# 2 □ LIMITS AND DERIVATIVES

## 2.1 The Tangent and Velocity Problems

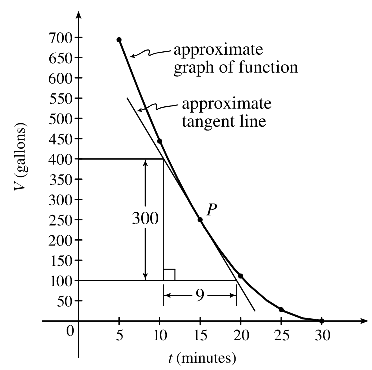
**1.** (a) Using $P(15, 250)$, we construct the following table:

| $t$ | $Q$ | slope $= m_{PQ}$ |
|---|---|---|
| 5 | $(5, 694)$ | $\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$ |
| 10 | $(10, 444)$ | $\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$ |
| 20 | $(20, 111)$ | $\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$ |
| 25 | $(25, 28)$ | $\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$ |
| 30 | $(30, 0)$ | $\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$ |

(b) Using the values of $t$ that correspond to the points closest to $P$ ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at $P$ to be $\frac{-300}{9} = -33.\overline{3}$.




**2.** (a) Slope $= \frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

**3.** (a) $y = \dfrac{1}{1-x}$, $P(2, -1)$

| | $x$ | $Q(x, 1/(1-x))$ | $m_{PQ}$ |
|---|---|---|---|
| (i) | 1.5 | $(1.5, -2)$ | 2 |
| (ii) | 1.9 | $(1.9, -1.111\,111)$ | 1.111 111 |
| (iii) | 1.99 | $(1.99, -1.010\,101)$ | 1.010 101 |
| (iv) | 1.999 | $(1.999, -1.001\,001)$ | 1.001 001 |
| (v) | 2.5 | $(2.5, -0.666\,667)$ | 0.666 667 |
| (vi) | 2.1 | $(2.1, -0.909\,091)$ | 0.909 091 |
| (vii) | 2.01 | $(2.01, -0.990\,099)$ | 0.990 099 |
| (viii) | 2.001 | $(2.001, -0.999\,001)$ | 0.999 001 |

(b) The slope appears to be 1.

(c) Using $m = 1$, an equation of the tangent line to the curve at $P(2, -1)$ is $y - (-1) = 1(x - 2)$, or $y = x - 3$.

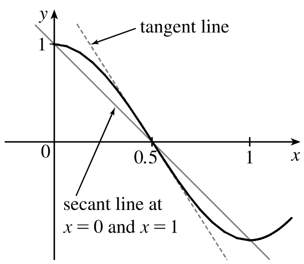
**4.** (a) $y = \cos \pi x$, $P(0.5, 0)$

| | $x$ | $Q$ | $m_{PQ}$ |
|---|---|---|---|
| (i) | 0 | $(0, 1)$ | $-2$ |
| (ii) | 0.4 | $(0.4, 0.309017)$ | $-3.090170$ |
| (iii) | 0.49 | $(0.49, 0.031411)$ | $-3.141076$ |
| (iv) | 0.499 | $(0.499, 0.003142)$ | $-3.141587$ |
| (v) | 1 | $(1, -1)$ | $-2$ |
| (vi) | 0.6 | $(0.6, -0.309017)$ | $-3.090170$ |
| (vii) | 0.51 | $(0.51, -0.031411)$ | $-3.141076$ |
| (viii) | 0.501 | $(0.501, -0.003142)$ | $-3.141587$ |

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)




**5.** (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ ($h$ approaches 0) is $-24$ ft/s.

**6.** (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{ave}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{ave}} = 4.42$ m/s

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = 5.35$ m/s

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = 6.094$ m/s

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{ave}} = 6.2614$ m/s

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{ave}} = 6.27814$ m/s

(b) The instantaneous velocity when $t = 1$ ($h$ approaches 0) is 6.28 m/s.

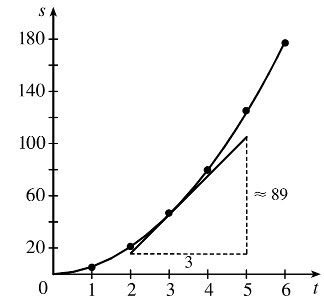
**7.** (a) (i) On the interval $[2, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(2)}{4 - 2} = \dfrac{79.2 - 20.6}{2} = 29.3$ ft/s.

(ii) On the interval $[3, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{79.2 - 46.5}{1} = 32.7$ ft/s.

(iii) On the interval $[4, 5]$, $v_{\text{ave}} = \dfrac{s(5) - s(4)}{5 - 4} = \dfrac{124.8 - 79.2}{1} = 45.6$ ft/s.

(iv) On the interval $[4, 6]$, $v_{\text{ave}} = \dfrac{s(6) - s(4)}{6 - 4} = \dfrac{176.7 - 79.2}{2} = 48.75$ ft/s.

(b) Using the points $(2, 16)$ and $(5, 105)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\dfrac{105 - 16}{5 - 2} = \dfrac{89}{3} \approx 29.7$ ft/s.




**8.** (a) (i) $s = s(t) = 2\sin \pi t + 3\cos \pi t$. On the interval $[1, 2]$, $v_{\text{ave}} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{3 - (-3)}{1} = 6$ cm/s.

(ii) On the interval $[1, 1.1]$, $v_{\text{ave}} = \dfrac{s(1.1) - s(1)}{1.1 - 1} \approx \dfrac{-3.471 - (-3)}{0.1} = -4.71$ cm/s.

(iii) On the interval $[1, 1.01]$, $v_{\text{ave}} = \dfrac{s(1.01) - s(1)}{1.01 - 1} \approx \dfrac{-3.0613 - (-3)}{0.01} = -6.13$ cm/s.

(iv) On the interval $[1, 1.001]$, $v_{\text{ave}} = \dfrac{s(1.001) - s(1)}{1.001 - 1} \approx \dfrac{-3.00627 - (-3)}{0.001} = -6.27$ cm/s.

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about $-6.3$ cm/s.

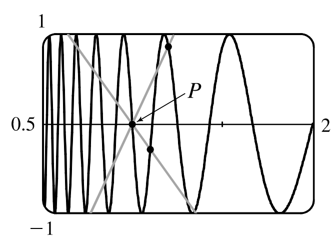
**9.** (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

| $x$ | $Q$ | $m_{PQ}$ |
|---|---|---|
| 2 | $(2, 0)$ | 0 |
| 1.5 | $(1.5, 0.8660)$ | 1.7321 |
| 1.4 | $(1.4, -0.4339)$ | $-1.0847$ |
| 1.3 | $(1.3, -0.8230)$ | $-2.7433$ |
| 1.2 | $(1.2, 0.8660)$ | 4.3301 |
| 1.1 | $(1.1, -0.2817)$ | $-2.8173$ |

| $x$ | $Q$ | $m_{PQ}$ |
|---|---|---|
| 0.5 | $(0.5, 0)$ | 0 |
| 0.6 | $(0.6, 0.8660)$ | $-2.1651$ |
| 0.7 | $(0.7, 0.7818)$ | $-2.6061$ |
| 0.8 | $(0.8, 1)$ | $-5$ |
| 0.9 | $(0.9, -0.3420)$ | 3.4202 |

As $x$ approaches 1, the slopes do not appear to be approaching any particular value.

(b)




We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at $P$ that we need to take $x$-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point $Q$ is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then $Q$ is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is $-31.4108$. So we estimate that the slope of the tangent line at $P$ is about $-31.4$.