2 Review

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

4. False. $\dfrac{x^2-9}{x-3}$ is not defined when $x=3$, but $x+3$ is.

5. True. $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x+3)(x-3)}{(x-3)}=\lim_{x\to 3}(x+3)$

6. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

7. False. Consider $\displaystyle\lim_{x\to 5}\frac{x(x-5)}{x-5}$ or $\displaystyle\lim_{x\to 5}\frac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 2.2.3, we know that the latter limit exists (and it is equal to 1).

8. False. If $f(x)=1/x$, $g(x)=-1/x$, and $a=0$, then $\displaystyle\lim_{x\to 0}f(x)$ does not exist, $\displaystyle\lim_{x\to 0}g(x)$ does not exist, but $\displaystyle\lim_{x\to 0}[f(x)+g(x)]=\lim_{x\to 0}0=0$ exists.

9. True. Suppose that $\displaystyle\lim_{x\to a}[f(x)+g(x)]$ exists. Now $\displaystyle\lim_{x\to a}f(x)$ exists and $\displaystyle\lim_{x\to a}g(x)$ does not exist, but $\displaystyle\lim_{x\to a}g(x)=\lim_{x\to a}\{[f(x)+g(x)]-f(x)\}=\lim_{x\to a}[f(x)+g(x)]-\lim_{x\to a}f(x)$ [by Limit Law 2], which exists, and we have a contradiction. Thus, $\displaystyle\lim_{x\to a}[f(x)+g(x)]$ does not exist.

10. False. Consider $\lim_{x\to 6}[f(x)g(x)] = \lim_{x\to 6}\left[(x-6)\,\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

11. True. A polynomial is continuous everywhere, so $\lim_{x\to b} p(x)$ exists and is equal to $p(b)$.

12. False. Consider $\lim_{x\to 0}[f(x) - g(x)] = \lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

13. True. See Figure 2.6.8.

14. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim_{x\to\infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

15. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

16. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0,3]$ with $f(c) = 0$.

17. True. Use Theorem 2.5.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

18. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

19. True, by the definition of a limit with $\varepsilon = 1$.

20. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 + 1\right) = 1$.

21. False. See the note after Theorem 2.8.4.

22. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x\to r} f(x) = f(r)$.

23. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

24. True. $f(x) = x^{10} - 10x^2 + 5$ is continuous on the interval $[0, 2]$, $f(0) = 5$, $f(1) = -4$, and $f(2) = 989$. Since $-4 < 0 < 5$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^{10} - 10x^2 + 5 = 0$ in the interval $(0, 1)$. Similarly, there is a root in $(1, 2)$.

25. True. See Exercise 2.5.72(b).

26. False See Exercise 2.5.72(b).

EXERCISES

1. (a) (i) $\lim_{x\to 2^+} f(x) = 3$ (ii) $\lim_{x\to -3^+} f(x) = 0$

(iii) $\lim_{x\to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x\to 4} f(x) = 2$

(v) $\lim_{x\to 0} f(x) = \infty$ (vi) $\lim_{x\to 2^-} f(x) = -\infty$

(vii) $\lim_{x\to \infty} f(x) = 4$ (viii) $\lim_{x\to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

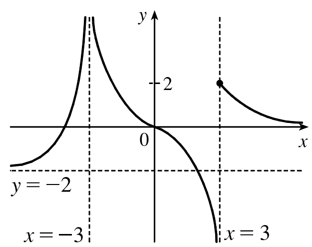
(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) f is discontinuous at $x = -3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2. $\lim_{x\to -\infty} f(x) = -2$, $\lim_{x\to \infty} f(x) = 0$, $\lim_{x\to -3} f(x) = \infty$,

$\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = 2$,

f is continuous from the right at 3



3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1$.

4. Since rational functions are continuous, $\lim_{x\to 3} \dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0$.

5. $$\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3} = \lim_{x\to -3} \frac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to -3} \frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \frac{3}{2}$$

6. $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3 \to 0^+$ as $x \to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $$\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \lim_{h\to 0} \frac{\left(h^3-3h^2+3h-1\right)+1}{h} = \lim_{h\to 0} \frac{h^3-3h^2+3h}{h} = \lim_{h\to 0} \left(h^2-3h+3\right) = 3$$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to 0} \frac{(h-1)^3+1}{h} &= \lim_{h\to 0} \frac{(h-1)^3+1^3}{h} = \lim_{h\to 0} \frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h} \\ &= \lim_{h\to 0} \left[(h-1)^2-h+2\right] = 1-0+2 = 3\end{aligned}$$

8. $$\lim_{t\to 2} \frac{t^2-4}{t^3-8} = \lim_{t\to 2} \frac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \frac{t+2}{t^2+2t+4} = \frac{2+2}{4+4+4} = \frac{4}{12} = \frac{1}{3}$$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0^+$ as $r \to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r \neq 9$.

10. $\displaystyle\lim_{v\to 4^+}\frac{4-v}{|4-v|}=\lim_{v\to 4^+}\frac{4-v}{-(4-v)}=\lim_{v\to 4^+}\frac{1}{-1}=-1$

11. $\displaystyle\lim_{u\to 1}\frac{u^4-1}{u^3+5u^2-6u}=\lim_{u\to 1}\frac{(u^2+1)(u^2-1)}{u(u^2+5u-6)}=\lim_{u\to 1}\frac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)}=\lim_{u\to 1}\frac{(u^2+1)(u+1)}{u(u+6)}=\frac{2(2)}{1(7)}=\frac{4}{7}$

12. $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}=\lim_{x\to 3}\left[\frac{\sqrt{x+6}-x}{x^2(x-3)}\cdot\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right]=\lim_{x\to 3}\frac{(\sqrt{x+6}\,)^2-x^2}{x^2(x-3)(\sqrt{x+6}+x)}$

$$=\lim_{x\to 3}\frac{x+6-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to 3}\frac{-(x^2-x-6)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}=\lim_{x\to 3}\frac{-(x-3)(x+2)}{x^2(x-3)\left(\sqrt{x+6}+x\right)}$$

$$=\lim_{x\to 3}\frac{-(x+2)}{x^2\left(\sqrt{x+6}+x\right)}=-\frac{5}{9(3+3)}=-\frac{5}{54}$$

13. Since x is positive, $\sqrt{x^2}=|x|=x$. Thus,

$$\lim_{x\to\infty}\frac{\sqrt{x^2-9}}{2x-6}=\lim_{x\to\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x}=\lim_{x\to\infty}\frac{\sqrt{1-9/x^2}}{2-6/x}=\frac{\sqrt{1-0}}{2-0}=\frac{1}{2}$$

14. Since x is negative, $\sqrt{x^2}=|x|=-x$. Thus,

$$\lim_{x\to -\infty}\frac{\sqrt{x^2-9}}{2x-6}=\lim_{x\to -\infty}\frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/(-x)}=\lim_{x\to -\infty}\frac{\sqrt{1-9/x^2}}{-2+6/x}=\frac{\sqrt{1-0}}{-2+0}=-\frac{1}{2}$$

15. Let $t=\sin x$. Then as $x\to\pi^-$, $\sin x\to 0^+$, so $t\to 0^+$. Thus, $\displaystyle\lim_{x\to\pi^-}\ln(\sin x)=\lim_{t\to 0^+}\ln t=-\infty$.

16. $\displaystyle\lim_{x\to -\infty}\frac{1-2x^2-x^4}{5+x-3x^4}=\lim_{x\to -\infty}\frac{(1-2x^2-x^4)/x^4}{(5+x-3x^4)/x^4}=\lim_{x\to -\infty}\frac{1/x^4-2/x^2-1}{5/x^4+1/x^3-3}=\frac{0-0-1}{0+0-3}=\frac{-1}{-3}=\frac{1}{3}$

17. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)=\lim_{x\to\infty}\left[\frac{\sqrt{x^2+4x+1}-x}{1}\cdot\frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}\right]=\lim_{x\to\infty}\frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}$

$$=\lim_{x\to\infty}\frac{(4x+1)/x}{(\sqrt{x^2+4x+1}+x)/x}\qquad\left[\text{divide by } x=\sqrt{x^2}\text{ for } x>0\right]$$

$$=\lim_{x\to\infty}\frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1}=\frac{4+0}{\sqrt{1+0+0}+1}=\frac{4}{2}=2$$

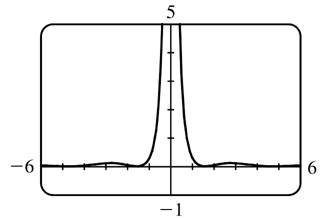
18. Let $t=x-x^2=x(1-x)$. Then as $x\to\infty$, $t\to-\infty$, and $\displaystyle\lim_{x\to\infty}e^{x-x^2}=\lim_{t\to-\infty}e^t=0$.

19. Let $t=1/x$. Then as $x\to 0^+$, $t\to\infty$, and $\displaystyle\lim_{x\to 0^+}\tan^{-1}(1/x)=\lim_{t\to\infty}\tan^{-1}t=\frac{\pi}{2}$.

20. $\displaystyle\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)=\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}\right]=\lim_{x\to 1}\left[\frac{x-2}{(x-1)(x-2)}+\frac{1}{(x-1)(x-2)}\right]$

$$=\lim_{x\to 1}\left[\frac{x-1}{(x-1)(x-2)}\right]=\lim_{x\to 1}\frac{1}{x-2}=\frac{1}{1-2}=-1$$

21. From the graph of $y = \left(\cos^2 x\right)/x^2$, it appears that $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote. Now $0 \le (\cos x)^2 \le 1 \;\Rightarrow$ $\dfrac{0}{x^2} \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2} \;\Rightarrow\; 0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$. But $\lim\limits_{x\to\pm\infty} 0 = 0$ and $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\pm\infty} \dfrac{\cos^2 x}{x^2} = 0$.



Thus, $y = 0$ is the horizontal asymptote. $\lim\limits_{x\to 0} \dfrac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \to 1$ and $x^2 \to 0^+$ as $x \to 0$, so $x = 0$ is the vertical asymptote.

22. From the graph of $y = f(x) = \sqrt{x^2+x+1} - \sqrt{x^2-x}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for f, let's multiply and divide it by its conjugate.

$$\begin{aligned} f_1(x) &= \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) \frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} = \frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} \\ &= \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} \end{aligned}$$

Now

$$\begin{aligned} \lim_{x\to\infty} f_1(x) &= \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} \\ &= \lim_{x\to\infty} \frac{2+(1/x)}{\sqrt{1+(1/x)+(1/x^2)} + \sqrt{1-(1/x)}} \qquad \left[\text{since } \sqrt{x^2} = x \text{ for } x > 0\right] \\ &= \frac{2}{1+1} = 1, \end{aligned}$$

so $y = 1$ is a horizontal asymptote. For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the denominator by x, with $x < 0$, we get

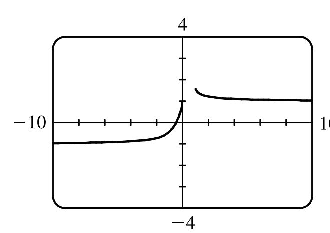
$$\frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{x} = -\frac{\sqrt{x^2+x+1} + \sqrt{x^2-x}}{\sqrt{x^2}} = -\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1-\frac{1}{x}}\right]$$

Therefore,

$$\begin{aligned} \lim_{x\to-\infty} f_1(x) &= \lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1} + \sqrt{x^2-x}} = \lim_{x\to\infty} \frac{2+(1/x)}{-\left[\sqrt{1+(1/x)+(1/x^2)} + \sqrt{1-(1/x)}\right]} \\ &= \frac{2}{-(1+1)} = -1, \end{aligned}$$

so $y = -1$ is a horizontal asymptote.

The domain of f is $(-\infty, 0] \cup [1, \infty)$. As $x \to 0^-$, $f(x) \to 1$, so $x = 0$ is *not* a vertical asymptote. As $x \to 1^+$, $f(x) \to \sqrt{3}$, so $x = 1$ is *not* a vertical asymptote and hence there are no vertical asymptotes.



23. Since $2x - 1 \le f(x) \le x^2$ for $0 < x < 3$ and $\lim_{x \to 1}(2x - 1) = 1 = \lim_{x \to 1} x^2$, we have $\lim_{x \to 1} f(x) = 1$ by the Squeeze Theorem.

24. Let $f(x) = -x^2$, $g(x) = x^2 \cos(1/x^2)$ and $h(x) = x^2$. Then since $\left|\cos(1/x^2)\right| \le 1$ for $x \ne 0$, we have $f(x) \le g(x) \le h(x)$ for $x \ne 0$, and so $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \quad \Rightarrow \quad \lim_{x \to 0} g(x) = 0$ by the Squeeze Theorem.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 2| < \delta$, then $|(14 - 5x) - 4| < \varepsilon$. But $|(14 - 5x) - 4| < \varepsilon \quad \Leftrightarrow$ $|-5x + 10| < \varepsilon \quad \Leftrightarrow \quad |-5|\,|x - 2| < \varepsilon \quad \Leftrightarrow \quad |x - 2| < \varepsilon/5$. So if we choose $\delta = \varepsilon/5$, then $0 < |x - 2| < \delta \quad \Rightarrow$ $|(14 - 5x) - 4| < \varepsilon$. Thus, $\lim_{x \to 2}(14 - 5x) = 4$ by the definition of a limit.

26. Given $\varepsilon > 0$ we must find $\delta > 0$ so that if $0 < |x - 0| < \delta$, then $|\sqrt[3]{x} - 0| < \varepsilon$. Now $|\sqrt[3]{x} - 0| = |\sqrt[3]{x}| < \varepsilon \quad \Rightarrow$ $|x| = |\sqrt[3]{x}|^3 < \varepsilon^3$. So take $\delta = \varepsilon^3$. Then $0 < |x - 0| = |x| < \varepsilon^3 \quad \Rightarrow \quad |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\varepsilon^3} = \varepsilon$. Therefore, by the definition of a limit, $\lim_{x \to 0} \sqrt[3]{x} = 0$.

27. Given $\varepsilon > 0$, we need $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $\left|x^2 - 3x - (-2)\right| < \varepsilon$. First, note that if $|x - 2| < 1$, then $-1 < x - 2 < 1$, so $0 < x - 1 < 2 \quad \Rightarrow \quad |x - 1| < 2$. Now let $\delta = \min\{\varepsilon/2, 1\}$. Then $0 < |x - 2| < \delta \quad \Rightarrow$ $\left|x^2 - 3x - (-2)\right| = |(x - 2)(x - 1)| = |x - 2|\,|x - 1| < (\varepsilon/2)(2) = \varepsilon$.

Thus, $\lim_{x \to 2}(x^2 - 3x) = -2$ by the definition of a limit.

28. Given $M > 0$, we need $\delta > 0$ such that if $0 < x - 4 < \delta$, then $2/\sqrt{x - 4} > M$. This is true $\quad \Leftrightarrow \quad \sqrt{x - 4} < 2/M \quad \Leftrightarrow$ $x - 4 < 4/M^2$. So if we choose $\delta = 4/M^2$, then $0 < x - 4 < \delta \quad \Rightarrow \quad 2/\sqrt{x - 4} > M$. So by the definition of a limit, $\lim_{x \to 4^+}\left(2/\sqrt{x - 4}\right) = \infty$.

29. (a) $f(x) = \sqrt{-x}$ if $x < 0$, $f(x) = 3 - x$ if $0 \le x < 3$, $f(x) = (x - 3)^2$ if $x > 3$.

(i) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+}(3 - x) = 3$

(ii) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{-x} = 0$

(iii) Because of (i) and (ii), $\lim_{x \to 0} f(x)$ does not exist.

(iv) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-}(3 - x) = 0$

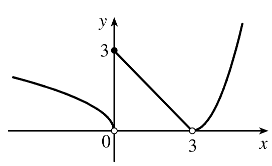
(v) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+}(x - 3)^2 = 0$

(vi) Because of (iv) and (v), $\lim_{x \to 3} f(x) = 0$.

(b) f is discontinuous at 0 since $\lim_{x \to 0} f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

(c)



30. (a) $g(x) = 2x - x^2$ if $0 \le x \le 2$, $g(x) = 2 - x$ if $2 < x \le 3$, $g(x) = x - 4$ if $3 < x < 4$, $g(x) = \pi$ if $x \ge 4$.

Therefore, $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-}\left(2x - x^2\right) = 0$ and $\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+}(2 - x) = 0$. Thus, $\lim_{x \to 2} g(x) = 0 = g(2)$, so g is continuous at 2. $\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-}(2 - x) = -1$ and $\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+}(x - 4) = -1$. Thus,

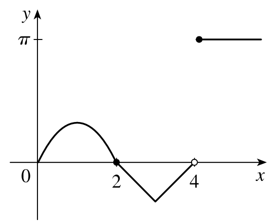
$\lim_{x\to 3} g(x) = -1 = g(3)$, so g is continuous at 3.

$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x-4) = 0$ and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \pi = \pi$.

Thus, $\lim_{x\to 4} g(x)$ does not exist, so g is discontinuous at 4. But

$\lim_{x\to 4^+} g(x) = \pi = g(4)$, so g is continuous from the right at 4.

(b)



31. $\sin x$ and e^x are continuous on $\mathbb{R}$ by Theorem 2.5.7. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 2.5.9. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 2.5.4.

32. $x^2 - 9$ is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0, \infty)$ by Theorem 2.5.7, so the composition $\sqrt{x^2-9}$ is continuous on $\{x \mid x^2 - 9 \geq 0\} = (-\infty, -3] \cup [3, \infty)$ by Theorem 2.5.9. Note that $x^2 - 2 \neq 0$ on this set and so the quotient function $g(x) = \dfrac{\sqrt{x^2-9}}{x^2-2}$ is continuous on its domain, $(-\infty, -3] \cup [3, \infty)$ by Theorem 2.5.4.

33. $f(x) = x^5 - x^3 + 3x - 5$ is continuous on the interval $[1, 2]$, $f(1) = -2$, and $f(2) = 25$. Since $-2 < 0 < 25$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^5 - x^3 + 3x - 5 = 0$ in the interval $(1, 2)$.

34. $f(x) = \cos\sqrt{x} - e^x + 2$ is continuous on the interval $[0, 1]$, $f(0) = 2$, and $f(1) \approx -0.2$. Since $-0.2 < 0 < 2$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos\sqrt{x} - e^x + 2 = 0$, or $\cos\sqrt{x} = e^x - 2$, in the interval $(0, 1)$.

35. (a) The slope of the tangent line at $(2, 1)$ is

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{9-2x^2-1}{x-2} = \lim_{x\to 2} \frac{8-2x^2}{x-2} = \lim_{x\to 2} \frac{-2(x^2-4)}{x-2} = \lim_{x\to 2} \frac{-2(x-2)(x+2)}{x-2}$$
$$= \lim_{x\to 2} [-2(x+2)] = -2 \cdot 4 = -8$$

(b) An equation of this tangent line is $y - 1 = -8(x-2)$ or $y = -8x + 17$.

36. For a general point with x-coordinate a, we have

$$m = \lim_{x\to a} \frac{2/(1-3x) - 2/(1-3a)}{x-a} = \lim_{x\to a} \frac{2(1-3a) - 2(1-3x)}{(1-3a)(1-3x)(x-a)} = \lim_{x\to a} \frac{6(x-a)}{(1-3a)(1-3x)(x-a)}$$
$$= \lim_{x\to a} \frac{6}{(1-3a)(1-3x)} = \frac{6}{(1-3a)^2}$$

For $a = 0$, $m = 6$ and $f(0) = 2$, so an equation of the tangent line is $y - 2 = 6(x-0)$ or $y = 6x + 2$. For $a = -1$, $m = \frac{3}{8}$ and $f(-1) = \frac{1}{2}$, so an equation of the tangent line is $y - \frac{1}{2} = \frac{3}{8}(x+1)$ or $y = \frac{3}{8}x + \frac{7}{8}$.

37. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h)-1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10+h}{4}$$

[continued]

So for the following intervals the average velocities are:

(i) $[1,3]$: $h=2$, $v_{\text{ave}}=(10+2)/4=3$ m/s (ii) $[1,2]$: $h=1$, $v_{\text{ave}}=(10+1)/4=2.75$ m/s

(iii) $[1,1.5]$: $h=0.5$, $v_{\text{ave}}=(10+0.5)/4=2.625$ m/s (iv) $[1,1.1]$: $h=0.1$, $v_{\text{ave}}=(10+0.1)/4=2.525$ m/s

(b) When $t=1$, the instantaneous velocity is $\lim\limits_{h\to0}\dfrac{s(1+h)-s(1)}{h}=\lim\limits_{h\to0}\dfrac{10+h}{4}=\dfrac{10}{4}=2.5$ m/s.

38. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V=250-200=50$ in^3, and since $P=800/V$,

$\Delta P=P(250)-P(200)=\dfrac{800}{250}-\dfrac{800}{200}=3.2-4=-0.8$ lb/in^2. So the average rate of change

is $\dfrac{\Delta P}{\Delta V}=\dfrac{-0.8}{50}=-0.016\ \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V=800/P$, the instantaneous rate of change of V with respect to P is

$$\begin{aligned}\lim_{h\to0}\frac{\Delta V}{\Delta P}&=\lim_{h\to0}\frac{V(P+h)-V(P)}{h}=\lim_{h\to0}\frac{800/(P+h)-800/P}{h}=\lim_{h\to0}\frac{800\,[P-(P+h)]}{h(P+h)P}\\&=\lim_{h\to0}\frac{-800}{(P+h)P}=-\frac{800}{P^2}\end{aligned}$$

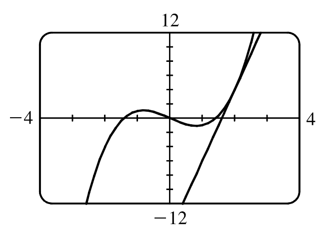
which is inversely proportional to the square of P.

39. (a) $f'(2)=\lim\limits_{x\to2}\dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to2}\dfrac{x^3-2x-4}{x-2}$

$$=\lim_{x\to2}\frac{(x-2)(x^2+2x+2)}{x-2}=\lim_{x\to2}(x^2+2x+2)=10$$

(b) $y-4=10(x-2)$ or $y=10x-16$

(c)



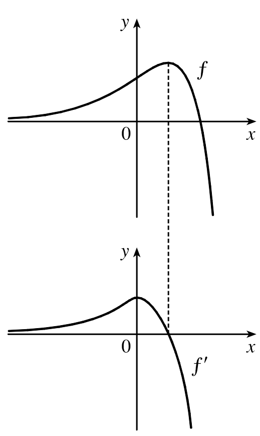
40. $2^6=64$, so $f(x)=x^6$ and $a=2$.

41. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

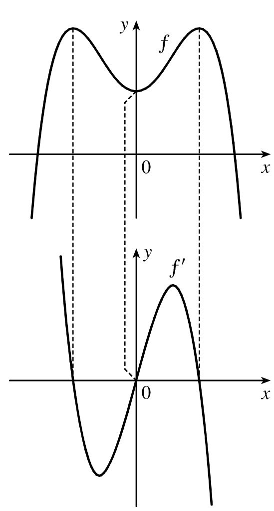
(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

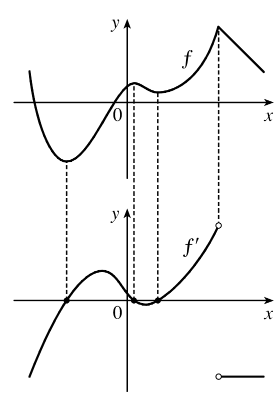
42.



43.



44.

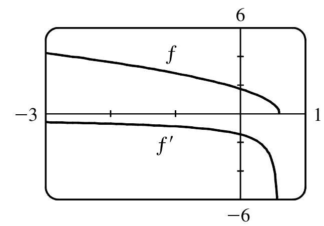


45. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)}+\sqrt{3-5x}}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$

$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

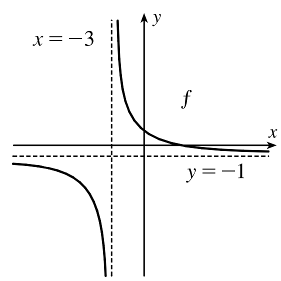
(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \;\Rightarrow$ $5x \leq 3 \;\Rightarrow\; x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero; $x \in \left(-\infty, \frac{3}{5}\right)$

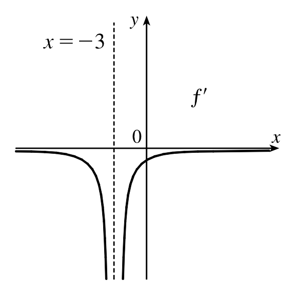


(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

46. (a) As $x \to \pm\infty$, $f(x) = (4-x)/(3+x) \to -1$, so there is a horizontal asymptote at $y=-1$. As $x \to -3^+$, $f(x) \to \infty$, and as $x \to -3^-$, $f(x) \to -\infty$. Thus, there is a vertical asymptote at $x=-3$.



(b) Note that f is decreasing on $(-\infty,-3)$ and $(-3,\infty)$, so f' is negative on those intervals. As $x \to \pm\infty$, $f' \to 0$. As $x \to -3^-$ and as $x \to -3^+$, $f' \to -\infty$.



(c) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h} = \lim\limits_{h\to 0} \dfrac{(3+x)\,[4-(x+h)] - (4-x)\,[3+(x+h)]}{h\,[3+(x+h)]\,(3+x)}$

$$= \lim_{h\to 0} \frac{(12-3x-3h+4x-x^2-hx)-(12+4x+4h-3x-x^2-hx)}{h[3+(x+h)](3+x)}$$

$$= \lim_{h\to 0} \frac{-7h}{h\,[3+(x+h)]\,(3+x)} = \lim_{h\to 0} \frac{-7}{[3+(x+h)]\,(3+x)} = -\frac{7}{(3+x)^2}$$

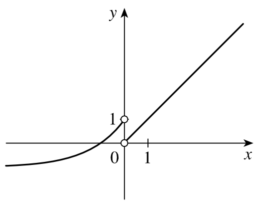
(d) The graphing device confirms our graph in part (b).

47. f is not differentiable: at $x=-4$ because f is not continuous, at $x=-1$ because f has a corner, at $x=2$ because f is not continuous, and at $x=5$ because f has a vertical tangent.

48. The graph of a has tangent lines with positive slope for $x<0$ and negative slope for $x>0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

49. Domain: $(-\infty, 0) \cup (0, \infty)$; $\lim_{x\to 0^-} f(x) = 1$; $\lim_{x\to 0^+} f(x) = 0$;

$f'(x) > 0$ for all x in the domain; $\lim_{x\to -\infty} f'(x) = 0$; $\lim_{x\to \infty} f'(x) = 1$



50. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0} \dfrac{P(t+h)-P(t)}{h} \approx \dfrac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) \approx \dfrac{P(1960)-P(1950)}{1960-1950} = \dfrac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

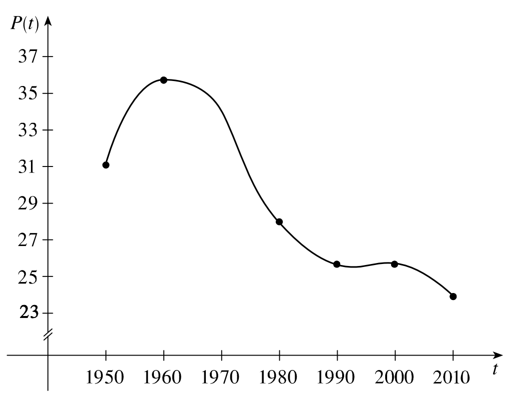
$h=-10 \quad \Rightarrow \quad P'(1960) \approx \dfrac{P(1950)-P(1960)}{1950-1960} = \dfrac{31.1-35.7}{-10} = 0.46$

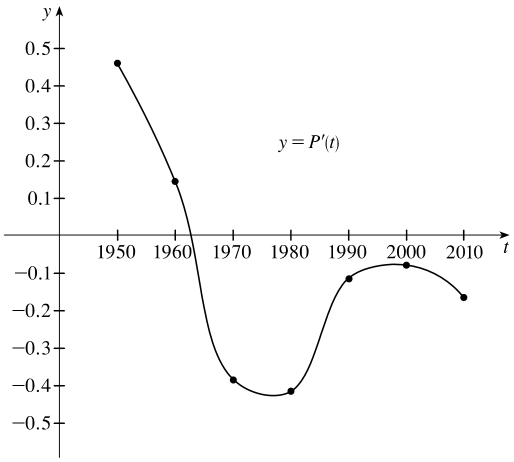
$h=10 \quad \Rightarrow \quad P'(1960) \approx \dfrac{P(1970)-P(1960)}{1970-1960} = \dfrac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46 + (-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000	2010
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	−0.085	−0.170

(c)





(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

51. $B'(t)$ is the rate at which the number of US \$20 bills in circulation is changing with respect to time. Its units are billions of bills per year. We use a symmetric difference quotient to estimate $B'(2000)$.

$B'(2000) \approx \dfrac{B(2005)-B(1995)}{2005-1995} = \dfrac{5.77-4.21}{10} = 0.156$ billions of bills per year (or 156 million bills per year).

52. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

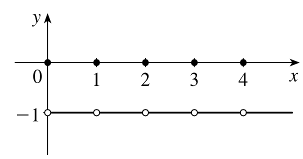
(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

53. $|f(x)| \leq g(x) \quad \Leftrightarrow \quad -g(x) \leq f(x) \leq g(x)$ and $\lim\limits_{x \to a} g(x) = 0 = \lim\limits_{x \to a} -g(x)$.

Thus, by the Squeeze Theorem, $\lim\limits_{x \to a} f(x) = 0$.

54. (a) Note that f is an even function since $f(x) = f(-x)$. Now for any integer n, $[\![n]\!] + [\![-n]\!] = n - n = 0$, and for any real number k which is not an integer, $[\![k]\!] + [\![-k]\!] = [\![k]\!] + (-[\![k]\!] - 1) = -1$. So $\lim\limits_{x \to a} f(x)$ exists (and is equal to -1) for all values of a.



(b) f is discontinuous at all integers.