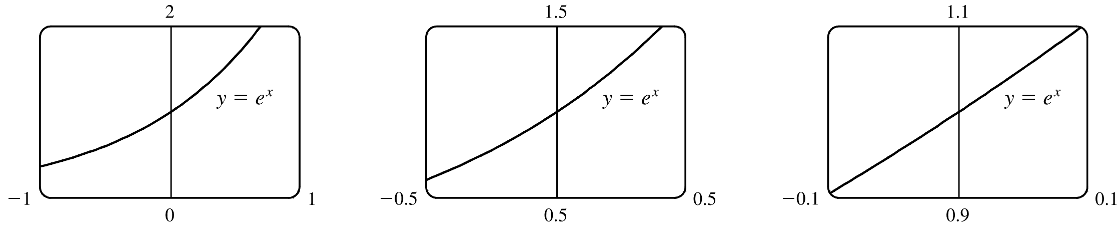
2.7 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



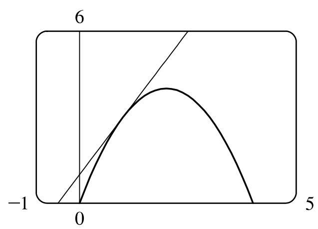
3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x \to 1} \frac{-(x-1)(x-3)}{x - 1}$$
$$= \lim_{x \to 1} (3 - x) = 3 - 1 = 2$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[4(1+h) - (1+h)^2\right] - 3}{h}$$
$$= \lim_{h \to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(-h+2)}{h} = \lim_{h \to 0} (-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c) 

The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

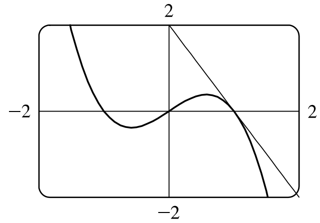
4. (a) (i) Using Definition 1 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{x \to 1} \frac{f(x) - 0}{x - 1} = \lim_{x \to 1} \frac{x - x^3}{x - 1} = \lim_{x \to 1} \frac{x(1 - x^2)}{x - 1} = \lim_{x \to 1} \frac{x(1+x)(1-x)}{x - 1}$$
$$= \lim_{x \to 1} \left[-x(1+x)\right] = -1(2) = -2$$

(ii) Using Equation 2 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1+h) - (1+h)^3\right] - 0}{h}$$
$$= \lim_{h \to 0} \frac{1 + h - (1 + 3h + 3h^2 + h^3)}{h} = \lim_{h \to 0} \frac{-h^3 - 3h^2 - 2h}{h} = \lim_{h \to 0} \frac{h(-h^2 - 3h - 2)}{h}$$
$$= \lim_{h \to 0} (-h^2 - 3h - 2) = -2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 0 = -2(x - 1)$, or $y = -2x + 2$.

(c) 

The graph of $y = -2x + 2$ is tangent to the graph of $y = x - x^3$ at the point $(1, 0)$. Now zoom in toward the point $(1, 0)$ until the cubic and the tangent line are indistinguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x \to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$
$$= \lim_{x \to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x \to 2} (-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \quad \Leftrightarrow \quad y + 4 = -8x + 16 \quad \Leftrightarrow \quad y = -8x + 12.$

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^3 - 3(2 + h) + 1 - 3}{h}$$
$$= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h - 2}{h} = \lim_{h \to 0} \frac{9h + 6h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(9 + 6h + h^2)}{h}$$
$$= \lim_{h \to 0} (9 + 6h + h^2) = 9$$

Tangent line: $y - 3 = 9(x - 2) \quad \Leftrightarrow \quad y - 3 = 9x - 18 \quad \Leftrightarrow \quad y = 9x - 15$

7. Using (1), $m = \lim_{x \to 1} \dfrac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \dfrac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \dfrac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1} \dfrac{1}{\sqrt{x} + 1} = \dfrac{1}{2}.$

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \quad \Leftrightarrow \quad y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x + 1}{x + 2}$ and $P(1, 1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x + 1}{x + 2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x + 1 - (x + 2)}{x + 2}}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 2)}$$
$$= \lim_{x \to 1} \frac{1}{x + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1) \quad \Leftrightarrow \quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$

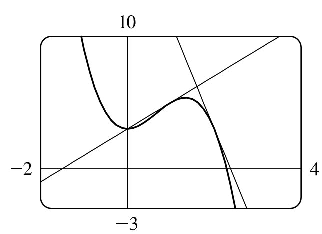
9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a + h)^2 - 2(a + h)^3 - (3 + 4a^2 - 2a^3)}{h}$$
$$= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$
$$= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$
$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$
$$= \lim_{h \to 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

(b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x - 1) \quad \Leftrightarrow \quad y = 2x + 3$.

At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x - 2) \quad \Leftrightarrow \quad y = -8x + 19$.

(c)



10. (a) Using (1),

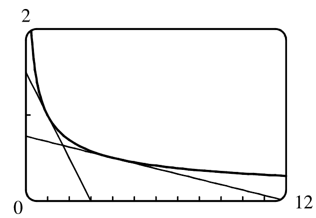
$$m = \lim_{x\to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x-a} = \lim_{x\to a} \frac{\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{ax}}}{x-a} = \lim_{x\to a} \frac{(\sqrt{a}-\sqrt{x})(\sqrt{a}+\sqrt{x})}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})} = \lim_{x\to a} \frac{a-x}{\sqrt{ax}\,(x-a)(\sqrt{a}+\sqrt{x})}$$

$$= \lim_{x\to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a}+\sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2} \quad [a>0]$$

(b) At $(1,1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

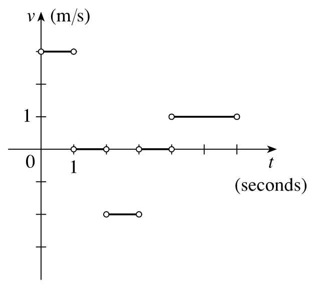
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0,1)$ and $(4,6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2,3)$. The particle is standing still when s is constant; that is, on the intervals $(1,2)$ and $(3,4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0,1)$, the slope is $\dfrac{3-0}{1-0} = 3$. On the interval $(2,3)$, the slope is $\dfrac{1-3}{3-2} = -2$. On the interval $(4,6)$, the slope is $\dfrac{3-1}{6-4} = 1$.



12. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant. **Runner B** starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

(b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

(c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t)-s(2)}{t-2} = \lim_{t\to 2} \frac{\left(40t-16t^2\right)-16}{t-2} = \lim_{t\to 2} \frac{-16t^2+40t-16}{t-2} = \lim_{t\to 2} \frac{-8\left(2t^2-5t+2\right)}{t-2}$$

$$= \lim_{t\to 2} \frac{-8(t-2)(2t-1)}{t-2} = -8\lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $H(t) = 10t - 1.86t^2$.

$$\begin{aligned} v(1) &= \lim_{h\to 0} \frac{H(1+h) - H(1)}{h} = \lim_{h\to 0} \frac{\left[10(1+h) - 1.86(1+h)^2\right] - (10 - 1.86)}{h} \\ &= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h} \\ &= \lim_{h\to 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h} \\ &= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h\to 0}(6.28 - 1.86h) = 6.28 \end{aligned}$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$\begin{aligned} v(a) &= \lim_{h\to 0} \frac{H(a+h) - H(a)}{h} = \lim_{h\to 0} \frac{\left[10(a+h) - 1.86(a+h)^2\right] - (10a - 1.86a^2)}{h} \\ &= \lim_{h\to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h} \\ &= \lim_{h\to 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h\to 0} \frac{10h - 3.72ah - 1.86h^2}{h} \\ &= \lim_{h\to 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h\to 0}(10 - 3.72a - 1.86h) = 10 - 3.72a \end{aligned}$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0 \quad\Leftrightarrow\quad 10t - 1.86t^2 = 0 \quad\Leftrightarrow\quad t(10 - 1.86t) = 0 \quad\Leftrightarrow\quad t = 0$ or $1.86t = 10$.

The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

15. $$\begin{aligned} v(a) &= \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2} \\ &= \lim_{h\to 0} \frac{-(2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a+h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a+h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s} \end{aligned}$$

So $v(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

16. (a) The average velocity between times t and $t + h$ is

$$\begin{aligned} \frac{s(t+h) - s(t)}{(t+h) - t} &= \frac{\frac{1}{2}(t+h)^2 - 6(t+h) + 23 - \left(\frac{1}{2}t^2 - 6t + 23\right)}{h} \\ &= \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h} \\ &= \frac{th + \frac{1}{2}h^2 - 6h}{h} = \frac{h\left(t + \frac{1}{2}h - 6\right)}{h} = \left(t + \tfrac{1}{2}h - 6\right) \text{ ft/s} \end{aligned}$$

(i) $[4, 8]$: $t = 4$, $h = 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ ft/s.

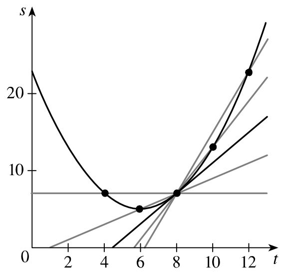
(ii) $[6, 8]$: $t = 6$, $h = 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ ft/s.

(iii) $[8, 10]$: $t = 8$, $h = 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ ft/s.

(iv) $[8, 12]$: $t = 8$, $h = 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ ft/s.

(b) $v(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = \lim\limits_{h \to 0} \left(t + \frac{1}{2}h - 6\right)$

$= t - 6, \quad \text{so } v(8) = 2 \text{ ft/s}.$

(c) 

17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

18. (a) On $[20, 60]$: $\dfrac{f(60) - f(20)}{60 - 20} = \dfrac{700 - 300}{40} = \dfrac{400}{40} = 10$

(b) Pick any interval that has the same y-value at its endpoints. $[0, 57]$ is such an interval since $f(0) = 600$ and $f(57) = 600$.

(c) On $[40, 60]$: $\dfrac{f(60) - f(40)}{60 - 40} = \dfrac{700 - 200}{20} = \dfrac{500}{20} = 25$

On $[40, 70]$: $\dfrac{f(70) - f(40)}{70 - 40} = \dfrac{900 - 200}{30} = \dfrac{700}{30} = 23\frac{1}{3}$

Since $25 > 23\frac{1}{3}$, the average rate of change on $[40, 60]$ is larger.

(d) $\dfrac{f(40) - f(10)}{40 - 10} = \dfrac{200 - 400}{30} = \dfrac{-200}{30} = -6\frac{2}{3}$

This value represents the slope of the line segment from $(10, f(10))$ to $(40, f(40))$.

19. (a) The tangent line at $x = 50$ appears to pass through the points $(43, 200)$ and $(60, 640)$, so

$f'(50) \approx \dfrac{640 - 200}{60 - 43} = \dfrac{440}{17} \approx 26.$

(b) The tangent line at $x = 10$ is steeper than the tangent line at $x = 30$, so it is larger in magnitude, but less in numerical value, that is, $f'(10) < f'(30)$.

(c) The slope of the tangent line at $x = 60$, $f'(60)$, is greater than the slope of the line through $(40, f(40))$ and $(80, f(80))$.

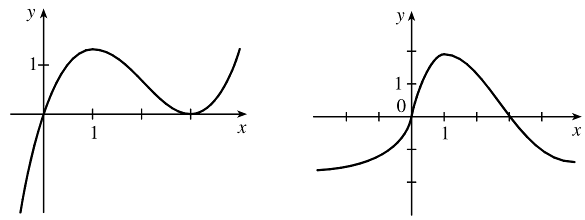
So yes, $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$.

20. Since $g(5) = -3$, the point $(5, -3)$ is on the graph of g. Since $g'(5) = 4$, the slope of the tangent line at $x = 5$ is 4. Using the point-slope form of a line gives us $y - (-3) = 4(x - 5)$, or $y = 4x - 23$.

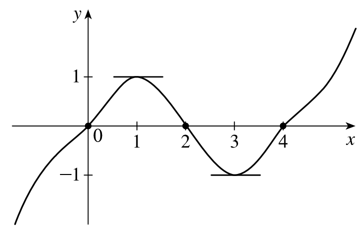
21. For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

22. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

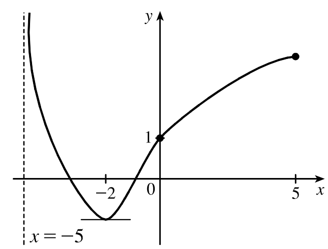
23. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.



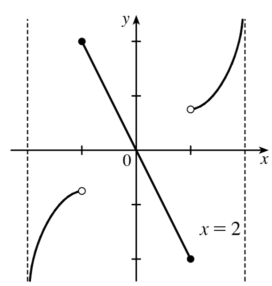
24. We begin by drawing a curve through the origin with a slope of 1 to satisfy $g(0) = 0$ and $g'(0) = 1$. We round off our figure at $x = 1$ to satisfy $g'(1) = 0$, and then pass through $(2, 0)$ with slope -1 to satisfy $g(2) = 0$ and $g'(2) = -1$. We round the figure at $x = 3$ to satisfy $g'(3) = 0$, and then pass through $(4, 0)$ with slope 1 to satisfy $g(4) = 0$ and $g'(4) = 1$. Finally we extend the curve on both ends to satisfy $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = -\infty$.



25. We begin by drawing a curve through $(0, 1)$ with a slope of 1 to satisfy $g(0) = 1$ and $g'(0) = 1$. We round off our figure at $x = -2$ to satisfy $g'(-2) = 0$. As $x \to -5^+$, $y \to \infty$, so we draw a vertical asymptote at $x = -5$. As $x \to 5^-$, $y \to 3$, so we draw a dot at $(5, 3)$ [the dot could be open or closed].



26. We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $f'(0) = -2$. Now draw a curve starting at $x = 1$ and increasing without bound as $x \to 2^-$ since $\lim_{x\to 2^-} f(x) = \infty$. Lastly, reflect the last curve through the origin (rotate 180°) since f is an odd function.



27. Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h\to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h\to 0} \frac{3h - h^3}{h} = \lim_{h\to 0} \frac{h(3 - h^2)}{h} \\ &= \lim_{h\to 0} (3 - h^2) = 3 - 0 = 3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

28. Using (5) with $g(x) = x^4 - 2$ and $a = 1$,

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x\to 1} \frac{x^4 - 1}{x - 1} = \lim_{x\to 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x\to 1} [(x^2 + 1)(x + 1)] = 2(2) = 4$$

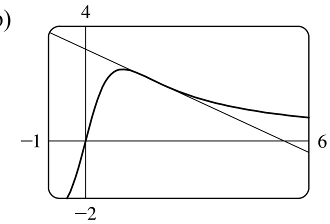
Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

29. (a) Using (4) with $F(x) = 5x/(1 + x^2)$ and the point $(2, 2)$, we have

$$F'(2) = \lim_{h\to 0} \frac{F(2 + h) - F(2)}{h} = \lim_{h\to 0} \frac{\dfrac{5(2 + h)}{1 + (2 + h)^2} - 2}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{5h + 10}{h^2 + 4h + 5} - 2}{h} = \lim_{h\to 0} \frac{\dfrac{5h + 10 - 2(h^2 + 4h + 5)}{h^2 + 4h + 5}}{h}$$
$$= \lim_{h\to 0} \frac{-2h^2 - 3h}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{h(-2h - 3)}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)

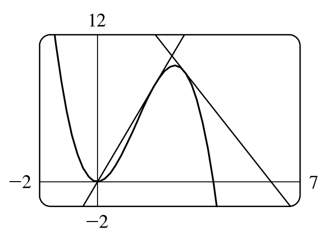


30. (a) Using (4) with $G(x) = 4x^2 - x^3$, we have

$$G'(a) = \lim_{h\to 0} \frac{G(a + h) - G(a)}{h} = \lim_{h\to 0} \frac{[4(a + h)^2 - (a + h)^3] - (4a^2 - a^3)}{h}$$
$$= \lim_{h\to 0} \frac{4a^2 + 8ah + 4h^2 - (a^3 + 3a^2h + 3ah^2 + h^3) - 4a^2 + a^3}{h}$$
$$= \lim_{h\to 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h} = \lim_{h\to 0} \frac{h(8a + 4h - 3a^2 - 3ah - h^2)}{h}$$
$$= \lim_{h\to 0} (8a + 4h - 3a^2 - 3ah - h^2) = 8a - 3a^2$$

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.

(b)



31. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{[3(a + h)^2 - 4(a + h) + 1] - (3a^2 - 4a + 1)]}{h}$$
$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h + 1 - 3a^2 + 4a - 1}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h}$$
$$= \lim_{h\to 0} \frac{h(6a + 3h - 4)}{h} = \lim_{h\to 0} (6a + 3h - 4) = 6a - 4$$

32. Use (4) with $f(t) = 2t^3 + t$.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[2(a+h)^3 + (a+h)] - (2a^3+a)}{h} \\
&= \lim_{h\to 0} \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 + a + h - 2a^3 - a}{h} = \lim_{h\to 0} \frac{6a^2h + 6ah^2 + 2h^3 + h}{h} \\
&= \lim_{h\to 0} \frac{h(6a^2 + 6ah + 2h^2 + 1)}{h} = \lim_{h\to 0}(6a^2 + 6ah + 2h^2 + 1) = 6a^2 + 1
\end{aligned}$$

33. Use (4) with $f(t) = (2t+1)/(t+3)$.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h} \\
&= \lim_{h\to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)} \\
&= \lim_{h\to 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)} \\
&= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}
\end{aligned}$$

34. Use (4) with $f(x) = x^{-2} = 1/x^2$.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2-(a+h)^2}{a^2(a+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-2ah - h^2}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{h(-2a-h)}{ha^2(a+h)^2} \\
&= \lim_{h\to 0} \frac{-2a-h}{a^2(a+h)^2} = \frac{-2a}{a^2(a^2)} = \frac{-2}{a^3}
\end{aligned}$$

35. Use (4) with $f(x) = \sqrt{1-2x}$.

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2 - \left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{(1-2a-2h)-(1-2a)}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} \\
&= \lim_{h\to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} \\
&= \frac{-2}{\sqrt{1-2a} + \sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}
\end{aligned}$$

36. Use (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$

$$= 4\lim_{h\to 0} \frac{\dfrac{\sqrt{1-a}-\sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4\lim_{h\to 0} \frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

$$= 4\lim_{h\to 0} \frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}} \cdot \frac{\sqrt{1-a}+\sqrt{1-a-h}}{\sqrt{1-a}+\sqrt{1-a-h}} = 4\lim_{h\to 0} \frac{(\sqrt{1-a})^2-(\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0} \frac{(1-a)-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\lim_{h\to 0} \frac{h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0} \frac{1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4 \cdot \frac{1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a})}$$

$$= \frac{4}{(1-a)(2\sqrt{1-a})} = \frac{2}{(1-a)^1(1-a)^{1/2}} = \frac{2}{(1-a)^{3/2}}$$

37. By (4), $\displaystyle\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x) = \sqrt{x}$ and $a = 9$.

38. By (4), $\displaystyle\lim_{h\to 0} \frac{e^{-2+h}-e^{-2}}{h} = f'(-2)$, where $f(x) = e^x$ and $a = -2$.

39. By Equation 5, $\displaystyle\lim_{x\to 2} \frac{x^6-64}{x-2} = f'(2)$, where $f(x) = x^6$ and $a = 2$.

40. By Equation 5, $\displaystyle\lim_{x\to 1/4} \frac{\dfrac{1}{x}-4}{x-\dfrac{1}{4}} = f'(4)$, where $f(x) = \dfrac{1}{x}$ and $a = \dfrac{1}{4}$.

41. By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By (4), $\displaystyle\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi+x)$ and $a = 0$.

42. By Equation 5, $\displaystyle\lim_{\theta\to \pi/6} \frac{\sin\theta-\frac{1}{2}}{\theta-\dfrac{\pi}{6}} = f'\left(\frac{\pi}{6}\right)$, where $f(\theta) = \sin\theta$ and $a = \dfrac{\pi}{6}$.

43. $\displaystyle v(4) = f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\left[80(4+h)-6(4+h)^2\right]-\left[80(4)-6(4)^2\right]}{h}$

$$= \lim_{h\to 0} \frac{(320+80h-96-48h-6h^2)-(320-96)}{h} = \lim_{h\to 0} \frac{32h-6h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(32-6h)}{h} = \lim_{h\to 0}(32-6h) = 32 \text{ m/s}$$

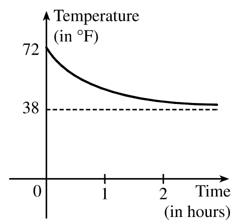
The speed when $t = 4$ is $|32| = 32$ m/s.

44. $v(4) = f'(4) = \lim\limits_{h\to 0} \dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0} \dfrac{\left(10 + \dfrac{45}{4+h+1}\right) - \left(10 + \dfrac{45}{4+1}\right)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{45}{5+h} - 9}{h}$

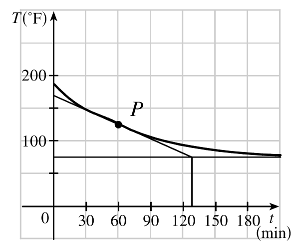
$= \lim\limits_{h\to 0} \dfrac{45 - 9(5+h)}{h(5+h)} = \lim\limits_{h\to 0} \dfrac{-9h}{h(5+h)} = \lim\limits_{h\to 0} \dfrac{-9}{5+h} = -\dfrac{9}{5}$ m/s.

The speed when $t = 4$ is $\left|-\frac{9}{5}\right| = \frac{9}{5}$ m/s.

45. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



46. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\dfrac{75 - 168}{132 - 0} \approx -0.7\,^\circ\mathrm{F/min}$.



47. (a) (i) $[1.0, 2.0]$: $\dfrac{C(2)-C(1)}{2-1} = \dfrac{0.18-0.33}{1} = -0.15\ \dfrac{\text{mg/mL}}{\text{h}}$

(ii) $[1.5, 2.0]$: $\dfrac{C(2)-C(1.5)}{2-1.5} = \dfrac{0.18-0.24}{0.5} = \dfrac{-0.06}{0.5} = -0.12\ \dfrac{\text{mg/mL}}{\text{h}}$

(iii) $[2.0, 2.5]$: $\dfrac{C(2.5)-C(2)}{2.5-2} = \dfrac{0.12-0.18}{0.5} = \dfrac{-0.06}{0.5} = -0.12\ \dfrac{\text{mg/mL}}{\text{h}}$

(iv) $[2.0, 3.0]$: $\dfrac{C(3)-C(2)}{3-2} = \dfrac{0.07-0.18}{1} = -0.11\ \dfrac{\text{mg/mL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $t = 2$ by averaging the average rates of change for $[1.5, 2.0]$ and $[2.0, 2.5]$:

$\dfrac{-0.12 + (-0.12)}{2} = -0.12\ \dfrac{\text{mg/mL}}{\text{h}}$. After 2 hours, the BAC is decreasing at a rate of $0.12\ (\text{mg/mL})/\text{h}$.

48. (a) (i) $[2006, 2008]$: $\dfrac{N(2008)-N(2006)}{2008-2006} = \dfrac{16{,}680 - 12{,}440}{2} = \dfrac{4240}{2} = 2120$ locations/year

(ii) $[2008, 2010]$: $\dfrac{N(2010)-N(2008)}{2010-2008} = \dfrac{16{,}858 - 16{,}680}{2} = \dfrac{178}{2} = 89$ locations/year.

The rate of growth decreased over the period from 2006 to 2010.

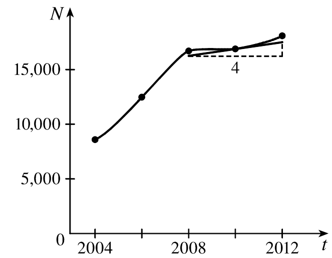
(b) $[2010, 2012]$: $\dfrac{N(2012)-N(2010)}{2012-2010} = \dfrac{18{,}066 - 16{,}858}{2} = \dfrac{1208}{2} = 604$ locations/year.

Using that value and the value from part (a)(ii), we have $\dfrac{89+604}{2} = \dfrac{693}{2} = 346.5$ locations/year.

(c) The tangent segment has endpoints $(2008, 16{,}250)$ and $(2012, 17{,}500)$.

An estimate of the instantaneous rate of growth in 2010 is

$$\frac{17{,}500 - 16{,}250}{2012 - 2008} = \frac{1250}{4} = 312.5 \text{ locations/year.}$$



49. (a) $[1990, 2005]$: $\dfrac{84{,}077 - 66{,}533}{2005 - 1990} = \dfrac{17{,}544}{15} = 1169.6$ thousands of barrels per day per year. This means that oil consumption rose by an average of 1169.6 thousands of barrels per day each year from 1990 to 2005.

(b) $[1995, 2000]$: $\dfrac{76{,}784 - 70{,}099}{2000 - 1995} = \dfrac{6685}{5} = 1337$

$[2000, 2005]$: $\dfrac{84{,}077 - 76{,}784}{2005 - 2000} = \dfrac{7293}{5} = 1458.6$

An estimate of the instantaneous rate of change in 2000 is $\frac{1}{2}\,(1337 + 1458.6) = 1397.8$ thousands of barrels per day per year.

50. (a) (i) $[4, 11]$: $\dfrac{V(11) - V(4)}{11 - 4} = \dfrac{9.4 - 53}{7} = \dfrac{-43.6}{7} \approx -6.23\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(ii) $[8, 11]$: $\dfrac{V(11) - V(8)}{11 - 8} = \dfrac{9.4 - 18}{3} = \dfrac{-8.6}{3} \approx -2.87\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iii) $[11, 15]$: $\dfrac{V(15) - V(11)}{15 - 11} = \dfrac{5.2 - 9.4}{4} = \dfrac{-4.2}{4} = -1.05\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iv) $[11, 22]$: $\dfrac{V(22) - V(11)}{22 - 11} = \dfrac{3.6 - 9.4}{11} = \dfrac{-5.8}{11} \approx -0.53\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(b) An estimate of $V'(11)$ is the average of the answers from part (a)(ii) and (iii).

$$V'(11) \approx \tfrac{1}{2}\,[-2.87 + (-1.05)] = -1.96\ \frac{\text{RNA copies/mL}}{\text{day}}.$$

$V'(11)$ measures the instantaneous rate of change of patient 303's viral load 11 days after ABT-538 treatment began.

51. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b)
$$\begin{aligned}\frac{C(100 + h) - C(100)}{h} &= \frac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \frac{20h + 0.05h^2}{h} \\ &= 20 + 0.05h,\ h \neq 0\end{aligned}$$

So the instantaneous rate of change is $\displaystyle\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0}(20 + 0.05h) = \$20/\text{unit}.$

52. $\Delta V = V(t+h) - V(t) = 100{,}000\left(1 - \frac{t+h}{60}\right)^2 - 100{,}000\left(1 - \frac{t}{60}\right)^2$

$$= 100{,}000\left[\left(1 - \frac{t+h}{30} + \frac{(t+h)^2}{3600}\right) - \left(1 - \frac{t}{30} + \frac{t^2}{3600}\right)\right] = 100{,}000\left(-\frac{h}{30} + \frac{2th}{3600} + \frac{h^2}{3600}\right)$$

$$= \frac{100{,}000}{3600}h\,(-120 + 2t + h) = \frac{250}{9}h\,(-120 + 2t + h)$$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100{,}000$
10	$-2777.\overline{7}$	$69{,}444.\overline{4}$
20	$-2222.\overline{2}$	$44{,}444.\overline{4}$
30	$-1666.\overline{6}$	$25{,}000$
40	$-1111.\overline{1}$	$11{,}111.\overline{1}$
50	$-555.\overline{5}$	$2{,}777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

53. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

54. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

55. (a) $H'(58)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is $58\,^\circ$F. The units are dollars/$\,^\circ$F.

(b) If the outside temperature increases, the building should require less heating, so we would expect $H'(58)$ to be negative.

56. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

57. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg/L)/$^\circ$C.

(b) For $T = 16^\circ$C, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \frac{6 - 14}{32 - 0} = -\frac{8}{32} = -0.25$ (mg/L)/$^\circ$C. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/$^\circ$C.

58. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are (cm/s)/°C.

(b) For $T = 15°\mathrm{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32 - 25}{20 - 10} = 0.7$ (cm/s)/°C. This tells us that at $T = 15°\mathrm{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 (cm/s)/°C. In a similar fashion for $T = 25°\mathrm{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25 - 35}{25 - 20} = -2$ (cm/s)/°C. As it gets warmer than 20°C, the maximum sustainable speed decreases rapidly.

59. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h\to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 2.2.4.)

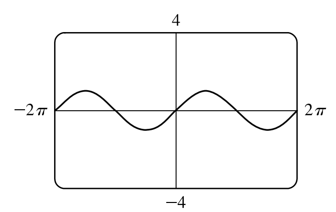
60. Since $f(x) = x^2\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2\sin(1/h) - 0}{h} = \lim\limits_{h\to 0} h\sin(1/h)$. Since $-1 \leq \sin\dfrac{1}{h} \leq 1$, we have

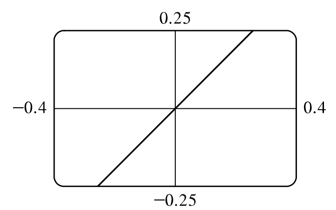
$-|h| \leq |h|\sin\dfrac{1}{h} \leq |h| \quad\Rightarrow\quad -|h| \leq h\sin\dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h\to 0}(-|h|) = 0$ and $\lim\limits_{h\to 0}|h| = 0$, we know that

$\lim\limits_{h\to 0}\left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

61. (a) The slope at the origin appears to be 1.



(b) The slope at the origin still appears to be 1.



(c) Yes, the slope at the origin now appears to be 0.