2.5 Continuity

1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

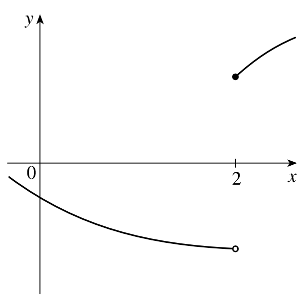
2. The graph of f has no hole, jump, or vertical asymptote.

3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

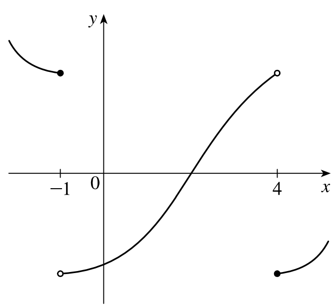
   (b) f is continuous from the left at -2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.

4. From the graph of g, we see that g is continuous on the intervals $[-3, -2)$, $(-2, -1)$, $(-1, 0]$, $(0, 1)$, and $(1, 3]$.

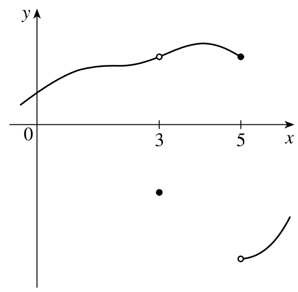
5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim_{x \to 2^+} f(x) = f(2)$.



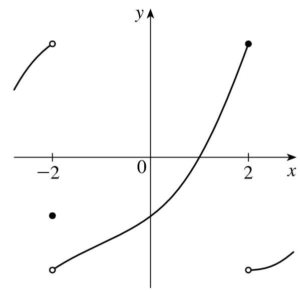
6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim_{x \to -1^-} f(x) = f(-1)$ and $\lim_{x \to 4^+} f(x) = f(4)$.



7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.

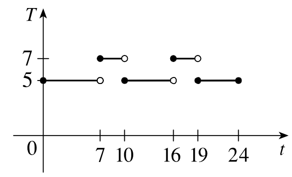


8. The graph of $y = f(x)$ must have a discontinuity at $x = -2$ with $\lim\limits_{x \to -2^-} f(x) \neq f(-2)$ and $\lim\limits_{x \to -2^+} f(x) \neq f(-2)$. It must also show that $\lim\limits_{x \to 2^-} f(x) = f(2)$ and $\lim\limits_{x \to 2^+} f(x) \neq f(2)$.



9. (a) The toll is \$7 between 7:00 AM and 10:00 AM and between 4:00 PM and 7:00 PM.

(b) The function T has jump discontinuities at $t = 7$, 10, 16, and 19. Their significance to someone who uses the road is that, because of the sudden jumps in the toll, they may want to avoid the higher rates between $t = 7$ and $t = 10$ and between $t = 16$ and $t = 19$ if feasible.



10. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \left(x + 2x^3\right)^4 = \left(\lim\limits_{x \to -1} x + 2 \lim\limits_{x \to -1} x^3\right)^4 = \left[-1 + 2(-1)^3\right]^4 = (-3)^4 = 81 = f(-1)$.

By the definition of continuity, f is continuous at $a = -1$.

12. $\lim\limits_{t \to 2} g(t) = \lim\limits_{t \to 2} \dfrac{t^2 + 5t}{2t + 1} = \dfrac{\lim\limits_{t \to 2}(t^2 + 5t)}{\lim\limits_{t \to 2}(2t + 1)} = \dfrac{\lim\limits_{t \to 2} t^2 + 5 \lim\limits_{t \to 2} t}{2 \lim\limits_{t \to 2} t + \lim\limits_{t \to 2} 1} = \dfrac{2^2 + 5(2)}{2(2) + 1} = \dfrac{14}{5} = g(2)$.

By the definition of continuity, g is continuous at $a = 2$.

13. $\lim_{v\to 1} p(v) = \lim_{v\to 1} 2\sqrt{3v^2+1} = 2\lim_{v\to 1}\sqrt{3v^2+1} = 2\sqrt{\lim_{v\to 1}(3v^2+1)} = 2\sqrt{3\lim_{v\to 1} v^2 + \lim_{v\to 1} 1}$

$$= 2\sqrt{3(1)^2+1} = 2\sqrt{4} = 4 = p(1)$$

By the definition of continuity, p is continuous at $a = 1$.

14. $\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(3x^4 - 5x + \sqrt[3]{x^2+4}\right) = 3\lim_{x\to 2} x^4 - 5\lim_{x\to 2} x + \sqrt[3]{\lim_{x\to 2}(x^2+4)}$

$$= 3(2)^4 - 5(2) + \sqrt[3]{2^2+4} = 48 - 10 + 2 = 40 = f(2)$$

By the definition of continuity, f is continuous at $a = 2$.

15. For $a > 4$, we have

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a}(x + \sqrt{x-4}) = \lim_{x\to a} x + \lim_{x\to a}\sqrt{x-4} && \text{[Limit Law 1]} \\ &= a + \sqrt{\lim_{x\to a} x - \lim_{x\to a} 4} && \text{[8, 11, and 2]} \\ &= a + \sqrt{a-4} && \text{[8 and 7]} \\ &= f(a) \end{aligned}$$

So f is continuous at $x = a$ for every a in $(4,\infty)$. Also, $\lim_{x\to 4^+} f(x) = 4 = f(4)$, so f is continuous from the right at 4.

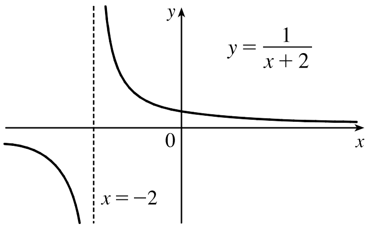
Thus, f is continuous on $[4,\infty)$.

16. For $a < -2$, we have

$$\begin{aligned} \lim_{x\to a} g(x) = \lim_{x\to a}\frac{x-1}{3x+6} &= \frac{\lim_{x\to a}(x-1)}{\lim_{x\to a}(3x+6)} && \text{[Limit Law 5]} \\ &= \frac{\lim_{x\to a} x - \lim_{x\to a} 1}{3\lim_{x\to a} x + \lim_{x\to a} 6} && \text{[2, 1, and 3]} \\ &= \frac{a-1}{3a+6} && \text{[8 and 7]} \end{aligned}$$

Thus, g is continuous at $x = a$ for every a in $(-\infty,-2)$; that is, g is continuous on $(-\infty,-2)$.

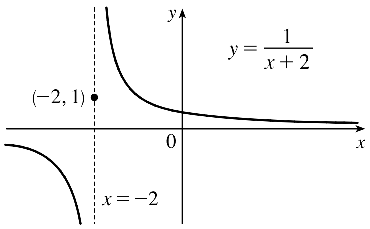
17. $f(x) = \dfrac{1}{x+2}$ is discontinuous at $a = -2$ because $f(-2)$ is undefined.



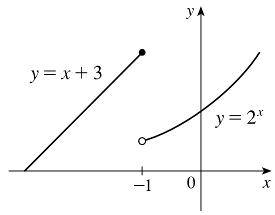
18. $f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$

Here $f(-2) = 1$, but $\lim_{x\to -2^-} f(x) = -\infty$ and $\lim_{x\to -2^+} f(x) = \infty$,

so $\lim_{x\to -2} f(x)$ does not exist and f is discontinuous at -2.



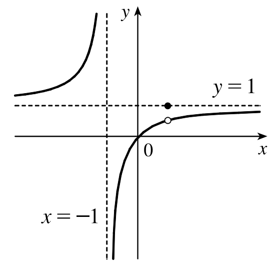
19. $f(x) = \begin{cases} x+3 & \text{if } x \le -1 \\ 2^x & \text{if } x > -1 \end{cases}$



$\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (x+3) = -1+3 = 2$ and

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} 2^x = 2^{-1} = \frac{1}{2}$. Since the left-hand and the right-hand limits of f at -1 are not equal, $\lim\limits_{x \to -1} f(x)$ does not exist, and f is discontinuous at -1.

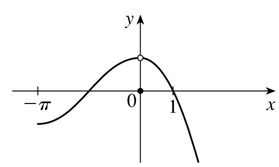
20. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \ne 1 \\ 1 & \text{if } x = 1 \end{cases}$



$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x \to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2},$$

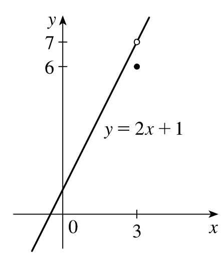
but $f(1) = 1$, so f is discontinous at 1.

21. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$



$\lim\limits_{x \to 0} f(x) = 1$, but $f(0) = 0 \ne 1$, so f is discontinuous at 0.

22. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \ne 3 \\ 6 & \text{if } x = 3 \end{cases}$



$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \to 3} \frac{(2x+1)(x-3)}{x-3} = \lim_{x \to 3} (2x+1) = 7,$$

but $f(3) = 6$, so f is discontinuous at 3.

23. $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{x-2} = x + 1$ for $x \ne 2$. Since $\lim\limits_{x \to 2} f(x) = 2 + 1 = 3$, define $f(2) = 3$. Then f is continuous at 2.

24. $f(x) = \dfrac{x^3 - 8}{x^2 - 4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \dfrac{x^2+2x+4}{x+2}$ for $x \ne 2$. Since $\lim\limits_{x \to 2} f(x) = \dfrac{4+4+4}{2+2} = 3$, define $f(2) = 3$. Then f is continuous at 2.

25. $F(x) = \dfrac{2x^2 - x - 1}{x^2 + 1}$ is a rational function, so it is continuous on its domain, $(-\infty, \infty)$, by Theorem 5(b).

26. $G(x) = \dfrac{x^2 + 1}{2x^2 - x - 1} = \dfrac{x^2 + 1}{(2x+1)(x-1)}$ is a rational function, so it is continuous on its domain, $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$, by Theorem 5(b).

27. $x^3 - 2 = 0 \quad \Rightarrow \quad x^3 = 2 \quad \Rightarrow \quad x = \sqrt[3]{2}$, so $Q(x) = \dfrac{\sqrt[3]{x} - 2}{x^3 - 2}$ has domain $\left(-\infty, \sqrt[3]{2}\right) \cup \left(\sqrt[3]{2}, \infty\right)$. Now $x^3 - 2$ is continuous everywhere by Theorem 5(a) and $\sqrt[3]{x} - 2$ is continuous everywhere by Theorems 5(a), 7, and 9. Thus, Q is continuous on its domain by part 5 of Theorem 4.

28. The domain of $R(t) = \dfrac{e^{\sin t}}{2 + \cos \pi t}$ is $(-\infty, \infty)$ since the denominator is never 0 $[\cos \pi t \geq -1 \quad \Rightarrow \quad 2 + \cos \pi t \geq 1]$. By Theorems 7 and 9, $e^{\sin t}$ and $\cos \pi t$ are continuous on $\mathbb{R}$. By part 1 of Theorem 4, $2 + \cos \pi t$ is continuous on $\mathbb{R}$ and by part 5 of Theorem 4, R is continuous on $\mathbb{R}$.

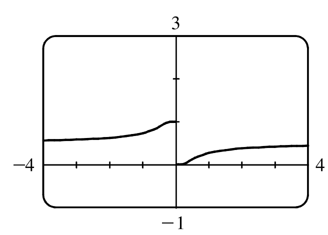
29. By Theorem 5(a), the polynomial $1 + 2t$ is continuous on $\mathbb{R}$. By Theorem 7, the inverse trigonometric function $\arcsin x$ is continuous on its domain, $[-1, 1]$. By Theorem 9, $A(t) = \arcsin(1 + 2t)$ is continuous on its domain, which is $\{t \mid -1 \leq 1 + 2t \leq 1\} = \{t \mid -2 \leq 2t \leq 0\} = \{t \mid -1 \leq t \leq 0\} = [-1, 0]$.

30. By Theorem 7, the trigonometric function $\tan x$ is continuous on its domain, $\left\{x \mid x \neq \frac{\pi}{2} + \pi n\right\}$. By Theorems 5(a), 7, and 9, the composite function $\sqrt{4 - x^2}$ is continuous on its domain $[-2, 2]$. By part 5 of Theorem 4, $B(x) = \dfrac{\tan x}{\sqrt{4 - x^2}}$ is continuous on its domain, $(-2, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 2)$.

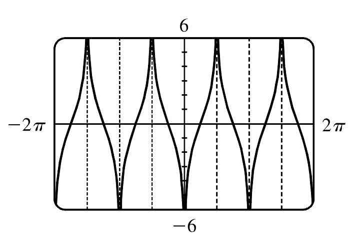
31. $M(x) = \sqrt{1 + \dfrac{1}{x}} = \sqrt{\dfrac{x + 1}{x}}$ is defined when $\dfrac{x + 1}{x} \geq 0 \quad \Rightarrow \quad x + 1 \geq 0$ and $x > 0$ or $x + 1 \leq 0$ and $x < 0 \quad \Rightarrow \quad x > 0$ or $x \leq -1$, so M has domain $(-\infty, -1] \cup (0, \infty)$. M is the composite of a root function and a rational function, so it is continuous at every number in its domain by Theorems 7 and 9.

32. By Theorems 7 and 9, the composite function e^{-r^2} is continuous on $\mathbb{R}$. By part 1 of Theorem 4, $1 + e^{-r^2}$ is continuous on $\mathbb{R}$. By Theorem 7, the inverse trigonometric function $\tan^{-1}$ is continuous on its domain, $\mathbb{R}$. By Theorem 9, the composite function $N(r) = \tan^{-1}\left(1 + e^{-r^2}\right)$ is continuous on its domain, $\mathbb{R}$.

33. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



34. The function $y = \tan^2 x$ is discontinuous at $x = \frac{\pi}{2} + \pi k$, where k is any integer. The function $y = \ln\left(\tan^2 x\right)$ is also discontinuous where $\tan^2 x$ is 0, that is, at $x = \pi k$. So $y = \ln\left(\tan^2 x\right)$ is discontinuous at $x = \frac{\pi}{2} n$, n any integer.



35. Because x is continuous on $\mathbb{R}$ and $\sqrt{20-x^2}$ is continuous on its domain, $-\sqrt{20} \le x \le \sqrt{20}$, the product $f(x) = x\sqrt{20-x^2}$ is continuous on $-\sqrt{20} \le x \le \sqrt{20}$. The number 2 is in that domain, so f is continuous at 2, and $\lim\limits_{x\to 2} f(x) = f(2) = 2\sqrt{16} = 8$.

36. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x\to\pi} f(x) = f(\pi) = \sin(\pi + \sin\pi) = \sin\pi = 0$.

37. The function $f(x) = \ln\left(\dfrac{5-x^2}{1+x}\right)$ is continuous throughout its domain because it is the composite of a logarithm function and a rational function. For the domain of f, we must have $\dfrac{5-x^2}{1+x} > 0$, so the numerator and denominator must have the same sign, that is, the domain is $(-\infty, -\sqrt{5}\,] \cup (-1, \sqrt{5}\,]$. The number 1 is in that domain, so f is continuous at 1, and $\lim\limits_{x\to 1} f(x) = f(1) = \ln\dfrac{5-1}{1+1} = \ln 2$.

38. The function $f(x) = 3^{\sqrt{x^2-2x-4}}$ is continuous throughout its domain because it is the composite of an exponential function, a root function, and a polynomial. Its domain is

$$\begin{aligned}\left\{x \mid x^2-2x-4 \ge 0\right\} &= \left\{x \mid x^2-2x+1 \ge 5\right\} = \left\{x \mid (x-1)^2 \ge 5\right\}\\ &= \left\{x \,\middle|\, |x-1| \ge \sqrt{5}\right\} = (-\infty, 1-\sqrt{5}\,] \cup [1+\sqrt{5}, \infty)\end{aligned}$$

The number 4 is in that domain, so f is continuous at 4, and $\lim\limits_{x\to 4} f(x) = f(4) = 3^{\sqrt{16-8-4}} = 3^2 = 9$.

39. $f(x) = \begin{cases} 1-x^2 & \text{if } x \le 1 \\ \ln x & \text{if } x > 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial $1-x^2$ on $(-\infty, 1]$, f is continuous on $(-\infty, 1]$.

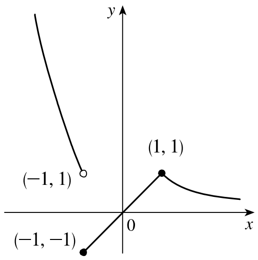
By Theorem 7, since $f(x)$ equals the logarithm function $\ln x$ on $(1, \infty)$, f is continuous on $(1, \infty)$.

At $x = 1$, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (1-x^2) = 1 - 1^2 = 0$ and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \ln x = \ln 1 = 0$. Thus, $\lim\limits_{x\to 1} f(x)$ exists and equals 0. Also, $f(1) = 1 - 1^2 = 0$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

40. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \ge \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

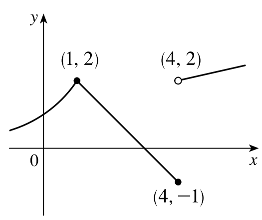
41. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \le x < 1 \\ 1/x & \text{if } x \ge 1 \end{cases}$



f is continuous on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, where it is a polynomial, a polynomial, and a rational function, respectively.

Now $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} x^2 = 1$ and $\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} x = -1$, so f is discontinuous at -1. Since $f(-1) = -1$, f is continuous from the right at -1. Also, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{1}{x} = 1 = f(1)$, so f is continuous at 1.

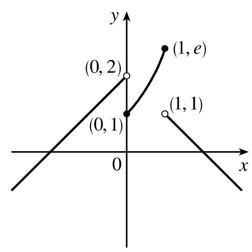
42. $f(x) = \begin{cases} 2^x & \text{if } x \le 1 \\ 3 - x & \text{if } 1 < x \le 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$



f is continuous on $(-\infty, 1)$, $(1, 4)$, and $(4, \infty)$, where it is an exponential, a polynomial, and a root function, respectively.

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2^x = 2$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (3 - x) = 2$. Since $f(1) = 2$ we have continuity at 1. Also, $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (3 - x) = -1 = f(4)$ and $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \sqrt{x} = 2$, so f is discontinuous at 4, but it is continuous from the left at 4.

43. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 2) = 2$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} e^x = e$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2 - x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

44. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$.

$\lim\limits_{r \to R^-} F(r) = \lim\limits_{r \to R^-} \dfrac{GMr}{R^3} = \dfrac{GM}{R^2}$ and $\lim\limits_{r \to R^+} F(r) = \lim\limits_{r \to R^+} \dfrac{GM}{r^2} = \dfrac{GM}{R^2}$, so $\lim\limits_{r \to R} F(r) = \dfrac{GM}{R^2}$. Since $F(R) = \dfrac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

45. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \ge 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left(cx^2 + 2x\right) = 4c + 4$ and

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^3 - cx) = 8 - 2c$. So f is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

46. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \le x < 3 \\ 2x - a + b & \text{if } x \ge 3 \end{cases}$

At $x = 2$: $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2^-} \dfrac{(x+2)(x-2)}{x-2} = \lim\limits_{x\to 2^-} (x+2) = 2 + 2 = 4$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - 2b + 3 = 4$, or $\boldsymbol{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} (2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\boldsymbol{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rcr} -8a + 4b &=& -2 \\ 10a - 4b &=& 3 \\ \hline 2a &=& 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

47. If f and g are continuous and $g(2) = 6$, then $\lim\limits_{x\to 2}[3f(x) + f(x)\,g(x)] = 36$ $\Rightarrow$

$3\lim\limits_{x\to 2} f(x) + \lim\limits_{x\to 2} f(x) \cdot \lim\limits_{x\to 2} g(x) = 36$ $\Rightarrow$ $3f(2) + f(2)\cdot 6 = 36$ $\Rightarrow$ $9f(2) = 36$ $\Rightarrow$ $f(2) = 4$.

48. (a) $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$, so $(f \circ g)(x) = f(g(x)) = f(1/x^2) = 1\,/(1/x^2) = x^2$.

(b) The domain of $f \circ g$ is the set of numbers x in the domain of g (all nonzero reals) such that $g(x)$ is in the domain of f (also all nonzero reals). Thus, the domain is $\left\{ x \,\middle|\, x \ne 0 \text{ and } \dfrac{1}{x^2} \ne 0 \right\} = \{x \mid x \ne 0\}$ or $(-\infty, 0) \cup (0, \infty)$. Since $f \circ g$ is the composite of two rational functions, it is continuous throughout its domain; that is, everywhere except $x = 0$.

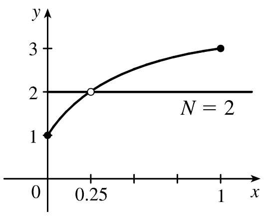
49. (a) $f(x) = \dfrac{x^4-1}{x-1} = \dfrac{(x^2+1)(x^2-1)}{x-1} = \dfrac{(x^2+1)(x+1)(x-1)}{x-1} = (x^2+1)(x+1)$ [or $x^3 + x^2 + x + 1$]

for $x \ne 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \ne 1$ and is continuous on $\mathbb{R}$.

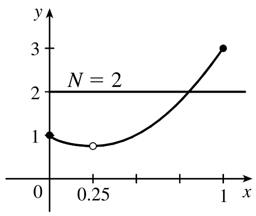
(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x-2} = \dfrac{x(x^2 - x - 2)}{x-2} = \dfrac{x(x-2)(x+1)}{x-2} = x(x+1)$ [or $x^2 + x$] for $x \ne 2$. The discontinuity is removable and $g(x) = x^2 + x$ agrees with f for $x \ne 2$ and is continuous on $\mathbb{R}$.

(c) $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} [\![\sin x]\!] = \lim\limits_{x\to\pi^-} 0 = 0$ and $\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{x\to\pi^+} [\![\sin x]\!] = \lim\limits_{x\to\pi^+} (-1) = -1$, so $\lim\limits_{x\to\pi} f(x)$ does not exist. The discontinuity at $x = \pi$ is a jump discontinuity.

50.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

51. $f(x) = x^2 + 10\sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

52. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

53. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

54. The equation $\ln x = x - \sqrt{x}$ is equivalent to the equation $\ln x - x + \sqrt{x} = 0$. $f(x) = \ln x - x + \sqrt{x}$ is continuous on the interval $[2, 3]$, $f(2) = \ln 2 - 2 + \sqrt{2} \approx 0.107$, and $f(3) = \ln 3 - 3 + \sqrt{3} \approx -0.169$. Since $f(2) > 0 > f(3)$, there is a number c in $(2, 3)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - x + \sqrt{x} = 0$, or $\ln x = x - \sqrt{x}$, in the interval $(2, 3)$.

55. The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$, $f(0) = -2$, and $f(1) = e - 1 \approx 1.72$. Since $-2 < 0 < e - 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.

56. The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

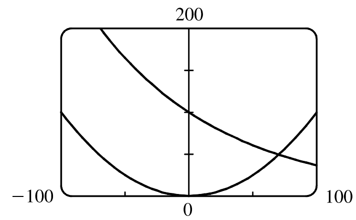
57. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

58. (a) $f(x) = \ln x - 3 + 2x$ is continuous on the interval $[1, 2]$, $f(1) = -1 < 0$, and $f(2) = \ln 2 + 1 \approx 1.7 > 0$. Since $-1 < 0 < 1.7$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - 3 + 2x = 0$, or $\ln x = 3 - 2x$, in the interval $(1, 2)$.

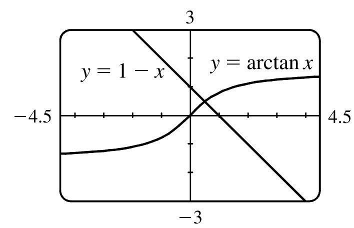
(b) $f(1.34) \approx -0.03 < 0$ and $f(1.35) \approx 0.0001 > 0$, so there is a root between 1.34 and 1.35, that is, in the interval $(1.34, 1.35)$.

59. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 100)$ such that $f(c) = 0$. This implies that $100e^{-c/100} = 0.01c^2$.



(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.

60. (a) Let $f(x) = \arctan x + x - 1$. Then $f(0) = -1 < 0$ and $f(1) = \frac{\pi}{4} > 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that $f(c) = 0$. This implies that $\arctan c = 1 - c$.



(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 0.520$, correct to three decimal places.

61. Let $f(x) = \sin x^3$. Then f is continuous on $[1, 2]$ since f is the composite of the sine function and the cubing function, both of which are continuous on $\mathbb{R}$. The zeros of the sine are at $n\pi$, so we note that $0 < 1 < \pi < \frac{3}{2}\pi < 2\pi < 8 < 3\pi$, and that the pertinent cube roots are related by $1 < \sqrt[3]{\frac{3}{2}\pi}$ [call this value A] < 2. [By observation, we might notice that $x = \sqrt[3]{\pi}$ and $x = \sqrt[3]{2\pi}$ are zeros of f.]

Now $f(1) = \sin 1 > 0$, $f(A) = \sin \frac{3}{2}\pi = -1 < 0$, and $f(2) = \sin 8 > 0$. Applying the Intermediate Value Theorem on $[1, A]$ and then on $[A, 2]$, we see there are numbers c and d in $(1, A)$ and $(A, 2)$ such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in $(1, 2)$.

62. Let $f(x) = x^2 - 3 + 1/x$. Then f is continuous on $(0, 2]$ since f is a rational function whose domain is $(0, \infty)$. By inspection, we see that $f\left(\frac{1}{4}\right) = \frac{17}{16} > 0$, $f(1) = -1 < 0$, and $f(2) = \frac{3}{2} > 0$. Appling the Intermediate Value Theorem on $\left[\frac{1}{4}, 1\right]$ and then on $[1, 2]$, we see there are numbers c and d in $\left(\frac{1}{4}, 1\right)$ and $(1, 2)$ such that $f(c) = f(d) = 0$. Thus, f has at least two x-intercepts in $(0, 2)$.

63. ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h) = a + h$, we have

$$\lim_{h\to 0} f(a+h) = f\left(\lim_{h\to 0}(a+h)\right) = f(a).$$

($\Leftarrow$) Let $\varepsilon > 0$. Since $\lim_{h\to 0} f(a+h) = f(a)$, there exists $\delta > 0$ such that $0 < |h| < \delta \Rightarrow$

$|f(a+h) - f(a)| < \varepsilon$. So if $0 < |x-a| < \delta$, then $|f(x) - f(a)| = |f(a + (x-a)) - f(a)| < \varepsilon$.

Thus, $\lim_{x\to a} f(x) = f(a)$ and so f is continuous at a.

64. $\lim_{h\to 0} \sin(a+h) = \lim_{h\to 0}(\sin a\cos h + \cos a \sin h) = \lim_{h\to 0}(\sin a \cos h) + \lim_{h\to 0}(\cos a \sin h)$

$$= \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right) + \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right) = (\sin a)(1) + (\cos a)(0) = \sin a$$

65. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\lim_{h\to 0}\cos(a+h) = \lim_{h\to 0}(\cos a\cos h - \sin a \sin h) = \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a \sin h)$$

$$= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right) - \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a$$

66. (a) Since f is continuous at a, $\lim_{x\to a} f(x) = f(a)$. Thus, using the Constant Multiple Law of Limits, we have

$\lim_{x\to a}(cf)(x) = \lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a.

(b) Since f and g are continuous at a, $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law

of Limits: $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$. Thus, $\dfrac{f}{g}$ is continuous at a.

67. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x\to a} f(x) \neq f(a)$. [In fact, $\lim_{x\to a} f(x)$ does not even exist.]

68. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0. To see why, note that $-|x| \le g(x) \le |x|$, so by the Squeeze Theorem $\lim_{x\to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a, there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x\to a} g(x) \neq g(a)$.

69. If there is such a number, it satisfies the equation $x^3 + 1 = x \Leftrightarrow x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

70. $\dfrac{a}{x^3 + 2x^2 - 1} + \dfrac{b}{x^3 + x - 2} = 0 \Rightarrow a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. Let $p(x)$ denote the left side of the last equation. Since p is continuous on $[-1, 1]$, $p(-1) = -4a < 0$, and $p(1) = 2b > 0$, there exists a c in $(-1, 1)$ such that

$p(c) = 0$ by the Intermediate Value Theorem. Note that the only root of either denominator that is in $(-1, 1)$ is $(-1 + \sqrt{5})/2 = r$, but $p(r) = (3\sqrt{5} - 9)a/2 \neq 0$. Thus, c is not a root of either denominator, so $p(c) = 0 \Rightarrow x = c$ is a root of the given equation.

71. $f(x) = x^4 \sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \leq \sin(1/x) \leq 1$, we have $-x^4 \leq x^4 \sin(1/x) \leq x^4$. Because $\lim_{x \to 0}(-x^4) = 0$ and $\lim_{x \to 0} x^4 = 0$, the Squeeze Theorem gives us $\lim_{x \to 0}(x^4 \sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

72. (a) $\lim_{x \to 0^+} F(x) = 0$ and $\lim_{x \to 0^-} F(x) = 0$, so $\lim_{x \to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} x = a = F(a)$. For $a < 0$, $\lim_{x \to a} F(x) = \lim_{x \to a}(-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim_{x \to a} |f(x)| = \left|\lim_{x \to a} f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

73. Define $u(t)$ to be the monk's distance from the monastery, as a function of time t (in hours), on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \Leftrightarrow u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.