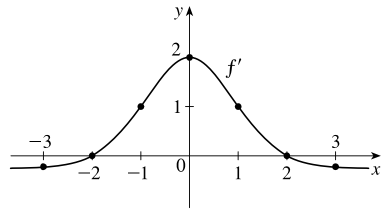
2.8 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

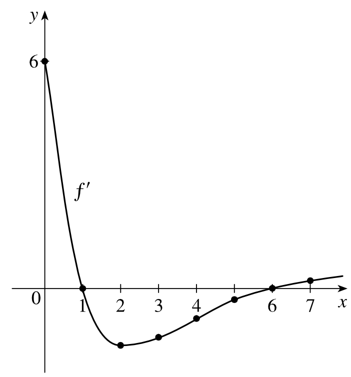


(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$

2. Your answers may vary depending on your estimates.



(a) *Note:* By estimating the slopes of tangent lines on the graph of f, it appears that $f'(0) \approx 6$.

(b) $f'(1) \approx 0$

(c) $f'(2) \approx -1.5$ (d) $f'(3) \approx -1.3$ (e) $f'(4) \approx -0.8$

(f) $f'(5) \approx -0.3$ (g) $f'(6) \approx 0$ (h) $f'(7) \approx 0.2$

3. (a)$' =$ II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

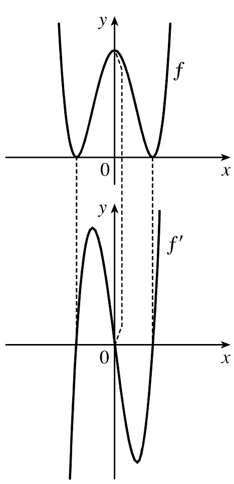
(b)$' =$ IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$' =$ I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$' =$ III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4 –11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f— there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

4.



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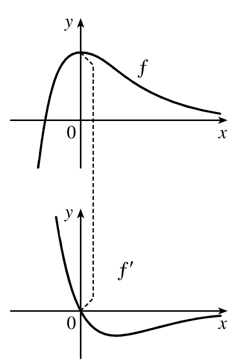

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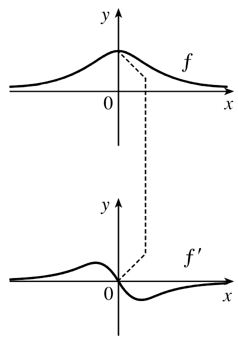

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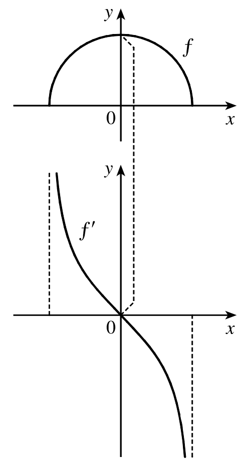

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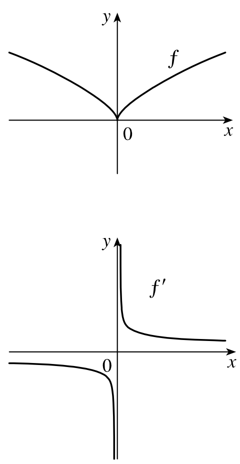

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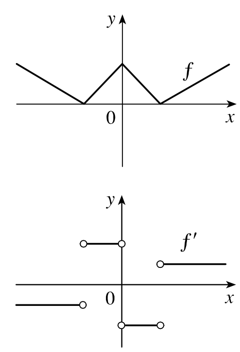

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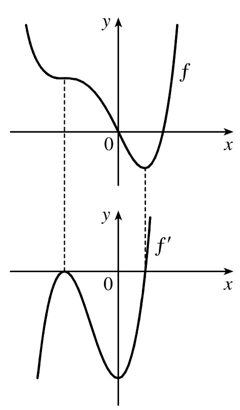

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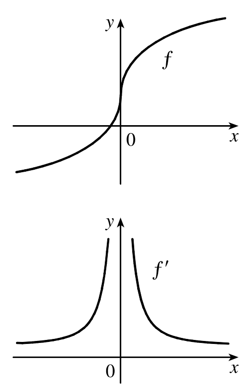
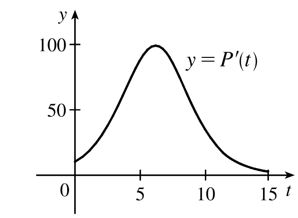
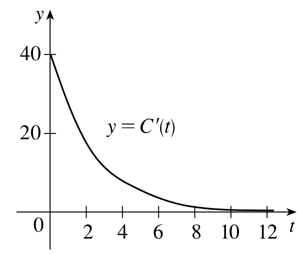


12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

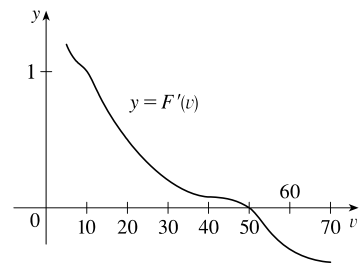
(b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



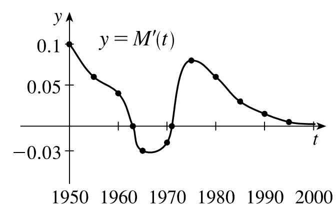
14. (a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

(b) Graphs will vary depending on estimates of F', but will change from positive to negative at about $v = 50$.

(c) To save on gas, drive at the speed where F is a maximum and F' is 0, which is about 50 mi/h.

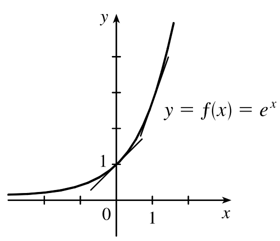


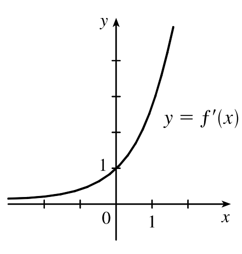
15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



16. See Figure 3.3.1.

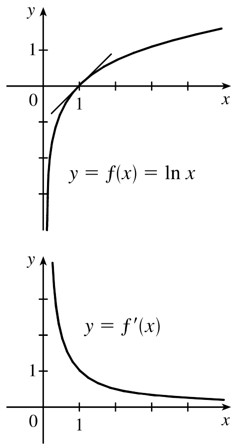
17.





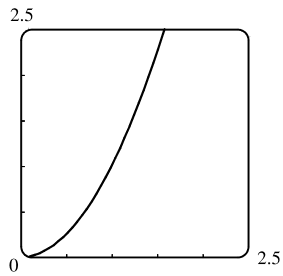
The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

18.



As x increases toward 1, $f'(x)$ decreases from very large numbers to 1. As x becomes large, $f'(x)$ gets closer to 0. As a guess, $f'(x) = 1/x^2$ or $f'(x) = 1/x$ makes sense.

19. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.



(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

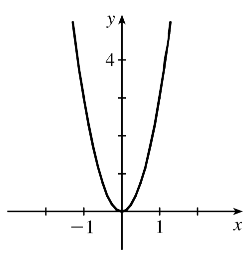
(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x$$

20. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2$$

21. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[3(x+h)-8]-(3x-8)}{h} = \lim\limits_{h\to 0} \dfrac{3x+3h-8-3x+8}{h}$

$= \lim\limits_{h\to 0} \dfrac{3h}{h} = \lim\limits_{h\to 0} 3 = 3$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[m(x+h)+b]-(mx+b)}{h} = \lim\limits_{h\to 0} \dfrac{mx+mh+b-mx-b}{h}$

$= \lim\limits_{h\to 0} \dfrac{mh}{h} = \lim\limits_{h\to 0} m = m$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(t) = \lim\limits_{h\to 0} \dfrac{f(t+h)-f(t)}{h} = \lim\limits_{h\to 0} \dfrac{\left[2.5(t+h)^2+6(t+h)\right]-\left(2.5t^2+6t\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2.5(t^2+2th+h^2)+6t+6h-2.5t^2-6t}{h} = \lim\limits_{h\to 0} \dfrac{2.5t^2+5th+2.5h^2+6h-2.5t^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{5th+2.5h^2+6h}{h} = \lim\limits_{h\to 0} \dfrac{h\,(5t+2.5h+6)}{h} = \lim\limits_{h\to 0} (5t+2.5h+6)$

$= 5t+6$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[4+8(x+h)-5(x+h)^2\right]-(4+8x-5x^2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{4+8x+8h-5(x^2+2xh+h^2)-4-8x+5x^2}{h} = \lim\limits_{h\to 0} \dfrac{8h-5x^2-10xh-5h^2+5x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{8h-10xh-5h^2}{h} = \lim\limits_{h\to 0} \dfrac{h(8-10x-5h)}{h} = \lim\limits_{h\to 0} (8-10x-5h)$

$= 8-10x$

Domain of f = domain of $f' = \mathbb{R}$.

25. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^2-2(x+h)^3]-(x^2-2x^3)}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2-2x^3-6x^2h-6xh^2-2h^3-x^2+2x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh+h^2-6x^2h-6xh^2-2h^3}{h} = \lim\limits_{h\to 0} \dfrac{h(2x+h-6x^2-6xh-2h^2)}{h}$

$= \lim\limits_{h\to 0} (2x+h-6x^2-6xh-2h^2) = 2x-6x^2$

Domain of f = domain of $f' = \mathbb{R}$.

26. $g'(t) = \lim\limits_{h\to 0} \dfrac{g(t+h)-g(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{\sqrt{t+h}}-\dfrac{1}{\sqrt{t}}}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h} = \lim\limits_{h\to 0} \left(\dfrac{\sqrt{t}-\sqrt{t+h}}{h\,\sqrt{t+h}\,\sqrt{t}} \cdot \dfrac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}\right)$

$= \lim\limits_{h\to 0} \dfrac{t-(t+h)}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim\limits_{h\to 0} \dfrac{-h}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim\limits_{h\to 0} \dfrac{-1}{\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}$

$= \dfrac{-1}{\sqrt{t}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t}\right)} = \dfrac{-1}{t\left(2\sqrt{t}\right)} = -\dfrac{1}{2t^{3/2}}$

Domain of g = domain of $g' = (0,\infty)$.

27. $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\left[\dfrac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right]$

$$= \lim_{h\to 0} \frac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

28. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)^2-1}{2(x+h)-3} - \dfrac{x^2-1}{2x-3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[(x+h)^2-1](2x-3)-[2(x+h)-3](x^2-1)}{[2(x+h)-3](2x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-1)(2x-3)-(2x+2h-3)(x^2-1)}{h[2(x+h)-3](2x-3)}$$

$$= \lim_{h\to 0} \frac{(2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3)-(2x^3+2x^2h-3x^2-2x-2h+3)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{4x^2h+2xh^2-6xh-3h^2-2x^2h+2h}{h(2x+2h-3)(2x-3)} = \lim_{h\to 0} \frac{h(2x^2+2xh-6x-3h+2)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{2x^2+2xh-6x-3h+2}{(2x+2h-3)(2x-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

Domain of f = domain of $f' = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

29. $G'(t) = \lim\limits_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1-2(t+h)}{3+(t+h)} - \dfrac{1-2t}{3+t}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[1-2(t+h)](3+t)-[3+(t+h)](1-2t)}{[3+(t+h)](3+t)}}{h}$$

$$= \lim_{h\to 0} \frac{3+t-6t-2t^2-6h-2ht-(3-6t+t-2t^2+h-2ht)}{h[3+(t+h)](3+t)} = \lim_{h\to 0} \frac{-6h-h}{h(3+t+h)(3+t)}$$

$$= \lim_{h\to 0} \frac{-7h}{h(3+t+h)(3+t)} = \lim_{h\to 0} \frac{-7}{(3+t+h)(3+t)} = \frac{-7}{(3+t)^2}$$

Domain of G = domain of $G' = (-\infty, -3) \cup (-3, \infty)$.

30. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^{3/2}-x^{3/2}}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^{3/2}-x^{3/2}][(x+h)^{3/2}+x^{3/2}]}{h\left[(x+h)^{3/2}+x^{3/2}\right]}$

$$= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h[(x+h)^{3/2}+x^{3/2}]}$$

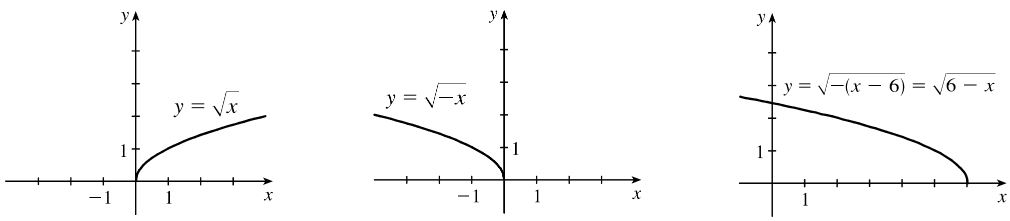
$$= \lim_{h\to 0} \frac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \tfrac{3}{2}x^{1/2}$$

Domain of f = domain of $f' = [0, \infty)$. Strictly speaking, the domain of f' is $(0, \infty)$ because the limit that defines $f'(0)$ does not exist (as a two-sided limit). But the right-hand derivative (in the sense of Exercise 64) does exist at 0, so in that sense one could regard the domain of f' to be $[0, \infty)$.

31. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0} \frac{\left(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4\right) - x^4}{h}$$
$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h\to 0} \left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 4x^3$$

Domain of f = domain of $f' = \mathbb{R}$.

32. (a)

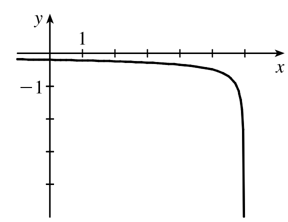


(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{6-(x+h)} - \sqrt{6-x}}{h} \left[\frac{\sqrt{6-(x+h)} + \sqrt{6-x}}{\sqrt{6-(x+h)} + \sqrt{6-x}}\right]$$
$$= \lim_{h\to 0} \frac{[6-(x+h)] - (6-x)}{h\left[\sqrt{6-(x+h)} + \sqrt{6-x}\right]} = \lim_{h\to 0} \frac{-h}{h\left(\sqrt{6-x-h} + \sqrt{6-x}\right)}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{6-x-h} + \sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}}$$

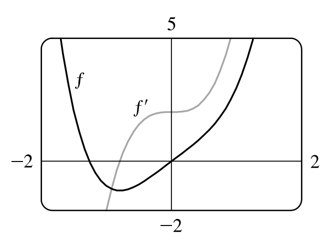
Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



33. (a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^4 + 2(x+h)] - (x^4 + 2x)}{h}$$
$$= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h}$$
$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h}$$
$$= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.

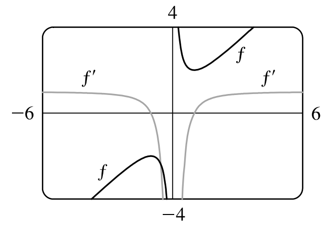


34. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)+1/(x+h)]-(x+1/x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)^2+1}{x+h} - \dfrac{x^2+1}{x}}{h}$

$$= \lim_{h\to 0} \frac{x[(x+h)^2+1]-(x+h)(x^2+1)}{h(x+h)x} = \lim_{h\to 0} \frac{(x^3+2hx^2+xh^2+x)-(x^3+x+hx^2+h)}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{hx^2+xh^2-h}{h(x+h)x} = \lim_{h\to 0} \frac{h(x^2+xh-1)}{h(x+h)x} = \lim_{h\to 0} \frac{x^2+xh-1}{(x+h)x} = \frac{x^2-1}{x^2}, \text{ or } 1-\frac{1}{x^2}$$

(b) Notice that $f'(x)=0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x=0$.



35. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent unemployed per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 2003: $U'(2003) \approx \dfrac{U(2004)-U(2003)}{2004-2003} = \dfrac{5.5-6.0}{1} = -0.5$

For 2004: We estimate $U'(2004)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \quad\Rightarrow\quad U'(2004) \approx \dfrac{U(2003)-U(2004)}{2003-2004} = \dfrac{6.0-5.5}{-1} = -0.5;$

$h=1 \quad\Rightarrow\quad U'(2004) \approx \dfrac{U(2005)-U(2004)}{2005-2004} = \dfrac{5.1-5.5}{1} = -0.4.$

So we estimate that $U'(2004) \approx \frac{1}{2}[-0.5+(-0.4)] = -0.45$. Other values for $U'(t)$ are calculated in a similar fashion.

t	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
$U'(t)$	-0.50	-0.45	-0.45	-0.25	0.60	2.35	1.90	-0.20	-0.75	-0.80

36. (a) $N'(t)$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $N'(t)$, we use $\lim\limits_{h\to 0} \dfrac{N(t+h)-N(t)}{h} \approx \dfrac{N(t+h)-N(t)}{h}$ for small values of h.

For 2000: $N'(2000) \approx \dfrac{N(2002)-N(2000)}{2002-2000} = \dfrac{4897-5500}{2} = -301.5$

For 2002: We estimate $N'(2002)$ by using $h=-2$ and $h=2$, and then average the two results to obtain a final estimate.

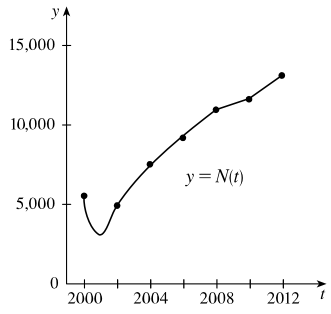
$h=-2 \quad\Rightarrow\quad N'(2002) \approx \dfrac{N(2000)-N(2002)}{2000-2002} = \dfrac{5500-4897}{-2} = -301.5$

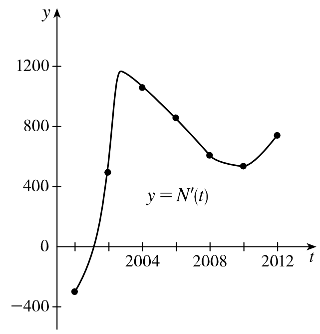
$h=2 \quad\Rightarrow\quad N'(2002) \approx \dfrac{N(2004)-N(2002)}{2004-2002} = \dfrac{7470-4897}{2} = 1286.5$

So we estimate that $N'(2002) \approx \frac{1}{2}[-301.5+1286.5] = 492.5$.

t	2000	2002	2004	2006	2008	2010	2012
$N'(t)$	-301.5	492.5	1060.25	856.75	605.75	534.5	737

(c)

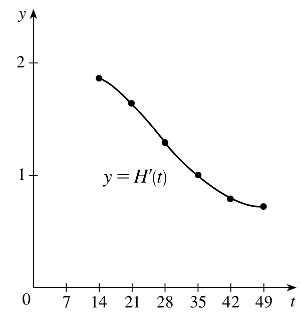




(d) We could get more accurate values for $N'(t)$ by obtaining data for more values of t.

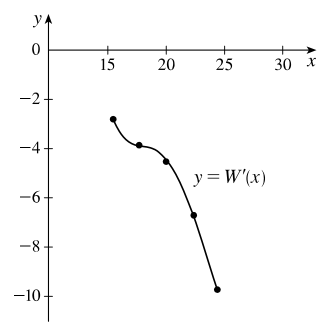
37. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

t	14	21	28	35	42	49
$H(t)$	41	54	64	72	78	83
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{18}{14}$	$\frac{14}{14}$	$\frac{11}{14}$	$\frac{5}{7}$



38. As in Exercise 35, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for $W'(x)$ are grams per degree (g/°C).

x	15.5	17.7	20.0	22.4	24.4
$W(x)$	37.2	31.0	19.8	9.7	−9.8
$W'(x)$	−2.82	−3.87	−4.53	−6.73	−9.75



39. (a) dP/dt is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time t, measured in percentage points per year.

(b) 2 years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

40. dN/dp is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect dN/dp to be negative.

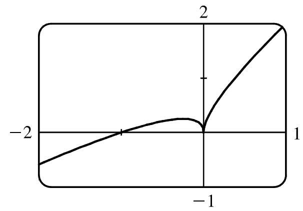
41. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

42. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

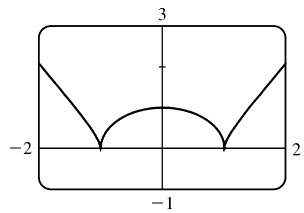
43. f is not differentiable at $x = 1$, because f is not defined there, and at $x = 5$, because the graph has a vertical tangent there.

44. f is not differentiable at $x = -2$ and $x = 3$, because the graph has corners there, and at $x = 1$, because there is a discontinuity there.

45. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



46. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so $g(x) = (x^2 - 1)^{2/3}$ is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So g is not differentiable at $x = \pm 1$.



47. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

48. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

49. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

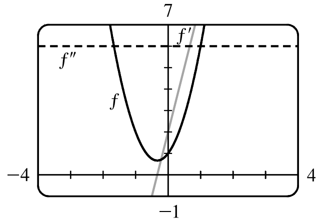
50. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

51. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

52. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

53.
$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h) + 1] - (3x^2 + 2x + 1)}{h} \\
&= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 1) - (3x^2 + 2x + 1)}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h} \\
&= \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2
\end{aligned}$$

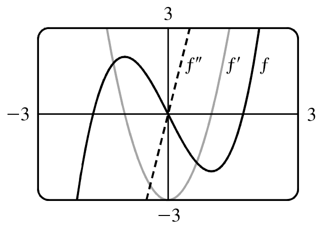
$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[6(x+h)+2] - (6x+2)}{h} = \lim_{h \to 0} \frac{(6x+6h+2) - (6x+2)}{h}$$
$$= \lim_{h \to 0} \frac{6h}{h} = \lim_{h \to 0} 6 = 6$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

54. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h}$$
$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$
$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 - 3] - (3x^2 - 3)}{h} = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - (3x^2 - 3)}{h}$$
$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{h(6x + 3h)}{h} = \lim_{h \to 0} (6x + 3h) = 6x$$



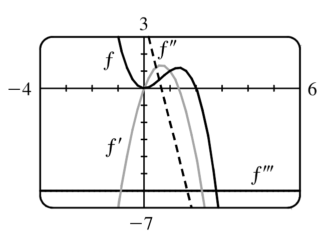
We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

55. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[2(x+h)^2 - (x+h)^3\right] - (2x^2 - x^3)}{h}$$
$$= \lim_{h \to 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h \to 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{\left[4(x+h) - 3(x+h)^2\right] - (4x - 3x^2)}{h} = \lim_{h \to 0} \frac{h(4 - 6x - 3h)}{h}$$
$$= \lim_{h \to 0} (4 - 6x - 3h) = 4 - 6x$$

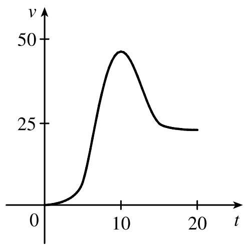
$$f'''(x) = \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \to 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h \to 0} \frac{-6h}{h} = \lim_{h \to 0} (-6) = -6$$

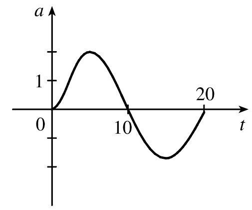
$$f^{(4)}(x) = \lim_{h \to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \to 0} \frac{-6 - (-6)}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} (0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

56. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by $10\ \text{ft/s}^2$ over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

57. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{h} - 0}{h} = \lim\limits_{h \to 0} \dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x \to 0} |f'(x)| = \lim\limits_{x \to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

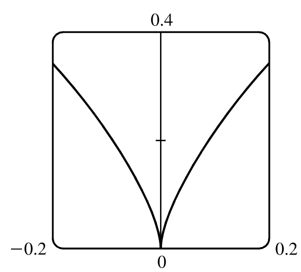
58. (a) $g'(0) = \lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^{2/3} - 0}{x} = \lim\limits_{x \to 0} \dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim\limits_{x \to 0} |g'(x)| = \lim\limits_{x \to 0} \dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

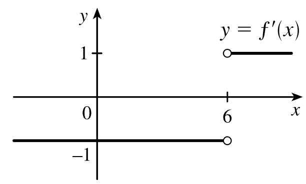
(d)



59. $$f(x) = |x - 6| = \begin{cases} x - 6 & \text{if } x - 6 \ge 6 \\ -(x - 6) & \text{if } x - 6 < 0 \end{cases} = \begin{cases} x - 6 & \text{if } x \ge 6 \\ 6 - x & \text{if } x < 6 \end{cases}$$

So the right-hand limit is $\lim\limits_{x \to 6^+} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x \to 6^+} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x \to 6^+} \dfrac{x - 6}{x - 6} = \lim\limits_{x \to 6^+} 1 = 1$, and the left-hand limit is $\lim\limits_{x \to 6^-} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x \to 6^-} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x \to 6^-} \dfrac{6 - x}{x - 6} = \lim\limits_{x \to 6^-} (-1) = -1$. Since these limits are not equal,

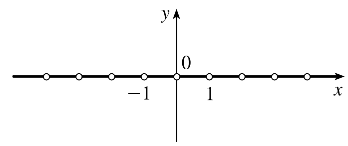
$f'(6) = \lim\limits_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.



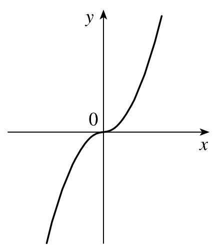
However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

Another way of writing the formula is $f'(x) = \dfrac{x - 6}{|x - 6|}$.

60. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



61. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 19(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x\,|x|}{x} = \lim_{x \to 0} |x| = 0.$$

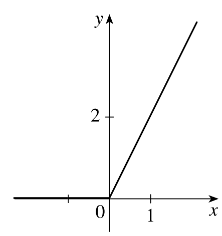
So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{matrix} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{matrix} \right\} = 2\,|x|$.

62. (a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

so $g(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



(b) g is not differentiable at $x = 0$ because the graph has a corner there, but is differentiable at all other values; that is, g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

(c) $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Another way of writing the formula is $g'(x) = 1 + \operatorname{sgn} x$ for $x \neq 0$.

63. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x - h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h} \\ &= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

64. (a) $f'_-(4) = \displaystyle\lim_{h\to 0^-}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-}\frac{5-(4+h)-1}{h}$

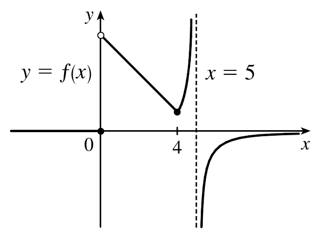
$= \displaystyle\lim_{h\to 0^-}\frac{-h}{h} = -1$

and

$f'_+(4) = \displaystyle\lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+}\frac{\dfrac{1}{5-(4+h)}-1}{h}$

$= \displaystyle\lim_{h\to 0^+}\frac{1-(1-h)}{h(1-h)} = \lim_{h\to 0^+}\frac{1}{1-h} = 1$

(b)



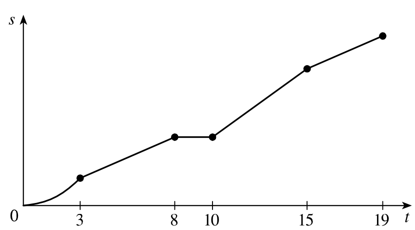
(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

At 4 we have $\displaystyle\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(5-x) = 1$ and $\displaystyle\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\frac{1}{5-x} = 1$, so $\displaystyle\lim_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5. These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5-x) = 5 \ne 0 = \lim_{x\to 0^-} f(x)$, $\displaystyle\lim_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

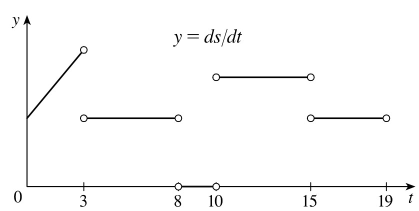
(d) From (a), f is not differentiable at 4 since $f'_-(4) \ne f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

65. These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.

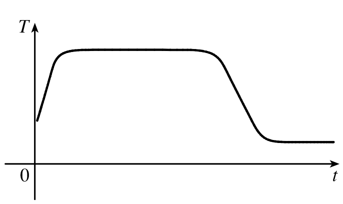
(a)



(b)

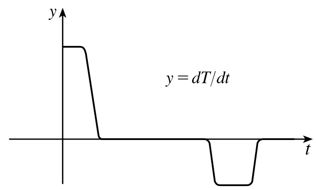


66. (a)

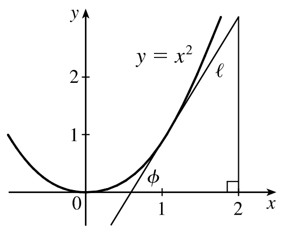


(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)



67.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent to angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 19) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63°$.