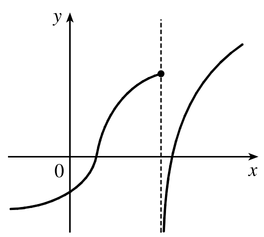
2.6 Limits at Infinity; Horizontal Asymptotes

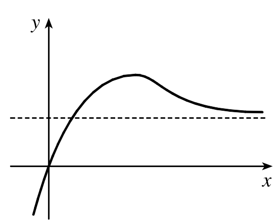
1. (a) As x becomes large, the values of $f(x)$ approach 5.

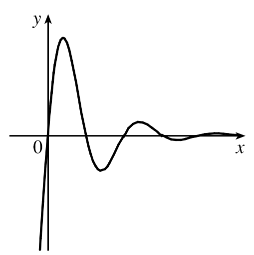
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

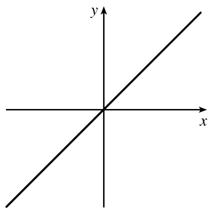


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

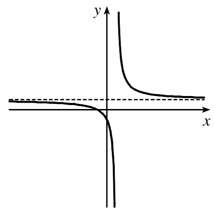




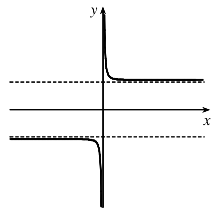
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

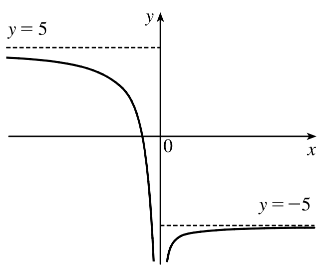
3. (a) $\lim\limits_{x\to\infty} f(x) = -2$ (b) $\lim\limits_{x\to-\infty} f(x) = 2$ (c) $\lim\limits_{x\to1} f(x) = \infty$

(d) $\lim\limits_{x\to3} f(x) = -\infty$ (e) Vertical: $x = 1$, $x = 3$; horizontal: $y = -2$, $y = 2$

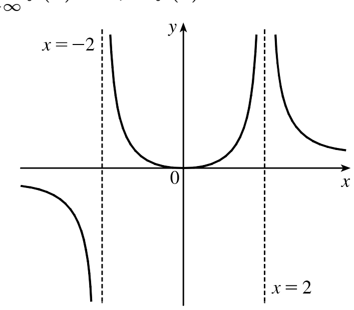
4. (a) $\lim\limits_{x\to\infty} g(x) = 2$ (b) $\lim\limits_{x\to-\infty} g(x) = -1$ (c) $\lim\limits_{x\to0} g(x) = -\infty$

(d) $\lim\limits_{x\to2^-} g(x) = -\infty$ (e) $\lim\limits_{x\to2^+} g(x) = \infty$ (f) Vertical: $x = 0$, $x = 2$;
horizontal: $y = -1$, $y = 2$

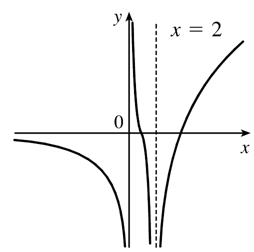
5. $\lim\limits_{x\to0} f(x) = -\infty$,
$\lim\limits_{x\to-\infty} f(x) = 5$,
$\lim\limits_{x\to\infty} f(x) = -5$



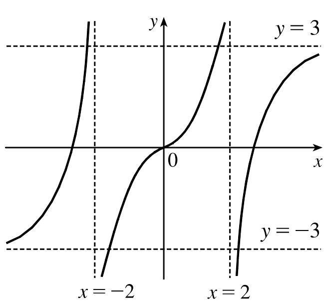
6. $\lim\limits_{x\to2} f(x) = \infty$, $\lim\limits_{x\to-2^+} f(x) = \infty$,
$\lim\limits_{x\to-2^-} f(x) = -\infty$, $\lim\limits_{x\to-\infty} f(x) = 0$,
$\lim\limits_{x\to\infty} f(x) = 0$, $f(0) = 0$



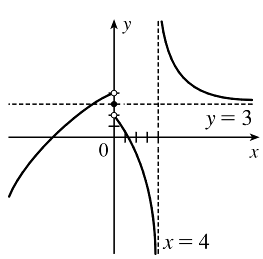
7. $\lim\limits_{x\to2} f(x) = -\infty$, $\lim\limits_{x\to\infty} f(x) = \infty$,
$\lim\limits_{x\to-\infty} f(x) = 0$, $\lim\limits_{x\to0^+} f(x) = \infty$,
$\lim\limits_{x\to0^-} f(x) = -\infty$



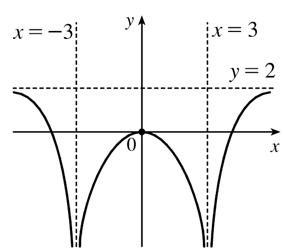
8. $\lim\limits_{x\to\infty} f(x) = 3$,
$\lim\limits_{x\to2^-} f(x) = \infty$,
$\lim\limits_{x\to2^+} f(x) = -\infty$,
f is odd



9. $f(0) = 3$, $\lim\limits_{x\to0^-} f(x) = 4$,
$\lim\limits_{x\to0^+} f(x) = 2$,
$\lim\limits_{x\to-\infty} f(x) = -\infty$, $\lim\limits_{x\to4^-} f(x) = -\infty$,
$\lim\limits_{x\to4^+} f(x) = \infty$, $\lim\limits_{x\to\infty} f(x) = 3$



10. $\lim\limits_{x\to3} f(x) = -\infty$, $\lim\limits_{x\to\infty} f(x) = 2$,
$f(0) = 0$, f is even



11. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$. It appears that $\lim\limits_{x\to\infty} \left(x^2/2^x\right) = 0$.

12. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135 308
100,000	0.135 333
1,000,000	0.135 335

From the table, we estimate that $\lim\limits_{x\to\infty} f(x) = 0.1353$ (to four decimal places.)

13.
$$\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3} = \lim_{x\to\infty} \frac{(2x^2 - 7)/x^2}{(5x^2 + x - 3)/x^2}$$
[Divide both the numerator and denominator by x^2 (the highest power of x that appears in the denominator)]
$$= \frac{\lim\limits_{x\to\infty} (2 - 7/x^2)}{\lim\limits_{x\to\infty} (5 + 1/x - 3/x^2)}$$
[Limit Law 5]
$$= \frac{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} (7/x^2)}{\lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty} (1/x) - \lim\limits_{x\to\infty} (3/x^2)}$$
[Limit Laws 1 and 2]
$$= \frac{2 - 7\lim\limits_{x\to\infty} (1/x^2)}{5 + \lim\limits_{x\to\infty} (1/x) - 3\lim\limits_{x\to\infty} (1/x^2)}$$
[Limit Laws 7 and 3]
$$= \frac{2 - 7(0)}{5 + 0 + 3(0)}$$
[Theorem 5]
$$= \frac{2}{5}$$

14.
$$\lim_{x\to\infty} \sqrt{\frac{9x^3 + 8x - 4}{3 - 5x + x^3}} = \sqrt{\lim_{x\to\infty} \frac{9x^3 + 8x - 4}{3 - 5x + x^3}}$$
[Limit Law 11]
$$= \sqrt{\lim_{x\to\infty} \frac{9 + 8/x^2 - 4/x^3}{3/x^3 - 5/x^2 + 1}}$$
[Divide by x^3]
$$= \sqrt{\frac{\lim\limits_{x\to\infty} (9 + 8/x^2 - 4/x^3)}{\lim\limits_{x\to\infty} (3/x^3 - 5/x^2 + 1)}}$$
[Limit Law 5]
$$= \sqrt{\frac{\lim\limits_{x\to\infty} 9 + \lim\limits_{x\to\infty} (8/x^2) - \lim\limits_{x\to\infty} (4/x^3)}{\lim\limits_{x\to\infty} (3/x^3) - \lim\limits_{x\to\infty} (5/x^2) + \lim\limits_{x\to\infty} 1}}$$
[Limit Laws 1 and 2]
$$= \sqrt{\frac{9 + 8\lim\limits_{x\to\infty} (1/x^2) - 4\lim\limits_{x\to\infty} (1/x^3)}{3\lim\limits_{x\to\infty} (1/x^3) - 5\lim\limits_{x\to\infty} (1/x^2) + 1}}$$
[Limit Laws 7 and 3]
$$= \sqrt{\frac{9 + 8(0) - 4(0)}{3(0) - 5(0) + 1}}$$
[Theorem 5]
$$= \sqrt{\frac{9}{1}} = \sqrt{9} = 3$$

15. $\displaystyle \lim_{x\to\infty}\frac{3x-2}{2x+1}=\lim_{x\to\infty}\frac{(3x-2)/x}{(2x+1)/x}=\lim_{x\to\infty}\frac{3-2/x}{2+1/x}=\frac{\lim\limits_{x\to\infty}3-2\lim\limits_{x\to\infty}1/x}{\lim\limits_{x\to\infty}2+\lim\limits_{x\to\infty}1/x}=\frac{3-2(0)}{2+0}=\frac{3}{2}$

16. $\displaystyle \lim_{x\to\infty}\frac{1-x^2}{x^3-x+1}=\lim_{x\to\infty}\frac{(1-x^2)/x^3}{(x^3-x+1)/x^3}=\lim_{x\to\infty}\frac{1/x^3-1/x}{1-1/x^2+1/x^3}$

$$=\frac{\lim\limits_{x\to\infty}1/x^3-\lim\limits_{x\to\infty}1/x}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}1/x^2+\lim\limits_{x\to\infty}1/x^3}=\frac{0-0}{1-0+0}=0$$

17. $\displaystyle \lim_{x\to-\infty}\frac{x-2}{x^2+1}=\lim_{x\to-\infty}\frac{(x-2)/x^2}{(x^2+1)/x^2}=\lim_{x\to-\infty}\frac{1/x-2/x^2}{1+1/x^2}=\frac{\lim\limits_{x\to-\infty}1/x-2\lim\limits_{x\to-\infty}1/x^2}{\lim\limits_{x\to-\infty}1+\lim\limits_{x\to-\infty}1/x^2}=\frac{0-2(0)}{1+0}=0$

18. $\displaystyle \lim_{x\to-\infty}\frac{4x^3+6x^2-2}{2x^3-4x+5}=\lim_{x\to-\infty}\frac{(4x^3+6x^2-2)/x^3}{(2x^3-4x+5)/x^3}=\lim_{x\to-\infty}\frac{4+6/x-2/x^3}{2-4/x^2+5/x^3}=\frac{4+0-0}{2-0+0}=2$

19. $\displaystyle \lim_{t\to\infty}\frac{\sqrt{t}+t^2}{2t-t^2}=\lim_{t\to\infty}\frac{(\sqrt{t}+t^2)/t^2}{(2t-t^2)/t^2}=\lim_{t\to\infty}\frac{1/t^{3/2}+1}{2/t-1}=\frac{0+1}{0-1}=-1$

20. $\displaystyle \lim_{t\to\infty}\frac{t-t\sqrt{t}}{2t^{3/2}+3t-5}=\lim_{t\to\infty}\frac{\left(t-t\sqrt{t}\right)/t^{3/2}}{\left(2t^{3/2}+3t-5\right)/t^{3/2}}=\lim_{t\to\infty}\frac{1/t^{1/2}-1}{2+3/t^{1/2}-5/t^{3/2}}=\frac{0-1}{2+0-0}=-\frac{1}{2}$

21. $\displaystyle \lim_{x\to\infty}\frac{(2x^2+1)^2}{(x-1)^2(x^2+x)}=\lim_{x\to\infty}\frac{(2x^2+1)^2/x^4}{[(x-1)^2(x^2+x)]/x^4}=\lim_{x\to\infty}\frac{[(2x^2+1)/x^2]^2}{[(x^2-2x+1)/x^2][(x^2+x)/x^2]}$

$$=\lim_{x\to\infty}\frac{(2+1/x^2)^2}{(1-2/x+1/x^2)(1+1/x)}=\frac{(2+0)^2}{(1-0+0)(1+0)}=4$$

22. $\displaystyle \lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+1}}=\lim_{x\to\infty}\frac{x^2/x^2}{\sqrt{x^4+1}/x^2}=\lim_{x\to\infty}\frac{1}{\sqrt{(x^4+1)/x^4}}$ [since $x^2=\sqrt{x^4}$ for $x>0$]

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^4}}=\frac{1}{\sqrt{1+0}}=1$$

23. $\displaystyle \lim_{x\to\infty}\frac{\sqrt{1+4x^6}}{2-x^3}=\lim_{x\to\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\frac{\lim\limits_{x\to\infty}\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to\infty}(2/x^3-1)}$ [since $x^3=\sqrt{x^6}$ for $x>0$]

$$=\frac{\lim\limits_{x\to\infty}\sqrt{1/x^6+4}}{\lim\limits_{x\to\infty}(2/x^3)-\lim\limits_{x\to\infty}1}=\frac{\sqrt{\lim\limits_{x\to\infty}(1/x^6)+\lim\limits_{x\to\infty}4}}{0-1}$$

$$=\frac{\sqrt{0+4}}{-1}=\frac{2}{-1}=-2$$

24. $\displaystyle \lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3}=\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\frac{\lim\limits_{x\to-\infty}-\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to-\infty}(2/x^3-1)}$ [since $x^3=-\sqrt{x^6}$ for $x<0$]

$$=\frac{\lim\limits_{x\to-\infty}-\sqrt{1/x^6+4}}{2\lim\limits_{x\to-\infty}(1/x^3)-\lim\limits_{x\to-\infty}1}=\frac{-\sqrt{\lim\limits_{x\to-\infty}(1/x^6)+\lim\limits_{x\to-\infty}4}}{2(0)-1}$$

$$=\frac{-\sqrt{0+4}}{-1}=\frac{-2}{-1}=2$$

25. $\displaystyle \lim_{x\to\infty} \frac{\sqrt{x+3x^2}}{4x-1} = \lim_{x\to\infty} \frac{\sqrt{x+3x^2}/x}{(4x-1)/x} = \frac{\lim\limits_{x\to\infty} \sqrt{(x+3x^2)/x^2}}{\lim\limits_{x\to\infty} (4-1/x)}$ [since $x = \sqrt{x^2}$ for $x > 0$]

$$= \frac{\lim\limits_{x\to\infty} \sqrt{1/x+3}}{\lim\limits_{x\to\infty} 4 - \lim\limits_{x\to\infty} (1/x)} = \frac{\sqrt{\lim\limits_{x\to\infty} (1/x) + \lim\limits_{x\to\infty} 3}}{4-0} = \frac{\sqrt{0+3}}{4} = \frac{\sqrt{3}}{4}$$

26. $\displaystyle \lim_{x\to\infty} \frac{x+3x^2}{4x-1} = \lim_{x\to\infty} \frac{(x+3x^2)/x}{(4x-1)/x} = \lim_{x\to\infty} \frac{1+3x}{4-1/x}$

$= \infty$ since $1 + 3x \to \infty$ and $4 - 1/x \to 4$ as $x \to \infty$.

27. $\displaystyle \lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right) = \lim_{x\to\infty} \frac{\left(\sqrt{9x^2+x} - 3x\right)\left(\sqrt{9x^2+x} + 3x\right)}{\sqrt{9x^2+x} + 3x} = \lim_{x\to\infty} \frac{\left(\sqrt{9x^2+x}\right)^2 - (3x)^2}{\sqrt{9x^2+x} + 3x}$

$$= \lim_{x\to\infty} \frac{(9x^2+x) - 9x^2}{\sqrt{9x^2+x} + 3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x} + 3x} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{x/x}{\sqrt{9x^2/x^2 + x/x^2} + 3x/x} = \lim_{x\to\infty} \frac{1}{\sqrt{9+1/x} + 3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

28. $\displaystyle \lim_{x\to-\infty} \left(\sqrt{4x^2+3x} + 2x\right) = \lim_{x\to-\infty} \left(\sqrt{4x^2+3x} + 2x\right)\left[\frac{\sqrt{4x^2+3x} - 2x}{\sqrt{4x^2+3x} - 2x}\right]$

$$= \lim_{x\to-\infty} \frac{(4x^2+3x) - (2x)^2}{\sqrt{4x^2+3x} - 2x} = \lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+3x} - 2x}$$

$\displaystyle = \lim_{x\to-\infty} \frac{3x/x}{\left(\sqrt{4x^2+3x} - 2x\right)/x} = \lim_{x\to-\infty} \frac{3}{-\sqrt{4+3/x} - 2}$ [since $x = -\sqrt{x^2}$ for $x < 0$]

$$= \frac{3}{-\sqrt{4+0} - 2} = -\frac{3}{4}$$

29. $\displaystyle \lim_{x\to\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right) = \lim_{x\to\infty} \frac{\left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax} + \sqrt{x^2+bx}\right)}{\sqrt{x^2+ax} + \sqrt{x^2+bx}}$

$$= \lim_{x\to\infty} \frac{(x^2+ax) - (x^2+bx)}{\sqrt{x^2+ax} + \sqrt{x^2+bx}} = \lim_{x\to\infty} \frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax} + \sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{a-b}{\sqrt{1+a/x} + \sqrt{1+b/x}} = \frac{a-b}{\sqrt{1+0} + \sqrt{1+0}} = \frac{a-b}{2}$$

30. For $x > 0$, $\sqrt{x^2+1} > \sqrt{x^2} = x$. So as $x \to \infty$, we have $\sqrt{x^2+1} \to \infty$, that is, $\lim\limits_{x\to\infty} \sqrt{x^2+1} = \infty$.

31. $\displaystyle \lim_{x\to\infty} \frac{x^4 - 3x^2 + x}{x^3 - x + 2} = \lim_{x\to\infty} \frac{(x^4 - 3x^2 + x)/x^3}{(x^3 - x + 2)/x^3}$ [divide by the highest power of x in the denominator] $\displaystyle = \lim_{x\to\infty} \frac{x - 3/x + 1/x^2}{1 - 1/x^2 + 2/x^3} = \infty$

since the numerator increases without bound and the denominator approaches 1 as $x \to \infty$.

32. $\lim\limits_{x\to\infty} (e^{-x} + 2\cos 3x)$ does not exist. $\lim\limits_{x\to\infty} e^{-x} = 0$, but $\lim\limits_{x\to\infty} (2\cos 3x)$ does not exist because the values of $2\cos 3x$ oscillate between the values of -2 and 2 infinitely often, so the given limit does not exist.

33. $\lim\limits_{x\to-\infty}(x^2+2x^7)=\lim\limits_{x\to-\infty}x^7\left(\dfrac{1}{x^5}+2\right)$ [factor out the largest power of x] $=-\infty$ because $x^7\to-\infty$ and $1/x^5+2\to 2$ as $x\to-\infty$.

Or: $\lim\limits_{x\to-\infty}\left(x^2+2x^7\right)=\lim\limits_{x\to-\infty}x^2\left(1+2x^5\right)=-\infty$.

34. $\lim\limits_{x\to-\infty}\dfrac{1+x^6}{x^4+1}=\lim\limits_{x\to-\infty}\dfrac{(1+x^6)/x^4}{(x^4+1)/x^4}$ [divide by the highest power of x in the denominator] $=\lim\limits_{x\to-\infty}\dfrac{1/x^4+x^2}{1+1/x^4}=\infty$

since the numerator increases without bound and the denominator approaches 1 as $x\to-\infty$.

35. Let $t=e^x$. As $x\to\infty$, $t\to\infty$. $\lim\limits_{x\to\infty}\arctan(e^x)=\lim\limits_{t\to\infty}\arctan t=\frac{\pi}{2}$ by (3).

36. Divide numerator and denominator by e^{3x}: $\lim\limits_{x\to\infty}\dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim\limits_{x\to\infty}\dfrac{1-e^{-6x}}{1+e^{-6x}}=\dfrac{1-0}{1+0}=1$

37. $\lim\limits_{x\to\infty}\dfrac{1-e^x}{1+2e^x}=\lim\limits_{x\to\infty}\dfrac{(1-e^x)/e^x}{(1+2e^x)/e^x}=\lim\limits_{x\to\infty}\dfrac{1/e^x-1}{1/e^x+2}=\dfrac{0-1}{0+2}=-\dfrac{1}{2}$

38. Since $0\le\sin^2x\le1$, we have $0\le\dfrac{\sin^2x}{x^2+1}\le\dfrac{1}{x^2+1}$. We know that $\lim\limits_{x\to\infty}0=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x^2+1}=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}\dfrac{\sin^2x}{x^2+1}=0$.

39. Since $-1\le\cos x\le1$ and $e^{-2x}>0$, we have $-e^{-2x}\le e^{-2x}\cos x\le e^{-2x}$. We know that $\lim\limits_{x\to\infty}(-e^{-2x})=0$ and $\lim\limits_{x\to\infty}\left(e^{-2x}\right)=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}(e^{-2x}\cos x)=0$.

40. Let $t=\ln x$. As $x\to0^+$, $t\to-\infty$. $\lim\limits_{x\to0^+}\tan^{-1}(\ln x)=\lim\limits_{t\to-\infty}\tan^{-1}t=-\frac{\pi}{2}$ by (4).

41. $\lim\limits_{x\to\infty}[\ln(1+x^2)-\ln(1+x)]=\lim\limits_{x\to\infty}\ln\dfrac{1+x^2}{1+x}=\ln\left(\lim\limits_{x\to\infty}\dfrac{1+x^2}{1+x}\right)=\ln\left(\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+x}{\frac{1}{x}+1}\right)=\infty$, since the limit in parentheses is ∞.

42. $\lim\limits_{x\to\infty}[\ln(2+x)-\ln(1+x)]=\lim\limits_{x\to\infty}\ln\left(\dfrac{2+x}{1+x}\right)=\lim\limits_{x\to\infty}\ln\left(\dfrac{2/x+1}{1/x+1}\right)=\ln\dfrac{1}{1}=\ln1=0$

43. (a) (i) $\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}\dfrac{x}{\ln x}=0$ since $x\to0^+$ and $\ln x\to-\infty$ as $x\to0^+$.

(ii) $\lim\limits_{x\to1^-}f(x)=\lim\limits_{x\to1^-}\dfrac{x}{\ln x}=-\infty$ since $x\to1$ and $\ln x\to0^-$ as $x\to1^-$.

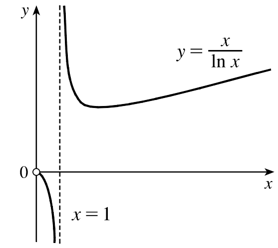
(iii) $\lim\limits_{x\to1^+}f(x)=\lim\limits_{x\to1^+}\dfrac{x}{\ln x}=\infty$ since $x\to1$ and $\ln x\to0^+$ as $x\to1^+$.

(b)

x	$f(x)$
10,000	1085.7
100,000	8685.9
1,000,000	72,382.4

It appears that $\lim\limits_{x\to\infty}f(x)=\infty$.

(c)



44. (a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = 0$

since $\dfrac{2}{x} \to 0$ and $\dfrac{1}{\ln x} \to 0$ as $x \to \infty$.

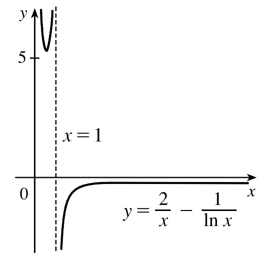
(b) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = \infty$

since $\dfrac{2}{x} \to \infty$ and $\dfrac{1}{\ln x} \to 0$ as $x \to 0^+$.

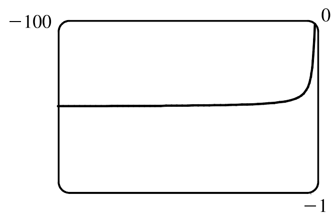
(c) $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = \infty$ since $\dfrac{2}{x} \to 2$ and $\dfrac{1}{\ln x} \to -\infty$ as $x \to 1^-$.

(d) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}\left(\dfrac{2}{x} - \dfrac{1}{\ln x}\right) = -\infty$ since $\dfrac{2}{x} \to 2$ and $\dfrac{1}{\ln x} \to \infty$ as $x \to 1^+$.

(e)



45. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim\limits_{x\to -\infty} f(x)$ to be -0.5.

(b)

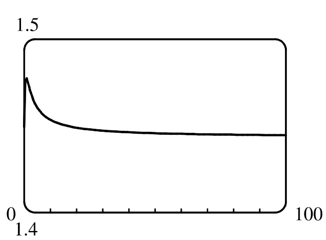
x	$f(x)$
$-10{,}000$	$-0.499\,962\,5$
$-100{,}000$	$-0.499\,996\,2$
$-1{,}000{,}000$	$-0.499\,999\,6$

From the table, we estimate the limit to be -0.5.

(c) $$\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right) = \lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\frac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right] = \lim_{x\to-\infty}\frac{\left(x^2+x+1\right)-x^2}{\sqrt{x^2+x+1}-x}$$
$$= \lim_{x\to-\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1}-x\right)(1/x)} = \lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1}$$
$$= \frac{1+0}{-\sqrt{1+0+0}-1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{x^2+x+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+(1/x^2)}$.

46. (a)



From the graph of $f(x) = \sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1}$, we estimate (to one decimal place) the value of $\lim\limits_{x\to\infty} f(x)$ to be 1.4.

(b)

x	$f(x)$
10,000	1.443 39
100,000	1.443 38
1,000,000	1.443 38

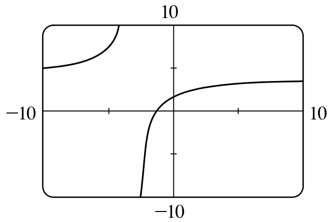
From the table, we estimate (to four decimal places) the limit to be 1.4434.

(c) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$

$$= \lim_{x\to\infty} \frac{\left(3x^2+8x+6\right)-\left(3x^2+3x+1\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}} = \lim_{x\to\infty} \frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$

$$= \lim_{x\to\infty} \frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

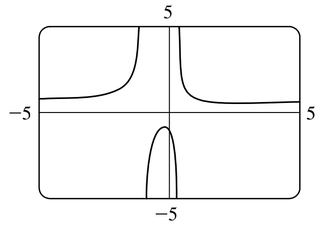
47. $\lim\limits_{x\to\pm\infty} \dfrac{5+4x}{x+3} = \lim\limits_{x\to\pm\infty} \dfrac{(5+4x)/x}{(x+3)/x} = \lim\limits_{x\to\pm\infty} \dfrac{5/x+4}{1+3/x} = \dfrac{0+4}{1+0} = 4$, so $y=4$ is a horizontal asymptote. $y = f(x) = \dfrac{5+4x}{x+3}$, so $\lim\limits_{x\to -3^+} f(x) = -\infty$ since $5+4x \to -7$ and $x+3 \to 0^+$ as $x \to -3^+$. Thus, $x=-3$ is a vertical asymptote. The graph confirms our work.



48. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+1}{3x^2+2x-1} = \lim\limits_{x\to\pm\infty} \dfrac{(2x^2+1)/x^2}{(3x^2+2x-1)/x^2}$

$$= \lim_{x\to\pm\infty} \frac{2+1/x^2}{3+2/x-1/x^2} = \frac{2}{3}$$

so $y = \dfrac{2}{3}$ is a horizontal asymptote. $y = f(x) = \dfrac{2x^2+1}{3x^2+2x-1} = \dfrac{2x^2+1}{(3x-1)(x+1)}$.

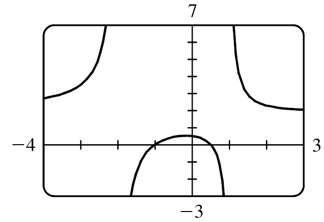


The denominator is zero when $x = \frac{1}{3}$ and -1, but the numerator is nonzero, so $x = \frac{1}{3}$ and $x = -1$ are vertical asymptotes. The graph confirms our work.

49. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+x-1}{x^2+x-2} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{2x^2+x-1}{x^2}}{\dfrac{x^2+x-2}{x^2}} = \lim\limits_{x\to\pm\infty} \dfrac{2+\dfrac{1}{x}-\dfrac{1}{x^2}}{1+\dfrac{1}{x}-\dfrac{2}{x^2}} = \dfrac{\lim\limits_{x\to\pm\infty}\left(2+\dfrac{1}{x}-\dfrac{1}{x^2}\right)}{\lim\limits_{x\to\pm\infty}\left(1+\dfrac{1}{x}-\dfrac{2}{x^2}\right)}$

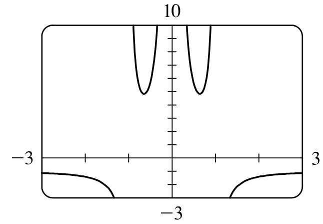
$$= \frac{\lim\limits_{x\to\pm\infty} 2 + \lim\limits_{x\to\pm\infty} \dfrac{1}{x} - \lim\limits_{x\to\pm\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\pm\infty} 1 + \lim\limits_{x\to\pm\infty} \dfrac{1}{x} - 2\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2}} = \frac{2+0-0}{1+0-2(0)} = 2,$$ so $y=2$ is a horizontal asymptote.

$y = f(x) = \dfrac{2x^2+x-1}{x^2+x-2} = \dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$, so $\lim\limits_{x\to -2^-} f(x) = \infty$, $\lim\limits_{x\to -2^+} f(x) = -\infty$, $\lim\limits_{x\to 1^-} f(x) = -\infty$, and $\lim\limits_{x\to 1^+} f(x) = \infty$. Thus, $x=-2$ and $x=1$ are vertical asymptotes. The graph confirms our work.



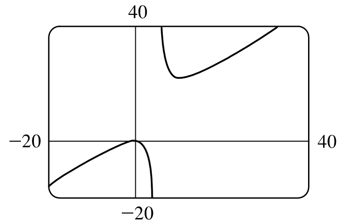
50. $\lim\limits_{x\to\pm\infty} \dfrac{1+x^4}{x^2-x^4} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1+x^4}{x^4}}{\dfrac{x^2-x^4}{x^4}} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1}{x^4}+1}{\dfrac{1}{x^2}-1} = \dfrac{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^4}+1\right)}{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^2}-1\right)} = \dfrac{\lim\limits_{x\to\pm\infty} \dfrac{1}{x^4} + \lim\limits_{x\to\pm\infty} 1}{\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} - \lim\limits_{x\to\pm\infty} 1}$

$$= \frac{0+1}{0-1} = -1,$$ so $y=-1$ is a horizontal asymptote.



$y = f(x) = \dfrac{1+x^4}{x^2 - x^4} = \dfrac{1+x^4}{x^2(1-x^2)} = \dfrac{1+x^4}{x^2(1+x)(1-x)}$. The denominator is zero when $x = 0, -1$, and 1, but the numerator is nonzero, so $x = 0$, $x = -1$, and $x = 1$ are vertical asymptotes. Notice that as $x \to 0$, the numerator and denominator are both positive, so $\lim\limits_{x \to 0} f(x) = \infty$. The graph confirms our work.

51. $y = f(x) = \dfrac{x^3 - x}{x^2 - 6x + 5} = \dfrac{x(x^2-1)}{(x-1)(x-5)} = \dfrac{x(x+1)(x-1)}{(x-1)(x-5)} = \dfrac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.

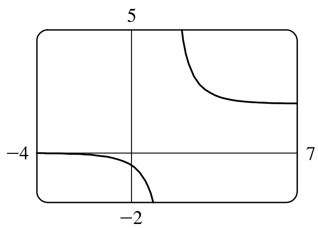


The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \dfrac{x^2 + x}{x - 5} = x + 6 + \dfrac{30}{x-5}$.

As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x = 5$. $\lim\limits_{x \to 5^-} g(x) = -\infty$ and $\lim\limits_{x \to 5^+} g(x) = \infty$, so $x = 5$ is a vertical asymptote. The graph confirms our work.

52. $\lim\limits_{x \to \infty} \dfrac{2e^x}{e^x - 5} = \lim\limits_{x \to \infty} \dfrac{2e^x}{e^x - 5} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x \to \infty} \dfrac{2}{1 - (5/e^x)} = \dfrac{2}{1-0} = 2$, so $y = 2$ is a horizontal asymptote.

$\lim\limits_{x \to -\infty} \dfrac{2e^x}{e^x - 5} = \dfrac{2(0)}{0-5} = 0$, so $y = 0$ is a horizontal asymptote. The denominator is zero (and the numerator isn't) when $e^x - 5 = 0 \quad \Rightarrow \quad e^x = 5 \quad \Rightarrow \quad x = \ln 5$.

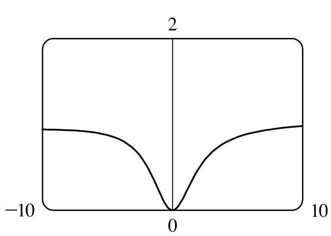


$\lim\limits_{x \to (\ln 5)^+} \dfrac{2e^x}{e^x - 5} = \infty$ since the numerator approaches 10 and the denominator approaches 0 through positive values as $x \to (\ln 5)^+$. Similarly, $\lim\limits_{x \to (\ln 5)^-} \dfrac{2e^x}{e^x - 5} = -\infty$. Thus, $x = \ln 5$ is a vertical asymptote. The graph confirms our work.

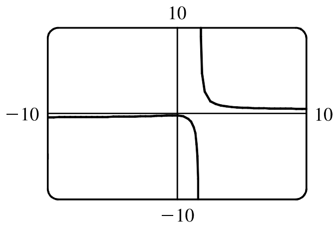
53. From the graph, it appears $y = 1$ is a horizontal asymptote.

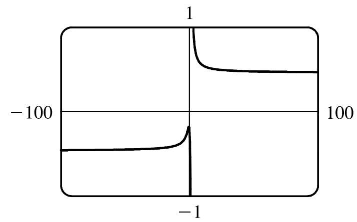
$$\begin{aligned}
\lim_{x \to \pm\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} &= \lim_{x \to \pm\infty} \frac{\dfrac{3x^3 + 500x^2}{x^3}}{\dfrac{x^3 + 500x^2 + 100x + 2000}{x^3}} \\
&= \lim_{x \to \pm\infty} \frac{3 + (500/x)}{1 + (500/x) + (100/x^2) + (2000/x^3)} \\
&= \frac{3+0}{1+0+0+0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}
\end{aligned}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000, 100{,}000]$ by $[-1, 4]$ to get a graph that lends credibility to our calculation that $y = 3$ is a horizontal asymptote.



54. (a)





From the graph, it appears at first that there is only one horizontal asymptote, at $y \approx 0$, and a vertical asymptote at $x \approx 1.7$. However, if we graph the function with a wider and shorter viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at $y \approx 0.5$ and one at $y \approx -0.5$. So we estimate that

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx 0.5 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx -0.5$$

(b) $f(1000) \approx 0.4722$ and $f(10{,}000) \approx 0.4715$, so we estimate that $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} \approx 0.47$.

$f(-1000) \approx -0.4706$ and $f(-10{,}000) \approx -0.4713$, so we estimate that $\lim\limits_{x\to-\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} \approx -0.47$.

(c) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim\limits_{x\to\infty} \dfrac{\sqrt{2+1/x^2}}{3-5/x}$ $\quad$ [since $\sqrt{x^2} = x$ for $x > 0$] $\quad = \dfrac{\sqrt{2}}{3} \approx 0.471404$.

For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the numerator by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{2x^2+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+1/x^2}$. Therefore,

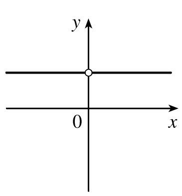
$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

55. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

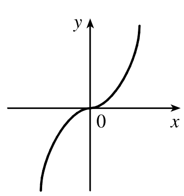
(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

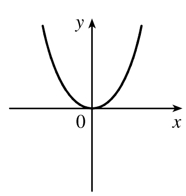
56.



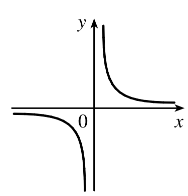
(i) $n = 0$



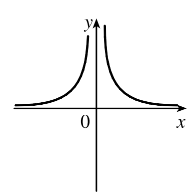
(ii) $n > 0$ $\quad$ (n odd)



(iii) $n > 0$ $\quad$ (n even)



(iv) $n < 0$ $\quad$ (n odd)



(v) $n < 0$ $\quad$ (n even)

From these sketches we see that

(a) $\lim\limits_{x\to0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$ $\qquad$ (b) $\lim\limits_{x\to0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim\limits_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$

(d) $\lim\limits_{x\to-\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

57. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \quad\Rightarrow\quad$ degree of numerator $<$ degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \quad\Rightarrow\quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \quad\Rightarrow\quad$ vertical asymptote at $x = 3$; there is a factor of $(x-3)$ in the denominator.

(4) $f(2) = 0 \quad\Rightarrow\quad$ 2 is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

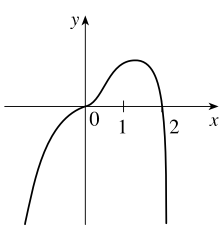
Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

58. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

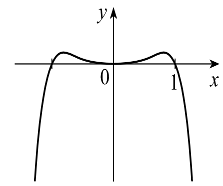
59. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x-1)(x+1)}{(x-4)(x+1)}$, where a is still to be determined. Then $\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \dfrac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim\limits_{x\to-1} \dfrac{a(x-1)}{x-4} = \dfrac{a(-1-1)}{(-1-4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and $a = 5$. Thus $f(x) = \dfrac{5(x-1)(x+1)}{(x-4)(x+1)}$ is a ratio of quadratic functions satisfying all the given conditions and $f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

(b) $\lim\limits_{x\to\infty} f(x) = 5\lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2-3x-4} = 5\lim\limits_{x\to\infty} \dfrac{(x^2/x^2)-(1/x^2)}{(x^2/x^2)-(3x/x^2)-(4/x^2)} = 5\dfrac{1-0}{1-0-0} = 5(1) = 5$

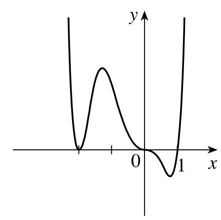
60. $y = f(x) = 2x^3 - x^4 = x^3(2-x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0 and 2. There are sign changes at 0 and 2 (odd exponents on x and $2-x$). As $x \to \infty$, $f(x) \to -\infty$ because $x^3 \to \infty$ and $2 - x \to -\infty$. As $x \to -\infty$, $f(x) \to -\infty$ because $x^3 \to -\infty$ and $2 - x \to \infty$. Note that the graph of f near $x = 0$ flattens out (looks like $y = x^3$).



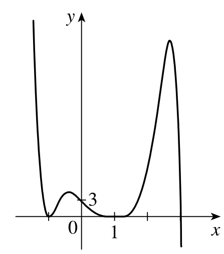
61. $y = f(x) = x^4 - x^6 = x^4(1 - x^2) = x^4(1 + x)(1 - x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1 - x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1 - x^2) \to -\infty$.



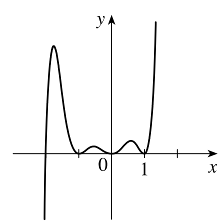
62. $y = f(x) = x^3(x + 2)^2(x - 1)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -2, and 1. There are sign changes at 0 and 1 (odd exponents on x and $x - 1$). There is no sign change at -2. Also, $f(x) \to \infty$ as $x \to \infty$ because all three factors are large. And $f(x) \to \infty$ as $x \to -\infty$ because $x^3 \to -\infty$, $(x + 2)^2 \to \infty$, and $(x - 1) \to -\infty$. Note that the graph of f at $x = 0$ flattens out (looks like $y = -x^3$).



63. $y = f(x) = (3 - x)(1 + x)^2(1 - x)^4$. The y-intercept is $f(0) = 3(1)^2(1)^4 = 3$. The x-intercepts are 3, -1, and 1. There is a sign change at 3, but not at -1 and 1. When x is large positive, $3 - x$ is negative and the other factors are positive, so $\lim\limits_{x\to\infty} f(x) = -\infty$. When x is large negative, $3 - x$ is positive, so $\lim\limits_{x\to-\infty} f(x) = \infty$.

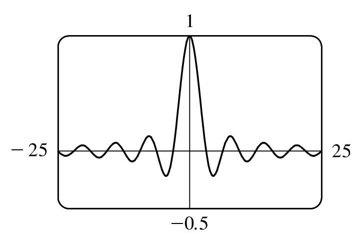


64. $y = f(x) = x^2(x^2 - 1)^2(x + 2) = x^2(x + 1)^2(x - 1)^2(x + 2)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, 1, and -2. There is a sign change at -2, but not at 0, -1, and 1. When x is large positive, all the factors are positive, so $\lim\limits_{x\to\infty} f(x) = \infty$. When x is large negative, only $x + 2$ is negative, so $\lim\limits_{x\to-\infty} f(x) = -\infty$.

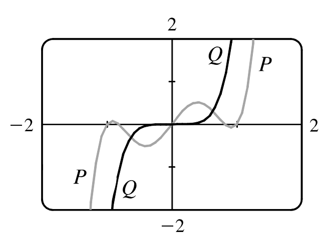


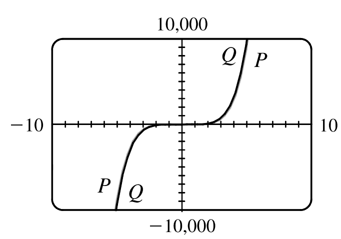
65. (a) Since $-1 \leq \sin x \leq 1$ for all x, $-\dfrac{1}{x} \leq \dfrac{\sin x}{x} \leq \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*



66. (a) In both viewing rectangles, $\lim\limits_{x\to\infty} P(x) = \lim\limits_{x\to\infty} Q(x) = \infty$ and $\lim\limits_{x\to-\infty} P(x) = \lim\limits_{x\to-\infty} Q(x) = -\infty$. In the larger viewing rectangle, P and Q become less distinguishable.





(b) $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim\limits_{x\to\infty} \left(1 - \dfrac{5}{3} \cdot \dfrac{1}{x^2} + \dfrac{2}{3} \cdot \dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \quad \Rightarrow$

P and Q have the same end behavior.

67. $\lim\limits_{x\to\infty} \dfrac{5\sqrt{x}}{\sqrt{x-1}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1-(1/x)}} = \dfrac{5}{\sqrt{1-0}} = 5$ and

$\lim\limits_{x\to\infty} \dfrac{10e^x - 21}{2e^x} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{10 - (21/e^x)}{2} = \dfrac{10-0}{2} = 5$. Since $\dfrac{10e^x - 21}{2e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x-1}}$,

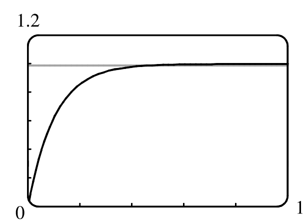
we have $\lim\limits_{x\to\infty} f(x) = 5$ by the Squeeze Theorem.

68. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \dfrac{750t}{5000 + 25t} = \dfrac{30t}{200 + t} \dfrac{\text{g}}{\text{L}}$.

(b) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \dfrac{30t}{200+t} = \lim\limits_{t\to\infty} \dfrac{30t/t}{200/t + t/t} = \dfrac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

69. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$

(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

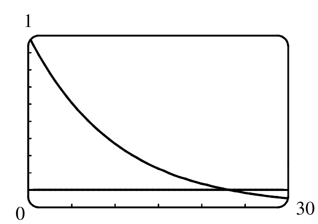


70. (a) $y = e^{-x/10}$ and $y = 0.1$ intersect at $x_1 \approx 23.03$.

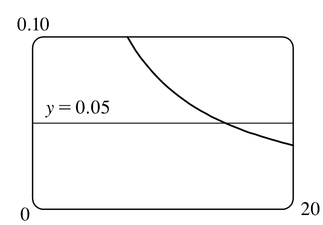
If $x > x_1$, then $e^{-x/10} < 0.1$.

(b) $e^{-x/10} < 0.1 \quad \Rightarrow \quad -x/10 < \ln 0.1 \quad \Rightarrow$

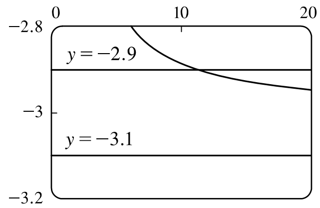
$x > -10 \ln \frac{1}{10} = -10 \ln 10^{-1} = 10 \ln 10 \approx 23.03$

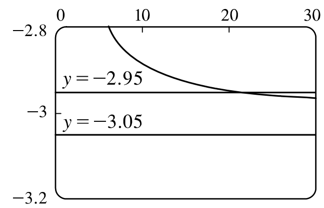


71. Let $g(x) = \dfrac{3x^2 + 1}{2x^2 + x + 1}$ and $f(x) = |g(x) - 1.5|$. Note that $\lim\limits_{x\to\infty} g(x) = \frac{3}{2}$ and $\lim\limits_{x\to\infty} f(x) = 0$. We are interested in finding the x-value at which $f(x) < 0.05$. From the graph, we find that $x \approx 14.804$, so we choose $N = 15$ (or any larger number).

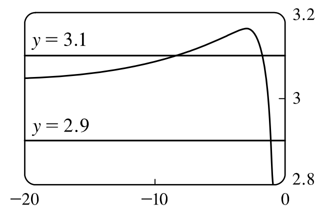


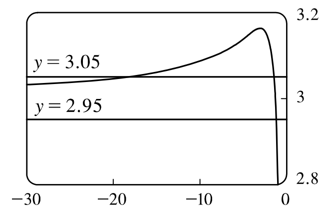
72. We want to find a value of N such that $x > N \quad \Rightarrow \quad \left| \dfrac{1-3x}{\sqrt{x^2+1}} - (-3) \right| < \varepsilon$, or equivalently, $-3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < -3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = -3.1$, and $y = -2.9$. From the graph, we find that $f(x) = -2.9$ at about $x = 11.283$, so we choose $N = 12$ (or any larger number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = -2.95$ at about $x = 21.379$, so we choose $N = 22$ (or any larger number).



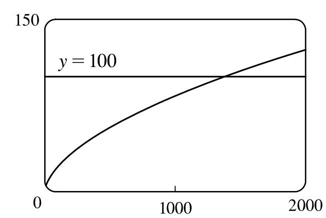


73. We want a value of N such that $x < N \Rightarrow \left| \dfrac{1-3x}{\sqrt{x^2+1}} - 3 \right| < \varepsilon$, or equivalently, $3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < 3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = 3.1$, and $y = 2.9$. From the graph, we find that $f(x) = 3.1$ at about $x = -8.092$, so we choose $N = -9$ (or any lesser number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = 3.05$ at about $x = -18.338$, so we choose $N = -19$ (or any lesser number).





74. We want to find a value of N such that $x > N \quad \Rightarrow \quad \sqrt{x \ln x} > 100$. We graph $y = f(x) = \sqrt{x \ln x}$ and $y = 100$. From the graph, we find that $f(x) = 100$ at about $x = 1382.773$, so we choose $N = 1383$ (or any larger number).



75. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10\,000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left| \dfrac{1}{x^2} - 0 \right| = \dfrac{1}{x^2} < \varepsilon$, so $\displaystyle\lim_{x \to \infty} \frac{1}{x^2} = 0$.

76. (a) $1/\sqrt{x} < 0.0001 \quad \Leftrightarrow \quad \sqrt{x} > 1/0.0001 = 10^4 \quad \Leftrightarrow \quad x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad \Leftrightarrow \quad \sqrt{x} > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\varepsilon^2} \quad \Rightarrow \quad \left| \dfrac{1}{\sqrt{x}} - 0 \right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\displaystyle\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$.

77. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.
Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\displaystyle\lim_{x \to -\infty} (1/x) = 0$.

78. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad x^3 > M$. Now $x^3 > M \quad \Leftrightarrow \quad x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \quad \Rightarrow \quad x^3 > M$, so $\displaystyle\lim_{x \to \infty} x^3 = \infty$.

79. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad e^x > M$. Now $e^x > M \quad \Leftrightarrow \quad x > \ln M$, so take $N = \max(1, \ln M)$. (This ensures that $N > 0$.) Then $x > N = \max(1, \ln M) \quad \Rightarrow \quad e^x > \max(e, M) \geq M$, so $\lim\limits_{x \to \infty} e^x = \infty$.

80. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x \to -\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$. Now we use the definition to prove that $\lim\limits_{x \to -\infty} \left(1 + x^3\right) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \quad \Rightarrow \quad 1 + x^3 < M$. Now $1 + x^3 < M \quad \Leftrightarrow \quad x^3 < M - 1 \quad \Leftrightarrow \quad x < \sqrt[3]{M-1}$. Thus, we take $N = \sqrt[3]{M-1}$ and find that $x < N \quad \Rightarrow \quad 1 + x^3 < M$. This proves that $\lim\limits_{x \to -\infty} \left(1 + x^3\right) = -\infty$.

81. (a) Suppose that $\lim\limits_{x \to \infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \quad \Leftrightarrow \quad 0 < 1/x < 1/N \quad \Leftrightarrow \quad 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\lim\limits_{t \to 0^+} f(1/t) = L = \lim\limits_{x \to \infty} f(x)$.

Now suppose that $\lim\limits_{x \to -\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \quad \Leftrightarrow \quad 1/N < 1/x < 0 \quad \Leftrightarrow \quad 1/N < t < 0$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t \to 0^-} f(1/t) = L = \lim\limits_{x \to -\infty} f(x)$.

(b)
$$
\begin{aligned}
\lim_{x \to 0^+} x \sin \frac{1}{x} &= \lim_{t \to 0^+} t \sin \frac{1}{t} && [\text{let } x = t] \\
&= \lim_{y \to \infty} \frac{1}{y} \sin y && [\text{part (a) with } y = 1/t] \\
&= \lim_{x \to \infty} \frac{\sin x}{x} && [\text{let } y = x] \\
&= 0 && [\text{by Exercise 65}]
\end{aligned}
$$