3.3 Derivatives of Trigonometric Functions

1. $f(x) = x^2 \sin x \quad \overset{\text{PR}}{\Rightarrow} \quad f'(x) = x^2 \cos x + (\sin x)(2x) = x^2 \cos x + 2x \sin x$

2. $f(x) = x \cos x + 2 \tan x \quad \Rightarrow \quad f'(x) = x(-\sin x) + (\cos x)(1) + 2 \sec^2 x = \cos x - x \sin x + 2 \sec^2 x$

3. $f(x) = e^x \cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$

4. $y = 2 \sec x - \csc x \quad \Rightarrow \quad y' = 2(\sec x \, \tan x) - (-\csc x \, \cot x) = 2 \sec x \, \tan x + \csc x \, \cot x$

5. $y = \sec \theta \, \tan \theta \quad \Rightarrow \quad y' = \sec \theta \, (\sec^2 \theta) + \tan \theta \, (\sec \theta \tan \theta) = \sec \theta \, (\sec^2 \theta + \tan^2 \theta)$. Using the identity $1 + \tan^2 \theta = \sec^2 \theta$, we can write alternative forms of the answer as $\sec \theta \, (1 + 2 \tan^2 \theta)$ or $\sec \theta \, (2 \sec^2 \theta - 1)$.

6. $g(\theta) = e^\theta (\tan \theta - \theta) \quad \Rightarrow \quad g'(\theta) = e^\theta (\sec^2 \theta - 1) + (\tan \theta - \theta) e^\theta = e^\theta (\sec^2 \theta - 1 + \tan \theta - \theta)$

7. $y = c\cos t + t^2 \sin t \quad \Rightarrow \quad y' = c(-\sin t) + t^2(\cos t) + \sin t\,(2t) = -c\sin t + t(t\cos t + 2\sin t)$

8. $f(t) = \dfrac{\cot t}{e^t} \quad \Rightarrow \quad f'(t) = \dfrac{e^t(-\csc^2 t) - (\cot t)e^t}{(e^t)^2} = \dfrac{e^t(-\csc^2 t - \cot t)}{(e^t)^2} = -\dfrac{\csc^2 t + \cot t}{e^t}$

9. $y = \dfrac{x}{2 - \tan x} \quad \Rightarrow \quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

10. $y = \sin\theta\,\cos\theta \quad \Rightarrow \quad y' = \sin\theta(-\sin\theta) + \cos\theta(\cos\theta) = \cos^2\theta - \sin^2\theta \quad$ [or $\cos 2\theta$]

11. $f(\theta) = \dfrac{\sin\theta}{1 + \cos\theta} \quad \Rightarrow$

$$f'(\theta) = \frac{(1 + \cos\theta)\cos\theta - (\sin\theta)(-\sin\theta)}{(1 + \cos\theta)^2} = \frac{\cos\theta + \cos^2\theta + \sin^2\theta}{(1 + \cos\theta)^2} = \frac{\cos\theta + 1}{(1 + \cos\theta)^2} = \frac{1}{1 + \cos\theta}$$

12. $y = \dfrac{\cos x}{1 - \sin x} \quad \Rightarrow$

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

13. $y = \dfrac{t\sin t}{1 + t} \quad \Rightarrow$

$$y' = \frac{(1 + t)(t\cos t + \sin t) - t\sin t(1)}{(1 + t)^2} = \frac{t\cos t + \sin t + t^2\cos t + t\sin t - t\sin t}{(1 + t)^2} = \frac{(t^2 + t)\cos t + \sin t}{(1 + t)^2}$$

14. $y = \dfrac{\sin t}{1 + \tan t} \quad \Rightarrow$

$$y' = \frac{(1 + \tan t)\cos t - (\sin t)\sec^2 t}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \dfrac{\sin t}{\cos^2 t}}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \tan t\sec t}{(1 + \tan t)^2}$$

15. Using Exercise 3.2.61(a) , $f(\theta) = \theta\cos\theta\,\sin\theta \quad \Rightarrow$

$$f'(\theta) = 1\cos\theta\,\sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta(\cos\theta) = \cos\theta\,\sin\theta - \theta\sin^2\theta + \theta\cos^2\theta$$
$$= \sin\theta\,\cos\theta + \theta(\cos^2\theta - \sin^2\theta) = \tfrac{1}{2}\sin 2\theta + \theta\cos 2\theta \quad \text{[using double-angle formulas]}$$

16. Using Exercise 3.2.61(a), $f(t) = te^t\cot t \quad \Rightarrow$

$$f'(t) = 1e^t\cot t + te^t\cot t + te^t(-\csc^2 t) = e^t(\cot t + t\cot t - t\csc^2 t)$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\,\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\,\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \quad \Rightarrow$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \, \cos h - \sin x \, \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left(\cos x \, \frac{\cos h - 1}{h} - \sin x \, \frac{\sin h}{h} \right) = \cos x \, \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \, \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

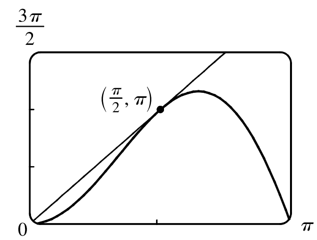
22. $y = e^x \cos x \quad \Rightarrow \quad y' = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x) \quad \Rightarrow$ the slope of the tangent line at $(0, 1)$ is $e^0(\cos 0 - \sin 0) = 1(1 - 0) = 1$ and an equation is $y - 1 = 1(x - 0)$ or $y = x + 1$.

23. $y = \cos x - \sin x \quad \Rightarrow \quad y' = -\sin x - \cos x$, so $y'(\pi) = -\sin \pi - \cos \pi = 0 - (-1) = 1$. An equation of the tangent line to the curve $y = \cos x - \sin x$ at the point $(\pi, -1)$ is $y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

24. $y = x + \tan x \quad \Rightarrow \quad y' = 1 + \sec^2 x$, so $y'(\pi) = 1 + (-1)^2 = 2$. An equation of the tangent line to the curve $y = x + \tan x$ at the point (π, π) is $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

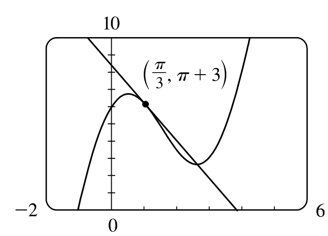
25. (a) $y = 2x \sin x \quad \Rightarrow \quad y' = 2(x \cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



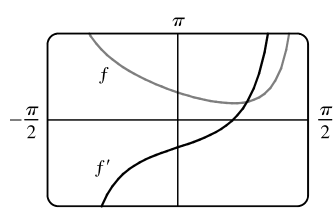
26. (a) $y = 3x + 6 \cos x \quad \Rightarrow \quad y' = 3 - 6 \sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6 \sin \frac{\pi}{3} = 3 - 6 \, \frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right) x + 3 + \pi\sqrt{3}$.

(b)



27. (a) $f(x) = \sec x - x \quad \Rightarrow \quad f'(x) = \sec x \, \tan x - 1$

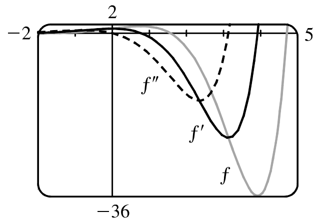
(b)



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

28. (a) $f(x) = e^x \cos x \quad \Rightarrow \quad f'(x) = e^x(-\sin x) + (\cos x)\, e^x = e^x(\cos x - \sin x) \quad \Rightarrow$

$$f''(x) = e^x(-\sin x - \cos x) + (\cos x - \sin x)\, e^x = e^x(-\sin x - \cos x + \cos x - \sin x) = -2e^x \sin x$$

(b) 

Note that $f' = 0$ where f has a minimum and $f'' = 0$ where f' has a minimum. Also note that f' is negative when f is decreasing and f'' is negative when f' is decreasing.

29. $H(\theta) = \theta \sin\theta \quad\Rightarrow\quad H'(\theta) = \theta\,(\cos\theta) + (\sin\theta)\cdot 1 = \theta\cos\theta + \sin\theta \quad\Rightarrow$

$H''(\theta) = \theta\,(-\sin\theta) + (\cos\theta)\cdot 1 + \cos\theta = -\theta\sin\theta + 2\cos\theta$

30. $f(t) = \sec t \quad\Rightarrow\quad f'(t) = \sec t\,\tan t \quad\Rightarrow\quad f''(t) = (\sec t)\sec^2 t + (\tan t)\sec t\,\tan t = \sec^3 t + \sec t\,\tan^2 t$, so

$f''\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}.$

31. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad\Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x\tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad\Rightarrow\quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

(c) From part (a), $f'(x) = \dfrac{1 + \tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

32. (a) $g(x) = f(x)\sin x \quad\Rightarrow\quad g'(x) = f(x)\cos x + \sin x \cdot f'(x)$, so

$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot(-2) = 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)} \quad\Rightarrow\quad h'(x) = \dfrac{f(x)\cdot(-\sin x) - \cos x\cdot f'(x)}{[f(x)]^2}$, so

$$h'\left(\tfrac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3} + 1}{16} = \frac{1 - 2\sqrt{3}}{16}$$

33. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad\Leftrightarrow\quad 1 + 2\cos x = 0 \quad\Leftrightarrow\quad \cos x = -\frac{1}{2} \quad\Leftrightarrow$

$x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

34. $f(x) = e^x\cos x$ has a horizontal tangent when $f'(x) = 0$. $f'(x) = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x)$.

$f'(x) = 0 \quad\Leftrightarrow\quad \cos x - \sin x = 0 \quad\Leftrightarrow\quad \cos x = \sin x \quad\Leftrightarrow\quad \tan x = 1 \quad\Leftrightarrow\quad x = \frac{\pi}{4} + n\pi$, n an integer.

35. (a) $x(t) = 8\sin t \quad\Rightarrow\quad v(t) = x'(t) = 8\cos t \quad\Rightarrow\quad a(t) = x''(t) = -8\sin t$

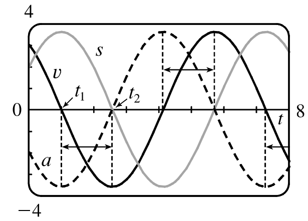
(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$,

and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

36. (a) $s(t) = 2\cos t + 3\sin t \quad \Rightarrow \quad v(t) = -2\sin t + 3\cos t \quad \Rightarrow$

$a(t) = -2\cos t - 3\sin t$

(b)

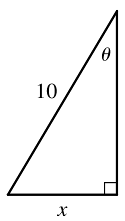


(c) $s = 0 \quad \Rightarrow \quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad \Rightarrow \quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

37.

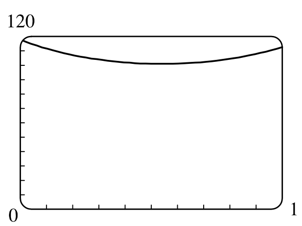


From the diagram we can see that $\sin\theta = x/10 \quad \Leftrightarrow \quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 10\sin\theta$, we get $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

38. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad \Leftrightarrow \quad \mu W(\sin\theta - \mu\cos\theta) = 0 \quad \Leftrightarrow \quad \sin\theta = \mu\cos\theta \quad \Leftrightarrow \quad \tan\theta = \mu \quad \Leftrightarrow \quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

39. $\displaystyle\lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{x\to 0} \frac{5}{3}\left(\frac{\sin 5x}{5x}\right) = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad [\theta = 5x] \quad = \frac{5}{3}\cdot 1 = \frac{5}{3}$

40. $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sin \pi x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\pi x}{\sin \pi x}\cdot\frac{1}{\pi} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{\theta\to 0}\frac{\theta}{\sin\theta}\cdot\frac{1}{\pi} \quad [\theta = \pi x]$

$$= 1\cdot\lim_{\theta\to 0}\frac{1}{\dfrac{\sin\theta}{\theta}}\cdot\frac{1}{\pi} = 1\cdot 1\cdot\frac{1}{\pi} = \frac{1}{\pi}$$

41. $\displaystyle\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}$

$$= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3$$

42. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\displaystyle\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta}}{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{0}{1} = 0$

43. $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}=\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{3}{5x^2-4}\right)=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{3}{5x^2-4}=1\cdot\left(\frac{3}{-4}\right)=-\frac{3}{4}$

44. $\displaystyle\lim_{x\to 0}\frac{\sin 3x\sin 5x}{x^2}=\lim_{x\to 0}\left(\frac{3\sin 3x}{3x}\cdot\frac{5\sin 5x}{5x}\right)=\lim_{x\to 0}\frac{3\sin 3x}{3x}\cdot\lim_{x\to 0}\frac{5\sin 5x}{5x}$

$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 5\lim_{x\to 0}\frac{\sin 5x}{5x}=3(1)\cdot 5(1)=15$$

45. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}=\frac{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

46. $\displaystyle\lim_{x\to 0}\csc x\,\sin(\sin x)=\lim_{x\to 0}\frac{\sin(\sin x)}{\sin x}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\quad[\text{As } x\to 0,\ \theta=\sin x\to 0.]\quad=1$

47. $\displaystyle\lim_{\theta\to 0}\frac{\cos\theta-1}{2\theta^2}=\lim_{\theta\to 0}\frac{\cos\theta-1}{2\theta^2}\cdot\frac{\cos\theta+1}{\cos\theta+1}=\lim_{\theta\to 0}\frac{\cos^2\theta-1}{2\theta^2(\cos\theta+1)}=\lim_{\theta\to 0}\frac{-\sin^2\theta}{2\theta^2(\cos\theta+1)}$

$$=-\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta+1}=-\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta+1}$$

$$=-\frac{1}{2}\cdot 1\cdot 1\cdot\frac{1}{1+1}=-\frac{1}{4}$$

48. $\displaystyle\lim_{x\to 0}\frac{\sin(x^2)}{x}=\lim_{x\to 0}\left[x\cdot\frac{\sin(x^2)}{x\cdot x}\right]=\lim_{x\to 0}x\cdot\lim_{x\to 0}\frac{\sin(x^2)}{x^2}=0\cdot\lim_{y\to 0^+}\frac{\sin y}{y}\quad\left[\text{where } y=x^2\right]\quad=0\cdot 1=0$

49. $\displaystyle\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}=\lim_{x\to\pi/4}\frac{\left(1-\dfrac{\sin x}{\cos x}\right)\cdot\cos x}{(\sin x-\cos x)\cdot\cos x}=\lim_{x\to\pi/4}\frac{\cos x-\sin x}{(\sin x-\cos x)\cos x}=\lim_{x\to\pi/4}\frac{-1}{\cos x}=\frac{-1}{1/\sqrt{2}}=-\sqrt{2}$

50. $\displaystyle\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}=\lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)}=\lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1}=\tfrac{1}{3}\cdot 1=\tfrac{1}{3}$

51. $\displaystyle\frac{d}{dx}(\sin x)=\cos x\ \Rightarrow\ \frac{d^2}{dx^2}(\sin x)=-\sin x\ \Rightarrow\ \frac{d^3}{dx^3}(\sin x)=-\cos x\ \Rightarrow\ \frac{d^4}{dx^4}(\sin x)=\sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99=4(24)+3$, we have $\dfrac{d^{99}}{dx^{99}}(\sin x)=\dfrac{d^3}{dx^3}(\sin x)=-\cos x$.

52. Let $f(x)=x\sin x$ and $h(x)=\sin x$, so $f(x)=xh(x)$. Then $f'(x)=h(x)+xh'(x)$,

$f''(x)=h'(x)+h'(x)+xh''(x)=2h'(x)+xh''(x)$,

$f'''(x)=2h''(x)+h''(x)+xh'''(x)=3h''(x)+xh'''(x),\cdots,f^{(n)}(x)=nh^{(n-1)}(x)+xh^{(n)}(x)$.

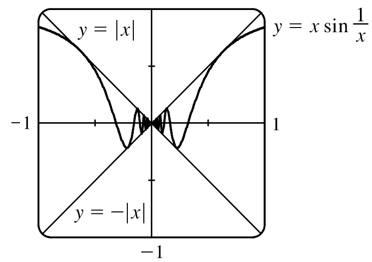
Since $34=4(8)+2$, we have $h^{(34)}(x)=h^{(2)}(x)=\dfrac{d^2}{dx^2}(\sin x)=-\sin x$ and $h^{(35)}(x)=-\cos x$.

Thus, $\dfrac{d^{35}}{dx^{35}}(x\sin x)=35h^{(34)}(x)+xh^{(35)}(x)=-35\sin x-x\cos x$.

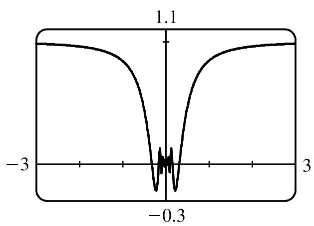
53. $y = A\sin x + B\cos x \quad\Rightarrow\quad y' = A\cos x - B\sin x \quad\Rightarrow\quad y'' = -A\sin x - B\cos x$. Substituting these expressions for y, y', and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us $(-A\sin x - B\cos x) + (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = \sin x \quad\Leftrightarrow$ $-3A\sin x - B\sin x + A\cos x - 3B\cos x = \sin x \quad\Leftrightarrow\quad (-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

54. (a) Let $\theta = \dfrac{1}{x}$. Then as $x \to \infty$, $\theta \to 0^+$, and $\displaystyle\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{\theta\to 0^+}\frac{1}{\theta}\sin\theta = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

(b) Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x\sin(1/x) \le |x|$. We know that $\displaystyle\lim_{x\to 0}(|x|) = 0$ and $\displaystyle\lim_{x\to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\displaystyle\lim_{x\to 0} x\sin(1/x) = 0$.



(c)



55. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \quad\Rightarrow\quad \sec^2 x = \dfrac{\cos x\,\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \quad\Rightarrow\quad \sec x\,\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\,\tan x = \dfrac{\sin x}{\cos^2 x}$.

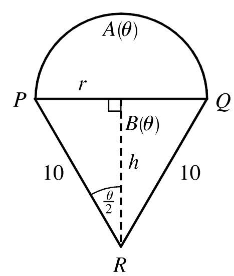
(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1 + \cot x}{\csc x} \quad\Rightarrow$

$$\cos x - \sin x = \frac{\csc x\,(-\csc^2 x) - (1 + \cot x)(-\csc x\,\cot x)}{\csc^2 x} = \frac{\csc x\,[-\csc^2 x + (1 + \cot x)\,\cot x]}{\csc^2 x}$$
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

56. We get the following formulas for r and h in terms of θ:

$$\sin\frac{\theta}{2} = \frac{r}{10} \quad\Rightarrow\quad r = 10\sin\frac{\theta}{2} \quad\text{and}\quad \cos\frac{\theta}{2} = \frac{h}{10} \quad\Rightarrow\quad h = 10\cos\frac{\theta}{2}$$



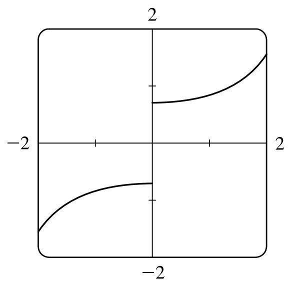
Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{10\sin(\theta/2)}{10\cos(\theta/2)}$$
$$= \frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0$$

57. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin\frac{\theta}{2}$. So $\lim\limits_{\theta\to 0^+}\frac{s}{d} = \lim\limits_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim\limits_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim\limits_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

58. (a)



It appears that $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$ has a jump discontinuity at $x = 0$.

(b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\dfrac{x}{\sqrt{1-\cos 2x}} = \dfrac{x}{\sqrt{1-(1-2\sin^2 x)}} = \dfrac{x}{\sqrt{2\sin^2 x}} = \dfrac{x}{\sqrt{2}\,|\sin x|}$.

Thus,

$$\begin{aligned}\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 2x}} &= \lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin x|} = \frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{x}{-(\sin x)} \\ &= -\frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{1} = -\frac{\sqrt{2}}{2}\end{aligned}$$

Evaluating $\lim\limits_{x\to 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1+\cos 2x}$.

3.4 The Chain Rule

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \dfrac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(\pi) = \pi\sec^2 \pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = 2 - e^x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{2}u^{-1/2})(-e^x) = -\dfrac{e^x}{2\sqrt{2-e^x}}$.

7. $F(x) = (5x^6 + 2x^3)^4 \Rightarrow F'(x) = 4(5x^6+2x^3)^3\cdot\dfrac{d}{dx}(5x^6+2x^3) = 4(5x^6+2x^3)^3(30x^5+6x^2)$.

We can factor as follows: $4(x^3)^3(5x^3+2)^3 6x^2(5x^3+1) = 24x^{11}(5x^3+2)^3(5x^3+1)$