2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x\to 2}[f(x)+5g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2}[5g(x)]$ [Limit Law 1]

$= \lim\limits_{x\to 2} f(x) + 5\lim\limits_{x\to 2} g(x)$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim\limits_{x\to 2}[g(x)]^3 = \left[\lim\limits_{x\to 2} g(x)\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim\limits_{x\to 2}\sqrt{f(x)} = \sqrt{\lim\limits_{x\to 2} f(x)}$ [Limit Law 11]

$= \sqrt{4} = 2$

(d) $\lim\limits_{x\to 2}\dfrac{3f(x)}{g(x)} = \dfrac{\lim\limits_{x\to 2}[3f(x)]}{\lim\limits_{x\to 2} g(x)}$ [Limit Law 5]

$= \dfrac{3\lim\limits_{x\to 2} f(x)}{\lim\limits_{x\to 2} g(x)}$ [Limit Law 3]

$= \dfrac{3(4)}{-2} = -6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim\limits_{x\to 2}\dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim\limits_{x\to 2}\dfrac{g(x)\,h(x)}{f(x)} = \dfrac{\lim\limits_{x\to 2}[g(x)\,h(x)]}{\lim\limits_{x\to 2} f(x)}$ [Limit Law 5]

$= \dfrac{\lim\limits_{x\to 2} g(x)\cdot\lim\limits_{x\to 2} h(x)}{\lim\limits_{x\to 2} f(x)}$ [Limit Law 4]

$= \dfrac{-2\cdot 0}{4} = 0$

2. (a) $\lim\limits_{x\to 2}[f(x)+g(x)] = \lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} g(x)$ [Limit Law 1]

$= -1 + 2$

$= 1$

(b) $\lim\limits_{x\to 0} f(x)$ exists, but $\lim\limits_{x\to 0} g(x)$ does not exist, so we cannot apply Limit Law 2 to $\lim\limits_{x\to 0}[f(x)-g(x)]$. The limit does not exist.

(c) $\lim\limits_{x\to -1}[f(x)\,g(x)] = \lim\limits_{x\to -1} f(x)\cdot\lim\limits_{x\to -1} g(x)$ [Limit Law 4]

$= 1\cdot 2$

$= 2$

(d) $\lim\limits_{x\to 3} f(x) = 1$, but $\lim\limits_{x\to 3} g(x) = 0$, so we cannot apply Limit Law 5 to $\lim\limits_{x\to 3}\dfrac{f(x)}{g(x)}$. The limit does not exist.

Note: $\lim\limits_{x\to 3^-}\dfrac{f(x)}{g(x)} = \infty$ since $g(x)\to 0^+$ as $x\to 3^-$ and $\lim\limits_{x\to 3^+}\dfrac{f(x)}{g(x)} = -\infty$ since $g(x)\to 0^-$ as $x\to 3^+$.

Therefore, the limit does not exist, even as an infinite limit.

(e) $\lim\limits_{x\to 2}\left[x^2 f(x)\right] = \lim\limits_{x\to 2} x^2\cdot\lim\limits_{x\to 2} f(x)$ [Limit Law 4]

$= 2^2\cdot(-1)$

$= -4$

(f) $f(-1) + \lim\limits_{x\to -1} g(x)$ is undefined since $f(-1)$ is not defined.

3. $\lim\limits_{x\to 3}(5x^3 - 3x^2 + x - 6) = \lim\limits_{x\to 3}(5x^3) - \lim\limits_{x\to 3}(3x^2) + \lim\limits_{x\to 3} x - \lim\limits_{x\to 3} 6$ [Limit Laws 2 and 1]

$= 5\lim\limits_{x\to 3} x^3 - 3\lim\limits_{x\to 3} x^2 + \lim\limits_{x\to 3} x - \lim\limits_{x\to 3} 6$ [3]

$= 5(3^3) - 3(3^2) + 3 - 6$ [9, 8, and 7]

$= 105$

4. $\displaystyle \lim_{x\to-1}(x^4-3x)(x^2+5x+3) = \lim_{x\to-1}(x^4-3x)\lim_{x\to-1}(x^2+5x+3)$ [Limit Law 4]

$$\begin{aligned}
\lim_{x\to-1}(x^4-3x)(x^2+5x+3) &= \lim_{x\to-1}(x^4-3x)\lim_{x\to-1}(x^2+5x+3) && \text{[Limit Law 4]}\\
&= \left(\lim_{x\to-1}x^4-\lim_{x\to-1}3x\right)\left(\lim_{x\to-1}x^2+\lim_{x\to-1}5x+\lim_{x\to-1}3\right) && \text{[2, 1]}\\
&= \left(\lim_{x\to-1}x^4-3\lim_{x\to-1}x\right)\left(\lim_{x\to-1}x^2+5\lim_{x\to-1}x+\lim_{x\to-1}3\right) && \text{[3]}\\
&= (1+3)(1-5+3) && \text{[9, 8, and 7]}\\
&= 4(-1)=-4
\end{aligned}$$

5.
$$\begin{aligned}
\lim_{t\to-2}\frac{t^4-2}{2t^2-3t+2} &= \frac{\lim\limits_{t\to-2}(t^4-2)}{\lim\limits_{t\to-2}(2t^2-3t+2)} && \text{[Limit Law 5]}\\
&= \frac{\lim\limits_{t\to-2}t^4-\lim\limits_{t\to-2}2}{2\lim\limits_{t\to-2}t^2-3\lim\limits_{t\to-2}t+\lim\limits_{t\to-2}2} && \text{[1, 2, and 3]}\\
&= \frac{16-2}{2(4)-3(-2)+2} && \text{[9, 7, and 8]}\\
&= \frac{14}{16}=\frac{7}{8}
\end{aligned}$$

6.
$$\begin{aligned}
\lim_{u\to-2}\sqrt{u^4+3u+6} &= \sqrt{\lim_{u\to-2}(u^4+3u+6)} && \text{[11]}\\
&= \sqrt{\lim_{u\to-2}u^4+3\lim_{u\to-2}u+\lim_{u\to-2}6} && \text{[1, 2, and 3]}\\
&= \sqrt{(-2)^4+3(-2)+6} && \text{[9, 8, and 7]}\\
&= \sqrt{16-6+6}=\sqrt{16}=4
\end{aligned}$$

7.
$$\begin{aligned}
\lim_{x\to8}(1+\sqrt[3]{x})(2-6x^2+x^3) &= \lim_{x\to8}(1+\sqrt[3]{x})\cdot\lim_{x\to8}(2-6x^2+x^3) && \text{[Limit Law 4]}\\
&= \left(\lim_{x\to8}1+\lim_{x\to8}\sqrt[3]{x}\right)\cdot\left(\lim_{x\to8}2-6\lim_{x\to8}x^2+\lim_{x\to8}x^3\right) && \text{[1, 2, and 3]}\\
&= (1+\sqrt[3]{8})\cdot(2-6\cdot8^2+8^3) && \text{[7, 10, 9]}\\
&= (3)(130)=390
\end{aligned}$$

8.
$$\begin{aligned}
\lim_{t\to2}\left(\frac{t^2-2}{t^3-3t+5}\right)^2 &= \left(\lim_{t\to2}\frac{t^2-2}{t^3-3t+5}\right)^2 && \text{[Limit Law 6]}\\
&= \left(\frac{\lim\limits_{t\to2}(t^2-2)}{\lim\limits_{t\to2}(t^3-3t+5)}\right)^2 && \text{[5]}\\
&= \left(\frac{\lim\limits_{t\to2}t^2-\lim\limits_{t\to2}2}{\lim\limits_{t\to2}t^3-3\lim\limits_{t\to2}t+\lim\limits_{t\to2}5}\right)^2 && \text{[1, 2, and 3]}\\
&= \left(\frac{4-2}{8-3(2)+5}\right)^2 && \text{[9, 7, and 8]}\\
&= \left(\frac{2}{7}\right)^2=\frac{4}{49}
\end{aligned}$$

9. $\displaystyle \lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}}$ [Limit Law 11]

$\displaystyle = \sqrt{\frac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(3x-2)}}$ [5]

$\displaystyle = \sqrt{\frac{2\lim\limits_{x\to 2}x^2+\lim\limits_{x\to 2}1}{3\lim\limits_{x\to 2}x-\lim\limits_{x\to 2}2}}$ [1, 2, and 3]

$\displaystyle = \sqrt{\frac{2(2)^2+1}{3(2)-2}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$ [9, 8, and 7]

10. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x\to a} f(x)$, we never consider $x = a$.

11. $\displaystyle \lim_{x\to 5}\frac{x^2-6x+5}{x-5} = \lim_{x\to 5}\frac{(x-5)(x-1)}{x-5} = \lim_{x\to 5}(x-1) = 5-1 = 4$

12. $\displaystyle \lim_{x\to -3}\frac{x^2+3x}{x^2-x-12} = \lim_{x\to -3}\frac{x(x+3)}{(x-4)(x+3)} = \lim_{x\to -3}\frac{x}{x-4} = \frac{-3}{-3-4} = \frac{3}{7}$

13. $\displaystyle \lim_{x\to 5}\frac{x^2-5x+6}{x-5}$ does not exist since $x-5 \to 0$, but $x^2-5x+6 \to 6$ as $x \to 5$.

14. $\displaystyle \lim_{x\to 4}\frac{x^2+3x}{x^2-x-12} = \lim_{x\to 4}\frac{x(x+3)}{(x-4)(x+3)} = \lim_{x\to 4}\frac{x}{x-4}$. The last limit does not exist since $\displaystyle \lim_{x\to 4^-}\frac{x}{x-4} = -\infty$ and $\displaystyle \lim_{x\to 4^+}\frac{x}{x-4} = \infty$.

15. $\displaystyle \lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3} = \lim_{t\to -3}\frac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t\to -3}\frac{t-3}{2t+1} = \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}$

16. $\displaystyle \lim_{x\to -1}\frac{2x^2+3x+1}{x^2-2x-3} = \lim_{x\to -1}\frac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim_{x\to -1}\frac{2x+1}{x-3} = \frac{2(-1)+1}{-1-3} = \frac{-1}{-4} = \frac{1}{4}$

17. $\displaystyle \lim_{h\to 0}\frac{(-5+h)^2-25}{h} = \lim_{h\to 0}\frac{(25-10h+h^2)-25}{h} = \lim_{h\to 0}\frac{-10h+h^2}{h} = \lim_{h\to 0}\frac{h(-10+h)}{h} = \lim_{h\to 0}(-10+h) = -10$

18. $\displaystyle \lim_{h\to 0}\frac{(2+h)^3-8}{h} = \lim_{h\to 0}\frac{\left(8+12h+6h^2+h^3\right)-8}{h} = \lim_{h\to 0}\frac{12h+6h^2+h^3}{h}$

$\displaystyle = \lim_{h\to 0}\left(12+6h+h^2\right) = 12+0+0 = 12$

19. By the formula for the sum of cubes, we have

$$\lim_{x\to -2}\frac{x+2}{x^3+8} = \lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)} = \lim_{x\to -2}\frac{1}{x^2-2x+4} = \frac{1}{4+4+4} = \frac{1}{12}.$$

20. We use the difference of squares in the numerator and the difference of cubes in the denominator.

$$\lim_{t\to 1}\frac{t^4-1}{t^3-1}=\lim_{t\to 1}\frac{(t^2-1)(t^2+1)}{(t-1)(t^2+t+1)}=\lim_{t\to 1}\frac{(t-1)(t+1)(t^2+1)}{(t-1)(t^2+t+1)}=\lim_{t\to 1}\frac{(t+1)(t^2+1)}{t^2+t+1}=\frac{2(2)}{3}=\frac{4}{3}$$

21. $$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}=\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\cdot\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}=\lim_{h\to 0}\frac{\left(\sqrt{9+h}\right)^2-3^2}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to 0}\frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}$$
$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{9+h}+3\right)}=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\frac{1}{3+3}=\frac{1}{6}$$

22. $$\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}=\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}\cdot\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}=\lim_{u\to 2}\frac{\left(\sqrt{4u+1}\right)^2-3^2}{(u-2)\left(\sqrt{4u+1}+3\right)}$$
$$=\lim_{u\to 2}\frac{4u+1-9}{(u-2)\left(\sqrt{4u+1}+3\right)}=\lim_{u\to 2}\frac{4(u-2)}{(u-2)\left(\sqrt{4u+1}+3\right)}$$
$$=\lim_{u\to 2}\frac{4}{\sqrt{4u+1}+3}=\frac{4}{\sqrt{9}+3}=\frac{2}{3}$$

23. $$\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}=\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x-3}\cdot\frac{3x}{3x}=\lim_{x\to 3}\frac{3-x}{3x(x-3)}=\lim_{x\to 3}\frac{-1}{3x}=-\frac{1}{9}$$

24. $$\lim_{h\to 0}\frac{(3+h)^{-1}-3^{-1}}{h}=\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}=\lim_{h\to 0}\frac{3-(3+h)}{h(3+h)3}=\lim_{h\to 0}\frac{-h}{h(3+h)3}$$
$$=\lim_{h\to 0}\left[-\frac{1}{3(3+h)}\right]=-\frac{1}{\lim\limits_{h\to 0}[3(3+h)]}=-\frac{1}{3(3+0)}=-\frac{1}{9}$$

25. $$\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}=\lim_{t\to 0}\frac{\sqrt{1+t}-\sqrt{1-t}}{t}\cdot\frac{\sqrt{1+t}+\sqrt{1-t}}{\sqrt{1+t}+\sqrt{1-t}}=\lim_{t\to 0}\frac{\left(\sqrt{1+t}\right)^2-\left(\sqrt{1-t}\right)^2}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}$$
$$=\lim_{t\to 0}\frac{(1+t)-(1-t)}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim_{t\to 0}\frac{2t}{t\left(\sqrt{1+t}+\sqrt{1-t}\right)}=\lim_{t\to 0}\frac{2}{\sqrt{1+t}+\sqrt{1-t}}$$
$$=\frac{2}{\sqrt{1}+\sqrt{1}}=\frac{2}{2}=1$$

26. $$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t(t+1)}\right)=\lim_{t\to 0}\frac{t+1-1}{t(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{0+1}=1$$

27. $$\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}$$
$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}=\frac{1}{16\left(4+\sqrt{16}\right)}=\frac{1}{16(8)}=\frac{1}{128}$$

28. $$\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}=\lim_{x\to 2}\frac{(x-2)^2}{(x^2-4)(x^2+1)}=\lim_{x\to 2}\frac{(x-2)^2}{(x+2)(x-2)(x^2+1)}$$
$$=\lim_{x\to 2}\frac{x-2}{(x+2)(x^2+1)}=\frac{0}{4\cdot 5}=0$$

29. $\displaystyle \lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\,\right)\left(1+\sqrt{1+t}\,\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\,\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\,\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\,\right)}=-\frac{1}{2}$$

30. $\displaystyle \lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$

$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

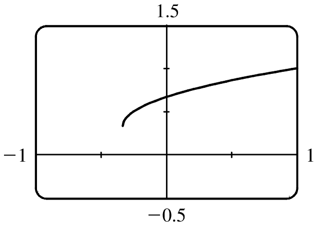
31. $\displaystyle \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$

$$=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}=\lim_{h\to 0}(3x^2+3xh+h^2)=3x^2$$

32. $\displaystyle \lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-h(2x+h)}{hx^2(x+h)^2}$

$$=\lim_{h\to 0}\frac{-(2x+h)}{x^2(x+h)^2}=\frac{-2x}{x^2\cdot x^2}=-\frac{2}{x^3}$$

33. (a)



$\displaystyle \lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$

(b)

x	$f(x)$
-0.001	$0.666\,166\,3$
$-0.000\,1$	$0.666\,616\,7$
$-0.000\,01$	$0.666\,661\,7$
$-0.000\,001$	$0.666\,666\,2$
$0.000\,001$	$0.666\,667\,2$
$0.000\,01$	$0.666\,671\,7$
$0.000\,1$	$0.666\,716\,7$
0.001	$0.667\,166\,3$

The limit appears to be $\frac{2}{3}$.

(c) $\displaystyle \lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$

$$=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right) \qquad \text{[Limit Law 3]}$$

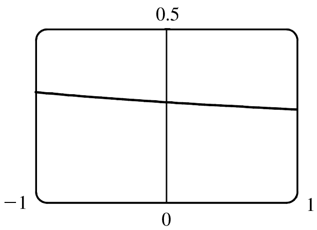
$$=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right] \qquad \text{[1 and 11]}$$

$$=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right) \qquad \text{[1, 3, and 7]}$$

$$=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right) \qquad \text{[7 and 8]}$$

$$=\frac{1}{3}(1+1)=\frac{2}{3}$$

34. (a)



$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$$

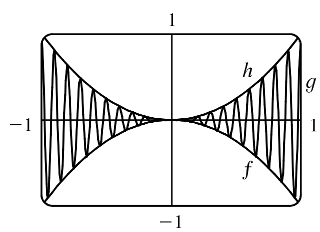
(b)

x	$f(x)$
-0.001	$0.288\,699\,2$
$-0.000\,1$	$0.288\,677\,5$
$-0.000\,01$	$0.288\,675\,4$
$-0.000\,001$	$0.288\,675\,2$
$0.000\,001$	$0.288\,675\,1$
$0.000\,01$	$0.288\,674\,9$
$0.000\,1$	$0.288\,672\,7$
0.001	$0.288\,651\,1$

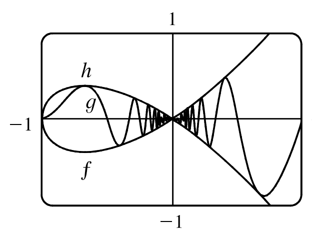
The limit appears to be approximately 0.2887.

(c)
$$\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$$
$$=\frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\sqrt{3+x}+\lim\limits_{x\to 0}\sqrt{3}}\qquad\text{[Limit Laws 5 and 1]}$$
$$=\frac{1}{\sqrt{\lim\limits_{x\to 0}(3+x)}+\sqrt{3}}\qquad\text{[7 and 11]}$$
$$=\frac{1}{\sqrt{3+0}+\sqrt{3}}\qquad\text{[1, 7, and 8]}$$
$$=\frac{1}{2\sqrt{3}}$$

35. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\le\cos 20\pi x\le 1\;\Rightarrow\;-x^2\le x^2\cos 20\pi x\le x^2\;\Rightarrow\;f(x)\le g(x)\le h(x)$. So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0}g(x)=0$.



36. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\le\sin(\pi/x)\le 1\;\Rightarrow\;-\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\;\Rightarrow\;f(x)\le g(x)\le h(x)$. So since $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim\limits_{x\to 0}g(x)=0$.



37. We have $\lim\limits_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim\limits_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\le f(x)\le x^2-4x+7$ for $x\ge 0$, $\lim\limits_{x\to 4}f(x)=7$ by the Squeeze Theorem.

38. We have $\lim\limits_{x\to 1}(2x)=2(1)=2$ and $\lim\limits_{x\to 1}(x^4-x^2+2)=1^4-1^2+2=2$. Since $2x\le g(x)\le x^4-x^2+2$ for all x, $\lim\limits_{x\to 1}g(x)=2$ by the Squeeze Theorem.

39. $-1\le\cos(2/x)\le 1\;\Rightarrow\;-x^4\le x^4\cos(2/x)\le x^4$. Since $\lim\limits_{x\to 0}\left(-x^4\right)=0$ and $\lim\limits_{x\to 0}x^4=0$, we have $\lim\limits_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

40. $-1 \le \sin(\pi/x) \le 1 \;\Rightarrow\; e^{-1} \le e^{\sin(\pi/x)} \le e^{1} \;\Rightarrow\; \sqrt{x}/e \le \sqrt{x}\,e^{\sin(\pi/x)} \le \sqrt{x}\,e$. Since $\lim\limits_{x\to 0^+} (\sqrt{x}/e) = 0$ and $\lim\limits_{x\to 0^+} (\sqrt{x}\,e) = 0$, we have $\lim\limits_{x\to 0^+} \left[\sqrt{x}\,e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

41. $|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Thus, $\lim\limits_{x\to 3^+} (2x + |x-3|) = \lim\limits_{x\to 3^+} (2x + x - 3) = \lim\limits_{x\to 3^+} (3x-3) = 3(3) - 3 = 6$ and $\lim\limits_{x\to 3^-} (2x + |x-3|) = \lim\limits_{x\to 3^-} (2x + 3 - x) = \lim\limits_{x\to 3^-} (x+3) = 3 + 3 = 6$. Since the left and right limits are equal, $\lim\limits_{x\to 3} (2x + |x-3|) = 6$.

42. $|x+6| = \begin{cases} x+6 & \text{if } x+6 \ge 0 \\ -(x+6) & \text{if } x+6 < 0 \end{cases} = \begin{cases} x+6 & \text{if } x \ge -6 \\ -(x+6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$$\lim_{x\to -6^+} \frac{2x+12}{|x+6|} = \lim_{x\to -6^+} \frac{2(x+6)}{x+6} = 2 \quad \text{and} \quad \lim_{x\to -6^-} \frac{2x+12}{|x+6|} = \lim_{x\to -6^-} \frac{2(x+6)}{-(x+6)} = -2$$

The left and right limits are different, so $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

43. $\left|2x^3 - x^2\right| = \left|x^2(2x-1)\right| = \left|x^2\right| \cdot |2x-1| = x^2\,|2x-1|$

$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \ge 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \ge 0.5 \\ -(2x-1) & \text{if } x < 0.5 \end{cases}$

So $\left|2x^3 - x^2\right| = x^2[-(2x-1)]$ for $x < 0.5$.

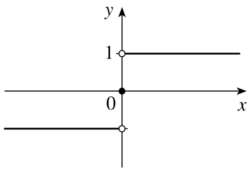
Thus, $\lim\limits_{x\to 0.5^-} \dfrac{2x-1}{|2x^3 - x^2|} = \lim\limits_{x\to 0.5^-} \dfrac{2x-1}{x^2[-(2x-1)]} = \lim\limits_{x\to 0.5^-} \dfrac{-1}{x^2} = \dfrac{-1}{(0.5)^2} = \dfrac{-1}{0.25} = -4$.

44. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x\to -2} \dfrac{2-(-x)}{2+x} = \lim\limits_{x\to -2} \dfrac{2+x}{2+x} = \lim\limits_{x\to -2} 1 = 1$.

45. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x\to 0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

46. Since $|x| = x$ for $x > 0$, we have $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{x}\right) = \lim\limits_{x\to 0^+} 0 = 0$.

47. (a)



(b) (i) Since $\operatorname{sgn} x = 1$ for $x > 0$, $\lim\limits_{x\to 0^+} \operatorname{sgn} x = \lim\limits_{x\to 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn} x = -1$ for $x < 0$, $\lim\limits_{x\to 0^-} \operatorname{sgn} x = \lim\limits_{x\to 0^-} -1 = -1$.

(iii) Since $\lim\limits_{x\to 0^-} \operatorname{sgn} x \ne \lim\limits_{x\to 0^+} \operatorname{sgn} x$, $\lim\limits_{x\to 0} \operatorname{sgn} x$ does not exist.

(iv) Since $|\operatorname{sgn} x| = 1$ for $x \ne 0$, $\lim\limits_{x\to 0} |\operatorname{sgn} x| = \lim\limits_{x\to 0} 1 = 1$.

48. (a) $g(x) = \operatorname{sgn}(\sin x) = \begin{cases} -1 & \text{if } \sin x < 0 \\ 0 & \text{if } \sin x = 0 \\ 1 & \text{if } \sin x > 0 \end{cases}$

(i) $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for small positive values of x.

(ii) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for small negative values of x.

(iii) $\lim_{x\to 0} g(x)$ does not exist since $\lim_{x\to 0^+} g(x) \neq \lim_{x\to 0^-} g(x)$.

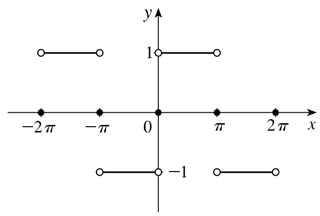
(iv) $\lim_{x\to \pi^+} g(x) = \lim_{x\to \pi^+} \operatorname{sgn}(\sin x) = -1$ since $\sin x$ is negative for values of x slightly greater than π.

(v) $\lim_{x\to \pi^-} g(x) = \lim_{x\to \pi^-} \operatorname{sgn}(\sin x) = 1$ since $\sin x$ is positive for values of x slightly less than π.

(vi) $\lim_{x\to \pi} g(x)$ does not exist since $\lim_{x\to \pi^+} g(x) \neq \lim_{x\to \pi^-} g(x)$.

(b) The sine function changes sign at every integer multiple of π, so the signum function equals 1 on one side and -1 on the other side of $n\pi$, n an integer. Thus, $\lim_{x\to a} g(x)$ does not exist for $a = n\pi$, n an integer.

(c)



49. (a) (i) $\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} \dfrac{x^2+x-6}{|x-2|} = \lim_{x\to 2^+} \dfrac{(x+3)(x-2)}{|x-2|}$

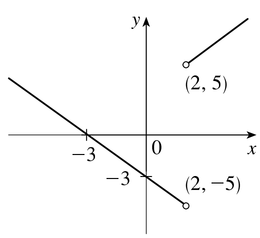
$$= \lim_{x\to 2^+} \frac{(x+3)(x-2)}{x-2} \quad [\text{since } x-2>0 \text{ if } x\to 2^+]$$

$$= \lim_{x\to 2^+} (x+3) = 5$$

(ii) The solution is similar to the solution in part (i), but now $|x-2| = 2-x$ since $x-2<0$ if $x\to 2^-$.
Thus, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} -(x+3) = -5$.

(b) Since the right-hand and left-hand limits of g at $x=2$ are not equal, $\lim_{x\to 2} g(x)$ does not exist.

(c)

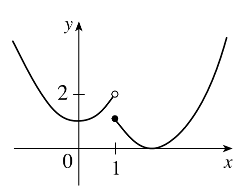


50. (a) $f(x) = \begin{cases} x^2+1 & \text{if } x<1 \\ (x-2)^2 & \text{if } x\geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+1) = 1^2+1 = 2,\quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2)^2 = (-1)^2 = 1$

(b) Since the right-hand and left-hand limits of f at $x=1$ are not equal, $\lim_{x\to 1} f(x)$ does not exist.

(c)



51. For the $\lim\limits_{t\to 2} B(t)$ to exist, the one-sided limits at $t = 2$ must be equal. $\lim\limits_{t\to 2^-} B(t) = \lim\limits_{t\to 2^-} \left(4 - \frac{1}{2}t\right) = 4 - 1 = 3$ and

$\lim\limits_{t\to 2^+} B(t) = \lim\limits_{t\to 2^+} \sqrt{t+c} = \sqrt{2+c}$. Now $3 = \sqrt{2+c} \;\Rightarrow\; 9 = 2 + c \;\Leftrightarrow\; c = 7$.

52. (a) (i) $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} x = 1$

(ii) $\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim\limits_{x\to 1^-} g(x) = 1$ and $\lim\limits_{x\to 1^+} g(x) = 1$, we have $\lim\limits_{x\to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

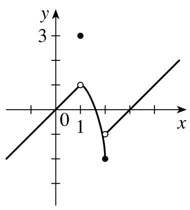
(iv) $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim\limits_{x\to 2} g(x)$ does not exist since $\lim\limits_{x\to 2^-} g(x) \neq \lim\limits_{x\to 2^+} g(x)$.

(b)

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$



53. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim\limits_{x\to -2^+} [\![x]\!] = \lim\limits_{x\to -2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to -2^-} [\![x]\!] = \lim\limits_{x\to -2^-} (-3) = -3$.

The right and left limits are different, so $\lim\limits_{x\to -2} [\![x]\!]$ does not exist.

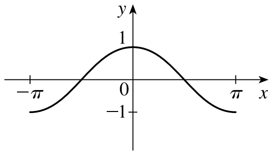
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to -2.4} [\![x]\!] = \lim\limits_{x\to -2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n - 1$ for $n - 1 \le x < n$, so $\lim\limits_{x\to n^-} [\![x]\!] = \lim\limits_{x\to n^-} (n - 1) = n - 1$.

(ii) $[\![x]\!] = n$ for $n \le x < n + 1$, so $\lim\limits_{x\to n^+} [\![x]\!] = \lim\limits_{x\to n^+} n = n$.

(c) $\lim\limits_{x\to a} [\![x]\!]$ exists $\;\Leftrightarrow\;$ a is not an integer.

54. (a) See the graph of $y = \cos x$.

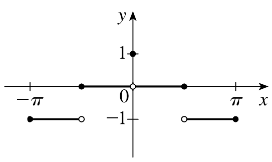


Since $-1 \le \cos x < 0$ on $[-\pi, -\pi/2)$, we have $y = f(x) = [\![\cos x]\!] = -1$ on $[-\pi, -\pi/2)$.

Since $0 \le \cos x < 1$ on $[-\pi/2, 0) \cup (0, \pi/2]$, we have $f(x) = 0$ on $[-\pi/2, 0) \cup (0, \pi/2]$.

Since $-1 \le \cos x < 0$ on $(\pi/2, \pi]$, we have $f(x) = -1$ on $(\pi/2, \pi]$.

Note that $f(0) = 1$.



(b) (i) $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 0$, so $\lim_{x\to 0} f(x) = 0$.

(ii) As $x \to (\pi/2)^-$, $f(x) \to 0$, so $\lim_{x\to(\pi/2)^-} f(x) = 0$.

(iii) As $x \to (\pi/2)^+$, $f(x) \to -1$, so $\lim_{x\to(\pi/2)^+} f(x) = -1$.

(iv) Since the answers in parts (ii) and (iii) are not equal, $\lim_{x\to\pi/2} f(x)$ does not exist.

(c) $\lim_{x\to a} f(x)$ exists for all a in the open interval $(-\pi, \pi)$ except $a = -\pi/2$ and $a = \pi/2$.

55. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim_{x\to 2^-} f(x) = -1$ and $\lim_{x\to 2^+} f(x) = -1$, so $\lim_{x\to 2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim_{x\to 2} f(x) \neq f(2)$.

56. $\lim_{v\to c^-}\left(L_0\sqrt{1-\dfrac{v^2}{c^2}}\right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

57. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = a_0 + a_1\lim_{x\to a} x + a_2\lim_{x\to a} x^2 + \cdots + a_n\lim_{x\to a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)\end{aligned}$$

Thus, for any polynomial p, $\lim_{x\to a} p(x) = p(a)$.

58. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Then

$$\lim_{x\to a} r(x) = \lim_{x\to a}\frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 57]} \quad = r(a).$$

59. $\lim_{x\to 1}[f(x) - 8] = \lim_{x\to 1}\left[\dfrac{f(x)-8}{x-1}\cdot(x-1)\right] = \lim_{x\to 1}\dfrac{f(x)-8}{x-1}\cdot\lim_{x\to 1}(x-1) = 10\cdot 0 = 0.$

Thus, $\lim_{x\to 1} f(x) = \lim_{x\to 1}\{[f(x)-8]+8\} = \lim_{x\to 1}[f(x)-8] + \lim_{x\to 1} 8 = 0 + 8 = 8$.

Note: The value of $\lim_{x\to 1}\dfrac{f(x)-8}{x-1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim_{x\to 1}\dfrac{f(x)-8}{x-1}$ exists.

60. (a) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\dfrac{f(x)}{x^2}\cdot x^2\right] = \lim_{x\to 0}\dfrac{f(x)}{x^2}\cdot\lim_{x\to 0} x^2 = 5\cdot 0 = 0$

(b) $\lim_{x\to 0}\dfrac{f(x)}{x} = \lim_{x\to 0}\left[\dfrac{f(x)}{x^2}\cdot x\right] = \lim_{x\to 0}\dfrac{f(x)}{x^2}\cdot\lim_{x\to 0} x = 5\cdot 0 = 0$

61. Observe that $0 \le f(x) \le x^2$ for all x, and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} x^2$. So, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = 0$.

62. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 3} g(x)$ do not exist [Example 10]

but $\lim\limits_{x\to 3}[f(x)+g(x)] = \lim\limits_{x\to 3}([\![x]\!] - [\![x]\!]) = \lim\limits_{x\to 3} 0 = 0$.

63. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59.

Thus, either f or g is 0 for any value of x. Then $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0}[f(x)g(x)] = \lim\limits_{x\to 0} 0 = 0$.

64.
$$\lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x\to 2}\left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}\cdot\frac{\sqrt{6-x}+2}{\sqrt{6-x}+2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}\right)$$
$$= \lim_{x\to 2}\left[\frac{\left(\sqrt{6-x}\right)^2-2^2}{\left(\sqrt{3-x}\right)^2-1^2}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right] = \lim_{x\to 2}\left(\frac{6-x-4}{3-x-1}\cdot\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2}\right)$$
$$= \lim_{x\to 2}\frac{(2-x)\left(\sqrt{3-x}+1\right)}{(2-x)\left(\sqrt{6-x}+2\right)} = \lim_{x\to 2}\frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{1}{2}$$

65. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches

0 as $x \to -2$. In order for this to happen, we need $\lim\limits_{x\to -2}\left(3x^2 + ax + a + 3\right) = 0 \quad\Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad\Leftrightarrow\quad 12 - 2a + a + 3 = 0 \quad\Leftrightarrow\quad a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to -2}\frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x\to -2}\frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

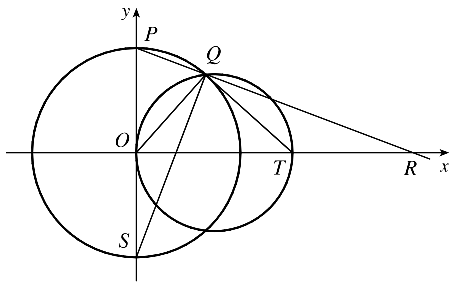
66. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad\Leftrightarrow\quad r^2 = 1 + 2x - 1 \quad\Leftrightarrow\quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad\Leftrightarrow\quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad\Leftrightarrow\quad y = r\sqrt{1-\frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1-\frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

$y - r = \dfrac{r\sqrt{1-\frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1-\frac{1}{4}r^2}-1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)}{1-\frac{1}{4}r^2-1} = 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right)$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r\to 0^+} x = \lim\limits_{r\to 0^+} 2\left(\sqrt{1-\frac{1}{4}r^2}+1\right) = \lim\limits_{r\to 0^+} 2\left(\sqrt{1}+1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.



Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90^\circ$ (subtended by diameter PS). So $\angle SQR = 90^\circ = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90^\circ - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.