3.2 The Product and Quotient Rules

1. Product Rule: $f(x) = (1 + 2x^2)(x - x^2) \quad \Rightarrow$

$$f'(x) = (1 + 2x^2)(1 - 2x) + (x - x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1 + 2x^2)(x - x^2) = x - x^2 + 2x^3 - 2x^4 \quad \Rightarrow \quad f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

2. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \quad \Rightarrow \quad F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $f(x) = (3x^2 - 5x)e^x \quad \Rightarrow$

$$f'(x) = (3x^2 - 5x)(e^x)' + e^x(3x^2 - 5x)' = (3x^2 - 5x)e^x + e^x(6x - 5)$$
$$= e^x[(3x^2 - 5x) + (6x - 5)] = e^x(3x^2 + x - 5)$$

4. By the Product Rule, $g(x) = (x + 2\sqrt{x}\,)\,e^x \Rightarrow$

$$g'(x) = (x + 2\sqrt{x}\,)(e^x)' + e^x(x + 2\sqrt{x}\,)' = (x + 2\sqrt{x}\,)e^x + e^x\left(1 + 2 \cdot \tfrac{1}{2}x^{-1/2}\right)$$
$$= e^x\left[(x + 2\sqrt{x}\,) + \left(1 + 1/\sqrt{x}\,\right)\right] = e^x\left(x + 2\sqrt{x} + 1 + 1/\sqrt{x}\,\right)$$

5. By the Quotient Rule, $y = \dfrac{x}{e^x} \quad \Rightarrow \quad y' = \dfrac{e^x(1) - x(e^x)}{(e^x)^2} = \dfrac{e^x(1 - x)}{(e^x)^2} = \dfrac{1 - x}{e^x}$.

6. By the Quotient Rule, $y = \dfrac{e^x}{1 - e^x} \quad \Rightarrow \quad y' = \dfrac{(1 - e^x)e^x - e^x(-e^x)}{(1 - e^x)^2} = \dfrac{e^x - e^{2x} + e^{2x}}{(1 - e^x)^2} = \dfrac{e^x}{(1 - e^x)^2}$.

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

7. $g(x) = \dfrac{1 + 2x}{3 - 4x} \quad \overset{\text{QR}}{\Rightarrow} \quad g'(x) = \dfrac{(3 - 4x)(2) - (1 + 2x)(-4)}{(3 - 4x)^2} = \dfrac{6 - 8x + 4 + 8x}{(3 - 4x)^2} = \dfrac{10}{(3 - 4x)^2}$

8. $G(x) = \dfrac{x^2 - 2}{2x + 1} \quad \overset{\text{QR}}{\Rightarrow} \quad G'(x) = \dfrac{(2x + 1)(2x) - (x^2 - 2)(2)}{(2x + 1)^2} = \dfrac{4x^2 + 2x - 2x^2 + 4}{(2x + 1)^2} = \dfrac{2x^2 + 2x + 4}{(2x + 1)^2}$

9. $H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) \quad \overset{\text{PR}}{\Rightarrow}$

$$H'(u) = (u - \sqrt{u}\,)\left(1 + \frac{1}{2\sqrt{u}}\right) + (u + \sqrt{u}\,)\left(1 - \frac{1}{2\sqrt{u}}\right) = u + \tfrac{1}{2}\sqrt{u} - \sqrt{u} - \tfrac{1}{2} + u - \tfrac{1}{2}\sqrt{u} + \sqrt{u} - \tfrac{1}{2} = 2u - 1.$$

An easier method is to simplify first and then differentiate as follows:

$$H(u) = (u - \sqrt{u}\,)(u + \sqrt{u}\,) = u^2 - (\sqrt{u}\,)^2 = u^2 - u \quad \Rightarrow \quad H'(u) = 2u - 1$$

10. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2}) \quad \overset{\text{PR}}{\Rightarrow}$

$$J'(v) = (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (v^{-4} + v^{-2})(3v^2 - 2)$$
$$= -4v^{-2} - 2v^0 + 8v^{-4} + 4v^{-2} + 3v^{-2} - 2v^{-4} + 3v^0 - 2v^{-2} = 1 + v^{-2} + 6v^{-4}$$

11. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \quad \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$
$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$
$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

12. $f(z) = (1 - e^z)(z + e^z) \quad \overset{\text{PR}}{\Rightarrow}$

$$f'(z) = (1 - e^z)(1 + e^z) + (z + e^z)(-e^z) = 1^2 - (e^z)^2 - ze^z - (e^z)^2 = 1 - ze^z - 2e^{2z}$$

13. $y = \dfrac{x^2+1}{x^3-1} \quad \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(x^3-1)(2x)-(x^2+1)(3x^2)}{(x^3-1)^2} = \frac{x\left[(x^3-1)(2)-(x^2+1)(3x)\right]}{(x^3-1)^2} = \frac{x(2x^3-2-3x^3-3x)}{(x^3-1)^2} = \frac{x(-x^3-3x-2)}{(x^3-1)^2}$$

14. $y = \dfrac{\sqrt{x}}{2+x} \quad \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(2+x)\left(\dfrac{1}{2\sqrt{x}}\right)-\sqrt{x}\,(1)}{(2+x)^2} = \frac{\dfrac{1}{\sqrt{x}}+\dfrac{\sqrt{x}}{2}-\sqrt{x}}{(2+x)^2} = \frac{\dfrac{2+x-2x}{2\sqrt{x}}}{(2+x)^2} = \frac{2-x}{2\sqrt{x}(2+x)^2}$$

15. $y = \dfrac{t^3+3t}{t^2-4t+3} \quad \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(t^2-4t+3)(3t^2+3)-(t^3+3t)(2t-4)}{(t^2-4t+3)^2}$$

$$= \frac{3t^4+3t^2-12t^3-12t+9t^2+9-(2t^4-4t^3+6t^2-12t)}{(t^2-4t+3)^2} = \frac{t^4-8t^3+6t^2+9}{(t^2-4t+3)^2}$$

16. $y = \dfrac{1}{t^3+2t^2-1} \quad \overset{\text{QR}}{\Rightarrow} \quad y' = \dfrac{(t^3+2t^2-1)(0)-1(3t^2+4t)}{(t^3+2t^2-1)^2} = -\dfrac{3t^2+4t}{(t^3+2t^2-1)^2}$

17. $y = e^p(p+p\sqrt{p}\,) = e^p(p+p^{3/2}) \quad \overset{\text{PR}}{\Rightarrow} \quad y' = e^p\left(1+\frac{3}{2}p^{1/2}\right)+(p+p^{3/2})e^p = e^p\left(1+\frac{3}{2}\sqrt{p}+p+p\sqrt{p}\right)$

18. $h(r) = \dfrac{ae^r}{b+e^r} \quad \overset{\text{QR}}{\Rightarrow} \quad h'(r) = \dfrac{(b+e^r)(ae^r)-(ae^r)(e^r)}{(b+e^r)^2} = \dfrac{abe^r+ae^{2r}-ae^{2r}}{(b+e^r)^2} = \dfrac{abe^r}{(b+e^r)^2}$

19. $y = \dfrac{s-\sqrt{s}}{s^2} = \dfrac{s}{s^2}-\dfrac{\sqrt{s}}{s^2} = s^{-1}-s^{-3/2} \quad \Rightarrow \quad y' = -s^{-2}+\frac{3}{2}s^{-5/2} = \dfrac{-1}{s^2}+\dfrac{3}{2s^{5/2}} = \dfrac{3-2\sqrt{s}}{2s^{5/2}}$

20. $y = (z^2+e^z)\sqrt{z} \quad \overset{\text{PR}}{\Rightarrow}$

$$y' = (z^2+e^z)\left(\frac{1}{2\sqrt{z}}\right)+\sqrt{z}\,(2z+e^z) = \frac{z^2}{2\sqrt{z}}+\frac{e^z}{2\sqrt{z}}+2z\sqrt{z}+\sqrt{z}\,e^z$$

$$= \frac{z^2+e^z+4z^2+2ze^z}{2\sqrt{z}} = \frac{5z^2+e^z+2ze^z}{2\sqrt{z}}$$

21. $f(t) = \dfrac{\sqrt[3]{t}}{t-3} \quad \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(t-3)\left(\frac{1}{3}t^{-2/3}\right)-t^{1/3}(1)}{(t-3)^2} = \frac{\frac{1}{3}t^{1/3}-t^{-2/3}-t^{1/3}}{(t-3)^2} = \frac{-\frac{2}{3}t^{1/3}-t^{-2/3}}{(t-3)^2} = \frac{\dfrac{-2t}{3t^{2/3}}-\dfrac{3}{3t^{2/3}}}{(t-3)^2} = \frac{-2t-3}{3t^{2/3}(t-3)^2}$$

22. $V(t) = \dfrac{4+t}{te^t} \quad \overset{\text{QR}}{\Rightarrow}$

$$V'(t) = \frac{te^t(1)-(4+t)(te^t+e^t(1))}{(te^t)^2} = \frac{te^t-4te^t-4e^t-t^2e^t-te^t}{t^2e^{2t}}$$

$$= \frac{-4te^t-4e^t-t^2e^t}{t^2e^{2t}} = \frac{-e^t(t^2+4t+4)}{t^2e^{2t}} = -\frac{(t+2)^2}{t^2e^t}$$

23. $f(x) = \dfrac{x^2e^x}{x^2+e^x} \overset{\text{QR}}{\Rightarrow}$

$$f'(x) = \frac{(x^2+e^x)\left[x^2e^x+e^x(2x)\right]-x^2e^x(2x+e^x)}{(x^2+e^x)^2} = \frac{x^4e^x+2x^3e^x+x^2e^{2x}+2xe^{2x}-2x^3e^x-x^2e^{2x}}{(x^2+e^x)^2}$$

$$= \frac{x^4e^x+2xe^{2x}}{(x^2+e^x)^2} = \frac{xe^x(x^3+2e^x)}{(x^2+e^x)^2}$$

24. $F(t) = \dfrac{At}{Bt^2+Ct^3} = \dfrac{A}{Bt+Ct^2} \overset{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt+Ct^2)(0)-A(B+2Ct)}{(Bt+Ct^2)^2} = \frac{-A(B+2Ct)}{(t)^2(B+Ct)^2} = -\frac{A(B+2Ct)}{t^2(B+Ct)^2}$$

25. $f(x) = \dfrac{x}{x+c/x} \quad\Rightarrow\quad f'(x) = \dfrac{(x+c/x)(1)-x(1-c/x^2)}{\left(x+\dfrac{c}{x}\right)^2} = \dfrac{x+c/x-x+c/x}{\left(\dfrac{x^2+c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2+c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2+c)^2}$

26. $f(x) = \dfrac{ax+b}{cx+d} \quad\Rightarrow\quad f'(x) = \dfrac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} = \dfrac{acx+ad-acx-bc}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$

27. $f(x) = (x^3+1)e^x \overset{\text{PR}}{\Rightarrow}$

$$f'(x) = (x^3+1)e^x + e^x(3x^2) = e^x\left[(x^3+1)+3x^2\right] = e^x(x^3+3x^2+1) \overset{\text{PR}}{\Rightarrow}$$

$$f''(x) = e^x(3x^2+6x)+(x^3+3x^2+1)e^x = e^x\left[(3x^2+6x)+(x^3+3x^2+1)\right] = e^x(x^3+6x^2+6x+1)$$

28. $f(x) = \sqrt{x}\,e^x \overset{\text{PR}}{\Rightarrow} f'(x) = \sqrt{x}\,e^x + e^x\left(\dfrac{1}{2\sqrt{x}}\right) = \left(\sqrt{x}+\dfrac{1}{2\sqrt{x}}\right)e^x = \dfrac{2x+1}{2\sqrt{x}}\,e^x.$

Using the Product Rule and $f'(x) = \left(x^{1/2}+\frac{1}{2}x^{-1/2}\right)e^x$, we get

$$f''(x) = \left(x^{1/2}+\tfrac{1}{2}x^{-1/2}\right)e^x + e^x\left(\tfrac{1}{2}x^{-1/2}-\tfrac{1}{4}x^{-3/2}\right) = \left(x^{1/2}+x^{-1/2}-\tfrac{1}{4}x^{-3/2}\right)e^x = \frac{4x^2+4x-1}{4x^{3/2}}\,e^x$$

29. $f(x) = \dfrac{x^2}{1+e^x} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(1+e^x)(2x)-x^2(e^x)}{(1+e^x)^2} = \dfrac{x[(1+e^x)2-xe^x]}{(1+e^x)^2} = \dfrac{x(2+2e^x-xe^x)}{(1+e^x)^2}.$

Using the Quotient and Product Rules and $f'(x) = \dfrac{2x+2xe^x-x^2e^x}{(1+e^x)^2}$, we get

$$f''(x) = \frac{(1+e^x)^2\left[2+2(xe^x+e^x)-(x^2e^x+2xe^x)\right]-(2x+2xe^x-x^2e^x)\left[(1+e^x)e^x+(1+e^x)e^x\right]}{[(1+e^x)^2]^2}$$

$$= \frac{(1+e^x)\left\{\left[(1+e^x)(2+2xe^x+2e^x-x^2e^x-2xe^x)\right]-(2x+2xe^x-x^2e^x)(2e^x)\right\}}{(1+e^x)^4}$$

$$= \frac{(1+e^x)(2+2e^x-x^2e^x)-4xe^x-4xe^{2x}+2x^2e^{2x}}{(1+e^x)^3}$$

$$= \frac{2+2e^x-x^2e^x+2e^x+2e^{2x}-x^2e^{2x}-4xe^x-4xe^{2x}+2x^2e^{2x}}{(1+e^x)^3}$$

$$= \frac{2+4e^x-x^2e^x-4xe^x+2e^{2x}+x^2e^{2x}-4xe^{2x}}{(1+e^x)^3}$$

30. $f(x) = \dfrac{x}{x^2 - 1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = \dfrac{x^2 - 1 - 2x^2}{(x^2 - 1)^2} = \dfrac{-x^2 - 1}{(x^2 - 1)^2} \quad \Rightarrow$

$$f''(x) = \frac{(x^2 - 1)^2(-2x) - (-x^2 - 1)(x^4 - 2x^2 + 1)'}{[(x^2 - 1)^2]^2} = \frac{(x^2 - 1)^2(-2x) + (x^2 + 1)(4x^3 - 4x)}{(x^2 - 1)^4}$$

$$= \frac{(x^2 - 1)^2(-2x) + (x^2 + 1)(4x)(x^2 - 1)}{(x^2 - 1)^4} = \frac{(x^2 - 1)[(x^2 - 1)(-2x) + (x^2 + 1)(4x)]}{(x^2 - 1)^4}$$

$$= \frac{-2x^3 + 2x + 4x^3 + 4x}{(x^2 - 1)^3} = \frac{2x^3 + 6x}{(x^2 - 1)^3}$$

31. $y = \dfrac{x^2 - 1}{x^2 + x + 1} \quad \Rightarrow$

$$y' = \frac{(x^2 + x + 1)(2x) - (x^2 - 1)(2x + 1)}{(x^2 + x + 1)^2} = \frac{2x^3 + 2x^2 + 2x - 2x^3 - x^2 + 2x + 1}{(x^2 + x + 1)^2} = \frac{x^2 + 4x + 1}{(x^2 + x + 1)^2}.$$

At $(1, 0)$, $y' = \dfrac{6}{3^2} = \dfrac{2}{3}$, and an equation of the tangent line is $y - 0 = \frac{2}{3}(x - 1)$, or $y = \frac{2}{3}x - \frac{2}{3}$.

32. $y = \dfrac{1 + x}{1 + e^x} \quad \Rightarrow \quad y' = \dfrac{(1 + e^x)(1) - (1 + x)e^x}{(1 + e^x)^2} = \dfrac{1 + e^x - e^x - xe^x}{(1 + e^x)^2} = \dfrac{1 - xe^x}{(1 + e^x)^2}.$

At $\left(0, \frac{1}{2}\right)$, $y' = \dfrac{1}{(1 + 1)^2} = \dfrac{1}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{1}{4}(x - 0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

33. $y = 2xe^x \quad \Rightarrow \quad y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x + 1)$.

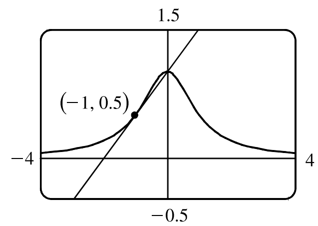
At $(0, 0)$, $y' = 2e^0(0 + 1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x - 0)$, or $y = 2x$. The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x - 0)$, or $y = -\frac{1}{2}x$.

34. $y = \dfrac{2x}{x^2 + 1} \quad \Rightarrow \quad y' = \dfrac{(x^2 + 1)(2) - 2x(2x)}{(x^2 + 1)^2} = \dfrac{2 - 2x^2}{(x^2 + 1)^2}$. At $(1, 1)$, $y' = 0$, and an equation of the tangent line is $y - 1 = 0(x - 1)$, or $y = 1$. The slope of the normal line is undefined, so an equation of the normal line is $x = 1$.

35. (a) $y = f(x) = \dfrac{1}{1 + x^2} \quad \Rightarrow$

$f'(x) = \dfrac{(1 + x^2)(0) - 1(2x)}{(1 + x^2)^2} = \dfrac{-2x}{(1 + x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x + 1)$ or $y = \frac{1}{2}x + 1$.

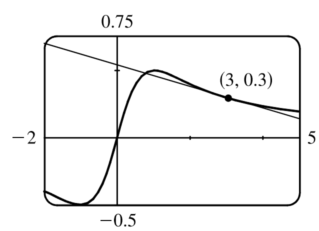
(b)



36. (a) $y = f(x) = \dfrac{x}{1 + x^2} \quad \Rightarrow$

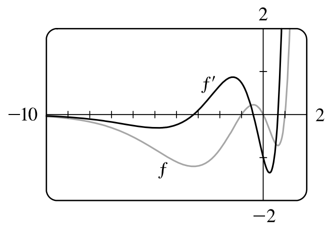
$f'(x) = \dfrac{(1 + x^2)1 - x(2x)}{(1 + x^2)^2} = \dfrac{1 - x^2}{(1 + x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x - 3)$ or $y = -0.08x + 0.54$.

(b)



37. (a) $f(x) = (x^3 - x)e^x \quad \Rightarrow \quad f'(x) = (x^3 - x)e^x + e^x(3x^2 - 1) = e^x(x^3 + 3x^2 - x - 1)$

(b)

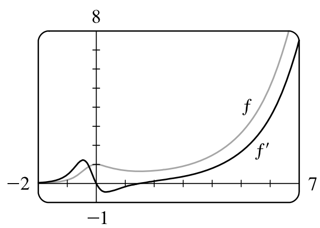


$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

38. (a) $f(x) = \dfrac{e^x}{2x^2 + x + 1} \quad \Rightarrow$

$$f'(x) = \frac{(2x^2 + x + 1)e^x - e^x(4x + 1)}{(2x^2 + x + 1)^2} = \frac{e^x(2x^2 + x + 1 - 4x - 1)}{(2x^2 + x + 1)^2} = \frac{e^x(2x^2 - 3x)}{(2x^2 + x + 1)^2}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

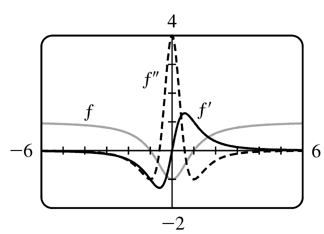
39. (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1} \quad \Rightarrow$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{(2x)[(x^2 + 1) - (x^2 - 1)]}{(x^2 + 1)^2} = \frac{(2x)(2)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} \quad \Rightarrow$$

$$f''(x) = \frac{(x^2 + 1)^2(4) - 4x(x^4 + 2x^2 + 1)'}{[(x^2 + 1)^2]^2} = \frac{4(x^2 + 1)^2 - 4x(4x^3 + 4x)}{(x^2 + 1)^4}$$

$$= \frac{4(x^2 + 1)^2 - 16x^2(x^2 + 1)}{(x^2 + 1)^4} = \frac{4(x^2 + 1)[(x^2 + 1) - 4x^2]}{(x^2 + 1)^4} = \frac{4(1 - 3x^2)}{(x^2 + 1)^3}$$

(b)

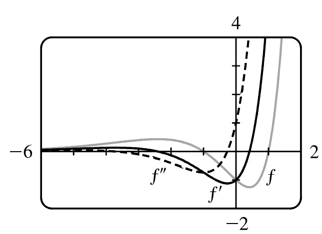


$f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

40. (a) $f(x) = (x^2 - 1)e^x \quad \Rightarrow \quad f'(x) = (x^2 - 1)e^x + e^x(2x) = e^x(x^2 + 2x - 1) \quad \Rightarrow$

$f''(x) = e^x(2x + 2) + (x^2 + 2x - 1)e^x = e^x(x^2 + 4x + 1)$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

41. $f(x) = \dfrac{x^2}{1+x} \quad \Rightarrow \quad f'(x) = \dfrac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \dfrac{2x + 2x^2 - x^2}{(1+x)^2} = \dfrac{x^2+2x}{x^2+2x+1} \quad \Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2) - (x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$
$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

42. $g(x) = \dfrac{x}{e^x} \quad \Rightarrow \quad g'(x) = \dfrac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x} \quad \Rightarrow$

$g''(x) = \dfrac{e^x \cdot (-1) - (1-x)e^x}{(e^x)^2} = \dfrac{e^x[-1-(1-x)]}{(e^x)^2} = \dfrac{x-2}{e^x} \quad \Rightarrow$

$g'''(x) = \dfrac{e^x \cdot 1 - (x-2)e^x}{(e^x)^2} = \dfrac{e^x[1-(x-2)]}{(e^x)^2} = \dfrac{3-x}{e^x} \quad \Rightarrow$

$g^{(4)}(x) = \dfrac{e^x \cdot (-1) - (3-x)e^x}{(e^x)^2} = \dfrac{e^x[-1-(3-x)]}{(e^x)^2} = \dfrac{x-4}{e^x}$.

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

43. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

44. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

(a) $h(x) = 3f(x) + 8g(x) \quad \Rightarrow \quad h'(x) = 3f'(x) + 8g'(x)$, so

$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6$.

(b) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, so

$h'(4) = f(4)\,g'(4) + g(4)\,f'(4) = 2(-3) + 5(6) = -6 + 30 = 24$.

(c) $h(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$, so

$h'(4) = \dfrac{g(4)\,f'(4) - f(4)\,g'(4)}{[g(4)]^2} = \dfrac{5(6) - 2(-3)}{5^2} = \dfrac{30+6}{25} = \dfrac{36}{25}$.

(d) $h(x) = \dfrac{g(x)}{f(x) + g(x)} \quad \Rightarrow$

$h'(4) = \dfrac{[f(4) + g(4)]\,g'(4) - g(4)\,[f'(4) + g'(4)]}{[f(4) + g(4)]^2} = \dfrac{(2+5)\,(-3) - 5\,[6 + (-3)]}{(2+5)^2} = \dfrac{-21-15}{7^2} = -\dfrac{36}{49}$

45. $f(x) = e^x g(x) \quad \Rightarrow \quad f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5+2) = 7$

46. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x)\cdot 1}{x^2} \quad \Rightarrow \quad \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

47. $g(x) = xf(x) \quad \Rightarrow \quad g'(x) = xf'(x) + f(x)\cdot 1$. Now $g(3) = 3f(3) = 3\cdot 4 = 12$ and $g'(3) = 3f'(3) + f(3) = 3(-2) + 4 = -2$. Thus, an equation of the tangent line to the graph of g at the point where $x = 3$ is $y - 12 = -2(x-3)$, or $y = -2x + 18$.

48. $f'(x) = x^2 f(x) \quad \Rightarrow \quad f''(x) = x^2 f'(x) + f(x)\cdot 2x$. Now $f'(2) = 2^2 f(2) = 4(10) = 40$, so $f''(2) = 2^2(40) + 10(4) = 200$.

49. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1,2)$ is on the graph of f; $g(1) = 1$ since the point $(1,1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0,0)$ and $(2,4)$ is $\dfrac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2,4)$ and $(2,0)$ is $\dfrac{0-4}{2-(-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2\cdot(-1) + 1\cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3\cdot\frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

50. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3\cdot\frac{2}{4} + 2\cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1\cdot\frac{1}{4} - 5\cdot\left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

51. (a) $y = xg(x) \quad \Rightarrow \quad y' = xg'(x) + g(x)\cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad \Rightarrow \quad y' = \dfrac{g(x)\cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad \Rightarrow \quad y' = \dfrac{xg'(x) - g(x)\cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

52. (a) $y = x^2 f(x) \quad \Rightarrow \quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad \Rightarrow \quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

$$= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x}\cdot\frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

53. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

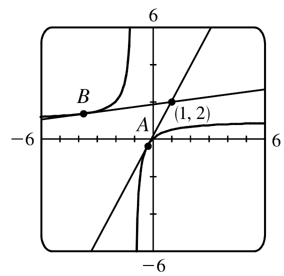
$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$f\left(-2 \pm \sqrt{3}\right) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$
$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

54. $y = \dfrac{x-1}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects

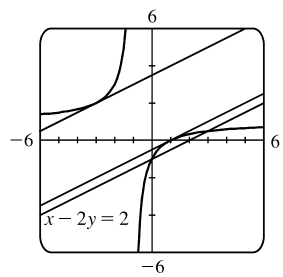
the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to

$x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \quad \Rightarrow$

$(a+1)^2 = 4 \quad \Rightarrow \quad a + 1 = \pm 2 \quad \Rightarrow \quad a = 1$ or -3. When $a = 1$, $y = 0$ and the

equation of the tangent is $y - 0 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and

the equation of the tangent is $y - 2 = \frac{1}{2}(x+3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.



55. $R = \dfrac{f}{g} \quad \Rightarrow \quad R' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = x - 3x^3 + 5x^5$, $f'(x) = 1 - 9x^2 + 25x^4$,

and for $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$, $g'(x) = 9x^2 + 36x^5 + 81x^8$.

Thus, $R'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = 1$.

56. $Q = \dfrac{f}{g} \quad \Rightarrow \quad Q' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = 1 + x + x^2 + xe^x$, $f'(x) = 1 + 2x + xe^x + e^x$,

and for $g(x) = 1 - x + x^2 - xe^x$, $g'(x) = -1 + 2x - xe^x - e^x$.

Thus, $Q'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 2 - 1 \cdot (-2)}{1^2} = \dfrac{4}{1} = 4$.

57. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad \Rightarrow$

$$T'(1999) = P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr})$$
$$= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

58. (a) $f(20) = 10{,}000$ means that when the price of the fabric is $\$20/\text{yard}$, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past $\$20/\text{yard}$, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \quad\Rightarrow\quad R'(p) = pf'(p) + f(p)\cdot 1 \quad\Rightarrow\quad R'(20) = 20f'(20) + f(20)\cdot 1 = 20(-350) + 10{,}000 = 3000.$

This means that as the price of the fabric increases past $\$20/\text{yard}$, the total revenue is increasing at $\$3000/(\$/\text{yard})$. Note that the Product Rule indicates that we will lose $\$7000/(\$/\text{yard})$ due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

59. $v = \dfrac{0.14[\text{S}]}{0.015 + [\text{S}]} \quad\Rightarrow\quad \dfrac{dv}{d[\text{S}]} = \dfrac{(0.015 + [\text{S}])(0.14) - (0.14[\text{S}])(1)}{(0.015 + [\text{S}])^2} = \dfrac{0.0021}{(0.015 + [\text{S}])^2}.$

$dv/d[\text{S}]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

60. $B(t) = N(t)\,M(t) \quad\Rightarrow\quad B'(t) = N(t)\,M'(t) + M(t)\,N'(t)$, so

$B'(4) = N(4)\,M'(4) + M(4)\,N'(4) = 820(0.14) + 1.2(50) = 174.8\ \text{g/week}.$

61. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x).$

(c) $\dfrac{d}{dx}\left(e^{3x}\right) = \dfrac{d}{dx}\left(e^x\right)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

62. (a) We use the Product Rule repeatedly: $F = fg \quad\Rightarrow\quad F' = f'g + fg' \quad\Rightarrow$

$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''.$

(b) $F''' = f'''g + f''g' + 2\,(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \quad\Rightarrow$

$$\begin{aligned} F^{(4)} &= f^{(4)}g + f'''g' + 3\,(f'''g' + f''g'') + 3\,(f''g'' + f'g''') + f'g''' + fg^{(4)} \\ &= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)} \end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2} f^{(n-2)}g'' + \cdots + \binom{n}{k} f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$

63. For $f(x) = x^2 e^x$, $f'(x) = x^2 e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$f''(x) = e^x(x^2 + 4x + 2)$$
$$f'''(x) = e^x(x^2 + 6x + 6)$$
$$f^{(4)}(x) = e^x(x^2 + 8x + 12)$$
$$f^{(5)}(x) = e^x(x^2 + 10x + 20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then
$$\begin{aligned} f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] &= e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$
This shows that S_{k+1} is true.
3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

64. (a) $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = \dfrac{g(x) \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$ [Quotient Rule] $= \dfrac{g(x) \cdot 0 - 1 \cdot g'(x)}{[g(x)]^2} = \dfrac{0 - g'(x)}{[g(x)]^2} = -\dfrac{g'(x)}{[g(x)]^2}$

(b) $\dfrac{d}{dt}\left(\dfrac{1}{t^3 + 2t^2 - 1}\right) = -\dfrac{\left(t^3 + 2t^2 - 1\right)'}{\left(t^3 + 2t^2 - 1\right)^2} = -\dfrac{3t^2 + 4t}{\left(t^3 + 2t^2 - 1\right)^2}$

(c) $\dfrac{d}{dx}\left(x^{-n}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = -\dfrac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $= -\dfrac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$

3.3 Derivatives of Trigonometric Functions

1. $f(x) = x^2 \sin x \overset{\text{PR}}{\Rightarrow} f'(x) = x^2 \cos x + (\sin x)(2x) = x^2 \cos x + 2x \sin x$

2. $f(x) = x \cos x + 2 \tan x \Rightarrow f'(x) = x(-\sin x) + (\cos x)(1) + 2\sec^2 x = \cos x - x \sin x + 2\sec^2 x$

3. $f(x) = e^x \cos x \Rightarrow f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$

4. $y = 2\sec x - \csc x \Rightarrow y' = 2(\sec x \tan x) - (-\csc x \cot x) = 2\sec x \tan x + \csc x \cot x$

5. $y = \sec\theta \tan\theta \Rightarrow y' = \sec\theta\,(\sec^2\theta) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\,(\sec^2\theta + \tan^2\theta)$. Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as $\sec\theta\,(1 + 2\tan^2\theta)$ or $\sec\theta\,(2\sec^2\theta - 1)$.

6. $g(\theta) = e^\theta(\tan\theta - \theta) \Rightarrow g'(\theta) = e^\theta(\sec^2\theta - 1) + (\tan\theta - \theta)e^\theta = e^\theta(\sec^2\theta - 1 + \tan\theta - \theta)$