2.4 The Precise Definition of a Limit

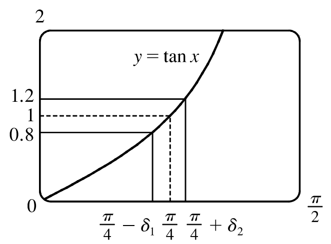
1. If $|f(x) - 1| < 0.2$, then $-0.2 < f(x) - 1 < 0.2 \quad \Rightarrow \quad 0.8 < f(x) < 1.2$. From the graph, we see that the last inequality is true if $0.7 < x < 1.1$, so we can choose $\delta = \min\{1 - 0.7, 1.1 - 1\} = \min\{0.3, 0.1\} = 0.1$ (or any smaller positive number).

2. If $|f(x) - 2| < 0.5$, then $-0.5 < f(x) - 2 < 0.5 \quad \Rightarrow \quad 1.5 < f(x) < 2.5$. From the graph, we see that the last inequality is true if $2.6 < x < 3.8$, so we can take $\delta = \min\{3 - 2.6, 3.8 - 3\} = \min\{0.4, 0.8\} = 0.4$ (or any smaller positive number). Note that $x \neq 3$.

3. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x - 4| < |2.56 - 4| = 1.44$. On the right side, we need $|x - 4| < |5.76 - 4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold—namely, $|x - 4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

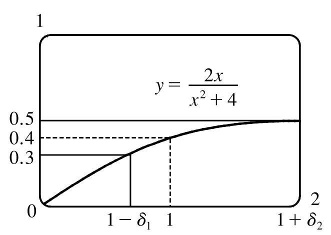
4. The leftmost question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the rightmost question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$ (rounding down to be safe). On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

5.

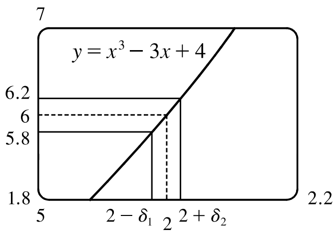


From the graph, we find that $y = \tan x = 0.8$ when $x \approx 0.675$, so $\frac{\pi}{4} - \delta_1 \approx 0.675 \quad \Rightarrow \quad \delta_1 \approx \frac{\pi}{4} - 0.675 \approx 0.1106$. Also, $y = \tan x = 1.2$ when $x \approx 0.876$, so $\frac{\pi}{4} + \delta_2 \approx 0.876 \quad \Rightarrow \quad \delta_2 = 0.876 - \frac{\pi}{4} \approx 0.0906$. Thus, we choose $\delta = 0.0906$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

6.

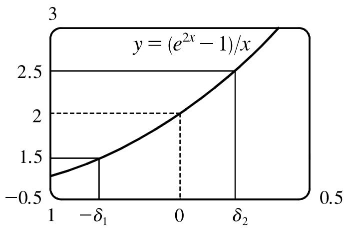


From the graph, we find that $y = 2x/(x^2 + 4) = 0.3$ when $x = \frac{2}{3}$, so $1 - \delta_1 = \frac{2}{3} \quad \Rightarrow \quad \delta_1 = \frac{1}{3}$. Also, $y = 2x/(x^2 + 4) = 0.5$ when $x = 2$, so $1 + \delta_2 = 2 \quad \Rightarrow \quad \delta_2 = 1$. Thus, we choose $\delta = \frac{1}{3}$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

7. 

From the graph with $\varepsilon = 0.2$, we find that $y = x^3 - 3x + 4 = 5.8$ when $x \approx 1.9774$, so $2 - \delta_1 \approx 1.9774 \Rightarrow \delta_1 \approx 0.0226$. Also, $y = x^3 - 3x + 4 = 6.2$ when $x \approx 2.022$, so $2 + \delta_2 \approx 2.0219 \Rightarrow \delta_2 \approx 0.0219$. Thus, we choose $\delta = 0.0219$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

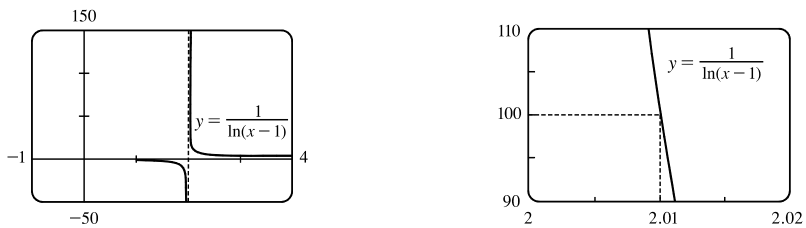
For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.0112$ and $\delta_2 \approx 0.0110$, so we choose $\delta = 0.011$ (or any smaller positive number).

8. 

From the graph with $\varepsilon = 0.5$, we find that $y = (e^{2x} - 1)/x = 1.5$ when $x \approx -0.303$, so $\delta_1 \approx 0.303$. Also, $y = (e^{2x} - 1)/x = 2.5$ when $x \approx 0.215$, so $\delta_2 \approx 0.215$. Thus, we choose $\delta = 0.215$ (or any smaller positive number) since this is the smaller of δ_1 and δ_2.

For $\varepsilon = 0.1$, we get $\delta_1 \approx 0.052$ and $\delta_2 \approx 0.048$, so we choose $\delta = 0.048$ (or any smaller positive number).

9. (a)

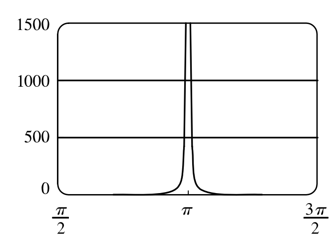


The first graph of $y = \dfrac{1}{\ln(x-1)}$ shows a vertical asymptote at $x = 2$. The second graph shows that $y = 100$ when $x \approx 2.01$ (more accurately, 2.01005). Thus, we choose $\delta = 0.01$ (or any smaller positive number).

(b) From part (a), we see that as x gets closer to 2 from the right, y increases without bound. In symbols,

$$\lim_{x \to 2^+} \frac{1}{\ln(x-1)} = \infty.$$

10. We graph $y = \csc^2 x$ and $y = 500$. The graphs intersect at $x \approx 3.186$, so we choose $\delta = 3.186 - \pi \approx 0.044$. Thus, if $0 < |x - \pi| < 0.044$, then $\csc^2 x > 500$. Similarly, for $M = 1000$, we get $\delta = 3.173 - \pi \approx 0.031$.



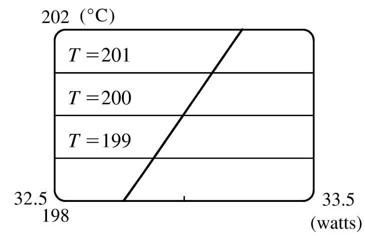
11. (a) $A = \pi r^2$ and $A = 1000\text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow r = \sqrt{\frac{1000}{\pi}}\quad (r > 0) \quad \approx 17.8412$ cm.

(b) $|A - 1000| \le 5 \Rightarrow -5 \le \pi r^2 - 1000 \le 5 \Rightarrow 1000 - 5 \le \pi r^2 \le 1000 + 5 \Rightarrow$ $\sqrt{\frac{995}{\pi}} \le r \le \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \le r \le 17.8858$. $\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and $\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412, the area will be within 5 cm^2 of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000 cm^2), ε is the magnitude of the error tolerance in the area (5 cm^2), and δ is the tolerance in the radius given in part (b).

12. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \quad \Rightarrow$

$0.1w^2 + 2.155w + 20 = 200 \quad \Rightarrow$ [by the quadratic formula or from the graph] $\quad w \approx 33.0$ watts $(w > 0)$



(b) From the graph, $199 \le T \le 201 \quad \Rightarrow \quad 32.89 < w < 33.11$.

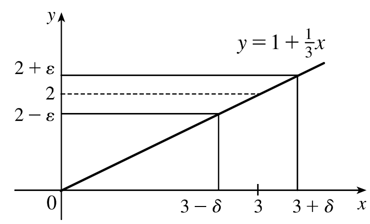
(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

13. (a) $|4x - 8| = 4\,|x - 2| < 0.1 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.1}{4}$, so $\delta = \dfrac{0.1}{4} = 0.025$.

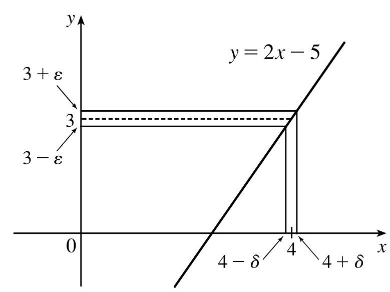
(b) $|4x - 8| = 4\,|x - 2| < 0.01 \quad \Leftrightarrow \quad |x - 2| < \dfrac{0.01}{4}$, so $\delta = \dfrac{0.01}{4} = 0.0025$.

14. $|(5x - 7) - 3| = |5x - 10| = |5(x - 2)| = 5\,|x - 2|$. We must have $|f(x) - L| < \varepsilon$, so $5\,|x - 2| < \varepsilon \quad \Leftrightarrow$ $|x - 2| < \varepsilon/5$. Thus, choose $\delta = \varepsilon/5$. For $\varepsilon = 0.1$, $\delta = 0.02$; for $\varepsilon = 0.05$, $\delta = 0.01$; for $\varepsilon = 0.01$, $\delta = 0.002$.

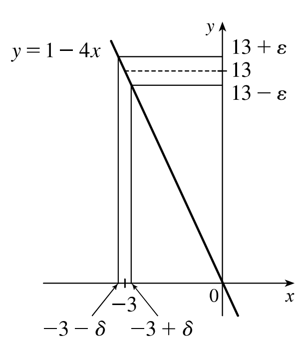
15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left|\left(1 + \frac{1}{3}x\right) - 2\right| < \varepsilon$. But $\left|\left(1 + \frac{1}{3}x\right) - 2\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{1}{3}x - 1\right| < \varepsilon \quad \Leftrightarrow$ $\left|\frac{1}{3}\right|\,|x - 3| < \varepsilon \quad \Leftrightarrow \quad |x - 3| < 3\varepsilon$. So if we choose $\delta = 3\varepsilon$, then $0 < |x - 3| < \delta \quad \Rightarrow \quad \left|\left(1 + \frac{1}{3}x\right) - 2\right| < \varepsilon$. Thus, $\displaystyle\lim_{x \to 3}\left(1 + \tfrac{1}{3}x\right) = 2$ by the definition of a limit.



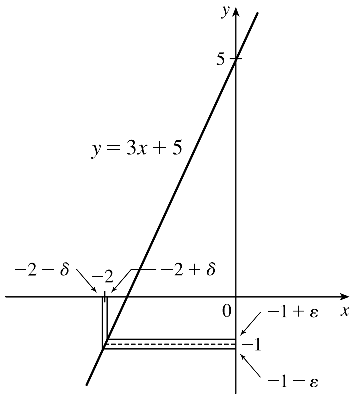
16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(2x - 5) - 3| < \varepsilon$. But $|(2x - 5) - 3| < \varepsilon \quad \Leftrightarrow \quad |2x - 8| < \varepsilon \quad \Leftrightarrow$ $|2|\,|x - 4| < \varepsilon \quad \Leftrightarrow \quad |x - 4| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x - 4| < \delta \quad \Rightarrow \quad |(2x - 5) - 3| < \varepsilon$. Thus, $\displaystyle\lim_{x \to 4}(2x - 5) = 3$ by the definition of a limit.



17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-3)| < \delta$, then $|(1 - 4x) - 13| < \varepsilon$. But $|(1 - 4x) - 13| < \varepsilon \quad \Leftrightarrow$ $|-4x - 12| < \varepsilon \quad \Leftrightarrow \quad |-4|\,|x + 3| < \varepsilon \quad \Leftrightarrow \quad |x - (-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x - (-3)| < \delta \quad \Rightarrow \quad |(1 - 4x) - 13| < \varepsilon$. Thus, $\displaystyle\lim_{x \to -3}(1 - 4x) = 13$ by the definition of a limit.



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(3x + 5) - (-1)| < \varepsilon$. But $|(3x + 5) - (-1)| < \varepsilon \quad \Leftrightarrow$ $|3x + 6| < \varepsilon \quad \Leftrightarrow \quad |3|\,|x + 2| < \varepsilon \quad \Leftrightarrow \quad |x + 2| < \varepsilon/3$. So if we choose $\delta = \varepsilon/3$, then $0 < |x + 2| < \delta \quad \Rightarrow \quad |(3x + 5) - (-1)| < \varepsilon$. Thus, $\lim\limits_{x \to -2}(3x + 5) = -1$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon$. But $\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon \quad \Leftrightarrow$

$\left|\dfrac{4x - 4}{3}\right| < \varepsilon \quad \Leftrightarrow \quad \left|\frac{4}{3}\right|\,|x - 1| < \varepsilon \quad \Leftrightarrow \quad |x - 1| < \frac{3}{4}\varepsilon$. So if we choose $\delta = \frac{3}{4}\varepsilon$, then $0 < |x - 1| < \delta \quad \Rightarrow$

$\left|\dfrac{2 + 4x}{3} - 2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 1} \dfrac{2 + 4x}{3} = 2$ by the definition of a limit.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 10| < \delta$, then $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. But $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon \quad \Leftrightarrow$ $\left|8 - \frac{4}{5}x\right| < \varepsilon \quad \Leftrightarrow \quad \left|-\frac{4}{5}\right|\,|x - 10| < \varepsilon \quad \Leftrightarrow \quad |x - 10| < \frac{5}{4}\varepsilon$. So if we choose $\delta = \frac{5}{4}\varepsilon$, then $0 < |x - 10| < \delta \quad \Rightarrow$ $\left|3 - \frac{4}{5}x - (-5)\right| < \varepsilon$. Thus, $\lim\limits_{x \to 10}(3 - \frac{4}{5}x) = -5$ by the definition of a limit.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $\left|\dfrac{x^2 - 2x - 8}{x - 4} - 6\right| < \varepsilon \quad \Leftrightarrow$

$\left|\dfrac{(x - 4)(x + 2)}{x - 4} - 6\right| < \varepsilon \quad \Leftrightarrow \quad |x + 2 - 6| < \varepsilon \quad [x \neq 4] \quad \Leftrightarrow \quad |x - 4| < \varepsilon$. So choose $\delta = \varepsilon$. Then

$0 < |x - 4| < \delta \quad \Rightarrow \quad |x - 4| < \varepsilon \quad \Rightarrow \quad |x + 2 - 6| < \varepsilon \quad \Rightarrow \quad \left|\dfrac{(x - 4)(x + 2)}{x - 4} - 6\right| < \varepsilon \quad [x \neq 4] \quad \Rightarrow$

$\left|\dfrac{x^2 - 2x - 8}{x - 4} - 6\right| < \varepsilon$. By the definition of a limit, $\lim\limits_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4} = 6$.

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x + 1.5| < \delta$, then $\left|\dfrac{9 - 4x^2}{3 + 2x} - 6\right| < \varepsilon \quad \Leftrightarrow$

$\left|\dfrac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6\right| < \varepsilon \quad \Leftrightarrow \quad |3 - 2x - 6| < \varepsilon \quad [x \neq -1.5] \quad \Leftrightarrow \quad |-2x - 3| < \varepsilon \quad \Leftrightarrow \quad |-2|\,|x + 1.5| < \varepsilon \quad \Leftrightarrow$

$|x + 1.5| < \varepsilon/2$. So choose $\delta = \varepsilon/2$. Then $0 < |x + 1.5| < \delta \quad \Rightarrow \quad |x + 1.5| < \varepsilon/2 \quad \Rightarrow \quad |-2|\,|x + 1.5| < \varepsilon \quad \Rightarrow$

$|-2x - 3| < \varepsilon \quad \Rightarrow \quad |3 - 2x - 6| < \varepsilon \quad \Rightarrow \quad \left|\dfrac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6\right| < \varepsilon \quad [x \neq -1.5] \quad \Rightarrow \quad \left|\dfrac{9 - 4x^2}{3 + 2x} - 6\right| < \varepsilon$.

By the definition of a limit, $\lim\limits_{x \to -1.5} \dfrac{9 - 4x^2}{3 + 2x} = 6$.

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$. But $|c - c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^2 - 0\right| < \varepsilon \quad \Leftrightarrow \quad x^2 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^2 - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\left|x^3 - 0\right| < \varepsilon \quad \Leftrightarrow \quad |x|^3 < \varepsilon \quad \Leftrightarrow \quad |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x - 0| < \delta \quad \Rightarrow \quad \left|x^3 - 0\right| < \delta^3 = \varepsilon$. Thus, $\lim\limits_{x \to 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 0| < \delta$, then $\big||x| - 0\big| < \varepsilon$. But $\big||x|\big| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim\limits_{x \to 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < x - (-6) < \delta$, then $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon$. But $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon \quad \Leftrightarrow$ $\sqrt[8]{6 + x} < \varepsilon \quad \Leftrightarrow \quad 6 + x < \varepsilon^8 \quad \Leftrightarrow \quad x - (-6) < \varepsilon^8$. So if we choose $\delta = \varepsilon^8$, then $0 < x - (-6) < \delta \quad \Rightarrow$ $\left|\sqrt[8]{6 + x} - 0\right| < \varepsilon$. Thus, $\lim\limits_{x \to -6^+} \sqrt[8]{6 + x} = 0$ by the definition of a right-hand limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|\left(x^2 - 4x + 5\right) - 1\right| < \varepsilon \quad \Leftrightarrow \quad \left|x^2 - 4x + 4\right| < \varepsilon \quad \Leftrightarrow$ $\left|(x - 2)^2\right| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x - 2| < \delta \quad \Leftrightarrow \quad |x - 2| < \sqrt{\varepsilon} \quad \Leftrightarrow \quad \left|(x - 2)^2\right| < \varepsilon$. Thus, $\lim\limits_{x \to 2} \left(x^2 - 4x + 5\right) = 1$ by the definition of a limit.

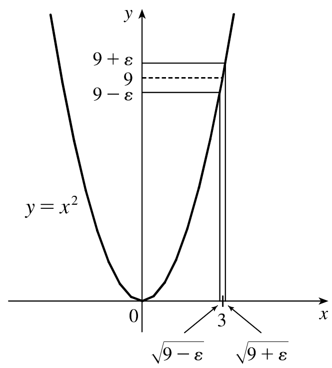
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon$. But $\left|(x^2 + 2x - 7) - 1\right| < \varepsilon \quad \Leftrightarrow$ $\left|x^2 + 2x - 8\right| < \varepsilon \quad \Leftrightarrow \quad |x + 4|\,|x - 2| < \varepsilon$. Thus our goal is to make $|x - 2|$ small enough so that its product with $|x + 4|$ is less than ε. Suppose we first require that $|x - 2| < 1$. Then $-1 < x - 2 < 1 \quad \Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 5 < x + 4 < 7 \quad \Rightarrow$ $|x + 4| < 7$, and this gives us $7\,|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \varepsilon/7$. Choose $\delta = \min\{1, \varepsilon/7\}$. Then if $0 < |x - 2| < \delta$, we have $|x - 2| < \varepsilon/7$ and $|x + 4| < 7$, so $\left|(x^2 + 2x - 7) - 1\right| = |(x + 4)(x - 2)| = |x + 4|\,|x - 2| < 7(\varepsilon/7) = \varepsilon$, as desired. Thus, $\lim\limits_{x \to 2}(x^2 + 2x - 7) = 1$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $\left|\left(x^2 - 1\right) - 3\right| < \varepsilon$ or upon simplifying we need $\left|x^2 - 4\right| < \varepsilon$ whenever $0 < |x + 2| < \delta$. Notice that if $|x + 2| < 1$, then $-1 < x + 2 < 1 \quad \Rightarrow \quad -5 < x - 2 < -3 \quad \Rightarrow$ $|x - 2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x + 2| < \delta \quad \Rightarrow \quad |x - 2| < 5$ and $|x + 2| < \varepsilon/5$, so $\left|\left(x^2 - 1\right) - 3\right| = |(x + 2)(x - 2)| = |x + 2|\,|x - 2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to -2}(x^2 - 1) = 3$.

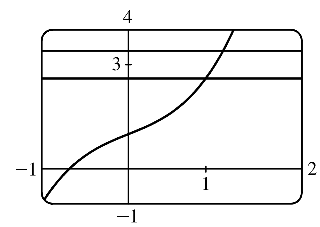
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $\left|x^3 - 8\right| < \varepsilon$. Now $\left|x^3 - 8\right| = \left|(x - 2)\left(x^2 + 2x + 4\right)\right|$. If $|x - 2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $\left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < 19\,|x - 2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x - 2| < \delta \quad \Rightarrow$ $\left|x^3 - 8\right| = |x - 2|\left(x^2 + 2x + 4\right) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim\limits_{x \to 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \;\Rightarrow\; -2 < x-3 < 2 \;\Rightarrow$ $4 < x+3 < 8 \;\Rightarrow\; |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $\left|x^2-9\right| = |x+3|\,|x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim\limits_{x\to 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9-\varepsilon}$ and $\delta_2 = \sqrt{9+\varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9+\varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1 - x_1$ and $x_2 - 1$. So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \varepsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\varepsilon) = \frac{\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\,\right)^{2/3} - 12}{6\left(216 + 108\varepsilon + 12\sqrt{336 + 324\varepsilon + 81\varepsilon^2}\,\right)^{1/3}}.$$ Thus, $\delta = x(\varepsilon) - 1$.

(c) If $\varepsilon = 0.4$, then $x(\varepsilon) \approx 1.093\,272\,342$ and $\delta = x(\varepsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. *1. Guessing a value for* δ Let $\varepsilon > 0$ be given. We have to find a number $\delta > 0$ such that $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$. But $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \left|\dfrac{2-x}{2x}\right| = \dfrac{|x-2|}{|2x|} < \varepsilon$. We find a positive constant C such that $\dfrac{1}{|2x|} < C \;\Rightarrow$ $\dfrac{|x-2|}{|2x|} < C\,|x-2|$ and we can make $C\,|x-2| < \varepsilon$ by taking $|x-2| < \dfrac{\varepsilon}{C} = \delta$. We restrict x to lie in the interval $|x-2| < 1 \;\Rightarrow\; 1 < x < 3$ so $1 > \dfrac{1}{x} > \dfrac{1}{3} \;\Rightarrow\; \dfrac{1}{6} < \dfrac{1}{2x} < \dfrac{1}{2} \;\Rightarrow\; \dfrac{1}{|2x|} < \dfrac{1}{2}$. So $C = \dfrac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\varepsilon\}$.

2. Showing that δ *works* Given $\varepsilon > 0$ we let $\delta = \min\{1, 2\varepsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \;\Rightarrow\; 1 < x < 3 \;\Rightarrow$ $\dfrac{1}{|2x|} < \dfrac{1}{2}$ (as in part 1). Also $|x-2| < 2\varepsilon$, so $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \dfrac{|x-2|}{|2x|} < \dfrac{1}{2} \cdot 2\varepsilon = \varepsilon$. This shows that $\lim\limits_{x\to 2}(1/x) = \frac{1}{2}$.

37. *1. Guessing a value for* δ Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x-a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \dfrac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then

$\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a. If $|x-a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x-a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a \Rightarrow \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a}+\sqrt{a}$ is a suitable choice for the constant. So $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$. This suggests that we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$.

2. Showing that δ works Given $\varepsilon > 0$, we let $\delta = \min\left\{\frac{1}{2}a, \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon\right\}$. If $0 < |x-a| < \delta$, then $|x-a| < \frac{1}{2}a \Rightarrow \sqrt{x}+\sqrt{a} > \sqrt{\frac{1}{2}a}+\sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a}+\sqrt{a}\right)\varepsilon$, so

$|\sqrt{x}-\sqrt{a}\,| = \dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{\left(\sqrt{a/2}+\sqrt{a}\right)\varepsilon}{\left(\sqrt{a/2}+\sqrt{a}\right)} = \varepsilon$. Therefore, $\lim\limits_{x\to a}\sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim\limits_{t\to 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |H(t)-L| < \frac{1}{2} \Leftrightarrow L-\frac{1}{2} < H(t) < L+\frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L+\frac{1}{2} \Rightarrow L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L-\frac{1}{2} < 0 \Rightarrow L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim\limits_{t\to 0} H(t)$ does not exist.

39. Suppose that $\lim\limits_{x\to 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |f(x)-L| < \frac{1}{2}$. Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0-L| < \frac{1}{2}$, so $L \le |L| < \frac{1}{2}$. Now take any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1-L| < \frac{1}{2}$. Hence, $1-L < \frac{1}{2}$, so $L > \frac{1}{2}$. This contradicts $L < \frac{1}{2}$, so $\lim\limits_{x\to 0} f(x)$ does not exist.

40. First suppose that $\lim\limits_{x\to a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$. Then $a-\delta < x < a \Rightarrow 0 < |x-a| < \delta$ so $|f(x)-L| < \varepsilon$. Thus, $\lim\limits_{x\to a^-} f(x) = L$. Also $a < x < a+\delta \Rightarrow 0 < |x-a| < \delta$ so $|f(x)-L| < \varepsilon$. Hence, $\lim\limits_{x\to a^+} f(x) = L$.

Now suppose $\lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to a^-} f(x) = L$, there exists $\delta_1 > 0$ so that $a-\delta_1 < x < a \Rightarrow |f(x)-L| < \varepsilon$. Since $\lim\limits_{x\to a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a+\delta_2 \Rightarrow |f(x)-L| < \varepsilon$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x-a| < \delta \Rightarrow a-\delta_1 < x < a$ or $a < x < a+\delta_2$ so $|f(x)-L| < \varepsilon$. Hence, $\lim\limits_{x\to a} f(x) = L$. So we have proved that $\lim\limits_{x\to a} f(x) = L \Leftrightarrow \lim\limits_{x\to a^-} f(x) = L = \lim\limits_{x\to a^+} f(x)$.

41. $\dfrac{1}{(x+3)^4} > 10{,}000 \Leftrightarrow (x+3)^4 < \dfrac{1}{10{,}000} \Leftrightarrow |x+3| < \dfrac{1}{\sqrt[4]{10{,}000}} \Leftrightarrow |x-(-3)| < \dfrac{1}{10}$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \Rightarrow 1/(x+3)^4 > M$. Now $\dfrac{1}{(x+3)^4} > M \Leftrightarrow (x+3)^4 < \dfrac{1}{M} \Leftrightarrow |x+3| < \dfrac{1}{\sqrt[4]{M}}$. So take $\delta = \dfrac{1}{\sqrt[4]{M}}$. Then $0 < |x+3| < \delta = \dfrac{1}{\sqrt[4]{M}} \Rightarrow \dfrac{1}{(x+3)^4} > M$, so $\lim\limits_{x\to -3}\dfrac{1}{(x+3)^4} = \infty$.

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$. This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit, $\lim\limits_{x \to 0^+} \ln x = -\infty$.

44. (a) Let M be given. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad \Rightarrow \quad f(x) > M + 1 - c$. Since $\lim\limits_{x \to a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad \Rightarrow \quad |g(x) - c| < 1 \quad \Rightarrow \quad g(x) > c - 1$. Let δ be the smaller of δ_1 and δ_2. Then $0 < |x - a| < \delta \quad \Rightarrow \quad f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim\limits_{x \to a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim\limits_{x \to a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad \Rightarrow$ $|g(x) - c| < c/2 \quad \Rightarrow \quad g(x) > c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad \Rightarrow$ $f(x) > 2M/c$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \quad \Rightarrow \quad f(x)\,g(x) > \dfrac{2M}{c}\,\dfrac{c}{2} = M$, so $\lim\limits_{x \to a} f(x)\,g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim\limits_{x \to a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \quad \Rightarrow$ $|g(x) - c| < -c/2 \quad \Rightarrow \quad g(x) < c/2$. Since $\lim\limits_{x \to a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \quad \Rightarrow$ $f(x) > 2N/c$. (Note that $c < 0$ and $N < 0 \quad \Rightarrow \quad 2N/c > 0$.) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - a| < \delta \quad \Rightarrow$ $f(x) > 2N/c \quad \Rightarrow \quad f(x)\,g(x) < \dfrac{2N}{c} \cdot \dfrac{c}{2} = N$, so $\lim\limits_{x \to a} f(x)\,g(x) = -\infty$.