Cayley Graphs

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Motivation

ÇAMPANALOGIA Improved:

OR, THE

ART OF RINGING MADE EASY,

BY

PLAIN and METHODICAL RULES

AND

DIRECTIONS,

WHEREBY

The INGENIOUS PRACTITIONER

MAY,

With a little PRACTICE and CARE,

ATTAIN TO THE

KNOWLEDGE of RINGING

ALL MANNER OF

DOUBLE, TRIPPLE,

AND

QUADRUPLE CHANGES.

On three Bells there are twice as many more; for by multiplying two by three, the additional Figure the Product is ſix, in Ringing whereof no Bell is peculiarly aſſigned as an Hunt, becauſe every Bell has one and the ſame Courſe, as may be here ſeen:

1	2	3
2	1	3
2	3	1
3	2	1
3	1	2
1	3	2
1	2	3

Theſe ſix Changes may be likewiſe rung another Way, that is, by hunting the Bell which lies behind, being third, firſt down; or, as it appears, by ringing the ſaid Changes backwards, thus:

1	2	3
1	3	2
3	1	2
3	2	1
2	3	1
2	1	3
1	2	3

On four Bells there are three Times as many more Changes as there are on three, as appears by

Group theory

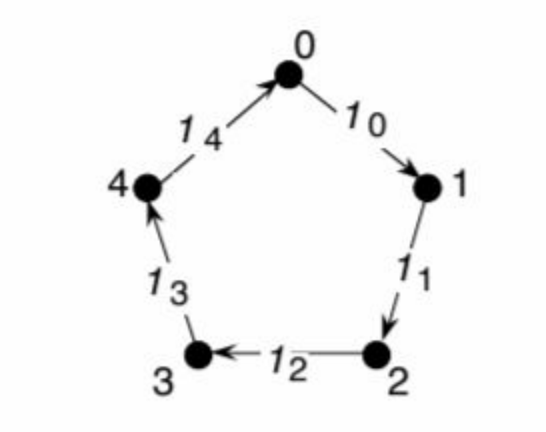
- A *group* is a set of objects together with an associative and invertible closed binary operation sending any two of its elements to a third.
- We will discuss the following famous groups, among others:
  - $\boldsymbol{Z}_5 = \{ 0, 1, 2, 3, 4 \}$
  - $D_4 = \{ e, a, b, ab, a^2, a^2b, a^3b, a^3b \}$
- A subset *S* of group *G* elements is called a *generating set* for *G* provided each element *g* of *G* can be written as a linear combination of *S*.
- A group *G* is called *cyclic* if there exists a singleton set generating it.

Cayley graphs

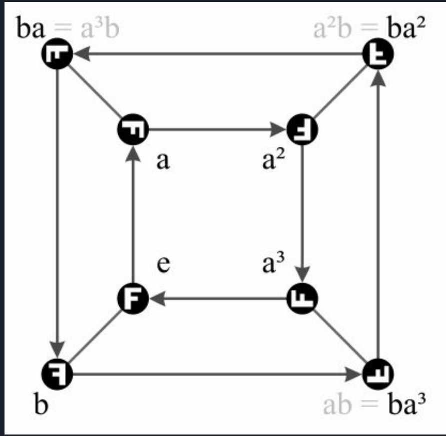
- A Cayley graph G is a directed graph formed from a group H and a generating set S. Each element $h \in H$ is assigned some vertex $g \in V(G)$ and the directed edges are defined by $(h, hs) \in E(G)$.
- Changing the generating set S can significantly change the Cayley graph, even though the underlying group remains the same.
- A Cayley graph is undirected if and only if its generating set is closed under inverses, i.e., $S = S^{-1}$.

Examples

Cayley graph of $\mathbf{Z}_5$ generated by {1}:



Cayley graph of D_4 generated by {a,b}:



Graph theory

- A *walk* in a Cayley graph represents how many times one must perform the binary operations and with which generators to get from one element to another.
- A *path* in a Cayley graph represents a series of elements obtained by performing the binary operation a certain number of times with certain generators such that no two elements in the series are the same.
- A *cycle* in a Cayley graph represents how many times one must perform the binary operations and with which generators to get from one element to itself, without repeating elements along the way.
- The *diameter* of a Cayley graph represents the two elements that require the most iterations of the binary operation to get from one element to the other.

Vertex-transitivity

- A graph is *vertex-transitive* provided there exists a graph automorphism taking any point to any other point.
- Every Cayley graph is vertex-transitive, but not every vertex-transitive graph is Cayley.
- The Petersen graph is the smallest such graph.

Vertex-transitivity (McKay & Prager 1996)

Theorem 1. *Let p and q be primes with $q < p$. Then $pq \in NC$ if and only if one of the following holds.*

(a) q^2 *divides* $p - 1$,

(b) $p = 2q - 1 > 3$ or $p = (q^2 + 1)/2$,

(c) $p = 2^t + 1$ *and either* q *divides* $2^t - 1$ *or* $q = 2^{t-1} - 1$,

(d) $p = 2^t - 1$ *and* $q = 2^{t-1} + 1$,

(e) $p = 11$ *and* $q = 7$.

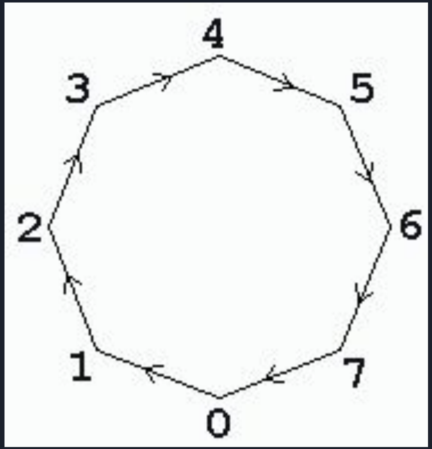
Theorem 2. *Let p and q be distinct primes. Then $p^2q \in NC$ unless $p^2q = 12$.*

One consequence of this theorem is a complete determination of membership of NC for non-square-free integers n.

Theorem 3. *Let n be a positive integer which is divisible by the square of some prime p. Then $n \in NC$ unless $n = p^2$ or $n = p^3$ or $n = 12$.*

Applications

- By far the most important usage of a Cayley graph is an intuitive visual on a group's structure.
  - More specifically, how the given generating set interacts with the group.
- An easy example is the number of perfect bridge shuffles to get back to the ordered state is 8.



Applications (cont.)

- The Cayley Graph of a 3x3x3 Rubik's cube has a diameter of 20. This means that when scrambled it takes 20 or less moves to solve a Rubik's cube no matter the state it starts.
  - This number is known as God's Number. This is powerful when knowing that there would be about $4.3*10^{19}$ different vertices on that Cayley Graph.

- The diameter of the Cayley Graph of a 2x2x2 Rubik's Cube when using only 90° rotations of the front, upper, and left faces was found to be 14 by a team at Northeastern.

Open problems

- A variant of the Lovász conjecture states that:
  Every finite undirected Cayley graph contains a Hamiltonian cycle.
  - For directed Cayley graphs, the Lovász conjecture is false.

- It has been shown that for finite groups the only groups that generate a planar Cayley Graph are $\mathbf{Z}_n$, $\mathbf{Z}_n$ x $\mathbf{Z}_2$, D_n, S_4, A_4, and A_5 (Maschke 1896).
  - There is ongoing research regarding which infinite groups generate planar Cayley Graphs.

References

1. B. D. McKay and C. E. Praeger, Vertex-transitive graphs which are not Cayley graphs, II., J. Graph Theory, 22 (1996) 321-334
2. “Cayley Graph.” Wikipedia, Wikimedia Foundation, 10 Nov. 2019, en.wikipedia.org/wiki/Cayley_graph.
3. “Coarse Structure.” Wikipedia, Wikimedia Foundation, 4 June 2019, en.wikipedia.org/wiki/Coarse_structure.
4. Cooperman, G., et al. “Applications of Cayley Graphs.” Northeastern University. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.431.1321&rep=rep1&type=pdf
5. “CS E6204 Lecture 6 Cayley Graphs.” Columbia University. http://www.cs.columbia.edu/~cs6204/files/Lec6-CayleyGraphs.pdf
6. “Eight Perfect Shuffles.” Eight Perfect Shuffles, jdgmiles.github.io/Riffle-Shuffles
7. “File:Dih 4 Cayley Graph; Generators b, C.svg.” File:Dih 4 Cayley Graph; Generators b, C.svg - Wikimedia Commons, 10 Nov. 2019, commons.wikimedia.org/wiki/File:Dih_4_Cayley_Graph;_generators_b,_c.svg.
8. “Geometric Group Theory.” Wikipedia, Wikimedia Foundation, 29 Sept. 2019, en.wikipedia.org/wiki/Geometric_group_theory.
9. “How to Build Groups: Semidirect Products.” Building Groups: Semidirect Products, www.weddslist.com/groups/building/sdp.html.
10. Pegg, Ed Jr.; Rowland, Todd; and Weisstein, Eric W. “Cayley Graph.” From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/CayleyGraph.html
11. “Rubik's Cube Group.” Wikipedia, Wikimedia Foundation, 27 Apr. 2019, en.wikipedia.org/wiki/Rubik%27s_Cube_group.