3 ☐ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim\limits_{h\to 0}\dfrac{e^h-1}{h}=1$.

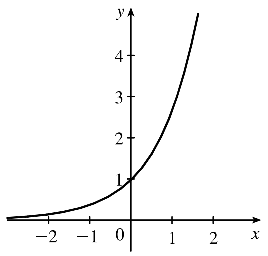
(b)

x	$\dfrac{2.7^x-1}{x}$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

x	$\dfrac{2.8^x-1}{x}$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),
$\lim\limits_{h\to 0}\dfrac{2.7^h-1}{h}=0.99$ and $\lim\limits_{h\to 0}\dfrac{2.8^h-1}{h}=1.03$.
Since $0.99<1<1.03$, $2.7<e<2.8$.

2. (a)



The function value at $x=0$ is 1 and the slope at $x=0$ is 1.

(b) $f(x)=e^x$ is an exponential function and $g(x)=x^e$ is a power function. $\dfrac{d}{dx}(e^x)=e^x$ and $\dfrac{d}{dx}(x^e)=ex^{e-1}$.

(c) $f(x)=e^x$ grows more rapidly than $g(x)=x^e$ when x is large.

3. $f(x)=2^{40}$ is a constant function, so its derivative is 0, that is, $f'(x)=0$.

4. $f(x)=e^5$ is a constant function, so its derivative is 0, that is, $f'(x)=0$.

5. $f(x)=5.2x+2.3 \quad\Rightarrow\quad f'(x)=5.2(1)+0=5.2$

6. $g(x)=\frac{7}{4}x^2-3x+12 \quad\Rightarrow\quad g'(x)=\frac{7}{4}(2x)-3(1)+0=\frac{7}{2}x-3$

7. $f(t)=2t^3-3t^2-4t \quad\Rightarrow\quad f'(t)=2(3t^2)-3(2t)-4(1)=6t^2-6t-4$

8. $f(t)=1.4t^5-2.5t^2+6.7 \quad\Rightarrow\quad f'(t)=1.4(5t^4)-2.5(2t)+0=7t^4-5t$

9. $g(x)=x^2(1-2x)=x^2-2x^3 \quad\Rightarrow\quad g'(x)=2x-2(3x^2)=2x-6x^2$

10. $H(u)=(3u-1)(u+2)=3u^2+5u-2 \quad\Rightarrow\quad H'(u)=3(2u)+5(1)-0=6u+5$

11. $g(t)=2t^{-3/4} \quad\Rightarrow\quad g'(t)=2\left(-\frac{3}{4}t^{-7/4}\right)=-\frac{3}{2}t^{-7/4}$

12. $B(y)=cy^{-6} \quad\Rightarrow\quad B'(y)=c(-6y^{-7})=-6cy^{-7}$

13. $F(r)=\dfrac{5}{r^3}=5r^{-3} \quad\Rightarrow\quad F'(r)=5(-3r^{-4})=-15r^{-4}=-\dfrac{15}{r^4}$

14. $y=x^{5/3}-x^{2/3} \quad\Rightarrow\quad y'=\frac{5}{3}x^{2/3}-\frac{2}{3}x^{-1/3}$

15. $R(a) = (3a+1)^2 = 9a^2 + 6a + 1 \quad \Rightarrow \quad R'(a) = 9(2a) + 6(1) + 0 = 18a + 6$

16. $h(t) = \sqrt[4]{t} - 4e^t = t^{1/4} - 4e^t \quad \Rightarrow \quad h'(t) = \frac{1}{4}t^{-3/4} - 4(e^t) = \frac{1}{4}t^{-3/4} - 4e^t$

17. $S(p) = \sqrt{p} - p = p^{1/2} - p \quad \Rightarrow \quad S'(p) = \frac{1}{2}p^{-1/2} - 1$ or $\dfrac{1}{2\sqrt{p}} - 1$

18. $y = \sqrt[3]{x}\,(2+x) = 2x^{1/3} + x^{4/3} \quad \Rightarrow \quad y' = 2\left(\frac{1}{3}x^{-2/3}\right) + \frac{4}{3}x^{1/3} = \frac{2}{3}x^{-2/3} + \frac{4}{3}x^{1/3}$ or $\dfrac{2}{3\sqrt[3]{x^2}} + \dfrac{4}{3}\sqrt[3]{x}$

19. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}} = 3e^x + 4x^{-1/3} \quad \Rightarrow \quad y' = 3(e^x) + 4(-\frac{1}{3})x^{-4/3} = 3e^x - \frac{4}{3}x^{-4/3}$

20. $S(R) = 4\pi R^2 \quad \Rightarrow \quad S'(R) = 4\pi(2R) = 8\pi R$

21. $h(u) = Au^3 + Bu^2 + Cu \quad \Rightarrow \quad h'(u) = A(3u^2) + B(2u) + C(1) = 3Au^2 + 2Bu + C$

22. $y = \dfrac{\sqrt{x} + x}{x^2} = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \quad \Rightarrow \quad y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

23. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad \Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ [note that $x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}$]

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

24. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}\,t^{1/2} + \sqrt{7}\,t^{-1} \quad \Rightarrow \quad G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}\left(-1t^{-2}\right) = \dfrac{\sqrt{5}}{2\sqrt{t}} - \dfrac{\sqrt{7}}{t^2}$

25. $j(x) = x^{2.4} + e^{2.4} \quad \Rightarrow \quad j'(x) = 2.4x^{1.4} + 0 = 2.4x^{1.4}$

26. $k(r) = e^r + r^e \quad \Rightarrow \quad k'(r) = e^r + er^{e-1}$

27. $G(q) = \left(1 + q^{-1}\right)^2 = 1 + 2q^{-1} + q^{-2} \quad \Rightarrow \quad G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

28. $F(z) = \dfrac{A + Bz + Cz^2}{z^2} = \dfrac{A}{z^2} + \dfrac{Bz}{z^2} + \dfrac{Cz^2}{z^2} = Az^{-2} + Bz^{-1} + C \quad \Rightarrow$

$F'(z) = A(-2z^{-3}) + B(-1z^{-2}) + 0 = -2Az^{-3} - Bz^{-2} = -\dfrac{2A}{z^3} - \dfrac{B}{z^2}$ or $-\dfrac{2A + Bz}{z^3}$

29. $f(v) = \dfrac{\sqrt[3]{v} - 2ve^v}{v} = \dfrac{\sqrt[3]{v}}{v} - \dfrac{2ve^v}{v} = v^{-2/3} - 2e^v \quad \Rightarrow \quad f'(v) = -\frac{2}{3}v^{-5/3} - 2e^v$

30. $D(t) = \dfrac{1 + 16t^2}{(4t)^3} = \dfrac{1 + 16t^2}{64t^3} = \frac{1}{64}t^{-3} + \frac{1}{4}t^{-1} \quad \Rightarrow$

$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2}$ or $-\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$

31. $z = \dfrac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y \quad \Rightarrow \quad z' = -10Ay^{-11} + Be^y = -\dfrac{10A}{y^{11}} + Be^y$

32. $y = e^{x+1} + 1 = e^x e^1 + 1 = e \cdot e^x + 1 \quad \Rightarrow \quad y' = e \cdot e^x = e^{x+1}$

33. $y = 2x^3 - x^2 + 2 \quad \Rightarrow \quad y' = 6x^2 - 2x$. At $(1, 3)$, $y' = 6(1)^2 - 2(1) = 4$ and an equation of the tangent line is $y - 3 = 4(x - 1)$ or $y = 4x - 1$.

34. $y = 2e^x + x \quad \Rightarrow \quad y' = 2e^x + 1$. At $(0, 2)$, $y' = 2e^0 + 1 = 3$ and an equation of the tangent line is $y - 2 = 3(x - 0)$ or $y = 3x + 2$.

35. $y = x + \dfrac{2}{x} = x + 2x^{-1} \quad \Rightarrow \quad y' = 1 - 2x^{-2}$. At $(2, 3)$, $y' = 1 - 2(2)^{-2} = \frac{1}{2}$ and an equation of the tangent line is $y - 3 = \frac{1}{2}(x - 2)$ or $y = \frac{1}{2}x + 2$.

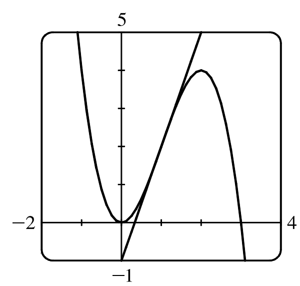
36. $y = \sqrt[4]{x} - x = x^{1/4} - x \quad \Rightarrow \quad y' = \frac{1}{4}x^{-3/4} - 1 = \dfrac{1}{4\sqrt[4]{x^3}} - 1$. At $(1, 0)$, $y' = \frac{1}{4} - 1 = -\frac{3}{4}$ and an equation of the tangent line is $y - 0 = -\frac{3}{4}(x - 1)$ or $y = -\frac{3}{4}x + \frac{3}{4}$.

37. $y = x^4 + 2e^x \quad \Rightarrow \quad y' = 4x^3 + 2e^x$. At $(0, 2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x - 0)$ or $y = 2x + 2$. The slope of the normal line is $-\frac{1}{2}$ (the negative reciprocal of 2) and an equation of the normal line is $y - 2 = -\frac{1}{2}(x - 0)$ or $y = -\frac{1}{2}x + 2$.

38. $y^2 = x^3 \quad \Rightarrow \quad y = x^{3/2}$ [since x and y are positive at $(1, 1)$] $\quad \Rightarrow \quad y' = \frac{3}{2}x^{1/2}$. At $(1, 1)$, $y' = \frac{3}{2}$ and an equation of the tangent line is $y - 1 = \frac{3}{2}(x - 1)$ or $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$ $\left(\text{the negative reciprocal of } \frac{3}{2}\right)$ and an equation of the normal line is $y - 1 = -\frac{2}{3}(x - 1)$ or $y = -\frac{2}{3}x + \frac{5}{3}$.

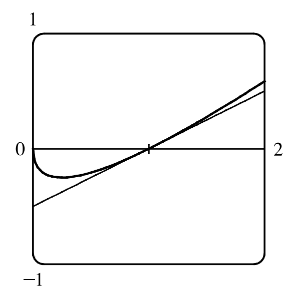
39. $y = 3x^2 - x^3 \quad \Rightarrow \quad y' = 6x - 3x^2$.

At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is $y - 2 = 3(x - 1)$ or $y = 3x - 1$.



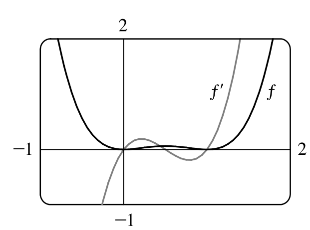
40. $y = x - \sqrt{x} \quad \Rightarrow \quad y' = 1 - \frac{1}{2}x^{-1/2} = 1 - \dfrac{1}{2\sqrt{x}}$.

At $(1, 0)$, $y' = \frac{1}{2}$, so an equation of the tangent line is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$.



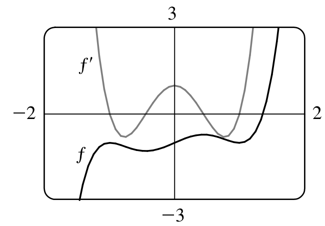
41. $f(x) = x^4 - 2x^3 + x^2 \quad \Rightarrow \quad f'(x) = 4x^3 - 6x^2 + 2x$

Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

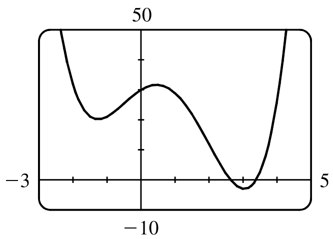


42. $f(x) = x^5 - 2x^3 + x - 1 \quad \Rightarrow \quad f'(x) = 5x^4 - 6x^2 + 1$

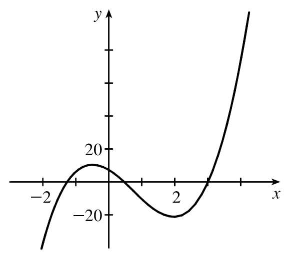
Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



43. (a)

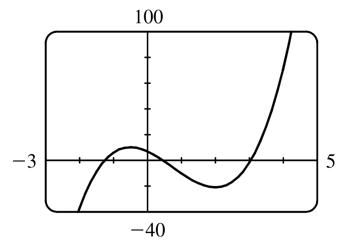


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

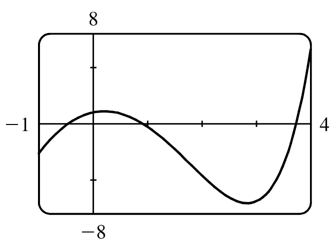


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

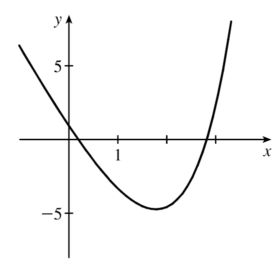
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



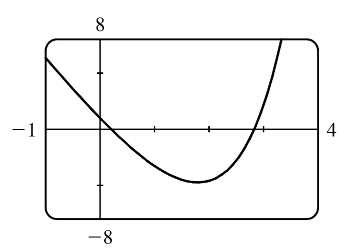
44. (a)



(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞). The slopes are negative (so f' is negative) on (x_1, x_2).



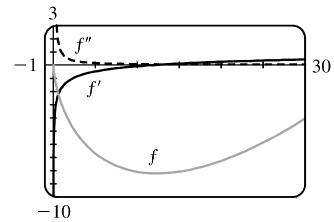
(c) $g(x) = e^x - 3x^2 \quad \Rightarrow \quad g'(x) = e^x - 6x$



45. $f(x) = 0.001x^5 - 0.02x^3 \quad \Rightarrow \quad f'(x) = 0.005x^4 - 0.06x^2 \quad \Rightarrow \quad f''(x) = 0.02x^3 - 0.12x$

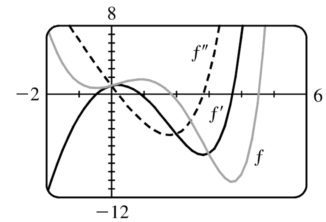
46. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad \Rightarrow \quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad \Rightarrow \quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

47. $f(x) = 2x - 5x^{3/4} \quad \Rightarrow \quad f'(x) = 2 - \frac{15}{4}x^{-1/4} \quad \Rightarrow \quad f''(x) = \frac{15}{16}x^{-5/4}$



Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.

48. $f(x) = e^x - x^3 \quad \Rightarrow \quad f'(x) = e^x - 3x^2 \quad \Rightarrow \quad f''(x) = e^x - 6x$



Note that $f'(x) = 0$ when f has a horizontal tangent and that $f''(x) = 0$ when f' has a horizontal tangent.

49. (a) $s = t^3 - 3t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 3 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12 \text{ m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ $[t \geq 0]$ and $a(1) = 6 \text{ m/s}^2$.

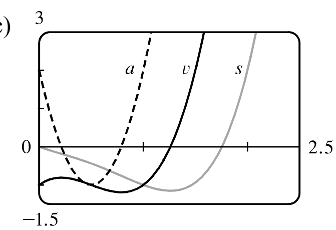
50. (a) $s = t^4 - 2t^3 + t^2 - t \quad \Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad \Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2 \text{ m/s}^2$

(c)



51. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \quad \Rightarrow \quad \dfrac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\left.\dfrac{dL}{dA}\right|_{A=12} = 0.0465(12)^2 - 0.744(12) + 3.95 = 1.718$. The derivative is the instantaneous rate of change of the length of an Alaskan rockfish with respect to its age when its age is 12 years.

52. $S(A) = 0.882A^{0.842} \quad \Rightarrow \quad S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

53. (a) $P = \dfrac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \dfrac{5.3}{V}$ and $V = \dfrac{5.3}{P}$.

(b) $V = 5.3P^{-1} \quad \Rightarrow \quad \dfrac{dV}{dP} = 5.3(-1P^{-2}) = -\dfrac{5.3}{P^2}$. When $P = 50$, $\dfrac{dV}{dP} = -\dfrac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at $25\,^\circ\text{C}$. Its units are m^3/kPa.

54. (a) $L = aP^2 + bP + c$, where $a \approx -0.275428$, $b \approx 19.74853$, and $c \approx -273.55234$.

(b) $\dfrac{dL}{dP} = 2aP + b$. When $P = 30$, $\dfrac{dL}{dP} \approx 3.2$, and when $P = 40$, $\dfrac{dL}{dP} \approx -2.3$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of miles)/(lb/in^2). When $\dfrac{dL}{dP}$ is positive, tire life is increasing, and when $\dfrac{dL}{dP} < 0$, tire life is decreasing.

55. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \quad\Leftrightarrow\quad 6(x^2 + x - 2) = 0 \quad\Leftrightarrow$ $6(x + 2)(x - 1) = 0 \quad\Leftrightarrow\quad x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

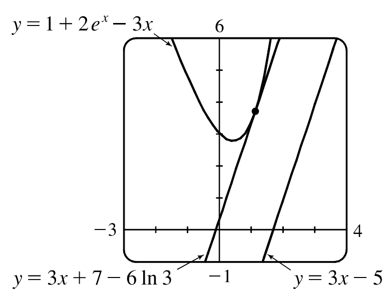
56. $f(x) = e^x - 2x \quad\Rightarrow\quad f'(x) = e^x - 2$. $f'(x) = 0 \quad\Rightarrow\quad e^x = 2 \quad\Rightarrow\quad x = \ln 2$, so f has a horizontal tangent when $x = \ln 2$.

57. $y = 2e^x + 3x + 5x^3 \quad\Rightarrow\quad y' = 2e^x + 3 + 15x^2$. Since $2e^x > 0$ and $15x^2 \geq 0$, we must have $y' > 0 + 3 + 0 = 3$, so no tangent line can have slope 2.

58. $y = x^4 + 1 \quad\Rightarrow\quad y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \quad\Leftrightarrow\quad 4x^3 = 32 \quad\Leftrightarrow\quad x^3 = 8 \quad\Leftrightarrow\quad x = 2$, which is the x-coordinate of the point on the curve at which the slope is 32. The y-coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

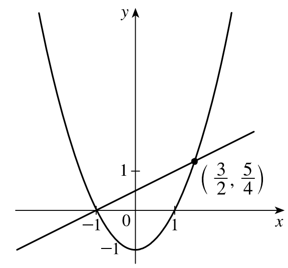
59. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3.
$y = x^3 - 3x^2 + 3x - 3 \quad\Rightarrow\quad y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$. Thus, $3(x - 1)^2 = 3 \quad\Rightarrow$ $(x - 1)^2 = 1 \quad\Rightarrow\quad x - 1 = \pm 1 \quad\Rightarrow\quad x = 0$ or 2, which are the x-coordinates at which the tangent lines have slope 3. The points on the curve are $(0, -3)$ and $(2, -1)$, so the tangent line equations are $y - (-3) = 3(x - 0)$ or $y = 3x - 3$ and $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

60. The slope of $y = 1 + 2e^x - 3x$ is given by $m = y' = 2e^x - 3$.
The slope of $3x - y = 5 \quad\Leftrightarrow\quad y = 3x - 5$ is 3.
$m = 3 \quad\Rightarrow\quad 2e^x - 3 = 3 \quad\Rightarrow\quad e^x = 3 \quad\Rightarrow\quad x = \ln 3$.
This occurs at the point $(\ln 3, 7 - 3\ln 3) \approx (1.1, 3.7)$.

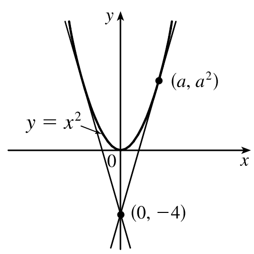


61. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2, so the desired normal line must have slope -2, and hence, the tangent line to the curve must have slope $\frac{1}{2}$. This occurs if $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} \quad\Rightarrow$ $\sqrt{x} = 1 \quad\Rightarrow\quad x = 1$. When $x = 1$, $y = \sqrt{1} = 1$, and an equation of the normal line is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

62. $y = f(x) = x^2 - 1 \quad\Rightarrow\quad f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is $y - 0 = \frac{1}{2}[x - (-1)]$ or $y = \frac{1}{2}x + \frac{1}{2}$. Substituting this into the equation of the parabola, we obtain $\frac{1}{2}x + \frac{1}{2} = x^2 - 1 \quad\Leftrightarrow\quad x + 1 = 2x^2 - 2 \quad\Leftrightarrow\quad 2x^2 - x - 3 = 0 \quad\Leftrightarrow\quad (2x - 3)(x + 1) = 0 \quad\Leftrightarrow\quad x = \frac{3}{2}$ or -1. Substituting $\frac{3}{2}$ into the equation of the normal line gives us $y = \frac{5}{4}$. Thus, the second point of intersection is $\left(\frac{3}{2}, \frac{5}{4}\right)$, as shown in the sketch.



63.



Let $\left(a, a^2\right)$ be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad\Leftrightarrow\quad y = 2ax - 4$. Since $\left(a, a^2\right)$ also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

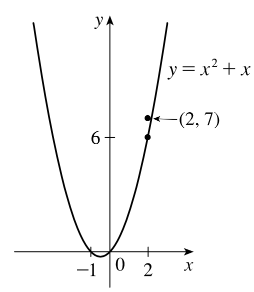
64. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $\left(a, a^2 + a\right)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad\Leftrightarrow\quad a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \quad\Leftrightarrow\quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \quad\Rightarrow\quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad\Leftrightarrow\quad a^2 - 4a + 5 = 0.$$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is “inside” the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



65. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h} = \lim\limits_{h \to 0} \dfrac{x - (x + h)}{hx(x + h)} = \lim\limits_{h \to 0} \dfrac{-h}{hx(x + h)} = \lim\limits_{h \to 0} \dfrac{-1}{x(x + h)} = -\dfrac{1}{x^2}$

66. (a) $f(x) = x^n \quad\Rightarrow\quad f'(x) = nx^{n-1} \quad\Rightarrow\quad f''(x) = n(n - 1)x^{n-2} \quad\Rightarrow\quad \cdots \quad\Rightarrow$

$f^{(n)}(x) = n(n - 1)(n - 2) \cdots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \quad\Rightarrow\quad f'(x) = (-1)x^{-2} \quad\Rightarrow\quad f''(x) = (-1)(-2)x^{-3} \quad\Rightarrow\quad \cdots \quad\Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3) \cdots (-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n\, n!}{x^{n+1}}$

67. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \;\Rightarrow\; 2a = 2 \;\Rightarrow\; a = 1$.

$P'(2) = 3 \;\Rightarrow\; 2(1)(2) + b = 3 \;\Rightarrow\; 4 + b = 3 \;\Rightarrow\; b = -1$.

$P(2) = 5 \;\Rightarrow\; 1(2)^2 + (-1)(2) + c = 5 \;\Rightarrow\; 2 + c = 5 \;\Rightarrow\; c = 3$. So $P(x) = x^2 - x + 3$.

68. $y = Ax^2 + Bx + C \;\Rightarrow\; y' = 2Ax + B \;\Rightarrow\; y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$\begin{aligned}(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) &= x^2\\ 2A + 2Ax + B - 2Ax^2 - 2Bx - 2C &= x^2\\ (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) &= (1)x^2 + (0)x + (0)\end{aligned}$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \;\Rightarrow\; A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \;\Rightarrow\; A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \;\Rightarrow\; -1 - \frac{1}{2} - 2C = 0 \;\Rightarrow\; C = -\frac{3}{4}$.

69. $y = f(x) = ax^3 + bx^2 + cx + d \;\Rightarrow\; f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \;\Rightarrow\; -8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \;\Rightarrow\; 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \;\Rightarrow\; 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \;\Rightarrow\; 12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \;\Rightarrow\; b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \;\Rightarrow\; 3 = 16a \;\Rightarrow\; a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

70. $y = ax^2 + bx + c \;\Rightarrow\; y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \;\Rightarrow\; 2a + b = 4$ **(1)** and $y'(-1) = -8 \;\Rightarrow\; -2a + b = -8$ **(2)**. Adding **(1)** and **(2)** gives us $2b = -4 \;\Leftrightarrow\; b = -2$. From **(1)**, $2a - 2 = 4 \;\Leftrightarrow\; a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \;\Rightarrow\; c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

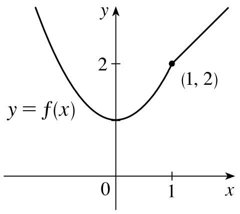
71. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

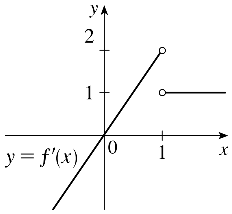
Calculate the left- and right-hand derivatives as defined in Exercise 2.8.64:

$$f'_-(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h + 2) = 2 \text{ and}$$

$$f'_+(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 1] - (1+1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Since the left and right limits are different, $\displaystyle\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





72. $g(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \ge 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$g'_-(0) = \lim\limits_{h \to 0^-} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{2h - 2(0)}{h} = 2$ and

$g'_+(0) = \lim\limits_{h \to 0^+} \dfrac{g(0+h) - g(0)}{h} = \lim\limits_{h \to 0^+} \dfrac{(2h - h^2) - 2(0)}{h} = \lim\limits_{h \to 0^+} (2 - h) = 2$, so g is differentiable at $x = 0$.

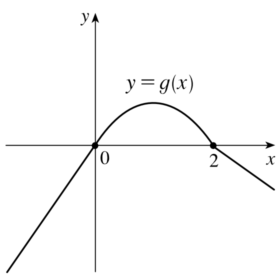
$g'_-(2) = \lim\limits_{h \to 0^-} \dfrac{g(2+h) - g(2)}{h} = \lim\limits_{h \to 0^-} \dfrac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim\limits_{h \to 0^-} \dfrac{-2h - h^2}{h} = \lim\limits_{h \to 0^-} (-2 - h) = -2$

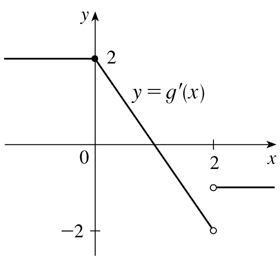
and

$g'_+(2) = \lim\limits_{h \to 0^+} \dfrac{g(2+h) - g(2)}{h} = \lim\limits_{h \to 0^+} \dfrac{[2 - (2+h)] - (2-2)}{h} = \lim\limits_{h \to 0^+} \dfrac{-h}{h} = \lim\limits_{h \to 0^+} (-1) = -1$,

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$g'(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$





73. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \iff |x| < 3 \iff -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \le -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \ge 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

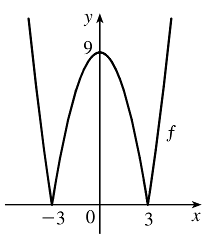
To show that $f'(3)$ does not exist we investigate $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.8.64.

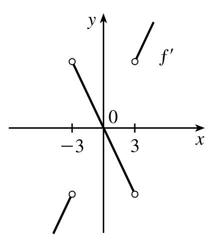
$f'_-(3) = \lim\limits_{h \to 0^-} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0^-} \dfrac{[-(3+h)^2 + 9] - 0}{h} = \lim\limits_{h \to 0^-} (-6 - h) = -6$ and

$f'_+(3) = \lim\limits_{h \to 0^+} \dfrac{f(3+h) - f(3)}{h} = \lim\limits_{h \to 0^+} \dfrac{\left[(3+h)^2 - 9\right] - 0}{h} = \lim\limits_{h \to 0^+} \dfrac{6h + h^2}{h} = \lim\limits_{h \to 0^+} (6 + h) = 6$.

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)





74. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x - 1 + x + 2 = 2x + 1$.

If $-2 < x < 1$, then $h(x) = -(x-1) + x + 2 = 3$.

If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x - 1$. Therefore,

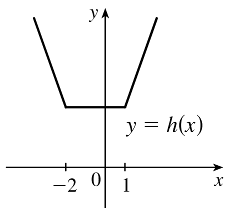
$$h(x) = \begin{cases} -2x-1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases} \quad \Rightarrow \quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

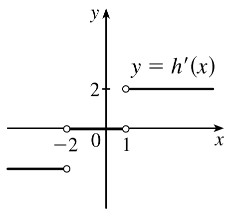
To see that $h'(1) = \lim\limits_{x \to 1} \dfrac{h(x) - h(1)}{x - 1}$ does not exist,

observe that $\lim\limits_{x \to 1^-} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^-} \dfrac{3 - 3}{3 - 1} = 0$ but

$\lim\limits_{x \to 1^+} \dfrac{h(x) - h(1)}{x - 1} = \lim\limits_{x \to 1^+} \dfrac{2x - 2}{x - 1} = 2$. Similarly,

$h'(-2)$ does not exist.





75. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \Rightarrow 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

76. $y = x^4 + ax^3 + bx^2 + cx + d \Rightarrow y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3 + 3ax^2 + 2bx + c \Rightarrow y'(0) = c$. Since the slope of the tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

77. $y = f(x) = ax^2 \Rightarrow f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \Leftrightarrow y = -2x + b$, is seen to be -2, so we must have $4a = -2 \Leftrightarrow a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \Leftrightarrow b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

78. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \Rightarrow c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so

$c\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow 3a = \frac{3}{2}a + 6 \Rightarrow \frac{3}{2}a = 6 \Rightarrow a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

79. The line $y = 2x + 3$ has slope 2. The parabola $y = cx^2 \Rightarrow y' = 2cx$ has slope $2ca$ at $x = a$. Equating slopes gives us $2ca = 2$, or $ca = 1$. Equating y-coordinates at $x = a$ gives us $ca^2 = 2a + 3 \Leftrightarrow (ca)a = 2a + 3 \Leftrightarrow 1a = 2a + 3 \Leftrightarrow a = -3$. Thus, $c = \dfrac{1}{a} = -\dfrac{1}{3}$.

80. $f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$. The slope of the tangent line at $x = p$ is $2ap + b$, the slope of the tangent line at $x = q$ is $2aq + b$, and the average of those slopes is $\dfrac{(2ap + b) + (2aq + b)}{2} = ap + aq + b$. The midpoint of the interval $[p, q]$ is $\dfrac{p + q}{2}$ and the slope of the tangent line at the midpoint is $2a\left(\dfrac{p + q}{2}\right) + b = a(p + q) + b$. This is equal to $ap + aq + b$, as required.

81. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

82. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

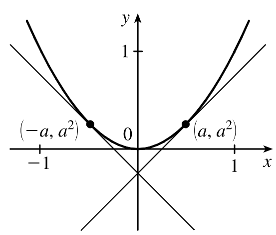
(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

83. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x \to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

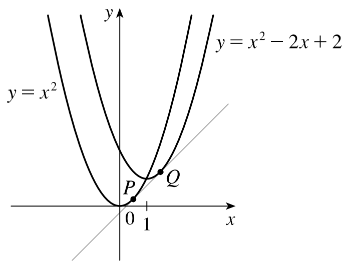
Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\begin{aligned}\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x \to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1) \\ &= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}\end{aligned}$$

84. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.



85. $y = x^2 \Rightarrow y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \Rightarrow$ $a^2 - c = -\frac{1}{2} \Rightarrow a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.

86. 

From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $[y' = 2x]$ and at Q is $2b - 2$ $[y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \Leftrightarrow a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \Rightarrow 2b - 2 = \dfrac{b^2 - 2b + 2 - (b-1)^2}{b - (b - 1)} \Rightarrow 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \Rightarrow$ $2b = 3 \Rightarrow b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or $y = x - \frac{1}{4}$.

APPLIED PROJECT Building a Better Roller Coaster

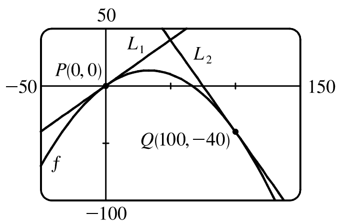
1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

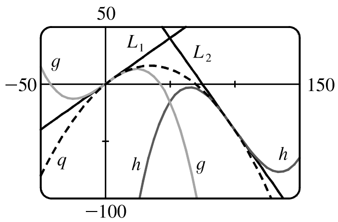
a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

Solving the system gives us the formulas for q, g, and h.

$$\left.\begin{array}{l} a=-0.01\overline{3}=-\frac{1}{75} \\ b=0.9\overline{3}=\frac{14}{15} \\ c=-0.\overline{4}=-\frac{4}{9} \end{array}\right\} q(x)=-\tfrac{1}{75}x^2+\tfrac{14}{15}x-\tfrac{4}{9} \qquad \left.\begin{array}{l} k=-0.000\overline{4}=-\frac{1}{2250} \\ l=0 \\ m=0.8=\frac{4}{5} \\ n=0 \end{array}\right\} g(x)=-\tfrac{1}{2250}x^3+\tfrac{4}{5}x$$

$$\left.\begin{array}{l} p=0.000\overline{4}=\frac{1}{2250} \\ q=-0.1\overline{3}=-\frac{2}{15} \\ r=11.7\overline{3}=\frac{176}{15} \\ s=-324.\overline{4}=-\frac{2920}{9} \end{array}\right\} h(x)=\tfrac{1}{2250}x^3-\tfrac{2}{15}x^2+\tfrac{176}{15}x-\tfrac{2920}{9}$$

(c) Graph of L_1, q, g, h, and L_2:

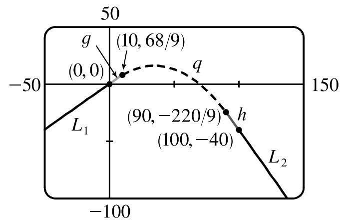


This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $\mathrm{Y}_2=L_1$, $\mathrm{Y}_6=g$, $\mathrm{Y}_5=q$, $\mathrm{Y}_7=h$, and $\mathrm{Y}_3=L_2$.

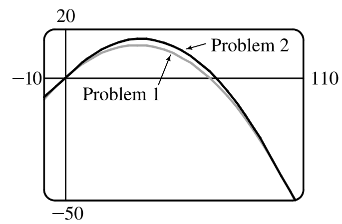
```
Plot1 Plot2 Plot3
\Y8=Y2*(X<0)+Y6*
(X≥0 and X<10)+Y
5*(X≥10 and X≤90
)+Y7*(X>90 and X
≤100)+Y3*(X>100)

\Y9=
```

The graph of the five functions as a piecewise-defined function:



A comparison of the graphs in part 1(c) and part 2(c):



3.2 The Product and Quotient Rules

1. Product Rule: $f(x)=(1+2x^2)(x-x^2) \quad \Rightarrow$

$$f'(x)=(1+2x^2)(1-2x)+(x-x^2)(4x)=1-2x+2x^2-4x^3+4x^2-4x^3=1-2x+6x^2-8x^3.$$

Multiplying first: $f(x)=(1+2x^2)(x-x^2)=x-x^2+2x^3-2x^4 \quad \Rightarrow \quad f'(x)=1-2x+6x^2-8x^3$ (equivalent).

2. Quotient Rule: $F(x)=\dfrac{x^4-5x^3+\sqrt{x}}{x^2}=\dfrac{x^4-5x^3+x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x)=\frac{x^2(4x^3-15x^2+\frac{1}{2}x^{-1/2})-(x^4-5x^3+x^{1/2})(2x)}{(x^2)^2}=\frac{4x^5-15x^4+\frac{1}{2}x^{3/2}-2x^5+10x^4-2x^{3/2}}{x^4}$$

$$=\frac{2x^5-5x^4-\frac{3}{2}x^{3/2}}{x^4}=2x-5-\tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x)=\dfrac{x^4-5x^3+\sqrt{x}}{x^2}=x^2-5x+x^{-3/2} \quad \Rightarrow \quad F'(x)=2x-5-\frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.