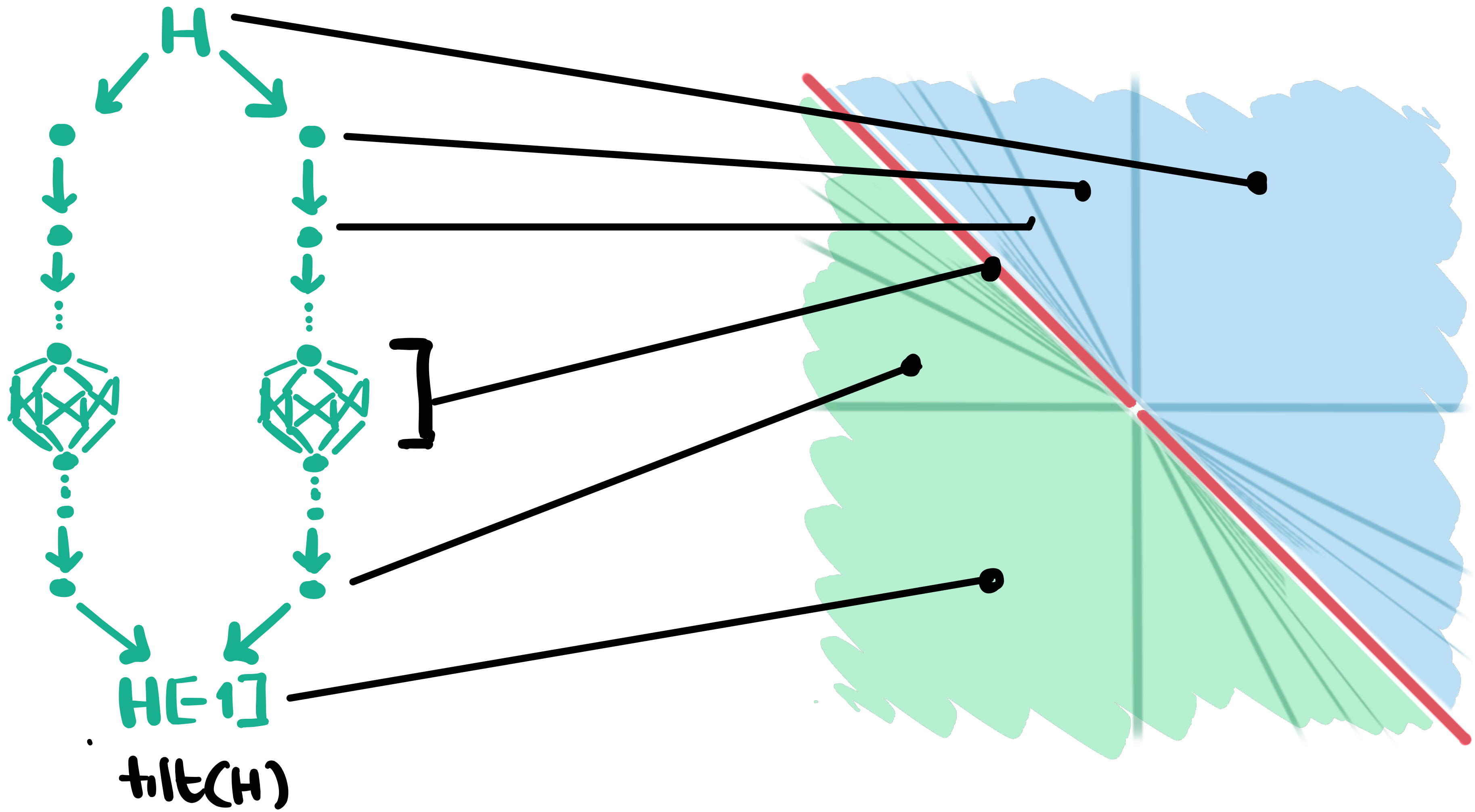
$H^{\theta} = \mathrm{per}(X/Y)$, $SS(\theta) = \langle$ skyscrapers in geom, all of algebraic $\rangle$

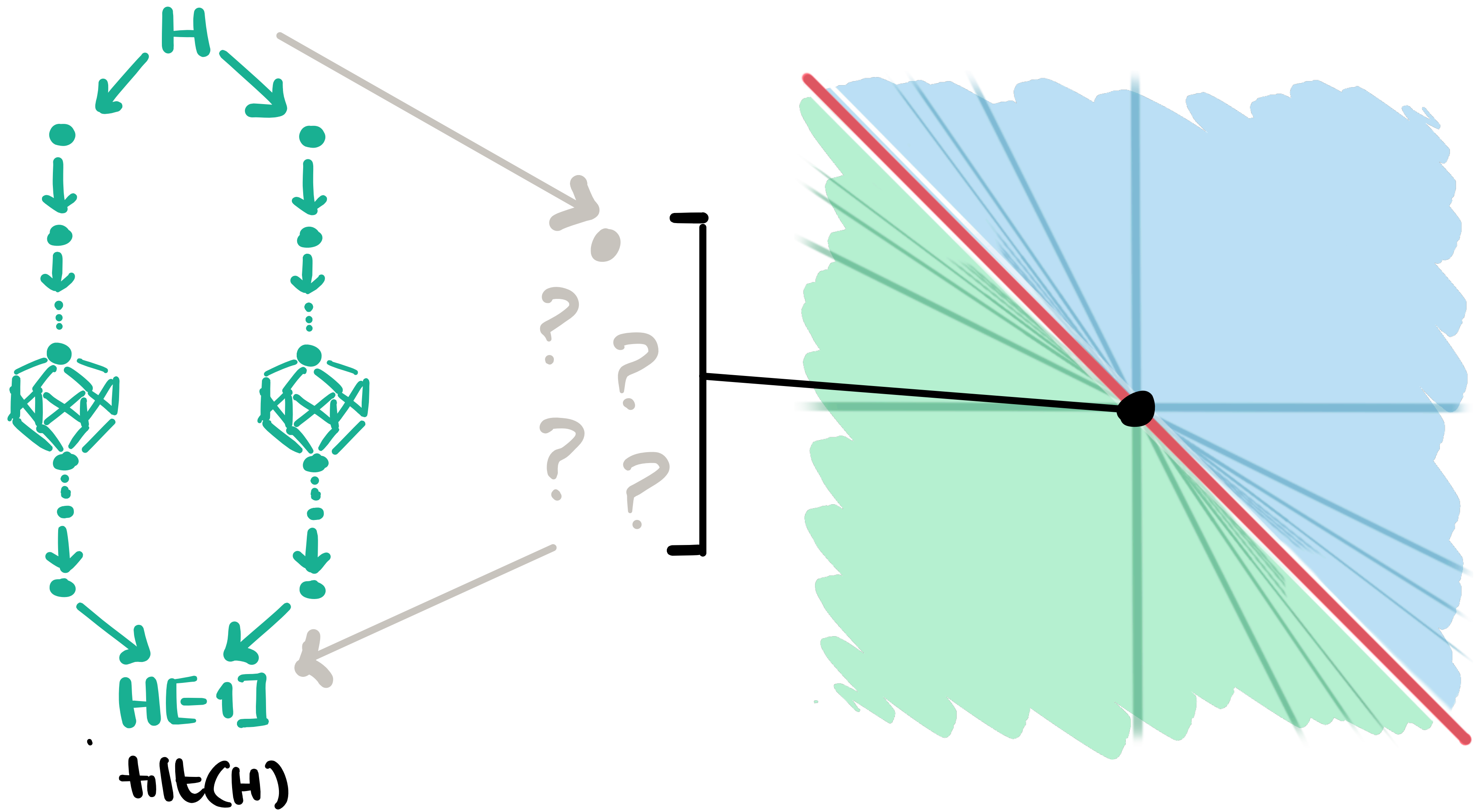
Poset is $\mathrm{tilt}(\mathrm{fmod}\,\Lambda_{\tilde{A}_1}) \times \mathrm{Bool}(C \setminus C_1)$



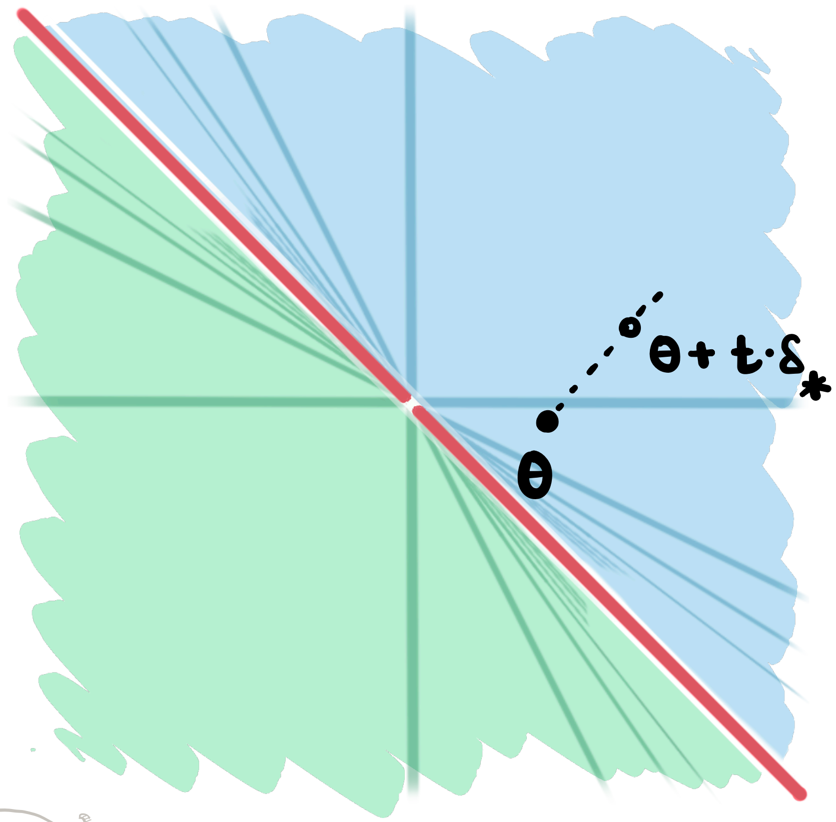


$\heartsuit$-fan of $SS(\theta) \cong \mathrm{mod}(\Lambda_1)$


H
H[-1]
tilt(H)


H
H[-1]
tilt(H)

Lemma. $\mathrm{coh}\,X$ is a limit of algebraic tilts.



$$\text{max-tilt}(\theta)$$

$$= \text{tilt in } \text{max-torf}(\theta)$$

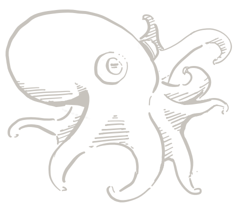
$$= \text{tilt in } \{h \in H \mid \theta(s) > 0 \ \forall\, s \overset{\neq 0}{\hookrightarrow} h\}$$

$$= \text{tilt in } \left\{h \in H \;\middle|\; \begin{array}{c} (\theta + t\delta_*)(s) \geq 0 \\ \forall t > 0 \\ \forall s \hookrightarrow h \end{array}\right\}$$

$$= \text{tilt in } \bigcap_{t>0} \text{min-torf}(\theta + t\delta_*)$$

$$= \inf\{\text{max-tilt}(\theta + t\delta_*) \mid t > 0\}$$

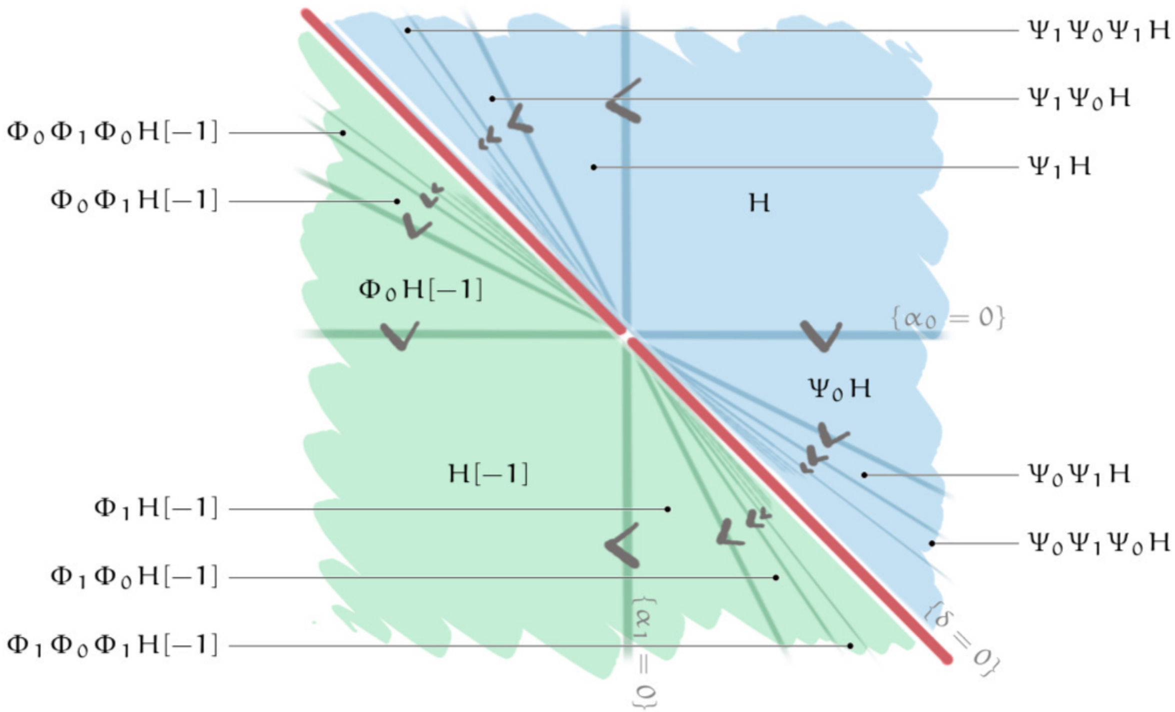
$$= \inf\{H, \psi_1 H, \psi_{01} H, \ldots\}$$



Theorem. 0 is not a heart cone

i.e. $\forall K \in \mathrm{tilt}(H)$, $\dot{C}K \neq 0$.

In single curve case, this is immediate from the fact that relations between algebraic hearts are covering.

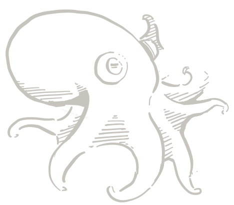


$(H > \Psi_1 H > \Psi_{01} H > \dots > K$

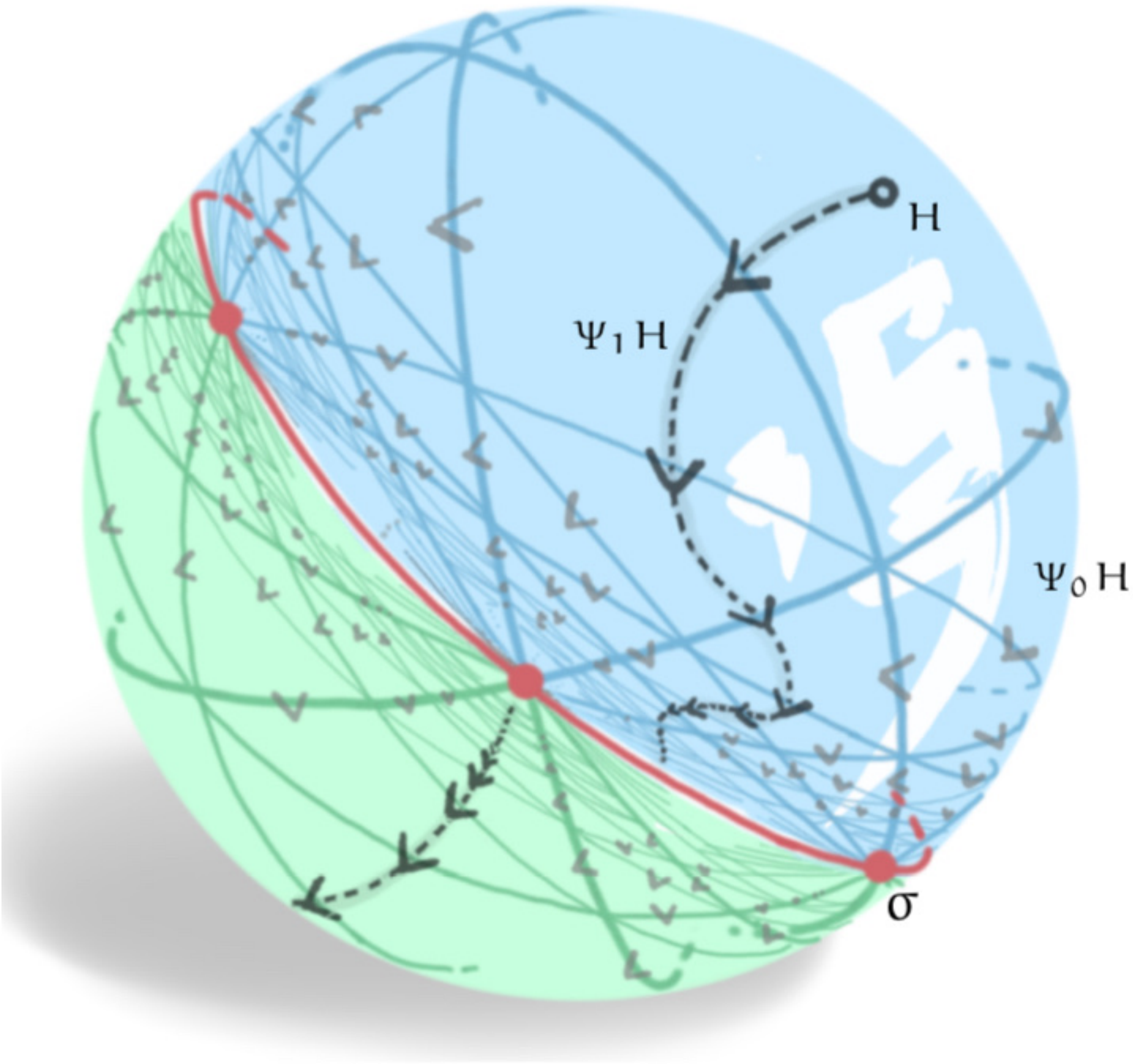
$\Rightarrow \mathrm{coh}\, X \geq K$

$K < \dots < \Phi_{10} H[-1] < \Phi_0 H[-1]$

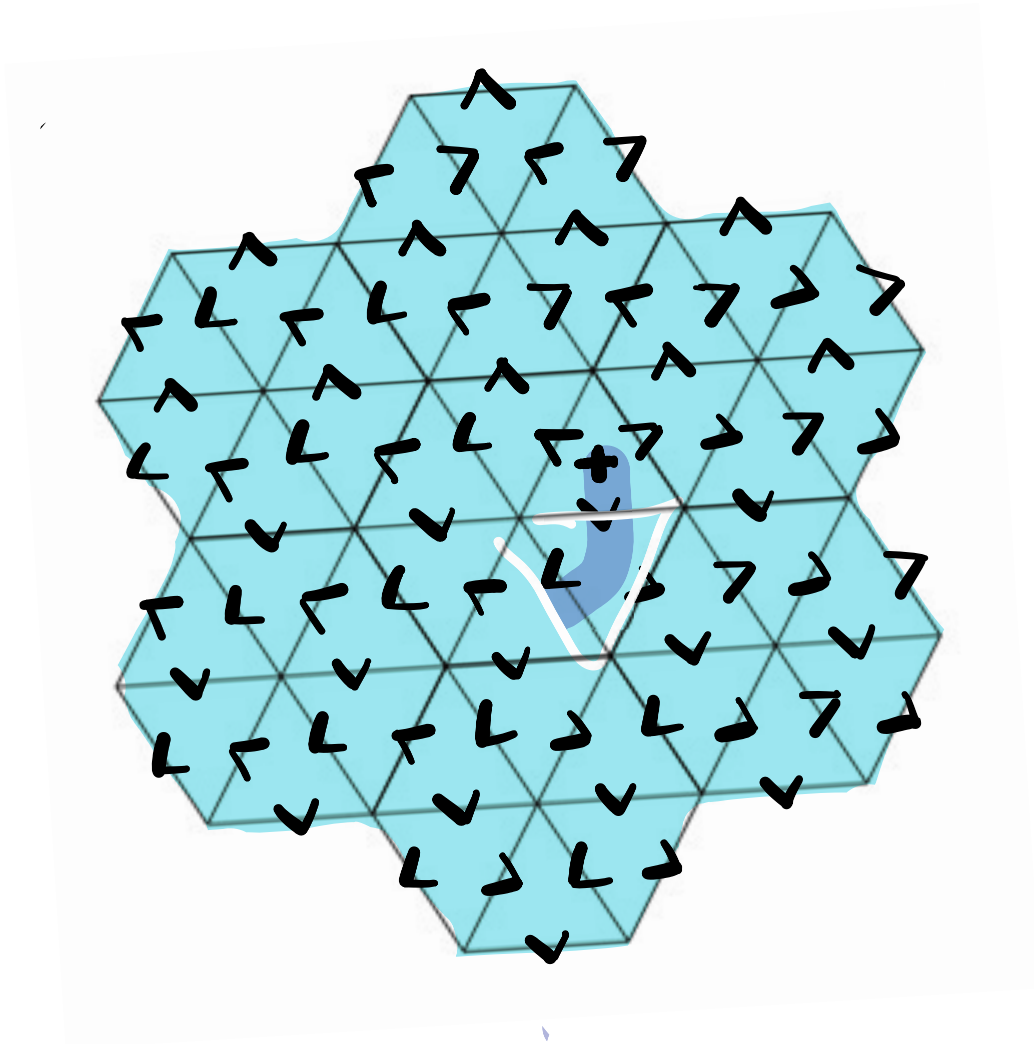
$\Rightarrow K \leq \overline{\mathrm{coh}}\, X)$



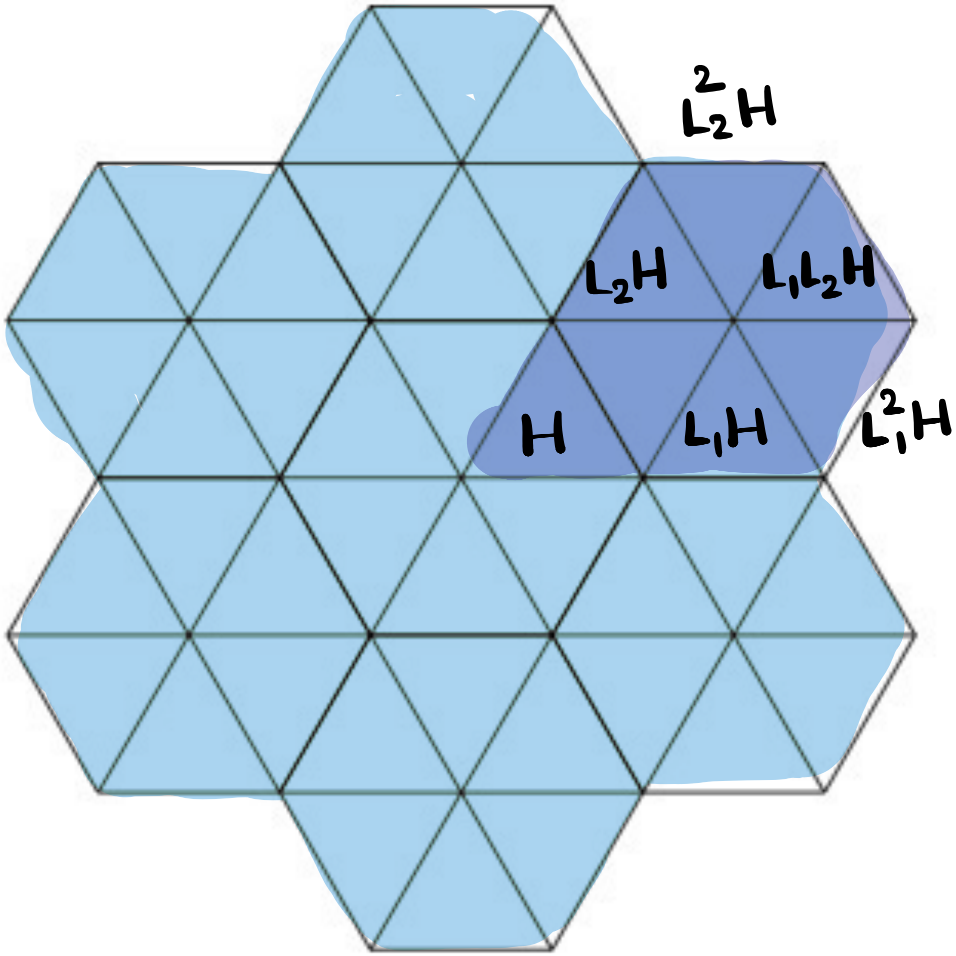
Need more control in dim $\geqslant 3$.



Instead of all algebraic,



Focus on orbit of H under $\mathrm{Pic}\,X \curvearrowright D^0 X$



$L_1, L_2 \in \mathrm{Pic}(X/Z)$

$L_i \cdot E_j = \delta_{ij}$

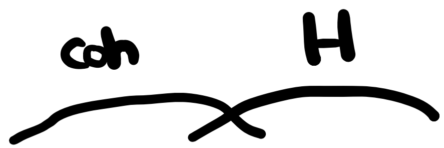
Theorem. $L \otimes H$ lies in $\mathrm{tilt}(H)$ if and only if $L^\vee$ is ample.

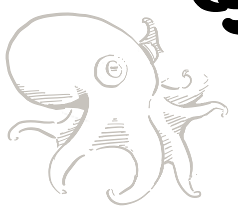
Further, $\lim\limits_{n\to\infty} (L^n \otimes H)$ is

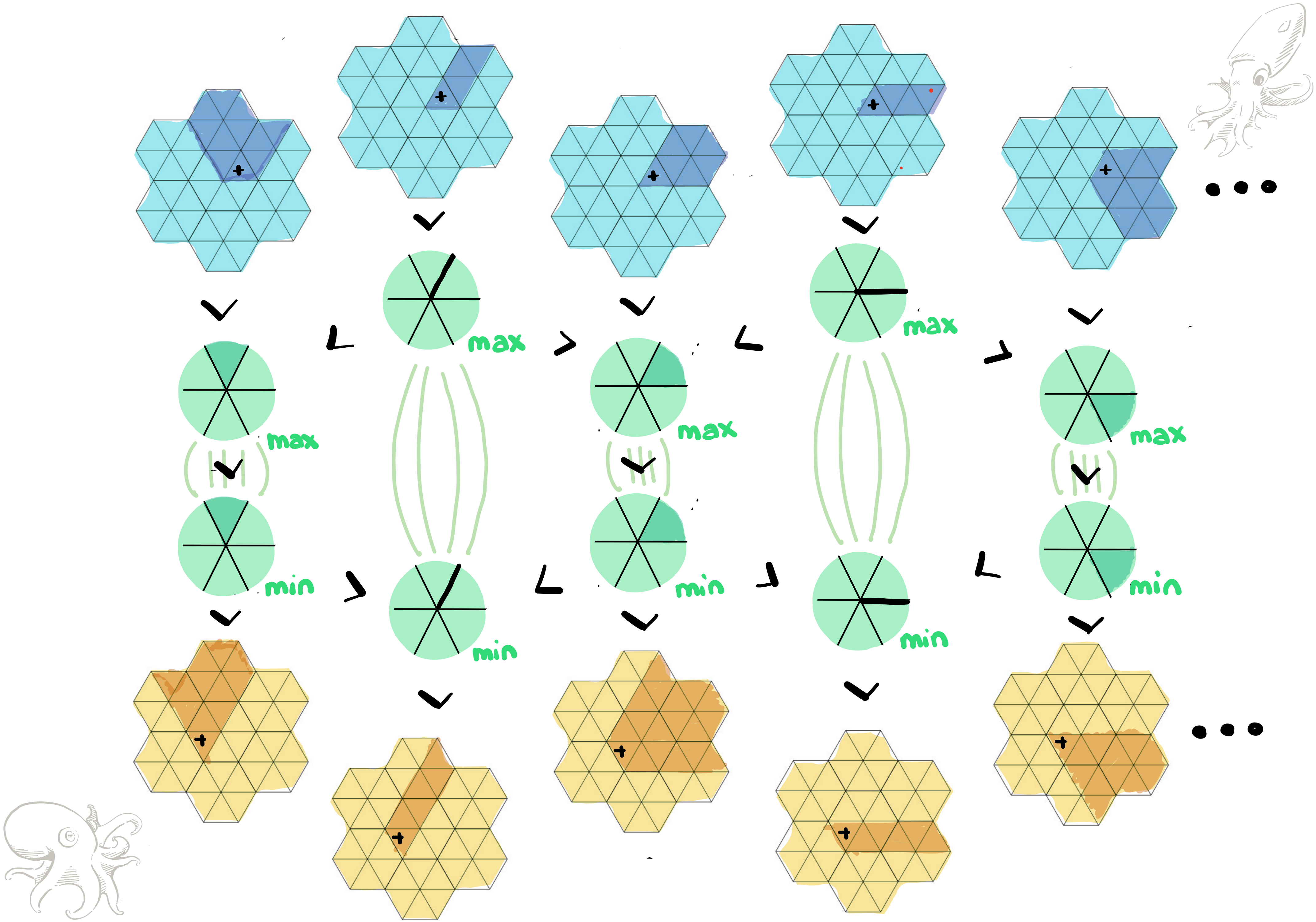
geometric on C_i if $L\cdot C_i \neq 0$

algebraic on C_i if $L\cdot C_i = 0$.

eg $\quad L_1^{-n} \otimes H \longrightarrow$






max
max
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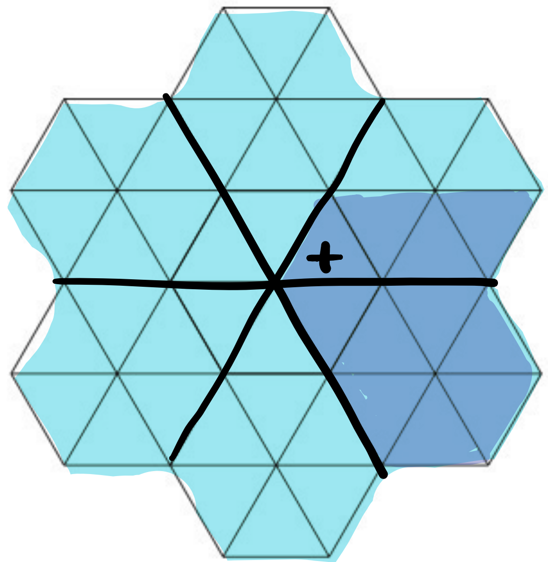
...
...

Lemma. For $K \in \mathrm{tilt}(H)$ not algebraic, let ν be the longest path such that

① ν does not contain 0

② $\Psi_2 H > K$.

Then for every $p \in X$, $\Psi_2 \mathcal{O}_p$ lies in K or $K[1]$.



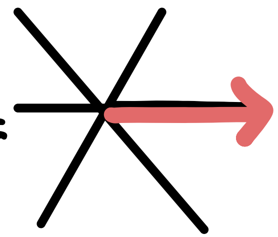
Lemma. For $K \in \mathrm{tilt}(H)$ not algebraic, let ν be the longest path such that ① ν does not contain 0
② $\Psi_2 H > K$.

Then for every $p \in X$, $\Psi_2 \mathcal{O}_p$ lies in K or $K[1]$.

- Replace K by $\Psi_2^{-1} K$ if necessary.

If $p \in C_1$ and $\mathcal{O}_p \in K$, then get $K \geqslant \overline{\mathrm{coh}}X \;\; H[-1]$

$\mathcal{O}_p \in K[1]$, then get $\mathrm{coh}X \;\; H \geqslant K$

so if $\exists\, p, q \in C_1$ with $\mathcal{O}_p \in K$, $\mathcal{O}_q \in K[1]$ then $CK =$ 

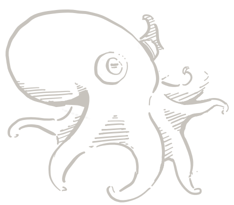
Else have (wlog) $\mathcal{O}_p \in K$ $\forall p \in C$, so that

$$K \geqslant \sup\left(\overline{\mathrm{coh}}X \quad H[-1],\; H[-1] \quad \overline{\mathrm{coh}}X\right) = \overline{\mathrm{coh}}X.$$

Chasing simple tilts gives $\mathrm{coh}X \quad H \geqslant K$ for some curve, so that

$$\mathrm{coh}X \quad H \geqslant K \geqslant \overline{\mathrm{coh}}X \geqslant \overline{\mathrm{coh}}X \quad H[-1].$$

Again, $CK \neq 0$.

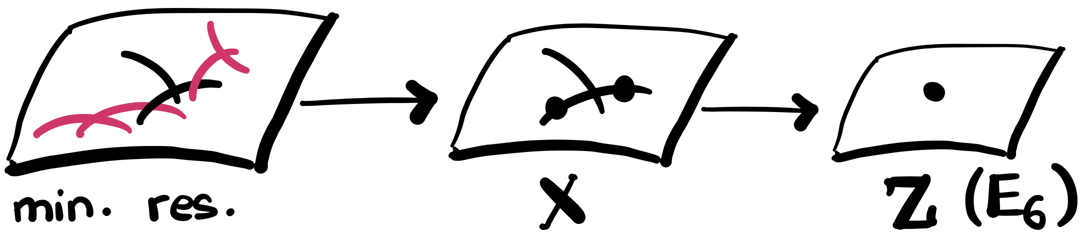


Scope: $H = {}^{0}\mathrm{per}(X/Z)$ for

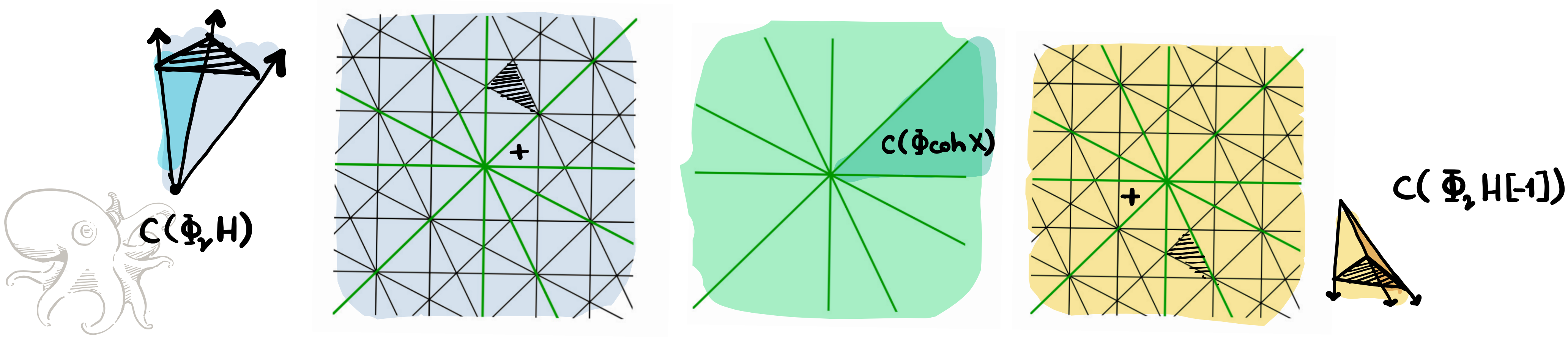
- Z any Kleinian surface singularity
  X a partial resolution
  $\rightsquigarrow$ $\Lambda = e\Pi e$ is a contracted preprojective algebra
  (Π affine preprojective algebra)

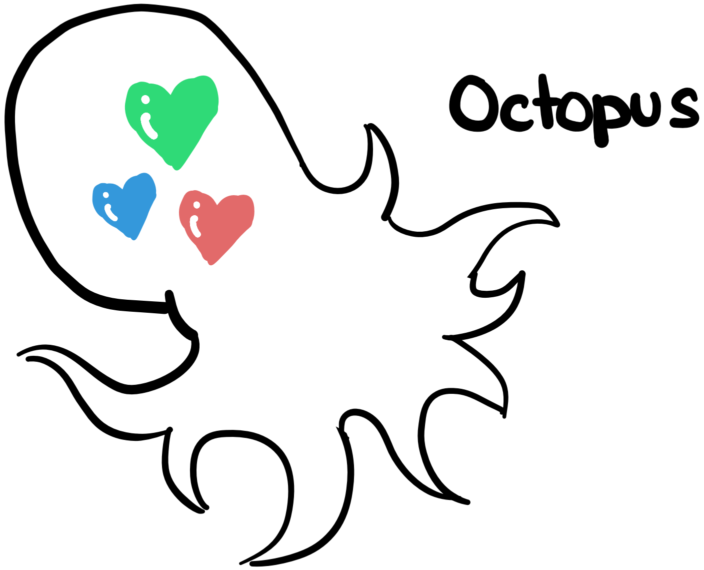
- $X \to Z$ a flopping contraction with Z the germ of a 3-fold point.

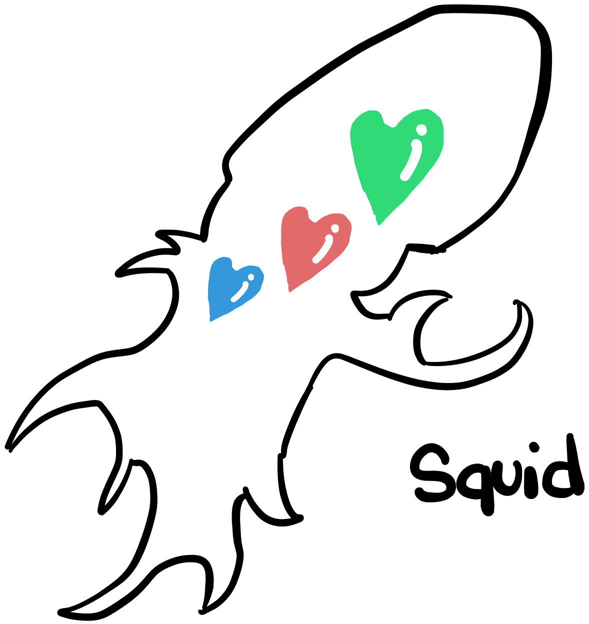
Eg

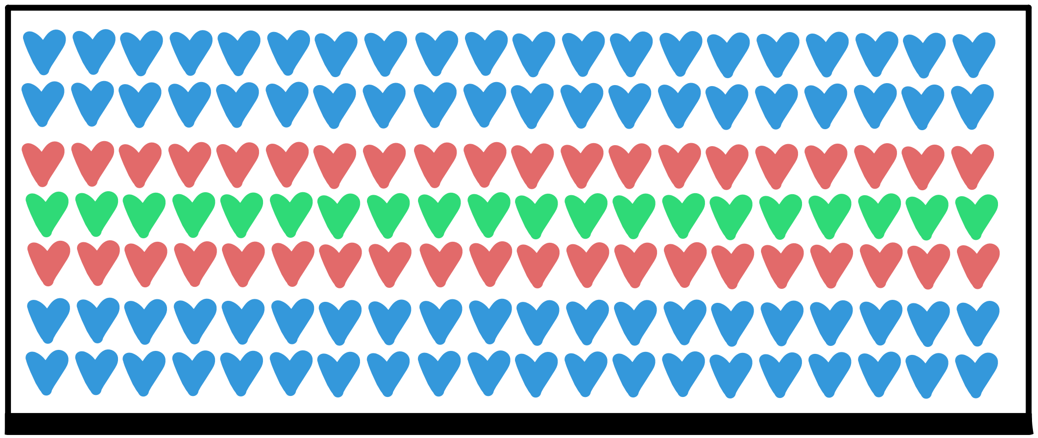


(or a 3-fold map with this generic slice)

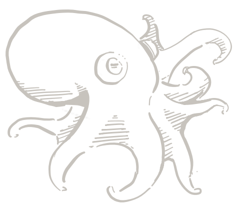








$D^{fl}\bmod\Lambda$



*Pictures may be anatomically inaccurate.