Classical World Theory

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$$\Omega_{\alpha\beta} = 0$$

Abstract

This theory is an attempt of a classical, geometric unification of gravitation, electromagnetism and dark energy. The tensorial formalism employed for that purpose is similar to a not widely known, alternative unification approach denoted as conformal geometrodynamics. However, the central difference is that conformal geometrodynamics is founded on Weyl spaces, whereas the present theory is a reinterpretation on a special category of non-Weyl spaces. The non-Weyl spaces appear to be more suitable from the epistemological perspective. Also, this paper goes in various aspects beyond the state of research in conformal geometrodynamics: Trying to give a solid justification for the tensorial formalism, it is shown that the entire theory can be derived by providing the simplest possible cure for an anomaly in the interaction of gravitation and electromagnetism as described by Einstein's field equation. The derivation is performed in an arbitrary number of dimensions, and that way a fundamentally new criterion is discovered which may be used to explain why our universe is 4-dimensional. Pure electric fields are possible in only that number of dimensions. There is nowadays not much interest in Weyl's original theory, because the rescaling of the metric under gauge transformations does not fit to observations, as first noted by Einstein. For the present theory, it is demonstrated that this problem does not exist, because of the additionally involved dark energy. Dark energy is realized as a scalar field and even allows the derivation of a novel equation, which is very similar to the Klein-Gordon equation, except for the mass-term. Pursuing the idea of geometry-based physics, the scalar field is reduced to a new, modified curvature such that the entire theory can be interpreted geometrically. To obtain nontrivial solutions of the theory, for the first time, a numeric approach is developed in this context, founded on the BSSN-formalism of numeric relativity. The approach is implemented in a new simulation code, which is applied to provide initial simulation results. For researchers intending to quickly explore the present theory numerically for themselves, the relevant evolution equations are summarized in a 2-page numeric formulary. Contrary to general relativity, the Hamiltonian, momentum and Gauss constraints are cured by the present theory, and also by conformal geometrodynamics, which makes setting up initial models a simple task.

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Contents

1 Introduction

1.1 Overview

Physics divides into two parts, classical physics and quantum physics. The following theory is an attempt to create a unified, geometric description of the fundamentals of the classical world. For that purpose, two successive unifications are performed. The first one is the geometric unification of electromagnetism and gravitation to "electrogravitation". This is accomplished by identifying electromagnetism as the "interior" torsion and nonmetricity of spacetime. The second one is the geometric unification of electrogravitation and a scalar "matter field". The impact of the scalar field mentioned here is visible in cosmology. It causes an exponential expansion of space and is therefore proposed as a geometric interpretation of dark energy.

The second unification above is a candidate for the true fundament of classical physics. The natural way to call that step is thus "classical world theory". Whenever not misleading, the word "classical" can also be omitted. This kind of shortening is common practice for classical theories and used throughout the paper. The name of the introduced theory reduces then to "world theory". So, the term "world" is applied not referring to earth but the universe as a whole. The theory itself is suggested as the successor of general relativity, which appears to have deficits. It is, however, important to bear in mind that the new theory is not an attempt to describe the fundamentals of all physics. For such a description, a proper inclusion of the quantum world would be unavoidable.

1.2 Contents

To guide the readers who do not want to advance through this paper continuously, let us first of all have a look at its structure (see also the table of contents at the beginning). We start with general remarks in Sections 1.3 and 1.4. They justify the approach of the paper and explain why we do not limit ourselves to the observable four dimensions. Then, we conduct a short repetition of electromagnetism and gravitation in Sections 1.5 and 1.6. In these two sections and the one following, we simultaneously introduce our mathematical notations and conventions. Section 1.7 deals with Einstein-Maxwell theory. This theory already contains a coupling of electromagnetism and gravitation but unfortunately no unification.

Experienced readers who want to immediately catch a glimpse of the trick to achieve a geometric unification can directly head to Section 2.1. The ansatz presented there allows us to derive the field equation of electrogravitation in Section 2.2 by curing an anomaly in Einstein's equation whose significance was overseen until now. Section 2.3 explains the geometric interpretation of electromagnetism in detail. Figure 1 illustrates, for instance, the geometry of an electromagnetic wave. At this point, it becomes clear that the geometry of electrogravitation is not very different from the one of Weyl theory, one of the early attempts to unify electromagnetism and gravitation. Yet, there are important discrepancies between both theories, illustrated in Section 2.4. The subsequent Section 2.5 shows that electrogravitation violates the gauge invariance underlying electromagnetism. This appears to be a severe problem, but in fact the contrary is true. Electrogravitation is governed by the more natural and unified "conformal gauge invariance" also known from Weyl theory. When trying to generalize electrogravitation to an arbitrary number of dimensions in Section 2.6, a fundamental criterion is revealed which explains why we live in just four dimensions. In Section 2.7, we realize a peculiarity of the field equation of electrogravitation which distinguishes it from the established field equations. It cannot be derived from a variation principle. This philosophically anyway not rigorously necessary principle is therefore regarded as obsolete. Eventually, we investigate geodesics and the parallel transport of spin in Sections 2.8 and 2.9. The fundamentals of electrogravitation are that way set up.

We have then the means to unify electrogravitation and the already mentioned matter field. This field is introduced in Section 3.1, and the subsequent Section 3.2 contains the unification to world theory, which turns out to be an easy endeavor. The result is the central equation of this paper, the "world equation", shown in Section 3.3 and on the front page. Note that this equation is merely a classical field equation. Its intention is not to describe the fundamentals of all physics. A hypothetical equation succeeding in that task is typically called world formula. Unfortunately, we do not know what the world formula is and thus also not its relationship with the world equation. Despite sounding similar, these two notions are not to be mixed up. The reason why we have not chosen a more distinctive notion for the world equation is explained in Section 3.3.5. As a pure tensor equation, i.e. without the geometric interpretation, the world equation does by the way not differ from the field equation of a unification approach known as conformal geometrodynamics, which is founded on the geometry of Weyl theory. However, the two field equations are still not the same, because they are based on different geometries. More on this issue can be found in Section 3.4. It is also important to know that the world equation incorporates Einstein's equation in the vacuum limit, i.e. in the case where only gravitation is present. However, it contradicts the full Einstein equation such that world theory is proposed as a partial replacement of general relativity. In this manner, we obtain a new approach for cosmology, which is shown in Section 3.6. Based on that cosmology, the universe did not start in a big bang. Instead, it is infinitely old and will exist for all times. The new cosmology also enables a geometric interpretation of dark energy in Section 3.7. And, we propose there that using the wrong field equation, namely the one of Einstein, may cause the appearance of the phenomenon dark matter. We proceed with Section 3.8, where we study different limits of the world equation. Particular limits are the linearization of world theory, which is shown in Section 3.9, and the split of the metric into a scalar and tensorial density in Section 3.10. The split leads to a classical, Klein-Gordon-like equation and a geometric interpretation of the associated wave function. The last Section 3.11 of the chapter about world theory generalizes various relations known from general relativity.

In Chapter 4, we investigate solutions of world theory. After an overview of the solutions which can just be taken over from general relativity in Section 4.1, we continue with the most important nontrivial, analytic solution: the stationary, spherically symmetric case. In Section 4.2, we see that only the Schwarzschild-de Sitter black hole is possible under that symmetry. So, in world theory, there are no stationary, spherically symmetric black holes which are charged, in contrast to general relativity. More complicated analytic solutions which are not already familiar from general relativity are currently not known. Section 4.3 and the subsequent ones therefore present a numeric approach to world theory. It takes the BSSN-formalism of numeric relativity and customizes it for the world equation. A striking feature of the world equation is that it is free of constraints. This means that there is no Hamiltonian constraint and so forth. I have also created a new code from scratch, the so-called "Alpha-code". It is introduced in Section 4.7 and demonstrates that there are no obstacles when trying to simulate world theory.

Chapter 5 concludes the paper. Readers who want to see a short way to summarize world theory are advised to visit Section 5.1. The subsequent Section 5.2 demonstrates why full general relativity, i.e. the case where gravitation is joined by other physical phenomena, should be replaced by world theory. Ambitious readers trying to start their own research in world theory will find initial aid in Section 5.3. The contents of this paper are thus illustrated, and we can begin our expedition.

1.3 Approach

It is an easy task to create an arbitrary new theory. It is, however, a hard undertaking to construct a theory that fulfills a sufficiently special purpose, especially in physics. The easiest approach in that field of science is induction. We possess knowledge about a phenomenon observed in nature and conclude that there is a general law behind it. As conclusions of such a kind are commonly ambiguous, we must always take into account that the discovered law may be wrong. It is then the task of experimental physics to decide whether the physical theory holds.

Unfortunately, fundamental physics is nowadays so intricate that induction is no longer a sufficient means to find a new theory there. Induction can at best be the reason to seek for a theory at all and narrow down where one has to look. The history of physics has in fact taught us that the best approach appears to be simplicity. It has turned out that the fundamental theories of physics are governed by rather simple laws. That insight is also the motivation for the theory presented in this paper. The central goal there is the unification of electromagnetism and gravitation. The resulting theory should be based on a law that is simpler than the ones of the two original theories. As simplicity luckily reduces the number of relevant starting points, it appears as if we just have to try out a few general laws and that way reach the goal. This approach is known as deduction. However, it is not that easy. We do not have a strict definition of simplicity such that we never really know how far we have to go.

Fortunately, there is a third way. We study the already sufficiently tested and thus generally accepted theories, and look for anomalies in them. Anomalies may be a sign that there is a hidden lack of simplicity. The starting point for a new theory is then the goal to cure a certain anomaly. This method is not straightforward. Most anomalies will be useless for finding a new theory. Hence, there is no guarantee that this approach works. Similar to simplicity, there is also no exact definition what an anomaly is at all. However, this approach successfully led to the discovery of world theory, and we therefore follow that way closely here. So, it is not the aim of this paper to present world theory in the shortest possible manner. For that purpose, a few axioms would be required, and all the rest could then be deduced. It is rather the goal to clearly show the reader the individual steps that start from a specific anomaly and eventually lead to the discovery of new axioms. These steps are practically unambiguous such that demanding a cure makes the new axioms a necessity.

1.4 Dimensions

Before studying any mathematical expressions, let us reflect on their framework. It is obvious that our universe possesses one temporal and three spatial dimensions. Observing anything else does not seem to be possible for us. Hence, theories with different dimensions are not falsifiable and thus not physical. This would be different for microscopic dimensions, but we refer to macroscopic ones throughout the paper. Anyway, whenever investigating observability in physics, one is advised to be careful. We are surely unable to observe the universes described by the theories mentioned above. However, we can at least study what is mathematically thinkable at all. The limits of our thoughts are set by physics, because we are governed by the laws of nature. So, investigating what happens if there are not one temporal or not three spatial dimensions tells us something about physics itself and must therefore be regarded as part of it.

No time or more than one temporal dimension will lead to theories that are so fundamentally different from the ones describing our universe that we are not interested in them. In this paper, we therefore always assume that there is one temporal dimension. Yet, looking at more or less than three spatial dimensions is interesting, because at first glance there does not seem to be any reason why we live in just three of them. A hint why this is the case may be found by studying what theories can be constructed and how they look like if there are not three spatial dimensions. Therefore, whenever not mentioned otherwise and up to Chapters 4 plus 5, the mathematical expressions in this paper are considered to hold for an arbitrary number of spatial dimensions. This is also a reason why we show intermediate computations, even if they may already be familiar from general relativity. We have to go sure that they hold for any number of dimensions. That way, we automatically reduce the requirements to understand this paper to a minimum.

1.5 Electromagnetism

Let us now repeat vacuum electromagnetism in flat spacetime in one of the shortest ways possible (for more details, see, e.g., Jackson 1999). To unambiguously define a classical field theory, two constituents are required, a field and a field equation.

In case of electromagnetism, the field is the electromagnetic vector potential

$$A_\alpha$$

We follow the convention that Greek indices run through the values $t, 1, ..., D-1$, where D is the number of spatial dimensions plus one. The one takes time into account, which is represented by the case $\alpha = t$. Having introduced the field, we can now proceed to the field equation. For that purpose, we define the electromagnetic field strength

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{1.1}$$

where $\partial_\alpha = \partial/\partial x^\alpha$ is the partial derivative with respect to the coordinate x^α. Using Einstein's sum convention and $\partial^\alpha = \pm\partial_\alpha$, in which the minus is applied for temporal and the plus for spatial dimensions, we eventually arrive at Maxwell's equation in flat spacetime without sources:

$$\partial^\beta F_{\beta\alpha} = 0 \tag{1.2}$$

This field equation has been discovered in the 19th century as a result of experiments and having observed nature. However, we do not have a deeper understanding of Maxwell's equation until now. We neither know why just this equation is realized in nature nor do we have a true philosophical insight into electromagnetism.

1.6 Gravitation

1.6.1 Field and field equation

For gravitation, the situation has once been similar. There, observations led to the discovery of Newtonian gravitation in the 17th century. This theory is able to approximately describe slowly changing, weak gravitational fields. Yet, for gravitation, a geometric understanding has been found by Einstein at the beginning of the 20th century. The starting point of Einstein's work was the realization of a weak spot in Newtonian gravitation. The problem with that theory was its foundation on absolute spacetime coordinates. However, coordinates do not exist in nature. They are just a means to describe it. Therefore, the laws of nature must be invariant under general coordinate transformations, which is also known as the principle of general covariance. By combining this principle with gravitation, Einstein was able to generalize Newtonian gravitation to his theory of general relativity. In general relativity, gravitation is then identified as the effect of the interior curvature of spacetime. Similar to electromagnetism, we will now give a quick review of vacuum gravitation, which is also known as vacuum general relativity (for more details, see, e.g., Misner *et al.* 2002).

The field of gravitation is the metric

$$g_{\alpha\beta}$$

which is a symmetric tensor, i.e. $g_{\alpha\beta} = g_{\beta\alpha}$, with signature $(-,+,+,...)$. Its inverse $g^{\alpha\beta}$ is defined by

$$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma} \tag{1.3}$$

where δ^{β}_{α} is the Kronecker tensor. This quantity is one for $\alpha = \beta$ and zero otherwise. In order to give the field equation, we will now introduce two ancillary quantities. These are the Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right) \tag{1.4}$$

and they enable us to define the Ricci tensor

$$R_{\alpha\beta} = \partial_\gamma \Gamma^{\gamma}_{\alpha\beta} - \partial_\beta \Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\delta\gamma}\Gamma^{\delta}_{\alpha\beta} - \Gamma^{\gamma}_{\delta\beta}\Gamma^{\delta}_{\alpha\gamma} \tag{1.5}$$

The field equation of vacuum gravitation is then Einstein's equation without sources:

$$R_{\alpha\beta} = 0 \tag{1.6}$$

1.6.2 Geometric interpretation

In contrast to electromagnetism, we do not have to stop here, but we can give a geometric explanation of the two ancillary quantities defined in the last section. Let us first look at the Christoffel symbols (1.4). Instead of just defining them, we can proceed in a different way and derive them from more fundamental equations. For that purpose, we introduce the covariant derivative

$$\nabla_\gamma T^{\beta\ldots}_{\alpha\ldots} = \partial_\gamma T^{\beta\ldots}_{\alpha\ldots} - \Gamma^{\delta}_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + \Gamma^{\beta}_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots - W \Gamma^{\delta}_{\delta\gamma} T^{\beta\ldots}_{\alpha\ldots} \tag{1.7}$$

where $T^{\beta\ldots}_{\alpha\ldots}$ is an arbitrary tensor density of coordinate weight W. Let us now for a moment forget definition (1.4) and assume that $\Gamma^{\alpha}_{\beta\gamma}$ is an arbitrary connection which has to obey

$$\begin{aligned} \Gamma^{\alpha}_{\beta\gamma} &= \Gamma^{\alpha}_{\gamma\beta} && (1.8) \\ \nabla_\gamma g_{\alpha\beta} &= 0 && (1.9) \end{aligned}$$

These two equations have a geometric meaning. The first one tells us that there is no torsion and the second one that there is no nonmetricity. We will now show that such a connection are just the Christoffel symbols (1.4). The first step to achieve this is to apply definition (1.7) on condition (1.9):

$$\partial_\gamma g_{\alpha\beta} - \Gamma^\delta_{\alpha\gamma} g_{\delta\beta} - \Gamma^\delta_{\beta\gamma} g_{\alpha\delta} = 0 \tag{1.10}$$

Hence,

$$\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma} = \left(\Gamma^\epsilon_{\delta\beta} - \Gamma^\epsilon_{\beta\delta}\right) g_{\epsilon\gamma} + \left(\Gamma^\epsilon_{\gamma\beta} + \Gamma^\epsilon_{\beta\gamma}\right) g_{\epsilon\delta} + \left(\Gamma^\epsilon_{\delta\gamma} - \Gamma^\epsilon_{\gamma\delta}\right) g_{\beta\epsilon}$$

such that condition (1.8) leads to

$$\frac{1}{2}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right) = \Gamma^\epsilon_{\beta\gamma} g_{\epsilon\delta}$$

Multiplying this equation by $g^{\alpha\delta}$ and using definition (1.3), we arrive at the Christoffel symbols (1.4).

Next, we define the Ricci tensor $R_{\alpha\beta}$ in a more fundamental manner than shown in equation (1.5). To this end, we consider an arbitrary vector X_α, and define the Riemann tensor $R^\alpha_{\ \beta\gamma\delta}$ via the Ricci identity

$$R^\alpha_{\ \beta\gamma\delta} X_\alpha = -\left[\nabla_\gamma, \nabla_\delta\right] X_\beta \tag{1.11}$$

The commutator appearing therein is defined in the usual way as $[X, Y] = XY - YX$, where X and Y are arbitrary expressions. In the above case, we can also use the index antisymmetrization $\nabla_{[\gamma}\nabla_{\delta]} = \left[\nabla_\gamma, \nabla_\delta\right]/2$ known from general relativity. The index antisymmetrization does not only work here but for any expression with at least two indices. If we enclose two indices of such an expression with squared brackets, then this stands for the following difference. The minuend is a half of the original expression without the squared brackets, and the subtrahend is equal to the minuend, except that the two indices considered here are exchanged. Note that if more than two indices are enclosed by the squared bracket, the antisymmetrization refers only to the two embraced indices closest to the bracket. Such cases will occur in the following, where we evaluate the Riemann tensor. The Riemann tensor by the way measures the curvature of spacetime and is also known as the curvature tensor. To evaluate it, we use definition (1.7) and the symmetry (1.8) such that

$$R^\alpha_{\ \beta\gamma\delta} X_\alpha = 2\nabla_{[\delta}\left(\partial_{\gamma]} X_\beta - \Gamma^\alpha_{\beta\gamma]} X_\alpha\right) = 2\left[\partial_{[\delta}\left(\partial_{\gamma]} X_\beta - \Gamma^\alpha_{\beta\gamma]} X_\alpha\right) - \Gamma^\epsilon_{\beta[\delta}\left(\partial_{\gamma]} X_\epsilon - \Gamma^\alpha_{\epsilon\gamma]} X_\alpha\right)\right]$$

In contrast to the covariant derivatives, the partial ones commute such that we can continue with

$$R^\alpha_{\ \beta\gamma\delta} X_\alpha = 2\left(\partial_{[\gamma}\Gamma^\alpha_{\beta\delta]} + \Gamma^\alpha_{\epsilon[\gamma}\Gamma^\epsilon_{\beta\delta]}\right) X_\alpha \tag{1.12}$$

As the vector X_α was assumed to be arbitrary, it can be left away and the Riemann tensor simply be written as

$$R^\alpha_{\ \beta\gamma\delta} = 2\left(\partial_{[\gamma}\Gamma^\alpha_{\beta\delta]} + \Gamma^\alpha_{\epsilon[\gamma}\Gamma^\epsilon_{\beta\delta]}\right) \tag{1.13}$$

The Ricci tensor is then given by the symmetric tensor

$$R_{\alpha\beta} = R^\gamma_{\ \alpha\gamma\beta} \tag{1.14}$$

and this is clearly equal to definition (1.5). So, we have given the geometric origin (1.8) and (1.9) of the Christoffel symbols, and we have shown that the rather complicated definition (1.5) of the Ricci tensor is in a simple manner coming from a consideration of how the covariant derivative commutes with itself in equation (1.11), which defines the curvature tensor $R^\alpha_{\ \beta\gamma\delta}$.

1.7 Einstein-Maxwell theory

After the review of vacuum electromagnetism in flat spacetime and vacuum gravitation, we will now study how the two forces can be merged. Yet, before looking at the geometric unification presented in this paper, we pay a visit to the established way. There, a non-geometric coupling attempt is used, namely Einstein-Maxwell theory. This theory does not alter the fields of electromagnetism and gravitation. So, we still have the electromagnetic vector potential A_α and the metric $g_{\alpha\beta}$. The non-geometric coupling is then performed in two independent steps.

The first one is generalizing Maxwell's equation (1.2) to curved spacetime such that gravitation influences electromagnetism. For that purpose, the only thing we have to do is to take equation (1.2) and replace the partial derivative with a covariant one:

$$\nabla^\beta F_{\beta\alpha} = 0 \tag{1.15}$$

Here, we have used the metric to raise and lower indices, i.e. $\nabla^\alpha = g^{\alpha\beta}\nabla_\beta$ and vice versa $\nabla_\alpha = g_{\alpha\beta}\nabla^\beta$. This procedure can be used on any other tensor index as well. Note that we do not have to replace the partial derivatives in the field strength (1.1) with covariant ones, because the symmetry (1.8) of the Christoffel symbols would make the outcome equal to definition (1.1).

The second coupling step is to let electromagnetism influence gravitation. To this end, we use the full Einstein equation, i.e. introducing the Ricci scalar

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{1.16}$$

and the symmetric Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \tag{1.17}$$

we have

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1.18}$$

In this equation, the quantity $T_{\alpha\beta}$ is the stress-energy tensor, where the speed of light and the gravitational constant are assumed to be unity: $c = G = 1$. For electromagnetism alone, i.e. without any other sources, the stress-energy tensor $T_{\alpha\beta}$ is known to be

$$T^{\mathrm{EM}}_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta}F^{\gamma\delta}\right) \tag{1.19}$$

such that Einstein's equation becomes

$$G_{\alpha\beta} = 2\left(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta}F^{\gamma\delta}\right) \tag{1.20}$$

So, Einstein-Maxwell theory consists of the two fields A_α and $g_{\alpha\beta}$, which are governed by Maxwell's equation (1.15) and Einstein's equation (1.20). Yet, there is neither a unified field nor a unified field equation such that Einstein-Maxwell theory is not a unification of electromagnetism and gravitation but merely a coupling.

2 Electrogravitation

2.1 Unification trick

In order to geometrically unify electromagnetism and gravitation, we will now study an odd behavior of the full Einstein equation (1.18). Realizing the importance of this anomaly is the ignition spark for all further theoretical steps required to eventually achieve a unification.

To see the anomaly, we look at Einstein-Maxwell theory and evaluate the covariant divergence of the stress-energy tensor $T^{\mathrm{EM}}_{\alpha\beta}$ used there. The outcome may be known to readers familiar with general relativity, but for a complete understanding we repeat the required steps here. So, due to equation (1.19) we get

$$4\pi\nabla^\beta T^{\mathrm{EM}}_{\beta\alpha} = \nabla^\beta\left(F_{\beta\gamma}F_\alpha{}^\gamma - \frac{1}{4}g_{\beta\alpha}F_{\gamma\delta}F^{\gamma\delta}\right) = F_\alpha{}^\gamma\nabla^\beta F_{\beta\gamma} + F_{\beta\gamma}\nabla^\beta F_\alpha{}^\gamma - \frac{1}{2}F^{\gamma\delta}\nabla_\alpha F_{\gamma\delta}$$

where we have used the Leibniz rule for the covariant derivative and equation (1.9). Renaming, raising and lowering some indices as well as taking the antisymmetry of the electromagnetic field strength $F_{\alpha\beta}$ into account, we can proceed with

$$4\pi\nabla^\beta T^{\mathrm{EM}}_{\beta\alpha} = F_\alpha{}^\beta\nabla^\gamma F_{\gamma\beta} - \frac{1}{2}F^{\beta\gamma}\left(\nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} + \nabla_\alpha F_{\beta\gamma}\right) \tag{2.1}$$

Let us now evaluate the round bracket with the help of the definition of the electromagnetic field strength (1.1):

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 2\left(\nabla_\alpha\nabla_{[\beta}A_{\gamma]} + \nabla_\beta\nabla_{[\gamma}A_{\alpha]} + \nabla_\gamma\nabla_{[\alpha}A_{\beta]}\right)$$

The covariant derivatives can be reordered such that the definition (1.11) of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ is applicable:

$$\begin{aligned}\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} &= \left[\nabla_\alpha, \nabla_\beta\right]A_\gamma + \left[\nabla_\beta, \nabla_\gamma\right]A_\alpha + \left[\nabla_\gamma, \nabla_\alpha\right]A_\beta \\ &= \left(R_{\delta\gamma\beta\alpha} + R_{\delta\alpha\gamma\beta} + R_{\delta\beta\alpha\gamma}\right)A^\delta\end{aligned} \tag{2.2}$$

The second line vanishes due to the first Bianchi identity

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} \equiv 0 \tag{2.3}$$

known from general relativity. Note that this relation holds for all choices of the metric $g_{\alpha\beta}$, i.e. independent of whether any field equation is effective, like Einstein's (1.18). We emphasize this by the three horizontal lines used above instead of the equality sign. The first Bianchi identity is actually an obvious consequence of definition (1.13). We merely have to apply a total antisymmetrization of the three indices β, γ, δ in that equation such that the squared brackets present there can be left away, and then the whole expression vanishes due to the symmetry of the Christoffel symbols. As the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ is antisymmetric in its last two indices, we thus arrive at identity (2.3). Equation (2.2) shows us then the new identity

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} \equiv 0$$

and relation (2.1) reveals yet another identity:

$$\nabla^\beta T^{\mathrm{EM}}_{\beta\alpha} \equiv \frac{1}{4\pi}F_\alpha{}^\beta\nabla^\gamma F_{\gamma\beta} \tag{2.4}$$

In flat spacetime, where the covariant derivatives are replaced by partial ones, identity (2.4) does not have any further consequences. However, in curved spacetime, we recall the contracted Bianchi identity from general relativity (later in the paper, we will repeat the derivation of this identity and see that it even holds in D dimensions):

$$\nabla^\beta G_{\beta\alpha} \equiv 0 \tag{2.5}$$

Applying it on Einstein's equation (1.18) leads to

$$\nabla^\beta T^{\text{EM}}_{\beta\alpha} = 0$$

such that identity (2.4) spews out the following surprising equation:

General relativistic anomaly:

Einstein's equation implies

$$F_\alpha{}^\beta \nabla^\gamma F_{\gamma\beta} = 0 \tag{2.6}$$

This is the anomaly of the full Einstein equation mentioned at the beginning, which will guide us to a new field equation at the end of the next section. The crucial point is now that the above equation is nearly Maxwell's equation (1.15). However, the factor $F_\alpha{}^\beta$ causes an important difference. The equation in box (2.6) is equal to Maxwell's equation only at all those points of spacetime where the determinant $|F_\alpha{}^\beta|$ is not zero. This difference is, for instance, investigated on p. 472f of Misner *et al.* (2002). However, as the discrepancy appears to be relevant only in rare special cases, its importance is not realized there.

So, just for the sake of all those points of spacetime where $|F_\alpha{}^\beta| = 0$, it is not sufficient to have only Einstein's equation (1.20), but we have to additionally demand Maxwell's equation (1.15). This is not elegant. As elegance was one of the driving forces that led to the discovery of vacuum general relativity, it is therefore natural to investigate whether there is any way to get rid of the factor $F_\alpha{}^\beta$ in the anomaly (2.6). However, all the steps performed above show us that there does not seem to be such a way as long as the stress-energy tensor $T^{\text{EM}}_{\alpha\beta}$ is used on the right hand side of Einstein's equation (1.18). Therefore, we will from now on assume that nature is not described by that equation but by a new theory called **electrogravitation** (=EG) which uses an alternative, yet unknown stress-energy tensor $t^{\text{EM}}_{\alpha\beta}$. Even Einstein himself confessed that there is no solid criterion which demands his stress-energy tensor $T^{\text{EM}}_{\alpha\beta}$ (see the sentence below equation (53) in Einstein 1916: "Es muß zugegeben werden, daß diese Einführung des Energietensors der Materie durch das Relativitätspostulat allein nicht gerechtfertigt wird..."). Already modifying Einstein's equation, we also omit the factor 8π appearing there such that the field equation belonging to the new theory has the simple form

$$\boxed{G_{\alpha\beta} = t^{\text{EM}}_{\alpha\beta}} \tag{2.7}$$

Let us call the quantity $t^{\text{EM}}_{\alpha\beta}$ introduced above **electrogravitational stress-energy tensor**, in contrast to the general relativistic one $T^{\text{EM}}_{\alpha\beta}$. As both stress-energy tensors must differ beyond the trivial difference 8π, electrogravitation is not consistent with full general relativity. Hence, one of the two theories has to be wrong. However, we demand that in the limit of vacuum gravitation the electrogravitational stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ must vanish such that electrogravitation will at least be compatible with vacuum general relativity, described by equation (1.6). In the following, we will derive the electrogravitational stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$.

2.2 Field equation

2.2.1 Ansatz

To derive the field equation belonging to electrogravitation, the only working method appears to be trial and error. Let us therefore investigate all forms possible for the unknown stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ such that equation (2.7) implies Maxwell's equation (1.15). I.e., there may be no factor like the one in the anomaly (2.6) present. For that purpose, we will now make an ansatz for the stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ composed of different terms. As the quantity $t^{\mathrm{EM}}_{\alpha\beta}$ is a tensor, this ansatz has to be covariant and of coordinate weight 0. This will fortunately reduce the possibilities drastically. Each term of the ansatz may then contain the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α, the covariant derivative ∇_α and the Levi-Civita tensor $\epsilon_{\alpha\beta...}$.

The dots in the Levi-Civita tensor highlight that the number of its indices is not fixed but equal to the number of dimensions D. This is a consequence of the Levi-Civita tensor being defined as $\epsilon_{\alpha\beta...} = \sqrt{-g}\,[\alpha\beta...]$ (see equation (8.10a) in Misner *et al.* 2002, in which only the 4-dimensional version $\epsilon_{\alpha\beta\gamma\delta}$ is shown). The first factor here uses the determinant $g = \det g_{\alpha\beta}$ and the second one is the completely antisymmetric symbol $[\alpha\beta...]$. The value of the latter symbol is given as follows. If $\alpha\beta...$ is an even / odd permutation of $t, 1, ..., D-1$, then the value is $+1$ / -1. In all other cases, the value is 0. By the way, note that the symbol $\epsilon_{\alpha\beta...}$ has the coordinate weight 0 as a tensor (in contrast to an also existing Levi-Civita tensor density $\varepsilon_{\alpha\beta...}$ of coordinate weight -1 encountered, .e.g, in equation (2.5.12) of Carmeli 2001). The Levi-Civita tensor $\epsilon_{\alpha\beta...}$ could even be left away in the following, because we would arrive at the same field equation at the end, but for completeness we include this tensor.

We do not, however, have to separately take the Kronecker tensor δ^{β}_{α} into account here, because it is just the metric $g_{\alpha\beta}$ with the index β risen. What about the more complicated, covariant quantities known from vacuum general relativity, like the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$? These quantities are composed of the metric $g_{\alpha\beta}$ and the partial derivative ∂_α. Yet, they are not already part of the above ansatz, because we cannot rewrite the partial derivatives occurring in them to covariant ones. The covariant derivatives would have to be applied on the metric, which is known to vanish due to equation (1.9). Hence, quantities like the Riemann tensor have to be treated separately. All these quantities have in common that they possess at least a second partial derivative. However, we do not have to include them in our ansatz, because it is reasonable to consider the Einstein tensor $G_{\alpha\beta}$ in equation (2.7) as a nonlinear wave operator applied on the metric $g_{\alpha\beta}$ and the stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ as its source, which is then limited to first derivatives. That way, our ansatz is composed of the following quantities:

$$g_{\alpha\beta}, A_\alpha, \nabla_\alpha, \epsilon_{\alpha\beta...}$$

To obtain a non-vanishing outcome for the covariant derivative, we have to apply it on the electromagnetic vector potential, i.e. $\nabla_\alpha A_\beta$. This is the only way to use the covariant derivative, because the limit to first derivatives prohibits us from applying it more than once. Then, the terms in our ansatz must be products of the factors $g_{\alpha\beta}$, A_α, $\nabla_\alpha A_\beta$ and

$\epsilon_{\alpha\beta\ldots}$. As the sought stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ contains an even number of indices, the factor A_α must always occur in pairs $A_\alpha A_\beta$ for terms that do not contain the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$. It is thus sufficient to consider the factors

$$g_{\alpha\beta}, A_\alpha A_\beta, \nabla_\alpha A_\beta, \epsilon_{\alpha\beta\ldots}\cdots$$

The dots at the end refer to an unknown number of occurrences of the electromagnetic vector potential, which may also occur with a covariant derivative applied on it.

Each term must at least contain the electromagnetic vector potential A_α once, because in the limit of vacuum gravitation the stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ is demanded to vanish. The Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$ can occur maximally a single time per term, because it is known that

$$\epsilon_{\alpha\beta\ldots}\epsilon^{\gamma\delta\ldots} = -\delta^{\gamma\delta\ldots}_{\alpha\beta\ldots} \tag{2.8}$$

(see the last line on p. 41 of Carmeli 2001 for the 4-dimensional version). The right hand side contains the generalized Kronecker tensor $\delta^{\gamma\delta\ldots}_{\alpha\beta\ldots}$, whose exact definition does not matter here. The only matter of moment is that this tensor can be decomposed into products of the Kronecker tensor δ^α_β (see equation (3) on p. 31 of Carmeli 2001, again in four dimensions) and is that way already taken into account by our ansatz. Unfortunately, despite all the above limitations, there are still too many possibilities left to consider them all. As a first trial, we therefore proceed in the following way.

Both the electromagnetic vector potential A_α and the covariant derivative ∇_α are vectors, the latter actually being a vector operator. In our trial, we neglect all orders which are third order or higher in these two vectors. In so doing, and taking the total antisymmetry of the Levi-Civita tensor into account, we arrive at the ansatz

$$t^{\mathrm{EM}}_{\alpha\beta} = a\nabla_\alpha A_\beta + b\nabla_\beta A_\alpha + cg_{\alpha\beta}\nabla_\gamma A^\gamma + dA_\alpha A_\beta + eg_{\alpha\beta}A^2 + f\delta^4_D\epsilon_{\alpha\beta\gamma\delta}\nabla^\gamma A^\delta + \cdots \tag{2.9}$$

where $a, \ldots, f \in \mathbb{R}$ are constants to be determined. Let us now look at the term with the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$. For $D = 4$, this is the only term constructible from the Levi-Civita tensor. The constant f plays by the way a role only for that number of dimensions. For $D \neq 4$, the Levi-Civita tensor does not have four but D indices. It will then contribute terms to ansatz (2.9) which are not shown there and represented by the dots at the end.

2.2.2 Derivation

Let us eventually derive the field equation by determining the constants in ansatz (2.9) (in conformal geometrodynamics, more on that theory in the later Section 3.4, see p. 19ff of Gorbatenko *et al.* 2002). For that purpose, we first look at equation (2.7), where the symmetry of the Einstein tensor $G_{\alpha\beta}$ tells us

$$0 = t^{\mathrm{EM}}_{[\alpha\beta]} = a\nabla_{[\alpha}A_{\beta]} + b\nabla_{[\beta}A_{\alpha]} + f\delta^4_D\epsilon_{\alpha\beta\gamma\delta}\nabla^\gamma A^\delta + \cdots = \frac{a-b}{2}F_{\alpha\beta} + \frac{f}{2}\delta^4_D\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} + \cdots$$

This is a constraint on the electromagnetic field strength $F_{\alpha\beta}$ and for $D = 4$ on its dual $\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta}/2$. No such relation is known to hold for electromagnetism such that we conclude $b = a$ and $f = 0$. So, the Levi-Civita tensor plays no role in four dimensions. What about the generalization of electrogravitation to an arbitrary number of dimensions? It is reasonable to keep this case as close as possible to the 4-dimensional one. We thus ignore all terms with a Levi-Civita tensor for $D \neq 4$, which means the dots at the end of ansatz (2.9).

Next, we apply the contracted Bianchi identity (2.5) on equation (2.7):

$$\boxed{\nabla^\beta t^{\mathrm{EM}}_{\beta\alpha} = 0}$$

Due to the additional covariant derivative ∇^β, we will obtain terms in the following which are of third order in the two vectors A_α and ∇_α. We are not allowed to neglect these terms, because the approach of the last section to neglect third or higher orders was exclusively valid for setting up ansatz (2.9). This ansatz then gives

$$a\nabla^\beta \left(\nabla_\beta A_\alpha + \nabla_\alpha A_\beta\right) + c\nabla_\alpha \nabla_\beta A^\beta + d\nabla_\beta \left(A^\beta A_\alpha\right) + e\nabla_\alpha \left(A_\beta A^\beta\right) = 0 \tag{2.10}$$

The only terms here which are also present in Maxwell's equation (1.15) are $\nabla^\beta \nabla_\beta A_\alpha$ and $\nabla^\beta \nabla_\alpha A_\beta$. The other three terms must hence vanish or be rewritable. In flat spacetime, the term $\nabla_\alpha \nabla_\beta A^\beta$ is actually rewritable, because it becomes $\partial_\alpha \partial_\beta A^\beta$, and we could replace the order of the partial derivatives. The result would then be part of Maxwell's equation (1.2) in flat spacetime. However, in curved spacetime, the order of the covariant derivatives matters, and we get

$$\nabla_\alpha \nabla_\beta A^\beta = \nabla_\beta \nabla_\alpha A^\beta + \left[\nabla_\alpha, \nabla_\beta\right] A^\beta = \nabla_\beta \nabla_\alpha A^\beta - R^{\gamma\beta}{}_{\alpha\beta} A_\gamma \tag{2.11}$$

due to relation (1.11). In order to continue here, we use the identity

$$R_{\alpha\beta\gamma\delta} \equiv -R_{\beta\alpha\gamma\delta} \tag{2.12}$$

of general relativity (like for the contracted Bianchi identity, the proof is repeated later in this paper) such that the obvious antisymmetry of the Riemann tensor (1.13) in its last two indices and definition (1.14) give

$$R^{\gamma\beta}{}_{\alpha\beta} A_\gamma = R^{\beta\gamma}{}_{\beta\alpha} A_\gamma = A^\beta R_{\beta\alpha} \tag{2.13}$$

Hence, equation (2.10) becomes

$$\nabla^\beta \left[a\nabla_\beta A_\alpha + (a+c)\nabla_\alpha A_\beta\right] - cA^\beta R_{\beta\alpha} + d\left(A^\beta \nabla_\beta A_\alpha + A_\alpha \nabla_\beta A^\beta\right) + 2eA^\beta \nabla_\alpha A_\beta = 0 \tag{2.14}$$

All terms after the squared bracket contain at least one occurrence of the electromagnetic vector potential which is not derived. Such terms are not found in Maxwell's equation (1.15). Consequently, we can split equation (2.14) into two parts. The first one is the term with the squared bracket, which must give Maxwell's equation (1.15). This allows us to deduce $c = -2a$ such that

$$\nabla^\beta \left[a\nabla_\beta A_\alpha + (a+c)\nabla_\alpha A_\beta\right] = a\nabla^\beta F_{\beta\alpha} \tag{2.15}$$

The second part is the rest of equation (2.14), which must vanish:

$$2aA^\beta R_{\beta\alpha} + A^\beta \left(d\nabla_\beta A_\alpha + 2e\nabla_\alpha A_\beta\right) + dA_\alpha \nabla_\beta A^\beta = 0 \tag{2.16}$$

Above, we can replace the Ricci tensor $R_{\alpha\beta}$. To find the form of that tensor, we first evaluate the trace $G = g^{\alpha\beta} G_{\alpha\beta}$ of the Einstein tensor $G_{\alpha\beta}$ (which may be called Einstein scalar). Note that the trace uses the same letter as the gravitational constant. This is no problem, because the gravitational constant is set to unity throughout the paper and does, up to statements of the form "$G = 1$", not explicitly occur anywhere. For the evaluation, we consider equation (1.17) and find

$$G = -\left(\frac{D}{2} - 1\right) R$$

where we have applied $g^{\alpha\beta} g_{\alpha\beta} = D$. The same equation shows

$$R_{\alpha\beta} \overset{D\neq 2}{=} G_{\alpha\beta} - \frac{1}{D-2} g_{\alpha\beta} G \tag{2.17}$$

which holds only for $D \neq 2$. We will therefore exclude this case in the following. The next step is to adopt equation (2.7) and our ansatz (2.9) such that

$$G_{\alpha\beta} = a\left(\nabla_\alpha A_\beta + \nabla_\beta A_\alpha - 2g_{\alpha\beta}\nabla_\gamma A^\gamma\right) + dA_\alpha A_\beta + e g_{\alpha\beta} A^2$$

and consequently

$$-\frac{G}{D-2} \overset{D\neq 2}{=} 2a\frac{D-1}{D-2}\nabla_\gamma A^\gamma - \frac{d+eD}{D-2} A^2$$

We use these two outcomes in equation (2.17) to evaluate the Ricci tensor $R_{\alpha\beta}$, and additionally multiply by the factor A^β:

$$A^\beta R_{\beta\alpha} \overset{D\neq 2}{=} aA^\beta\left(\nabla_\beta A_\alpha + \nabla_\alpha A_\beta\right) + \frac{2a}{D-2} A_\alpha \nabla_\beta A^\beta + \left(d - \frac{d+2e}{D-2}\right) A_\alpha A^2 \tag{2.18}$$

The last term is the only one here and in equation (2.16) which does not contain covariant derivatives. Hence, we conclude that for $D \neq 2$ the second round bracket in equation (2.18) must vanish, i.e. $e = (D-3)\,d/2$. We then insert the outcome (2.18) in relation (2.16):

$$\left(2a^2 + d\right) A^\beta \nabla_\beta A_\alpha + \left[2a^2 + (D-3)\,d\right] A^\beta \nabla_\alpha A_\beta + \left(\frac{4a^2}{D-2} + d\right) A_\alpha \nabla_\beta A^\beta \overset{D\neq 2}{=} 0 \tag{2.19}$$

All three terms are based on different contractions applied on $A_\alpha \nabla_\beta A_\gamma$ such that they must vanish independent of each other. The first round bracket above gives $d = -2a^2$ for $D \neq 2$. Inserting this result in the squared bracket leads to $2a^2(4-D) = 0$ for $D \neq 2$. Yet, equation (2.15) tells us that the only way to get Maxwell's equation (1.15) is a nonzero constant a. Hence, the squared bracket in equation (2.19) cannot vanish unless $D = 4$. For that case, also the third round bracket in equation (2.19) becomes zero, because it just reduces to the first round bracket $2a^2 + d$.

So, the above derivation works only in four dimensions. This is a first sign that $D = 4$ is a special case, and it justifies our initial choice to not limit ourselves to the observable four dimensions. However, at this stage, we do not yet have sufficient means to draw conclusions. Instead, we have to continue with the above derivation. For that purpose, we recall the values of the constants found out via that derivation:

$$(a, b, c, d, e, f) \overset{D=4}{=} \left(a, a, -2a, -2a^2, -a^2, 0\right)$$

Inserting this outcome in our ansatz (2.9) and renaming $a \to c_{\mathrm{EM}}$ leads to

$$t^{\mathrm{EM}}_{\alpha\beta} \overset{D=4}{=} c_{\mathrm{EM}}\left(\nabla_\alpha A_\beta + \nabla_\beta A_\alpha - 2g_{\alpha\beta}\nabla_\gamma A^\gamma\right) - c^2_{\mathrm{EM}}\left(2A_\alpha A_\beta + g_{\alpha\beta} A^2\right) \tag{2.20}$$

The value of the remaining constant c_{EM} cannot be deduced. It is an unknown coupling constant, which we want to call **electromagnetic constant**. However, there is exactly one occurrence of the constant c_{EM} per factor A_α in equation (2.20). We can therefore choose the units of the electromagnetic vector potential A_α such that $c_{\mathrm{EM}} = 1$. This fits to the natural units $c = G = 1$ introduced in Section 1.7. Then, using equation (2.7), we arrive at the **electrogravitational field equation**

$$\boxed{G_{\alpha\beta} \overset{D=4}{=} \nabla_\alpha A_\beta + \nabla_\beta A_\alpha - 2A_\alpha A_\beta - g_{\alpha\beta}\left(2\nabla_\gamma A^\gamma + A^2\right)} \tag{2.21}$$

in four dimensions.

The attentive reader may oppose that the electrogravitational field equation appears to violate the gauge invariance known from electromagnetism. We will ignore this issue in the following, and later in this paper it will be explained in detail how gauge invariance actually works for equation (2.21). It is now also clear that the electrogravitational field equation differs from the field equation of Einstein-Maxwell theory (1.20). This has the consequence that at least one of the two field equations cannot be realized in nature. Note that electrogravitation is still based on two separated fields, the electromagnetic vector potential A_α and the metric $g_{\alpha\beta}$, just like Einstein-Maxwell theory. So, there is no unification of these two tensor fields. However, in electrogravitation, there is a **unified field equation**, the electrogravitational field equation (2.21), which automatically contains Maxwell's equation (1.15) as a consequence of the contracted Bianchi identity (2.5). So, in contrast to Einstein-Maxwell theory, Maxwell's equation does not have to be demanded separately. It is also important to be aware that the derivation of equation (2.21) was unambiguous. The only unjustified limitation performed above was neglecting all terms where the sum of the occurrences of the electromagnetic vector potential A_α and the covariant derivative ∇_α was three or higher in ansatz (2.9). However, as we have arrived at a field equation working in the desired way without these orders, it is no longer relevant to take them into consideration. Instead, let us find the geometric origin of equation (2.21) in the following sections.

2.3 Geometric interpretation of electromagnetism

2.3.1 Electrogravitational connection

In order to obtain a geometric interpretation of electromagnetism, we first recall the electrogravitational field equation (2.21) and Maxwell's equation (1.15):

$$G_{\alpha\beta} \overset{D=4}{=} \nabla_\alpha A_\beta + \nabla_\beta A_\alpha - 2A_\alpha A_\beta - g_{\alpha\beta}\left(2\nabla_\gamma A^\gamma + A^2\right) \tag{2.22}$$

$$\nabla^\beta F_{\beta\alpha} \overset{D=4}{=} 0 \tag{2.23}$$

Due to the last section, we know that the second equation is a consequence of the first one. For that purpose, one has to apply the contracted Bianchi identity (2.5) on the first equation. So, in principle, it is sufficient to just consider equation (2.22). However, when thinking in terms of the nonlinear wave operator picture mentioned in Section 2.2.1, it is more reasonable to extract Maxwell's equation from the electrogravitational field equation and use the above two equations separately. In that case, the right hand side of equation (2.22) is the source of gravitation, and equation (2.23) shows us that electromagnetism does not have a source for $D = 4$.

The question is now, where do the terms on the right hand side of equation (2.22) truly come from? We have discovered them by demanding that the anomaly (2.6) must vanish. However, there should be a deeper reason and the anomaly disappear as a consequence of it. Whatever that reason is, the experience with the geometric interpretation of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ and the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ in Section 1.6.2 tells us that it is likely a simple mathematical concept. Let us therefore have a close look at the terms on the right hand side of equation (2.22). An eye-catching property is the simultaneous occurrence of terms with and without a derivative. This is, for instance, also the case in definition (1.5) of the Ricci tensor $R_{\alpha\beta}$. However, the striking difference is that in equation (2.22) the electromagnetic vector potential A_{α} is a fundamental field, whereas in definition (1.5) the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ are composed quantities which contain derivatives themselves, see equation (1.4). To investigate this more thoroughly, we count the number of partial derivatives ∂_{α} appearing effectively. For that purpose, we introduce a number that we want to call **derivative level**.

By definition, the metric $g_{\alpha\beta}$ has the derivative level 0. Looking at equation (1.4), we see that each term coming from an expansion of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ contains one partial derivative. Therefore, the Christoffel symbols have the derivative level 1. For the Ricci tensor $R_{\alpha\beta}$, definition (1.5) tells us that its terms are either partially derived Christoffel symbols or products of Christoffel symbols. Despite this mixture, which we mentioned already in the last paragraph, the Ricci tensor $R_{\alpha\beta}$ has now the unique derivative level 2, just like due to definition (1.17) also the Einstein tensor $G_{\alpha\beta}$ on the left hand side of equation (2.22). Let us next look at the right hand side of that equation. Based on definition (1.7), the covariant derivative has the derivative level 1. It is then natural to demand the same unique derivative level as on the left hand side. That way, we conclude that the electromagnetic vector potential A_{α} must have the derivative level 1. Note that this does not mean that there is a partial derivative hidden in the electromagnetic vector potential A_{α}, like for the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. The derivative level is just an instrument to better understand the mathematical structure of electrogravitation. So, the derivative level of the field of electromagnetism is one higher than that of the field of gravitation. This behavior is not restricted to the fields but also visible for the field equations. Knowing that equation (2.23) is the result of applying the covariant derivative ∇^{β} on equation (2.22), we see that in our nonlinear wave operator picture the electromagnetic field equation (2.23) is at a one higher derivative level than the gravitational field equation (2.22). Why does electromagnetism have a derivative level which is one higher than that of gravitation?

When looking at the possibilities where Riemannian geometry can be generalized, the only explanation seems to be that the electromagnetic vector potential A_{α} does not influence vacuum general relativity directly at the derivative level 0, i.e. at the metric $g_{\alpha\beta}$. Instead, the influence occurs at the derivative level 1, which makes us look for a generalized, non-Riemannian connection. We refer to this unknown, new quantity with the symbol $I^{\alpha}_{\beta\gamma}$ and call it **electrogravitational connection**. The new connection has to reduce to the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ for vacuum gravitation. So, we can split it into

$$\boxed{I^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + L^{\alpha}_{\beta\gamma}} \tag{2.24}$$

where the tensor $L^{\alpha}_{\beta\gamma}$, which we denote as **electrogravitational deviation**, must vanish for $A_{\alpha} = 0$. A side effect of this is that each term of the electrogravitational deviation $L^{\alpha}_{\beta\gamma}$ must at least contain the electromagnetic vector potential A_{α} once. On the other hand, the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ and thus also the electrogravitational deviation

$L^\alpha_{\beta\gamma}$ should have the same derivative level as the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, i.e. 1. That way, in each term of the electrogravitational deviation $L^\alpha_{\beta\gamma}$, the electromagnetic vector potential A_α must appear exactly once. All other quantities possible in the terms of the electrogravitational deviation $L^\alpha_{\beta\gamma}$ must thus have the derivative level 0. So, the easiest case is the metric $g_{\alpha\beta}$ with the derivative level 0, where also an index may be risen such that it becomes the Kronecker tensor δ^β_α. The only alternative is the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$. Due to equation (2.8), we know that a product of two such tensors is composed of Kronecker tensors δ^β_α, which have the derivative level 0. Therefore, we conclude that also the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$ has that derivative level and is thus allowed in the following. However, it may appear only once per term, because otherwise it could be rewritten as Kronecker tensors. The above limitations then allow us to make the ansatz

$$L^\alpha_{\beta\gamma} = a_2\delta^\alpha_\beta A_\gamma + b_2\delta^\alpha_\gamma A_\beta + c_2 A^\alpha g_{\beta\gamma} + d_2\delta^4_D \epsilon^\alpha{}_{\beta\gamma\delta}A^\delta + \cdots \tag{2.25}$$

with $a_2, \ldots, d_2 \in \mathbb{R}$ being constants. The dots have the same meaning as in ansatz (2.9) and represent the Levi-Civita terms for $D \neq 4$.

2.3.2 Electrogravitational derivative

To get the values for the constants of the electrogravitational deviation (2.25) at least for $D = 4$, we have to find out the field equation belonging to it and demand that it has the form (2.21). For that purpose, we first generalize the covariant derivative (1.7) to the **electrogravitational derivative**

$$\boxed{\Delta_\gamma T^{\beta\ldots}_{\alpha\ldots} = \partial_\gamma T^{\beta\ldots}_{\alpha\ldots} - I^\delta_{\alpha\gamma}T^{\beta\ldots}_{\delta\ldots} - \cdots + I^\beta_{\delta\gamma}T^{\delta\ldots}_{\alpha\ldots} + \cdots - W I^\delta_{\delta\gamma}T^{\beta\ldots}_{\alpha\ldots}} \tag{2.26}$$

Using the split (2.24), it is also possible to write this derivative in the manifestly covariant form

$$\boxed{\Delta_\gamma T^{\beta\ldots}_{\alpha\ldots} = \nabla_\gamma T^{\beta\ldots}_{\alpha\ldots} - L^\delta_{\alpha\gamma}T^{\beta\ldots}_{\delta\ldots} - \cdots + L^\beta_{\delta\gamma}T^{\delta\ldots}_{\alpha\ldots} + \cdots - W L^\delta_{\delta\gamma}T^{\beta\ldots}_{\alpha\ldots}} \tag{2.27}$$

Before we continue, it is necessary to clarify the notation used here. So, we have two derivatives, which we call covariant derivative ∇_γ and electrogravitational derivative Δ_γ. The difference is the connection underlying them. The covariant derivative ∇_γ is based on the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ and the electrogravitational one Δ_γ on the electrogravitational connection $I^\alpha_{\beta\gamma}$. As this is the only difference, the electrogravitational derivative Δ_γ is, from the mathematician's viewpoint, strictly speaking also a covariant derivative. A more accurate notation would therefore be to call ∇_γ general relativistic covariant derivative and Δ_γ electrogravitational covariant derivative. However, this notation is cumbersome such that we do not adopt it. We will proceed in a similar manner for several other quantities, for instance, when we come to curvature in the next section.

In contrast to the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, where definitions (1.8) and (1.9) demand that there is no torsion and nonmetricity, these two geometric properties do not have to vanish for the electrogravitational connection $I^\alpha_{\beta\gamma}$. Instead, torsion and nonmetricity may be present. We can use our new definition (2.26) as well as the electrogravitational connection itself to define them as the expressions

$$\boxed{\begin{aligned} T^\alpha_{\beta\gamma} &= 2I^\alpha_{[\beta\gamma]} \\ \Delta_\gamma g_{\alpha\beta} &= \partial_\gamma g_{\alpha\beta} - 2I_{(\alpha\beta)\gamma} \end{aligned}} \tag{2.28}$$

(see equation (13.30) in Straumann 2004 for the definition of torsion and equation (41) of Goenner 2004 for the one of nonmetricity), where

$$I_{\alpha\beta\gamma} = g_{\alpha\delta} I^{\delta}_{\beta\gamma}$$

The round brackets around the indices α and β in box (2.28) denote a symmetrization. This is the analog of the antisymmetrization with squared brackets, except that the minus appearing therein is replaced by a plus. Also, note that the torsion tensor $T^{\alpha}_{\beta\gamma}$ is antisymmetric in the indices β and γ. Using the split (2.24) and equation (1.10), we can bring the two definitions (2.28) in the manifestly covariant form

$$\boxed{\begin{aligned} T^{\alpha}_{\beta\gamma} &= 2L^{\alpha}_{[\beta\gamma]} \\ \Delta_{\gamma} g_{\alpha\beta} &= -2L_{(\alpha\beta)\gamma} \end{aligned}} \tag{2.29}$$

with

$$L_{\alpha\beta\gamma} = g_{\alpha\delta} L^{\delta}_{\beta\gamma} \tag{2.30}$$

2.3.3 Electrogravitational curvature

Having introduced the electrogravitational derivative Δ_{α}, we now use equation (2.27) and investigate how this derivative commutes with itself. In analogy to definition (1.11) of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$, we therefore compute

$$\begin{aligned} \Delta_{[\gamma}\Delta_{\delta]} X_{\beta} &= \nabla_{[\gamma}\Delta_{\delta]} X_{\beta} - L^{\alpha}_{[\delta\gamma]}\Delta_{\alpha} X_{\beta} - L^{\alpha}_{\beta[\gamma}\Delta_{\delta]} X_{\alpha} \\ &= \nabla_{[\gamma}\left(\nabla_{\delta]} X_{\beta} - L^{\alpha}_{\beta\delta]} X_{\alpha}\right) - \frac{1}{2} T^{\alpha}_{\delta\gamma}\Delta_{\alpha} X_{\beta} - L^{\alpha}_{\beta[\gamma}\left(\nabla_{\delta]} X_{\alpha} - L^{\epsilon}_{\alpha\delta]} X_{\epsilon}\right) \end{aligned}$$

where we have used the first equation in box (2.29) to obtain the torsion tensor $T^{\alpha}_{\delta\gamma}$. Applying definition (1.11), we then arrive at

$$\Delta_{[\gamma}\Delta_{\delta]} X_{\beta} = -\left(\frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta} + \nabla_{[\gamma} L^{\alpha}_{\beta\delta]}\right) X_{\alpha} + \frac{1}{2} T^{\alpha}_{\gamma\delta}\Delta_{\alpha} X_{\beta} + L^{\alpha}_{\beta[\gamma} L^{\epsilon}_{\alpha\delta]} X_{\epsilon}$$

The term with $\Delta_{\alpha} X_{\beta}$ prevents us from writing the right hand side as the product of a tensor that is no operator and the vector X_{α}, like in definition (1.11). Therefore, we have to move the term with $\Delta_{\alpha} X_{\beta}$ on the left hand side:

$$\left(\Delta_{[\gamma}\Delta_{\delta]} - \frac{1}{2} T^{\alpha}_{\gamma\delta}\Delta_{\alpha}\right) X_{\beta} = -\left(\frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta} + \nabla_{[\gamma} L^{\alpha}_{\beta\delta]} - L^{\epsilon}_{\beta[\gamma} L^{\alpha}_{\epsilon\delta]}\right) X_{\alpha} \tag{2.31}$$

This shows us that it is reasonable to generalize the definition (1.11) of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ to

$$\boxed{Z^{\alpha}{}_{\beta\gamma\delta} X_{\alpha} = \left(T^{\alpha}_{\gamma\delta}\Delta_{\alpha} - [\Delta_{\gamma}, \Delta_{\delta}]\right) X_{\beta}} \tag{2.32}$$

where $Z^{\alpha}{}_{\beta\gamma\delta}$ shall be called **electrogravitational curvature**. As the vector X_{α} is assumed to be arbitrary, equation (2.31) allows us also to write

$$\boxed{Z^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma} L^{\alpha}_{\beta\delta]} + L^{\alpha}_{\epsilon[\gamma} L^{\epsilon}_{\beta\delta]}\right)} \tag{2.33}$$

So, the electrogravitational curvature is the Riemann tensor plus a correction coming from electromagnetism.

2.3.4 Geometric form of field equation

Next, we generalize the Ricci tensor $R_{\alpha\beta}$ defined in equation (1.14) to the **electrogravitation tensor**

$$\boxed{Z_{\alpha\beta} = Z^{\gamma}{}_{\alpha\gamma\beta}} \tag{2.34}$$

The right hand side can be evaluated with the help of equations (2.33) and (1.14):

$$\boxed{Z_{\alpha\beta} = R_{\alpha\beta} + 2\left(\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} + L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]}\right)} \tag{2.35}$$

Hence, demanding that the electrogravitational field equation can be written in the compact, geometric form

$$\boxed{Z_{\alpha\beta} = 0} \tag{2.36}$$

we find

$$R_{\alpha\beta} = -2\left(\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} + L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]}\right) \tag{2.37}$$

The form (2.36) emphasizes that electrogravitation is based on a unified field equation despite the lack of a unified tensor field.

For $D = 4$, equation (2.37) should now produce the electrogravitational field equation (2.21), which can be brought in a similar form. For that purpose, we first take the trace of equation (2.21):

$$G \overset{D=4}{=} -6\left(\nabla_{\alpha}A^{\alpha} + A^2\right)$$

Then, equation (2.17) allows us to rewrite equation (2.21) to

$$R_{\alpha\beta} \overset{D=4}{=} \nabla_{\alpha}A_{\beta} + \nabla_{\beta}A_{\alpha} + g_{\alpha\beta}\nabla_{\gamma}A^{\gamma} + 2\left(g_{\alpha\beta}A^2 - A_{\alpha}A_{\beta}\right) \tag{2.38}$$

Comparing the right hand side with the one of equation (2.37), we can evaluate the constants $a_2, ..., d_2$ in ansatz (2.25) for $D = 4$.

2.3.5 Electrogravitational deviation

To evaluate the constants in ansatz (2.25), we first take that ansatz and compute

$$\begin{aligned} -2\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} \quad &= \quad -\nabla_{\gamma}\left(a_2\delta^{\gamma}_{\alpha}A_{\beta} + b_2\delta^{\gamma}_{\beta}A_{\alpha} + c_2A^{\gamma}g_{\alpha\beta} + d_2\delta^4_D\epsilon^{\gamma}{}_{\alpha\beta\delta}A^{\delta}\right) \\ &\quad +\nabla_{\beta}\left(a_2\delta^{\gamma}_{\alpha}A_{\gamma} + b_2\delta^{\gamma}_{\gamma}A_{\alpha} + c_2A^{\gamma}g_{\alpha\gamma} + d_2\delta^4_D\epsilon^{\gamma}{}_{\alpha\gamma\delta}A^{\delta}\right) + \cdots \end{aligned}$$

Recalling the total antisymmetry of the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$, we can continue with

$$-2\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} = -a_2\nabla_{\alpha}A_{\beta} + \left[a_2 + (D-1)\, b_2 + c_2\right]\nabla_{\beta}A_{\alpha} - c_2g_{\alpha\beta}\nabla_{\gamma}A^{\gamma} - d_2\delta^4_D\epsilon^{\gamma}{}_{\alpha\beta\delta}\nabla_{\gamma}A^{\delta} + \cdots \tag{2.39}$$

The right hand side must give the first three terms in equation (2.38), because the linearity of our ansatz (2.25) in the electromagnetic vector potential A_{α} makes all terms coming from the second term in the round bracket of equation (2.37) quadratic in the potential A_{α}. That way, we find $d_2 = 0$ such that the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$ does not only play no role for the electrogravitational field equation but also not for the geometric interpretation of electromagnetism. We also obtain $a_2 = c_2 = -1$ and $b_2 = 1$ for $D = 4$. Hence, our ansatz (2.25) gives the electrogravitational deviation

$$L^{\alpha}_{\beta\gamma} \overset{D=4}{=} -\delta^{\alpha}_{\beta}A_{\gamma} + \delta^{\alpha}_{\gamma}A_{\beta} - A^{\alpha}g_{\beta\gamma} \tag{2.40}$$

In case of $D \neq 4$, we have to be careful. This issue is investigated separately in the next section.

Let us eventually show that in fact the whole right hand side of equation (2.38) originates from the electrogravitational deviation (2.40). For that purpose, we generally (up to $d_2 = 0$ and the dots in ansatz (2.25)) compute

$$\begin{aligned} & \left(a_2 \delta^\alpha_\epsilon A_{[\gamma} + b_2 \delta^\alpha_{[\gamma} A_\epsilon + c_2 A^\alpha g_{\epsilon[\gamma}\right) \left(a_2 \delta^\epsilon_\beta A_{\delta]} + b_2 \delta^\epsilon_{\delta]} A_\beta + c_2 A^\epsilon g_{\beta\delta]}\right) \\ = \; & b_2^2 \delta^\alpha_{[\gamma} A_{\delta]} A_\beta + b_2 c_2 \delta^\alpha_{[\gamma} g_{\beta\delta]} A^2 + c_2^2 g_{\beta[\delta} A_{\gamma]} A^\alpha \end{aligned} \tag{2.41}$$

This relation will also be useful later in the paper. Then, ansatz (2.25) shows that

$$L^\gamma_{\delta[\gamma} L^\delta_{\alpha\beta]} = b_2^2 \delta^\gamma_{[\gamma} A_{\beta]} A_\alpha + b_2 c_2 \delta^\gamma_{[\gamma} g_{\alpha\beta]} A^2 + c_2^2 g_{\alpha[\beta} A_{\gamma]} A^\gamma$$

and hence

$$-2 L^\gamma_{\delta[\gamma} L^\delta_{\alpha\beta]} = -(D-1)\left(b_2^2 A_\alpha A_\beta + b_2 c_2 g_{\alpha\beta} A^2\right) + c_2^2 \left(A_\alpha A_\beta - g_{\alpha\beta} A^2\right) \tag{2.42}$$

such that in four dimensions

$$-2 L^\gamma_{\delta[\gamma} L^\delta_{\alpha\beta]} \overset{D=4}{=} 2\left(g_{\alpha\beta} A^2 - A_\alpha A_\beta\right)$$

This is the term with the round bracket in equation (2.38). That way, we clearly see that electromagnetism can truly be interpreted as a geometric effect of spacetime for $D = 4$. We will investigate this issue much more thoroughly after the next section.

2.3.6 Derivation of dimensional dependence

Now, we come to the electrogravitational deviation $L^\alpha_{\beta\gamma}$ for $D \neq 4$. Readers not interested in the derivation of the dimensional dependence of electrogravitation can skip this section or just look at the results in the two boxes at the end. It is, however, reasonable to derive that dependence immediately here, because otherwise we will later either have lots of equations which hold only in four dimensions or we will have to address the cases $D = 4$ and $D \neq 4$ separately.

To derive the dimensional dependence, we look at equation (2.42), which shows us that the second term in the round bracket of equation (2.37) is symmetric in the indices α and β, just like the Ricci tensor $R_{\alpha\beta}$ on the left hand side of that equation. That way, we see that also the first term in the round bracket must be symmetric and thus the right hand side of equation (2.39). This symmetry causes

$$0 = -a_2 \nabla_{[\alpha} A_{\beta]} + \left[a_2 + (D-1) b_2 + c_2\right] \nabla_{[\beta} A_{\alpha]} = \frac{1}{2}\left[2a_2 + (D-1) b_2 + c_2\right] F_{\beta\alpha} \tag{2.43}$$

In case of $D \neq 4$, we cannot reuse the values $a_2 = c_2 = -1$ and $b_2 = 1$ obtained in the last section for $D = 4$, because the squared bracket at the end of equation (2.43) would not be zero. Then, the electromagnetic field strength $F_{\alpha\beta}$ would have to vanish, which means that there would be no electromagnetism. Neglecting electromagnetism for $D \neq 4$ is the easiest way to generalize to an arbitrary number of dimensions, but our goal is to still have electromagnetism and stay as close as possible to the case $D = 4$. However that way looks like, it will incorporate the case without electromagnetism as a limit and will thus be more general.

So, the second squared bracket in relation (2.43) has to vanish. This can be achieved by properly setting one of the constants a_2, b_2, c_2. Independent of that, the units of the electromagnetic vector potential A_α in ansatz (2.25) can be chosen freely for $D \neq 4$. For $D = 4$, these units were by the way fixed using the choice $c_{\text{EM}} = 1$ in equation (2.21). And, we decide for units such that the value $b_2 = 1$ of $D = 4$ also holds for $D \neq 4$. Then, the second squared bracket in relation (2.43) is zero for $c_2 = 1 - D - 2a_2$. Now, we merely have to determine a reasonable value for the constant a_2. To this end, we insert equation (2.39) together with ansatz (2.25) in equation (2.37) and neglect all terms which are quadratic in the electromagnetic vector potential A_α:

$$R_{\alpha\beta} = -2a_2 \nabla_{(\alpha} A_{\beta)} + (2a_2 + D - 1)\, g_{\alpha\beta} \nabla_\gamma A^\gamma + \mathcal{O}\left(A_\alpha\right)^2$$

Then,

$$R = (D-1)(D + 2a_2) \nabla_\alpha A^\alpha + \mathcal{O}\left(A_\alpha\right)^2$$

such that definition (1.17) gives

$$G_{\alpha\beta} = -2a_2 \nabla_{(\alpha} A_{\beta)} + \left[2a_2 + (D-1)(1 - \frac{D}{2} - a_2)\right] g_{\alpha\beta} \nabla_\gamma A^\gamma + \mathcal{O}\left(A_\alpha\right)^2$$

Next, we go back to equation (2.7) such that ansatz (2.9) shows $a = -a_2$ and $c = 2a_2 + (D-1)(1 - D/2 - a_2)$. We recall that we were not able to derive Maxwell's equation (1.15) for $D \neq 4$ in Section 2.2.2. The obstacle there were the terms after the squared bracket in equation (2.14), which are the only ones that are not linear in the electromagnetic vector potential A_α. However, we have found that at least the term with the squared bracket alone leads to Maxwell's equation if $c = -2a$, irrespective of whether we have $D = 4$ or $D \neq 4$. We now demand that this condition has to hold for $D \neq 4$, too. That way, we will surely not exactly get Maxwell's equation (1.15) for $D \neq 4$, but we get it at least up to a correction which is of second or higher order in the electromagnetic vector potential A_α. The careful reader may oppose that this approach is ambiguous. It surely appears to be natural, but there might be alternatives. Further below in the paper, we will, however, demonstrate that our choice here is the only reasonable one. To eventually apply it, we recall our above conditions for a and c such that the constraint $c = -2a$ gives

$$(D-1)(1 - D/2 - a_2) = 0 \tag{2.44}$$

For $D = 1$, this condition is automatically obeyed, and we do not have a constraint on the constant a_2. The reason is that there can anyway be no electromagnetism in that case due to the antisymmetry present in the electromagnetic field strength $F_{\alpha\beta}$. To not have to treat this case separately, we assume that the result $a_2 = 1 - D/2$ of equation (2.44) for $D > 1$ also holds for $D = 1$. Going then back to the beginning of this paragraph, we see that the resulting constant $c_2 = -1$ has the same value as for $D = 4$. We also recall the value $b_2 = 1$ chosen above such that ansatz (2.25) is completely determined. So, the electrogravitational deviation (2.40) generalizes to an arbitrary number of dimensions in the following way:

$$\boxed{L^\alpha_{\beta\gamma} = -\left(\frac{D}{2} - 1\right) \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma}} \tag{2.45}$$

Then, equations (1.4) and (2.24) allow us to write the electrogravitational connection as

$$\boxed{I^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right) - \left(\frac{D}{2} - 1\right) \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma}} \tag{2.46}$$

This leads to equation (2.38) and that way to the electrogravitational field equation (2.21) for $D = 4$. We already know that the field equation has to somehow look differently for $D \neq 4$, but we will address that interesting issue later in detail.

2.3.7 Interpretation

In order to finally determine the exact geometric meaning of electromagnetism, we insert the electrogravitational deviation (2.45) in box (2.29), which gives

$$T^{\alpha}_{\beta\gamma} = 2\left[-\left(\frac{D}{2} - 1\right)\delta^{\alpha}_{[\beta}A_{\gamma]} + \delta^{\alpha}_{[\gamma}A_{\beta]}\right]$$

and

$$\Delta_\gamma g_{\alpha\beta} = -2\left[-\left(\frac{D}{2} - 1\right)g_{\alpha\beta}A_\gamma + g_{(\alpha\gamma}A_{\beta)} - A_{(\alpha}g_{\beta)\gamma}\right] \tag{2.47}$$

such that we get the **geometric interpretation of electromagnetism**:

$$\boxed{\begin{aligned} T^{\alpha}_{\beta\gamma} &= D\delta^{\alpha}_{[\gamma}A_{\beta]} \\ \Delta_\gamma g_{\alpha\beta} &= (D-2)\,A_\gamma g_{\alpha\beta} \end{aligned}} \tag{2.48}$$

This outcome tells us that electromagnetism is the result of a special combination of the torsion and nonmetricity of spacetime. Yet, not all degrees of freedom available for torsion and nonmetricity are used for electromagnetism. This is similar to gravitation, which is not the result of the full curvature of spacetime but only the interior one. The exterior curvature does not seem to play a role in physics. In analogy to curvature, we now split both torsion and nonmetricity into interior and exterior constituents. The **interior torsion and nonmetricity** are given by box (2.48), and the **exterior torsion and nonmetricity** are all degrees of freedom that are prohibited by these two equations.

Knowing that electromagnetism is caused by the torsion and nonmetricity in box (2.48), we eventually verify that this is the only cause. For that purpose, we take equations (2.29), where the deviation $L^{\alpha}_{\beta\gamma}$ is assumed to not necessarily be the electrogravitational one (2.45) for the rest of this paragraph. And, we evaluate

$$\begin{aligned} \Delta_\alpha g_{\beta\gamma} - \Delta_\beta g_{\alpha\gamma} - \Delta_\gamma g_{\beta\alpha} &= -L_{\beta\gamma\alpha} - L_{\gamma\beta\alpha} + L_{\alpha\gamma\beta} + L_{\gamma\alpha\beta} + L_{\beta\alpha\gamma} + L_{\alpha\beta\gamma} \\ &= -T_{\gamma\beta\alpha} - T_{\beta\gamma\alpha} + T_{\alpha\gamma\beta} + 2L_{\alpha\beta\gamma} \end{aligned}$$

where we have used the components

$$T_{\alpha\beta\gamma} = g_{\alpha\delta}T^{\delta}_{\beta\gamma}$$

as well as (2.30). Then,

$$\boxed{L_{\alpha\beta\gamma} = \frac{1}{2}\left(\Delta_\alpha g_{\beta\gamma} - \Delta_\beta g_{\alpha\gamma} - \Delta_\gamma g_{\alpha\beta} + T_{\alpha\beta\gamma} - T_{\beta\alpha\gamma} - T_{\gamma\alpha\beta}\right)} \tag{2.49}$$

which tells us that we can unambiguously compute the deviation $L_{\alpha\beta\gamma}$ from the torsion $T^{\alpha}_{\beta\gamma}$ and nonmetricity $\Delta_\gamma g_{\alpha\beta}$. Therefore, these two geometric features of spacetime must be the only cause of electromagnetism. This also cures an apparent philosophical shortcoming of our theory that we have neglected all the time. Our theory is based on a unified field equation but not on a unified tensor field. The electromagnetic vector potential A_α and the metric $g_{\alpha\beta}$ are still separated fields. However, this is no longer a philosophical obstacle, because both fields have a geometric fundament now such that actually **geometry itself is the unified field**. After this clarification, let us finally interpret in words:[1]

Geometric interpretation of electromagnetism:

Electromagnetism is the interior torsion and nonmetricity of spacetime. (2.50)

2.3.8 Illustration

The two boxes (2.48) and (2.50) explain the geometric meaning of electromagnetism in the language of mathematics and in common words. In this section, we also want to give a visual illustration of the encountered new geometry by looking at a simple example. The example is the parallel transport of an arbitrary vector X^α.

However, we have to first of all perform some mathematical computations. The parallel transport itself is given by

$$\Delta_\beta X^\alpha = 0 \tag{2.51}$$

such that definition (2.26) and the split (2.24) imply

$$\partial_\beta X^\alpha = -\left(\Gamma^\alpha_{\gamma\beta} + L^\alpha_{\gamma\beta}\right) X^\gamma$$

To simplify the illustration as strong as possible, it is best to get rid of the Christoffel symbols $\Gamma^\alpha_{\gamma\beta}$. For that purpose, we assume that spacetime is flat and the coordinates are Cartesian ones for the rest of this section. This means

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

where $\eta_{\alpha\beta} = \operatorname{diag}(-1, 1, 1, ...)$ denotes the Minkowski metric. The Christoffel symbols (1.4) then disappear and definition (2.45) yields

$$u^\beta \partial_\beta X^\alpha = \left(\frac{D}{2} - 1\right) A_\beta u^\beta X^\alpha - A_\beta X^\beta u^\alpha + X_\beta u^\beta A^\alpha \tag{2.52}$$

The vector u^α denotes the direction of the parallel transport here.

We do not have to consider all possible parallel transports to understand the meaning of equation (2.52). For a qualitative understanding, it is sufficient to adopt two restrictions. The first one is to look not at the whole spacetime but only at the vicinity of an arbitrary point x^α_{P}. On the point itself, we have the vectors $X^\alpha(x^\alpha_{\mathrm{P}})$ and $u^\alpha(x^\alpha_{\mathrm{P}})$. The second restriction is to normalize these two vectors and pick them only parallel or orthogonal to the electromagnetic vector potential $A^\alpha(x^\alpha_{\mathrm{P}})$. Let us therefore introduce the normalized direction vector

$$\hat{A}^\alpha = \frac{A^\alpha(x^\alpha_{\mathrm{P}})}{l} \tag{2.53}$$

along the electromagnetic vector potential $A^\alpha(x^\alpha_{\mathrm{P}})$, with $l = \sqrt{|A_\alpha(x^\alpha_{\mathrm{P}})\, A^\alpha(x^\alpha_{\mathrm{P}})|} \geq 0$ in this section. Note that

$$\hat{A}_\alpha \hat{A}^\alpha = \sigma_A \tag{2.54}$$

[1] Everyday speech and translation hint: Torsion and nonmetricity are mathematical concepts. In German, they are “Torsion” and “Nichtmetrizität”. While torsion is known in everyday speech as twisting, or into German “Verdrillung”, nonmetricity is unknown there. Hence, I recommend to use the notion nesting, or in German “Verschachtelung”.

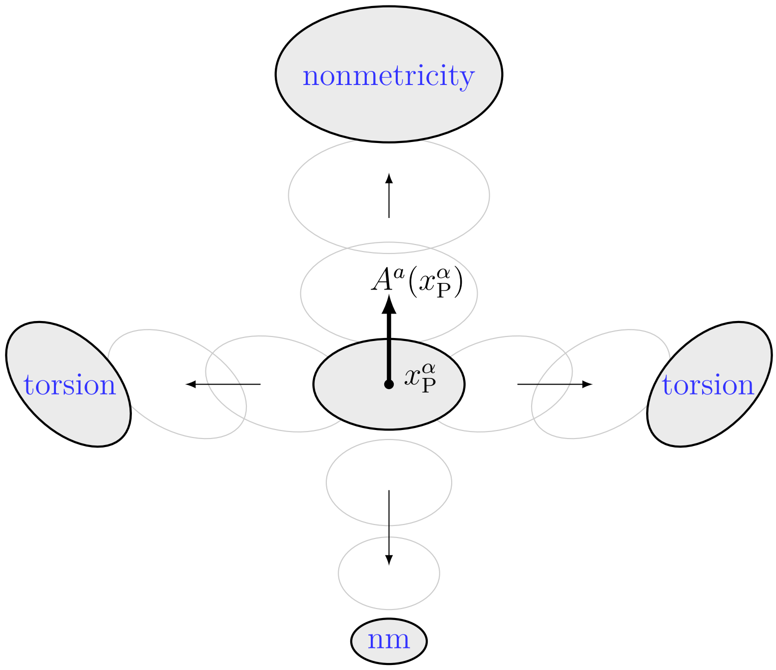

Figure 1: **Geometry of electromagnetic wave**: The drawing illustrates the torsion and nonmetricity of an electromagnetic wave, see the end of Section 2.3.8 for more details. The 2-dimensional visualization works as follows. We are interested only in the geometry around a single point x^α_P of spacetime. There, we have the spacelike (due to Weyl gauge) electromagnetic vector potential $A^\alpha\left(x^\alpha_\mathrm{P}\right)$. The figure plane represents an arbitrary slice through 3-dimensional space such that the polarization vector $A^a\left(x^\alpha_\mathrm{P}\right)$ points upwards as shown above. The choice of the slice does not matter because the geometry of the electromagnetic wave is locally axisymmetric around the vector $A^a\left(x^\alpha_\mathrm{P}\right)$. The geometry can then be understood by performing parallel transports in different directions. This is visualized best by studying how the ellipse in the center changes (see Section 2.3.8 for the explanation). Torsion causes a rotation of it when moving left or right, whereas nonmetricity (=nm) leads to a pure size change when going up or down.

where $\sigma_A = -1$ for timelike and $\sigma_A = 1$ for spacelike vectors $\hat{A}^\alpha$. If we now transport along the electromagnetic vector potential, we have to choose $u^\alpha\left(x^\alpha_\mathrm{P}\right) = \hat{A}^\alpha$ in equation (2.52):

$$\hat{A}^\beta\left(\partial_\beta X^\alpha\right)\left(x^\alpha_\mathrm{P}\right) = \left(\frac{D}{2} - 1\right)\sigma_A l X^\alpha\left(x^\alpha_\mathrm{P}\right)$$

This means that the direction of the vector X^α does not change. However, this vector becomes shorter / longer for timelike / spacelike electromagnetic vector potentials $A^\alpha\left(x^\alpha_\mathrm{P}\right)$ (for $D > 2$). The size change is an immediate consequence of the nonmetricity of spacetime.

On the other hand, let $\hat{B}^\alpha$ be an arbitrary normal vector orthogonal to the electromagnetic vector potential $A^\alpha\left(x^\alpha_\mathrm{P}\right)$, i.e.

$$\hat{B}_\alpha\hat{A}^\alpha = 0 \qquad \text{and} \qquad \hat{B}_\alpha\hat{B}^\alpha = \sigma_B \tag{2.55}$$

with $\sigma_B \in \{-1, 1\}$. Then, equations (2.52) and (2.53) yield for $u^\alpha\left(x^\alpha_\mathrm{P}\right) = \hat{B}^\alpha$:

$$\hat{B}^\beta\left(\partial_\beta X^\alpha\right)\left(x^\alpha_\mathrm{P}\right) = l\left(X_\beta\left(x^\alpha_\mathrm{P}\right)\hat{B}^\beta\hat{A}^\alpha - \hat{A}_\beta X^\beta\left(x^\alpha_\mathrm{P}\right)\hat{B}^\alpha\right) \tag{2.56}$$

The right hand side simplifies if we set the vector $X^\alpha(x_\mathrm{P}^\alpha)$ parallel or orthogonal to the electromagnetic vector potential $A^\alpha(x_\mathrm{P}^\alpha)$. Due to equations (2.54) and (2.55), we find that for $X^\alpha(x_\mathrm{P}^\alpha) = \hat{A}^\alpha$

$$\hat{B}^\beta\left(\partial_\beta X^\alpha\right)(x_\mathrm{P}^\alpha) = -l\sigma_A \hat{B}^\alpha \tag{2.57}$$

whereas $X^\alpha(x_\mathrm{P}^\alpha) = \hat{B}^\alpha$ leads to

$$\hat{B}^\beta\left(\partial_\beta X^\alpha\right)(x_\mathrm{P}^\alpha) = l\sigma_B \hat{A}^\alpha \tag{2.58}$$

The easiest possibility is now that the vector $X^\alpha(x_\mathrm{P}^\alpha)$ and the directions $\hat{A}^\alpha$ and $\hat{B}^\alpha$ are only spacelike. Then $\sigma_A = \sigma_B = 1$, which yields opposite signs on the right hand side of equations (2.57) and (2.58). This shows us that the vector X^α is spatially rotated. The rotation is actually the result of the torsion of spacetime[2]. Note that if we allow timelike vectors, i.e. σ_A and σ_B can have opposite signs, then the change of X^α is harder to imagine. We therefore neglect this case here.

So, the electromagnetic vector potential A^α has to be spacelike for an easy illustration. This is, for instance, the case if we consider an electromagnetic wave in the Weyl gauge, for which $A^t = 0$. The spatial components A^a are by the way known to give the polarization of the wave. We can then illustrate the geometric interpretation of electromagnetism with the drawing in Figure 1. Note that we do not visualize the full parallel transport of the vector X^α there, because there are infinitely many possibilities to choose the vector X^α. For an illustration of the parallel transport, it is, however, sufficient to consider a subset of the possible values of the vector X^α. We have decided for a set of vector values whose vector heads form an ellipse. Why an ellipse? We have to just choose any shape which is not invariant under rotations and size changes. A circle would, for instance, not be able to illustrate torsion. The ellipse is then also not transported along all possible directions in Figure 1 but only horizontally and vertically. What about the propagation direction k^a of the electromagnetic wave in that figure? From electrodynamics, this direction is known to be orthogonal to the polarization vector A^a. Figure 1 holds independent of how we choose the propagation k^a. We can set it, for example, perpendicular to the figure plane or horizontally lying inside of that plane.

2.3.9 Outlook

In Section 2.3.7, we have discovered a geometric interpretation of electromagnetism. We can surely not answer whether that interpretation is actually realized in nature. This is not the task of theoretical physics. For that purpose, observations or experiments have to be performed. And, it is at least necessary to falsify Einstein-Maxwell theory, which is based on the field equations (1.15) and (1.20). We also do not know whether our interpretation is the only reasonable one. However, it is at least astonishing that a working geometric interpretation of electromagnetism is mathematically thinkable at all.

So, we know that gravitation is a geometric phenomenon. For electromagnetism, it is at least possible to construct a well-defined theory which interprets it in a geometric

[2] The right hand side of equation (2.58) comes from the last term of equation (2.52) and thus the last one of relation (2.45). However, this term is symmetric in the indices β, γ and should thus not contribute to torsion. Yet, in Figure 1, it does. This is a discrepancy between the mathematical object torsion, described by the torsion tensor $T^\alpha_{\beta\gamma}$, and the illustrated one. Using the covariant vector X_α instead of the contravariant one X^α in equation (2.51) does also not help here, because box (2.48) yields $g^{\alpha\gamma}\Delta_\beta X_\gamma = g^{\alpha\gamma}\Delta_\beta\left(g_{\gamma\delta}X^\delta\right) = \Delta_\beta X^\alpha + (D-2)\,X^\alpha A_\beta$, where the second term vanishes if we multiply by $u^\beta(x_\mathrm{P}^\alpha) = \hat{B}^\beta$.

manner. It is therefore tempting to assume that all known forces of nature, i.e. also the weak and strong force, have a geometric origin, as well as matter itself. If this assumption were true, then the universe would be nothing else than just a spacetime. I.e., it would be empty and not filled with non-geometric constituents. This is an old idea of physics and not necessarily valid. However, it is a very short and easily understandable way to summarize the basic idea behind world theory. Let us therefore at least give a name to that idea such that we can easily refer to it:

World principle:

The universe is a spacetime. (2.59)

A consequence of the world principle is by the way that the electroweak unification breaks up. This does not mean that the mathematical structure of the electroweak theory is wrong. However, we have now a significant asymmetry in that theory. On the one hand, there is the electromagnetic part, for which electrogravitation provides a geometric interpretation. On the other hand, we have the weak part. We do not know a geometric interpretation for it, but the world principle demands such an interpretation. This discrepancy makes it no longer reasonable to consider the "electroweak unification" as a true unification. Instead, it is just a step in that direction. The true unification must then be a deeper understanding of the electroweak theory, where a geometric understanding of the weak force, in whatever manner, is included. Of course, all this is meaningful only if the world principle holds. Unfortunately, we do not know whether this is the case.

2.4 Weyl theory

2.4.1 Geometry

After having discovered the electrogravitational field equation in Section 2.2 and demonstrated that electrogravitation is a geometric unification of electromagnetism and gravitation in Section 2.3, it is now time to have a look at previous unification attempts in literature. These attempts are usually referred to as **unified field theories**. Also, electrogravitation is a unified field theory (as well as world theory). However, we have to be careful with the notation. The term unified field theory was coined in an era when only electromagnetism and gravitation were known. At that time, it was even possible to speak of "the unified field theory", although it was never clear how such a theory should truly look like in detail. However, nowadays we also know the weak and strong force. The notion unified field theory is therefore ambiguous, because it does not specify which fields are unified. This explains why we explicitly call our unified field theory electrogravitation. By the way, we do not give an overview of the previous unification trials in this paper. It is not always easy to see why these various approaches fail, and a thorough review is of historic interest only (a historic overview can be found in Goenner 2004 and Goenner 2014). However, there is one attempt worth to be mentioned, namely the one conducted by Weyl in the years after the discovery of general relativity (see Weyl 1918).

The reason why we want to pay a visit to Weyl theory is that from all previous unification trials it is the one closest to our theory (the conformal geometrodynamics of Section 3.4 will contain an additional scalar field). For that purpose, we look at the original paper Weyl (1918), where equation (5) clarifies that Weyl theory is based on a symmetric connection, i.e. the torsion tensor ${}^{\mathrm{W}}T^{\alpha}_{\beta\gamma}$ of that theory vanishes. However, the equation following relation (7) in the paper shows us that Weyl's connection uses our nonmetricity in box (2.48) up to the factor $D-2$, which is only a trivial rescaling of the electromagnetic vector potential A_α. So, writing Weyl's nonmetricity as ${}^{\mathrm{W}}\Delta_\gamma g_{\alpha\beta}$, the geometric difference to our theory (2.48) can be summarized as

$$\begin{aligned} {}^{\mathrm{W}}T^\alpha_{\beta\gamma} &= 0 \\ {}^{\mathrm{W}}\Delta_\gamma g_{\alpha\beta} &= -2A_\gamma g_{\alpha\beta} \end{aligned} \tag{2.60}$$

where in contrast to Weyl's original paper we have included a factor -2 here to have a better comparison with our theory. At first glance, it seems as if the factor $D-2$ of box (2.48) would be the more appropriate choice for such a comparison. However, in that case, equation (2.49) shows that we would get a deviation ${}^{\mathrm{W}}L^\alpha_{\beta\gamma}$ for Weyl theory which has only the first term of our electrogravitational deviation (2.45) in common. For the nonmetricity (2.60), Weyl's version of the electrogravitational deviation has instead the form

$${}^{\mathrm{W}}L^\alpha_{\beta\gamma} = \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma} \tag{2.61}$$

such that two terms agree, the last two ones. The difference to our theory for $D=4$ is then even more marginal. Looking at our electrogravitational deviation (2.40), we see that there is just a sign difference in front of the first term $\delta^\alpha_\beta A_\gamma$. A mere typo could have easily brought Weyl to our theory of electrogravitation. Yet, the missing torsion and also a far too cumbersome, not unified field equation (16) in Weyl (1918), derived from a Lagrangian, prohibited Weyl from getting to our theory.

2.4.2 Field equation

In the last section, we have mentioned that Weyl derived his field equation (16) in Weyl (1918) from a Lagrangian. This leads to the question why Weyl did not simply use our field equation (2.37) with his own deviation ${}^{\mathrm{W}}L^\alpha_{\beta\gamma}$ in place of our quantity $L^\alpha_{\beta\gamma}$? The field equation would then have the form

$$R_{\alpha\beta} = -2\left(\nabla_{[\gamma}{}^{\mathrm{W}}L^\gamma_{\alpha\beta]} + {}^{\mathrm{W}}L^\gamma_{\delta[\gamma}{}^{\mathrm{W}}L^\delta_{\alpha\beta]}\right) \tag{2.62}$$

To answer this question, we first look at the generally valid relation (2.42), i.e. this relation also holds if the quantity $L^\alpha_{\beta\gamma}$ appearing therein is replaced by Weyl's deviation ${}^{\mathrm{W}}L^\alpha_{\beta\gamma}$. Examining then the right hand side of that relation, we see that the expression ${}^{\mathrm{W}}L^\gamma_{\delta[\gamma}{}^{\mathrm{W}}L^\delta_{\alpha\beta]}$ in equation (2.62) is symmetric in the indices α and β. This is also the case for the Ricci tensor $R_{\alpha\beta}$. Thus, the expression $\nabla_{[\gamma}{}^{\mathrm{W}}L^\gamma_{\alpha\beta]}$ must be symmetric in those two indices. Let us now evaluate it using definition (2.61):

$$\begin{aligned} \nabla_{[\gamma}{}^{\mathrm{W}}L^\gamma_{\alpha\beta]} &= \nabla_{[\gamma}\left(\delta^\gamma_\alpha A_{\beta]} + \delta^\gamma_{\beta]}A_\alpha - A^\gamma g_{\alpha\beta]}\right) \\ &= \nabla_{[\alpha}A_{\beta]} + \frac{1}{2}\left[\nabla_\gamma\left(\delta^\gamma_\beta A_\alpha - A^\gamma g_{\alpha\beta}\right) - \nabla_\beta\left(\delta^\gamma_\gamma A_\alpha - A^\gamma g_{\alpha\gamma}\right)\right] \end{aligned}$$

The antisymmetric part has to vanish such that

$$0 = 4\nabla_{[\alpha}A_{\beta]} + 2\left(2-D\right)\nabla_{[\beta}A_{\alpha]} = DF_{\alpha\beta} \tag{2.63}$$

As $D \geq 1$, we clearly see the central problem of Weyl's theory. Equation (2.62) does not work for it, because it would lead to a disappearing electromagnetism. Therefore, Weyl had to go another way and approach his field equation (16) in Weyl (1918) from a Lagrangian, which, however, resulted in a theory that was eventually rejected.

2.4.3 Comparison

The central problem of equation (16) in Weyl (1918) is that it is a partial differential equation of fourth order. As Weyl himself realized in the second last paragraph of his paper (before Einstein's addendum), that field equation does not even contain Einstein's equation in the limit of vacuum gravitation (1.6). Einstein's equation is by the way second order in the partial derivatives. All in all, we can therefore summarize Weyl theory by the following comparison with electrogravitation:

$$\begin{array}{cc} \underline{\text{Weyl theory}} & \underline{\text{Electrogravitation}} \\ \text{no torsion} & \text{torsion} \\ \text{nonmetricity} & \text{nonmetricity} \\ \text{Lagrangian} & \text{no Lagrangian} \\ \text{4th order field equation} & \text{2nd order field equation} \end{array} \tag{2.64}$$

Aside of nonmetricity, Weyl's theory also has the underlying invariance in common with our theory of electrogravitation, which will be discussed in the following sections.

Weyl theory was not only studied by its discoverer, Weyl, but also by other authors, who have built on that theory. In two recent papers of such authors, I even came across the electrogravitation tensor $Z_{\alpha\beta}$. The electrogravitation tensor can be found in equation (15) of Kan *et al.* (2010) and equation (3.3) of Maki *et al.* (2012). However, the electrogravitation tensor is used there just as an intermediate quantity for other purposes and is not set to zero as in our equation (2.36). Anyway, the two papers are thus even closer to electrogravitation than Weyl's original work.

2.5 Invariance

2.5.1 Gauge invariance

We have already noted below equation (2.21) that the electrogravitational field equation appears to violate the gauge invariance

$$\begin{aligned} A_\alpha &\stackrel{\mathrm{g}}{\to} A_\alpha + \partial_\alpha \chi \\ g_{\alpha\beta} &\stackrel{\mathrm{g}}{\to} g_{\alpha\beta} \end{aligned} \tag{2.65}$$

known from electromagnetism, where $\chi \in \mathbb{R}$ is an arbitrary scalar field. We use the letter "g" on the arrows above to distinguish the transformation there from the ones encountered further below, which have their own unique letters. The left hand side of the electrogravitational field equation (2.21) is actually invariant under the above transformation. For the right hand side, we see that the two terms $-2A_\alpha A_\beta$ and $-g_{\alpha\beta}A^2$ are the only ones for which transformation (2.65) spews out expressions that are quadratic in the field χ. These quadratic expressions $-2\partial_\alpha\chi\partial_\beta\chi$ and $-g_{\alpha\beta}g^{\gamma\delta}\partial_\gamma\chi\partial_\delta\chi$ do not cancel. So, it is true that **electrogravitation violates gauge invariance** (later in this paper, in the 3rd last paragraph of Section 4.6.11, we will even find that it is possible to restore gauge invariance by using a so-called physical metric).

A violation of gauge invariance does not necessarily mean that electrogravitation is no gauge theory. We have to merely look for a proper modification of transformation (2.65) that keeps our field equation (2.21) invariant. Recalling that a general coordinate transformation changes both the electromagnetic vector potential A_α and the metric $g_{\alpha\beta}$, it would actually be odd that a gauge transformation should have an effect on only the electromagnetic vector potential A_α. So, we seek for an alternative version of transformation (2.65) which also manipulates the metric $g_{\alpha\beta}$ and thus accommodates the geometric unification electrogravitation is founded on.

2.5.2 Electrogravitational curvature via electrogravitational connection

To find out how transformation (2.65) has to be modified, we first look at the electrogravitational field equation in the geometric form (2.36), and write the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$, underlying it via definition (2.34), in a more unified manner than in equation (2.33). So, we do not want to express the electrogravitational curvature as the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ plus a correction based on the electrogravitational deviation $L^\alpha_{\beta\gamma}$. Instead, the electrogravitational curvature should be an expression which solely uses the electrogravitational connection $I^\alpha_{\beta\gamma}$. This is a goal that can be achieved easily. We merely have to redo the computations of Section 2.3.3 but this time via definition (2.26) and not the manifestly covariant one (2.27).

So, using the definition of the torsion tensor $T^\alpha_{\beta\gamma}$ in box (2.28), we evaluate

$$\begin{aligned}
\left(T^\alpha_{\gamma\delta}\Delta_\alpha - 2\Delta_{[\gamma}\Delta_{\delta]}\right) X_\beta &= -2\left(\partial_{[\gamma}\Delta_{\delta]}X_\beta - I^\alpha_{\beta[\gamma}\Delta_{\delta]}X_\alpha\right) \\
&= 2\left[\partial_{[\gamma}\left(I^\alpha_{\beta\delta]}X_\alpha\right) + I^\alpha_{\beta[\gamma}\left(\partial_{\delta]}X_\alpha - I^\epsilon_{\alpha\delta]}X_\epsilon\right)\right]
\end{aligned}$$

Then, definition (2.32) and the arbitrariness of the vector X_α gives the electrogravitational curvature in the form

$$\boxed{Z^\alpha{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma}I^\alpha_{\beta\delta]} + I^\alpha_{\epsilon[\gamma}I^\epsilon_{\beta\delta]}\right)} \tag{2.66}$$

This form is structurally equivalent to the definition (1.13) of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ and thus its natural generalization. It is also the reason why the quantity (2.66) is called Riemann tensor from the mathematician's viewpoint (see the previous comment on this issue in the 2nd paragraph of Section 2.3.2). However, writing the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$ in the general relativistic notation (2.33), where the only unfamiliar quantity $L^\alpha_{\beta\gamma}$ is defined in equation (2.45), may be easier to comprehend for readers used to Riemannian connections. On the other hand, the outcome (2.66) allows us to understand the electrogravitational field equation (2.36) with as few as possible ancillary quantities. We just need the definition (2.46) of the electrogravitational connection $I^\alpha_{\beta\gamma}$, compute the electrogravitational curvature (2.66) and perform two trivial steps, a contracting of the indices α, γ as well as setting the final outcome to zero.

2.5.3 Gauge derivative

The preceding section tells us that the only nontrivial steps to arrive at the electrogravitational field equation are evaluating the electrogravitational connection $I^\alpha_{\beta\gamma}$ and with that quantity the electrogravitational curvature (2.66). So, the starting point is the electrogravitational connection (2.46), which can also be written as

$$I^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left\{\left(\partial_\beta + 2A_\beta\right)g_{\delta\gamma} + \left[\partial_\gamma - (D-2)\,A_\gamma\right]g_{\beta\delta} - \left(\partial_\delta + 2A_\delta\right)g_{\beta\gamma}\right\} \tag{2.67}$$

If there were a factor 2 instead of the factor $-(D-2)$ in the squared bracket, the electrogravitational connection $I^\alpha_{\beta\gamma}$ would be the result of replacing

$$\partial_\alpha \to D_\alpha$$

in the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$, with the **gauge derivative**

$$\boxed{D_\alpha = \partial_\alpha + wA_\alpha} \tag{2.68}$$

The constant $w \in \mathbb{R}$ in this derivative is the gauge weight of the operand. In our case, the operand is the metric $g_{\alpha\beta}$, and it has the gauge weight 2. For the first and third round bracket in equation (2.67), where the proposed replacement would anyway work, this is actually a result of having chosen $c_{\text{EM}} = 1$ in equation (2.20) such that the electrogravitational field equation (2.21) becomes as simple as possible. We could have also chosen a different value for the electromagnetic constant c_{EM} and thus get a gauge weight for the metric $g_{\alpha\beta}$ which differs from 2.

For many readers the gauge derivative will be a quantity familiar from quantum theory (see, e.g., Peskin & Schroeder 1997 for more on quantum theory, a theory not required to understand this paper, and equation (4.5) in that book). There, it is also called "gauge covariant derivative" or "covariant derivative". We do not use this notion here, because it could be mistaken for the covariant derivative ∇_α, which has nothing to do with the gauge derivative D_α. A marginal difference to the gauge derivative in quantum theory is also that the gauge weight is imaginary there. It is just the imaginary unit $i = \sqrt{-1}$ times a quantized charge q: $w = iq$.

2.5.4 Conformal gauge invariance

Why is the gauge derivative so important? The first and third round bracket in equation (2.67) cause terms in the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ which are based on the expression

$$g^{\alpha\beta} D_\gamma g_{\delta\epsilon} = g^{\alpha\beta} \left(\partial_\gamma + 2A_\gamma\right) g_{\delta\epsilon} \tag{2.69}$$

Such an expression is obviously invariant under the transformation

$$\boxed{\begin{aligned} A_\alpha &\overset{\text{cg}}{\to} A_\alpha + \partial_\alpha \chi \\ g_{\alpha\beta} &\overset{\text{cg}}{\to} e^{-2\chi} g_{\alpha\beta} \end{aligned}} \tag{2.70}$$

because the general invariance of the Kronecker tensor δ^{β}_{α} in equation (1.3) causes

$$g^{\alpha\beta} \overset{\text{cg,c}}{\to} e^{2\chi} g^{\alpha\beta} \tag{2.71}$$

The replacement (2.70) is actually a unification of two known transformations, the gauge transformation (2.65) and the conformal transformation

$$\begin{aligned} A_\alpha &\overset{\text{c}}{\to} A_\alpha \\ g_{\alpha\beta} &\overset{\text{c}}{\to} e^{-2\chi} g_{\alpha\beta} \end{aligned} \tag{2.72}$$

Therefore, we call the replacement (2.70) **conformal gauge transformation**, and it underlies the so-called **conformal gauge invariance**. This also explains the letters "cg" in box (2.70), whereas we have used the letter "c" for transformation (2.72). The simultaneous presence of the letters "cg" and "c" in replacement (2.71) highlights the validity of it for both transformations. Note that the factor $e^{-2\chi}$ in the second line of box (2.70) is actually the true definition of the gauge weight $w = 2$ of the metric $g_{\alpha\beta}$. Or in other words, a quantity which has the gauge weight w is multiplied by the factor $e^{-w\chi}$ if it undergoes a conformal gauge transformation.

The conformal gauge invariance is also a cornerstone of Weyl theory (see equation (9) in Weyl 1918, and take care of the factor -2 included in Section 2.4.1). Let us denote the connection of Weyl theory as ${}^{\mathrm{W}}I^{\alpha}_{\beta\gamma}$. Then, Weyl's deviation (2.61) tells us

$$ {}^{\mathrm{W}}I^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left[\left(\partial_\beta + 2A_\beta \right) g_{\delta\gamma} + \left(\partial_\gamma + 2A_\gamma \right) g_{\beta\delta} - \left(\partial_\delta + 2A_\delta \right) g_{\beta\gamma} \right] \tag{2.73} $$

So, there is a factor 2 in the second round bracket in contrast to the electrogravitational connection (2.67), and we can use the gauge derivative (2.68) to obtain the manifestly conformally gauge invariant expression

$$ {}^{\mathrm{W}}I^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(D_\beta g_{\delta\gamma} + D_\gamma g_{\beta\delta} - D_\delta g_{\beta\gamma} \right) \tag{2.74} $$

2.5.5 World connection

In our theory of electrogravitation, expression (2.73) does not seem to play a role, because of the factor issue in the second round bracket of equation (2.67) mentioned above. However, the experience with the factor $F_\alpha{}^\beta$ in the anomaly (2.6) tells us that we should not give up so lightly. So, let us now introduce expression (2.74) in electrogravitation in the form of what we call the **world connection**

$$ \boxed{ i^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(D_\beta g_{\delta\gamma} + D_\gamma g_{\beta\delta} - D_\delta g_{\beta\gamma} \right) } \tag{2.75} $$

This is a symmetric quantity which is conformally gauge invariant:

$$ \boxed{ i^{\alpha}_{\beta\gamma} \overset{\mathrm{cg}}{\to} i^{\alpha}_{\beta\gamma} } \tag{2.76} $$

We can also perform a split

$$ \boxed{ i^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + l^{\alpha}_{\beta\gamma} } \tag{2.77} $$

similar to the one in equation (2.24). The **world deviation** $l^{\alpha}_{\beta\gamma}$, a tensorial quantity, is our way of calling Weyl's deviation (2.61), which tells us

$$ \boxed{ l^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{\beta} A_\gamma + \delta^{\alpha}_{\gamma} A_\beta - A^\alpha g_{\beta\gamma} } \tag{2.78} $$

The world connection $i^{\alpha}_{\beta\gamma}$ differs from the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ of equation (2.67). So, we are not allowed to use the world connection $i^{\alpha}_{\beta\gamma}$ in equation (2.66) in place of the electrogravitational one $I^{\alpha}_{\beta\gamma}$. Instead, if there is a way to express the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ in terms of the world connection $i^{\alpha}_{\beta\gamma}$, it must be a fundamentally new one. This is emphasized by the letter "i", which is lowercase in contrast to the capital letter "Γ" used for the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ of vacuum general relativity. It must also be a way not taken into account by Weyl, for otherwise he would have found our theory. So, we see again the vicinity to Weyl theory. Either, we use the electrogravitational connection $I^{\alpha}_{\beta\gamma}$, which differs from Weyl's connection ${}^{\mathrm{W}}I^{\alpha}_{\beta\gamma}$ by the factor $-(D-2)$ in the squared bracket of equation (2.67), or we actually use Weyl's connection, but then it must be applied in a yet unknown way. In the following three sections, we will derive that way, where we have to be careful to preserve conformal gauge invariance.

2.5.6 World derivative

Having introduced the world connection $i^{\alpha}_{\beta\gamma}$ in equation (2.75), we can now proceed in a manner similar to Section 2.3.2 and use it to define a new derivative. The easiest way would be to just take equation (2.26) and replace the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ with the world connection $i^{\alpha}_{\beta\gamma}$, or in the notation of Weyl theory with the identical quantity ${}^{\mathrm{W}}I^{\alpha}_{\beta\gamma}$. The result would be Weyl's version of the electrogravitational derivative:

$$ {}^{\mathrm{W}}\Delta_{\gamma} T^{\beta\ldots}_{\alpha\ldots} = \partial_{\gamma} T^{\beta\ldots}_{\alpha\ldots} - {}^{\mathrm{W}}I^{\delta}_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + {}^{\mathrm{W}}I^{\beta}_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots - W\, {}^{\mathrm{W}}I^{\delta}_{\delta\gamma} T^{\beta\ldots}_{\alpha\ldots} \tag{2.79} $$

(see the equation preceding equation (12) in Weyl 1918). However, Weyl was not careful enough here and has not drawn conclusions from the anomaly that this derivative is not conformally gauge invariant. So, he demanded conformal gauge invariance for the connection ${}^{\mathrm{W}}I^{\alpha}_{\beta\gamma}$ but he did not transfer that invariance to the derivative ${}^{\mathrm{W}}\Delta_{\alpha}$. To see this, we merely have to look at a scalar T of coordinate weight $W = 0$, for which definition (2.79) gives

$$ {}^{\mathrm{W}}\Delta_{\alpha} T = \partial_{\alpha} T $$

Then, the gauge weight w of the scalar T causes the change

$$ T \overset{\mathrm{cg}}{\to} e^{-w\chi} T $$

under a conformal gauge transformation such that

$$ {}^{\mathrm{W}}\Delta_{\alpha} T \overset{\mathrm{cg}}{\to} e^{-w\chi} \left({}^{\mathrm{W}}\Delta_{\alpha} - w\partial_{\alpha}\chi \right) T $$

The second term in the round bracket prevents Weyl's derivative ${}^{\mathrm{W}}\Delta_{\alpha}$ from being conformally gauge invariant. Yet, there is a simple cure, and it is the sought way mentioned in the last section. We just have to replace the partial derivative ∂_{γ} in equation (2.79) with the gauge derivative D_{γ}. In the notation of our theory of electrogravitation, where we denote Weyl's connection ${}^{\mathrm{W}}I^{\alpha}_{\beta\gamma}$ as the world connection $i^{\alpha}_{\beta\gamma}$, we then arrive at what we call the **world derivative**

$$ \boxed{\delta_{\gamma} T^{\beta\ldots}_{\alpha\ldots} = D_{\gamma} T^{\beta\ldots}_{\alpha\ldots} - i^{\delta}_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + i^{\beta}_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots - W i^{\delta}_{\delta\gamma} T^{\beta\ldots}_{\alpha\ldots}} \tag{2.80} $$

Similar to equation (2.27), we can also use the split (2.77) to write the world derivative in a manifestly covariant form. For that purpose, we expand the gauge derivative D_{γ} appearing above via equation (2.68). Moreover, we use relation (2.78) to evaluate

$$ \boxed{l^{\beta}_{\beta\alpha} = DA_{\alpha}} \tag{2.81} $$

which is relevant for the last term in definition (2.80). By the way, for the electrogravitational deviation (2.45), we find the differing result

$$ \boxed{L^{\beta}_{\beta\alpha} = -\left(\frac{D}{2} - 1\right) DA_{\alpha}} \tag{2.82} $$

Going now back to the world derivative (2.80), we arrive at

$$ \boxed{\delta_{\gamma} T^{\beta\ldots}_{\alpha\ldots} = \nabla_{\gamma} T^{\beta\ldots}_{\alpha\ldots} - l^{\delta}_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + l^{\beta}_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots + (w - WD)\, A_{\gamma} T^{\beta\ldots}_{\alpha\ldots}} \tag{2.83} $$

In analogy to box (2.28), the world derivative δ_α and the world connection $i^\alpha_{\beta\gamma}$ allow us to define a torsion and nonmetricity given by the expressions

$$\boxed{\begin{aligned} t^\alpha_{\beta\gamma} &= 2i^\alpha_{[\beta\gamma]} \\ \delta_\gamma g_{\alpha\beta} &= D_\gamma g_{\alpha\beta} - 2i_{(\alpha\beta)\gamma} \end{aligned}} \tag{2.84}$$

where

$$i_{\alpha\beta\gamma} = g_{\alpha\delta} i^\delta_{\beta\gamma}$$

We refer to these quantities as **world torsion** and **world nonmetricity**. Let us now examine them more closely. The already mentioned symmetry of the world connection $i^\alpha_{\beta\gamma}$ causes the world torsion $t^\alpha_{\beta\gamma}$ to vanish. To evaluate the world nonmetricity $\delta_\gamma g_{\alpha\beta}$, we use the split (2.77) with

$$l_{\alpha\beta\gamma} = g_{\alpha\delta} l^\delta_{\beta\gamma}$$

take care of the gauge weight 2 of the metric $g_{\alpha\beta}$ and apply relation (1.10) such that the second equation in box (2.84) becomes

$$\delta_\gamma g_{\alpha\beta} = 2\left(A_\gamma g_{\alpha\beta} - l_{(\alpha\beta)\gamma}\right)$$

Due to equation (2.78), this quantity disappears, and we arrive at

$$\boxed{t^\alpha_{\beta\gamma} = \delta_\gamma g_{\alpha\beta} = 0} \tag{2.85}$$

So, both the world torsion and world nonmetricity of electrogravitation vanish such that they seem to be useless when speaking about the geometric interpretation of electromagnetism. For that purpose, it appears to be necessary to use the two alternative expressions $T^\alpha_{\beta\gamma}$ and $\Delta_\gamma g_{\alpha\beta}$, whose values are given in box (2.48). These two quantities truly describe the geometric phenomena torsion and nonmetricity. However, the world torsion $t^\alpha_{\beta\gamma}$ and world nonmetricity $\delta_\gamma g_{\alpha\beta}$ are just another, more abstract way of expressing one and the same property of spacetime. We have already seen that equation (2.66) is an equivalent, but more natural and thus elegant way of writing equation (2.33). The outcome (2.85) can therefore be considered as the natural, less familiar way of expressing torsion and nonmetricity, which is structurally equivalent to its Riemannian analog (1.8) and (1.9).

2.5.7 Electrogravitational versus world view

All in all, it is now clear that electrogravitation can be described in different, but equivalent ways. We can use the electrogravitational connection $I^\alpha_{\beta\gamma}$ and the electrogravitational derivative Δ_α. The electrogravitational connection $I^\alpha_{\beta\gamma}$ itself can be split into the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ and the electrogravitational deviation $L^\alpha_{\beta\gamma}$. This approach is more familiar from the viewpoint of vacuum general relativity, and we call it **electrogravitational view**.

On the other hand, we can use the world connection $i^\alpha_{\beta\gamma}$ and the world derivative δ_α. Similar to the split above, the world connection $i^\alpha_{\beta\gamma}$ produces also the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, but now they are joined by the world deviation $l^\alpha_{\beta\gamma}$. The second approach, denoted as **world view**, is more abstract and natural, which is emphasized by the adjective "world" used to refer to the quantities in it.

Let us now summarize both views, where we use equations (2.24), (2.26), (2.45), (2.77), (2.78) and (2.80):

Electrogravitational view	World view
$L^{\alpha}_{\beta\gamma} = -\left(\frac{D}{2}-1\right)\delta^{\alpha}_{\beta}A_{\gamma} + \delta^{\alpha}_{\gamma}A_{\beta} - A^{\alpha}g_{\beta\gamma}$	$l^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{\beta}A_{\gamma} + \delta^{\alpha}_{\gamma}A_{\beta} - A^{\alpha}g_{\beta\gamma}$
$I^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + L^{\alpha}_{\beta\gamma}$	$i^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + l^{\alpha}_{\beta\gamma}$
$\Delta_{\gamma}T^{\beta\dots}_{\alpha\dots} = \partial_{\gamma}T^{\beta\dots}_{\alpha\dots} - I^{\delta}_{\alpha\gamma}T^{\beta\dots}_{\delta\dots} - \cdots + I^{\beta}_{\delta\gamma}T^{\delta\dots}_{\alpha\dots} + \cdots - W I^{\delta}_{\delta\gamma}T^{\beta\dots}_{\alpha\dots}$	$\delta_{\gamma}T^{\beta\dots}_{\alpha\dots} = D_{\gamma}T^{\beta\dots}_{\alpha\dots} - i^{\delta}_{\alpha\gamma}T^{\beta\dots}_{\delta\dots} - \cdots + i^{\beta}_{\delta\gamma}T^{\delta\dots}_{\alpha\dots} + \cdots - W i^{\delta}_{\delta\gamma}T^{\beta\dots}_{\alpha\dots}$

2.5.8 Electrogravitational curvature via world connection

Curvature cannot be expressed by alternative quantities. We have to always use the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$. However, there are different ways to write this tensor. When we first encountered it in equation (2.32), we have directly employed the electrogravitational derivative Δ_{α}. Then, we have found a way in equation (2.33) to describe the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ in terms of the electrogravitational deviation $L^{\alpha}_{\beta\gamma}$. Eventually, in equation (2.66), the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ was utilized. Let us now repeat these three steps in the world view.

Due to the result (2.85), we know that the world torsion $t^{\alpha}_{\beta\gamma}$ vanishes. Hence, the analog of equation (2.32) in the world view must be based on a consideration of

$$\begin{aligned}\delta_{[\gamma}\delta_{\delta]}X_{\beta} &= D_{[\gamma}\delta_{\delta]}X_{\beta} - i^{\alpha}_{\beta[\gamma}\delta_{\delta]}X_{\alpha} \\ &= D_{[\gamma}\left(D_{\delta]}X_{\beta} - i^{\alpha}_{\beta\delta]}X_{\alpha}\right) - i^{\alpha}_{\beta[\gamma}\left(D_{\delta]}X_{\alpha} - i^{\epsilon}_{\alpha\delta]}X_{\epsilon}\right)\end{aligned} \tag{2.86}$$

in which definition (2.80) and the first line of box (2.84) came into play. Note that the gauge derivative D_{α} does not commute with itself. To show this, we first make ourselves aware that all conformally gauge invariant quantities like the gauge derivative itself have the gauge weight 0. Then, with definition (2.68), we can evaluate

$$D_{[\alpha}D_{\beta]} = \left(\partial_{[\alpha} + wA_{[\alpha}\right)\left(\partial_{\beta]} + wA_{\beta]}\right) = w\partial_{[\alpha}A_{\beta]}$$

where w is the gauge weight of the expression the whole operator is applied on. Let us summarize this as

$$\boxed{[D_{\alpha}, D_{\beta}] = wF_{\alpha\beta}} \tag{2.87}$$

a result familiar from quantum theory (see equation (15.16) of Peskin & Schroeder 1997). Using this outcome and the Leibniz rule, which is also known to hold for the gauge derivative, we can continue evaluating equation (2.86) to

$$\delta_{[\gamma}\delta_{\delta]}X_{\beta} = \frac{w_X}{2}F_{\gamma\delta}X_{\beta} - D_{[\gamma}i^{\alpha}_{\beta\delta]}X_{\alpha} + i^{\alpha}_{\beta[\gamma}i^{\epsilon}_{\alpha\delta]}X_{\epsilon}$$

where w_X is the currently unknown gauge weight of the vector X_{α}. The world connection $i^{\alpha}_{\beta\gamma}$ is conformally gauge invariant due to box (2.76). It has thus the gauge weight 0 such that we arrive at

$$\delta_{[\gamma}\delta_{\delta]}X_{\beta} = -\left(\partial_{[\gamma}i^{\alpha}_{\beta\delta]} + i^{\alpha}_{\epsilon[\gamma}i^{\epsilon}_{\beta\delta]} - \frac{w_X}{2}\delta^{\alpha}_{\beta}F_{\gamma\delta}\right)X_{\alpha} \tag{2.88}$$

with definition (2.68). As no derivative is applied on the vector X_{α} on the right hand side, the round bracket can be used to define a tensor similar to equation (2.32).

The open question is now the relationship of the above round bracket with the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$. For that purpose, we first use the split (2.77) such that equation (1.13) yields

$$\partial_{[\gamma} i^\alpha_{\beta\delta]} + i^\alpha_{\epsilon[\gamma} i^\epsilon_{\beta\delta]} = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} + \partial_{[\gamma} l^\alpha_{\beta\delta]} + \Gamma^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]} - \Gamma^\epsilon_{\beta[\gamma} l^\alpha_{\epsilon\delta]} + l^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]}$$

We recall that the world deviation $l^\alpha_{\beta\gamma}$ is a tensor, and proceed with

$$\partial_{[\gamma} i^\alpha_{\beta\delta]} + i^\alpha_{\epsilon[\gamma} i^\epsilon_{\beta\delta]} = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} + \nabla_{[\gamma} l^\alpha_{\beta\delta]} + l^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]} \tag{2.89}$$

Using equations (2.45) and (2.78), the second term on the right hand side can be written as

$$\nabla_{[\gamma} l^\alpha_{\beta\delta]} = \nabla_{[\gamma} L^\alpha_{\beta\delta]} + \frac{D}{4} \delta^\alpha_\beta F_{\gamma\delta}$$

Equations (2.45) and (2.78) also tell us that the deviations $l^\alpha_{\beta\gamma}$ and $L^\alpha_{\beta\gamma}$ differ only in the factor in their first term, which is based on $\delta^\alpha_\beta A_\gamma$. Hence, the general relation (2.41) immediately shows us

$$l^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]} = L^\alpha_{\epsilon[\gamma} L^\epsilon_{\beta\delta]}$$

because in that relation the mentioned factor is written as the constant a_2, which does not appear on the right hand side there. Looking then back to equation (2.89), we see that definition (2.33) allows us to write

$$\boxed{Z^\alpha{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma} i^\alpha_{\beta\delta]} + i^\alpha_{\epsilon[\gamma} i^\epsilon_{\beta\delta]}\right) - \frac{D}{2} \delta^\alpha_\beta F_{\gamma\delta}} \tag{2.90}$$

That way, we have found the sought relationship of the round bracket in equation (2.88) with the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$. So, demanding that the gauge weight of the vector X_α in that equation is $w_X = D/2$, we eventually arrive at an alternative definition of the electrogravitational curvature

$$\boxed{Z^\alpha{}_{\beta\gamma\delta} X_\alpha = -[\delta_\gamma, \delta_\delta] X_\beta} \tag{2.91}$$

which is actually the shortest one possible and thus the most natural one.

We could have explained the whole theory of electrogravitation in a different, more abstract order, not starting with the unification trick in Section 2.1 but directly with the fundamentals of the world view and the above natural way to define the electrogravitational curvature. This is the simple, axiomatic approach mentioned but rejected at the very beginning of this paper in Section 1.3. However, this is plausible only in retrospect, not initially.

Back to the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$. The third way to write it in the world view is the analog of equation (2.33), for which equations (2.89) and (2.90) tell us

$$\boxed{Z^\alpha{}_{\beta\gamma\delta} = R^\alpha{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma} l^\alpha_{\beta\delta]} + l^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]}\right) - \frac{D}{2} \delta^\alpha_\beta F_{\gamma\delta}} \tag{2.92}$$

In total, we have thus encountered six different ways to write the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$, which are equations (2.90),(2.91) plus (2.92) in the world view and equations (2.32),(2.33) as well as (2.66) in the electrogravitational view:

Electrogravitational view	World view
$Z^{\alpha}{}_{\beta\gamma\delta}X_{\alpha} = \left(T^{\alpha}_{\gamma\delta}\Delta_{\alpha} - [\Delta_{\gamma}, \Delta_{\delta}]\right) X_{\beta}$	$Z^{\alpha}{}_{\beta\gamma\delta}X_{\alpha} = -[\delta_{\gamma}, \delta_{\delta}]X_{\beta}$
$Z^{\alpha}{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma}I^{\alpha}_{\beta\delta]} + I^{\alpha}_{\epsilon[\gamma}I^{\epsilon}_{\beta\delta]}\right)$	$Z^{\alpha}{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma}i^{\alpha}_{\beta\delta]} + i^{\alpha}_{\epsilon[\gamma}i^{\epsilon}_{\beta\delta]}\right) - \frac{D}{2}\delta^{\alpha}_{\beta}F_{\gamma\delta}$
$= R^{\alpha}{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma}L^{\alpha}_{\beta\delta]} + L^{\alpha}_{\epsilon[\gamma}L^{\epsilon}_{\beta\delta]}\right)$	$= R^{\alpha}{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma}l^{\alpha}_{\beta\delta]} + l^{\alpha}_{\epsilon[\gamma}l^{\epsilon}_{\beta\delta]}\right) - \frac{D}{2}\delta^{\alpha}_{\beta}F_{\gamma\delta}$

2.5.9 Trivial derivation of electrogravitational curvature

Having found equation (2.90), we can now address an interesting side question. Is it not simply sufficient to take the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ and perform the replacement

$$\partial_{\alpha} \to D_{\alpha} \tag{2.93}$$

in it to obtain the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$?

To answer this question, we look at the definition (1.13) of the Riemann tensor. The replacement (2.93) substitutes the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ in that definition with the world connection $i^{\alpha}_{\beta\gamma}$ due to box (2.75). That way, we obtain

$$R^{\alpha}{}_{\beta\gamma\delta} \to 2\left(\partial_{[\gamma}i^{\alpha}_{\beta\delta]} + i^{\alpha}_{\epsilon[\gamma}i^{\epsilon}_{\beta\delta]}\right) \tag{2.94}$$

Note that the partial derivative $\partial_{[\gamma}$ seen above is not affected by the replacement (2.93), because the world connection has the gauge weight 0. So, using the gauge derivative $D_{[\gamma}$ there just reduces to the partial derivative $\partial_{[\gamma}$. Comparing the result (2.94) with equation (2.90), it is clear that the replacement (2.93) does not spew out the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$. Otherwise, electrogravitation might have been discovered much earlier.

However, let us not give up so lightly. Combining equations (2.87) and (2.90), we can write

$$Z^{\alpha}{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma}i^{\alpha}_{\beta\delta]} + i^{\alpha}_{\epsilon[\gamma}i^{\epsilon}_{\beta\delta]}\right) - \frac{D}{2w}\delta^{\alpha}_{\beta}\left[D_{\gamma}, D_{\delta}\right]$$

It is then possible to perform the opposite replacement

$$D_{\alpha} \to \partial_{\alpha}$$

such that

$$R^{\alpha}{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma}\Gamma^{\alpha}_{\beta\delta]} + \Gamma^{\alpha}_{\epsilon[\gamma}\Gamma^{\epsilon}_{\beta\delta]}\right) - \frac{D}{2w}\delta^{\alpha}_{\beta}\left[\partial_{\gamma}, \partial_{\delta}\right] \tag{2.95}$$

As partial derivatives commute, this result is equal to definition (1.13). So, the crucial point here is to write the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ in the right manner. A deficit of equation (2.95) is still that the gauge weight w appears there. As we will anyway apply the whole equation on the vector X_{α} and know $w_X = D/2$ from the last section, we write

$$R^{\alpha}{}_{\beta\gamma\delta} = 2\left(\partial_{[\gamma}\Gamma^{\alpha}_{\beta\delta]} + \Gamma^{\alpha}_{\epsilon[\gamma}\Gamma^{\epsilon}_{\beta\delta]}\right) - \delta^{\alpha}_{\beta}\left[\partial_{\gamma}, \partial_{\delta}\right]$$

Then, relation (1.11) gives

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta}X_{\alpha} = -\left(\left[\partial_{\gamma}, \partial_{\delta}\right] + \left[\nabla_{\gamma}, \nabla_{\delta}\right]\right)X_{\beta}} \tag{2.96}$$

So, this outcome is equal to definition (1.11). However, if we apply the replacement (2.93) on it, then we obtain the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$, or more exactly the expression $Z^{\alpha}{}_{\beta\gamma\delta}X_{\alpha}$. The electrogravitational curvature can therefore be derived from the Riemann tensor in a truly trivial manner. Unfortunately, the form (2.96) is not evident from the viewpoint of general relativity.

2.5.10 World invariance

To study whether the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ is subject to any invariance, we look at version (2.90). Due to definition (1.1) of the electromagnetic field strength $F_{\alpha\beta}$, it is obvious that this quantity is not only gauge invariant but also conformally gauge invariant. Both invariances also hold for the Kronecker tensor δ^{α}_{β}. However, the world connection $i^{\alpha}_{\beta\gamma}$ is only conformally gauge invariant, an invariance thus also valid for the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ and due to definition (2.34) for the electrogravitation tensor $Z_{\alpha\beta}$:

$$\boxed{\begin{aligned} F_{\alpha\beta} &\stackrel{\text{cg}}{\to} F_{\alpha\beta} \\ Z^{\alpha}{}_{\beta\gamma\delta} &\stackrel{\text{cg}}{\to} Z^{\alpha}{}_{\beta\gamma\delta} \\ Z_{\alpha\beta} &\stackrel{\text{cg}}{\to} Z_{\alpha\beta} \end{aligned}} \tag{2.97}$$

So, the electrogravitational field equation (2.36) is conformally gauge invariant. We recall that this field equation allows us to derive Maxwell's equation (1.15) for $D = 4$, which means that also Maxwell's equation is conformally gauge invariant. However, this holds only for $D = 4$. To have conformal gauge invariance in the other cases, we have to look at the true generalization of Maxwell's equation to an arbitrary number of dimensions, which differs from equation (1.15) and is derived further below.

As a tensor equation, relation (2.36) is also covariant. Hence, the electrogravitational field equation is both covariant and conformally gauge invariant. Let us call the simultaneous presence of these two invariances **world invariance**. We can then sum up:

> **Invariance of electrogravitation**:
> Electrogravitation is world invariant.

2.5.11 Conformal invariance

In the last section, we have encountered that electrogravitation is invariant under the conformal gauge transformation (2.70). We have already mentioned earlier that this kind of transformation unifies the gauge transformation (2.65) and the conformal transformation (2.72). What does a unification mean here? A conformal gauge transformation is nothing else than a gauge transformation followed by a conformal transformation. The order does actually not matter. That way, we can interestingly conclude that **electrogravitation violates conformal invariance**. Why? If it were invariant under a conformal transformation, then it would still be unchanged under an additional conformal gauge transformation which undoes the influence of the conformal one. The net effect would be an invariance under a gauge transformation, which was shown to be broken in Section 2.5.1.

What about Einstein-Maxwell theory? It is obvious that this theory is gauge invariant. To determine whether Einstein-Maxwell theory is also conformally invariant, we have to perform a few computations below. Some readers may already know the outcome, but for completeness we do not skip the required steps. So, the starting point are transformations (2.71) and (2.72) such that the Christoffel symbols (1.4) transform as

$$\Gamma^{\alpha}_{\beta\gamma} \stackrel{\text{c}}{\to} \Gamma^{\alpha}_{\beta\gamma} - \left(\delta^{\alpha}_{\beta}\partial_{\gamma} + \delta^{\alpha}_{\gamma}\partial_{\beta} - g^{\alpha\delta}g_{\beta\gamma}\partial_{\delta}\right)\chi$$

In the notation of electrogravitation, the right hand side is just the world connection (2.77) where the electromagnetic vector potential in the world deviation (2.78) has the form

$A_\alpha = -\partial_\alpha \chi$. This allows us to quickly find the transformation behavior of the Riemann tensor (1.13). We merely have to look at equation (2.90) and immediately see

$$R^{\alpha}{}_{\beta\gamma\delta} \overset{\text{c}}{\to} Z^{\alpha}{}_{\beta\gamma\delta}\Big|_{A_\alpha=-\partial_\alpha\chi}$$

where the $F_{\gamma\delta}$-term vanishes due to $A_\alpha = -\partial_\alpha \chi$. Hence, definition (2.34) gives

$$R_{\alpha\beta} \overset{\text{c}}{\to} Z_{\alpha\beta}\big|_{A_\alpha=-\partial_\alpha\chi} \tag{2.98}$$

To rewrite the right hand side above, we use equation (2.35). Surely, the field equation (2.36) cannot hold in this context, because then the electrogravitation tensor $Z_{\alpha\beta}$ in replacement (2.98) would vanish. However, if it did, we would get relation (2.38) for $D = 4$. This allows us to conclude that the expression after the Ricci tensor $R_{\alpha\beta}$ in equation (2.35) is the negative of the right hand side of relation (2.38) such that

$$R_{\alpha\beta} \overset{\text{c},D=4}{\to} R_{\alpha\beta} + 2\nabla_\alpha\nabla_\beta\chi + g_{\alpha\beta}\nabla^\gamma\nabla_\gamma\chi + \mathcal{O}\left(\chi\right)^2 \tag{2.99}$$

Looking then at Einstein's equation (1.6), we see that vacuum general relativity violates conformal invariance. Let us next write the Einstein tensor (1.17) as

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} R_{\gamma\delta}$$

That way, replacement (2.99) shows

$$G_{\alpha\beta} \overset{\text{c},D=4}{\to} G_{\alpha\beta} + 2\left(\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\nabla^\gamma\nabla_\gamma\right)\chi + \mathcal{O}\left(\chi\right)^2$$

It is, however, obvious that the right hand side of the field equation (1.20) cannot produce terms which are linear in the scalar χ under a conformal transformation. Therefore, also Einstein-Maxwell theory violates conformal invariance. As a conformal gauge transformation is a gauge transformation followed by a conformal transformation, Einstein-Maxwell theory also violates conformal gauge invariance.

All in all, we then come to the following conclusion. Conformal invariance works neither for Einstein-Maxwell theory nor for electrogravitation. However, for electrogravitation, conformal transformations at least lead to an invariance if they are combined with a gauge transformation. That way, electrogravitation actually activates the dormant conformal invariance of general relativity.

2.6 Cause of four dimensions

2.6.1 Alternative derivation of dimensional dependence

It is already visible in equation (2.45) that the mathematical form of electrogravitation depends on the number of dimensions D, in contrast to gravitation. In the following, we will study the dimensional dependence of electrogravitation more thoroughly. For that purpose, we first give another, additional criterion which shows that equation (2.45) is the correct generalization of the electrogravitational deviation $L^{\alpha}_{\beta\gamma}$ to D dimensions. The first criterion was encountered in Section 2.3.6. And, it demanded that Maxwell's equation (1.15) at least describes the linear limit of the D-dimensional electromagnetic field equation properly, i.e. the case when second or higher orders in the electromagnetic vector potential A_α are neglected. The alternative approach is based on the world view and explained in the following.

The consideration of the world view started in Section 2.5.5. From there on, we have used the dimensional dependence of the electrogravitational deviation (2.45) not until after equation (2.89), except for the here irrelevant side computation (2.82). The goal is now to not assume that dimensional dependence but to derive it. For that purpose, we use equations (2.88), (2.89) and also relation (2.91), a relation which should hold independent of how the electrogravitational deviation $L^\alpha_{\beta\gamma}$ looks like, such that

$$Z^\alpha{}_{\beta\gamma\delta} = R^\alpha{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma} l^\alpha_{\beta\delta]} + l^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]}\right) - w_X \delta^\alpha_\beta F_{\gamma\delta}$$

Then, definition (2.34) and the field equation (2.36) lead to

$$R_{\alpha\beta} + 2\left(\nabla_{[\gamma} l^\gamma_{\alpha\beta]} + l^\gamma_{\delta[\gamma} l^\delta_{\alpha\beta]}\right) - w_X F_{\alpha\beta} = 0 \tag{2.100}$$

Using equation (2.78), we can rewrite the first term in the round bracket as

$$\begin{aligned} 2\nabla_{[\gamma} l^\gamma_{\alpha\beta]} &= \nabla_\gamma\left(\delta^\gamma_\alpha A_\beta + \delta^\gamma_\beta A_\alpha - A^\gamma g_{\alpha\beta}\right) - \nabla_\beta\left(\delta^\gamma_\alpha A_\gamma + \delta^\gamma_\gamma A_\alpha - A^\gamma g_{\alpha\gamma}\right) \\ &= \nabla_\alpha A_\beta + (1-D)\nabla_\beta A_\alpha - g_{\alpha\beta}\nabla_\gamma A^\gamma \end{aligned}$$

such that

$$2\nabla_{[\gamma} l^\gamma_{\alpha\beta]} - w_X F_{\alpha\beta} = (2-D)\nabla_{(\alpha} A_{\beta)} - g_{\alpha\beta}\nabla_\gamma A^\gamma + \left(\frac{D}{2} - w_X\right) F_{\alpha\beta} \tag{2.101}$$

Let us now have a closer look at the symmetry of equation (2.100). The Ricci tensor $R_{\alpha\beta}$ is obviously symmetric. For the second term in the round bracket, we apply relation (2.42). This relation uses the electrogravitational deviation $L^\alpha_{\beta\gamma}$, but at that stage the deviation was still a general expression described by ansatz (2.25), with $d_2 = 0$ and without the dots there. Hence, we can also use the world deviation $l^\alpha_{\beta\gamma}$ with the form (2.78) in place of it. Then, the right hand side of relation (2.42) shows that the second term in the round bracket of equation (2.100) is symmetric in the indices α and β. That way, equation (2.101) must be symmetric, which clearly shows that there is no alternative to $w_X = D/2$. On the other hand, equations (2.37) and (2.100) reveal

$$2\nabla_{[\gamma} L^\gamma_{\alpha\beta]} = 2\nabla_{[\gamma} l^\gamma_{\alpha\beta]} - w_X F_{\alpha\beta}$$

where we have used the fact that both quantities $l^\alpha_{\beta\gamma}$ and $L^\alpha_{\beta\gamma}$ are linear in the electromagnetic vector potential A_α to remove the two products of them. Then, relation (2.101) gives

$$-2\nabla_{[\gamma} L^\gamma_{\alpha\beta]} = (D-2)\nabla_{(\alpha} A_{\beta)} + g_{\alpha\beta}\nabla_\gamma A^\gamma \tag{2.102}$$

We compare this with equation (2.39) and eventually find the constants $a_2 = 1 - D/2$, $c_2 = -1$ and thus $b_2 = 1$. These constants describe just the electrogravitational deviation (2.45) such that it is the only reasonable generalization to D dimensions.

2.6.2 Dimensional dependence of field equation

Being convinced that the electrogravitational deviation (2.45) describes the only reasonable generalization of electromagnetism to D dimensions, we can now derive the dimensional dependence of the field equation. For that purpose, we merely have to evaluate the right hand side of equation (2.37). The first term in the round bracket was already evaluated in equation (2.102) and the second one partially in equation (2.42). Using the constants $b_2 = 1$ and $c_2 = -1$ of the last section, we can finish that evaluation to

$$-2L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]} = (D-2)\left(g_{\alpha\beta}A^2 - A_\alpha A_\beta\right)$$

such that equation (2.37) leads to

$$\boxed{R_{\alpha\beta} = g_{\alpha\beta}\nabla_\gamma A^\gamma + (D-2)\left(\nabla_{(\alpha}A_{\beta)} - A_\alpha A_\beta + g_{\alpha\beta}A^2\right)} \tag{2.103}$$

This is one way to write the field equation. Performing a contraction, we obtain the Ricci scalar

$$\boxed{R = (D-1)\left[2\nabla_\alpha A^\alpha + (D-2)\,A^2\right]} \tag{2.104}$$

Hence, relation (1.17) spews out the natural way to write the ***D*-dimensional electro-gravitational field equation**:

$$\boxed{G_{\alpha\beta} = (D-2)\left\{\nabla_{(\alpha}A_{\beta)} - A_\alpha A_\beta - g_{\alpha\beta}\left[\nabla_\gamma A^\gamma + \tfrac{1}{2}(D-3)\,A^2\right]\right\}} \tag{2.105}$$

The above unified field equation is a complete description of the dynamics of gravitation and electromagnetism. Similar to equation (2.23), we can now extract the electromagnetic field equation hidden in it. For that purpose, we have to apply the contracted Bianchi identity (2.5), which requires the evaluation of

$$\begin{aligned}
&\nabla^\beta\left\{\nabla_{(\beta}A_{\alpha)} - A_\beta A_\alpha - g_{\beta\alpha}\left[\nabla_\gamma A^\gamma + \frac{1}{2}(D-3)\,A^2\right]\right\}\\
=\;&\frac{1}{2}\left(\nabla^\beta\nabla_\beta A_\alpha + \nabla^\beta\nabla_\alpha A_\beta\right) - A_\alpha\nabla_\beta A^\beta - A^\beta\nabla_\beta A_\alpha - \nabla_\alpha\left[\nabla_\beta A^\beta + \frac{1}{2}(D-3)\,A^2\right]
\end{aligned}$$

such that we get

$$(D-2)\left[\frac{1}{2}\nabla^\beta F_{\beta\alpha} + [\nabla_\beta,\nabla_\alpha]A^\beta - A_\alpha\nabla_\beta A^\beta - A^\beta\nabla_\beta A_\alpha - \frac{1}{2}(D-3)\,\nabla_\alpha\left(A_\beta A^\beta\right)\right] = 0 \tag{2.106}$$

Due to equations (2.11) as well as (2.13) for the following first step and the field equation in the form (2.103) for the second one, it is possible to rewrite

$$[\nabla_\beta,\nabla_\alpha]A^\beta = A^\beta R_{\beta\alpha} = A_\alpha\nabla_\beta A^\beta + (D-2)\,A^\beta\nabla_{(\beta}A_{\alpha)} \tag{2.107}$$

That way, we arrive at

$$(D-2)\left\{\frac{1}{2}\nabla^\beta F_{\beta\alpha} + A^\beta\left[(D-2)\,\nabla_{(\beta}A_{\alpha)} - \nabla_\beta A_\alpha - (D-3)\,\nabla_\alpha A_\beta\right]\right\} = 0$$

such that the ***D*-dimensional electromagnetic field equation** is

$$\boxed{(D-2)\left[\nabla^\beta + (D-4)\,A^\beta\right]F_{\beta\alpha} = 0} \tag{2.108}$$

For $D = 4$, this field equation reduces to Maxwell's equation (1.15). However, in all other cases, we have found a field equation which differs from the naive generalization of Maxwell's equation to D dimensions in equation (1.15). This striking discrepancy is the true reason why this paper is not limited to four dimensions, and it motivates a more thorough dimensional analysis, which we perform in the following sections.

2.6.3 Gravitation

Let us first review the dimensional dependence of gravitation before investigating this issue for electromagnetism. Due to Section 1.4, we always assume that there is one temporal dimension such that $D \geq 1$. For $D = 1$, all indices can take only the temporal value t and thus identity (2.12) causes

$$R_{\alpha\beta\gamma\delta} \overset{D=1}{\equiv} 0$$

For $D = 2$ and $D = 3$, it is known from general relativity that

$$R_{\alpha\beta\gamma\delta} \overset{D=2}{\equiv} R g_{\alpha[\gamma} g_{\delta]\beta}$$

and

$$R_{\alpha\beta\gamma\delta} \overset{D=3}{\equiv} 2\left(g_{\alpha[\gamma} R_{\delta]\beta} - g_{\beta[\gamma} R_{\delta]\alpha}\right) - R g_{\alpha[\gamma} g_{\delta]\beta}$$

(see exercise 4 on p. 54 of Wald 1984). Applying now Einstein's equation (1.6) gives

$$\boxed{R_{\alpha\beta\gamma\delta} \overset{D\leq 3}{\equiv} 0}$$

which means that the curvature tensor vanishes and spacetime is flat. So, there is no gravitation for $D \leq 3$, i.e. our universe with $D = 4$ has the lowest dimension in which general relativistic gravitation is possible. Unfortunately, this does not tell us why we live in four dimensions, because gravitation works for any one of the infinitely many choices $D \geq 4$.

2.6.4 Electromagnetism

Now we come to electromagnetism, where we know that the case $D = 1$ does not allow electromagnetism, because the antisymmetry of the electromagnetic field strength $F_{\alpha\beta}$ causes that tensor to vanish. One might oppose that one can still have the electromagnetic vector potential A_α in that case. However, this vector then consists of only a single component, which can be transformed to zero via a conformal gauge transformation.

For $D = 2$, these obstacles are not present, but in the field equation (2.105) the factor $D - 2$ shows that electromagnetism does not couple to gravitation. In addition to that, the same factor in equation (2.108) prevents an electromagnetic field equation for $D = 2$ such that the phenomenon electromagnetism does not occur in that case.

In all other cases, the electromagnetic field equation (2.108) reduces to

$$\nabla^\beta F_{\beta\alpha} \overset{D\geq 3}{\equiv} (D-4)\, F_{\alpha\beta} A^\beta \tag{2.109}$$

where the right hand side is now the source of electromagnetism. That way, vacuum electromagnetism in flat spacetime for D dimensions is actually not described by the linear field equation (1.2) but by the nonlinear one

$$\partial^\beta F_{\beta\alpha} \overset{D\geq 3}{\equiv} (D-4)\, F_{\alpha\beta} A^\beta$$

So, electromagnetism couples to itself for $D = 3$ and $D \geq 5$. For gravitation, such a self-coupling is a well-known phenomenon. There, it is the result of the nonlinearity of the Ricci tensor $R_{\alpha\beta}$ in Einstein's equation (1.6). Let us investigate the consequences of the source in equation (2.109) in the following section.

2.6.5 Dimensional constraint of electromagnetism

Looking at equation (2.109), it is tempting to apply a covariant derivative on it similar to how we derived the electromagnetic field equation (2.108) from the electrogravitational one (2.105). For that purpose, we have to evaluate the expression

$$\nabla^\alpha \nabla^\beta F_{\alpha\beta} = \frac{1}{2}\left[\nabla^\alpha, \nabla^\beta\right] F_{\alpha\beta} \tag{2.110}$$

where we used the antisymmetry of the electromagnetic field strength $F_{\alpha\beta}$ to arrive at the right hand side. Later in this paper, we will need the outcome of similar expressions such that we now replace the electromagnetic field strength $F_{\alpha\beta}$ with an arbitrary tensor $Y_{\alpha\beta}$ and repeat the evaluation of

$$\begin{aligned}
\nabla_{[\gamma}\nabla_{\delta]}Y_{\alpha\beta} &= \partial_{[\gamma}\nabla_{\delta]}Y_{\alpha\beta} - \Gamma^\epsilon_{\alpha[\gamma}\nabla_{\delta]}Y_{\epsilon\beta} - \Gamma^\epsilon_{\beta[\gamma}\nabla_{\delta]}Y_{\alpha\epsilon} \\
&= \partial_{[\gamma}\left(\partial_{\delta]}Y_{\alpha\beta} - \Gamma^\epsilon_{\alpha\delta]}Y_{\epsilon\beta} - \Gamma^\epsilon_{\beta\delta]}Y_{\alpha\epsilon}\right) - \Gamma^\epsilon_{\alpha[\gamma}\left(\partial_{\delta]}Y_{\epsilon\beta} - \Gamma^\zeta_{\epsilon\delta]}Y_{\zeta\beta} - \Gamma^\zeta_{\beta\delta]}Y_{\epsilon\zeta}\right) \\
&\quad -\Gamma^\epsilon_{\beta[\gamma}\left(\partial_{\delta]}Y_{\alpha\epsilon} - \Gamma^\zeta_{\alpha\delta]}Y_{\zeta\epsilon} - \Gamma^\zeta_{\epsilon\delta]}Y_{\alpha\zeta}\right)
\end{aligned}$$

known from general relativity, for completeness. Using then definition (1.13) of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$, we arrive at

$$\left[\nabla_\gamma, \nabla_\delta\right] Y_{\alpha\beta} = -R^\epsilon{}_{\alpha\gamma\delta}Y_{\epsilon\beta} - R^\epsilon{}_{\beta\gamma\delta}Y_{\alpha\epsilon} \tag{2.111}$$

This is the extension of relation (1.11) to an operand with two indices. Next, identity (2.12), the antisymmetry of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ in its last two indices and definition (1.14) give

$$\left[\nabla^\alpha, \nabla^\beta\right] Y_{\alpha\beta} = 2R^{\alpha\beta}Y_{[\alpha\beta]} \tag{2.112}$$

such that the symmetry of the Riemann tensor $R_{\alpha\beta}$ leads to the identity

$$\left[\nabla^\alpha, \nabla^\beta\right] Y_{\alpha\beta} \equiv 0 \tag{2.113}$$

Returning now to equation (2.110), the result is a second identity:

$$\nabla^\alpha \nabla^\beta F_{\alpha\beta} \equiv 0 \tag{2.114}$$

So, we have the means to continue with the field equation (2.109). Using the just derived identity (2.114) in it, we obtain

$$(D-4)\left(\nabla^\alpha F_{\alpha\beta}A^\beta + F_{\alpha\beta}\nabla^\alpha A^\beta\right) \overset{D\geq 3}{\equiv} 0$$

For the first term in the long bracket, we use the field equation again such that

$$(D-4)\left[(D-4)\,F_{\beta\alpha}A^\alpha A^\beta + \frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\right] \overset{D\geq 3}{\equiv} 0$$

The first term in the squared bracket vanishes due to the antisymmetry of the electromagnetic field strength $F_{\alpha\beta}$ such that we eventually arrive at what we call the **dimensional constraint of electromagnetism**:

$$\boxed{F_{\alpha\beta}F^{\alpha\beta} \overset{D=3,5,6,\dots}{=} 0} \tag{2.115}$$

This constraint occurs for $D = 3$ and $D \geq 5$, and we have a closer look at it in the following sections.

2.6.6 Electric field

To understand the meaning of the constraint (2.115), we have to decompose the electromagnetic field $F_{\alpha\beta}$ into the electric and magnetic field. For that purpose, we perform a $1 + (D-1)$-foliation of spacetime (see Section 21.4 of Misner *et al.* 2002 for a more thorough explanation of the case $D = 4$). Then, following the convention that lowercase Roman indices run through the values $1, ..., D-1$, we can write the metric $g_{\alpha\beta}$ as

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_c\beta^c & \beta_b \\ \beta_a & \gamma_{ab} \end{pmatrix} \tag{2.116}$$

where the quantities α, β^a and γ_{ab} are the lapse, shift and $(D-1)$-metric, and $\beta_a = \gamma_{ab}\beta^b$. The timelike unit vector orthogonal to the hypersurfaces caused by the foliation is known to be

$$n_\alpha = (-\alpha, 0, 0, ...) \tag{2.117}$$

In general relativity, which is emphasized by the letters "GR" in the following, the electric field is now defined as

$$^{\mathrm{GR}}E^\alpha = F^{\alpha\beta}n_\beta \tag{2.118}$$

(see equation (5.115) of Baumgarte & Shapiro 2010 for $D = 4$).

Unfortunately, the above definition has a deficit from the viewpoint of electrogravitation and is thus replaced in that theory. To see the problem, we need the contravariant form

$$g^{\alpha\beta} = \begin{pmatrix} -\dfrac{1}{\alpha^2} & \dfrac{\beta^b}{\alpha^2} \\ \dfrac{\beta^a}{\alpha^2} & \gamma^{ab} - \dfrac{\beta^a\beta^b}{\alpha^2} \end{pmatrix} \tag{2.119}$$

of the metric, which is just the inverse of the matrix (2.116). To verify this for oneself, the reader may validate the equation $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$, where $\gamma^{ab}\gamma_{bc} = \delta^a_c$. Anyway, we now see

$$\alpha = \frac{1}{\sqrt{-g^{tt}}} \tag{2.120}$$

such that adopting definition (2.117) the electric field (2.118) can be written as

$$^{\mathrm{GR}}E^\alpha = \alpha F^{t\alpha} = \frac{1}{\sqrt{-g^{tt}}}F^{t\alpha}$$

Next, we perform a conformal gauge transformation. Due to box (2.97), we recall that the covariant electromagnetic field strength $F_{\alpha\beta}$ is unaffected by such a transformation. However, the replacement (2.71) shows that the contravariant version transforms as

$$F^{\alpha\beta} \overset{\mathrm{cg}}{\to} e^{4\chi}F^{\alpha\beta}$$

and thus

$$^{\mathrm{GR}}E^\alpha \overset{\mathrm{cg}}{\to} e^{3\chi}\,{}^{\mathrm{GR}}E^\alpha$$

So, the **general relativistic definition of the electric field is not conformally gauge invariant**.

To cure this issue, we look at the replacement (2.70) and find that the determinant g of the metric $g_{\alpha\beta}$ transforms as

$$g \overset{\mathrm{cg}}{\to} e^{-2D\chi}g$$

Hence, defining the electric field as

$$\boxed{E^\alpha = \sqrt{-g}^{\frac{4}{D}} F^{t\alpha}} \tag{2.121}$$

it becomes a conformally gauge invariant quantity. Note that this definition is not obligatory. For instance, due to the replacement (2.71), we know $g^{tt} \stackrel{\text{cg}}{\to} e^{2\chi} g^{tt}$ such that the factor $-g^{tt}(-g)^{1/D}$ is conformally gauge invariant. Hence, multiplying definition (2.121) by this factor would lead to an alternative conformally gauge invariant quantity. Still, our definition (2.121) is the most reasonable one. In the 4-dimensional case, only the choice (2.121) leads to the simple constraint $\partial_a E^a = 0$ encountered later in this paper (see equation (4.63)). And, in the D-dimensional case, definition (2.121) is the most straightforward generalization of the 4-dimensional version. Finally, it is worth to realize that the difference between the electrogravitational definition E^α and the general relativistic one ${}^{\text{GR}}E^\alpha$ anyway vanishes in flat spacetime.

2.6.7 Magnetic field

In equation (2.121), the electric field is defined as a D-dimensional vector. However, due to the antisymmetry of the electromagnetic field strength $F^{\alpha\beta}$, the temporal component E^t vanishes. For $D = 4$, the electric field consists then of the three familiar components E^a. In four dimensions, the magnetic field is also a vector, namely the general relativistic definition

$${}^{\text{GR}}B^\alpha \stackrel{D=4}{=} \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} n_\beta F_{\delta\gamma} \tag{2.122}$$

(see equation (5.115) of Baumgarte & Shapiro 2010). Similar to the electric field, the component ${}^{\text{GR}}B^t$ disappears, because in that case $\alpha = t$ in definition (2.122). Then, either $\beta = t$ such that the Levi-Civita tensor $\epsilon^{\alpha\beta\gamma\delta}$ is zero due to its total antisymmetry. Or $\beta \neq t$, which makes the quantity n_β disappear, because it has only zeros in its spatial components due to definition (2.117).

The non-vanishing spatial components of the magnetic field (2.122) can be written in a more comprehensible manner. For that purpose, we recall the definition of the covariant Levi-Civita tensor in the second paragraph of Section 2.2.1. In four dimensions, the contravariant counterpart is known to be $\epsilon^{\alpha\beta\gamma\delta} = -[\alpha\beta\gamma\delta]/\sqrt{-g}$ (see equation (8.10a) in Misner *et al.* 2002). Also adopting equations (2.117) and (2.120), we then find

$${}^{\text{GR}}B^a \stackrel{D=4}{=} \frac{B^a}{\sqrt{g^{tt}g}} \tag{2.123}$$

where

$$\boxed{B^a \stackrel{D=4}{=} (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1)} \tag{2.124}$$

(see equation (21.106) of Misner *et al.* 2002, i.e. our fields E^a and B^a are the calligraphic quantities in that book). The quantity B^a is manifestly conformally gauge invariant. It is thus, however, obvious that the **general relativistic definition of the magnetic field is not conformally gauge invariant**, similar to the electric field. We therefore consider the quantity B^a as the magnetic field of electrogravitation in four dimensions. In D dimensions, the magnetic field does not have $D-1$ non-vanishing components like the electric one. Instead, it is described by the quantity F_{ab}. Note that similar to the electric field the choices F_{ab} and (2.124) are also not obligatory but the most reasonable ones.

In flat spacetime, the difference between definitions (2.123) and (2.124) vanishes. And, there are also no alternatives. Using a factor like $-g^{tt}\,(-g)^{1/D}$ mentioned in the last section would be useless, because that factor is unity in flat spacetime. So, the form (2.124), which can also be written as $\vec{B} = \mathrm{rot}\vec{A}$, is inevitable. This has the noteworthy consequence that $\mathrm{div}\vec{B} = 0$, which means that there are no magnetic monopoles in world theory. The existence of such monopoles, for which we would have $\mathrm{div}\vec{B} \neq 0$, was not clear in the past. Even though they were not observed in nature until now, magnetic monopoles could also not be excluded for sure theoretically (their consequences are, e.g., investigated on p. 273ff of Jackson 1999). The reason is that there was not a true understanding of electromagnetism and only the magnetic field $\vec{B}$ can be measured. As the field $\vec{A}$ cannot be directly observed, it was thus not clear whether $\vec{B} = \mathrm{rot}\vec{A}$ is obligatory. However, world theory changes all this. We now know the geometric interpretation of electromagnetism and the mathematical structure founded on it. This structure excludes magnetic monopoles.

2.6.8 Fundamental cause

Having introduced the electric and magnetic field in electrogravitation, we now return to the constraint (2.115). To simplify the following considerations, we consider only a single point of spacetime in this section. For such a point, it is known that we can perform a coordinate transformation which implies

$$g_{\alpha\beta} \stackrel{\star}{=} \eta_{\alpha\beta} \tag{2.125}$$

The asterisk in the above equation emphasizes that this equation does not simultaneously hold at all points of spacetime but only at a single one and for certain coordinates. Due to equation (2.125), we also have

$$g^{\alpha\beta} \stackrel{\star}{=} \eta_{\alpha\beta}$$

such that

$$F^{ab} \stackrel{\star}{=} F_{ab}$$

Moreover, definition (2.121) tells us

$$F^{ta} \stackrel{\star}{=} -F_{ta} \stackrel{\star}{=} E^a$$

That way, we find

$$F_{\alpha\beta}F^{\alpha\beta} \stackrel{\star}{=} 2F_{ta}F^{ta} + F_{ab}F^{ab} = -2E^aE^a + F_{ab}F_{ab}$$

Then, the constraint (2.115) implies

$$\boxed{E^aE^a \overset{D=3,5,6,\dots}{\stackrel{\star}{=}} \frac{1}{2}F_{ab}F_{ab}} \tag{2.126}$$

The above outcome is very important. It tells us that for $D = 3$ and $D \geq 5$ the absolute value of the electric field E^a is determined by the magnetic field F_{ab}. This has the interesting consequence that if the magnetic field F_{ab} vanishes, then also the electric field E^a has to disappear. Or in other words, there are **no pure electric fields** for $D = 3$ and $D \geq 5$. However, pure electric fields are an integral constituent of the electromagnetism as we know it in four dimensions. One merely has to conjure up, for instance, resting charges. Therefore, we consider the lack of pure electric fields for $D = 3$ and $D \geq 5$ due to box (2.126) as well as for $D = 1$ and $D = 2$ due to the first two paragraphs of Section 2.6.4 as the reason why we live in four dimensions. This does not mean that the other cases do not work in their own way. However, it is now obvious that the case $D = 4$ is special, and all remaining ones are based on fundamentally different dynamics. Let us summarize this result in words:

Cause of four dimensions:

Pure electric fields are possible in only four dimensions. (2.127)

As there is no strict way to exclude the cases $D \neq 4$, we will still take them into account in the computations coming in the rest of this paper, except of course where we constrain ourselves to $D = 4$ for other reasons.

2.6.9 Alternative causes

Notice that the cause (2.127) is not the only one which tries to explain why we live in four dimensions. The most important alternative is the finding that stable, circular orbits in gravitational fields are possible in just four dimensions (see Tangherlini 1963). In the following, we will give a short review of that alternative cause.

Let us consider the spherically symmetric gravitational field caused by a heavy point mass M. As we are interested merely in orbits here, it is sufficient to limit ourselves to a plane through the point mass. Next, we put a test particle of mass m in that plane such that the distance to the point mass M is r and the angular momentum around it l. The radial effective potential of the test particle is then

$$U_{\text{eff}} = \frac{l^2}{2mr^2} - \frac{mM}{r^{D-3}} \tag{2.128}$$

in D dimensions, where the first term is the centrifugal potential and the second one the gravitational potential energy.

Note that the centrifugal force is dimension-independent. This can be understood easily. The centrifugal force is the result of a constant rotation. And, a rotation changes the coordinates of only two dimensions, while the remaining ones are unaffected. Hence, it does not matter for the centrifugal potential whether there are additional dimensions. So, the mathematical expression for it does not depend on the number of dimensions D. The gravitational potential energy of relation (2.128), in contrast, has a dimension-dependence r^{D-3}. To find that dependence, one has to take D-dimensional vacuum general relativity and evaluate the corresponding Schwarzschild metric. This was done by Tangherlini (1963) and the metric is therefore called Schwarzschild-Tangherlini metric.

Knowing now the effective potential (2.128), let us quickly repeat how one finds the stable orbits. So, we evaluate

$$\frac{\mathrm{d}U_{\text{eff}}}{\mathrm{d}r} = -\frac{l^2}{mr^3} + (D-3)\,\frac{mM}{r^{D-2}} \tag{2.129}$$

and the only zero r_0 obeys $r_0^{D-5} = (D-3)\, m^2 M / l^2$. As

$$\left.\frac{\mathrm{d}^2 U_{\text{eff}}}{\mathrm{d}r^2}\right|_{r=r_0} = \frac{1}{r_0^4}\left[3\frac{l^2}{m} - (D-3)\,(D-2)\,\frac{mM}{r_0^{D-5}}\right] = (5-D)\,\frac{l^2}{mr_0^4}$$

has to be greater than zero for a stable orbit, we see $D < 5$. Combining this with the constraint $D \geq 4$ found at the end of Section 2.6.3, we eventually arrive at $D = 4$.

Due to the above repeated computations, it appears as if the cause of Tangherlini (1963) is a very short and simple way to show that our universe is 4-dimensional. However, we have to be careful and reflect on what is actually required to reach that cause. First, we must evaluate the Schwarzschild-Tangherlini metric. So, in contrast to our dimensional constraint of electromagnetism (2.115), it is not enough to stay at the mathematical level of the field equation. Instead, we have to descend to the application level and evaluate a solution of the field equation. Second, it is not sufficient to use just fields. We also need particles, namely the test particle of mass m. Therefore, we have to additionally prove that test particles move along geodesics in D dimensions. This cannot simply be postulated. The proof can be done like the 4-dimensional version on p. 267-275 of Carmeli (2001).

All in all, we that way see that Tangherlini's cause of four dimensions involves far more mathematical steps than those written out in this section. Our cause (2.127) instead requires just the computations done in Section 2.6.5. Later in this paper when we do not only study electrogravitation but full world theory, we will arrive at a "material field equation" from which one can even directly read off the dimensional constraint of electromagnetism (2.115) (see the later equation (3.10) for $\lambda = 0$). It is therefore reasonable to consider the result (2.127) as the true reason for our 4-dimensional universe, while other causes are additional, less fundamental ones.

A final remark about the above considerations. Both our cause (2.127) and the one of Tangherlini are based on finding a significant difference between the cases $D = 4$ and $D \neq 4$. So, we live in an exceptional number of dimensions. However, why could we not also live in one of the cases $D \neq 4$? To address this question, we would have to investigate the conditions for life in these cases. Note that we do not mean our 4-dimensional life here but life in general, i.e. dimension-independent. Such a consideration is of course beyond the scope of this paper. However, if it could be shown that life is sufficiently suppressed for $D \neq 4$, the anthropic principle could be used to explain why we live in the exceptional case $D = 4$.

2.7 Variation principle

2.7.1 Einstein-Maxwell theory

A common procedure in physics is to derive field equations from the variation principle. We investigate this approach throughout Section 2.7 in detail. However, the outcome at the end will be that the variation principle is not applicable to electrogravitation and thus considered as an obsolete concept in physics. Readers not interested in the details can therefore skip this section. Its main purpose is to explain why the variation principle fails.

The starting point for the variation principle is a Lagrangian. For Einstein-Maxwell theory, it has, for instance, the known form

$$L = R - F_{\alpha\beta}F^{\alpha\beta} \tag{2.130}$$

If we multiply the Lagrangian by the scalar density $\sqrt{-g}$, we arrive at the Lagrangian density

$$\mathcal{L} = \sqrt{-g}L \tag{2.131}$$

An integration then yields the action

$$A = \int \mathrm{d}^4 x \mathcal{L} \tag{2.132}$$

Applying a variation derivative $\delta/\delta...$ on this integral for one of the fields incorporated in it we retrieve the field equation belonging to the respective field. So, for Einstein-Maxwell theory, the fields are the electromagnetic vector potential A_α and the metric $g_{\alpha\beta}$. The electromagnetic field equation is thus

$$\frac{\delta A}{\delta A_\alpha} = 0 \tag{2.133}$$

and the gravitational one

$$\frac{\delta A}{\delta g_{\alpha\beta}} = 0 \tag{2.134}$$

Let us repeat the validity of these statements to help the reader understand what is going on in electrogravitation.

2.7.2 Maxwell's equation

We begin with the electromagnetic field equation. For that purpose, we first use equations (2.130) and (2.131) as well as the antisymmetry of the electromagnetic field strength $F_{\alpha\beta}$ to write the Lagrangian density as

$$\mathcal{L} = \sqrt{-g}\left[R - 2g^{\alpha\beta}g^{\gamma\delta}\partial_\alpha A_\gamma\left(\partial_\beta A_\delta - \partial_\delta A_\beta\right)\right] \tag{2.135}$$

Then, we have to evaluate the variation derivative in equation (2.133), which can be rewritten to partial derivatives in the following, known manner:

$$\frac{\delta A}{\delta A_\alpha} = \frac{\partial \mathcal{L}}{\delta A_\alpha} = \frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha} + \cdots \tag{2.136}$$

The operator $\partial/\delta...$ appearing above is the Euler-Lagrange derivative, which is neither a variation derivative nor a partial one. It is just an abbreviation of the derivatives on the right hand side of equation (2.136). We utilize the Euler-Lagrange derivative in the context of the variation principle, because it is better manageable than the variation derivative. The reason is that we do not have to care for the action then. Instead, we can work with the Lagrangian density all the time.

Now, in our case, only the second term on the right hand side of equation (2.136) plays a role such that equations (2.133) and (2.135) give

$$0 = \partial_\beta \frac{\partial \mathcal{L}}{\partial \partial_\beta A_\alpha} = -2\partial_\beta \left\{\sqrt{-g}g^{\gamma\delta}g^{\epsilon\zeta}\frac{\partial}{\partial \partial_\beta A_\alpha}\left[\partial_\gamma A_\epsilon\left(\partial_\delta A_\zeta - \partial_\zeta A_\delta\right)\right]\right\}$$

The derivatives appearing here can be evaluated via

$$\frac{\partial \partial_\delta A_\gamma}{\partial \partial_\beta A_\alpha} = \delta^\alpha_\gamma \delta^\beta_\delta$$

such that

$$0 = \partial_\beta\left(\sqrt{-g}g^{\beta\delta}g^{\alpha\zeta}F_{\delta\zeta}\right) = \partial_\beta\left(\sqrt{-g}F^{\beta\alpha}\right)$$

This is equal to Maxwell's equation (1.15), because the relation

$$\Gamma^{\beta}_{\beta\alpha} = \partial_\alpha \ln \sqrt{-g} \tag{2.137}$$

familiar from general relativity (see equation (8.51a) of Misner *et al.* 2002, an equation valid even in D dimensions) leads to

$$\frac{1}{\sqrt{-g}}\partial_\beta\left(\sqrt{-g}F^{\beta\alpha}\right) = \Gamma^{\gamma}_{\gamma\beta}F^{\beta\alpha} + \partial_\beta F^{\beta\alpha} + \Gamma^{\alpha}_{\gamma\beta}F^{\beta\gamma} = \nabla_\beta F^{\beta\alpha} \tag{2.138}$$

Note that we were able to add the term before the second equality sign, because it is zero due to the symmetry of the Christoffel symbols $\Gamma^{\alpha}_{\gamma\beta}$ in the two lower indices and the antisymmetry of the electromagnetic field strength $F^{\beta\gamma}$.

2.7.3 Einstein's equation

Now, we come to the gravitational field equation. So, we have to look at equation (2.134) and find out the variation derivative

$$\frac{\delta A}{\delta g_{\alpha\beta}} = \frac{\partial \mathcal{L}}{\delta g_{\alpha\beta}} = \frac{\partial \mathcal{L}}{\partial g_{\alpha\beta}} - \partial_\gamma \frac{\partial \mathcal{L}}{\partial\partial_\gamma g_{\alpha\beta}} + \cdots \tag{2.139}$$

For that purpose, we compute

$$\frac{\partial}{\delta g_{\alpha\beta}}\left(\sqrt{-g}g^{\gamma\delta}g^{\epsilon\zeta}F_{\gamma\epsilon}F_{\delta\zeta}\right) = \frac{\partial\sqrt{-g}}{\delta g_{\alpha\beta}}F_{\gamma\delta}F^{\gamma\delta} + 2\sqrt{-g}\frac{\partial g^{\gamma\delta}}{\delta g_{\alpha\beta}}F_{\gamma\epsilon}F_{\delta}{}^{\epsilon}$$

In the first term on the right hand side, we use the relation

$$\frac{\partial\sqrt{-g}}{\partial g_{\alpha\beta}} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta} \tag{2.140}$$

known from general relativity (see exercise 21.1 of Misner *et al.* 2002, where the considered relation holds even in D dimensions). For the second one, we take the differential of equation (1.3) such that

$$0 = \mathrm{d}g^{\gamma\alpha}g_{\alpha\beta} + g^{\gamma\alpha}\mathrm{d}g_{\alpha\beta}$$

and multiply the result by $g^{\beta\delta}$, i.e.

$$\mathrm{d}g^{\gamma\delta} = -g^{\alpha\gamma}g^{\beta\delta}\mathrm{d}g_{\alpha\beta} \tag{2.141}$$

which yields

$$\frac{\partial g^{\gamma\delta}}{\partial g_{\alpha\beta}} = -g^{(\alpha\gamma}g^{\beta)\delta} \tag{2.142}$$

Hence,

$$\frac{\partial}{\delta g_{\alpha\beta}}\left(\sqrt{-g}F_{\gamma\delta}F^{\gamma\delta}\right) = \sqrt{-g}\left(\frac{1}{2}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} - 2F^{(\alpha}{}_{\epsilon}F^{\beta)\epsilon}\right)$$

Then, equations (2.130), (2.131), (2.134) and (2.139) lead to

$$-\frac{1}{\sqrt{-g}}\frac{\partial}{\delta g_{\alpha\beta}}\left(\sqrt{-g}R\right) = 2\left(F^{\alpha}{}_{\gamma}F^{\beta\gamma} - \frac{1}{4}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right) \tag{2.143}$$

We could now also show that the left hand side is equal to the Einstein tensor $G^{\alpha\beta}$. However, for our purposes, it is sufficient to see that the right hand side in the above equation is just the one in Einstein's equation (1.20) (up to now using contravariant indices). That way, we have repeated the derivation of those terms in Maxwell's equation (1.15) and Einstein's equation (1.20) which contain the electromagnetic vector potential A_α.

2.7.4 Electrogravitation

For electrogravitation, we expect a similar behavior as for Einstein-Maxwell theory. So, a variation with respect to the metric should give the electrogravitational field equation (2.105). For simplicity, we proceed like in equation (2.143) and look only at the terms with an electromagnetic vector potential in them, i.e. the right hand side of equation (2.105). There, the constant factor $D-2$ can be left away in the following. If the variation principle worked, we would just have to rescale the resulting Lagrangian at the end by that factor and everything would be fine. So, without the mentioned factor, we seek for a Lagrangian which spews out

$$\nabla_{(\alpha}A_{\beta)} - A_\alpha A_\beta - g_{\alpha\beta}\left[\nabla_\gamma A^\gamma + \frac{1}{2}\left(D-3\right)A^2\right] \tag{2.144}$$

This expression (or its counterpart with contravariant indices α, β) should now be the result of a variation derivative like in equation (2.134), i.e. $\delta/\delta g_{\alpha\beta}$. Due to relation (2.139), we can also use the Euler-Lagrange derivative $\partial/\delta g_{\alpha\beta}$, which has to be applied on the associated Lagrangian density. However, we cannot be sure initially whether the derivative $\partial/\delta g_{\alpha\beta}$ is the right choice. Therefore, we perform a general search and allow the Euler-Lagrange derivatives

$$\frac{\partial}{\delta\left(\sqrt{-g}^{n_1} g_{\alpha\beta}\right)}, \frac{\partial}{\delta\left(\sqrt{-g}^{n_2} g^{\alpha\beta}\right)} \tag{2.145}$$

with arbitrary constants $n_1, n_2 \in \mathbb{R}$. These derivatives constitute all reasonable ways to perform a variation with respect to the metric. To simplify the relevant forms of the Lagrangian, we notice that the above derivatives conserve the number of occurrences of the electromagnetic vector potential per term. Hence, the terms in expression (2.144) that are linear in the electromagnetic vector potential, i.e.

$$\nabla_{(\alpha}A_{\beta)} - g_{\alpha\beta}\nabla_\gamma A^\gamma \tag{2.146}$$

must be the result of terms in the Lagrangian which are also linear. The only term possible here is obviously

$$\nabla_\alpha A^\alpha \tag{2.147}$$

Let us still be careful and verify that there are no other terms. The Lagrangian is a covariant scalar with coordinate weight zero. To construct terms in it, we can thus use the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α, the covariant derivative ∇_α and the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$. It is obvious that the Euler-Lagrange derivatives (2.145) also conserve the derivative level. Therefore, similar to our ansatz in Section 2.2.1, we can exclude quantities like the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$, which have a derivative level ≥ 2. If such quantities were present, the obligatory electromagnetic vector potential, which has the derivative level 1, would lead to a total derivative level ≥ 3. This is larger than the derivative level 2 of expression (2.146). Due to this derivative level, expression (2.146) must actually come from a term in the Lagrangian with one covariant derivative. So, the electromagnetic vector potential and the covariant derivative have to occur once in the form $\nabla_\alpha A_\beta$. Next, we look at the Levi-Civita tensor. Due to Section 2.6.4, we know that electromagnetism plays a role only if $D \geq 3$. The Levi-Civita tensor must therefore have at least three indices, i.e. $\epsilon_{\alpha\beta\gamma\ldots}$. Due to relation (2.8), it may not appear more than once. So, if it is present, we have to use the other available quantities, i.e. the metric $g_{\alpha\beta}$ and the expression $\nabla_\alpha A_\beta$, to contract the indices of the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\ldots}$ such that we obtain a scalar term in the Lagrangian. The mentioned other quantities have two indices.

We can thus not retrieve a scalar term for $D = 3$, because in that case the Levi-Civita tensor has three indices. The Levi-Civita tensor plays that way a role only for $D \geq 4$ such that it has at least four indices, i.e. $\epsilon_{\alpha\beta\gamma\delta\ldots}$. Two of its indices can be contracted by using the expression $\nabla_\alpha A_\beta$, i.e. $\epsilon_{\alpha\beta\gamma\delta\ldots}\nabla^\alpha A^\beta$. Then, at least two other indices remain, for which only the metric $g_{\alpha\beta}$ can be utilized. However, contracting indices with the metric leads to a vanishing term due to the total antisymmetry of the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta\ldots}$. This tensor does therefore not play a role here. So, we merely have the metric $g_{\alpha\beta}$ and the once occurring expression $\nabla_\alpha A_\beta$. The only way to then construct a scalar is our initial expression (2.147).

The question is now whether the term (2.147) can produce expression (2.146) if we perform a variation with respect to the metric. Hence, we have to multiply the term (2.147) by the factor $\sqrt{-g}$ to get a density like in equation (2.131) and then evaluate

$$\frac{\partial}{\delta\ldots}\left(\sqrt{-g}\nabla_\gamma A^\gamma\right)$$

Note that we use dots here to simultaneously treat both cases (2.145). Similar to this ambiguity, we unfortunately do not know the independent variable of the electromagnetic field. We therefore take care of all possibilities, which are

$$\sqrt{-g}^{n_3}A_\alpha, \sqrt{-g}^{n_4}A^\alpha$$

where $n_3, n_4 \in \mathbb{R}$ are another pair of constants.

2.7.5 Contravariant electromagnetic vector potential

Let us consider the independent variables $\sqrt{-g}^{n_4}A^\alpha$ first, because then the evaluation of the covariant derivative is easier. In that case, we have

$$\nabla_\gamma A^\gamma = \frac{1}{\sqrt{-g}^{n_4}}\nabla_\gamma\left(\sqrt{-g}^{n_4}A^\gamma\right) = \frac{1}{\sqrt{-g}^{n_4}}\left[\partial_\gamma\left(\sqrt{-g}^{n_4}A^\gamma\right) + (1-n_4)\,\Gamma^\delta_{\delta\gamma}\sqrt{-g}^{n_4}A^\gamma\right]$$

where the term n_4 in the last round bracket is required, because $\sqrt{-g}^{n_4}$ is a scalar density of coordinate weight n_4 (see 2nd last paragraph on p. 36 of Carmeli 2001). Due to equation (2.137), we then obtain

$$\frac{\partial}{\delta\ldots}\left(\sqrt{-g}\nabla_\gamma A^\gamma\right) = \partial_\gamma\left(\sqrt{-g}^{n_4}A^\gamma\right)\frac{\partial}{\delta\ldots}\sqrt{-g}^{1-n_4} + (1-n_4)\frac{\partial}{\delta\ldots}\left(A^\gamma\partial_\gamma\sqrt{-g}\right) \tag{2.148}$$

This should now yield expression (2.146). To simplify things, we split the desired outcome into terms where the partial derivative is applied on the electromagnetic vector potential and terms where it is applied on the metric. The latter are abbreviated by dots in the following:

$$\partial_{(\alpha}A_{\beta)} - g_{\alpha\beta}\partial_\gamma A^\gamma + \cdots \tag{2.149}$$

The expression $g_{\alpha\beta}\partial_\gamma A^\gamma$ appearing here may be produced by the first term on the right hand side of equation (2.148), because there a partial derivative ∂_γ and an electromagnetic vector potential A^γ occur. Yet, the expression $\partial_{(\alpha}A_{\beta)}$ can surely not be generated by that term. Thus, it must come from the second term of equation (2.148). At first glance, this seems to be impossible, because the partial derivative ∂_γ there is not applied on the electromagnetic vector potential A^γ. However, we have to be careful. Similar to relation (2.139), the Euler-Lagrange derivative $\partial/\delta...$ in the second term of equation (2.148) can be rewritten to a variation derivative $\delta/\delta...$. The operand is then no longer the Lagrangian density but the action. Looking at the relationship (2.132), this means an additional integral $\int d^4x$ in the operand. And, this integral allows us now to perform a partial integration (this works by the way only, because a fundamental assumption of the variation principle is that its variations are assumed to vanish at spatial and temporal infinity). That way, we can drag the partial derivative ∂_γ in the second term of equation (2.148) onto the electromagnetic vector potential A^γ there. Unfortunately, this generates just the contraction $\partial_\gamma A^\gamma$. All in all, the whole right hand side of equation (2.148) does therefore not produce the expression $\partial_{(\alpha}A_{\beta)}$. So, we see that the ansatz $\sqrt{-g}^{n_4}A^\alpha$ does not contain a working independent variable.

2.7.6 Covariant electromagnetic vector potential

Now, we come to the independent variables $\sqrt{-g}^{n_3}A_\alpha$. In that case, we write

$$\nabla_\gamma A^\gamma = \frac{1}{\sqrt{-g}^{n_3}}g^{\gamma\delta}\nabla_\gamma\left(\sqrt{-g}^{n_3}A_\delta\right) = g^{\gamma\delta}\left[\frac{1}{\sqrt{-g}^{n_3}}\partial_\gamma\left(\sqrt{-g}^{n_3}A_\delta\right) - \Gamma^\epsilon_{\delta\gamma}A_\epsilon - n_3\Gamma^\epsilon_{\epsilon\gamma}A_\delta\right] \tag{2.150}$$

Then, we apply definition (1.4) such that

$$g^{\gamma\delta}\Gamma^\epsilon_{\delta\gamma} = \frac{1}{2}g^{\gamma\delta}g^{\epsilon\zeta}\left(\partial_\delta g_{\zeta\gamma} + \partial_\gamma g_{\delta\zeta} - \partial_\zeta g_{\delta\gamma}\right) = g^{\gamma\delta}g^{\epsilon\zeta}\left(\partial_\gamma g_{\delta\zeta} - \frac{1}{2}\partial_\zeta g_{\delta\gamma}\right)$$

and

$$\Gamma^\epsilon_{\epsilon\gamma} = \frac{1}{2}g^{\epsilon\zeta}\left(\partial_\epsilon g_{\zeta\gamma} + \partial_\gamma g_{\epsilon\zeta} - \partial_\zeta g_{\epsilon\gamma}\right) = \frac{1}{2}g^{\epsilon\zeta}\partial_\gamma g_{\epsilon\zeta}$$

That way,

$$g^{\gamma\delta}\left(\Gamma^\epsilon_{\delta\gamma}A_\epsilon + n_3\Gamma^\epsilon_{\epsilon\gamma}A_\delta\right) = g^{\gamma\delta}g^{\epsilon\zeta}\left(\partial_\gamma g_{\delta\zeta} + \frac{n_3-1}{2}\partial_\zeta g_{\delta\gamma}\right)A_\epsilon \tag{2.151}$$

Hence, equation (2.150) leads to

$$\begin{aligned}\frac{\partial}{\delta...}\left(\sqrt{-g}\nabla_\gamma A^\gamma\right) \quad &= \quad \frac{\partial}{\delta...}\left(\sqrt{-g}^{1-n_3}g^{\gamma\delta}\right)\partial_\gamma\left(\sqrt{-g}^{n_3}A_\delta\right) \\ &\quad -\frac{\partial}{\delta...}\left[\sqrt{-g}g^{\gamma\delta}\left(\partial_\gamma g_{\delta\epsilon} + \frac{n_3-1}{2}\partial_\epsilon g_{\delta\gamma}\right)A^\epsilon\right]\end{aligned} \tag{2.152}$$

Similar to the last section, we are now interested in the question whether the term $\partial_{(\alpha}A_{\beta)}$ of the sum (2.149) can be produced. For the first term on the right hand side of equation (2.152), this is possible only if the Euler-Lagrange derivative $\partial/\delta...$ is applied on the second factor of its operand, i.e. using relation (2.141) we have

$$\sqrt{-g}^{1-n_3}\frac{\partial g^{\gamma\delta}}{\delta...}\partial_\gamma\left(\sqrt{-g}^{n_3}A_\delta\right) = \sqrt{-g}\frac{\partial g^{\gamma\delta}}{\delta...}\partial_\gamma A_\delta + \cdots = -\sqrt{-g}g^{\gamma\epsilon}g^{\delta\zeta}\frac{\partial g_{\epsilon\zeta}}{\delta...}\partial_\gamma A_\delta + \cdots \tag{2.153}$$

The dots are terms like in expression (2.149), i.e. no derivative is applied on the electromagnetic vector potential. For the second term in equation (2.152), we proceed like in the last section. So, we look at the operand of the Euler-Lagrange derivative. There, no partial derivative is applied on the electromagnetic vector potential A^ϵ. However, we can again use a partial integration. For the second term in the last round bracket of equation (2.152), this will merely produce a term with the factor $\partial_\epsilon A^\epsilon$. Hence, we have to look at only the first term in that bracket:

$$-\frac{\partial}{\delta\ldots}\left(\sqrt{-g}g^{\gamma\delta}\partial_\gamma g_{\delta\epsilon}A^\epsilon\right) = \sqrt{-g}g^{\gamma\delta}\frac{\delta g_{\delta\epsilon}}{\delta\ldots}\partial_\gamma A^\epsilon + \cdots \tag{2.154}$$

So, the only way to produce a term $\partial_{(\alpha}A_{\beta)}$ from equation (2.152) is to use the expressions shown on the right hand sides of relations (2.153) and (2.154). However, in relation (2.153), we can write

$$g^{\delta\zeta}\partial_\gamma A_\delta = \partial_\gamma A^\zeta + \cdots$$

such that adding the right hand sides of relations (2.153) and (2.154) leads to an expression without derivatives applied on the electromagnetic vector potential. This means that equation (2.152) is unable to produce a term $\partial_{(\alpha}A_{\beta)}$ like in expression (2.149). That way, we diagnose that also the independent variables $\sqrt{-g}^{n_3}A_\alpha$ fail.

2.7.7 Obsolescence

In the last two sections, we have tried to apply the variation principle on electrogravitation with the following independent variables

$$\begin{gathered}\sqrt{-g}^{n_1}g_{\alpha\beta}, \sqrt{-g}^{n_3}A_\alpha\\ \sqrt{-g}^{n_1}g_{\alpha\beta}, \sqrt{-g}^{n_4}A^\alpha\\ \sqrt{-g}^{n_2}g^{\alpha\beta}, \sqrt{-g}^{n_3}A_\alpha\\ \sqrt{-g}^{n_2}g^{\alpha\beta}, \sqrt{-g}^{n_4}A^\alpha\end{gathered}$$

Allowing densities and raising indices, instead of just taking the two fields $g_{\alpha\beta}$ and A_α like in Einstein-Maxwell theory, we have performed a very general search. However, in all four above cases we have obtained a failure. In principle, it may now be that there is some yet unknown way to apply the variation principle. It can also not be excluded for sure that there is a method to generalize the variation principle such that it is applicable on electrogravitation (in conformal geometrodynamics, addressed in Section 3.4, there is also no regular Lagrangian, see point 4 on p. 12 of Gorbatenko *et al.* 2002; however, there appears to be an exotic variation procedure in Gorbatenko & Pushkin 2002; yet, I have not studied it in detail such that I cannot judge it). Unfortunately, our current state of knowledge tells us that the variation principle and electrogravitation do not fit together.

This appears to be a severe problem. The variation method has a long tradition over the centuries in physics. For all established theories, it has turned out to be a viable way to derive the field equations belonging to them. Therefore, the variation principle is regarded as one of the fundaments of physics. And, it is employed as a constraint to simplify the search for new theories. So, we cannot trifle with the variation principle. However, when new territories are explored, it is not uncommon that some of the previously accepted foundations have to be given up. General relativity, for instance, has introduced curved spacetime and thus removed the restriction to flat spacetime, which was a base for the physics before that theory. This experience together with the realization that there is in fact no solid philosophical criterion why the variation principle should hold at all makes us proceed as follows. We consider the variation principle as a limitation which has to be given up in physics. Let us sum up this result in the following box:

$$\boxed{\begin{array}{l}\textbf{Obsolescence of variation principle:} \\ \qquad \text{The variation principle is not applicable on electrogravitation} \\ \qquad \text{and thus considered as an obsolete concept in physics.}\end{array}} \tag{2.155}$$

We recall from comparison (2.64) that Weyl derived his field equation from a Lagrangian. So, Weyl still demanded that the variation principle should hold universally in physics. This was another obstacle besides the missing torsion that prevented Weyl from arriving at our theory of electrogravitation.

2.8 Geodesics

2.8.1 General relativity

Having found a geometric interpretation of electromagnetism in Section 2.3, an interesting question is also how geodesics look like in electrogravitation. For that purpose, we consider a set of curves which fill spacetime tightly, i.e. every point of spacetime is hit by one curve. Such a set of curves is known as a congruence. As every point is hit exactly once, the congruence defines a tangent vector field u^α. This vector field can then be used to define geodesics. Geodesics are curves where the tangent vector u^α is transported parallelly to itself if one advances along the respective curve. In general relativity, geodesics are therefore described by

$$u^\beta \nabla_\beta u^\alpha = 0 \tag{2.156}$$

which is the familiar geodesic equation. Using

$$\nabla_\beta u^\alpha = \partial_\beta u^\alpha + \Gamma^\alpha_{\gamma\beta} u^\gamma$$

and defining

$$a^\alpha = u^\beta \partial_\beta u^\alpha \tag{2.157}$$

it can also be brought in the form

$$a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$$

Timelike geodesics are used in general relativity to describe the motion of massive, neutral particles. In that case, the vector field u^α is the velocity of the particles and a^α their acceleration.

2.8.2 Electrogravitational derivative

In electrogravitation, the covariant derivative ∇_β appearing in equation (2.156) is replaced. It can either be the electrogravitational derivative Δ_β encountered in definition (2.26) or the world derivative δ_β defined in equation (2.80). We study the electrogravitational derivative first, for which equation (2.156) becomes

$$\boxed{u^\beta \Delta_\beta u^\alpha = 0} \tag{2.158}$$

We call this result **electrogravitational geodesic equation**. Using the alternative definition (2.27), we obtain here

$$\Delta_\beta u^\alpha = \nabla_\beta u^\alpha + L^\alpha_{\gamma\beta} u^\gamma$$

Equation (2.45) allows us then to rewrite the second resulting term to

$$L^{\alpha}_{\gamma\beta}u^{\gamma} = \left[-\left(\frac{D}{2}-1\right)\delta^{\alpha}_{\gamma}A_{\beta} + \delta^{\alpha}_{\beta}A_{\gamma} - A^{\alpha}g_{\gamma\beta}\right]u^{\gamma} = -\left(\frac{D}{2}-1\right)u^{\alpha}A_{\beta} + \delta^{\alpha}_{\beta}A_{\gamma}u^{\gamma} - A^{\alpha}u_{\beta}$$

Hence, equation (2.158) becomes

$$u^{\beta}\nabla_{\beta}u^{\alpha} - \frac{1}{2}(D-4)\,u^{\alpha}u^{\beta}A_{\beta} - A^{\alpha}u^{2} = 0 \tag{2.159}$$

Remembering now definition (2.157), we arrive at

$$a^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}u^{\beta}u^{\gamma} - \frac{1}{2}(D-4)\,u^{\alpha}u^{\beta}A_{\beta} - A^{\alpha}u^{2} = 0$$

which reduces to

$$a^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}u^{\beta}u^{\gamma} \overset{D=4}{=} A^{\alpha}u^{2}$$

in four dimensions.

2.8.3 World derivative

An important question is what happens if we use the world derivative δ_{β} instead of the electrogravitational one Δ_{β}. In that case, equation (2.156) becomes

$$\boxed{u^{\beta}\delta_{\beta}u^{\alpha} = 0} \tag{2.160}$$

To evaluate this relation, we use the manifestly covariant form (2.83) of the world derivative such that

$$\delta_{\beta}u^{\alpha} = \nabla_{\beta}u^{\alpha} + l^{\alpha}_{\gamma\beta}u^{\gamma} + w_{u}A_{\beta}u^{\alpha} \tag{2.161}$$

where w_u is the gauge weight of the vector u^{α}. Unfortunately, we do not yet know this weight, a problem which was not present when using the electrogravitational derivative in the last section. We ignore this issue for the moment and continue evaluating equation (2.161). To this end, we apply definition (2.78) such that

$$l^{\alpha}_{\gamma\beta}u^{\gamma} = A_{\beta}u^{\alpha} + \delta^{\alpha}_{\beta}A_{\gamma}u^{\gamma} - A^{\alpha}u_{\beta}$$

Then, equation (2.160) can be written as

$$u^{\beta}\nabla_{\beta}u^{\alpha} + (2+w_{u})\,u^{\alpha}u^{\beta}A_{\beta} - A^{\alpha}u^{2} = 0 \tag{2.162}$$

Comparing this with equation (2.159), we see that $w_u = -D/2$.

However, we have a severe problem now. We know from general relativity that, for instance, massive particles should obey $u^2 = -1$. This equation can also be written as

$$g_{\alpha\beta}u^{\alpha}u^{\beta} = -1 \tag{2.163}$$

Moreover, due to Section 2.5.3, we recall that the gauge weight of the metric is 2. The conformal gauge invariance of the right hand side of relation (2.163) then demands that $w_u = -1$. Note that we would obtain the same gauge weight even if the right hand side of equation (2.163) were positive for spacelike and zero for lightlike geodesics. Anyway, the finding -1 contradicts the value $-D/2$ obtained from equation (2.162). To solve this issue, we have to study geodesics more thoroughly.

2.8.4 Electrogravitation

Let us address the problem of the last section in the following. For that purpose, we first realize that both equations (2.159) and (2.162) are of the form

$$a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma + x u^\alpha u^\beta A_\beta - A^\alpha u^2 = 0 \tag{2.164}$$

where $x \in \mathbb{R}$ is a constant and we have used definition (2.157). Then, we know from general relativity that we can change our view from considering a congruence of curves to studying only a single curve. How does that work exactly? We parametrize the congruence of curves with the coordinate mapping $x^\alpha_C(\lambda)$. The scalar $\lambda \in \mathbb{R}$ in that mapping is the curve parameter, and the index C, a quantity which we do not have to specify in more detail here, distinguishes the individual curves. The introduced mapping defines the tangent vectors

$$u^\alpha_C(\lambda) = \frac{\mathrm{d}x^\alpha_C(\lambda)}{\mathrm{d}\lambda} \tag{2.165}$$

which are correlated to the vector field u^α via

$$u^\alpha_C(\lambda) = u^\alpha\left(x^\alpha_C(\lambda)\right)$$

For the acceleration (2.157), we similarly have

$$a^\alpha_C(\lambda) = a^\alpha\left(x^\alpha_C(\lambda)\right)$$

such that

$$a^\alpha_C(\lambda) = u^\beta\left(x^\alpha_C(\lambda)\right)\partial_\beta u^\alpha\left(x^\alpha_C(\lambda)\right) = \frac{\mathrm{d}x^\alpha_C(\lambda)}{\mathrm{d}\lambda}\frac{\partial}{\partial x^\alpha_C(\lambda)} u^\alpha\left(x^\alpha_C(\lambda)\right) = \frac{\mathrm{d}u^\alpha_C(\lambda)}{\mathrm{d}\lambda} \tag{2.166}$$

where we have used the chain rule in the last step. For simplicity, we write $x^\alpha = x^\alpha_C(\lambda)$, $u^\alpha = u^\alpha_C(\lambda)$ and $a^\alpha = a^\alpha_C(\lambda)$ from now on such that the form of equation (2.164) does not change if we look at a single curve instead of a congruence. We just have to always bear in mind that there are two ways to understand that equation. It can either be a vector field equation, i.e. the fields in it depend on the considered point of spacetime, or it is parametrized by the scalar λ. Note that the quantity C does not play a role in that view. It is considered as fixed, because we look at only a single curve of the congruence.

The next step is to reparametrize equation (2.164). We limit ourselves to timelike geodesics in the following such that we choose a reparametrization to the coordinate time $x^t = t$. For geodesics which are not timelike, one has to select one of the other coordinates. It may then also be that the same coordinate cannot be chosen for all pieces of the considered geodesic. Anyway, the basic mathematical steps do not change such that the reparametrization to the time t is not a real reduction of generality. To actually reparametrize, we use the chain rule again and compute

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} v^\alpha + \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} \tag{2.167}$$

with

$$v^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} \tag{2.168}$$

Note that the quantity introduced here has only $D-1$ degrees of freedom, because $v^t = 1$. It is also possible to write equation (2.167) as

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 \left(\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} - v^\alpha \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}\right)$$

Moreover, equations (2.165) and (2.166) show us

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = a^\alpha$$

Then, equation (2.164) leads to

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 \left[v^\alpha \left(\Gamma^t_{\beta\gamma} u^\beta u^\gamma + x u^t u^\beta A_\beta - A^t u^2\right) - \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma - x u^\alpha u^\beta A_\beta + A^\alpha u^2\right]$$

We can simplify this lengthy expression by using

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} u^\alpha = \frac{\mathrm{d}\lambda}{\mathrm{d}t} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda} = v^\alpha \tag{2.169}$$

and $v^t = 1$ such that

$$\boxed{\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} = \left[v^\alpha \left(\Gamma^t_{\beta\gamma} - A^t g_{\beta\gamma}\right) - \Gamma^\alpha_{\beta\gamma} + A^\alpha g_{\beta\gamma}\right] v^\beta v^\gamma} \tag{2.170}$$

This is an alternative way to write the geodesic equation. Note that this equation is an identity for $\alpha = t$, because in that case both sides of it vanish due to $v^t = 1$. So, geodesics are described by merely $D-1$ and not D individual equations. This is also known from general relativity, i.e. it does not constitute a problem. In the timelike case, like chosen above, the purpose of the geodesic equation is to tell us how the spatial components x^a change when proceeding along the geodesics. To this end, $D-1$ equations are sufficient. Why does then the form (2.164) of the geodesic equation have D components? The reason is that equation (2.164) does not only inform us how the geodesics look like but also how the curve parameter λ changes along them. This knowledge is encoded in the component $\alpha = t$ of equation (2.164), which tells us what the nonzero derivative $\mathrm{d}^2 t/\mathrm{d}\lambda^2$ is. However, the curve parameter λ can be chosen arbitrarily, i.e. it has no influence on the shape of the geodesics. Therefore, equation (2.170) is a valid alternative way to describe geodesics, or more exactly timelike ones.

The crucial point is now that in the form (2.170) of the geodesic equation the constant x has disappeared. This would by the way also have been the case for non-timelike geodesics. We that way see that the constant x in equation (2.164) does not influence the shape of the geodesics but only how the curve parameter λ looks like. From the last section, we then recall that due to equation (2.162) and the finding $w_u = -1$ below relation (2.163), it is reasonable to choose a curve parameter such that $x = 1$. That way, we arrive at

$$u^\beta \nabla_\beta u^\alpha + u^\alpha u^\beta A_\beta - A^\alpha u^2 = 0 \tag{2.171}$$

This equation does not use the curve parameter of equation (2.159). However, it still describes the same geodesics as that equation. Therefore, both relations (2.158) and (2.160) are valid alternative ways to express the electrogravitational geodesic equation.

Let us use definition (2.157) again such that the geodesic equation (2.171) can be brought in the form

$$\boxed{a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = \left(A^\alpha u^\beta - u^\alpha A^\beta\right) u_\beta} \tag{2.172}$$

which is best comparable to general relativity.

We will now have a short look at the invariance of the above equation. Similar to the electrogravitational field equation (2.105), the geodesic equation does not appear to be conformally gauge invariant. However, the manifestly conformally gauge invariant form (2.160) immediately shows that all encountered possibilities to write the geodesic equation must be conformally gauge invariant. Another such possibility can by the way be obtained by using relations (2.77) and (2.78) such that

$$i^\alpha_{\beta\gamma} v^\beta v^\gamma = \left(\Gamma^\alpha_{\beta\gamma} + 2\delta^\alpha_\beta A_\gamma - A^\alpha g_{\beta\gamma}\right) v^\beta v^\gamma = \left(\Gamma^\alpha_{\beta\gamma} - A^\alpha g_{\beta\gamma}\right) v^\beta v^\gamma + 2A_\beta v^\alpha v^\beta$$

Then, equation (2.170) becomes

$$\boxed{\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} = \left(v^\alpha i^t_{\beta\gamma} - i^\alpha_{\beta\gamma}\right) v^\beta v^\gamma}$$

Here, the conformal gauge invariance is evident due to the one of the world connection $i^\alpha_{\beta\gamma}$, see the result (2.76).

2.8.5 No Lagrangian

From box (2.155), we know that the variation principle fails for electrogravitation and is therefore considered as obsolete. However, we have merely investigated the field equation for that purpose. So, an open issue is whether there is a Lagrangian for the electrogravitational geodesic equation. Note that even if there were such a Lagrangian, the variation principle would still have to be considered as obsolete, because only the field equation is deciding. It is merely a matter of interest to find out whether the variation principle is applicable for the geodesic equation.

Let us first repeat the variation principle for the geodesics in Riemannian spaces. Such geodesics are governed by the Lagrangian

$$L_{\mathrm{GE}} = g_{\alpha\beta} u^\alpha u^\beta \tag{2.173}$$

where the letters "GE" abbreviate "geodesic equation". A variation with respect to the coordinates x^α gives the familiar Euler-Lagrange equation

$$\partial_\alpha L_{\mathrm{GE}} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L_{\mathrm{GE}}}{\partial u^\alpha} \tag{2.174}$$

This equation leads to

$$\partial_\alpha g_{\beta\gamma} u^\beta u^\gamma = 2\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{\alpha\beta} u^\beta\right) = 2\left(u^\gamma \partial_\gamma g_{\alpha\beta} u^\beta + g_{\alpha\beta} a^\beta\right)$$

where we have used the chain rule as well as equations (2.165) and (2.166). Adopting definition (1.4), we then find

$$a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0$$

which is the general relativistic geodesic equation and the limit $A_\alpha = 0$ of equation (2.172).

The question is therefore now whether we can add terms to the Lagrangian (2.173) such that we obtain the full equation (2.172). These additional terms can be composed of the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α, the tangent vector u^α and the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$:

$$g_{\alpha\beta}, A_\alpha, u^\alpha, \epsilon_{\alpha\beta\ldots}$$

For simplicity, we ignore a dependence on the acceleration a^α or higher λ-derivatives of the tangent vector u^α. We do not have to be so strict here, because the variation principle is anyway already obsolete due to the field equation.

Unfortunately, we have the following problem now. The partial derivative ∂_α on the left hand side of equation (2.174) leads to a nonzero contribution only if it is applied on the metric $g_{\alpha\beta}$ or the electromagnetic vector potential A_α. However, such contributions do not occur on the right hand side of equation (2.172) such that they must be canceled by the right hand side of equation (2.174). The operator $\mathrm{d}/\mathrm{d}\lambda$ appearing there can either be applied on the tangent vector u^α. Then, it leads to an acceleration a^α, which is useless to get to equation (2.172). The alternative is that we have to use the chain rule. So, we have $\mathrm{d}/\mathrm{d}\lambda = u^\alpha \partial_\alpha$ such that we have again a partial derivative ∂_α and the same problem as mentioned earlier. That way, we see that there is no way to obtain the terms on the right hand side of equation (2.172). The variation principle is therefore not only not applicable to the field equation but also not to the geodesic equation of electrogravitation.

2.8.6 Interpretation

Let us eventually study the meaning of the electrogravitational geodesic equation (2.172). It is not very different from the equation of motion

$$a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = \frac{q}{m} F^{\alpha\beta} u_\beta \tag{2.175}$$

of charged particles in general relativity, where m is the mass and q the charge. This equation can in fact be written as

$$a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = -\frac{q}{m} g^{\alpha\beta} \left(\partial_\gamma A_\beta - \partial_\beta A_\gamma\right) u^\gamma$$

and equation (2.172) in the form

$$a^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = g^{\alpha\beta} \left(u_\gamma A_\beta - u_\beta A_\gamma\right) u^\gamma \tag{2.176}$$

That way, we interestingly see that we have to just replace

$$u_\alpha \to -\frac{q}{m} \partial_\alpha$$

in the round bracket of equation (2.176) to arrive at the equation of motion (2.175). Unfortunately, there is currently no deeper understanding of this curiosity. We also do not know whether the equation of motion (2.175) is still valid in electrogravitation or whether it has to be replaced by a yet unknown relation.

Whatever the equation of motion in electrogravitation is, it is also not clear whether the geodesic equation (2.172) has something to do with it. In the limit of vacuum gravitation, equation (2.172) at least describes the motion of particles. Moreover, the equation of motion in electrogravitation should somehow be derivable from the electrogravitational field equation (2.105). However, we do not know whether the geodesic equation (2.172) is usable as an intermediate step for that purpose. One outstanding possibility is that equation (2.172) itself is actually the equation of motion in electrogravitation. Then, a severe obstacle is that neither the mass m nor the charge q appear.

It is now obvious that we reach the realm of speculations here. A central problem is that it may also be that quantum theory is required to find the equation of motion in electrogravitation. That way, we definitely leave the scope of this paper such that we are eventually done with geodesics.

2.9 Spin

2.9.1 Parallel transport

Geodesics are the parallel transport of a vector along itself. However, it is also possible to transport that vector along an arbitrary curve in spacetime. In that case, we have two different vectors, the already encountered vector u^α tangent to the curve and the parallelly transported vector S^α. Similar to equations (2.158) and (2.160), there are then two ways to express the parallel transport mathematically, namely

$$u^\beta \Delta_\beta S^\alpha = 0 \tag{2.177}$$

and

$$\boxed{u^\beta \delta_\beta S^\alpha = 0} \tag{2.178}$$

In the following, we will study whether the above two equations are equal. For that purpose, we take definitions (2.26), (2.68) and (2.80) such that we can write equations (2.177) and (2.178) as

$$u^\beta \left(\partial_\beta S^\alpha + I^\alpha_{\gamma\beta} S^\gamma \right) = 0 \tag{2.179}$$

and

$$u^\beta \left(\partial_\beta S^\alpha + i^\alpha_{\gamma\beta} S^\gamma + w_S A_\beta S^\alpha \right) = 0 \tag{2.180}$$

with the gauge weight w_S of the vector S^α. Next, we combine equations (2.46) and (2.75), which implies

$$\boxed{I^\alpha_{\beta\gamma} = i^\alpha_{\beta\gamma} - \frac{D}{2} \delta^\alpha_\beta A_\gamma}$$

This shows us that both equations (2.179) and (2.180) are of the form

$$u^\beta \left(\partial_\beta S^\alpha + i^\alpha_{\gamma\beta} S^\gamma + x A_\beta S^\alpha \right) = 0 \tag{2.181}$$

where $x \in \mathbb{R}$ is a constant. For equation (2.179), this constant is $x = -D/2$, and for equation (2.180), it is $x = w_S$. Using relation (2.169) and the chain rule, we can also write

$$\frac{\mathrm{d}S^\alpha}{\mathrm{d}t} + \left(i^\alpha_{\gamma\beta} S^\gamma + x A_\beta S^\alpha \right) v^\beta = 0 \tag{2.182}$$

This limits the transport curve to a timelike one. For non-timelike ones, one merely has to change the definition of the coordinate t.

2.9.2 Direction

We proceed by introducing the direction

$$s^\alpha = \frac{S^\alpha}{S^t} \tag{2.183}$$

Hence,

$$\frac{\mathrm{d}s^\alpha}{\mathrm{d}t} = \frac{1}{S^t}\left(\frac{\mathrm{d}S^\alpha}{\mathrm{d}t} - s^\alpha \frac{\mathrm{d}S^t}{\mathrm{d}t}\right) = \frac{1}{S^t}\left[-\left(i^\alpha_{\gamma\beta}S^\gamma + xA_\beta S^\alpha\right)v^\beta + s^\alpha\left(i^t_{\gamma\beta}S^\gamma + xA_\beta S^t\right)v^\beta\right]$$

which simplifies to

$$\boxed{\frac{\mathrm{d}s^\alpha}{\mathrm{d}t} = \left(s^\alpha i^t_{\gamma\beta} - i^\alpha_{\gamma\beta}\right)v^\beta s^\gamma}$$

This equation does not contain the constant x, similar to equation (2.170) for geodesics. Unfortunately, this does not tell us that equations (2.177) and (2.178) are equal. They surely transport the direction s^α in the same manner. However, an open issue is still the length of the vector S^α.

2.9.3 Length

For the length of the vector S^α, we evaluate

$$u^\alpha\partial_\alpha S^2 = u^\alpha\partial_\alpha\left(g_{\beta\gamma}S^\beta S^\gamma\right) = u^\alpha\left(S^\beta S^\gamma\partial_\alpha g_{\beta\gamma} + 2S_\beta\partial_\alpha S^\beta\right)$$

Relation (2.181) then gives

$$u^\alpha\partial_\alpha S^2 = u^\alpha S_\beta\left[S^\gamma g^{\beta\delta}\partial_\alpha g_{\delta\gamma} - 2\left(i^\beta_{\gamma\alpha}S^\gamma + xA_\alpha S^\beta\right)\right]$$

or with the split (2.77)

$$u^\alpha\partial_\alpha S^2 = u^\alpha\left\{S_\beta\left[g^{\beta\delta}\left(\partial_\gamma g_{\delta\alpha} + \partial_\alpha g_{\gamma\delta} - \partial_\delta g_{\gamma\alpha}\right) - 2\left(\Gamma^\beta_{\gamma\alpha} + l^\beta_{\gamma\alpha}\right)\right]S^\gamma - 2xA_\alpha S^2\right\}$$

Next, we adopt definition (1.4) such that we arrive at

$$u^\alpha\partial_\alpha S^2 = -2u^\alpha\left(l^\beta_{\gamma\alpha}S_\beta S^\gamma + xA_\alpha S^2\right)$$

Eventually, definitions (2.78), (2.165) and (2.169) produce

$$\boxed{\frac{\mathrm{d}S^2}{\mathrm{d}t} = -2v^\alpha\left(1+x\right)A_\alpha S^2} \tag{2.184}$$

That way, we see that the constant x in fact matters.

This leads to two possibilities. We can choose the gauge weight $w_S = -D/2$. For that case, equations (2.177) and (2.178) describe the same parallel transport (recall equation (2.181) and the sentence below it). However, then the length of the vector S^α is not constant. The alternative is explained in the following section.

2.9.4 Spin transport

We demand that

$$S^2 = \text{const} \tag{2.185}$$

like, for instance, for a spin in general relativity. Then, relation (2.184) shows $x = -1$. Unfortunately, this turns out to be a problem for equation (2.177). From Section 2.9.1, we recall that equation (2.177) leads to $x = -D/2$. Therefore, this equation cannot describe constant lengths (2.185) unless $D = 2$. However, this case is irrelevant, because we live in four dimensions. The described limitation of equation (2.177) is by the way the reason why we have not put a box around it. So, we have to use equation (2.178) and, due to Section 2.9.1, the gauge weight $w_S = -1$ of the vector S^α. Looking at equation (2.182), we then find

$$\frac{\mathrm{d}S^\alpha}{\mathrm{d}t} + \left(i^\alpha_{\gamma\beta} S^\gamma - A_\beta S^\alpha\right) v^\beta = 0$$

Using the split (2.77) and definition (2.78) we can also write

$$\frac{\mathrm{d}S^\alpha}{\mathrm{d}t} + \left(\Gamma^\alpha_{\gamma\beta} S^\gamma + \delta^\alpha_\beta A_\gamma S^\gamma - A^\alpha S_\beta\right) v^\beta = 0$$

Next, we adopt relation (2.169) and arrive at

$$\boxed{\frac{\mathrm{d}S^\alpha}{\mathrm{d}\lambda} + \Gamma^\alpha_{\beta\gamma} S^\beta u^\gamma = \left(A^\alpha u^\beta - u^\alpha A^\beta\right) S_\beta} \tag{2.186}$$

This is the analog of equation (2.172).

Let us now assume that the vector S^α is in fact the spin of a massive particle. In the rest frame of that particle, the velocity and spin are $u^\alpha = (u^t, 0)$ and $S^\alpha = (0, S^a)$, which shows that for any coordinate choice

$$u^\alpha S_\alpha = 0 \tag{2.187}$$

That way, equation (2.186) simplifies to the **electrogravitational spin equation**

$$\boxed{\frac{\mathrm{d}S^\alpha}{\mathrm{d}\lambda} + \left(\Gamma^\alpha_{\beta\gamma} u^\gamma + A_\beta u^\alpha\right) S^\beta = 0} \tag{2.188}$$

To see that this equation is reasonable, we may also proceed in the following way. We define

$$\nabla_u = u^\alpha \nabla_\alpha$$

such that equations (2.172) and (2.186) can be written as

$$\nabla_u u^\alpha = \left(A^\alpha u^\beta - u^\alpha A^\beta\right) u_\beta$$

and

$$\nabla_u S^\alpha = \left(A^\alpha u^\beta - u^\alpha A^\beta\right) S_\beta \tag{2.189}$$

Then, relations (2.163) and (2.187) show

$$\nabla_u S^\alpha = u^\alpha S_\beta \nabla_u u^\beta$$

which is Fermi-Walker transport (see equation (40.27) of Misner *et al.* 2002). This would by the way have been an alternative procedure to derive the spin equation (2.188). However, to give the reader a better intuition, we have followed an approach in analogy to the geodesics.

One final remark is required here. We have limited ourselves to spins of the gauge weight $w_S = -1$ in this section. That way, condition (2.185) is obeyed. However, electrogravitation is, in principle, able to describe spins of arbitrary gauge weights. An important question is therefore which of these spins are in fact realized in nature. To find an answer, it is probably necessary to not just describe spins as vectors but as dynamic properties of extended objects. Unfortunately, such an investigation is beyond the scope of this paper. So, it cannot be excluded that the spins of gauge weight $w_S = -1$ described by equation (2.188) may not even be realized in nature. This case was just the most conservative one from the viewpoint of general relativity. It could, e.g., also be that the spins realized in nature have to just change their length according to the last paragraph of the preceding section. In that case, both equations (2.177) and (2.178) could be used for spin transport just as equations (2.158) and (2.160) can be adopted for geodesics.

We are that way done with the fundamentals of electrogravitation. There are some additional issues left like, for instance, how can the contracted Bianchi identity and other such relations be generalized to electrogravitation? However, we postpone all these questions for the moment and instead introduce a special form of matter in the following sections first. This will generalize electrogravitation to a more natural and complete theory. We will afterwards return to the open issues of electrogravitation but immediately study them together with the introduced matter. That way, we do not have to consider the same topics multiply. So, all aspects investigated up to now did not require matter, and they were exclusively electrogravitational issues.

3 Classical world theory

3.1 Unification trick

3.1.1 Hierarchy

In Chapter 2, we have unified electromagnetism and gravitation to electrogravitation. Unfortunately, electrogravitation does not contain a description of matter such that a generalization of that theory seems unavoidable. We will therefore perform another unification in this chapter. It will merge electrogravitation with a special form of classical matter, explained thoroughly in the next section. It is currently unclear whether there is a connection between this form of matter and elementary particle physics, our standard description of matter. In order to avoid any misunderstanding, we have to keep this distinction in mind throughout the rest of this paper. The result of the inclusion of the classical matter is a new theory which we call **world theory** (=WT). The two unifications leading to it are visualized in the following:

$$
\begin{tikzcd}
\text{Gravitation} \arrow[dr] & & \text{Electromagnetism} \arrow[dl] \arrow[dr] & & \text{Matter} \arrow[dl] \\
& \text{Electrogravitation} \arrow[dr] & & \text{Electromatter} \arrow[dl] & \\
& & \text{World theory} & &
\end{tikzcd}
$$

Instead of unifying electromagnetism first with gravitation and then with the mentioned type of matter, it is also possible to directly start with matter. This is shown in the above drawing, and we refer to the outcome as **electromatter**. So, the drawing contains three unifications: electrogravitation, electromatter and world theory. Both electrogravitation and electromatter are completely independent steps. This means that they are two theories on their own. They may respectively be a valid, yet incomplete description of the fundamentals of classical physics even if their generalization to world theory were not realized in nature.

3.1.2 Matter field

To generalize electrogravitation, we merely have to study where features can be added to that theory. Electrogravitation is based on the metric $g_{\alpha\beta}$ and the electromagnetic vector potential A_α. The two fields are governed by the tensor equation (2.105). From the last chapter, we recall that this equation incorporates the vector equation (2.108), which exclusively describes the dynamics of electromagnetism. To extract the vector equation from the tensorial one, we had to apply a covariant divergence. The generalization of electrogravitation is then based on the idea to repeat these steps with a new field. To find out how that field looks like, we make ourselves aware that going from the metric $g_{\alpha\beta}$ of gravitation to the vector potential A_α of electromagnetism one index is removed. It is therefore natural to introduce a new field where yet another index is removed. That way, we obtain a scalar $\lambda \in \mathbb{R}$ which we call **matter field**.

The critical reader will now wonder what that field has to do with matter at all. To clarify this, we remember that physics incorporates two distinct phenomena, forces and matter. As only two classical forces are known, gravitation and electromagnetism, the field λ must have something to do with matter. The reason why this seems so weird is that matter is typically described in a much more complicated manner. In general relativity, this is done by general relativistic hydrodynamics. Even the most ideal form of hydrodynamics must contain a vector field which describes the motion of the considered fluid. Therefore, hydrodynamics cannot be reduced to a single scalar field, like the matter field λ. Yet, general relativistic hydrodynamics is not a fundamental description of matter. It is an approximative, statistical description of classical particles. The classical particles themselves are again an approximation (via the Ehrenfest theorem), for which quantum theory gives a more accurate picture. The important point is now that quantum theory uses wave functions to describe particles. The easiest type of wave function is a scalar field $\psi \in \mathbb{C}$ such that it now seems much less peculiar to use the name "matter field" for the scalar λ. The only remaining difference is that the wave function ψ is complex, whereas the matter field λ is real. This discrepancy cannot be resolved yet. We have to first dive into the mathematical formalism of world theory and then perform some additional computations. Later in this paper (see Section 3.10.3), we will return to this topic and clarify the relationship between the two fields.

It is also possible to adopt another viewpoint. The only difference between the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α and the matter field λ is the number of indices. And, we know that the first two of these fields are associated with forces, namely the gravitational and the electromagnetic one. Hence, it is reasonable to consider the effect of the third field, the matter field λ, as a force, too. This force is then a third classical force or a fifth force if we include the weak and strong one. We want to call it **matter force**. Hence, the distinction between forces and matter mentioned in the last paragraph actually refers to non-material forces and matter. To make all even more clear, we conclude with the following summary:

Phenomenon	Force	Field
Gravitation	Gravitational force	Metric $g_{\alpha\beta}$
Electromagnetism	Electromagnetic force	Electromagnetic vector potential A_α
Matter	Matter force	Matter field λ

3.2 Field equation

3.2.1 World equation

The matter field λ must now somehow modify the electrogravitational field equation (2.105). For that purpose, we remember the derivative level introduced in Section 2.3.1. It tells us that the metric $g_{\alpha\beta}$ and the electromagnetic vector potential A_α have the derivative levels 0 and 1. It is then natural to assume that the matter field λ has the derivative level 2. That way, we can reduce the number of possible modifications of equation (2.105) drastically. These modifications are based on terms which contain the matter field λ at least a single time. In contrast to the metric $g_{\alpha\beta}$ and the electromagnetic vector potential A_α, the matter field λ has no index. Therefore, it appears as if we could also allow $1/\lambda$. However, this expression can be excluded, because it is reasonable to demand electrogravitation in the limit $\lambda = 0$. More general functional dependencies on λ do not have to be considered here, because either they can be Taylor-expanded in terms of λ or they are surely too exotic. Now, due to Section 2.3.1 we know that the field equation must have the derivative level 2. Hence, the matter field is not allowed to occur more than once in the modification terms. As the electromagnetic vector potential A_α and the covariant derivative ∇_α have the derivative level 1, the only other quantities relevant in a covariant modification are the metric $g_{\alpha\beta}$ and the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$. The Levi-Civita tensor may occur only once per term, because otherwise it could be rewritten to Kronecker tensors according to equation (2.8). However, due to its total antisymmetry, an individual Levi-Civita tensor is obviously useless here. That way, we may use the matter field λ a single time and additionally the metric $g_{\alpha\beta}$, which gives $g_{\alpha\beta}\lambda$ times a constant factor as the only possible modification. The constant factor appearing here can be chosen freely. Changing it merely rescales the matter field λ and does thus not produce a new field equation. We choose the rather complicated value $-(D-1)(D-2)/6$ for that factor. This choice will become understandable further below when we interpret matter geometrically (text below equation (3.21) and paragraph below equation (3.65)). All in all, we then have the modification

$$\boxed{G_{\alpha\beta} = (D-2)\left\{\nabla_{(\alpha}A_{\beta)} - A_\alpha A_\beta - g_{\alpha\beta}\left[\nabla_\gamma A^\gamma + \frac{1}{2}(D-3)A^2 + \frac{1}{6}(D-1)\lambda\right]\right\}} \tag{3.1}$$

or in four dimensions

$$G_{\alpha\beta} \overset{D=4}{=} \nabla_\alpha A_\beta + \nabla_\beta A_\alpha - 2A_\alpha A_\beta - g_{\alpha\beta}\left(2\nabla_\gamma A^\gamma + A^2 + \lambda\right) \tag{3.2}$$

We call this modification of electrogravitation **world equation** (in conformal geometrodynamics, see Section 3.4, equation (3.2) as a pure tensor equation can, for example, be found in equations (1) and (2) of Gorbatenko 2010).

Looking at the world equation (3.1), we also see that world theory is, like electrogravitation, governed by conformal gauge invariance. Why? We recall from box (2.70) that the metric $g_{\alpha\beta}$ is modified by the factor $e^{-2\chi}$ under such a transformation. World theory is then conformally gauge invariant, because that factor can be canceled by

$$\boxed{\lambda \overset{\mathrm{cg}}{\to} e^{2\chi}\lambda} \tag{3.3}$$

In addition to the above, the world equation (3.1) is also covariant, because it is a tensor equation. Hence, world theory is world invariant.

3.2.2 Electromagnetic

Similar to what we know from electrogravitation, the world equation (3.1) should be a unified field equation that simultaneously describes the dynamics of all three fields $g_{\alpha\beta}$, A_α and λ. Let us now verify that this idea truly works. For that purpose, we first derive the electromagnetic field equation again like in Section 2.6.2, but this time the matter field λ will be included. That field causes extra terms in the already performed computations.

We begin by looking at the world equation (3.1), which can be abbreviated as

$$G_{\alpha\beta} = \cdots - \frac{1}{6}(D-1)(D-2)\, g_{\alpha\beta}\lambda \tag{3.4}$$

where the dots represent in this context terms that contain the electromagnetic vector potential. A contraction of the two indices present above leads to

$$G = \cdots - \frac{D}{6}\,(D-1)\,(D-2)\,\lambda$$

such that we can evaluate the Ricci tensor (2.17)

$$R_{\alpha\beta} \overset{D\neq 2}{=} \cdots + \frac{1}{3}\,(D-1)\,g_{\alpha\beta}\lambda \tag{3.5}$$

It does not matter here that the case $D=2$ is excluded. The factor $D-2$ in the world equation (3.1) anyway shows that electromagnetism and matter work only for $D \neq 2$.

Now, we return to Section 2.6.2 and look at the computations below the electrogravitational field equation (2.105). That equation is just relation (3.4) without the λ-term. Relation (2.106) then shows the outcome when applying the covariant derivative ∇^β on equation (2.105). In world theory, this relation therefore generalizes to

$$\cdots + 2\,[\nabla_\beta, \nabla_\alpha]\,A^\beta - \cdots \overset{D\neq 2}{=} \frac{1}{3}\,(D-1)\,\nabla_\alpha\lambda \tag{3.6}$$

where we have left away the overall factor $(D-2)\,/2$. The term on the left hand side is intentionally not abbreviated by dots. This term is affected by the matter field λ. To see the influence, we look at relation (2.107), for which equation (3.5) tells us

$$[\nabla_\beta, \nabla_\alpha]A^\beta = A^\beta R_{\beta\alpha} \overset{D\neq 2}{=} \cdots + \frac{1}{3}\,(D-1)\,A_\alpha\lambda$$

That way, equation (3.6) becomes

$$\cdots \overset{D\neq 2}{=} \frac{1}{3}\,(D-1)\,(\nabla_\alpha - 2A_\alpha)\,\lambda$$

It is then clear how equation (2.108) has to be modified. The electromagnetic field equation in world theory is

$$\boxed{\left[\nabla^\beta + (D-4)\,A^\beta\right] F_{\beta\alpha} \overset{D\neq 2}{=} \frac{1}{3}\,(D-1)\,(\nabla_\alpha - 2A_\alpha)\,\lambda} \tag{3.7}$$

or in four dimensions

$$\nabla^\beta F_{\beta\alpha} \overset{D=4}{=} (\nabla_\alpha - 2A_\alpha)\,\lambda \tag{3.8}$$

(in conformal geometrodynamics, see Section 3.4, the pure tensor equation (3.8) is, for instance, given in equation (28) of Gorbatenko 2010).

3.2.3 Material

For matter, we proceed in the same way as in the last section for electromagnetism. So, this time we take the electromagnetic field equation (3.7) and evaluate an additional covariant divergence. This means the application of the operator ∇^α such that identity (2.114), which can be considered as an electromagnetic analog of the contracted Bianchi identity (2.5), gives

$$(D-4)\,\nabla^\alpha\left(A^\beta F_{\beta\alpha}\right) \overset{D\neq 2}{=} \frac{1}{3}\,(D-1)\,\nabla^\alpha\,(\nabla_\alpha - 2A_\alpha)\,\lambda \tag{3.9}$$

On the left hand side, we compute

$$\nabla^\alpha \left(A^\beta F_{\beta\alpha}\right) = \frac{1}{2} F^{\alpha\beta} F_{\beta\alpha} - A^\alpha \nabla^\beta F_{\beta\alpha}$$

Then, equation (3.7) leads to

$$\nabla^\alpha \left(A^\beta F_{\beta\alpha}\right) \overset{D\neq 2}{=} \frac{1}{2} F^{\alpha\beta} F_{\beta\alpha} - \frac{1}{3} (D-1) A^\alpha (\nabla_\alpha - 2A_\alpha) \lambda$$

where the second term in the squared bracket of equation (3.7) has no influence here, because the expression $A^\alpha A^\beta F_{\beta\alpha}$ vanishes due to the antisymmetry of the electromagnetic field strength $F_{\beta\alpha}$. The above outcome allows us to write equation (3.9) as

$$\boxed{(D-1)\left[\nabla^\alpha + (D-4) A^\alpha\right] (\nabla_\alpha - 2A_\alpha) \lambda \overset{D\neq 2}{=} \frac{3}{2} (D-4) F^{\alpha\beta} F_{\beta\alpha}} \tag{3.10}$$

or in four dimensions

$$\nabla^\alpha (\nabla_\alpha - 2A_\alpha) \lambda \overset{D=4}{=} 0 \tag{3.11}$$

which is the **material field equation**. Without gravitation and electromagnetism, this equation is the easiest relativistic field equation known, the scalar wave equation

$$\Box \lambda \overset{D=4}{=} 0 \tag{3.12}$$

with the d'Alembertian $\Box = \partial^\alpha \partial_\alpha$.

So, two field equations are hidden in the world equation (3.1): the electromagnetic field equation (3.7) and the material field equation (3.10). The world equation (3.1) itself can actually be considered as the gravitational field equation. That way, we have a separate field equation for each of the three fields $g_{\alpha\beta}$, A_α and λ or just the unified field equation (3.1). When looking at the right hand sides of the mentioned equations, we see the sources of gravitation, electromagnetism and matter. Equation (3.11) shows us that matter does not have a source in four dimensions. So, world theory is closed there, and it is not possible to perform another covariant divergence, because there are no running indices left. The right hand side of equation (3.11) has actually another purpose. As already pointed out in the 2nd last paragraph of Section 2.6.9, the right hand side of the D-dimensional version (3.10) contains the dimensional constraint of electromagnetism (2.115). Hence, the source of matter fixes the number of dimensions.

Vice versa, we can also not go in the other direction and derive the gravitational field equation (3.1) from the field equation of a new fundamental field $g_{\alpha\beta\gamma}$ with three indices, besides the already known ones $g_{\alpha\beta}$, A_α and λ. If such an approach worked like for electromagnetism and matter, the gravitational field equation (3.1) would have to be writable as $\nabla^\gamma X_{\gamma\alpha\beta} = 0$, where $X_{\gamma\alpha\beta}$ is an unknown covariant expression. The derivative level of that expression would have to be 1, because the one of equation (3.1) is 2. In addition to that, for $g_{\alpha\beta\gamma} = A_\alpha = \lambda = 0$, the expression $X_{\gamma\alpha\beta}$ would have to be composed of only the metric $g_{\alpha\beta}$ and the partial derivative ∂_α. However, the only possible such expression is a combination of covariant derivatives $\nabla_\alpha g_{\beta\gamma}$, which vanish due to equation (1.9). It is therefore not reasonable to study fundamental fields with more than two indices, like the above suggestion $g_{\alpha\beta\gamma}$.

3.3 Geometric interpretation of matter

3.3.1 World curvature

The steps with matter performed until now are basically similar to those of electromagnetism. This is especially visible in how we obtained the material field equation in the last section. It is therefore natural to proceed like for electromagnetism and investigate whether our form of matter can be interpreted geometrically. For that purpose, we recall the derivative levels of the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α and the matter field λ, which are 0, 1 and 2. From the last paragraph in Section 2.3.1, we additionally remember that electromagnetism does not influence vacuum general relativity directly at the derivative level 0 of the metric $g_{\alpha\beta}$ but at the derivative level 1 of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$. The matter field λ has the derivative level 2 and should therefore influence electrogravitation at the level of curvature. So, the matter field λ does not occur in the connection such that both the electrogravitational connection $I^\alpha_{\beta\gamma}$ and the world connection $i^\alpha_{\beta\gamma}$ do not have to be modified. However, we look for a generalization of the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$, i.e. the form (2.33) is not sufficient for world theory.

To obtain a generalized curvature, we use the same approach as for the world equation (3.1). So, we add terms to the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$ which must contain the matter field λ at least once. Due to the derivative level 2 of the curvature, it is then obvious that the matter field λ can be used only once per term, and the other possibilities are the metric $g_{\alpha\beta}$ and the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$. Based on relation (2.8), the Levi-Civita tensor $\epsilon_{\alpha\beta\ldots}$ may appear only a single time per term. For $D = 4$, we may therefore generalize the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$ by adding the term $\epsilon^\alpha{}_{\beta\gamma\delta}\lambda$ multiplied with an arbitrary real constant. However, when going to the field equation, we will contract the indices α and γ such that the total antisymmetry of the Levi-Civita tensor makes the mentioned term vanish. So, this term would play a role only for curvature but not in the field equation such that we can omit it. Then, the Levi-Civita tensor is irrelevant for $D = 4$, and thus it is not reasonable to use it in the other cases. That way, the new terms consist of the metric $g_{\alpha\beta}$ and a single time the matter field λ. This enables us to make an ansatz for the **world curvature**

$$\Omega^\alpha{}_{\beta\gamma\delta} = Z^\alpha{}_{\beta\gamma\delta} + \left(a_3\delta^\alpha_\beta g_{\gamma\delta} + b_3\delta^\alpha_\gamma g_{\beta\delta} + c_3\delta^\alpha_\delta g_{\beta\gamma}\right)\lambda \tag{3.13}$$

where $a_3, b_3, c_3 \in \mathbb{R}$ are constants whose values we have to find in the following.

A central property of the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$ is its antisymmetry in the last two indices γ and δ:

$$Z^\alpha{}_{\beta\gamma\delta} \equiv -Z^\alpha{}_{\beta\delta\gamma} \tag{3.14}$$

This is a direct consequence of definition (2.91). As this definition is the shortest and most natural one to specify the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$, it is reasonable to demand the identity

$$\boxed{\Omega^\alpha{}_{\beta\gamma\delta} \equiv -\Omega^\alpha{}_{\beta\delta\gamma}} \tag{3.15}$$

Using this relation in ansatz (3.13) gives

$$a_3\delta^\alpha_\beta g_{\gamma\delta} + (b_3 + c_3)\,\delta^\alpha_{(\gamma} g_{\beta\delta)} = 0$$

This shows us $a_3 = 0$ and $b_3 = -c_3$. The open question is then the choice of the constant c_3. To postpone the answer, we introduce the new field

$$h = c_3\lambda$$

such that the world curvature becomes

$$\boxed{\Omega^{\alpha}{}_{\beta\gamma\delta} = Z^{\alpha}{}_{\beta\gamma\delta} - 2\delta^{\alpha}_{[\gamma}g_{\beta\delta]}h} \tag{3.16}$$

This quantity is by the way conformally gauge invariant due to boxes (2.70), (2.97) and (3.3):

$$\boxed{\Omega^{\alpha}{}_{\beta\gamma\delta} \xrightarrow{\text{cg}} \Omega^{\alpha}{}_{\beta\gamma\delta}} \tag{3.17}$$

3.3.2 Geometric form of world equation

Having generalized the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ to the world curvature $\Omega^{\alpha}{}_{\beta\gamma\delta}$, we can now study whether matter can really be interpreted geometrically. To this end, we continue like in Section 2.3.4 for electrogravitation. So, we generalize the electrogravitation tensor $Z_{\alpha\beta}$ to the conformally gauge invariant **world tensor**

$$\boxed{\Omega_{\alpha\beta} = \Omega^{\gamma}{}_{\alpha\gamma\beta}} \tag{3.18}$$

such that equation (3.16) gives

$$\boxed{\Omega_{\alpha\beta} = Z_{\alpha\beta} - (D-1)\, g_{\alpha\beta}h} \tag{3.19}$$

Then, we demand that the world equation (3.1) can be written in the following compact form[3]:

World equation:

The field equation of world theory is

$$\Omega_{\alpha\beta} = 0 \tag{3.20}$$

This is the most natural way to write the world equation. It is also the central part of the paper and the counterpart of the anomaly (2.6), the ignition spark where all begun.

To continue with the interpretation of matter, we insert the world tensor (3.19) in the world equation (3.20) such that

$$Z_{\alpha\beta} = (D-1)\, g_{\alpha\beta}h$$

The electrogravitation tensor $Z_{\alpha\beta}$ appearing here can be rewritten with equation (2.35):

$$R_{\alpha\beta} = \cdots + (D-1)\, g_{\alpha\beta}h$$

The dots represent terms which contain the electromagnetic vector potential, similar to Section 3.2.2. Next, we perform a contraction

$$R = \cdots + (D-1)\, Dh$$

and then definition (1.17) gives

$$G_{\alpha\beta} = \cdots - \frac{1}{2}g_{\alpha\beta}\,(D-1)\,(D-2)\,h \tag{3.21}$$

[3]Found on the 12th of January 2012.

The two factors $D-1$ and $D-2$ appearing here are a partial explanation why we had to choose the expression $-(D-1)(D-2)/6$ in equation (3.1). The remaining factor $-1/6$ is a choice that causes the term $-g_{\alpha\beta}\lambda$ in equation (3.2). This will become understandable further below when we study cosmology (paragraph below equation (3.65)). For the moment, we compare the result (3.21) with equation (3.1) and find

$$\boxed{h = \frac{\lambda}{3}} \tag{3.22}$$

i.e. the sought factor is $c_3 = 1/3$.

Note that the field h is not just an ancillary field here. Both fields h and λ may be used in world theory. When studying, for instance, the world curvature $\Omega^{\alpha}{}_{\beta\gamma\delta}$, it is more natural to use the field h, because then no factor $1/3$ is present (see definition (3.16)). On the other hand, for the world equation in four dimensions, it is most appropriate to use the matter field λ, because otherwise an additional factor 3 would be required in equation (3.2). So, the two fields are just different ways to express the same phenomenon. This is similar to electrogravitation, where we encountered multiple possible ways, too. For instance, we were able to write the electrogravitational curvature in six different ways (see end of Section 2.5.8). It is reasonable to study all of these possibilities, because none is really exceptional and more views give a better understanding of world theory.

3.3.3 Interpretation

In definition (3.16), we have generalized the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ to the world curvature $\Omega^{\alpha}{}_{\beta\gamma\delta}$. The last section has then shown that this step extends the electrogravitational field equation to the world equation. The missing issue is still the geometric interpretation of matter. World theory does not alter the metric $g_{\alpha\beta}$ nor the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ such that it appears as if matter is not a geometric phenomenon. It is actually impossible to geometrically interpret matter in any known manner. However, the naturalness of how matter fits to the functioning of electrogravitation makes another step reasonable. We have to extend the range of phenomena which are known to be geometric. If a modification of the first derivative level, i.e. the connection, is a geometric appearance, why should an altered curvature not also be geometric? Therefore, we consider the change performed in definition (3.16) as a new geometric property of spacetime. We refer to this property as the geometric phenomenon **inflation**

$$\boxed{I^{\alpha}{}_{\beta\gamma\delta} = \Omega^{\alpha}{}_{\beta\gamma\delta} - Z^{\alpha}{}_{\beta\gamma\delta} = -2\delta^{\alpha}_{[\gamma}g_{\beta\delta]}h} \tag{3.23}$$

Why did we choose the word "inflation" above? The influence of the geometric phenomenon inflation in the world equation (3.1) is a term of the form $\lambda g_{\alpha\beta}$. This expression is very similar to the familiar term $\Lambda g_{\alpha\beta}$ caused by a cosmological constant Λ in Einstein's equation. In case of $\Lambda > 0$, the cosmological constant causes an expansion of space. Let us now look at cosmology, where two different expansions of space are known. There is the initial, rapid expansion described by inflation theory (see Guth 1981) and the lasting, weak expansion caused by dark energy (the details of dark energy are addressed further below in Section 3.7.1). None of the two expansion processes is truly understood at the moment. Although dark energy has already been observed, for inflation theory, it is not even clear whether that theory is realized in nature at all. Anyway, inflation theory calls its expansion process "inflation". As this is a very natural and fitting expression, we borrow it for the geometric phenomenon (3.23). However, we always have to bear in mind that the geometric phenomenon inflation is not to be mixed up with the hypothetical expansion process inflation.

Following the distinction between interior and exterior curvature, torsion and nonmetricity, we also want to perform such a discrimination for the geometric phenomenon inflation now. For that purpose, we look at the electrogravitational curvature (2.66). There, the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ appears, which uses the interior torsion and nonmetricity. Including also an unspecified exterior torsion and nonmetricity, we would have the same expression as in equation (2.66), but the quantity $I^{\alpha}_{\beta\gamma}$ would then be much more general. However, the resulting curvature would not be an arbitrary tensor with four indices. It would be constrained by equation (2.66). The missing degrees of freedom required to reach arbitrariness are just given by inflation. Definition (3.23) does not use all of these degrees of freedom. Therefore, we call the inflation present there the **interior inflation**. The remaining degrees of freedom are then the **exterior inflation**. That way, we can sum up our understanding of the kind of matter used in world theory:

> **Geometric interpretation of matter**:
> Matter is the interior inflation of spacetime. (3.24)

3.3.4 World field

So, world theory is based on a manifold which contains interior curvature, torsion, nonmetricity and inflation. We want to call such a manifold a **world space**[4]. The manifold of electrogravitation shall by the way be referred to as an **electrogravitational space**. Thus, we arrive at the following development of geometry in physics. The first geometry used there was a Euclidean space in Newtonian gravitation. That type of manifold describes only space but not time such that its signature is $(+,+,\cdots)$. Space and time were then merged to spacetime in special relativity. The resulting manifold was a Minkowskian space, which has the signature $(-,+,+,\cdots)$. Then, the manifold underlying physics was made dynamic in general relativity by allowing it to have an interior curvature. The manifolds of general relativity are known as Riemannian spaces. World theory now generalizes these manifolds to world spaces. All in all, we can therefore visualize the development of geometry as:

$$\begin{array}{rcc} & & \text{Euclidean space} \\ \text{Special relativity} & \longrightarrow & \downarrow \\ & & \text{Minkowskian space} \\ \text{General relativity} & \longrightarrow & \downarrow \\ & & \text{Riemannian space} \\ \text{World theory} & \longrightarrow & \downarrow \\ & & \text{World space} \end{array} \tag{3.25}$$

At the end of Section 2.3.7, we have found out that geometry itself is the unified field of electrogravitation. This is also valid for world theory. The difference is only that we

[4] Translation hint: I recommend to translate world space as “Weltraum” in German, even though that notion is already used there for outer space.

have another geometry in that theory, namely a world space. From the mathematical viewpoint, we can unfortunately not directly work with the world space. Instead, we have to use the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α and the matter field λ describing its properties:

World field:

The field of world theory is a world space, described by the tensor fields $g_{\alpha\beta}, A_\alpha, \lambda$. (3.26)

3.3.5 World formula

The world equation (3.20) is an attempt of a unified, geometric description of gravitation, electromagnetism and matter. Is this now a complete description of our universe? For that purpose, one requirement is that the universe can be represented mathematically in each of its aspects. The history of the last centuries in physics has taught us that this is very probably the case. However, there is no way to prove that. Let us ignore this obstacle and assume that the universe can in fact be described mathematically. The next issue is then the information carrier. For a long time, particles were an unavoidable constituent of physics. Yet, nowadays we know that they can be reduced to fields (due to quantum field theory). So, it appears as if the whole information of the universe is encoded in fields. Still, we have to be careful again. We do not know whether fields may one day have to be replaced like particles. As there is currently no sign that this is required, we want to assume that the universe is really a certain unified field. That field must be specified by multiple components, like those of the three fundamental tensor fields of world theory together. The possible configurations of the unified field in nature are then set by a field equation. Following the usual conventions in physics, we call that field equation world formula. So, the actual question is whether the world equation (due to the minimalism demanded by Occam's razor, we refer to the 4-dimensional version here) is equal to the world formula?

Unfortunately, it is not possible to answer that question with a yes, and this will also never be possible. The reason is a philosophical one. It is actually impossible to know what the world formula is. If we really did, then we would know that the theory containing the world formula is true. However, physical theories cannot be verified. They can only be falsified (see the philosophy of Karl Popper). All this is simply the result of having explicitly defined the world formula as a complete description of the universe.

Fortunately, if it really exists, it is still possible to discover the world formula. The problem is only that we can never know whether we have truly found it. We can, however, increase the probability that this is the case by trying to falsify the considered candidate in every imaginable manner, though the probability can never become unity. Having become aware of all this, let us now return to the world equation.

Based on our current understanding of physics, it appears to be a very simple task to show that the world equation cannot be the world formula. The most important reason is that world theory does not include the weak and strong force. It is also not a quantized theory. And, the elementary particles are missing. Instead, world theory adopts a very simple description of matter resting upon the matter field λ. It is therefore reasonable to deeply question any relationship between the world equation and the world formula.

However, in science, one is advised to not carelessly rule out options just because they appear very improbable. Let us therefore also speculate how the above difficulties might be overcome. Then, we first of all see that world theory is special in a certain sense. It is the first theory that tries to give a natural, unified, geometric description of the universe that is complete at least at certain scales, the macroscopic ones. This property distinguishes world theory from other theories. So, let us assume that world theory really holds in the macrocosm. Whether this is true or not has to be decided by observations. The question is then what happens in the microcosm? There, gravitation does not play a role in the current description of physics. The influence of gravitation appears to be too small such that it is typically neglected. One possibility which cannot be ruled out at the moment is therefore that the weak and strong force can somehow be reduced to gravitation. They could also be a mixture of the three fields present in world theory. However, it is totally unclear how this could work in detail. Anyway, elementary particles may then be special solutions of the theory. Note that general relativistic hydrodynamics would in that case be hidden in world theory. Finally, there is quantum theory. The violation of Bell's inequality in experiments tells us that nature does not appear to be classical. However, as not all loopholes of this issue are finally closed, one is recommended to be careful here. The crucial point is now that all the above ideas can currently not be excluded for sure. Consequently, there is at least a marginal chance that the world equation might be the world formula. Taking care of this possibility is a side reason why the notation used in this paper is extensively based on the word "world".

However, it would not be sound to choose the notation of a theory just because of a rare option. Therefore, we need a more solid and conservative reason for our notation. To understand it, we divide all the phenomena encountered in physics into three categories, depending on how they behave in the macrocosm. The first category does not play a role there at all, like short lived elementary particles. The second category has an impact on the macrocosm and can also be explained by macroscopic concepts, like gravitation. Finally, there is the third category, which also influences the macrocosm. However, the phenomena in that category cannot be understood by macroscopic descriptions. Instead, they originate from the microcosm. An example may be the weak and the strong force. If these two forces are not in a yet unknown way part of world theory, they will be purely microscopic phenomena. However, they still have a macroscopic influence, because they determine the behavior of matter, which is macroscopically present everywhere. Now we come back to the world formula. We do not know that equation, but we can reason about its classical limit. An exact classical limit will contain the second and the third of the above categories. Yet, there is also a weak classical limit, namely the one dealing with only the second category. The mentioned more conservative reason for having provided the field equation of world theory with a name that sounds very similar to world formula, namely world equation, is then the following one. The world equation, whose full name is classical world equation, is proposed as a candidate of at least the weak classical limit of the world formula.

3.3.6 Outlook

What happens if world theory is not a complete description of the universe? In that case, the easiest way is to proceed like in Einstein's equation (1.18) and use a stress-energy tensor. For that purpose, we have to first generalize the Einstein tensor $G_{\alpha\beta}$ of general relativity to world theory. This is done in analogy to relation (1.17). The Ricci tensor $R_{\alpha\beta}$ appearing there becomes just the world tensor $\Omega_{\alpha\beta}$, because combining equations (2.35) and (3.19) gives

$$\Omega_{\alpha\beta} = R_{\alpha\beta} + 2\left(\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} + L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]}\right) - (D-1)\, g_{\alpha\beta}h$$

So, the world tensor $\Omega_{\alpha\beta}$ is the Ricci tensor $R_{\alpha\beta}$ plus a correction coming from electromagnetism and matter. In addition to that, the Ricci scalar (1.16) generalizes to the **world scalar**

$$\boxed{\Omega = g^{\alpha\beta}\Omega_{\alpha\beta}} \tag{3.27}$$

That way, the sought generalization of the Einstein tensor $G_{\alpha\beta}$ is the **dual world tensor**

$$\boxed{\overline{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\Omega} \tag{3.28}$$

This quantity allows us to write the world equation (3.20) as

$$\overline{\Omega}_{\alpha\beta} = 0 \tag{3.29}$$

From the viewpoint of vacuum general relativity, this equation already contains a stress-energy tensor. To see this, we merely have to split

$$\overline{\Omega}_{\alpha\beta} = G_{\alpha\beta} - t_{\alpha\beta}$$

where $t_{\alpha\beta}$ is the **world stress-energy tensor**. Then, equation (3.29) gives

$$G_{\alpha\beta} = t_{\alpha\beta} \tag{3.30}$$

The world stress-energy tensor $t_{\alpha\beta}$ is therefore just the right hand side of equation (3.1).

However, from the viewpoint of world theory, we have to look at equation (3.29). There, the right hand side is zero such that we can generalize world theory by adding a stress-energy tensor $T^{\mathrm{R}}_{\alpha\beta}$ there, like in Einstein's equation (1.18):

$$\overline{\Omega}_{\alpha\beta} = 8\pi T^{\mathrm{R}}_{\alpha\beta} \tag{3.31}$$

The stress-energy tensor $T^{\mathrm{R}}_{\alpha\beta}$ cannot be the one of Einstein's equation (1.18), because electromagnetism and the matter field λ are already part of the left hand side of equation (3.31). So, the stress-energy tensor $T^{\mathrm{R}}_{\alpha\beta}$ is a reduced form of Einstein's stress-energy tensor $T_{\alpha\beta}$, emphasized by the letter "R". It could, for instance, contain the stress-energy tensor of a fluid. If general relativistic hydrodynamics is not hidden in world theory in a manner that we do not yet know, this appears to be the natural step to include it. Yet, one has to be careful that conformal gauge invariance is still conserved. A cosmological term $\Lambda g_{\alpha\beta}$ like in general relativity is therefore forbidden (Why: $\Lambda \overset{\mathrm{cg}}{\to} \Lambda$ and equation (2.70) cause $\Lambda g_{\alpha\beta} \overset{\mathrm{cg}}{\to} e^{-2\chi}\Lambda g_{\alpha\beta}$, which does not fit to $T^{\mathrm{R}}_{\alpha\beta} \overset{\mathrm{cg}}{\to} T^{\mathrm{R}}_{\alpha\beta}$ coming from equations (3.31), (3.28), (3.27), (2.70), (2.71), (3.18) and (3.17)). Unfortunately, we then arrive at the same problem as at the beginning of this paper. We will wonder what the geometric meaning of the reduced stress-energy tensor $T^{\mathrm{R}}_{\alpha\beta}$ is. The only known stress-energy tensor with a geometric interpretation is the one of world theory $t_{\alpha\beta}$. The quantity $T^{\mathrm{R}}_{\alpha\beta}$ should therefore either be provisional or an unknown form of stress-energy tensor. Otherwise, equation (3.31) will very probably be incorrect, even if it may be a more complete description of the universe than the world equation (3.20). This situation is then similar to the field equation of Einstein-Maxwell theory (1.20). Therefore, we do not proceed like in general relativity and propose that equation (3.31) should be used in physics if world theory is incomplete. Instead, we act in the only reasonable manner here. We keep this issue open.

3.4 Conformal geometrodynamics[5]

3.4.1 Comparison

The world equation (3.2) as a pure (=without the geometric interpretation and thus the connection) tensor equation is actually not new. It can also be found in, for instance, equation (2) of Gorbatenko (2010). The theory considered there is called conformal geometrodynamics, which builds on Weyl theory. Additional references which are not mentioned somewhere else here are Gorbatenko (2003) (equations (1)-(4) therein), Gorbatenko (2002) (equations (1) and (2) there) and Pushkin (2000) (equations (1) and (2) in it) (there are also some references, mainly Russian ones, which I was unable to access, see, for example, the references in Gorbatenko 2010; I apologize for this inconvenience and the mere indirect citation). However, the equation of Gorbatenko (2010) is still not the world equation. Why? To write down a field equation, two constituents are required, a field and an operator acting on that field. The operator together with the field is then set to zero. For the world equation (3.2), the operator can be found in box (3.20). We have to just consider the world tensor there as a function of the world field. Conformal geometrodynamics and world theory have that operator in common. However, the fields used in the two theories differ. Due to box (3.26), world theory is based on a world space. Conformal geometrodynamics, in contrast, uses a Weyl space (=the manifold of Weyl theory), which can be seen in equation (5) of Gorbatenko (2010). Note that although world and Weyl spaces are described by the same tensor fields, the three fields $g_{\alpha\beta}, A_\alpha, \lambda$, we are not allowed to consider these tensors as the field of the respective theory. That field has to be a unified one and cannot be a set of tensor fields. Instead, the field must be geometry itself. Therefore, **the fields of conformal geometrodynamics and world theory differ and thus also the field equations**.

Mathematically, the difference between world theory and conformal geometrodynamics is the following one. Due to equation (2.46), the connection of world theory is the electrogravitational connection

$$I^\alpha_{\beta\gamma} \overset{D=4}{=} \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right) - \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma} \tag{3.32}$$

And, equations (2.34), (2.66), (3.19) and (3.22) yield the world tensor

$$\Omega_{\alpha\beta} \overset{D=4}{=} \partial_\gamma I^\gamma_{\alpha\beta} - \partial_\beta I^\gamma_{\alpha\gamma} + I^\gamma_{\delta\gamma} I^\delta_{\alpha\beta} - I^\gamma_{\delta\beta} I^\delta_{\alpha\gamma} - g_{\alpha\beta}\lambda = 0 \tag{3.33}$$

which is set to zero because of the world equation (3.20). In conformal geometrodynamics, we instead have Weyl's connection (2.73)

$${}^{\mathrm{W}}I^\alpha_{\beta\gamma} \overset{D=4}{=} \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right) + \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma} \tag{3.34}$$

where the red sign differs from relation (3.32). This leads to the geometric difference between the two theories. Note that we have to limit ourselves to $D = 4$ here, because

[5] Chronological remark: I did not know of conformal geometrodynamics until the 8th of September 2015. I came across it when this paper was already practically finished and when I performed some final study of literature. So, I found world theory independently of conformal geometrodynamics. The latter is therefore taken into account just in Sections 3.4 and 4.6.11 as well as in minor remarks and citations spread across the paper. The choice of the letter λ for the matter field of world theory was by the way not inspired by conformal geometrodynamics. Instead, it was a surprising chance that both theories use the same letter for their scalar fields (there is, however, a sign difference between the two scalars).

there does not seem to be a generalization of conformal geometrodynamics to an arbitrary number of dimensions in literature. The second difference occurs if we introduce a world tensor ${}^{\mathrm{CGD}}\Omega_{\alpha\beta}$ for conformal geometrodynamics (="CGD") in analogy to the one of world theory. Such a tensor does not explicitly occur in the nomenclature of conformal geometrodynamics, but it is reasonable to use it for the comparison. And, then we have

$$ {}^{\mathrm{CGD}}\Omega_{\alpha\beta} \overset{D=4}{=} \partial_\gamma {}^{\mathrm{W}}\!I^\gamma_{\alpha\beta} - \partial_{(\beta} {}^{\mathrm{W}}\!I^\gamma_{\alpha)\gamma} + {}^{\mathrm{W}}\!I^\gamma_{\delta\gamma} {}^{\mathrm{W}}\!I^\delta_{\alpha\beta} - {}^{\mathrm{W}}\!I^\gamma_{\delta\beta} {}^{\mathrm{W}}\!I^\delta_{\alpha\gamma} - g_{\alpha\beta}\lambda = 0 \tag{3.35} $$

due to equation (4) of Gorbatenko (2010). There, not equation (2.62) is used (in case of $\lambda = 0$), which leads to the issue (2.63), but instead the symmetric part of equation (2.62). So, above the difference to world theory are the red, round brackets, which make the world tensor of conformal geometrodynamics slightly more complicated than the one of world theory (3.33). However, this is true only if we write the two world tensors in terms of merely the connections as above, without writing the connections out in terms of the metric $g_{\alpha\beta}$ and the electromagnetic vector potential A_α. As the purely tensorial field equations of the two theories are the same, we have, of course,

$$ \Omega_{\alpha\beta} \overset{D=4}{=} {}^{\mathrm{CGD}}\Omega_{\alpha\beta} $$

By the way, there is in fact also a sign difference in front of the term $g_{\alpha\beta}\lambda$ in conformal geometrodynamics (see also footnote 5). However, we have removed it in equation (3.35) by a renaming of the scalar field of conformal geometrodynamics. The comparison with world theory is easier that way. In contrast, the difference marked in red in relations (3.34) and (3.35) cannot be removed by such a procedure. So, here the two theories are mathematically and geometrically different.

3.4.2 Reinterpretation

Conformal geometrodynamics gives rise to a possible abstraction of world theory. The starting point of the abstraction is that in electrogravitation we have carelessly assumed that the electrogravitational field equation must have the form $Z_{\alpha\beta} = Z^\gamma{}_{\alpha\gamma\beta} = 0$ in equations (2.34) and (2.36). However, actually we do not know the right choice. Therefore, it is reasonable to investigate the general case here. And, this is to assume

$$ Z_{\alpha\beta} = a_4 Z^\gamma{}_{\alpha\gamma\beta} + b_4 Z^\gamma{}_{\beta\gamma\alpha} + c_4 Z_\alpha{}^\gamma{}_{\gamma\beta} + d_4 Z_\beta{}^\gamma{}_{\gamma\alpha} \tag{3.36} $$

with the constants $a_4, b_4, c_4, d_4 \in \mathbb{R}$, whereas equation (2.36) is still valid. So, for the electrogravitation of Chapter 2, $a_4 = 1$ and $b_4 = c_4 = d_4 = 0$. Is the ansatz (3.36) complete? Due to identity (3.14), we know that the electrogravitational curvature $Z^\alpha{}_{\beta\gamma\delta}$ is antisymmetric in the last two indices. Therefore, it is not reasonable to contract these indices. The result would be zero. In the limit $A_\alpha = 0$, equation (2.33) shows that the electrogravitational curvature becomes the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$. The Riemann tensor is antisymmetric in the first two indices due to identity (2.12). Hence, it is useless to contract these indices and thus also the first two indices of the electrogravitation tensor. So, the contraction must use one index from the pair α, β and one from γ, δ in $Z^\alpha{}_{\beta\gamma\delta}$. Due to the antisymmetry in γ, δ, we can always use the index γ. That way, we arrive at ansatz (3.36). By the way, the constants a_4, b_4, c_4, d_4 have to obey the constraint $a_4 + b_4 - c_4 - d_4 = 1$. The two minuses here are required because of the antisymmetry of the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ in its first two indices, which was mentioned above. Finally, in addition to ansatz (3.36), we return to ansatz (2.25), i.e.

$$ L^\alpha_{\beta\gamma} = a_2 \delta^\alpha_\beta A_\gamma + b_2 \delta^\alpha_\gamma A_\beta + c_2 A^\alpha g_{\beta\gamma} + d_2 \delta^4_D \epsilon^\alpha{}_{\beta\gamma\delta} A^\delta + \cdots \tag{3.37} $$

where we may now encounter new values for the constants $a_2, ..., d_2$.

For the following consideration, it is sufficient to assume $c_4 = d_4 = 0$ and $d_2 = 0$. Then equations (3.36), (2.36), (2.33) and (1.14) yield

$$R_{\alpha\beta} = -2\left[a_4\left(\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} + L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]}\right) + b_4\left(\nabla_{[\gamma}L^{\gamma}_{\beta\alpha]} + L^{\gamma}_{\delta[\gamma}L^{\delta}_{\beta\alpha]}\right)\right] \tag{3.38}$$

This is the generalization of equation (2.37). So, we have to compare the right hand side of equation (3.38) now with the one of equation (2.38) to find out the new values of the constants $a_2, ..., d_2$ for $D = 4$. Due to relation (2.42), we know that $L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]} = L^{\gamma}_{\delta[\gamma}L^{\delta}_{\beta\alpha]}$ for our choice $d_2 = 0$. Hence, equation (3.38) reduces to

$$R_{\alpha\beta} = -2\left(a_4\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} + b_4\nabla_{[\gamma}L^{\gamma}_{\beta\alpha]}\right) - 2L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]} \tag{3.39}$$

and we have a term without the constants a_4, b_4. In four dimensions, the term is due to equation (2.42)

$$-2L^{\gamma}_{\delta[\gamma}L^{\delta}_{\alpha\beta]} \overset{D=4}{\equiv} \left(c_2^2 - 3b_2^2\right) A_\alpha A_\beta - \left(3b_2 + c_2\right) c_2 g_{\alpha\beta} A^2$$

A comparison with the right hand side of equation (2.38) then yields

$$c_2^2 - 3b_2^2 \overset{D=4}{\equiv} -2 \tag{3.40}$$

and

$$c_2^2 + 3b_2c_2 \overset{D=4}{\equiv} -2 \tag{3.41}$$

Hence, $b_2\left(b_2 + c_2\right) = 0$. Due to equation (3.40), $b_2 \neq 0$ such that $c_2 = -b_2$. Then, equations (3.40) and (3.41) become both $b_2^2 = 1$, and we see $b_2 = \pm 1$. All in all, we thus have $b_2 = \pm 1$ and $c_2 = \mp 1$ for $D = 4$.

Next, we use the above finding and $d_2 = 0$ in equation (2.39):

$$-2\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} \overset{D=4}{\equiv} -a_2\nabla_\alpha A_\beta + \left(a_2 \pm 2\right)\nabla_\beta A_\alpha \pm g_{\alpha\beta}\nabla_\gamma A^\gamma$$

Hence, in equation (3.39),

$$\begin{aligned} & -2\left(a_4\nabla_{[\gamma}L^{\gamma}_{\alpha\beta]} + b_4\nabla_{[\gamma}L^{\gamma}_{\beta\alpha]}\right) \\ \overset{D=4}{\equiv} \; & \left[\left(a_2 \pm 2\right)b_4 - a_2a_4\right]\nabla_\alpha A_\beta + \left[\left(a_2 \pm 2\right)a_4 - a_2b_4\right]\nabla_\beta A_\alpha \pm g_{\alpha\beta}\nabla_\gamma A^\gamma \end{aligned}$$

due to $a_4 + b_4 = 1$. A comparison with the right hand side of equation (2.38) then yields the upper sign and

$$\left(a_2 + 2\right)b_4 - a_2a_4 \overset{D=4}{\equiv} 1$$

as well as

$$\left(a_2 + 2\right)a_4 - a_2b_4 \overset{D=4}{\equiv} 1$$

The sum of these two equations is obeyed automatically via $a_4 + b_4 = 1$, and the difference gives

$$\left(a_2 + 1\right)\left(a_4 - b_4\right) \overset{D=4}{\equiv} 0 \tag{3.42}$$

or

$$\left(a_2 + 1\right)\left(2a_4 - 1\right) \overset{D=4}{\equiv} 0 \tag{3.43}$$

Now, we see something interesting going on here. In conformal geometrodynamics, $a_4 = \frac{1}{2}$, which causes ansatz (3.36) to become $Z_{\alpha\beta} = Z^\gamma{}_{(\alpha\gamma\beta)}$, and we have the symmetrization as in equation (4) of Gorbatenko (2010). However, then the second round bracket of equation (3.43) vanishes such that we are unable to determine the constant a_2. For all other cases of a_4, like the $a_4 = 1$ as in the electrogravitation of Chapter 2, we unambiguously find $a_2 = -1$. This has an important consequence. We have to make ourselves aware here what we can observe in nature. We cannot observe geometry itself. The only entity we can principally find out empirically is a pure tensor equation, like equation (3.2). The nature of geometry has to then be concluded from such an equation. I.e., geometry can be observed only indirectly. However, conformal geometrodynamics uses the only choice out of the possible values for the constant a_4, namely $a_4 = \frac{1}{2}$, for which such a conclusion is impossible. Instead, the nature of geometry is given artificially and without an empirical basis. Trying to justify the geometry $a_2 = 1$ of conformal geometrodynamics (see equation (2.61)) by using some fundamental principle (see Weyl 1918) does also not help here, because one may invent another principle which yields $a_2 \neq 1$. One could, for example, demand that nonmetricity is forbidden and only torsion is allowed, which means a Riemann-Cartan space. Then, the above finding $b_2 = 1$, $c_2 = -1$ as well as $d_2 = 0$ in ansatz (3.37) yields $a_2 = 0$ due to box (2.29). This is the easiest connection possible here. Yet, whenever we go beyond an empirical justification, we have to apply Occam's razor. In the case $a_4 = \frac{1}{2}$ investigated here, this prevents us from giving an explicit value for the constant a_2 such that the entire geometric interpretation breaks down. However, a geometric interpretation is obligatory, because without it we do not have a unified field and we are not at the utter fundament of physics. I.e., conformal geometrodynamics does not appear to be the right way. Instead, we have to choose $a_4 \neq \frac{1}{2}$. The choice does not matter, as equation (3.43) always yields $a_2 = -1$.

What about ansatz (3.36)? There, our initial assumption $c_4 = d_4 = 0$ as well as $a_4 + b_4 = 1$ yield

$$Z_{\alpha\beta} = a_4 Z^\gamma{}_{\alpha\gamma\beta} + (1 - a_4)\, Z^\gamma{}_{\beta\gamma\alpha} \tag{3.44}$$

So, the electrogravitation tensor depends on the constant a_4. Hence, the exclusion $a_4 \neq \frac{1}{2}$ alone is not sufficient for it. We need an explicit value for the constant a_4. However, there is no empirical justification for such a choice such that it appears to be in conflict with Occam's razor. The solution to this problem is that it is not the electrogravitation tensor $Z_{\alpha\beta}$ which matters. Instead, we have to look at the field equation (2.36), which allows us to split equation (3.44) into

$$\begin{aligned} Z^\gamma{}_{(\alpha\gamma\beta)} &= 0 \\ (2a_4 - 1)\, Z^\gamma{}_{[\alpha\gamma\beta]} &= 0 \end{aligned}$$

For any value $a_4 \neq \frac{1}{2}$, we can divide by the round bracket above such that the dependency on the constant a_4 disappears. In so doing, we see that we actually do not have to specify the constant a_4 beyond the exclusion $a_4 \neq \frac{1}{2}$. Hence, there is no violation of Occam's razor. However, it is still reasonable to specify a value for the constant a_4, because without it we do not have an explicit form for the electrogravitation tensor $Z_{\alpha\beta}$. We have to just always bear in mind that this tensor is only an ancillary quantity to write down the field equation and that the choice is arbitrary. Hence, we have in fact an entire set of possible electrogravitation tensors (3.44) and we decide for the easiest case $a_4 = 1$ as in the electrogravitation of Chapter 2.

Is the mere usage of ansatz (3.36) not itself a fundamental principle which has no empirical justification and could thus be removed by Occam's razor? To go at the utter fundament of physics, we remember that we need a geometric interpretation. In electrogravitation, this means the presence of the metric $g_{\alpha\beta}$, the electrogravitational connection $I^{\alpha}_{\beta\gamma}$ and the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$. That way, we have a set of manifolds. However, the geometric interpretation on its own is not enough. We also need a field equation which rules out as many manifolds of the set as possible, except of course our own universe. This is obligatory to comply with Occam's razor. Otherwise, if the world principle (2.59) holds, it would be sufficient to consider the entire set of thinkable manifolds as the world formula. This is obviously a great deal too much. Now, we have enough knowledge here to be able to say that the field equation must have the form $Z_{\alpha\beta} = 0$ in this context. The only open issue is the relationship between the electrogravitation tensor $Z_{\alpha\beta}$ and the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$. However, this relationship cannot be left away. It cannot be removed by Occam's razor. That way, we see that ansatz (3.36) is unavoidable (up to, of course, the degree of freedom a_4 in relation (3.44))[6].

3.4.3 Review

All in all, we thus see that the geometry of world theory appears to be more suitable than the one of conformal geometrodynamics. However, both theories share the same purely tensorial equation (3.2). Therefore, without new, unknown physics beyond that equation, it is not possible to empirically discern both theories. World theory is then just a reinterpretation of conformal geometrodynamics in such a case. And, many considerations of the one theory also work for the other, like the numeric approach investigated later in this paper. If there is fundamental physics beyond the two theories, the situation may be different. However, there is no way to investigate this issue with our current state of knowledge.

So, we have now Weyl theory, conformal geometrodynamics and world theory. Weyl theory has introduced the idea to put the electromagnetic vector potential into a non-Riemannian connection. Conformal geometrodynamics has added a scalar field and provided the right purely tensorial equation. And, world theory has introduced an empirically more suitable non-Riemannian connection than the one of Weyl. By the way, in Weyl theory, where only the electromagnetic vector potential A_α is present aside from the metric $g_{\alpha\beta}$, the indeterminateness of the metric due to $g_{\alpha\beta} \overset{\text{cg}}{\to} e^{-2\chi} g_{\alpha\beta}$ led to serious criticism from Einstein (see addendum in Weyl 1918). The interest in that theory therefore decayed. In world theory, we will later in this paper (see Section 4.6.11) encounter a way to fix the metric, which yields a so-called physical metric. Therefore, Einstein's objection is not applicable to world theory. However, without the matter field λ, it does not appear to be possible to construct a physical metric. So, the matter field is an essential ingredient missing in Weyl theory. In conformal geometrodynamics, the matter field is also present. However, it seems as if in that theory the vectorial quantity is not explicitly interpreted as the electromagnetic vector potential, despite being denoted as A_α due to its origin from Weyl theory (see the title, point 5 on p. 3 and Section 8 of Gorbatenko 2010, the text

[6]If the assumption $c_4 = d_4 = d_2 = 0$ is given up, a lengthier computation which is not shown here yields $b_2 = 1$, $c_2 = -1$, $d_2 = 0$ and $(a_2+1)(a_4-b_4) + (a_2-1)(c_4-d_4) = 0$. So, $d_2 \neq 0$ is impossible. And, giving up $c_4 = d_4 = 0$ makes the value of a_2 again ambiguous. However, one can argue here as follows. Due to equation (2.66), which holds even for the general ansatz (3.37), the expressions $Z_{\alpha}{}^{\gamma}{}_{\gamma\beta}$ and $Z_{\beta}{}^{\gamma}{}_{\gamma\alpha}$ in ansatz (3.36) do not only depend on the connection but also on the metric. For $Z^{\gamma}{}_{\alpha\gamma\beta}$ and $Z^{\gamma}{}_{\beta\gamma\alpha}$, in contrast, only the connection is required such that these two contractions are simpler and more natural. Occam's razor then causes $c_4 = d_4 = 0$. *(This comment is separated from the remaining text in a footnote, because it does not contain intermediate mathematical steps)*

below equation (2) in Gorbatenko & Kochemasov 2007, Section 5 of Gorbatenko 2009 or point 5 in the abstract of Gorbatenko 2005).

3.5 Conservation laws

3.5.1 Two different stress-energy tensors

By having denoted the right hand side $t_{\alpha\beta}$ in equation (3.30) as world stress-energy tensor, we have now three stress-energy tensors in total. The first stress-energy tensor encountered in this paper was the general relativistic one

$$\boxed{T^{\mathrm{EM}}_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right)}$$

of equation (1.19). Then, we came across the electrogravitational stress-energy tensor

$$t^{\mathrm{EM}}_{\alpha\beta} = (D-2)\left\{\nabla_{(\alpha}A_{\beta)} - A_{\alpha}A_{\beta} - g_{\alpha\beta}\left[\nabla_{\gamma}A^{\gamma} + \frac{1}{2}(D-3)A^{2}\right]\right\} \tag{3.45}$$

in equation (2.20) for $D = 4$ (where $c_{\mathrm{EM}} = 1$). Note that the right hand side of equation (2.105) contains the above tensor in D dimensions. Eventually, there is the world stress-energy tensor

$$\boxed{t_{\alpha\beta} = (D-2)\left\{\nabla_{(\alpha}A_{\beta)} - A_{\alpha}A_{\beta} - g_{\alpha\beta}\left[\nabla_{\gamma}A^{\gamma} + \frac{1}{2}(D-3)A^{2} + \frac{1}{6}(D-1)\lambda\right]\right\}} \tag{3.46}$$

which is the right hand side of the gravitational field equation (3.1).

The first finding is then that the difference between the quantities $t^{\mathrm{EM}}_{\alpha\beta}$ and $t_{\alpha\beta}$ comes exclusively from the matter field λ. Therefore, the world stress-energy tensor $t_{\alpha\beta}$ is the generalization of the electrogravitational stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$ to world theory. In the following, we can therefore omit the latter tensor, for simplicity.

3.5.2 Energy-momentum conservation

An important question is now the relationship between the general relativistic stress-energy tensor $T^{\mathrm{EM}}_{\alpha\beta}$ and the world stress-energy tensor $t_{\alpha\beta}$. For that purpose, we evaluate the covariant divergence of both tensors. Applying identity (2.4) on equation (3.7), we find

$$\boxed{\nabla^{\beta}T^{\mathrm{EM}}_{\beta\alpha} \overset{D\neq 2}{=} \frac{1}{4\pi}F_{\alpha}{}^{\beta}\left[(D-4)F_{\beta\gamma}A^{\gamma} + \frac{1}{3}(D-1)(\nabla_{\beta} - 2A_{\beta})\lambda\right]}$$

or in four dimensions

$$\nabla^{\beta}T^{\mathrm{EM}}_{\beta\alpha} \overset{D=4}{=} \frac{1}{4\pi}F_{\alpha}{}^{\beta}(\nabla_{\beta} - 2A_{\beta})\lambda \tag{3.47}$$

On the other hand, the contracted Bianchi identity (2.5) and equation (3.30) reveal

$$\boxed{\nabla^{\beta}t_{\beta\alpha} = 0} \tag{3.48}$$

Let us now look at $D = 4$ and electrogravitation, for which the matter field λ disappears in equation (3.47). We have then two different quantities $T^{\mathrm{EM}}_{\alpha\beta}$ and $t_{\alpha\beta}$ whose covariant divergence vanishes. This shows us that both quantities are governed by a conservation law. And, that is the reason why we do not only call $T^{\mathrm{EM}}_{\alpha\beta}$ stress-energy tensor but also

$t_{\alpha\beta}$. One has to now ask, what is the true stress-energy tensor? In general relativity, we know that only the general relativistic stress-energy tensor $T^{\mathrm{EM}}_{\alpha\beta}$ is conserved. The tensor $t_{\alpha\beta}$ is thus not associated with stress-energy there. However, in electrogravitation, the situation is ambiguous. There, we have two conserved tensors such that we are unable to make a decision. Fortunately, there is a way to solve this issue. We just have to go from electrogravitation to world theory. Then, the right hand side of equation (3.47) shows us that the general relativistic energy-momentum conservation is violated. It is therefore reasonable to consider the quantity $t_{\alpha\beta}$ as the true stress-energy tensor. The general relativistic one $T^{\mathrm{EM}}_{\alpha\beta}$ is instead a limited concept, which is not universally applicable.

When looking at definition (3.46), we see that the world stress-energy tensor $t_{\alpha\beta}$ contains multiple terms with the electromagnetic vector potential A_α and one term with the matter field λ. These terms are the true stress-energy tensor of electromagnetism and matter, respectively. The stress-energy "tensor" of gravitation, which is actually not a tensor but a pseudotensor, is instead not directly found in the tensor $t_{\alpha\beta}$. Instead, one has to extract it from the covariant divergence of the conservation law (3.48). The result is the Landau-Lifshitz pseudotensor $t^{\alpha\beta}_{\mathrm{G}}$ known from general relativity. Readers not familiar with this idea and interested in more details are referred to Section 96 of Landau & Lifshitz (2003). There, equation (96.10) shows us that the covariant form (3.48) can be rewritten to

$$\boxed{\partial_\beta \left[(-g)\left(t^{\beta\alpha}_{\mathrm{G}} + t^{\beta\alpha}\right)\right] = 0} \tag{3.49}$$

with

$$t^{\alpha\beta}_{\mathrm{G}} = -G^{\alpha\beta} + \frac{1}{(-g)}\partial_{\gamma\delta}\left[(-g)\, g^{\alpha[\beta} g^{\gamma]\delta}\right] \tag{3.50}$$

(Landau & Lifshitz 2003 actually denotes $t^{\alpha\beta}_{\mathrm{LL}} = t^{\alpha\beta}_{\mathrm{G}}/\left(8\pi\right)$ as the stress-energy pseudotensor of gravitation, but this would lead to an unwanted factor 8π in equation (3.49)). That way, we have a conservation law of the energy-momentum of gravitation, electromagnetism and matter. The stress-energy "tensor" of gravitation is given by the Landau-Lifshitz pseudotensor $t^{\alpha\beta}_{\mathrm{G}}$ and the one of electromagnetism together with matter by the world stress-energy tensor $t^{\alpha\beta}$.

3.5.3 Charge conservation

Besides energy-momentum conservation, there is also charge conservation. In general relativity, the presence of charges requires a generalization of Maxwell's equation (1.15) to

$$\nabla^\beta F_{\beta\alpha} = -4\pi J_\alpha \tag{3.51}$$

with the general relativistic current J_α (see equation (22.17a) of Misner *et al.* 2002). However, we do not want to take over the above equation in world theory without modification. In fact, we have to compare equation (3.51) with Einstein's equation (1.18). The world theoretical counterpart of Einstein's equation is equation (2.7), where we have omitted the factor 8π to improve the mathematical elegance. Similarly, we leave the factor -4π away now such that in world theory

$$\nabla^\beta F_{\beta\alpha} \overset{D\neq 2}{=} j_\alpha \tag{3.52}$$

where j_α shall be called **world current**.

Looking at the electromagnetic field equation (3.7), we recognize that

$$\boxed{j_\alpha \overset{D\neq 2}{=} (D-4)\, F_{\alpha\beta} A^\beta + \frac{1}{3}(D-1)\,(\nabla_\alpha - 2A_\alpha)\,\lambda} \tag{3.53}$$

There is also the **electrogravitational current**

$$\boxed{j^{\mathrm{EM}}_\alpha \overset{D\neq 2}{=} (D-4)\, F_{\alpha\beta} A^\beta} \tag{3.54}$$

which is just the vector (3.53) without the matter field λ. Why do we use the abbreviation "EM" in the electrogravitational current j^{EM}_α and not "EG"? The notion "electrogravitational current" shows that the quantity j^{EM}_α is the current of electrogravitation, and the abbreviation "EM" denotes that we have to take the electromagnetic part of the whole current j_α alone to get the electrogravitational one. This is the same convention as for the electrogravitational stress-energy tensor $t^{\mathrm{EM}}_{\alpha\beta}$.

The electrogravitational current j^{EM}_α refers to a self-charge of the electromagnetic field. However, in our universe with four dimensions, this vector vanishes such that it is not interesting. Let us therefore return to the world current j_α. Applying identity (2.114) on equation (3.52) leads to the charge conservation law

$$\boxed{\nabla^\alpha j_\alpha \overset{D\neq 2}{=} 0} \tag{3.55}$$

3.5.4 Hidden field equations

All in all, we have two conservation laws in world theory, the conservation of energy-momentum (3.48) and the conservation of charge (3.55). It is then also clear why the electromagnetic field equation (3.7) and the material field equation (3.10) are hidden in the world equation (3.1). These two field equations are just the energy-momentum and charge conservation laws.

The conservation laws and the associated hidden field equations are an integral part of world theory. It is therefore reasonable to summarize this concept such that it is visible at a single glance. For that purpose, we use identities (2.5) and (2.114) together with the field equations in the forms (3.30), (3.52) and (3.11):

$$\begin{tikzcd}
\underline{\text{Identity}} & \underline{\text{Conservation law}} & \underline{\text{Field equation}} \\
 & & G_{\alpha\beta} = t_{\alpha\beta} \arrow[dl] \\
\nabla^\beta G_{\beta\alpha} \equiv 0 \arrow[r] & \nabla^\beta t_{\beta\alpha} = 0 \arrow[r, phantom, "\Longleftrightarrow"] & \nabla^\beta F_{\beta\alpha} \overset{D\neq 2}{=} j_\alpha \arrow[dl] \\
\nabla^\alpha \nabla^\beta F_{\alpha\beta} \equiv 0 \arrow[r] & \nabla^\alpha j_\alpha \overset{D\neq 2}{=} 0 \arrow[r, phantom, "\Longleftrightarrow"] & \nabla^\alpha (\nabla_\alpha - 2A_\alpha)\,\lambda \overset{D=4}{=} 0
\end{tikzcd}$$

Note that we have limited ourselves to the 4-dimensional case for the material field equation above to keep our summary short.

3.6 Cosmology

3.6.1 Spatial cosmological principle

Having discovered an alternative to general relativity, one of the interesting questions is its influence on cosmology. Let us call the cosmology of world theory **world cosmology**. Studying it will also explain the factor $-1/6$ mentioned in Section 3.3.2.

The starting point in cosmology is the familiar cosmological principle. This principle is the result of observations, and it alleges the following. At large enough spatial scales, the universe is approximately spatially homogeneous and spatially isotropic. Note that the cosmological principle refers to properties of only space. Further below, we will encounter a similar principle that refers to just time. To discern both principles, we denote the spatial version as **spatial cosmological principle**.

The spatial cosmological principle reduces the possible forms of the metric $g_{\alpha\beta}$ drastically. We can constrain the metric even further by assuming that the universe is spatially flat. This hypothesis is strongly supported by current observations (see, e.g., Planck Collaboration 2014). It also simplifies the following computations, especially for $D \neq 4$. The metric $g_{\alpha\beta}$ is then the well-known Robertson-Walker metric

$$g_{\alpha\beta} = \mathrm{diag}\left(-1, R(t)^2, R(t)^2, ...\right) \tag{3.56}$$

where $R(t)$ is the scale factor. For the electromagnetic vector potential A_α and the matter field λ, the spatial cosmological principle obviously leads to the limitations

$$A_\alpha = (A_t(t), 0, 0, ...) \tag{3.57}$$

and

$$\lambda = \lambda(t) \tag{3.58}$$

3.6.2 Cosmological fields

Using the conformal gauge transformation

$$\chi(t) = -\int_0^t \mathrm{d}\tau A_t(\tau)$$

the electromagnetic vector potential (3.57) can even be made vanishing. To show this, we compute

$$\partial_\alpha \chi(t) = -\delta^t_\alpha A_t(t)$$

such that box (2.70) gives

$$A_\alpha \overset{\text{cg}}{\to} A_\alpha - \delta^t_\alpha A_t(t) = 0 \tag{3.59}$$

However, we have to be careful now, because also the metric $g_{\alpha\beta}$ and the matter field λ are affected by that transformation. For the matter field λ, this is clearly not a problem. Box (3.3) tells us that

$$\lambda \overset{\text{cg}}{\to} e^{2\chi(t)}\lambda$$

which is still of the form (3.58). Unfortunately, box (2.70) shows us that for the metric

$$g_{\alpha\beta} \overset{\text{cg}}{\to} e^{-2\chi(t)} g_{\alpha\beta} = g^{\bullet}_{\alpha\beta} \tag{3.60}$$

Note that we use dots $^\bullet$ to denote conformal gauge transformations and primes $'$ for coordinate transformations. The above outcome does not fit to the Robertson-Walker metric (3.56), because the component $g^{\bullet}_{tt}$ is not -1. So, it appears as if the simplification (3.59) is useless, because it has made the metric $g_{\alpha\beta}$ more complicated.

Fortunately, there is a solution to this issue. We have to just perform a coordinate transformation that changes the time t according to

$$t' = \int_0^t \mathrm{d}\tau e^{-\chi(\tau)} \tag{3.61}$$

and leaves all other coordinates unchanged. This is a reasonable coordinate transformation, because we see $e^{-\chi(\tau)} > 0$ such that the new time t' increases strictly monotonously with the old one t. The transformation is relevant only for the metric $g_{\alpha\beta}$. It does not matter for the electromagnetic vector potential (3.59), because a coordinate transformation cannot make a vanishing vector nonzero. And, the matter field λ is anyway not affected, because it is a scalar.

Let us now evaluate how the coordinate transformation (3.61) changes the metric $g^{\bullet}_{\alpha\beta}$ spewed out by the conformal gauge transformation (3.60). To this end, the computations

$$g^{\bullet\prime}_{ab} = \frac{\partial x^\gamma}{\partial x'^a}\frac{\partial x^\delta}{\partial x'^b} g^{\bullet}_{\gamma\delta} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g^{\bullet}_{cd} = g^{\bullet}_{ab}$$

and

$$g^{\bullet\prime}_{at} = \frac{\partial x^\gamma}{\partial x'^a}\frac{\partial x^\delta}{\partial x'^t} g^{\bullet}_{\gamma\delta} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^t}{\partial x'^t} g^{\bullet}_{ct} = 0$$

show us that only the component $g^{\bullet}_{tt}$ is affected. For that component, we obtain

$$g^{\bullet\prime}_{tt} = \frac{\partial x^\gamma}{\partial x'^t}\frac{\partial x^\delta}{\partial x'^t} g^{\bullet}_{\gamma\delta} = \left(\frac{\partial t}{\partial t'}\right)^2 g^{\bullet}_{tt} = \left(\frac{\partial t'}{\partial t}\right)^{-2} g^{\bullet}_{tt} = e^{2\chi(t)} g^{\bullet}_{tt} = g_{tt}$$

So, after the coordinate transformation, the metric has again the form (3.56). World cosmology is therefore based on the cosmological fields

$$\boxed{\begin{aligned} g_{\alpha\beta} &= \mathrm{diag}\left(-1, R(t)^2, R(t)^2, \ldots\right) \\ A_\alpha &= 0 \\ \lambda &= \lambda(t) \end{aligned}} \tag{3.62}$$

3.6.3 Cosmological constant

Having found the cosmological fields, we can now insert them in the field equations. For the matter field λ, we use equation (3.7), which produces

$$\partial_\alpha \lambda(t) \overset{D\geqq 3}{\equiv} 0 \tag{3.63}$$

This equation holds identically for the spatial components, and the temporal one gives

$$\lambda(t) \overset{D\geqq 3}{\equiv} \Lambda \tag{3.64}$$

It is reasonable to use the cosmological constant Λ here, because then the world equation (3.1) reduces to

$$\boxed{G_{\alpha\beta} + \frac{1}{6}(D-1)(D-2)\, g_{\alpha\beta}\Lambda \overset{D\geqq 3}{\equiv} 0}$$

such that in four dimensions

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda \overset{D=4}{=} 0 \tag{3.65}$$

This is just Einstein's vacuum equation with a cosmological constant (see equation (17.11) of Misner *et al.* 2002).

Now, two conventions adopted in world theory become clear. The first one is the letter "λ" chosen for the matter field. Due to equation (3.64), it was reasonable to select a letter similar to the letter "Λ" of the cosmological constant. In addition to that, equation (3.65) explains the factor $-1/6$ used in Section 3.3.2. Only that factor leads exactly to Einstein's vacuum equation with a cosmological constant if we apply equation (3.64). In total, we thus understand the origin of the factor $-(D-1)(D-2)/6$ appearing in the world equation (3.1). Note that there is actually also a coupling constant like the electromagnetic constant c_{EM} in equation (2.20). We want to call this quantity **material constant** c_{M}. Throughout the paper, we choose $c_{\mathrm{M}} = 1$ such that the material constant does not explicitly occur anywhere, except of course in this paragraph. The choice $c_{\mathrm{M}} = 1$ constrains the units of the matter field λ. For general units, the material constant must therefore be included, and then we get the factor $-(D-1)(D-2)c_{\mathrm{M}}/6$.

The outcome (3.65) is astonishing, but it is a beautiful, intrinsic property of world theory. So, for $D = 4$, world cosmology is just defined by equation (3.65). The solutions of this equation are known from general relativity. For $\Lambda = 0$, we have a Minkowski spacetime, for $\Lambda > 0$ a de Sitter spacetime and for $\Lambda < 0$ an anti-de Sitter spacetime. Cosmological observations now tell us that our universe has a positive cosmological constant Λ (see, e.g., Perlmutter *et al.* 1999). In this paper, we therefore limit ourselves to $\Lambda > 0$ and thus to de Sitter spacetimes. They are given by the scale factor

$$R(t) = R(0)\, e^{Ht} \tag{3.66}$$

where

$$H = \sqrt{\frac{\Lambda}{3}} \tag{3.67}$$

is the Hubble constant (see equation (27.76) of Misner *et al.* 2002). Let us sum up this result:

> **World cosmology**:
>
> The cosmology of world theory is de Sitter cosmology.

Note that equation (3.67) has a close similarity to relation (3.22). If we had chosen the expression h^2 on the left hand side there, the Hubble constant would be the cosmological value of the field h. However, this was not possible, because then the matter field λ would be unnaturally restricted to non-negative values. Thus, the Hubble constant H is the cosmological value of the field $\sqrt{h}$. By the way, we therefore want to call h **Hubble field**.

3.6.4 Temporal cosmological principle

De Sitter cosmology has an interesting side effect. It has the same shape at all spatial slices. In the following paragraph, I try to make that clear to readers not aware of this (an alternative way to show this is given later in Section 4.2.1). The remaining readers can skip that paragraph.

To recognize that all spatial slices of a de Sitter spacetime have the same form, we look at an arbitrary spatial line element comoving with spacetime. Such a line element is characterized by two lengths, the temporally growing physical length $\mathrm{d}s_{1\mathrm{p}}(t)$ and the constant coordinate length $\mathrm{d}s_{1\mathrm{c}}$. Due to the metric (3.62) and the scale factor (3.66), the lengths are coupled by the relation

$$\mathrm{d}s_{1\mathrm{p}}(t) = R(t)\,\mathrm{d}s_{1\mathrm{c}} = R(0)\,e^{Ht}\mathrm{d}s_{1\mathrm{c}} \tag{3.68}$$

If we now look at the growing physical length $\mathrm{d}s_{1\mathrm{p}}(t)$, one is easily misguided into believing that de Sitter spacetimes undergo an evolution. Why? We look at different points in time and see that the quantity $\mathrm{d}s_{1\mathrm{p}}(t)$ has different values. However, we have to be careful. It is important to compare the right entities at different times. Actually, we have to introduce a second line element like the above one and use it for the comparison. This second line element shall be described by the physical and coordinate lengths $\mathrm{d}s_{2\mathrm{p}}(t)$ and $\mathrm{d}s_{2\mathrm{c}}$, for which

$$\mathrm{d}s_{2\mathrm{p}}(t) = R(0)\,e^{Ht}\mathrm{d}s_{2\mathrm{c}} \tag{3.69}$$

Our goal is now to compare the time $t = t_1$ of the first line element with the time $t = t_2$ of the second one. Then, the same physical length has to be selected for the two line elements at their respective times. Mathematically, this stands for

$$\mathrm{d}s_{1\mathrm{p}}(t_1) = \mathrm{d}s_{2\mathrm{p}}(t_2)$$

Hence, equations (3.68) and (3.69) imply

$$e^{Ht_1}\mathrm{d}s_{1\mathrm{c}} = e^{Ht_2}\mathrm{d}s_{2\mathrm{c}}$$

and thus

$$\mathrm{d}s'_{1\mathrm{p}}(t_1) = HR(0)\,e^{Ht_1}\mathrm{d}s_{1\mathrm{c}} = HR(0)\,e^{Ht_2}\mathrm{d}s_{2\mathrm{c}} = \mathrm{d}s'_{2\mathrm{p}}(t_2)$$

What does that mean? We have two line elements of the same physical length at respectively different times. Still, they have the same growth rates. Therefore, de Sitter spacetimes look the same at all points in time.

The fact that all spatial slices of a de Sitter spacetime have the same shape can be summarized in what we call the **temporal cosmological principle**:

> **Temporal cosmological principle**:
>
> At large enough temporal scales, the universe is approximately temporally homogeneous. (3.70)

Note that in contrast to the spatial cosmological principle there is no temporal isotropy. As time is 1-dimensional, temporal isotropy would represent a time reversal invariance. However, there is no such invariance, because the expansion of space induces an arrow of time.

3.6.5 End of time

Where are we now? We know that in world theory both the spatial and the temporal cosmological principle hold. This case is also known as the **perfect cosmological principle**. However, we have to be careful with the evolution processes encountered in the universe. Actually, such processes appear to exist throughout the currently observed cosmological timescales. We do not want to go into details, because it is totally irrelevant for the argumentation performed here how these processes look like[7]. The important point is just that there is no observational evidence of the temporal cosmological principle at the moment. This does not mean that it is not realized. Yet, if it is, then it must be found beyond the observed timescales. And, this constraint has prevented the introduction of the temporal cosmological principle in general relativity. Why should we make a theory artificially more complicated to postulate something that is anyway not yet observed?

In world theory, the situation is different. In contrast to general relativity, we do not have to postulate the temporal cosmological principle there. Instead, we can derive it from the spatial cosmological principle. And, then it does not matter whether we have already observed the temporal cosmological principle. As we have not, a realization of world theory in nature would mean just that our observations have not yet reached the right temporal scales. The derivation of the temporal cosmological principle would then allow us to mathematically look beyond our current observational range. This is comparable to the spatial cosmological principle. We can witness that principle only within the observable universe. However, as soon as we accept it (due to Occam's razor), we get indirect knowledge about what is beyond the observable universe. The universe is spatially infinite.

The temporal cosmological principle allows us now to glance at the end of time in world theory. And, we see that there is no end, neither in the past nor in the future direction. The universe is temporally infinite. So, there is no big bang. The universe has existed since eternal times and will exist forever in that theory:

End of time:

The universe has no beginning and end.

Unfortunately, world theory neither tells us the scales where the temporal cosmological principle begins to hold nor does it give any constraints on the evolution processes occurring below. However, we are at least no longer plagued by the cosmological singularity of general relativity, the big bang.

3.6.6 Inhomogeneous cosmology

There is one important caveat to the cosmological results found above. Although there is no doubt about the validity of the spatial cosmological principle, the inhomogeneities present at small scales may become so strong that there are backreactions to the cosmological equation (3.65). This case is known as inhomogeneous cosmology (currently, there is a dispute about whether inhomogeneous cosmology is relevant, see Buchert *et al.* 2015 and the references therein). In the following, we will repeat the mathematical foundations of inhomogeneous cosmology and study its implications on world cosmology.

The starting point is to write the world equation (3.20) as an initial value problem. Later in the paper (see Section 4.6.12), we will prove that this is possible and show the mathematical details. However, this will be done only for $D = 4$ such that we limit ourselves to this case in the rest of this section. Anyway, for inhomogeneous cosmology,

[7] In fact, we have to also be careful here. The standard model of cosmology is founded on general relativity. However, if world theory is the right description of nature, general relativity can no longer be used for that task. Then, the standard model of cosmology is incorrect in parts. With world cosmology being de Sitter cosmology, the big bang can, for instance, not be taken over. And, there might be other modifications required, which cannot be foreseen at the moment.

we can omit mathematical details and adopt a simple, abstract viewpoint. Having an initial value problem means that there are N evolution variables v^A, with $A = 1, ..., N$. These variables are governed by the evolution equations

$$\partial_t v^A = f^A \left(v^1, ..., v^N \right) \tag{3.71}$$

As we will see in the mentioned later part of the paper, the functions f^A are nonlinear. This is the key problem of inhomogeneous cosmology. Why?

In Sections 3.6.1 and 3.6.2, we have found that the spatial cosmological principle reduces the form of the fields $g_{\alpha\beta}$, A_α and λ to the cosmological fields (3.62). This reduction is actually the result of applying a spatial average $\overline{...}$. An example for such an average may be

$$\overline{...} = \lim_{V\to\infty} \frac{\int_V \mathrm{d}^3x ...}{\int_V \mathrm{d}^3x}$$

where V is the integration volume and $\mathrm{d}^3x = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. A side difficulty in this context is that the above example depends on the chosen coordinates x^α. However, there does not seem to be a way to construct a coordinate-invariant average. Fortunately, we do not have to worry about such details here. Instead, we focus on the following problem.

Due to equation (3.71), we find that for nonlinear functions f^A

$$\partial_t \overline{v^A} = \overline{\partial_t v^A} = \overline{f^A \left(v^1, ..., v^N \right)} = \overline{f^A \left(\overline{v^1} + \Delta v^1, ..., \overline{v^N} + \Delta v^N \right)} \neq f^A \left(\overline{v^1}, ..., \overline{v^N} \right) \tag{3.72}$$

where $\Delta v^A = v^A - \overline{v^A}$. Hence, the evolution of the averages $\overline{v^A}$ is defined not just by these quantities themselves as stated in the first sentence of Section 3.6.3. Instead, we have to also take the inhomogeneities Δv^A into account.

The usual solution to the problem (3.72) is to assume that the inhomogeneities Δv^A are either weak such that a perturbative approach is possible. The background of the perturbations is then described by

$$\partial_t \overline{v^A} = f^A \left(\overline{v^1}, ..., \overline{v^N} \right) \tag{3.73}$$

which is what we used in Section 3.6.3. Or, the inhomogeneities are strong, but then they are localized in regions so small that they can be neglected on cosmological scales. An example for the second case are black holes. The inhomogeneity caused by a black hole is obviously very strong. However, all black holes known in our universe are of subgalactic size.

Based on these considerations, it appears as if we do not have to worry about the issue (3.72) in world cosmology. Unfortunately, world cosmology tries to look at the end of time and thus far beyond the observed cosmological timescales. It is therefore not really clear whether we can take the approach (3.73). However, we can argue as follows. If the evolution (3.73) is violated by strong cosmological inhomogeneities, then there are two possibilities. Either the expansion speed of space is lower or higher than the standard expansion speed described by equation (3.73). In case of a lower speed, the region covered by that speed will reduce over time compared to the surrounding space, because that space expands with the higher standard speed. Hence, such an inhomogeneity will eventually become cosmologically irrelevant. In case of a higher expansion speed than the surrounding one, an effect as described by inflation theory (see Guth 1981) will take place. The inhomogeneities will be smoothed out and become perturbative. Could this actually just be the explanation for inflation theory? Unfortunately, we cannot answer this question, because we are at a too speculative level here. Anyway, based on the above considerations, it appears as if the approach (3.73) is truly acceptable. Going sure is possible only by performing cosmological simulations based on world theory. We will address the foundations of this topic in Chapter 4.

3.7 Explanations

3.7.1 Dark energy

In the following two sections, we have a closer look at two phenomena known from astrophysics, dark energy and dark matter. Both phenomena are nowadays unsolved riddles and investigated intensively by the physics community.

We begin with dark energy (see Perlmutter *et al.* 1999 and Riess *et al.* 1998). Observations tell us that the universe does not only expand, the expansion even accelerates. So, there must be some mechanism responsible for this acceleration, and one uses the name "dark energy" for it. This is by the way only a notion. We do not know whether a yet unknown energy is the cause. Actually, such names can easily be misleading and prevent us from looking in the right direction for answers.

Anyway, the accelerated expansion of the universe is described by a cosmological constant Λ. That way, we come to world theory. We remember that the world equation reduces to equation (3.65) in world cosmology. In addition to that, equation (3.64) tells us that the cosmological constant Λ appearing there is the cosmological value of the matter field λ. So, in world theory, the accelerated expansion of the universe originates from the matter field λ. Therefore, we have the phenomenon known as quintessence (see, e.g., Caldwell *et al.* 1998). However, quintessence occurs in a fundamentally new manner here, described by the material field equation (3.10). For the matter field λ, we even know a geometric interpretation from box (3.24) such that I want to make the following proposal:

Geometric interpretation of dark energy:
Dark energy is the matter field λ. (3.74)

We can even argue vice versa. World theory was developed without the aim to explain dark energy. Instead, the goal was to unify electromagnetism and gravitation by curing the anomaly (2.6) in Einstein's equation. The result of this approach is a theory whose cosmology is based on de Sitter spacetimes. That way, we can consider the accelerated cosmic expansion as a first observational hint that world theory may be more than just a theory.

3.7.2 Dark matter

Now, we come to the other unsolved riddle of astrophysics, dark matter (see, e.g., the review Bertone *et al.* 2005). For that purpose, we have to look at the motion of stars inside of a galaxy. The stars move around the center of the galaxy with a speed that depends on the distance to the center. We do not go into details here. The important point is just that the observed radial speed profile is not consistent with general relativity. However, it can be made consistent by assuming that galaxies do not only harbor observable objects like stars, but also some constituents which are invisible to us. These constituents are referred to as "dark matter". Dark matter increases the mass of a galaxy such that the stars move faster around the center. This leads to the question what dark matter actually is. That is one way to address the dark matter problem. The other known approach is to assume that there is no dark matter at all. Instead, Einstein's equation does not hold and has to somehow be modified.

We have actually discovered such a modification. Yet, it was not motivated by observations but by philosophical considerations. The modification is just the world equation (3.2). I therefore want to propose our modification as a possible explanation for the phenomenon known as dark matter. How does that actually work? World theory does not modify vacuum general relativity. Electromagnetism will probably also not play a role when looking at the motion of stars in a galaxy, although this is not sure. We use, however, a completely different approach to matter in world theory. It is not based on general relativistic hydrodynamics but on the matter field λ. This difference could explain why stars do not follow the laws of general relativity. Unfortunately, working out the details goes beyond the scope of this paper. We also do not know whether that explanation is a complete one. It could in fact be that the riddle dark matter has two causes. The first one is that not the world equation (3.2) is applied on the visible matter but instead the supposedly incorrect field equation (1.18) of Einstein. And, the second one is some invisible form of matter. This kind of matter may be included in world theory, but not radiate electromagnetically or in some other detectable way. The alternative is that the invisible matter may be something yet unknown. So, we can merely suggest:

Source of phenomenon dark matter:

A potential source is using Einstein's equation instead of the world equation. (3.75)

Note that we have not given a comprehensive explanation for dark matter. We do not know whether our above suggestion explains just a fraction of that phenomenon or even all of it. So, we have only provided a proposal for dark matter, which appears reasonable in the context of world theory. This should be considered as the starting point for future investigations.

3.8 Limits

3.8.1 Separated fields

Let us now return to a less speculative topic. In the following, we will study various limits possible for the world equation (3.1). The easiest limits are those where only one of the three fields $g_{\alpha\beta}$, A_α and λ is used. For that purpose, we look at equations (3.1), (3.7) and (3.10) such that we get

$$\boxed{\begin{aligned} G_{\alpha\beta} &= 0 \\ \left[\partial^\beta + (D-4)\, A^\beta\right] F_{\beta\alpha} &\overset{D\neq 2}{=} 0 \\ \Box\lambda &\overset{D\geq 3}{=} 0 \end{aligned}} \tag{3.76}$$

or in four dimensions

$$\left.\begin{matrix} G_{\alpha\beta} \\ \partial^\beta F_{\beta\alpha} \\ \Box\lambda \end{matrix}\right\} \overset{D=4}{=} 0$$

These are Einstein's equation in the limit of vacuum gravitation, the electromagnetic field equation in flat spacetime without sources and the material field equation without non-material forces.

That way, we also see that all three limits are based on a wave operator. To make that even more clear, we apply the Lorentz gauge

$$\partial^\alpha A_\alpha = 0 \tag{3.77}$$

which is familiar from electrodynamics. Then, the electromagnetic field equation becomes

$$\Box A_\alpha \overset{D=4}{=} 0$$

To also obtain a d'Alembertian $\Box$ for Einstein's equation, we have to linearize gravitation. This is a well-known procedure in general relativity and repeated further below.

3.8.2 Electromatter

The only other possible limits are those where two fields are present. Having three fields would be just the full world equation (3.1) again. Let us first look at the case where the electromagnetic vector potential A_α vanishes. In that case, the electromagnetic field equation (3.7) becomes

$$\partial_\alpha \lambda \overset{D\geq 3}{=} 0$$

such that

$$\lambda \overset{D\geq 3}{=} \Lambda$$

just like in cosmology, see equation (3.64) there. This means that matter is not given by a dynamic field, which makes this limit exceptional. We have therefore also not included it in the drawing shown in Section 3.1.1. The other limits with two fields are the electrogravitational and the electromaterial ones. We do not have to repeat electrogravitation here. This theory was studied in Chapter 2, and the outcome was the electrogravitational field equation (2.105). That way, only electromatter remains.

So, we assume that

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

for the rest of this section. Then, the electromagnetic and material field equations (3.7) and (3.10) become

$$\boxed{\begin{aligned} \left[\partial^\beta + (D-4)\,A^\beta\right] F_{\beta\alpha} &\overset{D\neq 2}{=} \frac{1}{3}\,(D-1)\,(\partial_\alpha - 2A_\alpha)\,\lambda \\ (D-1)\left[\partial^\alpha + (D-4)\,A^\alpha\right](\partial_\alpha - 2A_\alpha)\,\lambda &\overset{D\neq 2}{=} \frac{3}{2}\,(D-4)\,F^{\alpha\beta}F_{\beta\alpha} \end{aligned}} \tag{3.78}$$

or in four dimensions

$$\begin{aligned} \partial^\beta F_{\beta\alpha} &\overset{D=4}{=} (\partial_\alpha - 2A_\alpha)\,\lambda \\ \partial^\alpha\,(\partial_\alpha - 2A_\alpha)\,\lambda &\overset{D=4}{=} 0 \end{aligned} \tag{3.79}$$

These two pairs of equations are the **electromaterial field equations in flat spacetime**. Note that the respective second equation is hidden in the first one. We have explicitly written these additional two equations down for completeness.

The above limit is by the way not a strict one. The problem is that we also have to look at the gravitational field equation (3.1), which spews out

$$\partial_{(\alpha}A_{\beta)} - A_\alpha A_\beta - \eta_{\alpha\beta}\left[\partial_\gamma A^\gamma + \frac{1}{2}(D-3)A^2 + \frac{1}{6}(D-1)\lambda\right] \overset{D\neq 2}{=} 0 \tag{3.80}$$

where $A^2 = \eta^{\alpha\beta}A_\alpha A_\beta$ here, with $\eta^{\alpha\beta} = \eta_{\alpha\beta}$. This equation is ignored for electromatter. Note that the second and third limits shown in box (3.76) are affected by that issue, too.

Another point is that the electromaterial limit violates conformal gauge invariance. The transformation laws (2.70) and (3.3) yield

$$(\partial_\alpha - 2A_\alpha)\lambda \overset{\text{cg}}{\to} e^{2\chi}(\partial_\alpha - 2A_\alpha)\lambda$$

where the factor $e^{2\chi}$ is the problem. Thus, for instance, equation (3.79) is obviously not conformally gauge invariant. So, in the electromaterial limit, we have the $D+1$ degrees of freedom coming from the electromagnetic vector potential A_α and the matter field λ. However, as the material field equation can be derived from the electromagnetic one (see Section 3.2.3), there are only D independent field equation components. Therefore, one degree of freedom is undetermined. Due to the lack of conformal gauge invariance, it is not a gauge degree of freedom. The undetermined degree of freedom has instead to be set manually. Hence, there is actually not a single electromaterial limit but one such limit for each choice made here.

3.8.3 Field equation hierarchy

Due to the limits studied in the last two sections, we now know the field equation belonging to each of the cases shown in the drawing of Section 3.1.1. For gravitation, electromagnetism and matter, the limits are found in box (3.76). With respect to electrogravitation and electromatter, we have to look at the electrogravitational field equation (2.36) and the first line in box (3.78). Eventually, world theory is governed by the world equation (3.20). That way, we obtain the following hierarchy of field equations:

$$\begin{tikzcd}
G_{\alpha\beta} = 0 \arrow[dr] & & [(D-4)A^\beta + \partial^\beta]F_{\beta\alpha} \overset{D\neq 2}{=} 0 \arrow[dl] \arrow[dr] & & \Box\lambda \overset{D\geq 3}{=} 0 \arrow[dl] \\
& Z_{\alpha\beta} = 0 \arrow[dr] & & [(D-4)A^\beta + \partial^\beta]F_{\beta\alpha} \overset{D\neq 2}{=} \frac{1}{3}(D-1)(\partial_\alpha - 2A_\alpha)\lambda \arrow[dl] & \\
& & \Omega_{\alpha\beta} = 0 & &
\end{tikzcd} \tag{3.81}$$

To simultaneously show the D- and 4-dimensional cases in this box, we have visualized some expressions above with gray color. The gray expressions disappear for $D = 4$.

3.9 Linearization

3.9.1 Gravitation

Let us now address the open issue at the end of Section 3.8.1 and repeat the linearization of gravitation in a flat background spacetime $\eta_{\alpha\beta}$. This repetition will simplify understanding what is going on if the other two fields A_α and λ of world theory are included. Readers familiar with general relativity can skip this section.

The linearization of gravitation is based on splitting

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{3.82}$$

under the assumption that $|h_{\alpha\beta}| \ll 1$ (see p. 435ff of Misner *et al.* 2002 for the case $D = 4$). In addition to that,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + \mathcal{O}\left(h_{\alpha\beta}\right)^2 \tag{3.83}$$

with $h^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta}h_{\gamma\delta}$, because then equation (1.3) holds if we neglect second and higher orders in $h_{\alpha\beta}$. Using now relations (3.82) and (3.83) for the Christoffel symbols (1.4), we get

$$\Gamma^\alpha_{\beta\gamma} - \mathcal{O}\left(h_{\alpha\beta}\right)^2 = \frac{1}{2}\eta^{\alpha\delta}\left(\partial_\beta h_{\delta\gamma} + \partial_\gamma h_{\beta\delta} - \partial_\delta h_{\beta\gamma}\right) = \frac{1}{2}\left(\partial_\beta h^\alpha_\gamma + \partial_\gamma h^\alpha_\beta - \partial^\alpha h_{\beta\gamma}\right) \tag{3.84}$$

The Ricci tensor (1.5) that way becomes

$$R_{\alpha\beta} - \mathcal{O}\left(h_{\alpha\beta}\right)^2 = \partial_{[\gamma}\left(\partial_\alpha h^\gamma_{\beta]} + \partial_{\beta]} h^\gamma_\alpha - \partial^\gamma h_{\alpha\beta]}\right) = \frac{1}{2}\left(\partial_\alpha{}^\gamma h_{\beta\gamma} + \partial_\beta{}^\gamma h_{\alpha\gamma} - \Box h_{\alpha\beta} - \partial_{\alpha\beta}h\right)$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ and a partial derivative with two indices abbreviates two successive such derivatives. Note that the quantity h utilized in this section is not to be mixed up with the Hubble field (3.22) used in the rest of this paper, which is referred to by the same letter. Next, we evaluate the Ricci scalar (1.16):

$$R = \partial^{\alpha\beta}h_{\alpha\beta} - \Box h + \mathcal{O}\left(h_{\alpha\beta}\right)^2$$

Then, we can compute the Einstein tensor (1.17), which is

$$G_{\alpha\beta} = \frac{1}{2}\left[\partial_\alpha{}^\gamma h_{\beta\gamma} + \partial_\beta{}^\gamma h_{\alpha\gamma} - \Box h_{\alpha\beta} - \partial_{\alpha\beta}h - \eta_{\alpha\beta}\left(\partial^{\gamma\delta}h_{\gamma\delta} - \Box h\right)\right] + \mathcal{O}\left(h_{\alpha\beta}\right)^2 \tag{3.85}$$

Eventually, we introduce the quantity

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \tag{3.86}$$

Due to

$$\bar{h} = \eta^{\alpha\beta}\bar{h}_{\alpha\beta} = -\frac{1}{2}\left(D-2\right)h \tag{3.87}$$

we can therefore rewrite

$$h_{\alpha\beta} \overset{D\neq 2}{=} \bar{h}_{\alpha\beta} - \frac{1}{D-2}\eta_{\alpha\beta}\bar{h}$$

This allows us to write the Einstein tensor (3.85) as

$$G_{\alpha\beta} \overset{D\neq 2}{=} \frac{1}{2}\left(\partial_\alpha{}^\gamma \bar{h}_{\beta\gamma} + \partial_\beta{}^\gamma \bar{h}_{\alpha\gamma} - \Box\bar{h}_{\alpha\beta} - \eta_{\alpha\beta}\partial^{\gamma\delta}\bar{h}_{\gamma\delta}\right) + \mathcal{O}\left(h_{\alpha\beta}\right)^2 \tag{3.88}$$

The next step is to use the coordinate transformation

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}$$

with the coordinate change $|\xi^{\alpha}| \ll 1$. Then,

$$g_{\alpha\beta} = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g'_{\gamma\delta} = \left(\delta^{\gamma}_{\alpha} + \partial_{\alpha}\xi^{\gamma}\right) \left(\delta^{\delta}_{\beta} + \partial_{\beta}\xi^{\delta}\right) g'_{\gamma\delta} = g'_{\alpha\beta} + 2g'_{\gamma(\alpha}\partial_{\beta)}\xi^{\gamma} + \mathcal{O}\left(\xi^{\alpha}\right)^2$$

Let us now assume that the coordinate change ξ^{α} has the same order of magnitude as the metric perturbation $h_{\alpha\beta}$. We denote this order of magnitude as ϵ. Then, the split (3.82) gives

$$h'_{\alpha\beta} = h_{\alpha\beta} - 2\partial_{(\alpha}\xi_{\beta)} + \mathcal{O}\left(\epsilon\right)^2$$

with $\xi_{\alpha} = \eta_{\alpha\beta}\xi^{\beta}$. Hence, equation (3.86) leads to

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - 2\partial_{(\alpha}\xi_{\beta)} + \eta_{\alpha\beta}\partial_{\gamma}\xi^{\gamma} + \mathcal{O}\left(\epsilon\right)^2$$

This implies

$$\partial'^{\beta}\bar{h}'_{\beta\alpha} = \partial^{\beta}\bar{h}_{\alpha\beta} - \Box\xi_{\alpha} + \mathcal{O}\left(\epsilon\right)^2$$

Note that also the partial derivative ∂^{β} is affected by the coordinate transformation. However, it does not produce a visible influence above, because the quantity $\bar{h}_{\alpha\beta}$ is already of the order ϵ such that the influence disappears inside of the expression $\mathcal{O}\left(\epsilon\right)^2$. The above outcome now shows us that it is possible to choose a coordinate change ξ^{α} such that the left hand side vanishes there. Leaving the prime away, we can thus impose what is known as the Lorentz gauge

$$\partial^{\beta}\bar{h}_{\alpha\beta} = 0$$

That way, equation (3.88) simplifies to

$$G_{\alpha\beta} \overset{D\neq 2}{=} -\frac{1}{2}\Box\bar{h}_{\alpha\beta} + \mathcal{O}\left(\epsilon\right)^2 \tag{3.89}$$

and we see that even gravitation is based on a d'Alembertian $\Box$.

3.9.2 Electromagnetism

Now, we come to electromagnetism, which means linearizing electrogravitation. To this end, we look at the field equation of that theory (2.105). For the Einstein tensor $G_{\alpha\beta}$ on the left hand side there, we can use the result (3.89). For the right hand side, an open issue is the order of magnitude of the electromagnetic vector potential A_{α}. To limit the possible answers, we write the first term in the curly bracket of equation (2.105) out here:

$$\nabla_{(\alpha}A_{\beta)} = \partial_{(\alpha}A_{\beta)} - \Gamma^{\gamma}_{\alpha\beta}A_{\gamma}$$

The term with the partial derivative shows us that the electromagnetic vector potential A_{α} may not be larger than the order ϵ, for otherwise it cannot be handled by the small quantity $\bar{h}_{\alpha\beta}$ in relation (3.89). For Einstein-Maxwell theory, equation (1.20) tells us that this limit would by the way be $\sqrt{\epsilon}$. This leads to the question whether the electromagnetic vector potential A_{α} must vice versa have a minimal order to be able to deal with the metric perturbation $\bar{h}_{\alpha\beta}$. For that purpose, we look at the electromagnetic field equation (2.108). The second term in the squared bracket there together with the factor $F_{\beta\alpha}$

is of the order $\mathcal{O}(\epsilon)^2$, because the electromagnetic vector potential A_α is maximally of the order ϵ. For the first term, we can reduce the covariant derivative ∇^β to a partial one. We merely have to rewrite $\nabla^\beta = g^{\beta\gamma}\nabla_\gamma$ and take into account that this expression is applied on an electromagnetic vector potential of the maximal order ϵ. Due to relation (3.84), the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ have the order ϵ such that if we neglect the order $\mathcal{O}(\epsilon)^2$, we can replace the covariant derivative ∇_γ with the partial one ∂_γ. That way, we have $\nabla^\beta = g^{\beta\gamma}\partial_\gamma$. Recalling again the order of the electromagnetic vector potential and relation (3.83), we see that up to the order $\mathcal{O}(\epsilon)^2$ we can raise the index γ here with the Minkowski metric $\eta^{\beta\gamma}$. This metric can by the way also be used to raise and lower indices of the electromagnetic vector potential in this context. All in all, we then find

$$\boxed{\partial^\beta F_{\beta\alpha} \overset{D\neq 2}{=} \mathcal{O}(\epsilon)^2} \tag{3.90}$$

So, here there is no limitation for the electromagnetic vector potential A_α, and we decide to make it as large as possible, i.e. of the order ϵ. That way, the linearization of electrogravitation is based on

$$\boxed{|h_{\alpha\beta}|, |A_\alpha| \ll 1}$$

For Einstein-Maxwell theory, this choice would decouple electromagnetism from gravitation, because then the right hand side of equation (1.20) is only of the neglected order $\mathcal{O}(\epsilon)^2$. In electrogravitation, this is different. There, the field equation (2.105) tells us

$$G_{\alpha\beta} = (D-2)\left(\nabla_{(\alpha}A_{\beta)} - g_{\alpha\beta}\nabla_\gamma A^\gamma\right) + \mathcal{O}(\epsilon)^2$$

Using then relations (3.82) and (3.84) for the metric $g_{\alpha\beta}$ and the Christoffel symbols inside of the covariant derivatives, together with the result (3.89), we obtain

$$\boxed{\Box\bar{h}_{\alpha\beta} \overset{D\neq 2}{=} 2(D-2)\left(\eta_{\alpha\beta}\partial_\gamma A^\gamma - \partial_{(\alpha}A_{\beta)}\right) + \mathcal{O}(\epsilon)^2} \tag{3.91}$$

This outcome and equation (3.90) are the **field equations of linearized electrogravitation**.

The above linearization is important in two ways. First, it shows us that there is a method to avoid the problem mentioned at the end of Section 3.8.2. So, we can obtain strict limits in world theory by using linearizations. That way, no equations are simply ignored. Instead, the full world equation (3.1) is properly taken into account, and all neglected expressions are clearly marked by the symbol $\mathcal{O}$. The second point is equation (3.90). As no metric perturbation $\bar{h}_{\alpha\beta}$ appears there, electromagnetism is not influenced by gravitation. This equation is therefore just Maxwell's equation in flat spacetime without sources (1.2). The only requirement for that is $|A_\alpha| \ll 1$, similar to if we looked at general relativity. This means that neither in world theory nor in Einstein-Maxwell theory it is possible to derive Maxwell's equation in flat spacetime without sources with an arbitrarily strong electromagnetic vector potential A_α.

3.9.3 Matter

With gravitation and electromagnetism being linearized, merely matter remains. For that purpose, we have to look at the world equation (3.1). The only difference to the electrogravitational field equation (2.105) is the term that contains

$$g_{\alpha\beta}\lambda = \eta_{\alpha\beta}\lambda + \mathcal{O}(\epsilon)$$

where we have used relation (3.82). The λ-term on the right hand side above limits the order of magnitude of the matter field λ. This is similar to what we have found in the last section for the electromagnetic vector potential A_α. The metric perturbation $\bar{h}_{\alpha\beta}$ in relation (3.89) must be able to handle the above λ-term such that the matter field λ may not be larger than the order ϵ. Then, equation (3.91) generalizes to

$$\boxed{\Box\bar{h}_{\alpha\beta} \overset{D\neq 2}{=} 2\left(D-2\right)\left\{\eta_{\alpha\beta}\left[\partial_\gamma A^\gamma + \frac{1}{6}\left(D-1\right)\lambda\right] - \partial_{(\alpha}A_{\beta)}\right\} + \mathcal{O}\left(\epsilon\right)^2} \tag{3.92}$$

or in four dimensions

$$\Box\bar{h}_{\alpha\beta} \overset{D=4}{=} 4\left[\eta_{\alpha\beta}\left(\partial_\gamma A^\gamma + \frac{\lambda}{2}\right) - \partial_{(\alpha}A_{\beta)}\right] + \mathcal{O}\left(\epsilon\right)^2$$

To also generalize equation (3.90), we look at the electromagnetic field equation (3.7), which shows us

$$\boxed{\partial^\beta F_{\beta\alpha} \overset{D\neq 2}{=} \frac{1}{3}\left(D-1\right)\partial_\alpha\lambda + \mathcal{O}\left(\epsilon\right)^2} \tag{3.93}$$

or again in four dimensions

$$\partial^\beta F_{\beta\alpha} \overset{D=4}{=} \partial_\alpha\lambda + \mathcal{O}\left(\epsilon\right)^2$$

This is a particularly simple source for electromagnetism. The source is just the gradient of the matter field λ. Yet, for strong matter fields λ, the source has the more complicated form in the electromagnetic field equation (3.7). By the way, equation (3.93) also tells us that a matter field λ of the order ϵ can even be handled by the electromagnetic vector potential A_α. That way, merely the material field equation (3.10) remains, which can be linearized as

$$\boxed{\Box\lambda \overset{D\geq 3}{=} \mathcal{O}\left(\epsilon\right)^2}$$

This result together with equations (3.92) and (3.93) are the **field equations of linearized world theory**.

All in all, we encountered no other constraint on the order of magnitude of the matter field λ. Therefore, we can in fact use the order ϵ such that the linearization of world theory is based on the simply assumption

$$\boxed{|h_{\alpha\beta}|, |A_\alpha|, |\lambda| \ll 1}$$

3.10 Metric split

3.10.1 Einstein's equation

Another interesting special case of world theory can be obtained by studying the consequences of splitting the metric into

$$g_{\alpha\beta} = \Phi^{2n}\varphi_{\alpha\beta} \tag{3.94}$$

with $n \in \mathbb{R}\setminus\{0\}$. By definition, the **tensorial gravitation density** $\varphi_{\alpha\beta}$ is given such that

$$\det\varphi_{\alpha\beta} = -1 \tag{3.95}$$

which implies that the **scalar gravitation density** Φ has the form

$$\Phi = \left(-g\right)^{\frac{1}{2nD}} > 0 \tag{3.96}$$

So, we can unambiguously compute both densities from the metric.

In what follows, we will first repeat the consequences of the above split for Einstein's equation (1.6). For that purpose, we also need the contravariant metric

$$g^{\alpha\beta} = \Phi^{-2n}\varphi^{\alpha\beta} \tag{3.97}$$

in which the quantity $\varphi^{\alpha\beta}$ is given by

$$\varphi^{\alpha\beta}\varphi_{\beta\gamma} = \delta^{\alpha}_{\gamma} \tag{3.98}$$

That way, the Christoffel symbols (1.4) become

$$\Gamma^{\alpha}_{\beta\gamma} = {}^{\varphi}\Gamma^{\alpha}_{\beta\gamma} + \frac{n}{\Phi}\left(\delta^{\alpha}_{\beta}\partial_{\gamma} + \delta^{\alpha}_{\gamma}\partial_{\beta} - \varphi^{\alpha\delta}\varphi_{\beta\gamma}\partial_{\delta}\right)\Phi \tag{3.99}$$

with

$${}^{\varphi}\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\varphi^{\alpha\delta}\left(\partial_{\beta}\varphi_{\delta\gamma} + \partial_{\gamma}\varphi_{\beta\delta} - \partial_{\delta}\varphi_{\beta\gamma}\right) \tag{3.100}$$

For the Ricci tensor (1.5), we first compute

$$\begin{aligned}\partial_{[\gamma}\Gamma^{\gamma}_{\alpha\beta]} \quad = \quad & \partial_{[\gamma}\,{}^{\varphi}\Gamma^{\gamma}_{\alpha\beta]} - \frac{n}{\Phi^2}\partial_{[\gamma}\Phi\left(\delta^{\gamma}_{\beta]}\partial_{\alpha} - \varphi^{\gamma\delta}\varphi_{\alpha\beta]}\partial_{\delta}\right)\Phi \\ & + \frac{n}{\Phi}\left(\delta^{\gamma}_{[\beta}\partial_{\gamma]\alpha} + \varphi^{\gamma\epsilon}\varphi^{\delta\zeta}\partial_{[\gamma}\varphi_{\epsilon\delta}\varphi_{\alpha\beta]}\partial_{\zeta} - \varphi^{\gamma\delta}\partial_{[\gamma}\varphi_{\alpha\beta]}\partial_{\delta} - \varphi^{\gamma\delta}\varphi_{\alpha[\beta}\partial_{\gamma]\delta}\right)\Phi\end{aligned} \tag{3.101}$$

where we have used

$$\partial_{\gamma}\varphi^{\alpha\beta} = -\varphi^{\alpha\delta}\varphi^{\beta\epsilon}\partial_{\gamma}\varphi_{\delta\epsilon}$$

This can be worked out in the same way as relation (2.141), with the exception that we have to replace the differential symbol d and the metric $g_{\alpha\beta}$ by the partial derivative ∂_{α} and the tensorial gravitation density $\varphi_{\alpha\beta}$. Moreover, we evaluate the Christoffel symbols product

$$\begin{aligned}\Gamma^{\gamma}_{\delta[\gamma}\Gamma^{\delta}_{\alpha\beta]} = \; & \frac{n}{\Phi}\left[{}^{\varphi}\Gamma^{\delta}_{\alpha[\beta}\left(\delta^{\gamma}_{\delta}\partial_{\gamma]} + \delta^{\gamma}_{\gamma]}\partial_{\delta} - \varphi^{\gamma\epsilon}\varphi_{\delta\gamma]}\partial_{\epsilon}\right) + {}^{\varphi}\Gamma^{\gamma}_{\delta[\gamma}\left(\delta^{\delta}_{\alpha}\partial_{\beta]} + \delta^{\delta}_{\beta]}\partial_{\alpha} - \varphi^{\delta\zeta}\varphi_{\alpha\beta]}\partial_{\zeta}\right)\right]\Phi \\ & + {}^{\varphi}\Gamma^{\gamma}_{\delta[\gamma}\,{}^{\varphi}\Gamma^{\delta}_{\alpha\beta]} + \left(\frac{n}{\Phi}\right)^2\left[\delta^{\gamma}_{[\gamma}\left(\partial_{\beta]}\Phi\partial_{\alpha} - \varphi_{\alpha\beta]}\varphi^{\delta\zeta}\partial_{\delta}\Phi\partial_{\zeta}\right) + \varphi^{\gamma\epsilon}\partial_{\epsilon}\Phi\varphi_{\alpha[\beta}\partial_{\gamma]}\right]\Phi\end{aligned} \tag{3.102}$$

We will now merge the above result with expression (3.101). In that expression, the second term in the lower round bracket can be easily combined with the last term in the second round bracket of equation (3.102) by using definition (3.100) such that

$$\varphi^{\delta\zeta}\left(\varphi^{\gamma\epsilon}\partial_{\eta}\varphi_{\epsilon\delta} - {}^{\varphi}\Gamma^{\gamma}_{\delta\eta}\right) = \frac{1}{2}\varphi^{\delta\zeta}\varphi^{\gamma\epsilon}\left(-\partial_{\delta}\varphi_{\epsilon\eta} + \partial_{\eta}\varphi_{\delta\epsilon} + \partial_{\epsilon}\varphi_{\delta\eta}\right) = \varphi^{\gamma\epsilon}\,{}^{\varphi}\Gamma^{\zeta}_{\epsilon\eta}$$

For the third term in the already considered bracket (3.101) and the last one in the first round bracket of equation (3.102), we see that

$${}^{\varphi}\Gamma^{\delta}_{\alpha[\beta}\varphi_{\delta\gamma]} = \partial_{[\beta}\varphi_{\alpha\gamma]}$$

such that the two terms considered here wear off. Hence, defining

$${}^{\varphi}R_{\alpha\beta} = 2\left(\partial_{[\gamma}\,{}^{\varphi}\Gamma^{\gamma}_{\alpha\beta]} + {}^{\varphi}\Gamma^{\gamma}_{\delta[\gamma}\,{}^{\varphi}\Gamma^{\delta}_{\alpha\beta]}\right) \tag{3.103}$$

we obtain the Ricci tensor

$$
\begin{aligned}
R_{\alpha\beta} &= {}^{\varphi}R_{\alpha\beta} + \frac{2n}{\Phi}\left(\delta^{\gamma}_{[\beta}\partial_{\gamma]\alpha} - \varphi^{\gamma\delta}\varphi_{\alpha[\beta}\partial_{\gamma]\delta} + {}^{\varphi}\Gamma^{\delta}_{\alpha[\beta}\delta^{\gamma}_{\gamma]}\partial_{\delta} + \varphi^{\gamma\epsilon}\varphi_{\alpha[\beta}\,{}^{\varphi}\Gamma^{\zeta}_{\epsilon\gamma]}\partial_{\zeta}\right)\Phi \\
&\quad + \frac{2n}{\Phi^{2}}\left[(n+1)\left(\delta^{\gamma}_{[\gamma}\partial_{\beta]}\Phi\partial_{\alpha} + \varphi^{\gamma\epsilon}\partial_{\epsilon}\Phi\varphi_{\alpha[\beta}\partial_{\gamma]}\right) - n\varphi_{\alpha[\beta}\varphi^{\delta\zeta}\delta^{\gamma}_{\gamma]}\partial_{\delta}\Phi\partial_{\zeta}\right]\Phi
\end{aligned}
$$

The next step is removing the index antisymmetrizations, and we then arrive at

$$
\begin{aligned}
R_{\alpha\beta} &= {}^{\varphi}R_{\alpha\beta} + \frac{n}{\Phi}\left[(2-D)\left(\partial_{\alpha\beta} - {}^{\varphi}\Gamma^{\delta}_{\alpha\beta}\partial_{\delta}\right) - \varphi_{\alpha\beta}\varphi^{\gamma\delta}\left(\partial_{\gamma\delta} - {}^{\varphi}\Gamma^{\epsilon}_{\gamma\delta}\partial_{\epsilon}\right)\right]\Phi \\
&\quad + \frac{n}{\Phi^{2}}\left[(n+1)(D-2)\,\partial_{\alpha}\Phi\partial_{\beta} + (2n-nD+1)\,\varphi_{\alpha\beta}\varphi^{\delta\zeta}\partial_{\delta}\Phi\partial_{\zeta}\right]\Phi \qquad (3.104)
\end{aligned}
$$

This motivates the introduction of a covariant derivative ${}^{\varphi}\nabla_{\alpha}$, which is defined like the common covariant derivative (1.7), but the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ appearing therein are replaced with the quantity ${}^{\varphi}\Gamma^{\alpha}_{\beta\gamma}$. In addition to that, care is required with respect to the scalar gravitation density Φ. For the covariant derivative ∇_{α}, this quantity is a scalar density of coordinate weight $1/(nD)$. To see this, we have to recall from general relativity that the expression $\sqrt{-g}$ has the coordinate weight 1 and then comply with relation (3.96). However, for the derivative ${}^{\varphi}\nabla_{\alpha}$, the quantity Φ shall by definition be treated as a scalar, i.e. it wears the coordinate weight 0. Additionally demanding that the index of the derivative ${}^{\varphi}\nabla_{\alpha}$ is not risen with the metric $g_{\alpha\beta}$ but with the tensorial gravitation density $\varphi_{\alpha\beta}$, we can then simplify equation (3.104) to

$$
\begin{aligned}
R_{\alpha\beta} &= {}^{\varphi}R_{\alpha\beta} + \frac{n}{\Phi}\left[(2-D)\,{}^{\varphi}\nabla_{\alpha}\,{}^{\varphi}\nabla_{\beta} - \varphi_{\alpha\beta}\,{}^{\varphi}\nabla^{\gamma}\,{}^{\varphi}\nabla_{\gamma}\right]\Phi \\
&\quad + \frac{n}{\Phi^{2}}\left[(n+1)(D-2)\,{}^{\varphi}\nabla_{\alpha}\Phi\,{}^{\varphi}\nabla_{\beta} + (2n-nD+1)\,\varphi_{\alpha\beta}\,{}^{\varphi}\nabla^{\gamma}\Phi\,{}^{\varphi}\nabla_{\gamma}\right]\Phi \qquad (3.105)
\end{aligned}
$$

For Einstein's equation (1.6), the above expression without doubt vanishes. The symmetry of the Ricci tensor $R_{\alpha\beta}$ then leads to $D(D+1)/2$ individual equations. An equation of particular interest is the one where the indices α and β are contracted. In that case, relations (3.97) and (3.105) give

$$
\Phi^{2n}R = {}^{\varphi}R + \frac{2n}{\Phi}(1-D)\,{}^{\varphi}\nabla^{\alpha}\,{}^{\varphi}\nabla_{\alpha}\Phi + \frac{n}{\Phi^{2}}(D-1)\left[2-n(D-2)\right]{}^{\varphi}\nabla^{\alpha}\Phi\,{}^{\varphi}\nabla_{\alpha}\Phi \qquad (3.106)
$$

with

$$
{}^{\varphi}R = \varphi^{\alpha\beta}\,{}^{\varphi}R_{\alpha\beta} \qquad (3.107)
$$

Choosing now

$$
n \overset{D\neq 2}{=} \frac{2}{D-2} \qquad (3.108)
$$

such that the lengthy last term in relation (3.106) vanishes, we see that Einstein's equation (1.6) leads to

$$
{}^{\varphi}\nabla^{\alpha}\,{}^{\varphi}\nabla_{\alpha}\Phi \overset{D\geqq 3}{=} \frac{1}{4}\frac{D-2}{D-1}\,{}^{\varphi}R\Phi
$$

For the tensorial gravitation density $\varphi_{\alpha\beta} = \eta_{\alpha\beta}$, we even find

$$
\Box\Phi \overset{D\geqq 3}{=} 0 \qquad (3.109)
$$

However, this is not a strict limit of Einstein's equation (1.6), because only the trace of that equation is guaranteed to hold. The remaining $D(D+1)/2-1$ individual equations may contradict our assumption $\varphi_{\alpha\beta} = \eta_{\alpha\beta}$.

3.10.2 World equation

The simplicity of the outcome (3.109) is the motivation for its generalization to world theory. For that purpose, we first look at the trace of equation (3.19)

$$\Omega = Z - (D-1)\, Dh \tag{3.110}$$

where we have used definition (3.27) and the **electrogravitation scalar**

$$\boxed{Z = g^{\alpha\beta} Z_{\alpha\beta}} \tag{3.111}$$

If this scalar vanished, we would obtain equation (2.104). For world theory, this scalar does not vanish, and we that way see

$$Z = R - (D-1)\left[2\nabla_\alpha A^\alpha + (D-2)\, A^2\right]$$

where we have simultaneously taken into account definitions (3.111) and (2.35). So, the world scalar (3.110) is

$$\Omega = R - (D-1)\left[2\nabla_\alpha A^\alpha + (D-2)\, A^2 + Dh\right] \tag{3.112}$$

and then equations (3.106) and (3.108) give

$$\Omega \overset{D\neq 2}{=} \Phi^{-2n}\, {}^{\varphi}R - \frac{D-1}{D-2}\frac{4}{\Phi^{1+2n}}\left\{ {}^{\varphi}\nabla^\alpha\, {}^{\varphi}\nabla_\alpha + \frac{D-2}{4}\Phi^{2n}\left[2\nabla_\alpha A^\alpha + (D-2)\, A^2 + Dh\right]\right\}\Phi \tag{3.113}$$

Let us now have a closer look at the curly bracket. The first term in that bracket is

$${}^{\varphi}\nabla^\alpha\, {}^{\varphi}\nabla_\alpha = \varphi^{\alpha\beta}\, {}^{\varphi}\nabla_\alpha\, {}^{\varphi}\nabla_\beta \tag{3.114}$$

For the second term in the squared bracket of equation (3.113), we can proceed in two different ways. We can either use the contravariant electromagnetic vector potential A^α and the split (3.94) such that

$$\Phi^{2n} A^2 = \Phi^{2n} g_{\alpha\beta} A^\alpha A^\beta = \Phi^{4n} \varphi_{\alpha\beta} A^\alpha A^\beta$$

However, this does not fit to expression (3.114) because of the presence of the scalar gravitation density Φ. Therefore, we decide for the covariant electromagnetic vector potential A_α and the split (3.97):

$$\Phi^{2n} A^2 = \Phi^{2n} g^{\alpha\beta} A_\alpha A_\beta = \varphi^{\alpha\beta} A_\alpha A_\beta \tag{3.115}$$

For the first term in the squared bracket of equation (3.113), we then use relations (3.99) and (3.108) such that

$$\nabla_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma^\gamma_{\beta\alpha} A_\gamma \overset{D\neq 2}{=} \partial_\alpha A_\beta - \left[{}^{\varphi}\Gamma^\gamma_{\beta\alpha} + \frac{2}{\Phi\,(D-2)}\left(\delta^\gamma_\beta \partial_\alpha + \delta^\gamma_\alpha \partial_\beta - \varphi^{\gamma\delta}\varphi_{\beta\alpha}\partial_\delta\right)\Phi\right] A_\gamma$$

and thus with the split (3.97)

$$\Phi^{2n}\nabla_\alpha A^\alpha \overset{D\neq 2}{=} \varphi^{\alpha\beta}\left(\partial_\alpha A_\beta - {}^{\varphi}\Gamma^\gamma_{\beta\alpha} A_\gamma + \frac{2}{\Phi}\partial_\alpha \Phi A_\beta\right)$$

To simplify this outcome further, we assume that the covariant electromagnetic vector potential A_α is a vector for the covariant derivative ${}^\varphi\nabla_\alpha$. Note that this is then not the case for the contravariant one A^α, because

$$A^\alpha = g^{\alpha\beta} A_\beta = \Phi^{-2n} \varphi^{\alpha\beta} A_\beta$$

That way, we arrive at

$$\Phi^{2n} \nabla_\alpha A^\alpha \overset{D\neq 2}{=} \varphi^{\alpha\beta} \left({}^\varphi\nabla_\alpha A_\beta + \frac{2}{\Phi} {}^\varphi\nabla_\alpha \Phi A_\beta \right) \tag{3.116}$$

Using this together with the outcomes (3.114) and (3.115) as well as the value (3.108) in equation (3.113), we finally get

$$\begin{aligned} \Omega \quad \overset{D\geqq 3}{=} \quad & -\frac{D-1}{D-2} \frac{4}{\Phi^{\frac{D+2}{D-2}}} \left\{ \varphi^{\alpha\beta} \left[{}^\varphi\nabla_\alpha + \left(\frac{D}{2} - 1 \right) A_\alpha \right] \left[{}^\varphi\nabla_\beta + \left(\frac{D}{2} - 1 \right) A_\beta \right] \right. \\ & \left. -\frac{1}{4} (D-2) \left(\frac{{}^\varphi R}{D-1} - Dh \Phi^{\frac{4}{D-2}} \right) \right\} \Phi \end{aligned} \tag{3.117}$$

Let us now apply the world equation (3.20) and use

$$\begin{aligned} {}^\varphi A_\alpha &= A_\alpha \\ {}^\varphi A^\alpha &= \varphi^{\alpha\beta} \, {}^\varphi A_\alpha \end{aligned}$$

such that

$$\boxed{\left[{}^\varphi\nabla_\alpha + \left(\frac{D}{2} - 1 \right) {}^\varphi A_\alpha \right]^2 \Phi \overset{D\geqq 3}{=} \frac{1}{4} (D-2) \left(\frac{{}^\varphi R}{D-1} - Dh \Phi^{\frac{4}{D-2}} \right) \Phi} \tag{3.118}$$

or in four dimensions

$$\left({}^\varphi\nabla_\alpha + {}^\varphi A_\alpha \right)^2 \Phi \overset{D=4}{=} \left(\frac{{}^\varphi R}{6} - 2h\Phi^2 \right) \Phi$$

For the tensorial gravitation density $\varphi_{\alpha\beta} = \eta_{\alpha\beta}$, equation (3.118) becomes

$$\left[\partial_\alpha + \left(\frac{D}{2} - 1 \right) A_\alpha \right]^2 \Phi \overset{D\geqq 3}{=} -\frac{1}{4} (D-2) \, Dh \Phi^{\frac{D+2}{D-2}}$$

and again in four dimensions

$$\boxed{\left(\partial_\alpha + A_\alpha \right)^2 \Phi \overset{D=4}{=} -2h\Phi^3} \tag{3.119}$$

This is the generalization of equation (3.109) for $D = 4$ to world theory.

3.10.3 Klein-Gordon equation

Now, we see why the metric split (3.94) is so interesting. Equation (3.119) is very similar to the Klein-Gordon equation

$$\left(\partial_\alpha + iqA_\alpha \right)^2 \psi = m^2 \psi \tag{3.120}$$

in flat spacetime (see equation (1.121) of Greiner 2000), where the reduced Planck constant is set to unity: $\hbar = 1$. A remark about the notation adopted here. So, $\hbar$ is the reduced Planck constant, which is correlated to the Planck constant h_{P} by $\hbar = h_{\mathrm{P}}/2\pi$. In literature, the Planck constant is usually written just as "h". However, we cannot do this, because that letter already stands for the Hubble field. We have to therefore be careful with our notation in this context. It is also important to bear in mind that we consider the wave function ψ in equation (3.120) as a pure classical field. The statistical features occurring in quantum physics are ignored here[8]. Anyway, let us now compare equations (3.119) and (3.120). The central discrepancy is that the wave function ψ is complex and not real like the scalar gravitation density Φ. In the following, we will resolve this difference. For that purpose, a closer look at the invariance of the two equations is required.

Let us begin with the Klein-Gordon equation (3.120). Quantum physics is invariant under gauge transformations. The behavior of the electromagnetic vector potential A_α for such transformations was already encountered in the list (2.65). Now, we additionally have the wave function ψ, which transforms according to the following summary:

$$\begin{aligned} A_\alpha &\overset{\mathrm{g}}{\rightarrow} A_\alpha + \partial_\alpha \chi \\ \psi &\overset{\mathrm{g}}{\rightarrow} e^{-iq\chi}\psi \end{aligned} \tag{3.121}$$

For completeness, we repeat that equation (3.120) is truly invariant under these replacements. We begin with

$$\left(\partial_\alpha + iqA_\alpha\right)\psi \overset{\mathrm{g}}{\rightarrow} \left(\partial_\alpha + iqA_\alpha + iq\partial_\alpha\chi\right)\left(e^{-iq\chi}\psi\right) = e^{-iq\chi}\left(\partial_\alpha + iqA_\alpha\right)\psi$$

such that obviously

$$\left(\partial_\alpha + iqA_\alpha\right)^2\psi \overset{\mathrm{g}}{\rightarrow} e^{-iq\chi}\left(\partial_\alpha + iqA_\alpha\right)^2\psi$$

This is already sufficient to see the gauge invariance of equation (3.120).

Now, let us split the wave function according to $\psi = \sqrt{\rho}e^{i\varphi}$, with $\rho, \varphi \in \mathbb{R}$ and $\rho \geq 0$. Then, we can choose the gauge transformation $\chi = \varphi/q$ such that the wave function ψ becomes a real, non-negative quantity. What about the case $q = 0$, where a division by zero occurs? Here, we replace the vanishing charge q by a very small charge $\Delta q \neq 0$. The replacement Δq has to be chosen so small that it does not produce a significant and thus observable effect in quantum physics. Due to our limited observation capabilities, this is, of course, possible. Hence, we see that the wave function ψ can in any case be made real. This allows us to split equation (3.120), which can also be written as

$$\left(\Box + iq\partial_\alpha A^\alpha + 2iqA^\alpha\partial_\alpha - q^2A^2\right)\psi = m^2\psi \tag{3.122}$$

into the real part

$$\left(\Box - q^2A^2\right)\psi = m^2\psi \tag{3.123}$$

and the imaginary one

$$\left(\partial_\alpha A^\alpha + 2A^\alpha\partial_\alpha\right)\psi = 0 \tag{3.124}$$

[8] So, we do not refer to the wave function $\psi\left(t, x^a\right)\big|_{\text{quantum}} \in \mathbb{C}$ of one-particle, spin 0 quantum mechanics. This wave function is no classical field but a statistical description of microscopic phenomena according to the Kopenhagen interpretation. However, quantum mechanics is not fundamental. It is based on quantum field theory. For spin 0, the statistical description there is given by the wave functional $\psi\left(t\right)\left[\lambda_1, \lambda_2\right] \in \mathbb{C}$, where $\lambda_{1/2} \in \mathbb{R}$ are spatial fields, i.e. $\lambda_{1/2} = \lambda_{1/2}\left(x^a\right) = \lambda_{1/2}\left(t = 0, x^a\right)$. The spatial fields originate from the classical wave function $\psi\left(t, x^a\right)\big|_{\text{classical}} \in \mathbb{C}$ underlying spin 0 quantum field theory. The relationship is simply $\psi|_{\text{classical}} = \sqrt{m}\left(\lambda_1 + i\lambda_2\right)$, where m is the mass. We refer to just this wave function. *(This comment is separated from the remaining text in a footnote and without a detailed explanation, because it concerns quantum field theory)*

Equation (3.123) is an alternative form of the Klein-Gordon equation, for which manifest gauge invariance is broken. The breaking is the result of our above gauge transformation. The second equation (3.124) is a gauge condition. It is in fact very similar to the Lorentz gauge (3.77). However, we have to be careful here with the interpretation. It is not possible to start with equation (3.120) and then impose condition (3.124). The problem is that the wave function ψ in equation (3.120) was still complex. However, the electromagnetic vector potential A_α is real such that relation (3.124) alone is not a valid gauge condition. The only working way would be to additionally demand that the wave function ψ is real. Unfortunately, there is no justification for this before our above choice $\chi = \varphi/q$. However, as soon as we use this transformation, we automatically obtain equation (3.124). So, from the viewpoint of the Klein-Gordon equation (3.120), relation (3.124) is not a reasonable gauge condition. If we had just that relation alone with an arbitrary real function ψ, it would be a valid gauge condition.

We proceed with equation (3.119). According to our earlier findings (2.70), (3.3), (3.22), (3.94), (3.95) and (3.108), we have an invariance under conformal gauge transformations, i.e.

$$\begin{aligned} A_\alpha &\stackrel{\text{cg}}{\rightarrow} A_\alpha + \partial_\alpha \chi \\ \Phi &\stackrel{\text{cg}}{\rightarrow} e^{-\chi}\Phi \\ h &\stackrel{\text{cg}}{\rightarrow} e^{2\chi} h \end{aligned} \tag{3.125}$$

in four dimensions. We will limit ourselves to this number of dimensions in the rest of Section 3.10.3, without denoting this explicitly. Let us also explicitly demonstrate that equation (3.119) is invariant under the above replacements. For that purpose, we compute

$$\left(\partial_\alpha + A_\alpha\right)\Phi \stackrel{\text{cg}}{\rightarrow} \left(\partial_\alpha + A_\alpha + \partial_\alpha \chi\right)\left(e^{-\chi}\Phi\right) = e^{-\chi}\left(\partial_\alpha + A_\alpha\right)\Phi$$

such that

$$\left(\partial_\alpha + A_\alpha\right)^2 \Phi \stackrel{\text{cg}}{\rightarrow} e^{-\chi}\left(\partial_\alpha + A_\alpha\right)^2 \Phi$$

As

$$h\Phi^3 \stackrel{\text{cg}}{\rightarrow} e^{-\chi} h\Phi^3$$

conformal gauge invariance is in fact given. And, now we write equation (3.119) out as

$$\left(\Box + \partial_\alpha A^\alpha + 2A^\alpha \partial_\alpha + A^2\right)\Phi = -2h\Phi^3 \tag{3.126}$$

This is very similar to relation (3.122). The only problem is that we cannot split equation (3.126) into something like the components (3.123) and (3.124), because we have just a real part. However, in contrast to the Klein-Gordon equation (3.120), the operand Φ is already real. Hence, it is now possible to impose the gauge condition

$$\left(\partial_\alpha A^\alpha + 2A^\alpha \partial_\alpha\right)\Phi = 0 \tag{3.127}$$

such that

$$\left(\Box + A^2\right)\Phi = -2h\Phi^3 \tag{3.128}$$

For the Klein-Gordon equation (3.120), we remember that condition (3.124) was a consequence of the gauge transformation $\chi = \varphi/q$, which made the wave function ψ real.

In contrast, from the perspective of world theory, gauge condition (3.127) appears arbitrary at first glance. However, according to equation (3.116), we can rewrite the gauge condition to the covariant Lorentz gauge

$$\nabla_\alpha A^\alpha = 0 \tag{3.129}$$

That way, we have a natural gauge condition such that it would be strange to get it without at least some relationship between equations (3.119) and (3.120). This brings us back to equation (3.128). The only remaining step to arrive at the form (3.123) is to employ

$$\boxed{m^2 = -2h\Phi^2 - \left(1 + q^2\right) A^2} \tag{3.130}$$

and write

$$\left(\Box - q^2 A^2\right) \Phi = m^2 \Phi \tag{3.131}$$

All in all, this means the following. If the Hubble field h is chosen such that condition (3.130) is obeyed or due to relation (3.22)

$$\lambda = -\frac{3}{2\Phi^2} \left[m^2 + \left(1 + q^2\right) A^2\right] \tag{3.132}$$

then equation (3.119) is the spin 0 Klein-Gordon equation. Whether such a choice is reasonable cannot be answered in this paper. However, one should anyway expect some still unknown mechanism behind the mass square m^2 in the Klein-Gordon equation. Somehow, the masses of the elementary particles have to come into being. The only matter of moment we can and have to address here is the sign $\lambda < 0$ demanded by the form (3.132). This appears to contradict the cosmological average (3.64), which is positive. We cannot get a positive average if the matter field λ is negative everywhere. The solution to this issue is the following one. The Klein-Gordon equation is observed only in microscopical experiments performed on earth and thus in the vicinity of matter. Hence, one can conclude that the matter field takes negative values when matter is present and positive ones everywhere else to imply the positive average. This together with the fact that the mass m, the charge q and the electromagnetic vector potential A_α enter relation (3.132) is the final justification to call the quantity λ matter field. And, by comparing now equations (3.123) and (3.131), we arrive at the following important proposal:

Geometric interpretation of wave function:

The real part of the wave function is proportional to the scalar gravitation density: $\mathrm{Re}\,\psi = \mathrm{const}\,\Phi$. (3.133)

Of course, this is valid only for the spin 0 wave function. It is not clear here how the other wave functions for spin $\frac{1}{2}$ and so forth have to be interpreted. However, we have now at least a bridge head into microphysics and also to the most important case, namely spin 0. Unfortunately, we cannot answer the functioning of the statistical nature of quantum physics in this paper. The scope of the paper is classical physics, and we have now already gone beyond it a little bit.

It is at least interesting to see that in quantum physics the gauge invariance appears to be implemented in a distorted way due to the imaginary number i in replacement (3.121). This results in the presence of complex numbers and the phase φ. In world theory in contrast, we have the replacement (3.125), which comes from the conformal constituent of conformal gauge invariance. This results, for instance, in a different handling of squares in both cases. For the wave function ψ, the square ψ^2 is not a typical quantum mechanical expression, because it is complex. Therefore, $\psi^\star\psi \in \mathbb{R}$ is utilized instead, where the asterisk denotes the complex conjugate. In world theory, the corresponding expression is Φ^2. However, it is not sufficient to just write down this expression. We have to additionally apply the Lorentz gauge (3.129). Then, we can identify $\psi^\star\psi = \text{const}\,\Phi^2$. So, the Lorentz gauge fulfills the task of the asterisk.[9]

Despite all the above considerations, the suggestion (3.133) may still appear somewhat arbitrary. We have employed the rather exotic quantity $\varphi_{\alpha\beta}$ and set it equal to the Minkowski metric $\eta_{\alpha\beta}$ to get to equation (3.119). Moreover, this equation comes exclusively from the world scalar Ω, see relation (3.117), and does not take the full world tensor $\Omega_{\alpha\beta}$ into account. The limitation to the world scalar Ω can be reproduced easily. We have a scalar Φ and need a scalar field equation. Hence, not all the components of the world tensor $\Omega_{\alpha\beta}$ can be used, and the trace Ω is a natural choice. However, the tensorial gravitation density $\varphi_{\alpha\beta}$ requires a deeper justification. For that purpose, we first

[9]It is even possible to provide a relationship between equation (3.119) and the Klein-Gordon equation (3.120) without adopting a gauge condition. For that purpose, we have to make the real equation (3.119) complex by attaching the also real material field equation (3.11). How does that work? Setting $D = 4$ and $\varphi_{\alpha\beta} = \eta_{\alpha\beta}$ like in equation (3.119), relations (3.108) and (3.94) tell us $g_{\alpha\beta} = \Phi^2\eta_{\alpha\beta}$. It is then a straightforward task to rewrite the material field equation (3.11) to

$$\partial_\alpha \left[\Phi^2 \left(\partial^\alpha - 2A^\alpha\right)\lambda\right] = 0 \tag{3.134}$$

So, the $h = \lambda/3$ equality due to relation (3.22) shows us that equations (3.119) and (3.134) constitute field equations for the fields Φ, λ.

The next step is to find a relationship between the fields $\Phi, \lambda \in \mathbb{R}$ and the wave function $\psi \in \mathbb{C}$. For that purpose, we do not just consider the Klein-Gordon equation (3.120) but also the Lagrangian $L = \sigma[|(\partial_\alpha + iqA_\alpha)\,\psi|^2 + m^2\,|\psi|^2]/(2m) - F_{\alpha\beta}F^{\alpha\beta}/4$ it comes from, where $\sigma = \pm 1$ is a sign explained further below and $m > 0$. Note that the metric of the Lagrangian is not $g_{\alpha\beta} = \Phi^2\eta_{\alpha\beta}$ but simply the Minkowski metric $\eta_{\alpha\beta}$, because we do not compare equation (3.119) with the Klein-Gordon equation in curved but in flat spacetime. Anyway, a variation of the Lagrangian with respect to A_α yields $\partial^\beta F_{\beta\alpha} = \sigma q[i\,(\psi^\star\partial_\alpha\psi - \psi\partial_\alpha\psi^\star)\,/2 - qA_\alpha|\psi|^2]/m$, where the asterisk denotes the complex conjugate. Applying $\psi = \sqrt{\rho}e^{i\varphi}$, we can also write $\partial^\beta F_{\beta\alpha} = -\sigma q\rho\,(\partial_\alpha\varphi + qA_\alpha)\,/m$. On the other hand, using $g_{\alpha\beta} = \Phi^2\eta_{\alpha\beta}$ in the electromagnetic field equation (3.8) immediately implies

$$\partial^\beta F_{\beta\alpha} = \Phi^2\left(\partial_\alpha - 2A_\alpha\right)\lambda$$

A comparison of the two $\partial^\beta F_{\beta\alpha}$-results leads to $\Phi^2\partial_\alpha\lambda = -\sigma q\rho\partial_\alpha\varphi/m$ and $2\Phi^2\lambda = \sigma q^2\rho/m$ or

$$\rho = \frac{2\sigma m}{q^2}\Phi^2\lambda, \qquad \varphi = -\frac{q}{2}\ln|\lambda| \tag{3.135}$$

It is now clear why we need the sign σ. As $\rho, m \geq 0$, we have to choose $\sigma\lambda \geq 0$. Depending on the choice of the sign σ, this implies that everywhere either $\lambda \geq 0$ or $\lambda \leq 0$. Yet, $\lambda \gtreqless 0$ is not possible.

Now, if we solve the relationship (3.135) for Φ, λ and insert the result in equations (3.119) and (3.134), we get, after some computations that are not shown here,

$$(\partial_\alpha + iqA_\alpha)^2\psi = m\,(\psi, A_\alpha)^2\,\psi, \qquad m\,(\psi, A_\alpha)^2 = -\left(1 + \frac{1}{q^2}\right)(\partial_\alpha\varphi + qA_\alpha)^2 - \frac{\sigma q^2\rho}{3m}$$

This is a classical, nonlinear Klein-Gordon-like equation with a strange mass term. The mass term by the way enforces $\sigma = -1$ and if we insert the ρ-value (3.135) in it the constraint $\lambda < 0$ everywhere, see also the text below equation (3.132). *(This comment is separated from the remaining text in a footnote and without intermediate mathematical steps, because it concerns microphysics)* look at Einstein-Maxwell theory. There, flat spacetime is given by the case $g_{\alpha\beta} = \eta_{\alpha\beta}$. The important point is now to recall from general relativity that this is a special way to express flat spacetime. It is based on Cartesian coordinates. The strict condition underlying flat spacetime is the vanishing of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta} = 0$. This condition is obeyed for $g_{\alpha\beta} = \eta_{\alpha\beta}$, but there are lots of alternatives $g'_{\alpha\beta} \neq \eta_{\alpha\beta}$, one for each choice of non-Cartesian coordinates. The difference between the tensors $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is a coordinate transformation. Now, back to world theory. There, the metric $g_{\alpha\beta}$ is not only affected by coordinate transformations as in Einstein-Maxwell theory, but also by conformal gauge transformations according to the replacement (2.70). This leads to four possibilities how to describe flat spacetime. The easiest possibility is to use $g_{\alpha\beta} = \eta_{\alpha\beta}$, based on Cartesian coordinates. Applying an arbitrary coordinate transformation on the metric $g_{\alpha\beta}$, we arrive at the second possibility, the tensors $g'_{\alpha\beta}$. The third option is to employ an arbitrary conformal gauge transformation on the Minkowski metric $\eta_{\alpha\beta}$ such that we get $\overset{\bullet}{g}_{\alpha\beta} = e^{-2\chi}\eta_{\alpha\beta}$. And, the final case is the combination of an arbitrary coordinate and conformal gauge transformation, i.e. $\overset{\bullet}{g}{}'_{\alpha\beta}$. Looking then at the usual form of the Klein-Gordon equation in flat spacetime (3.120) found in literature, we see that there Cartesian coordinates are used but no gauge condition is employed. Hence, it is natural to perform a comparison with the case $\overset{\bullet}{g}_{ab}$. According to the split (3.94), this means $\varphi_{\alpha\beta} = \eta_{\alpha\beta}$. That way, we see that equation (3.119) is the natural candidate for a comparison with the Klein-Gordon equation (3.120).

3.11 Relations[10]

3.11.1 Commutator of electrogravitational derivative

In the last sections, we have studied the form of the world equation in various special cases. The rest of this chapter is devoted to examine how several relations known from general relativity generalize to world theory. We have already encountered one such case. Our demand (3.15) generalizes the antisymmetry

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv -R^{\alpha}{}_{\beta\delta\gamma} \tag{3.136}$$

of the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$. Identity (2.12) tells us that there is also an antisymmetry for the first two indices. In the following, we will first repeat the proof of this identity, seeing that it also holds for D dimensions. Afterwards, we will generalize it to world theory.

For the proof, we have to just look at relation (2.111). Setting $Y_{\alpha\beta} = g_{\alpha\beta}$ there, we obtain

$$[\nabla_\gamma, \nabla_\delta]\, g_{\alpha\beta} = -R_{\beta\alpha\gamma\delta} - R_{\alpha\beta\gamma\delta} \tag{3.137}$$

[10]Conformal geometrodynamics vs Riemann-Cartan space: Let us assume in this footnote that we do not have deviation (2.45) but the general deviation allowed by footnote 6, i.e. $L^{\alpha}_{\beta\gamma} = a_2\delta^{\alpha}_{\beta}A_\gamma + \delta^{\alpha}_{\gamma}A_\beta - A^{\alpha}g_{\beta\gamma}$, where the constant $a_2 \in \mathbb{R}$ is arbitrary. It can then be shown via computations similar to those in Section 3.11 that the general counterparts of several relations there are easiest for respectively certain values of the constant a_2. The general version of relation (3.143) is easiest for the choice $a_2 = 1$, which speaks in favor of conformal geometrodynamics. However, the general variants of relations (3.140) and (3.146) are shortest for $a_2 = 0$. Due to the 3rd last paragraph of Section 3.4.2, this means a Riemann-Cartan space. The general counterparts of relations (3.150), (3.159) and (3.166) do not prefer any value of a_2. All in all, we thus see that the geometric relations studied in Section 3.11 favor a Riemann-Cartan space and not conformal geometrodynamics. This supports the statement of the 3rd last paragraph of Section 3.4.2 that the choice $a_2 = 1$ of conformal geometrodynamics is not fundamentally justified. *(This comment is separated from the remaining text in a footnote, because it does not contain intermediate mathematical steps)*

As the covariant derivative of the metric vanishes due to equation (1.9), we immediately get identity (2.12).

To extend general relativistic relations to world theory, we merely have to repeat the derivations known from general relativity with the respective quantities of world theory. For identity (2.12), we therefore first generalize relation (2.111). To this end, we replace the covariant derivatives appearing on the left hand side of that equation with electrogravitational ones. To evaluate the resulting expression, we use definition (2.27) such that

$$\Delta_{[\gamma}\Delta_{\delta]}Y_{\alpha\beta} = \nabla_{[\gamma}\Delta_{\delta]}Y_{\alpha\beta} - L^{\epsilon}_{[\delta\gamma]}\Delta_{\epsilon}Y_{\alpha\beta} - L^{\epsilon}_{\alpha[\gamma}\Delta_{\delta]}Y_{\epsilon\beta} - L^{\epsilon}_{\beta[\gamma}\Delta_{\delta]}Y_{\alpha\epsilon}$$

The expression $L^{\epsilon}_{[\delta\gamma]}$ can be replaced by the torsion tensor due to the first equation in box (2.29) such that we can continue with

$$\begin{aligned}\Delta_{[\gamma}\Delta_{\delta]}Y_{\alpha\beta} \;=\; & \nabla_{[\gamma}\left(\nabla_{\delta]}Y_{\alpha\beta} - L^{\epsilon}_{\alpha\delta]}Y_{\epsilon\beta} - L^{\epsilon}_{\beta\delta]}Y_{\alpha\epsilon}\right) - \frac{1}{2}T^{\epsilon}_{\delta\gamma}\Delta_{\epsilon}Y_{\alpha\beta} \\ & -L^{\epsilon}_{\alpha[\gamma}\left(\nabla_{\delta]}Y_{\epsilon\beta} - L^{\zeta}_{\epsilon\delta]}Y_{\zeta\beta} - L^{\zeta}_{\beta\delta]}Y_{\epsilon\zeta}\right) - L^{\epsilon}_{\beta[\gamma}\left(\nabla_{\delta]}Y_{\alpha\epsilon} - L^{\zeta}_{\alpha\delta]}Y_{\zeta\epsilon} - L^{\zeta}_{\epsilon\delta]}Y_{\alpha\zeta}\right)\end{aligned}$$

As all those single covariant derivatives vanish which are applied on the quantity Y, except the one with the torsion tensor $T^{\epsilon}_{\delta\gamma}$, we put the respective term on the left hand side such that we have an operator like in equation (2.32). In addition to that, we apply relation (2.111):

$$\begin{aligned}\left(\Delta_{[\gamma}\Delta_{\delta]} - \frac{1}{2}T^{\epsilon}_{\gamma\delta}\Delta_{\epsilon}\right)Y_{\alpha\beta} \;=\; & -\left(\frac{1}{2}R^{\zeta}{}_{\alpha\gamma\delta} + \nabla_{[\gamma}L^{\zeta}_{\alpha\delta]} + L^{\zeta}_{\epsilon[\gamma}L^{\epsilon}_{\alpha\delta]}\right)Y_{\zeta\beta} \\ & -\left(\frac{1}{2}R^{\zeta}{}_{\beta\gamma\delta} + \nabla_{[\gamma}L^{\zeta}_{\beta\delta]} + L^{\zeta}_{\epsilon[\gamma}L^{\epsilon}_{\beta\delta]}\right)Y_{\alpha\zeta}\end{aligned}$$

Note that we have used the antisymmetry of the torsion tensor in the indices γ and δ for that step. Using then equation (2.33), we find

$$\boxed{\left([\Delta_{\gamma}, \Delta_{\delta}] - T^{\epsilon}_{\gamma\delta}\Delta_{\epsilon}\right)Y_{\alpha\beta} = -Z^{\epsilon}{}_{\alpha\gamma\delta}Y_{\epsilon\beta} - Z^{\epsilon}{}_{\beta\gamma\delta}Y_{\alpha\epsilon}} \tag{3.138}$$

3.11.2 Commutator of world derivative

Let us for a moment stay at the outcome of the last section. One question is what happens if we had used world derivatives instead of electrogravitational ones. In that case, we have to use definition (2.83) and evaluate

$$\delta_{[\gamma}\delta_{\delta]}Y_{\alpha\beta} = \nabla_{[\gamma}\delta_{\delta]}Y_{\alpha\beta} - l^{\epsilon}_{[\delta\gamma]}\delta_{\epsilon}Y_{\alpha\beta} - l^{\epsilon}_{\alpha[\gamma}\delta_{\delta]}Y_{\epsilon\beta} - l^{\epsilon}_{\beta[\gamma}\delta_{\delta]}Y_{\alpha\epsilon} + w_Y A_{[\gamma}\delta_{\delta]}Y_{\alpha\beta}$$

where w_Y is the gauge weight of the tensor $Y_{\alpha\beta}$. The second term on the right hand side vanishes, because the world deviation (2.78) is symmetric in its lower two indices. We can therefore continue with

$$\begin{aligned}\delta_{[\gamma}\delta_{\delta]}Y_{\alpha\beta} \;=\; & \left(\nabla_{[\gamma} + w_Y A_{[\gamma}\right)\left(\nabla_{\delta]}Y_{\alpha\beta} - l^{\epsilon}_{\alpha\delta]}Y_{\epsilon\beta} - l^{\epsilon}_{\beta\delta]}Y_{\alpha\epsilon} + w_Y A_{\delta]}Y_{\alpha\beta}\right) \\ & -l^{\epsilon}_{\alpha[\gamma}\left(\nabla_{\delta]}Y_{\epsilon\beta} - l^{\zeta}_{\epsilon\delta]}Y_{\zeta\beta} - l^{\zeta}_{\beta\delta]}Y_{\epsilon\zeta} + w_Y A_{\delta]}Y_{\epsilon\beta}\right) \\ & -l^{\epsilon}_{\beta[\gamma}\left(\nabla_{\delta]}Y_{\alpha\epsilon} - l^{\zeta}_{\alpha\delta]}Y_{\zeta\epsilon} - l^{\zeta}_{\epsilon\delta]}Y_{\alpha\zeta} + w_Y A_{\delta]}Y_{\alpha\epsilon}\right)\end{aligned}$$

All single covariant derivatives applied on the quantity Y disappear such that we get

$$\delta_{[\gamma}\delta_{\delta]}Y_{\alpha\beta} = \nabla_{[\gamma}\nabla_{\delta]}Y_{\alpha\beta} - \left(\nabla_{[\gamma}l^{\zeta}_{\alpha\delta]} + l^{\zeta}_{\epsilon[\gamma}l^{\epsilon}_{\alpha\delta]}\right)Y_{\zeta\beta} - \left(\nabla_{[\gamma}l^{\zeta}_{\beta\delta]} + l^{\zeta}_{\epsilon[\gamma}l^{\epsilon}_{\beta\delta]}\right)Y_{\alpha\zeta} + w_Y\nabla_{[\gamma}A_{\delta]}Y_{\alpha\beta}$$

Then, we use relation (2.111) again as well as the electromagnetic field strength (1.1) such that

$$\begin{aligned}\left[\delta_{[\gamma}\delta_{\delta]} + \frac{1}{2}\left(D - w_Y\right) F_{\gamma\delta}\right] Y_{\alpha\beta} \;=\; & -\left(\frac{1}{2} R^{\zeta}{}_{\alpha\gamma\delta} + \nabla_{[\gamma} l^{\zeta}_{\alpha\delta]} + l^{\zeta}_{\epsilon[\gamma} l^{\epsilon}_{\alpha\delta]} - \frac{D}{4}\delta^{\zeta}_{\alpha} F_{\gamma\delta}\right) Y_{\zeta\beta} \\ & -\left(\frac{1}{2} R^{\zeta}{}_{\beta\gamma\delta} + \nabla_{[\gamma} l^{\zeta}_{\beta\delta]} + l^{\zeta}_{\epsilon[\gamma} l^{\epsilon}_{\beta\delta]} - \frac{D}{4}\delta^{\zeta}_{\beta} F_{\gamma\delta}\right) Y_{\alpha\zeta}\end{aligned}$$

Eventually, we see that equation (2.92) allows us to write

$$\boxed{\left[\left[\delta_\gamma, \delta_\delta\right] + \left(D - w_Y\right) F_{\gamma\delta}\right] Y_{\alpha\beta} = -Z^{\epsilon}{}_{\alpha\gamma\delta} Y_{\epsilon\beta} - Z^{\epsilon}{}_{\beta\gamma\delta} Y_{\alpha\epsilon}} \tag{3.139}$$

This is the analog of relation (3.138) in the world view. For $w_Y = D$, the left hand side above would simplify significantly. Then, it would look like the right hand side of definition (2.91), except that the operand has two indices here and up to the sign. However, further below, we will have to choose $w_Y \neq D$ such that we keep relation (3.139) the way it is.

Another question is, what happens if we use the world curvature $\Omega^{\alpha}{}_{\beta\gamma\delta}$ instead of the electrogravitational one $Z^{\alpha}{}_{\beta\gamma\delta}$? For that purpose, we adopt equation (3.16) and evaluate

$$Z^{\epsilon}{}_{\alpha\gamma\delta} Y_{\epsilon\beta} = \left(\Omega^{\epsilon}{}_{\alpha\gamma\delta} + 2\delta^{\epsilon}_{[\gamma} g_{\alpha\delta]} h\right) Y_{\epsilon\beta} = \Omega^{\epsilon}{}_{\alpha\gamma\delta} Y_{\epsilon\beta} + 2 g_{\alpha[\delta} h Y_{\gamma]\beta}$$

This clearly shows us that then relation (3.139) would become more complicated, and we therefore do not change it.

3.11.3 First two indices of world curvature

Having found relations (3.138) and (3.139), we can now pick $Y_{\alpha\beta} = g_{\alpha\beta}$ like in equation (3.137). The fastest way to arrive at a generalization of identity (2.12) is to decide for relation (3.139). Recalling then the gauge weight 2 of the metric $g_{\alpha\beta}$ from Section 2.5.3, we obtain

$$\left[\left[\delta_\gamma, \delta_\delta\right] + \left(D - 2\right) F_{\gamma\delta}\right] g_{\alpha\beta} = -Z_{\beta\alpha\gamma\delta} - Z_{\alpha\beta\gamma\delta}$$

The world nonmetricity appearing on the left hand side vanishes due to box (2.85) such that

$$Z_{\alpha\beta\gamma\delta} \equiv -Z_{\beta\alpha\gamma\delta} - \left(D - 2\right) g_{\alpha\beta} F_{\gamma\delta} \tag{3.140}$$

So, in contrast to the Riemann tensor $R_{\alpha\beta\gamma\delta}$, the electrogravitational curvature $Z_{\alpha\beta\gamma\delta}$ is not antisymmetric in its first two indices.

The final step is now to take identity (3.140) and replace the electrogravitational curvature $Z_{\alpha\beta\gamma\delta}$ with the world curvature $\Omega_{\alpha\beta\gamma\delta}$. For that purpose, we use definition (3.16) such that

$$Z_{\alpha\beta\gamma\delta} = \Omega_{\alpha\beta\gamma\delta} + \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right) h \tag{3.141}$$

Placing this in identity (3.140) eventually gives

$$\boxed{\Omega_{\alpha\beta\gamma\delta} \equiv -\Omega_{\beta\alpha\gamma\delta} - \left(D - 2\right) g_{\alpha\beta} F_{\gamma\delta}} \tag{3.142}$$

3.11.4 First world identity

Another identity known from general relativity is the first Bianchi identity (2.3). We have not only used this identity already, the text below its definition (2.3) even gives a quick proof. To generalize the first Bianchi identity to electrogravitation, we have to introduce the total antisymmetrization of multiple indices. We have already encountered the special case with two indices. Looking, for instance, at the tensor $Y_{\alpha\beta}$, the index antisymmetrization is $Y_{[\alpha\beta]}$. To address more than two indices, one way used in literature is to define the squared bracket in such a manner that all indices enclosed by it are affected by the total antisymmetrization. Unfortunately, this has drawbacks for many mathematical computations. It would not be possible to have indices inside of the squared bracket that are not affected by the total antisymmetrization. We would therefore have to shift these indices outside of the squared bracket, for example, by changing the order of factors in an expression. Yet, such a procedure cannot be performed in all cases such that it is better to explicitly mark the indices affected by the total antisymmetrization. For that purpose, we use the symbol $\mathcal{A}$ and highlight the relevant indices with subscripts. The index antisymmetrization mentioned above is then $Y_{[\alpha\beta]} = \mathcal{A}_{\alpha\beta} Y_{\alpha\beta}$. Having more than two indices, the total antisymmetrization is the sum of all even permutations minus the sum of all odd ones. In addition to that, a normalization factor is used, which is $1/N!$ for N indices.

That way, we can write the first Bianchi identity (2.3) in a shorter manner. Knowing that the Riemann tensor ${R^\alpha}_{\beta\gamma\delta}$ is antisymmetric in its last two indices due to relation (3.136), we first write

$$ {R^\alpha}_{\beta\gamma\delta} - {R^\alpha}_{\beta\delta\gamma} + {R^\alpha}_{\gamma\delta\beta} - {R^\alpha}_{\gamma\beta\delta} + {R^\alpha}_{\delta\beta\gamma} - {R^\alpha}_{\delta\gamma\beta} \equiv 0 $$

Then, we have the three even and three odd permutations required for a total antisymmetrization, and we arrive at the compact form

$$ \mathcal{A}_{\beta\gamma\delta} {R^\alpha}_{\beta\gamma\delta} \equiv 0 $$

Having introduced the total antisymmetrization $\mathcal{A}$, we can now go to relation (2.66) and write

$$ \mathcal{A}_{\beta\gamma\delta} {Z^\alpha}_{\beta\gamma\delta} = 2\mathcal{A}_{\beta\gamma\delta} \left(\partial_\gamma I^\alpha_{\beta\delta} + I^\alpha_{\epsilon\gamma} I^\epsilon_{\beta\delta} \right) = 2\mathcal{A}_{\beta\gamma\delta} \left(\partial_\gamma I^\alpha_{[\beta\delta]} + I^\alpha_{\epsilon\gamma} I^\epsilon_{[\beta\delta]} \right) $$

So, for indices already affected by a total antisymmetrization of three indices, we can leave the index antisymmetrization with a squared bracket away or apply it however we want. Using then the torsion tensor in box (2.28) and its value (2.48), we arrive at

$$ \mathcal{A}_{\beta\gamma\delta} {Z^\alpha}_{\beta\gamma\delta} = \mathcal{A}_{\beta\gamma\delta} \left(\partial_\gamma T^\alpha_{\beta\delta} + I^\alpha_{\epsilon\gamma} T^\epsilon_{\beta\delta} \right) = D\mathcal{A}_{\beta\gamma\delta} \left(\delta^\alpha_{[\delta} \partial_\gamma + I^\alpha_{[\delta\gamma} \right) A_{\beta]} $$

The second term in the last round bracket vanishes, because we can reorder the index antisymmetrization with the squared bracket again and then apply boxes (2.28) and (2.48):

$$ \mathcal{A}_{\beta\gamma\delta} I^\alpha_{[\delta\gamma} A_{\beta]} = \mathcal{A}_{\beta\gamma\delta} I^\alpha_{[\delta\gamma]} A_\beta = \frac{1}{2} \mathcal{A}_{\beta\gamma\delta} T^\alpha_{\delta\gamma} A_\beta = \frac{D}{2} \mathcal{A}_{\beta\gamma\delta} \delta^\alpha_{[\gamma} A_{\delta]} A_\beta = \frac{D}{2} \mathcal{A}_{\beta\gamma\delta} \delta^\alpha_\gamma A_\beta A_\delta = 0 $$

That way, we find

$$ \mathcal{A}_{\beta\gamma\delta} {Z^\alpha}_{\beta\gamma\delta} = D\mathcal{A}_{\beta\gamma\delta} \delta^\alpha_{[\delta} \partial_\gamma A_{\beta]} = D\mathcal{A}_{\beta\gamma\delta} \delta^\alpha_\delta \partial_{[\gamma} A_{\beta]} $$

where we have reordered the squared bracket once again. We can then use definition (1.1) and eventually arrive at the **first electrogravitational identity**

$$\boxed{\mathcal{A}_{\beta\gamma\delta}\left(Z^{\alpha}{}_{\beta\gamma\delta}+\frac{D}{2}\delta^{\alpha}_{\beta}F_{\gamma\delta}\right)\equiv 0} \tag{3.143}$$

Eventually, we generalize the above result to world theory. For that purpose, we apply relation (3.141) such that we get the **first world identity**

$$\boxed{\mathcal{A}_{\beta\gamma\delta}\left(\Omega^{\alpha}{}_{\beta\gamma\delta}+\frac{D}{2}\delta^{\alpha}_{\beta}F_{\gamma\delta}\right)\equiv 0} \tag{3.144}$$

3.11.5 Index pairs of world curvature

The first world identity (3.144) allows us to generalize another identity which changes the indices of the Riemann tensor, namely

$$R_{\alpha\beta\gamma\delta}\equiv R_{\gamma\delta\alpha\beta} \tag{3.145}$$

To achieve this, we first write identity (3.144) as

$$\Omega_{\alpha\beta\gamma\delta}=-\Omega_{\alpha\gamma\delta\beta}-\Omega_{\alpha\delta\beta\gamma}-\frac{D}{2}\left(g_{\alpha\beta}F_{\gamma\delta}+g_{\alpha\gamma}F_{\delta\beta}+g_{\alpha\delta}F_{\beta\gamma}\right)$$

where we have utilized the antisymmetry (3.15) and the one of the electromagnetic field strength $F_{\alpha\beta}$. Then,

$$2\Omega_{[\alpha\beta]\gamma\delta}=-\Omega_{\alpha\gamma\delta\beta}-\Omega_{\alpha\delta\beta\gamma}+\Omega_{\beta\gamma\delta\alpha}+\Omega_{\beta\delta\alpha\gamma}-\frac{D}{2}\left(g_{\alpha\gamma}F_{\delta\beta}+g_{\alpha\delta}F_{\beta\gamma}-g_{\beta\gamma}F_{\delta\alpha}-g_{\beta\delta}F_{\alpha\gamma}\right)$$

and

$$2\Omega_{[\delta\gamma]\alpha\beta}=-\Omega_{\delta\alpha\beta\gamma}-\Omega_{\delta\beta\gamma\alpha}+\Omega_{\gamma\alpha\beta\delta}+\Omega_{\gamma\beta\delta\alpha}-\frac{D}{2}\left(g_{\delta\alpha}F_{\beta\gamma}+g_{\delta\beta}F_{\gamma\alpha}-g_{\gamma\alpha}F_{\beta\delta}-g_{\gamma\beta}F_{\delta\alpha}\right)$$

Let us now look at the right hand sides of the above two equations. There is a minus in front of the respective first two world curvatures. Each of these signs can be changed to a plus by applying identity (3.15). That way, we get

$$\begin{aligned}\Omega_{[\alpha\beta]\gamma\delta}+\Omega_{[\delta\gamma]\alpha\beta} &= \Omega_{(\alpha\gamma)\beta\delta}+\Omega_{(\alpha\delta)\gamma\beta}+\Omega_{(\beta\gamma)\delta\alpha}+\Omega_{(\beta\delta)\alpha\gamma}\\ &\quad +\frac{D}{2}\left(g_{\alpha\gamma}F_{\beta\delta}+g_{\alpha\delta}F_{\gamma\beta}+g_{\beta\gamma}F_{\delta\alpha}+g_{\beta\delta}F_{\alpha\gamma}\right)\end{aligned}$$

Next, we use identity (3.142), which tells us

$$\Omega_{(\alpha\beta)\gamma\delta}=-\left(\frac{D}{2}-1\right)g_{\alpha\beta}F_{\gamma\delta}$$

and

$$\Omega_{[\alpha\beta]\gamma\delta}=\Omega_{\alpha\beta\gamma\delta}+\left(\frac{D}{2}-1\right)g_{\alpha\beta}F_{\gamma\delta}$$

Hence, we finally arrive at the identity

$$\boxed{\Omega_{\alpha\beta\gamma\delta}\equiv\Omega_{\gamma\delta\alpha\beta}+g_{\alpha\gamma}F_{\beta\delta}+g_{\alpha\delta}F_{\gamma\beta}+g_{\beta\gamma}F_{\delta\alpha}+g_{\beta\delta}F_{\alpha\gamma}+\left(\frac{D}{2}-1\right)\left(g_{\gamma\delta}F_{\alpha\beta}-g_{\alpha\beta}F_{\gamma\delta}\right)} \tag{3.146}$$

Note that we just have to replace the world curvatures with electrogravitational ones to get the analog in electrogravitation. In addition to that, the above outcome also proves the validity of identity (3.145) for an arbitrary number of dimensions, which is contained as a special case.

3.11.6 Symmetry of world tensor

Knowing what happens if the two index pairs of the world curvature $\Omega_{\alpha\beta\gamma\delta}$ are exchanged, we are now able to determine the outcome of an exchange of the two indices of the world tensor $\Omega_{\alpha\beta}$. For that purpose, we merely have to contract the indices α and γ in identity (3.146) such that definition (3.18) yields

$$\Omega_{\beta\delta} = \Omega_{\delta\beta} + DF_{\beta\delta} + F_{\delta\beta} + F_{\delta\beta} + \left(\frac{D}{2} - 1\right)(F_{\delta\beta} - F_{\beta\delta})$$

or

$$\boxed{\Omega_{\alpha\beta} \equiv \Omega_{\beta\alpha}} \tag{3.147}$$

So, the world tensor $\Omega_{\alpha\beta}$ is symmetric, just like the Ricci tensor $R_{\alpha\beta}$. Looking at the electrogravitational limit $\lambda = 0$, the world tensor in identity (3.147) becomes the electrogravitation tensor. That way, we see that the electrogravitation tensor $Z_{\alpha\beta}$ is also symmetric.

3.11.7 Second world identity

We proceed with the second Bianchi identity familiar from general relativity:

$$\mathcal{A}_{\epsilon\gamma\delta} \nabla_\epsilon R^\alpha{}_{\beta\gamma\delta} \equiv 0 \tag{3.148}$$

To generalize this identity to world theory, the first step is to use identity (3.139) and set $Y_{\alpha\beta} = \delta_\beta X_\alpha$. Note that the gauge weight of the vector X_α is $w_X = D/2$ due to the third paragraph of Section 2.5.8. As the world derivative δ_β has the gauge weight 0, we see $w_Y = D/2$, which implies

$$\left([\delta_\gamma, \delta_\delta] + \frac{D}{2} F_{\gamma\delta}\right) \delta_\beta X_\alpha = -Z^\epsilon{}_{\alpha\gamma\delta} \delta_\beta X_\epsilon - Z^\epsilon{}_{\beta\gamma\delta} \delta_\epsilon X_\alpha$$

On the other hand, definition (2.91) tells us

$$\delta_\beta [\delta_\gamma, \delta_\delta] X_\alpha = -\delta_\beta Z^\epsilon{}_{\alpha\gamma\delta} X_\epsilon - Z^\epsilon{}_{\alpha\gamma\delta} \delta_\beta X_\epsilon$$

Since

$$\mathcal{A}_{\beta\gamma\delta} [\delta_\gamma, \delta_\delta] \delta_\beta X_\alpha = \mathcal{A}_{\beta\gamma\delta} \delta_\beta [\delta_\gamma, \delta_\delta] X_\alpha \tag{3.149}$$

we then find

$$\mathcal{A}_{\beta\gamma\delta} \left(\delta_\beta Z^\epsilon{}_{\alpha\gamma\delta} X_\epsilon\right) = \mathcal{A}_{\beta\gamma\delta} \left(Z^\epsilon{}_{\beta\gamma\delta} \delta_\epsilon X_\alpha + \frac{D}{2} F_{\gamma\delta} \delta_\beta X_\alpha\right)$$

The right hand side vanishes due to the first electrogravitational identity (3.143). As the vector X_ϵ is arbitrary, we therefore get the **second electrogravitational identity**

$$\boxed{\mathcal{A}_{\epsilon\gamma\delta} \delta_\epsilon Z^\alpha{}_{\beta\gamma\delta} \equiv 0} \tag{3.150}$$

The last step is to use relation (3.16) and box (2.85), which gives the generalization of identity (3.148) to world theory, the **second world identity**

$$\boxed{\mathcal{A}_{\epsilon\gamma\delta} \left(\delta_\epsilon \Omega^\alpha{}_{\beta\gamma\delta} + 2\delta^\alpha_\gamma g_{\beta\delta} \delta_\epsilon h\right) \equiv 0} \tag{3.151}$$

Finally, let us have a closer look at the world gradient $\delta_\epsilon h$ appearing above. The gauge weight of the Hubble field h is -2 due to equations (3.3) and (3.22). Equation (2.83) then tells us that

$$\delta_\alpha h = (\nabla_\alpha - 2A_\alpha)\, h \tag{3.152}$$

3.11.8 Derivatives of metric

In the next section, we will encounter some terms of the form $g^{\alpha\beta}\delta_\gamma \cdots$ and we then have to move the metric $g^{\alpha\beta}$ inside of the operand $\cdots$ of the world derivative δ_γ. Let us therefore use a separate section here and study all possible ways to apply a derivative on the metric. From general relativity, we know

$$\nabla_\gamma g_{\alpha\beta} = \nabla_\gamma \delta^\beta_\alpha = \nabla_\gamma g^{\alpha\beta} = 0$$

such that the metric can be moved inside of the operand of the covariant derivative ∇_γ without any problems. For the electrogravitational derivative Δ_γ, this does not work. To see this, we merely have to look at the second line of box (2.48). The only vanishing outcome occurs if we use the Kronecker tensor, for which definition (2.26) tells us

$$\Delta_\gamma \delta^\beta_\alpha = \partial_\gamma \delta^\beta_\alpha - I^\delta_{\alpha\gamma}\delta^\beta_\delta + I^\beta_{\delta\gamma}\delta^\delta_\alpha$$

such that

$$\boxed{\Delta_\gamma \delta^\beta_\alpha = 0} \tag{3.153}$$

Using then relation (1.3), the Leibniz rule and box (2.48), we find

$$0 = \Delta_\gamma \left(g^{\beta\alpha} g_{\alpha\delta} \right) = \left[\Delta_\gamma g^{\beta\alpha} + (D-2)\, g^{\beta\alpha} A_\gamma \right] g_{\alpha\delta} \tag{3.154}$$

which implies

$$\boxed{\Delta_\gamma g^{\alpha\beta} = -\,(D-2)\, A_\gamma g^{\alpha\beta}} \tag{3.155}$$

Having studied the three ways the electrogravitational derivative can be applied on the metric, we proceed with the world derivative. The first way there is given in box (2.85). For the Kronecker tensor, which has the gauge weight 0, we this time use definition (2.83) such that

$$\delta_\gamma \delta^\beta_\alpha = \nabla_\gamma \delta^\beta_\alpha - l^\delta_{\alpha\gamma}\delta^\beta_\delta + l^\beta_{\delta\gamma}\delta^\delta_\alpha$$

and thus

$$\boxed{\delta_\gamma \delta^\beta_\alpha = 0} \tag{3.156}$$

Eventually, we proceed like in equation (3.154) and evaluate

$$0 = \delta_\gamma \left(g^{\beta\alpha} g_{\alpha\delta} \right) = \delta_\gamma g^{\beta\alpha} g_{\alpha\delta}$$

So, we also have a zero for

$$\boxed{\delta_\gamma g^{\alpha\beta} = 0} \tag{3.157}$$

such that the metric can be moved inside of the operand of the world derivative δ_γ however the indices of the metric are positioned.

3.11.9 First contracted world identity

The identity which will be generalized in this section is the contracted Bianchi identity (2.5). For that purpose, we start with the second electrogravitational identity (3.150) and write it out, using the antisymmetry (3.14):

$$\delta_\epsilon Z^\alpha{}_{\beta\gamma\delta} + \delta_\gamma Z^\alpha{}_{\beta\delta\epsilon} + \delta_\delta Z^\alpha{}_{\beta\epsilon\gamma} = 0$$

Then, we contract the indices α and γ such that definition (2.34) and the antisymmetry of the electrogravitational curvature $Z^{\alpha}{}_{\beta\epsilon\gamma}$ in its last two indices give

$$\delta_\epsilon Z_{\beta\delta} + \delta_\alpha Z^{\alpha}{}_{\beta\delta\epsilon} - \delta_\delta Z_{\beta\epsilon} = 0$$

To also reduce the remaining electrogravitational curvature $Z^{\alpha}{}_{\beta\delta\epsilon}$ to an electrogravitation tensor, we apply identity (3.140) such that

$$\delta_\epsilon Z_{\beta\delta} + \delta_\alpha \left[-Z_{\beta}{}^{\alpha}{}_{\delta\epsilon} - (D-2)\,\delta^{\alpha}_{\beta} F_{\delta\epsilon}\right] - \delta_\delta Z_{\beta\epsilon} = 0$$

Due to the last section, we know that the metric commutes with the world derivative. Therefore, we can multiply this equation by the metric $g^{\beta\delta}$ and move the metric inside of the operands of the world derivatives. Using the electrogravitation scalar (3.111), we then arrive at

$$\delta_\epsilon Z + \delta_\alpha \left[-Z^{\alpha}{}_{\epsilon} - (D-2)\,F^{\alpha}{}_{\epsilon}\right] - \delta_\delta Z^{\delta}{}_{\epsilon} = 0$$

So, defining the **dual electrogravitation tensor**

$$\boxed{\overline{Z}_{\alpha\beta} = Z_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} Z} \tag{3.158}$$

in analogy to the quantity (3.28), we obtain the **first contracted electrogravitational identity**

$$\boxed{\delta^\beta \overline{Z}_{\beta\alpha} \equiv \frac{1}{2}(D-2)\,\delta^\beta F_{\alpha\beta}} \tag{3.159}$$

where the index of the world derivative is risen in the same way as for other tensors, namely with the metric. The above outcome by the way proves the validity of the contracted Bianchi identity (2.5) in D dimensions.

For the generalization to world theory, we use equations (3.19) and (3.110) such that definitions (3.28) and (3.158) give

$$\boxed{\overline{\Omega}_{\alpha\beta} = \overline{Z}_{\alpha\beta} + \frac{1}{2}(D-1)(D-2)\,g_{\alpha\beta} h}$$

That way, identity (3.159) leads to the **first contracted world identity**

$$\boxed{\delta^\beta \overline{\Omega}_{\beta\alpha} \equiv \frac{1}{2}(D-2)\left[\delta^\beta F_{\alpha\beta} + (D-1)\,\delta_\alpha h\right]} \tag{3.160}$$

Let us eventually have a closer look at the right hand side of the above outcome. The electromagnetic field strength $F_{\alpha\beta}$ is conformally gauge invariant due to box (2.97) such that it has the gauge weight 0. However, as the contravariant metric $g^{\alpha\beta}$ with the gauge weight -2 is used twice to obtain the quantity $F^{\alpha\beta}$, this way of writing the electromagnetic field strength has the gauge weight -4. Then, definition (2.83) shows

$$\delta_\beta F^{\alpha\beta} = (\nabla_\beta - 4A_\beta)\,F^{\alpha\beta} + l^{\alpha}_{\gamma\beta} F^{\gamma\beta} + l^{\beta}_{\gamma\beta} F^{\alpha\gamma}$$

The second term vanishes due to the symmetry of the world connection $l^{\alpha}_{\gamma\beta}$ and the antisymmetry of the electromagnetic field strength $F^{\gamma\beta}$. For the last term, we use equation (2.81) such that

$$\delta_\beta F^{\alpha\beta} = \left[\nabla_\beta + (D-4)\,A_\beta\right] F^{\alpha\beta} \tag{3.161}$$

Using this together with relations (3.22) and (3.152) in the world identity (3.160) allows us to write

$$\delta^{\beta}\overline{\Omega}_{\alpha\beta} \equiv \frac{1}{2}(D-2)\left\{\left[\nabla^{\beta}+(D-4)A^{\beta}\right]F_{\alpha\beta}+\frac{1}{3}(D-1)(\nabla_{\alpha}-2A_{\alpha})\lambda\right\} \tag{3.162}$$

In this manner, we have encountered an alternative way to derive the electromagnetic field equation (3.7). We merely have to apply the world equation in the form (3.29). This by the way explains why all the identities of world theory studied in the last sections are not just their general relativistic counterparts with the Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ being replaced by the world tensor $\Omega^{\alpha}{}_{\beta\gamma\delta}$ and so on. Instead, additional terms appear like the right hand side in the above identity (3.162). These terms must be present, for otherwise the electromagnetic field equation would not be incorporated in the world equation. Finally, applying the world equation (3.29) on identity (3.160), we can also write the electromagnetic field equation (3.7) in the compact form

$$\boxed{\delta^{\beta}F_{\alpha\beta} \overset{D\neq 2}{=} -(D-1)\,\delta_{\alpha}h}$$

3.11.10 Second contracted world identity

Knowing that the first contracted world identity allows us to directly see the electromagnetic field equation, the question is now whether this is also possible for the material field equation (3.10). For that purpose, we begin with the first contracted electrogravitational identity (3.159) and apply a second world derivative on it:

$$\delta^{\alpha}\delta^{\beta}\overline{Z}_{\beta\alpha} = \frac{1}{2}(D-2)\,\delta^{\alpha}\delta^{\beta}F_{\alpha\beta} \tag{3.163}$$

To evaluate the right hand side, we adopt relation (3.139) and the knowledge of Section 3.11.8 that the metric commutes with the world derivative such that

$$\left[\left[\delta^{\alpha},\delta^{\beta}\right]+(D-w_Y)\,F^{\alpha\beta}\right]Y_{\alpha\beta} = -Z^{\epsilon}{}_{\alpha}{}^{\alpha\beta}Y_{\epsilon\beta} - Z^{\epsilon}{}_{\beta}{}^{\alpha\beta}Y_{\alpha\epsilon}$$

Using the antisymmetry of the electrogravitational curvature $Z^{\alpha}{}_{\beta\gamma\delta}$ in its last two indices, we can also write

$$\left[\delta^{\alpha},\delta^{\beta}\right]Y_{\alpha\beta} = 2Z^{\epsilon}{}_{\alpha}{}^{\alpha\beta}Y_{[\beta\epsilon]} - (D-w_Y)\,F^{\alpha\beta}Y_{\alpha\beta}$$

Next, we apply identity (3.140) and definition (2.34):

$$\left[\delta^{\alpha},\delta^{\beta}\right]Y_{\alpha\beta} = -2\left[Z^{\epsilon\beta}+(D-2)\,\delta^{\epsilon}_{\alpha}F^{\alpha\beta}\right]Y_{[\beta\epsilon]} - (D-w_Y)\,F^{\alpha\beta}Y_{\alpha\beta}$$

The symmetry of the electrogravitation tensor $Z^{\epsilon\beta}$ and the antisymmetry of the electromagnetic field strength $F^{\alpha\beta}$ then lead to the identity

$$\boxed{\left[\delta^{\alpha},\delta^{\beta}\right]Y_{\alpha\beta} \equiv (D-4+w_Y)\,F^{\alpha\beta}Y_{\alpha\beta}} \tag{3.164}$$

This is the generalization of identity (2.113) to world theory.

The result (3.164) implies

$$\delta^{\alpha}\delta^{\beta}F_{\alpha\beta} = \frac{1}{2}(D-4)\,F^{\alpha\beta}F_{\alpha\beta} \tag{3.165}$$

because the gauge weight of the electromagnetic field strength $F_{\alpha\beta}$ is 0 due to box (2.97). Relation (3.163) then simplifies to the **second contracted electrogravitational identity**

$$\boxed{\delta^{\alpha}\delta^{\beta}\overline{Z}_{\alpha\beta} \equiv \frac{1}{4}(D-2)(D-4)\,F^{\alpha\beta}F_{\alpha\beta}} \tag{3.166}$$

where we have also used the symmetry of the dual electrogravitation tensor $\overline{Z}_{\alpha\beta}$. The above identity shows us in a direct manner that the dimensional constraint of electromagnetism (2.115) is a consequence of the electromagnetic field equation (2.36). To make that even more evident, we merely have to use definition (3.158) and write the electromagnetic field equation as

$$\overline{Z}_{\alpha\beta} = 0$$

We can now address the material field equation. To this end, we take the first contracted world identity (3.160) and apply another world derivative on it:

$$\delta^{\alpha}\delta^{\beta}\overline{\Omega}_{\alpha\beta} = \frac{1}{2}(D-2)\left[\delta^{\alpha}\delta^{\beta}F_{\alpha\beta} + (D-1)\,\delta^{\alpha}\delta_{\alpha}h\right]$$

Equation (3.165) then gives the **second contracted world identity**

$$\boxed{\delta^{\alpha}\delta^{\beta}\overline{\Omega}_{\alpha\beta} \equiv \frac{1}{2}(D-2)\left[(D-1)\,\delta^{\alpha}\delta_{\alpha}h + \frac{1}{2}(D-4)\,F^{\alpha\beta}F_{\alpha\beta}\right]} \tag{3.167}$$

To see the material field equation in it, we adopt definition (2.83) and evaluate

$$\delta^{\alpha}\delta_{\alpha}h = g^{\alpha\beta}\delta_{\beta}\delta_{\alpha}h = g^{\alpha\beta}\left(\nabla_{\beta}\delta_{\alpha}h - l^{\gamma}_{\alpha\beta}\delta_{\gamma}h - 2A_{\beta}\delta_{\alpha}h\right)$$

where we have again used the gauge weight -2 of the Hubble field h. Relation (3.152) and definition (2.78) then imply

$$\delta^{\alpha}\delta_{\alpha}h = g^{\alpha\beta}\left[\nabla_{\beta}(\nabla_{\alpha} - 2A_{\alpha})\,h - (2\delta^{\gamma}_{\alpha}A_{\beta} - A^{\gamma}g_{\alpha\beta})(\nabla_{\gamma} - 2A_{\gamma})\,h - 2A_{\beta}(\nabla_{\alpha} - 2A_{\alpha})\,h\right]$$

This simplifies to

$$\delta^{\alpha}\delta_{\alpha}h = \left[\nabla^{\alpha} + (D-4)\,A^{\alpha}\right](\nabla_{\alpha} - 2A_{\alpha})\,h$$

such that using equation (3.22) the second contracted world identity (3.167) can be written as

$$\delta^{\alpha}\delta^{\beta}\overline{\Omega}_{\alpha\beta} \equiv \frac{1}{6}(D-2)\left\{(D-1)\left[\nabla^{\alpha} + (D-4)\,A^{\alpha}\right](\nabla_{\alpha} - 2A_{\alpha})\,\lambda - \frac{3}{2}(D-4)\,F^{\alpha\beta}F_{\beta\alpha}\right\}$$

This form clearly shows us that the material field equation (3.10) is an immediate consequence of the world equation (3.29). And, the form (3.167) allows us to write the material field equation compactly as

$$\boxed{\delta^{\alpha}\delta_{\alpha}h \overset{D\neq 1,2}{\equiv} -\frac{1}{2}(D-4)\,F^{\alpha\beta}F_{\alpha\beta}}$$

Similar to general relativity, it would be possible to investigate lots of additional, but less relevant relations. However, the goal of this paper is not to present all possible details of world theory, but only its fundamentals. We are therefore now done with that theory. In the following chapter, we will study solutions of the world equation.

4 Solutions

4.1 Overview

In this chapter, we investigate how solutions can be found in world theory. For that purpose, we **limit ourselves to four dimensions throughout this chapter without denoting this explicitly**.

The most trivial solution of the world equation (3.2) is obviously

$$\begin{aligned} g_{\alpha\beta} &= \eta_{\alpha\beta} \\ A_\alpha &= 0 \\ \lambda &= 0 \end{aligned} \tag{4.1}$$

Let us call such a world space an **undeformed spacetime**. Due to the metric $g_{\alpha\beta}$ being equal to the Minkowski metric $\eta_{\alpha\beta}$, an undeformed spacetime is also a flat spacetime. However, not every flat spacetime is undeformed. Flatness means only $g_{\alpha\beta} = \eta_{\alpha\beta}$ such that the electromagnetic vector potential A_α and the matter field λ do not necessarily have to vanish. For vacuum general relativity, the latter two fields are not present such that we do not have to discern between undeformed and flat there. As the notion undeformed spacetime arises from world theory and is not present in the general relativistic nomenclature, we can by the way keep in mind: The trivial solution of vacuum general relativity is a flat spacetime and the trivial one of world theory is an undeformed spacetime.

What solutions are there beyond the undeformed spacetime? As world theory has vacuum general relativity and vacuum electrodynamics in flat spacetime as limits, all solutions of these two cases are also part of world theory. Let us have a closer look at those solutions. The trivial solutions of the two cases are already taken into account by the undeformed spacetime (4.1). Then, there are gravitational and electromagnetic waves, as well as matter waves, which are waves of the matter field λ. Another important category of solutions are black holes. These kind of objects are still allowed in world theory. Knowing that vacuum general relativity is a limit, Schwarzschild and Kerr black holes are allowed, i.e. non-rotating and rotating black holes that are uncharged. Eventually, we have already encountered another kind of solutions, de Sitter spacetimes. Due to box (3.62) and equations (3.64), (3.66) plus (3.67), we remember that

$$\begin{aligned} g_{\alpha\beta} &= \operatorname{diag}\left(-1, R(t)^2, R(t)^2, R(t)^2\right) \\ A_\alpha &= 0 \\ \lambda &= \Lambda \end{aligned} \tag{4.2}$$

is allowed, with $\Lambda > 0$ and the scale factor

$$R(t) = R(0)\, e^{\sqrt{\frac{\Lambda}{3}} t}$$

In de Sitter spacetimes, the two black hole categories mentioned above become the Schwarzschild-de Sitter and Kerr-de Sitter black holes. To see this, we look at an arbitrary world space and set $A_\alpha = 0$. Then, we have a situation comparable to Section 3.6.3. This time, equation (3.8) shows

$$\partial_\alpha \lambda = 0$$

such that

$$\lambda = \Lambda$$

So, the world equation (3.2) reduces to

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda = 0 \tag{4.3}$$

This equation has the same form as equation (3.65). However, we have to be careful here. Equation (4.3) is based on the arbitrary metric $g_{\alpha\beta}(x^\alpha)$, whereas equation (3.65) uses the cosmological fields (3.62), for which the related metric $g_{\alpha\beta}(t)$ has just a time-dependence. Equation (4.3) is now familiar from general relativity. Therefore, all solutions known there are automatically solutions of world theory. This includes the Schwarzschild-de Sitter and Kerr-de Sitter black holes mentioned above. If we allow $\Lambda < 0$, there are also anti-de Sitter spacetimes and the corresponding black holes. So, in principle, we have to discern the three cases $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$. As this would be unnecessarily complicated and also not required, because cosmological observations tell us that our universe possesses a positive cosmological constant Λ (see, e.g., Perlmutter *et al.* 1999), we limit ourselves to $\Lambda > 0$ in the following.

World theory does not only contain the above already known solutions but also new ones. In the following sections, we will study these solutions, which divide into analytic and numeric ones. However, the solutions of nontrivial theories like world theory are so manifold that we can only show the first steps in this paper.

4.2 Stationary, spherical symmetry

4.2.1 Background spacetime

We begin with the analytic solutions, where we want to focus on the case of a stationary and spherically symmetric spacetime. From the last section, we recall that then Schwarzschild-de Sitter black holes are possible. However, the question is now whether there are additional solutions in world theory (de Sitter is considered as a special case of Schwarzschild-de Sitter here). In Einstein-Maxwell theory, for instance, charged black holes, the Reissner-Nordstrøm spacetimes, are known.

When studying solutions, an important issue are the chosen boundary conditions. In our case, this means clarifying how the solutions behave at spatial infinity. The easiest case would be to assume that the solutions converge to an undeformed spacetime. However, in the last section, we have limited ourselves to the case $\Lambda > 0$ such that we have to demand a convergence to the de-Sitter spacetime (4.2).

So, the background of our sought stationary and spherically symmetric solution is the de Sitter spacetime (4.2). This appears to be contradictory. How can we find a stationary solution in a time-dependent background (4.2)? The answer is simple. Though the fields (4.2) have an explicit time-dependence, this solution is actually time-independent. Choosing different coordinates, the time-dependence can be removed from the fields. We do not show how that works. This is not new and known from general relativity. The result is, for example, found in equation (8.34) of Stephani *et al.* (2006). To write it down here, we apply spherical coordinates, i.e.

$$x^\alpha = (t, r, \theta, \phi) \tag{4.4}$$

which are also used for the rest of Section 4.2. The metric of the de Sitter spacetime is then

$$g_{\alpha\beta} = \operatorname{diag}\left(-\left(1 - \frac{\Lambda}{3}r^2\right), \left(1 - \frac{\Lambda}{3}r^2\right)^{-1}, r^2, r^2 \sin^2\theta\right) \tag{4.5}$$

Note that for $r = \sqrt{3/\Lambda}$ we have a division by zero above. Due to definition (3.67), we can also write $r = 1/H$ such that we see that this is the Hubble radius. So, the problematic division occurs on the boundary of the observable universe. As this boundary and all beyond is not observable from earth, we can ignore the division by zero. The other two fields of the list (4.2) are unaffected by the new coordinates. Why? A coordinate transformation cannot make a vanishing vector, like A_α, nonzero. And, the matter field λ is a scalar. So, the time-dependence of the fields (4.2) can really be removed. This is by the way an easy, alternative way to see that the temporal cosmological principle (3.70) holds for world theory.

4.2.2 Symmetry

Knowing the background spacetime as a constraint on the sought solutions, the next question is how stationarity and spherical symmetry themselves limit the degrees of freedom of the three fields $g_{\alpha\beta}$, A_α and λ of world theory. From general relativity, we recall (see equation (23.7) of Misner *et al.* 2002) that a stationary and spherically symmetric metric can be written as

$$g_{\alpha\beta} = \mathrm{diag}\left(-A\left(r\right), B\left(r\right), r^2, r^2 \sin^2\theta\right) \tag{4.6}$$

where $A(r), B(r) > 0$ are unknown real functions. Remember here that a comparison with the de Sitter metric (4.5) works only within the Hubble radius. For the electromagnetic vector potential A_α, it is clear that spherical symmetry prohibits nonzero components A_θ and A_ϕ. In addition to that, it is obvious that only a radial dependence is allowed for the remaining components such that we have

$$A_\alpha = \left(A_t\left(r\right), A_r\left(r\right), 0, 0\right) \tag{4.7}$$

The same dependence is given for the matter field λ, i.e. we can write

$$\lambda = \lambda\left(r\right) \tag{4.8}$$

The three constraints (4.6), (4.7) and (4.8) are the stationary, spherically symmetric counterpart of equations (3.56), (3.57) and (3.58), which are the result of the spatial cosmological principle. In analogy to Section 3.6.2, we will now try to reduce the degrees of freedom of the electromagnetic vector potential (4.7) in the following section.

4.2.3 Symmetric fields

To reduce the degrees of freedom, we apply the conformal gauge transformation

$$\chi\left(r\right) = -\int_{r_0}^{r} \mathrm{d}\rho A_r\left(\rho\right) \tag{4.9}$$

Note that we start the integration from $r_0 > 0$ and not from 0 above. The reason is that a limit of the solutions obtained in the following must be Schwarzschild-de Sitter black holes. However, these solutions have a singularity at the position $r = 0$ and also an event horizon. This would make an integration there problematic. So, we have to choose the starting point r_0 somewhere outside of the event horizon and the singularity but not beyond the observable universe. Where exactly does not matter here.

The conformal gauge transformation (4.9) leads to

$$\partial_\alpha\chi\left(r\right) = -\delta_\alpha^r A_r\left(r\right)$$

Then, box (2.70) and the constraint (4.7) give

$$A_\alpha \overset{\text{cg}}{\to} A_\alpha - \delta^r_\alpha A_r(r) = (A_t(r), 0, 0, 0)$$

That way, the electromagnetic vector potential A_α does not vanish, but there is at least only a single unknown function $A_t(r)$ left. Similar to Section 3.6.2, we also do not have to worry about the consequences of the conformal gauge transformation for the matter field λ. Box (3.3) shows

$$\lambda \overset{\text{cg}}{\to} e^{2\chi(r)}\lambda$$

such that the form (4.8) is preserved.

However, we have to be careful again with the metric $g_{\alpha\beta}$. Due to box (2.70), the conformal gauge transformation causes

$$g_{\alpha\beta} \overset{\text{cg}}{\to} e^{-2\chi(r)} g_{\alpha\beta} = g^{\bullet}_{\alpha\beta}$$

where

$$g^{\bullet}_{\alpha\beta} = \operatorname{diag}\left(-e^{-2\chi(r)}A(r), e^{-2\chi(r)}B(r), e^{-2\chi(r)}r^2, e^{-2\chi(r)}r^2\sin^2\theta\right)$$

which does not fit to the constraint (4.6). Fortunately, we can again solve the problem by using a coordinate transformation. This time, it changes the radial coordinate r according to

$$r' = e^{-\chi(r)} r \tag{4.10}$$

while all other coordinates are unaffected. That way, we obtain

$$g^{\bullet\prime}_{\alpha\beta} = \operatorname{diag}\left(-A^{\bullet\prime}(r'), B^{\bullet\prime}(r'), r'^2, r'^2\sin^2\theta\right) \tag{4.11}$$

(see p. 594f of Misner *et al.* 2002). Notice that the primes of the new functions $A^{\bullet\prime}$ and $B^{\bullet\prime}$ do not represent derivatives with respect to the radial coordinate r' here but denote the coordinate transformation (4.10). The result (4.11) is now of the form (4.6). Therefore, we can use the following fields for stationary, spherically symmetric solutions of world theory:

$$\boxed{\begin{aligned} g_{\alpha\beta} &= \operatorname{diag}\left(-A(r), B(r), r^2, r^2\sin^2\theta\right) \\ A_\alpha &= (A_t(r), 0, 0, 0) \\ \lambda &= \lambda(r) \end{aligned}} \tag{4.12}$$

4.2.4 Christoffel symbols

To use the symmetric fields (4.12) eventually in the world equation, we have to first perform intermediate computations in this and the subsequent section. We do not refer to general relativity here, but repeat the computations for completeness. Readers familiar with general relativity can skim over these computations.

In this section, we address the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$. For that purpose, we need the contravariant metric

$$g^{\alpha\beta} = \operatorname{diag}\left(-\frac{1}{A(r)}, \frac{1}{B(r)}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\theta}\right) \tag{4.13}$$

We can then evaluate the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ by means of equation (1.4). In that equation, we now consider the factor $g^{\alpha\delta}$. Due to the constraint (4.13), we see that

$g^{\alpha\delta} = 0$ for $\delta \neq \alpha$. It is therefore sufficient to merely look at the values $\delta = \alpha$. However, we cannot just replace the index δ in equation (1.4) with the index α. Then, the index α would appear multiply per term such that Einstein's sum convention would have to be applied. To exclude Einstein's sum convention, we replace the index δ with the underlined index $\underline{\alpha}$. That way, equation (1.4) simplifies to

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\underline{\alpha}} \left(\partial_\beta g_{\underline{\alpha}\gamma} + \partial_\gamma g_{\beta\underline{\alpha}} - \partial_{\underline{\alpha}} g_{\beta\gamma} \right)$$

Looking then at the metrics (4.12) and (4.13), we find the following nonzero components, where the primes now denote derivatives with respect to the radial coordinate r:

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{A'}{2A} & \Gamma^r_{\theta\theta} &= -\frac{r}{B} & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^r_{tt} &= \frac{A'}{2B} & \Gamma^r_{\phi\phi} &= -\frac{r\sin^2\theta}{B} & \Gamma^\phi_{r\phi} &= \frac{1}{r} \\
\Gamma^r_{rr} &= \frac{B'}{2B} & \Gamma^\theta_{r\theta} &= \frac{1}{r} & \Gamma^\phi_{\theta\phi} &= \cot\theta
\end{aligned} \tag{4.14}$$

The symmetry of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ implies that also $\Gamma^t_{rt} = \Gamma^t_{tr}, \ldots$ are nonzero. For simplicity, we have left these components away above.

4.2.5 Ricci tensor

The next quantity which has to be evaluated is the Ricci tensor (1.5). For that purpose, we first take the Christoffel symbols (4.14) and find

$$\begin{aligned}
\Gamma^\alpha_{t\alpha} &= 0 \\
\Gamma^\alpha_{r\alpha} &= \frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r} \\
\Gamma^\alpha_{\theta\alpha} &= \cot\theta \\
\Gamma^\alpha_{\phi\alpha} &= 0
\end{aligned} \tag{4.15}$$

Then, it is easy to see that

$$\partial_\gamma \Gamma^\gamma_{\alpha\beta} = \partial_\beta \Gamma^\gamma_{\alpha\gamma} = 0$$

for $\alpha \neq \beta$. Moreover, for $\alpha \neq \beta$ the only non-vanishing case of the expression

$$\Gamma^\gamma_{\delta\gamma}\Gamma^\delta_{\alpha\beta} = \Gamma^\gamma_{\theta\gamma}\Gamma^\theta_{\alpha\beta}$$

in the Ricci tensor (1.5) is

$$\Gamma^\gamma_{\theta\gamma}\Gamma^\theta_{r\theta} = \frac{\cot\theta}{r} \tag{4.16}$$

Still assuming $\alpha \neq \beta$, we next look at the expression

$$\Gamma^\gamma_{\delta\beta}\Gamma^\delta_{\alpha\gamma} = \Gamma^\phi_{\delta\beta}\Gamma^\delta_{\alpha\phi} \tag{4.17}$$

for which the sole nonzero case is

$$\Gamma^\phi_{\delta\theta}\Gamma^\delta_{r\phi} = \frac{\cot\theta}{r}$$

This just cancels the term (4.16) in the Ricci tensor (1.5) such that we see

$$R_{\alpha\beta} = 0 \qquad \text{for } \alpha \neq \beta \tag{4.18}$$

For $\alpha = \beta$, the Ricci tensor $R_{\alpha\beta}$ does not vanish. However, we do not have to evaluate all four diagonal components here. In the following, we will only need the components R_{tt} and R_{rr}. So, using the Christoffel symbols (4.14) and (4.15), we compute

$$R_{tt} = \partial_\gamma \Gamma^\gamma_{tt} - \partial_t \Gamma^\gamma_{t\gamma} + \Gamma^\gamma_{\delta\gamma}\Gamma^\delta_{tt} - \Gamma^\gamma_{\delta t}\Gamma^\delta_{t\gamma} = \partial_r\left(\frac{A'}{2B}\right) + \left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right)\frac{A'}{2B} - 2\frac{A'}{2A}\frac{A'}{2B}$$

and

$$\begin{aligned} R_{rr} &= \partial_\gamma \Gamma^\gamma_{rr} - \partial_r \Gamma^\gamma_{r\gamma} + \Gamma^\gamma_{\delta\gamma}\Gamma^\delta_{rr} - \Gamma^\gamma_{\delta r}\Gamma^\delta_{r\gamma} \\ &= \partial_r\left(\frac{B'}{2B}\right) + \left(\frac{B'}{2B} - \partial_r\right)\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right) - \left(\frac{A'}{2A}\right)^2 - \left(\frac{B'}{2B}\right)^2 - \frac{2}{r^2} \end{aligned}$$

The result is then

$$\begin{aligned} R_{tt} &= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} \\ R_{rr} &= -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB} \end{aligned} \tag{4.19}$$

4.2.6 Field equation

Now, we have the means to look at the world equation. As we have merely computed the Ricci tensor $R_{\alpha\beta}$ in the last section and not the Einstein tensor $G_{\alpha\beta}$, we take equations (2.103) and (3.5) such that the world equation can be written as

$$R_{\alpha\beta} = g_{\alpha\beta}\nabla_\gamma A^\gamma + 2\left(\nabla_{(\alpha}A_{\beta)} - A_\alpha A_\beta + g_{\alpha\beta}g^{\gamma\delta}A_\gamma A_\delta\right) + g_{\alpha\beta}\lambda \tag{4.20}$$

in four dimensions.

For $\alpha \neq \beta$, the above way to write the world equation reduces drastically. Taking box (4.12) and the Ricci tensor (4.18), we find

$$\nabla_{(\alpha}A_{\beta)} = 0 \tag{4.21}$$

To continue here, we use the electromagnetic vector potential (4.12) and the Christoffel symbols (4.14) such that

$$\nabla_{(\alpha}A_{\beta)} = \partial_{(\alpha}A_{\beta)} - \Gamma^\gamma_{(\alpha\beta)}A_\gamma = \delta^t_{(\beta}\partial_{\alpha)}A_t - \Gamma^t_{(\alpha\beta)}A_t = \delta^r_{(\alpha}\delta^t_{\beta)}\left(A_t' - \frac{A'}{A}A_t\right) \tag{4.22}$$

Then, equation (4.21) leads to the simple relation

$$A_t' = \frac{A'}{A}A_t \tag{4.23}$$

for $\alpha = r$ and $\beta = t$.

In addition to the above case, we look at the components $\alpha = \beta = t$ and $\alpha = \beta = r$ of the world equation (4.20). To evaluate these components, we first take the metric (4.13) and relation (4.22) such that

$$\nabla_\gamma A^\gamma = g^{\alpha\beta} \nabla_{(\alpha} A_{\beta)} = 0$$

and due to the electromagnetic vector potential (4.12) also

$$g^{\gamma\delta} A_\gamma A_\delta = -\frac{A_t^2}{A}$$

Then, box (4.12), the Ricci tensor components (4.19) and equation (4.22) lead to

$$\begin{aligned} \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} &= -A\lambda \\ -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB} &= B\lambda - 2\frac{B}{A}A_t^2 \end{aligned}$$

We can combine the above two equations such that the matter field λ disappears. For that purpose, we divide the first equation by A and the second one by B. The sum of the outcomes is then

$$\frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right) = -\frac{2}{A}A_t^2$$

This equation can be simplified to

$$\partial_r \left(\frac{1}{AB}\right) = 2r\left(\frac{A_t}{A}\right)^2 \tag{4.24}$$

In the following section, we merely need the above equation and relation (4.23).

4.2.7 Charged black holes

Let us now have a closer look at relation (4.23). Due to Section 4.2.2, we know that $A > 0$. At first glance, this appears to hold throughout spacetime. However, we have to be careful here. We allow black holes in this context and assume a de Sitter background spacetime. Our choice of coordinates (4.4) has therefore limits. For small radii r, we may run into an event horizon or a singularity, and for large radii r, we will somewhere reach the boundary of the observable universe and thus the division by zero issue (4.5). Let us thus restrict ourselves to an open r-interval (R_1, R_2) where the above problems do not occur. This interval is chosen as large as possible. Note that $r = 0$ is excluded even if we have no black hole, because there we have a coordinate singularity for spherical coordinates.

So, within the interval (R_1, R_2), we have $A > 0$ such that equation (4.23) is well-defined and can also be written as

$$A_t' = A_t \partial_r \ln A \tag{4.25}$$

Next, we investigate the component A_t. There are two possibilities. Either the component vanishes everywhere in the considered interval:

$$A_t = 0 \tag{4.26}$$

Or, there exist one or more r-intervals (r_1, r_2) within the interval (R_1, R_2), i.e. $R_1 \leq r_1$ and $r_2 \leq R_2$, where $A_t \neq 0$. The intervals (r_1, r_2) are chosen as large as possible and have to be open, because the component A_t is continuous. Let us now focus on one of these intervals. There, either $A_t > 0$ or $A_t < 0$ everywhere such that we can divide by A_t in equation (4.25) and obtain

$$\partial_r \ln |A_t| = \partial_r \ln A$$

The solution is

$$|A_t| = aA \tag{4.27}$$

where $a > 0$ is a real constant (this is not the one of ansatz (2.9) but a new one).

If we have more than one interval (r_1, r_2), we look at two neighboring such intervals. Let us denote them by (r_1, r_2) and (r_1', r_2'), where $r_2 \leq r_1'$. It is now sufficient to look, for instance, at the boundary r_2. There, $A_t(r_2) = 0$. However, due to the continuity of the component A_t, we can also evaluate its value at the radius r_2 by means of equation (4.27) and the left-handed limit:

$$A_t(r_2) = \lim_{r \to r_2 -} aA(r) = aA(r_2) > 0 \tag{4.28}$$

This is a contradiction to the earlier result. That way, we see that there can be only a single interval (r_1, r_2). This interval has to actually be equal to the interval (R_1, R_2). Otherwise, if $r_2 < R_2$, the result (4.28) would cause a contradiction again, and likewise for $R_1 < r_1$. So, the only alternative to the case (4.26) is that the solution (4.27) holds in the entire interval (R_1, R_2).

Let us now insert the solution (4.27) in equation (4.24):

$$a^2 = \frac{1}{2r} \partial_r \left(\frac{1}{AB} \right) \tag{4.29}$$

The crucial point is here to look at the boundary condition at spatial infinity. Due to the metric (4.5) and the notation (4.6), the boundary condition is

$$\begin{aligned} A &= 1 - \frac{\Lambda}{3} r^2 \\ B &= \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} \end{aligned} \tag{4.30}$$

Unfortunately, there is the division by zero issue at the Hubble radius $r = 1/H$ such that we have to actually constrain ourselves to that radius. To make that radius an asymptotic boundary for a potential black hole here, we have to assume that such a black hole is much smaller than the radius of the observable universe. Of course, this is a reasonable assumption. Then, equations (4.30) yield

$$\lim_{r \to \frac{1}{H} -} \frac{1}{2r} \partial_r \left(\frac{1}{AB} \right) = 0$$

Hence, relation (4.29) leads to $a = 0$. This implies $A_t = 0$ due to equation (4.27) such that we are back at the case (4.26). So, due to box (4.12) we have $A_\alpha = 0$ and thus equation (4.3) again. As we have assumed stationarity and spherical symmetry, the only solutions are the Schwarzschild-de Sitter black holes. The interpretation of this unexpected result is given in the following box:

> **Charged black holes**:
>
> There are no charged, stationary, spherically symmetric black holes in world theory. (4.31)

Notice that this does not rule out charged black holes in general. Giving up stationarity may allow charged, spherically symmetric black holes that evolve in time (the case $\lambda = 0$ has been studied by Gorbatenko 2007, where the limitation to $\lambda = 0$ can be found on the bottom of p. 2 there). Except, if Birkhoff's theorem (see p. 843 of Misner *et al.* 2002) could be taken over from general relativity, which has not been investigated. Vice versa, without spherical symmetry, charged, stationary black holes may be present in world theory. These black holes will then at best be axisymmetric. In case of axisymmetry, they will be the world theoretical analog of the Kerr-Newman-de Sitter black holes familiar from general relativity. However, it is currently not known whether such black holes in fact exist in world theory (this statement is also supported by Example 2 on p. 8 of Gorbatenko & Kochemasov 2007).

4.3 Numerics

4.3.1 Partially working Lagrangians

The world equation has about the same mathematical complexity as Einstein's equation. Therefore, finding analytic solutions in world theory requires an amount of work comparable to general relativity. And similar to that theory, the limitation to analytic solutions excludes a huge amount of interesting spacetimes such that we will now perform a numeric approach to world theory. In analogy to numeric relativity, we call that approach **numeric world theory**.

For numeric world theory, we have to rewrite the world equation (3.20) in four dimensions. The first step is to look at the form (3.2). We recall that this equation describes the dynamics of gravitation, electromagnetism and matter. So, in principle, it would be sufficient to use the form (3.2). However, for numerics it is more appropriate to split that equation into the gravitational field equation, which is equation (3.2) itself, the electromagnetic field equation (3.8) and the material field equation (3.11).

The mentioned three equations are second order in time. To demonstrate this, we, for instance, look at the special case (3.12), which can be rewritten to

$$\partial_{tt}\lambda = \Delta\lambda$$

where $\Delta = \partial_{aa}$ is the Laplacian. For a numeric implementation of the above equation, we have to split it into equations that are first order in time. To this end, we introduce the ancillary field $\dot{\lambda} = \partial_t \lambda$ such that we get the two evolution equations

$$\begin{aligned} \partial_t \lambda &= \dot{\lambda} \\ \partial_t \dot{\lambda} &= \Delta\lambda \end{aligned}$$

Unfortunately, matter is not described by such simple evolution equations. We have to also take care of how it is influenced by gravitation and electromagnetism. An important numeric question is then the choice of the ancillary field used besides the matter field λ. The easiest case would be to just reuse the field $\dot{\lambda} = \partial_t \lambda$. However, there is an alternative, better way that reduces the length of the resulting evolution equations. If the matter field λ were governed by a Lagrangian density $\mathcal{L}_{\mathrm{M}}$, then we could use the conjugate momentum

$$\mu = \frac{\partial \mathcal{L}_{\mathrm{M}}}{\partial \dot{\lambda}} \tag{4.32}$$

This momentum is the natural companion of the matter field λ. The problem is now only that we have shown in Section 2.7 that the variation principle fails for world theory.

Yet, the variation principle does not fail completely. We can at least apply it partially. Surely, it does not work for electrogravitation, i.e. the case where the electromagnetic vector potential A_α does not disappear. Let us therefore assume that $A_\alpha = 0$. In that case, the gravitational field equation (3.2) becomes

$$G_{\alpha\beta} = -g_{\alpha\beta}\lambda \tag{4.33}$$

for which there is the Lagrangian

$$\boxed{L_{\mathrm{G}} = R - 2\lambda}$$

Let us now show that this Lagrangian truly works. For that purpose, we look at equations (1.20) and (2.143), where we see that

$$\frac{\partial}{\delta g_{\alpha\beta}}\left(\sqrt{-g}R\right) = -\sqrt{-g}G^{\alpha\beta}$$

Then,

$$0 = \frac{\partial \mathcal{L}_{\mathrm{G}}}{\delta g_{\alpha\beta}} = -\sqrt{-g}G^{\alpha\beta} - 2\lambda\frac{\partial\sqrt{-g}}{\delta g_{\alpha\beta}}$$

such that relation (2.140) and the Euler-Lagrange derivative (2.139) give equation (4.33). So, the variation principle is in fact applicable partially. We merely have to use $A_\alpha = 0$, and accept that a variation with respect to the matter field λ does not give the material field equation (3.11). This leads to the question, to what degree does the variation principle work for the electromagnetic field equation (3.8) and the material one (3.11).

For the electromagnetic field equation (3.8), there is even a complete Lagrangian, namely

$$\boxed{L_{\mathrm{EM}} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + A^\alpha\left(A_\alpha - \nabla_\alpha\right)\lambda} \tag{4.34}$$

To see this, we recall the computations performed in Section 2.7.2, which imply

$$0 = \frac{\partial \mathcal{L}_{\mathrm{EM}}}{\delta A_\alpha} = \partial_\beta\left(\sqrt{-g}F^{\beta\alpha}\right) + \sqrt{-g}\left(2A^\alpha - \nabla^\alpha\right)\lambda$$

This is already the electromagnetic field equation (3.8) due to relation (2.138). The only problem with the Lagrangian L_{EM} is that variations with respect to the metric $g_{\alpha\beta}$ and the matter field λ do not give the gravitational and material field equations (3.2) and (3.11), respectively.

Eventually, we look at the material field equation (3.11). For $A_\alpha = 0$, this equation reads

$$\nabla^\alpha\nabla_\alpha\lambda = 0 \tag{4.35}$$

In that case, there exists the Lagrangian

$$\boxed{L_{\mathrm{M}} = -\frac{1}{2}\left(\nabla_\alpha \lambda\right)^2} \tag{4.36}$$

We merely have to evaluate the Euler-Lagrange derivative (which is defined analogously to the earlier ones (2.136) and (2.139))

$$0 = \frac{\partial \mathcal{L}_{\mathrm{M}}}{\delta \lambda} = \frac{1}{2}\partial_\alpha \frac{\partial}{\partial \partial_\alpha \lambda}\left(\sqrt{-g} g^{\beta\gamma} \partial_\beta \lambda \partial_\gamma \lambda\right) = \partial_\alpha \left(\sqrt{-g} \nabla^\alpha \lambda\right)$$

As relation (2.137) leads to

$$\frac{1}{\sqrt{-g}}\partial_\alpha \left(\sqrt{-g}\nabla^\alpha \lambda\right) = \nabla_\alpha \nabla^\alpha \lambda$$

we then arrive at equation (4.35).

If the electromagnetic vector potential A_α is present, then we additionally have a term of the form

$$\nabla^\alpha \left(A_\alpha \lambda\right) = \lambda \nabla_\alpha A^\alpha + A^\alpha \partial_\alpha \lambda$$

from equation (3.11). For simplicity, we now assume $g_{\alpha\beta} = \eta_{\alpha\beta}$ such that we seek for an expression in the Lagrangian that leads to

$$\lambda \partial_\alpha A^\alpha + A^\alpha \partial_\alpha \lambda \tag{4.37}$$

This expression must be composed of one partial derivative, one electromagnetic vector potential and two matter fields per term, because the Euler-Lagrange derivative $\partial/\delta\lambda$ merely reduces the number of matter fields per term by one, but keeps the number of the other two quantities unchanged. We therefore have to look at the ansatz

$$l = a_5 \lambda A^\alpha \partial_\alpha \lambda + b_5 \lambda^2 \partial_\alpha A^\alpha \tag{4.38}$$

with the constants $a_5, b_5 \in \mathbb{R}$. Then,

$$\frac{\partial l}{\delta \lambda} = \frac{\partial l}{\partial \lambda} - \partial_\alpha \frac{\partial l}{\partial \partial_\alpha \lambda} = a_5 A^\alpha \partial_\alpha \lambda + 2b_5 \lambda \partial_\alpha A^\alpha - a_5 \partial_\alpha \left(\lambda A^\alpha\right) = \left(2b_5 - a_5\right) \lambda \partial_\alpha A^\alpha$$

which shows us that we cannot obtain the second term in expression (4.37). Therefore, the material field equation (3.11) cannot be deduced from a Lagrangian if the electromagnetic vector potential does not disappear.

All in all, we thus have the three Lagrangians L_{G}, L_{EM} and L_{M}. These three Lagrangians lead to the field equations of gravitation (3.2), electromagnetism (3.8) and matter (3.11) if a variation with respect to the metric $g_{\alpha\beta}$, the electromagnetic vector potential A_α and the matter field λ is performed, respectively. The only restrictions are that the electromagnetic vector potential A_α has to vanish for the Lagrangians L_{G} and L_{M}. In addition to that, there is no unified Lagrangian from which all three field equations could be derived. Therefore, the variation principle works only partially. As it does not work completely, it is still considered as obsolete.

4.3.2 Conjugate momenta

Though the variation principle is obsolete, we can now use the partially working Lagrangian (4.36). We recall that this Lagrangian does not work completely, because it requires $A_\alpha = 0$. However, it at least includes the metric $g_{\alpha\beta}$ such that definition (4.32) produces the conjugate momentum

$$\mu = \frac{\partial}{\partial \dot{\lambda}} \left(-\frac{1}{2}\sqrt{-g} g^{\alpha\beta} \partial_\alpha \lambda \partial_\beta \lambda \right)$$

which can be evaluated to

$$\boxed{\mu = -\sqrt{-g}\nabla^t \lambda} \tag{4.39}$$

We proceed with the conjugate momentum of the electromagnetic vector potential A_α. For the metric $g_{\alpha\beta}$, we by the way do not have to seek for a conjugate momentum, because all evolution variables are anyway determined by the BSSN-formalism, encountered further below. Using the Lagrangian (4.34), we compute the conjugate momentum

$$\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial \partial_t A_\alpha} = \frac{\partial}{\partial_t A_\alpha} \left\{ \sqrt{-g} \left[-\frac{1}{2} F^{\beta\gamma} \partial_\beta A_\gamma + A^\beta \left(A_\beta - \nabla_\beta \right) \lambda \right] \right\} = \sqrt{-g} F^{\alpha t} = -E^\alpha$$

where we have used definition (2.121). So, the conjugate momentum of the electromagnetic vector potential A_α is just the negative electric field $-E^\alpha$, which is familiar from electrodynamics[11].

4.4 Material evolution equations

4.4.1 λ-evolution

We have now the means to write down the evolution equations, where we begin with the material field equation (3.11). The evolution variables belonging to that field equation are the matter field λ and the conjugate momentum μ. In this section, we derive the evolution equation of the matter field λ. For that purpose, we use the 1+3-split (2.119) such that definition (4.39) leads to

$$\mu = -\sqrt{-g} g^{t\alpha} \partial_\alpha \lambda = \frac{\sqrt{-g}}{\alpha^2} \left(\partial_t \lambda - \beta^a \partial_a \lambda \right) \tag{4.40}$$

Let us now have a closer look at the factor $\sqrt{-g}$. It is known from general relativity that

$$\sqrt{-g} = \alpha \sqrt{\gamma} \tag{4.41}$$

where $\gamma = \det \gamma_{ab}$ (see equation (2.124) of Baumgarte & Shapiro 2010). We rewrite this factor even further, because we will adopt the BSSN-formalism for the gravitational field equation (see p. 390 of Baumgarte & Shapiro 2010). This formalism still uses the lapse α and the shift β^a, but the 3-metric γ_{ab} is split into

$$\gamma_{ab} = e^{4B} \overline{\gamma}_{ab} \tag{4.42}$$

[11] By replacing $L_{\mathrm{EM}} \to -L_{\mathrm{EM}}$ we could even manage to get the electric field E^α itself as the conjugate momentum, i.e. without a minus in front. However, this would also cause the Hamiltonian H_{EM} to become $-H_{\mathrm{EM}}$ and thus produce unwanted negative values for it. To see this, we have to just look at the case $g_{\alpha\beta} = \eta_{\alpha\beta}$ and $A_t = \lambda = 0$, where equation (E.2.16) of Wald (1984) shows that the original Hamiltonian has the form $H_{\mathrm{EM}} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \geq 0$.

where the determinant of the BSSN-metric $\overline{\gamma}_{ab}$ (which is symmetric) obeys

$$\overline{\gamma} = \det \overline{\gamma}_{ab} = 1 \tag{4.43}$$

(for Cartesian coordinates). Note that we cannot use the letter "ϕ" instead of the exponent B like in Baumgarte & Shapiro 2010, because we will need that letter further below for the electric potential. The above split then leads to

$$\sqrt{\gamma} = e^{6B} \tag{4.44}$$

such that

$$\sqrt{-g} = \alpha e^{6B} \tag{4.45}$$

That way, equation (4.40) implies the **$\boldsymbol{\lambda}$-evolution equation**

$$\boxed{\partial_t \lambda = \alpha e^{-6B} \mu + \beta^a \partial_a \lambda} \tag{4.46}$$

The above equation is the first one of the series of evolution equations that constitute the fundamentals of numeric world theory. In the following sections, we will encounter the remaining such equations. They are all highlighted with blue boxes such that readers not interested in their derivation have to just focus on these boxes or visit the Numeric formulary, where all the boxes are summarized.

4.4.2 μ-evolution

For the λ-evolution equation (4.46), we have not yet used the material field equation (3.11). The λ-evolution equation was merely a result of the definition of the conjugate momentum μ. However, for the evolution equation of that momentum, we have to finally apply the field equation. The first step to do this is to write the field equation as

$$\nabla_\alpha \nabla^\alpha \lambda = 2\left(\lambda \nabla_\alpha A^\alpha + A_\alpha \nabla^\alpha \lambda\right) \tag{4.47}$$

Let us then look at the covariant derivative $\nabla^\alpha \lambda$ appearing on both sides. We can expand that quantity according to

$$\nabla^a \lambda = g^{a\beta} \partial_\beta \lambda = g^{at} \partial_t \lambda + g^{ab} \partial_b \lambda$$

Next, we apply the 1+3-split (2.119) and equation (4.40) such that

$$\nabla^a \lambda = \frac{\beta^a}{\alpha^2}\left(\frac{\alpha^2}{\sqrt{-g}}\mu + \beta^b \partial_b \lambda\right) + \left(\gamma^{ab} - \frac{\beta^a \beta^b}{\alpha^2}\right)\partial_b \lambda = \frac{\beta^a}{\sqrt{-g}}\mu + \gamma^{ab}\partial_b \lambda$$

The contravariant counterpart of relation (4.42) is

$$\gamma^{ab} = e^{-4B} \overline{\gamma}^{ab} \tag{4.48}$$

where $\overline{\gamma}^{ab}\overline{\gamma}_{bc} = \delta^a_c$. Applying this together with equation (4.45), we get to

$$\sqrt{-g}\nabla^a \lambda = \beta^a \mu + \alpha e^{2B} \overline{\gamma}^{ab} \partial_b \lambda \tag{4.49}$$

We can now use the above outcome on the left hand side of equation (4.47). For that purpose, we take care of the coordinate weight 1 of the root $\sqrt{-g}$ and write

$$\sqrt{-g}\nabla^\alpha \nabla_\alpha \lambda = \nabla_\alpha\left(\sqrt{-g}\nabla^\alpha \lambda\right) = \partial_\alpha\left(\sqrt{-g}\nabla^\alpha \lambda\right) = \partial_t\left(\sqrt{-g}\nabla^t \lambda\right) + \partial_a\left(\sqrt{-g}\nabla^a \lambda\right)$$

For the first term at the end, we apply definition (4.39), and we that way arrive at

$$\sqrt{-g}\nabla^\alpha\nabla_\alpha\lambda = -\partial_t\mu + \partial_a\left(\beta^a\mu + \alpha e^{2B}\overline{\gamma}^{ab}\partial_b\lambda\right) \tag{4.50}$$

We proceed with the right hand side of equation (4.47). There, we have the expression

$$A_\alpha\nabla^\alpha\lambda = A_t\nabla^t\lambda + A_a\nabla^a\lambda \tag{4.51}$$

However, we do not use the component A_t as an evolution variable in our approach to numeric world theory. Instead, we take a more natural choice, the electrogravitational electric potential

$$\phi = \sqrt{-g}A^t \tag{4.52}$$

Using relation (4.41) and the 1+3-split (2.119), we can also write

$$\phi = \alpha\sqrt{\gamma}\left(g^{tt}A_t + g^{ta}A_a\right) = \frac{\sqrt{\gamma}}{\alpha}\left(-A_t + A_a\beta^a\right) \tag{4.53}$$

(see the line above equation (31) of Baumgarte & Shapiro 2003 for the general relativistic electric potential ${}^{\mathrm{GR}}\phi = \phi/\sqrt{\gamma}$, where the factor $\sqrt{\gamma}$ was introduced in analogy to the difference between the general relativistic fields ${}^{\mathrm{GR}}E^a, {}^{\mathrm{GR}}B^a$ and our fields E^a, B^a, see Sections 2.6.6 and 2.6.7). From definition (4.53) and relation (4.44), we then see

$$A_t = -\alpha e^{-6B}\phi + A_a\beta^a \tag{4.54}$$

Using this together with definition (4.39) and relation (4.49), we can write equation (4.51) as

$$\sqrt{-g}A_\alpha\nabla^\alpha\lambda = \left(\alpha e^{-6B}\phi - A_a\beta^a\right)\mu + A_a\left(\beta^a\mu + \alpha e^{2B}\overline{\gamma}^{ab}\partial_b\lambda\right) = \alpha e^{-6B}\phi\mu + \alpha e^{2B}\overline{\gamma}^{ab}A_a\partial_b\lambda$$

Eventually, we insert the above outcome and result (4.50) in equation (4.47):

$$\partial_t\mu = \partial_a\left(\beta^a\mu + \alpha e^{2B}\overline{\gamma}^{ab}\partial_b\lambda\right) - 2\alpha\left(e^{-6B}\phi\mu + e^{2B}\overline{\gamma}^{ab}A_a\partial_b\lambda\right) - 2\lambda\sqrt{-g}\nabla_\alpha A^\alpha$$

For a numeric implementation, it is best to apply the derivative ∂_a directly on the evolution variables. Using

$$\partial_a\overline{\gamma}^{bc} = -\overline{\gamma}^{bd}\overline{\gamma}^{ce}\partial_a\overline{\gamma}_{de} \tag{4.55}$$

which is an obvious consequence of $\overline{\gamma}^{ab}\overline{\gamma}_{bc} = \delta^a_c$, we then get the **$\mu$-evolution equation**

$$\boxed{\begin{aligned}\partial_t\mu =& \mu\partial_a\beta^a + \beta^a\partial_a\mu - 2\alpha e^{-6B}\phi\mu - 2\lambda\sqrt{-g}\nabla_\alpha A^\alpha \\ &+ \alpha e^{2B}\overline{\gamma}^{ab}\left[\partial_{ab}\lambda + \partial_a\lambda\left(\frac{1}{\alpha}\partial_b\alpha - \overline{\gamma}^{cd}\partial_c\overline{\gamma}_{bd} + 2\partial_b B - 2A_b\right)\right]\end{aligned}} \tag{4.56}$$

Note that we have not expanded the last term in the first line based on our evolution variables, because we will later use a gauge condition that sets $\nabla_\alpha A^\alpha$ to zero. The mentioned term will then vanish, which is emphasized by the gray font chosen for it above. However, we have not simply left the term away. Readers interested in other gauge conditions can then just take equation (4.56) and expand the term shown grayed out there in the most appropriate way for them.

4.5 Electromagnetic evolution equations

4.5.1 A_a-evolution

The next set of evolution equations are those of electromagnetism. For that purpose, we have to rewrite the electromagnetic field equation (3.8). In principle, we could take the full electromagnetic vector potential A_α and the conjugate momentum belonging to it, which is the negative electric field $-E^\alpha$, as evolution variables. However, it is best to proceed somewhat differently. First, we do not use the negative electric field but the electric field E^α itself. Looking at equation (2.121), we also see that $E^t = 0$ such that we merely have E^a as an evolution variable. Eventually, we do not take the component A_t, but the electric potential ϕ. In total, we then have the evolution variables A_a, E^a and ϕ.

In this section, we seek for the evolution equation of the components A_a. We can obtain them in analogy to the evolution equation of the matter field λ. So, we have to look at the definition of the negative conjugate momentum (2.121), which implies

$$E^a = \sqrt{-g} F^{ta} \tag{4.57}$$

For the electromagnetic field strength, we can write

$$F^{ta} = g^{t\beta} g^{a\gamma} F_{\beta\gamma} = \left(g^{tt} g^{ab} - g^{at} g^{bt} \right) F_{tb} + g^{tb} g^{ac} F_{bc}$$

The 1+3-split (2.119) then leads to

$$\alpha^2 F^{ta} = -\gamma^{ab} \left(\partial_t A_b - \partial_b A_t \right) + \beta^b \gamma^{ac} F_{bc}$$

where we have used the antisymmetry of the components F_{bc}. Next, we multiply the above equation by the metric γ_{ad} such that

$$\partial_t A_a = -\alpha^2 \gamma_{ab} F^{tb} + \partial_a A_t + \beta^b F_{ba} \tag{4.58}$$

Eventually, we use equations (4.42), (4.45) and (4.57) for the first term on the right hand side and equation (4.54) for the second one. Then, we obtain the **A_a-evolution equation**

$$\boxed{\partial_t A_a = -\alpha \left\{ e^{-2B} \overline{\gamma}_{ab} E^b + e^{-6B} \left[\partial_a \phi + \phi \left(\frac{1}{\alpha} \partial_a \alpha - 6 \partial_a B \right) \right] \right\} + \beta^b \partial_b A_a + A_b \partial_a \beta^b} \tag{4.59}$$

4.5.2 Electromagnetic constraint

Up to this point, we have not yet used the electromagnetic field equation (3.8). Let us now write that equation contravariantly and use relation (2.138):

$$\partial_\beta \left(\sqrt{-g} F^{\beta\alpha} \right) = \sqrt{-g} g^{\alpha\beta} \left(\partial_\beta - 2A_\beta \right) \lambda \tag{4.60}$$

In this section, we investigate the temporal component $\alpha = t$. For that case, the left hand side becomes

$$\partial_\beta \left(\sqrt{-g} F^{\beta t} \right) = \partial_b \left(\sqrt{-g} F^{bt} \right) = -\partial_a E^a$$

where we have applied relation (4.57). Hence, equation (4.60) and the 1+3-split (2.119) lead to

$$\partial_a E^a = -\sqrt{-g} \left[g^{tt} \left(\partial_t - 2A_t \right) + g^{ta} \left(\partial_a - 2A_a \right) \right] \lambda = \frac{\sqrt{-g}}{\alpha^2} \left[\partial_t - 2A_t - \beta^a \left(\partial_a - 2A_a \right) \right] \lambda$$

Then, we adopt relations (4.46) and (4.54), which imply

$$(\partial_t - 2A_t)\,\lambda = \alpha e^{-6B}\left(2\phi\lambda + \mu\right) + \beta^a\left(\partial_a - 2A_a\right)\lambda \tag{4.61}$$

With the help of relation (4.45), we that way arrive at the **electromagnetic constraint** (also known as Gauss constraint)

$$\boxed{\mu = \partial_a E^a - 2\phi\lambda} \tag{4.62}$$

Without matter, this constraint takes the form

$$\partial_a E^a = 0 \tag{4.63}$$

which is familiar from vacuum electrodynamics.

4.5.3 E^a-evolution

We proceed with the evolution equation of the electric field E^a. For this variable, we have to look at the spatial components $\alpha = a$ of equation (4.60)

$$\partial_\beta\left(\sqrt{-g}F^{\beta a}\right) = \sqrt{-g}g^{a\beta}\left(\partial_\beta - 2A_\beta\right)\lambda \tag{4.64}$$

Using definition (4.57), the left hand side above becomes

$$\partial_\beta\left(\sqrt{-g}F^{\beta a}\right) = \partial_t E^a + \partial_b\left(\sqrt{-g}F^{ba}\right) \tag{4.65}$$

For the right hand side of equation (4.64), we find

$$g^{a\beta}\left(\partial_\beta - 2A_\beta\right)\lambda = g^{at}\left(\partial_t - 2A_t\right)\lambda + g^{ab}\left(\partial_b - 2A_b\right)\lambda$$

such that the 1+3-split (2.119), equation (4.61) and the constraint (4.62) give

$$g^{a\beta}\left(\partial_\beta - 2A_\beta\right)\lambda = \frac{\beta^a}{\alpha^2}\alpha e^{-6B}\partial_b E^b + \gamma^{ab}\left(\partial_b - 2A_b\right)\lambda$$

Eventually, we use relations (4.45) and (4.48) such that

$$\sqrt{-g}g^{a\beta}\left(\partial_\beta - 2A_\beta\right)\lambda = \beta^a\partial_b E^b + \alpha e^{2B}\overline{\gamma}^{ab}\left(\partial_b - 2A_b\right)\lambda$$

We can now insert the above outcome and the result (4.65) in equation (4.64), which implies

$$\partial_t E^a = \partial_b\left(\sqrt{-g}F^{ab}\right) + \beta^a\partial_b E^b + \alpha e^{2B}\overline{\gamma}^{ab}\left(\partial_b - 2A_b\right)\lambda \tag{4.66}$$

The last step required for a numeric implementation of this equation is to write the first term on the right hand side out. However, without matter, i.e. $\lambda = 0$, the constraint (4.63) reduces equation (4.66) to the form

$$\partial_t E^a = \partial_b\left(\sqrt{-g}F^{ab}\right) \tag{4.67}$$

This form is numerically constraint conservative. The reason is that the right hand side is explicitly written as a divergence. For a numeric implementation, it would be easier to write the right hand side out and apply the partial derivative ∂_b directly on each single evolution variable hidden in the expression $\sqrt{-g}F^{ab}$. This requires the usage of the

Leibniz rule. However, the Leibniz rule holds only analytically but not numerically, where derivatives are computed via finite differences. Therefore, during the numeric evolution, the constraint (4.63) would then be gradually violated. For the form (4.67), we see

$$\partial_a \left(\partial_t E^a\right) = \partial_{ab} \left(\sqrt{-g} F^{ab}\right) = 0$$

due to the antisymmetry of the electromagnetic field strength F^{ab}. Hence, starting an evolution with $\partial_a E^a = 0$ initially, this constraint will be conserved during the numeric evolution.

For the electromaterial case (4.66), which means the inclusion of the matter field λ, the above considerations do not hold. However, it is still reasonable to stay as close as possible to the form (4.67), because that form is the limit $\lambda = 0$ of electromatter. So, we keep the partial derivative ∂_b in equation (4.66) around the expression $\sqrt{-g}F^{ab}$. However, we can at least write that expression out in terms of our evolution variables. For that purpose, we begin with

$$F^{ab} = g^{a\gamma} g^{b\delta} F_{\gamma\delta} = \left(g^{at} g^{bc} - g^{ac} g^{bt}\right) F_{tc} + g^{ac} g^{bd} F_{cd}$$

The 1+3-split (2.119) and the antisymmetry of the electromagnetic field strength F_{cd} then lead to

$$F^{ab} = \frac{1}{\alpha^2} \left(\beta^a \gamma^{bc} - \beta^b \gamma^{ac}\right) \left(\partial_t A_c - \partial_c A_t\right) + \left[\gamma^{ac} \gamma^{bd} - \frac{1}{\alpha^2} \left(\gamma^{ac} \beta^b \beta^d + \gamma^{bd} \beta^a \beta^c\right)\right] F_{cd}$$

Next, we apply equations (4.57) and (4.58) such that

$$\partial_t A_c - \partial_c A_t = -\beta^d F_{cd} - \frac{\alpha^2}{\sqrt{-g}} \gamma_{cd} E^d$$

That way, we arrive at

$$F^{ab} = \frac{1}{\sqrt{-g}} \left(E^a \beta^b - E^b \beta^a\right) + \gamma^{ac} \gamma^{bd} F_{cd}$$

Using this together with relations (4.45) and (4.48) in equation (4.66), we finally obtain the ***E*^{a}-evolution equation**

$$\boxed{\partial_t E^a = \partial_b \left(E^a \beta^b - E^b \beta^a + \alpha e^{-2B} \overline{\gamma}^{ac} \overline{\gamma}^{bd} F_{cd}\right) + \beta^a \partial_b E^b + \alpha e^{2B} \overline{\gamma}^{ab} \left(\partial_b - 2A_b\right) \lambda} \tag{4.68}$$

4.5.4 Electromagnetic constraint cure

Let us now pause the search for the evolution equations of world theory for a moment and consider the two equations (4.56) and (4.62). Both equations are a means to compute the conjugate momentum μ. Equation (4.56) is an evolution equation. So, we have to specify the momentum μ initially and can then evolve it in time. In contrast, equation (4.62) allows us to compute the momentum μ directly at each timestep. Note that the initial value of the momentum μ chosen for the evolution equation (4.56) must be just the one given by equation (4.62).

This leads to an important question that we have to address now. It could in principle be that equations (4.56) and (4.62) are inconsistent, which means that they lead to different evolutions of the momentum μ. In that case, electromatter and thus world theory would have a severe problem and could fail. We have to therefore find out whether equation (4.62) fits to the evolution equation (4.56). Fortunately, this is an easy task. We recall that equation (4.56) is a consequence of the material field equation (3.11) and equation (4.62) comes from the electromagnetic field equation (3.8). As the material field equation can be deduced from the electromagnetic field equation due to Section 3.2.3, we see that equation (4.56) must be consistent with equation (4.62).

So, we can really either choose equation (4.56) or equation (4.62) to compute the evolution of μ. This has a striking consequence for electromatter. If we take equation (4.62), then we not only see that equation (4.56) becomes superfluous. In addition to that, the meaning of the electromagnetic constraint (4.62) changes. It is no longer a constraint in the traditional sense, but an equation that allows us to compute the conjugate momentum μ. As we can evaluate that quantity directly via equation (4.62), the conjugate momentum μ is also not anymore an evolution variable but an ancillary quantity. Inserting it in the evolution equation (4.46) such that

$$\partial_t \lambda = \alpha e^{-6B} \left(\partial_a E^a - 2\phi\lambda \right) + \beta^a \partial_a \lambda$$

we can even remove the conjugate momentum μ completely from our numeric approach to electromatter. In that case, we have then merely the above λ-evolution equation together with equations (4.59) and (4.68). So, we have the three evolution variables λ, A_a and E^a, and there is no constraint. We can therefore say that the electromagnetic constraint (4.63) is cured by the matter field λ. Let us sum up this result:

Electromagnetic constraint cure:

The matter field λ cures the electromagnetic constraint $\partial_a E^a = 0$. (4.69)

Knowing this cure, we can vice versa state that the electromagnetic constraint is a sign that electromagnetism alone is incomplete. The matter field λ is then the natural way to round off electromagnetism.

4.5.5 Gauge condition

The last step required for a numeric implementation of electromatter is the specification of the electric potential ϕ. This quantity is unavoidable for the evolution of electromatter. We merely have to, for instance, look at the evolution equation (4.59), where the potential ϕ appears on the right hand side. However, we have already addressed all equations relevant for electromatter, the material field equation (3.11) and the electromagnetic one (3.8). The electric potential ϕ is therefore not governed by any field equation and can be chosen freely. This is a consequence of the conformal gauge invariance of world theory. For Einstein-Maxwell theory, such a behavior is also known. There, it is the result of gauge invariance. Similar to Einstein-Maxwell theory, we can thus fix the electric potential ϕ by choosing a gauge condition. The easiest choice would be to just set $\phi = 0$, which is called Weyl gauge. However, this gauge condition is not covariant. We therefore decide for the more commonly used **Lorentz gauge**

$$\nabla_\alpha A^\alpha = 0 \tag{4.70}$$

already encountered in relation (3.129). Note that this is by the way the reason why we have grayed out the last term in the first line of equation (4.56).

In the following, we will take the Lorentz gauge (4.70) and rewrite it in such a way that we can compute the electric potential ϕ from it. The first step to do this is to make use of the coordinate weight 1 of the root $\sqrt{-g}$, which allows us to write

$$0 = \sqrt{-g}\nabla_\alpha A^\alpha = \nabla_\alpha\left(\sqrt{-g}A^\alpha\right) = \partial_\alpha\left(\sqrt{-g}A^\alpha\right) = \partial_t\left(\sqrt{-g}A^t\right) + \partial_a\left(\sqrt{-g}A^a\right)$$

For the first term, we can apply definition (4.52). For the second one, we utilize the 1+3-split (2.119) and relation (4.54) such that

$$A^a = g^{at}A_t + g^{ab}A_b = \frac{\beta^a}{\alpha^2}\left(-\alpha e^{-6B}\phi + A_b\beta^b\right) + \left(\gamma^{ab} - \frac{\beta^a\beta^b}{\alpha^2}\right)A_b = \gamma^{ab}A_b - \frac{e^{-6B}}{\alpha}\beta^a\phi \tag{4.71}$$

Taking also equations (4.45) and (4.48) into account, we then arrive at

$$\partial_t\phi = \partial_a\left(\beta^a\phi - \alpha e^{2B}\overline{\gamma}^{ab}A_b\right)$$

Writing this out with relation (4.55) leads to the **ϕ-evolution equation**

$$\boxed{\partial_t\phi = \beta^a\partial_a\phi + \phi\partial_a\beta^a + \alpha e^{2B}\overline{\gamma}^{ab}\left[A_a\left(\overline{\gamma}^{cd}\partial_c\overline{\gamma}_{bd} - 2\partial_b B - \frac{1}{\alpha}\partial_b\alpha\right) - \partial_a A_b\right]} \tag{4.72}$$

So, the Lorentz gauge implies that the electric field ϕ can be computed by an evolution equation, like the other evolution variables.

4.5.6 Summary

With the gauge condition of the last section, we have the following evolution variables for electromatter:

$$\lambda, (\mu), A_a, E^a, [\phi]$$

The brackets have the following meaning. The round bracket emphasizes that the evolution variable μ can alternatively also be treated as an ancillary variable with equation (4.62). The squared bracket in contrast denotes that the electric potential is an evolution variable only with our gauge condition. In principle, it could also be a completely arbitrary function. The evolution equations of the above variables are then equations (4.46), (4.56), (4.59), (4.68) and (4.72).

4.6 Gravitational evolution equations

4.6.1 BSSN-formalism

We can now eventually address the evolution equations of the gravitational field equation (3.2). We could, in principle, proceed like in the last sections and derive the evolution equations on our own. However, this would not only be much more tedious for gravitation. It would also not be reasonable. An important problem of numeric relativity and thus world theory is numeric stability. For gravitation, the stability problem has been studied intensively during the last decades and it finally resulted in a stable, standard approach, the BSSN-formalism (for more details, see, e.g., p. 390 of Baumgarte & Shapiro 2010).

The BSSN-formalism is flexible and works for any stress-energy tensor $T_{\alpha\beta}$. Recalling Einstein's equation (1.18) and the world equation (3.30), i.e.

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{4.73}$$

$$G_{\alpha\beta} = t_{\alpha\beta} \tag{4.74}$$

we see that using $T_{\alpha\beta} = t_{\alpha\beta}/(8\pi)$ allows us to apply the BSSN-formalism also to world theory. However, we are advised to be careful here. It could be that the derivation of the BSSN-formalism is based on some not explicitly mentioned assumptions for the general relativistic stress-energy tensor $T_{\alpha\beta}$, which do not hold for the world stress-energy tensor $t_{\alpha\beta}$. To rule such problems out, I have proceeded in the following way for all equations resulting from the BSSN-formalism further below. I have derived these equations on my own without adopting any relations known from the BSSN-formalism. These derivations are not shown in this paper, as they are pretty lengthy. Fortunately, the resulting equations are the same ones as those further below such that the BSSN-formalism can in fact be applied on world theory.

Let us now look at the evolution variables of the BSSN-formalism. We have already encountered two of them, namely the fields B and $\overline{\gamma}_{ab}$. In addition to that, there are the gauge variables α and β^a. For the BSSN-formalism, three more evolution variables are missing. For that purpose, we introduce the exterior curvature

$$K_{ab} = \frac{1}{\alpha}\left(\partial_{(a}\beta_{b)} - \Gamma_{cab}\beta^c - \frac{1}{2}\partial_t\gamma_{ab}\right) \tag{4.75}$$

(see equation (2.128) and the 2nd last paragraph on p. 27 of Baumgarte & Shapiro 2010 as well as the 1+3-split (2.116)), with the Christoffel symbols of the first kind

$$\Gamma_{\alpha\beta\gamma} = g_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} \tag{4.76}$$

The quantities $\Gamma^{\alpha}_{\beta\gamma}$ are by the way also referred to as the Christoffel symbols of the second kind.

Note that the exterior curvature introduced here is not to be mixed up with the one mentioned in Section 2.3.7. The exterior curvature (4.75) is 3-dimensional, whereas the one of Section 2.3.7 is 4-dimensional. The 3-dimensional version K_{ab} refers to the exterior curvature of spatial slices with respect to the 4-dimensional spacetime. As these slices are caused by the foliation (2.116) and have nothing to do with a potential 4-dimensional exterior curvature, the quantity K_{ab} may be nonzero even if the 4-dimensional version vanishes, as in our case.

The exterior curvature (4.75) is now split according to

$$K_{ab} = e^{4B}\bar{A}_{ab} + \frac{1}{3}\gamma_{ab}K \tag{4.77}$$

(see equations (11.47) and (2.59) of Baumgarte & Shapiro 2010), where by definition

$$\bar{A} = \overline{\gamma}^{ab}\bar{A}_{ab} = 0 \tag{4.78}$$

We then see that two additional evolution variables are the trace

$$K = \gamma^{ab}K_{ab} \tag{4.79}$$

and the symmetric quantity

$$\bar{A}_{ab} = e^{-4B}\left(K_{ab} - \frac{1}{3}\gamma_{ab}K\right) \tag{4.80}$$

In addition to that, looking at the Christoffel symbols

$$\overline{\Gamma}^a_{bc} = \frac{1}{2}\overline{\gamma}^{ad}\left(\partial_b\overline{\gamma}_{dc} + \partial_c\overline{\gamma}_{bd} - \partial_d\overline{\gamma}_{bc}\right) \tag{4.81}$$

of the BSSN-metric $\overline{\gamma}_{ab}$, the quantity

$$\overline{\Gamma}^a = \overline{\gamma}^{bc}\overline{\Gamma}^a_{bc} \tag{4.82}$$

is treated as an independent variable. Note that we are not allowed to write the evolution variable $\overline{\Gamma}^a$ out in terms of the other evolution variable $\overline{\gamma}_{ab}$ according to definitions (4.81) and (4.82) wherever the quantity $\overline{\Gamma}^a$ occurs in the BSSN-formalism. Proceeding so would make the resulting evolution equations unstable. All in all, we then have to look at the variables

$$B, \overline{\gamma}_{ab}, K, \bar{A}_{ab}, \overline{\Gamma}^a, \alpha, \beta^a \tag{4.83}$$

in the BSSN-formalism.

For the first two of the above evolution variables, we can even write the evolution equations down without any further computations. For the field B, the BSSN-formalism gives us the ***B*-evolution equation**

$$\boxed{\partial_t B = \frac{1}{6}\left(\partial_a\beta^a - \alpha K\right) + \beta^a\partial_a B} \tag{4.84}$$

and for the BSSN-metric $\overline{\gamma}_{ab}$, the **$\overline{\gamma}_{ab}$-evolution equation**

$$\boxed{\partial_t\overline{\gamma}_{ab} = -2\alpha\bar{A}_{ab} + \beta^c\partial_c\overline{\gamma}_{ab} + \overline{\gamma}_{ac}\partial_b\beta^c + \overline{\gamma}_{bc}\partial_a\beta^c - \frac{2}{3}\overline{\gamma}_{ab}\partial_c\beta^c} \tag{4.85}$$

(see equations (11.50) and (11.51) of Baumgarte & Shapiro 2010). Just like equation (4.72), the above two equations do not contain any new content. However, they belong to our numeric approach of world theory and are therefore still emphasized by blue boxes. The reader has to then at the end merely look at the blue boxes in Sections 4.4, 4.5 and 4.6 to see all of our evolution equations at one go.

4.6.2 K-evolution

We proceed with the first equation of the BSSN-formalism for which we have to actually perform some ancillary computations, the evolution equation of the trace K of the exterior curvature. This equation has the form

$$\partial_t K = -\,{}^3\nabla^a\,{}^3\nabla_a\alpha + \alpha\left(\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}K^2\right) + \frac{\alpha}{2}\left(\rho + S\right) + \beta^a\partial_a K \tag{4.86}$$

(see equation (11.52) of Baumgarte & Shapiro 2010). Let us now go through the terms on the right hand side such that their meaning becomes clear in the following. The 3-covariant derivative ${}^3\nabla_a$ is the covariant derivative with respect to the 3-metric γ_{ab}. For this operator, the index a is risen and lowered with the 3-metric γ_{ab}, and it treats the lapse α as a scalar. Hence,

$${}^3\nabla_a\,{}^3\nabla_b\alpha = {}^3\nabla_a\partial_b\alpha = \partial_{ab}\alpha - {}^3\Gamma^c_{ab}\partial_c\alpha \tag{4.87}$$

with the Christoffel symbols

$${}^3\Gamma^a_{bc} = \frac{1}{2}\gamma^{ad}\left(\partial_b\gamma_{dc} + \partial_c\gamma_{bd} - \partial_d\gamma_{bc}\right) \tag{4.88}$$

of the 3-metric γ_{ab}. These Christoffel symbols can be rewritten to BSSN-variables with relations (4.42) and (4.48) such that

$$ {}^{3}\Gamma^{a}_{bc} = \overline{\Gamma}^{a}_{bc} + 2\left(\delta^{a}_{b}\partial_{c} + \delta^{a}_{c}\partial_{b} - \overline{\gamma}^{ad}\overline{\gamma}_{bc}\partial_{d}\right) B \tag{4.89} $$

Hence, equation (4.87) becomes

$$ {}^{3}\nabla_{a}\,{}^{3}\nabla_{b}\alpha = \partial_{ab}\alpha - \overline{\Gamma}^{c}_{ab}\partial_{c}\alpha - 2\left(\partial_{a}B\partial_{b} + \partial_{b}B\partial_{a} - \overline{\gamma}_{ab}\overline{\gamma}^{cd}\partial_{c}B\partial_{d}\right)\alpha \tag{4.90} $$

and thus

$$ {}^{3}\nabla^{a}\,{}^{3}\nabla_{a}\alpha = e^{-4B}\left[\overline{\gamma}^{ab}\left(\partial_{ab} + 2\partial_{a}B\partial_{b}\right) - \overline{\Gamma}^{a}\partial_{a}\right]\alpha \tag{4.91} $$

where we have used relation (4.48), definition (4.82) and $\overline{\gamma}^{ab}\overline{\gamma}_{bc} = \delta^{a}_{c}$.

For the second term in equation (4.86), we merely have to mention that the indices of the quantity $\bar{A}_{ab}$ are risen and lowered with the BSSN-metric $\overline{\gamma}_{ab}$ such that

$$ \bar{A}^{ab} = \overline{\gamma}^{ac}\overline{\gamma}^{bd}\bar{A}_{cd} \tag{4.92} $$

(see 3rd paragraph on p. 387 of Baumgarte & Shapiro 2010). The only open issue in equation (4.86) are then the quantities ρ and S. They are defined as

$$ \rho = n_{\alpha}n_{\beta}t^{\alpha\beta} \tag{4.93} $$

and

$$ S = \gamma^{ab}S_{ab} \tag{4.94} $$

with

$$ S_{ab} = \gamma_{a\gamma}\gamma_{b\delta}t^{\gamma\delta} \tag{4.95} $$

(see equations (2.138) of Baumgarte & Shapiro 2010, where the factor 8π left away in the world equation (4.74) in comparison to Einstein's equation (4.73) is taken into account in the evolution equation (4.86) by having used the factor $\alpha/2$ instead of $4\pi\alpha$ there, a procedure that we will also adopt further below) and

$$ \gamma_{\alpha\beta} = g_{\alpha\beta} + n_{\alpha}n_{\beta} $$

(see equation (2.29) of Baumgarte & Shapiro 2010). Using definition (2.117), we can write equation (4.93) as

$$ \rho = \alpha^{2}t^{tt} \tag{4.96} $$

and that definition also tells us

$$ \gamma_{ab} = g_{ab} $$

Hence, the quantity (4.95) is just

$$ S_{ab} = g_{a\gamma}g_{b\delta}t^{\gamma\delta} = t_{ab} \tag{4.97} $$

In the following, we have to therefore evaluate the world stress-energy tensor components t^{tt} and t_{ab} in

$$ \rho + S = \alpha^{2}t^{tt} + \gamma^{ab}t_{ab} $$

The world stress-energy tensor $t_{\alpha\beta}$ in four dimensions is due to equation (3.46)

$$ \boxed{t_{\alpha\beta} = \nabla_{\alpha}A_{\beta} + \nabla_{\beta}A_{\alpha} - 2A_{\alpha}A_{\beta} - g_{\alpha\beta}\left(2\nabla_{\gamma}A^{\gamma} + A^{2} + \lambda\right)} \tag{4.98} $$

Using the 1+3-split (2.116) and (2.119), which will also be applied frequently in the following without explicitly mentioning it, we find

$$\rho + S = \alpha^2 \left[2\nabla^t A^t - 2\left(A^t\right)^2 \right] + \gamma^{ab} \left(2\nabla_a A_b - 2A_a A_b\right) - 2\left(2\nabla_\alpha A^\alpha + A^2 + \lambda\right) \tag{4.99}$$

Let us now look at the terms quadratic in the electromagnetic vector potential. For these terms, we evaluate

$$\left(A^t\right)^2 = g^{t\alpha} g^{t\beta} A_\alpha A_\beta = \left(g^{tt} A_t\right)^2 + 2g^{tt} g^{ta} A_t A_a + g^{ta} g^{tb} A_a A_b$$

and

$$A^2 = g^{tt} \left(A_t\right)^2 + 2g^{ta} A_t A_a + g^{ab} A_a A_b \tag{4.100}$$

such that

$$\alpha^2 \left(A^t\right)^2 + \gamma^{ab} A_a A_b + A^2 = 2\gamma^{ab} A_a A_b$$

Equation (4.99) then simplifies to

$$\frac{1}{2}\left(\rho + S\right) = \alpha^2 \nabla^t A^t + \gamma^{ab} \nabla_a A_b - 2\gamma^{ab} A_a A_b - \lambda - 2\nabla_\alpha A^\alpha \tag{4.101}$$

We do not have to worry about the last term. It anyway disappears for our gauge condition. We have to therefore merely address the first two terms on the right hand side. The first one can be rewritten as

$$\alpha^2 \nabla^t A^t = \alpha^2 \left(g^{tt} \nabla_t A^t + g^{ta} \nabla_a A^t\right) = \left(g^{a\beta} + \beta^a g^{t\beta}\right) \nabla_a A_\beta - \nabla_\alpha A^\alpha = \gamma^{ab} \nabla_a A_b - \nabla_\alpha A^\alpha \tag{4.102}$$

such that rewriting equation (4.101) reduces to the evaluation of the covariant derivative

$$\nabla_a A_b = \partial_a A_b - \Gamma^\gamma_{ab} A_\gamma \tag{4.103}$$

To compute the second term here, we use the Christoffel symbols of the first kind (4.76) in the following together with the component

$$A^t = \frac{1}{\alpha} e^{-6B} \phi \tag{4.104}$$

which comes from relations (4.45) and (4.52). We also need the component (4.71) such that the Christoffel symbols (4.88) allow us to write

$$\Gamma^\gamma_{ab} A_\gamma = \Gamma_{tab} A^t + \Gamma_{cab} A^c = \frac{1}{\alpha} e^{-6B} \left(\Gamma_{tab} - \Gamma_{cab}\beta^c\right) \phi + {}^3\Gamma^c_{ab} A_c \tag{4.105}$$

We now evaluate

$$\Gamma_{tab} = \frac{1}{2}\left(\partial_a g_{tb} + \partial_b g_{at} - \partial_t g_{ab}\right) = \partial_{(a}\beta_{b)} - \frac{1}{2}\partial_t \gamma_{ab}$$

and use definition (4.75), which implies

$$\frac{1}{2}\partial_t \gamma_{ab} = \partial_{(a}\beta_{b)} - \Gamma_{cab}\beta^c - \alpha K_{ab} \tag{4.106}$$

Then,

$$\Gamma_{tab} = \alpha K_{ab} + \Gamma_{cab}\beta^c$$

We use this result in equation (4.105) such that the covariant derivative (4.103) becomes

$$\nabla_a A_b = \partial_a A_b - e^{-6B} K_{ab}\phi - {}^3\Gamma^c_{ab} A_c \tag{4.107}$$

or with relation (4.89):

$$\nabla_a A_b = \partial_a A_b - e^{-6B} K_{ab}\phi - \left[\overline{\Gamma}^c_{ab} + 2\left(\delta^c_a\partial_b + \delta^c_b\partial_a - \overline{\gamma}^{cd}\overline{\gamma}_{ab}\partial_d\right) B\right] A_c \tag{4.108}$$

Hence, relation (4.48) makes expression (4.102) look like

$$\alpha^2\nabla^t A^t = e^{-4B}\left[\overline{\gamma}^{ab}\left(\partial_b + 2\partial_b B\right) - \overline{\Gamma}^a\right] A_a - e^{-6B} K\phi - \nabla_\alpha A^\alpha \tag{4.109}$$

where we have used the trace (4.79) and definition (4.82). This eventually allows us to write equation (4.101) as

$$\frac{1}{2}\left(\rho + S\right) = 2\left\{e^{-4B}\left[\overline{\gamma}^{ab}\left(\partial_b + 2\partial_b B - A_b\right) - \overline{\Gamma}^a\right] A_a - e^{-6B} K\phi - \frac{\lambda}{2}\right\} - 3\nabla_\alpha A^\alpha$$

That way, we have the means to write down the final form of equation (4.86). Using the above outcome together with the result (4.91), we retrieve the ***K*-evolution equation**

$$\boxed{\begin{aligned}\partial_t K =& \alpha\left(\bar{A}_{ab}\bar{A}^{ab} + \frac{1}{3}K^2\right) + \beta^a\partial_a K - e^{-4B}\left[\overline{\gamma}^{ab}\left(\partial_{ab} + 2\partial_a B\partial_b\right) - \overline{\Gamma}^a\partial_a\right]\alpha \\ &+ 2\alpha\left\{e^{-4B}\left[\overline{\gamma}^{ab}\left(\partial_b + 2\partial_b B - A_b\right) - \overline{\Gamma}^a\right] A_a - e^{-6B} K\phi - \frac{\lambda}{2}\right\} - 3\alpha\nabla_\alpha A^\alpha\end{aligned}} \tag{4.110}$$

The last term is shown grayed out again, because it vanishes for our gauge condition.

4.6.3 $\bar{A}_{ab}$-evolution

The next equation of the BSSN-formalism is the evolution equation of the quantity $\overline{A}_{ab}$, which has the form

$$\begin{aligned}\partial_t \bar{A}_{ab} &= e^{-4B}\left[-\left({}^3\nabla_a {}^3\nabla_b\alpha\right)^{\mathrm{TF}} + \alpha\left({}^3R^{\mathrm{TF}}_{ab} - S^{\mathrm{TF}}_{ab}\right)\right] + \alpha\left(K\bar{A}_{ab} - 2\bar{A}_{ac}\bar{A}^c_b\right) \\ &\quad + \beta^c\partial_c\bar{A}_{ab} + \bar{A}_{ac}\partial_b\beta^c + \bar{A}_{bc}\partial_a\beta^c - \frac{2}{3}\bar{A}_{ab}\partial_c\beta^c\end{aligned} \tag{4.111}$$

(see equation (11.53) of Baumgarte & Shapiro 2010). In this equation, we merely have to look at the squared bracket. The remaining terms on the right hand side are anyway clear and do not have to be rewritten any further. In the squared bracket, the letters "TF" stand for the tracefree part of the expression the letters are attached to.

For the first term in the squared bracket, this, for instance, means

$$\left({}^3\nabla_a {}^3\nabla_b\alpha\right)^{\mathrm{TF}} = {}^3\nabla_a {}^3\nabla_b\alpha - \frac{1}{3}\gamma_{ab} {}^3\nabla^c {}^3\nabla_c\alpha$$

The above expression is tracefree, because a multiplication with the contravariant metric γ^{ab} shows us that its trace in fact vanishes. Using equations (4.90) and (4.91), we immediately find

$$\left({}^3\nabla_a {}^3\nabla_b\alpha\right)^{\mathrm{TF}} = \left\{\partial_{ab} - \overline{\Gamma}^c_{ab}\partial_c - 2\left(\partial_a B\partial_b + \partial_b B\partial_a\right) - \frac{1}{3}\overline{\gamma}_{ab}\left[\overline{\gamma}^{cd}\left(\partial_{cd} - 4\partial_c B\partial_d\right) - \overline{\Gamma}^c\partial_c\right]\right\}\alpha \tag{4.112}$$

where we have used relation (4.42).

We proceed with the term

$$ {}^3R^{\mathrm{TF}}_{ab} = {}^3R_{ab} - \frac{1}{3}\gamma_{ab}\gamma^{cd}\,{}^3R_{cd} = {}^3R_{ab} - \frac{1}{3}\overline{\gamma}_{ab}\overline{\gamma}^{cd}\,{}^3R_{cd} \tag{4.113} $$

of equation (4.111) (see the respective text line below equations (11.53) and (2.46) of Baumgarte & Shapiro 2010), in which we have combined relations (4.42) and (4.48). The 3-Ricci tensor ${}^3R_{ab}$ above is given by

$$ {}^3R_{ab} = 2\left(\partial_{[c}{}^3\Gamma^c_{ab]} + {}^3\Gamma^c_{d[c}{}^3\Gamma^d_{ab]}\right) \tag{4.114} $$

This quantity can be split according to

$$ {}^3R_{ab} = \overline{R}_{ab} + R^B_{ab} \tag{4.115} $$

where

$$ \overline{R}_{ab} = -\frac{1}{2}\overline{\gamma}^{cd}\partial_{cd}\overline{\gamma}_{ab} + \overline{\gamma}_{c(a}\partial_{b)}\overline{\Gamma}^c + \overline{\Gamma}^c\overline{\Gamma}_{(ab)c} + \overline{\gamma}^{cd}\left(2\overline{\Gamma}^e_{c(a}\overline{\Gamma}_{b)de} + \overline{\Gamma}^e_{ac}\overline{\Gamma}_{ebd}\right) \tag{4.116} $$

with the Christoffel symbols of the first kind $\overline{\Gamma}_{abc} = \overline{\gamma}_{ad}\overline{\Gamma}^d_{bc}$, and

$$ R^B_{ab} = -2\left(\overline{\nabla}_a\overline{\nabla}_b B + \overline{\gamma}_{ab}\overline{\gamma}^{cd}\overline{\nabla}_c\overline{\nabla}_d B\right) + 4\left(\overline{\nabla}_a B\overline{\nabla}_b B - \overline{\gamma}_{ab}\overline{\gamma}^{cd}\overline{\nabla}_c B\overline{\nabla}_d B\right) \tag{4.117} $$

(see equations (3.10) and (11.54) as well as the text line above equation (11.32) of Baumgarte & Shapiro 2010). The BSSN-derivative $\overline{\nabla}_a$ is the covariant derivative with respect to the BSSN-metric $\overline{\gamma}_{ab}$. It treats the quantity B as a scalar such that

$$ \overline{\nabla}_a B = \partial_a B $$

and thus

$$ \overline{\nabla}_a\overline{\nabla}_b B = \partial_{ab}B - \overline{\Gamma}^c_{ab}\partial_c B $$

(see last sentence on p. 56 of Baumgarte & Shapiro 2010). That way, we obtain

$$ R^B_{ab} = -2\left[\partial_{ab}B - \overline{\Gamma}^c_{ab}\partial_c B + \overline{\gamma}_{ab}\overline{\gamma}^{cd}\left(\partial_{cd}B - \overline{\Gamma}^e_{cd}\partial_e B\right)\right] + 4\left(\partial_a B\partial_b B - \overline{\gamma}_{ab}\overline{\gamma}^{cd}\partial_c B\partial_d B\right) $$

Using this outcome and expression (4.116) in equation (4.115), we arrive at

$$ \boxed{\begin{aligned} {}^3R_{ab} = &-\frac{1}{2}\overline{\gamma}^{cd}\partial_{cd}\overline{\gamma}_{ab} + \overline{\gamma}_{c(a}\partial_{b)}\overline{\Gamma}^c + \overline{\Gamma}^c\overline{\Gamma}_{(ab)c} + \overline{\gamma}^{cd}\left(2\overline{\Gamma}^e_{c(a}\overline{\Gamma}_{b)de} + \overline{\Gamma}^e_{ac}\overline{\Gamma}_{ebd}\right) \\ &- 2\left[\partial_{ab}B - \overline{\Gamma}^c_{ab}\partial_c B + \overline{\gamma}_{ab}\left(\overline{\gamma}^{cd}\partial_{cd}B - \overline{\Gamma}^c\partial_c B\right)\right] + 4\left(\partial_a B\partial_b B - \overline{\gamma}_{ab}\overline{\gamma}^{cd}\partial_c B\partial_d B\right) \end{aligned}} \tag{4.118} $$

where we have applied definition (4.82). We will not write this quantity explicitly out when returning to the evolution equation at the end of this section, because then that equation would become unnecessarily lengthy. For a numeric implementation, the reader has to merely look at the above box when encountering the quantity ${}^3R_{ab}$. Note that we do not also highlight the definitions of the Christoffel symbols $\overline{\Gamma}^a_{bc}$ and $\overline{\Gamma}_{abc}$ with boxes, because they can anyway be remembered far easier, see definition (4.81).

Eventually, we come to the term

$$ S^{\mathrm{TF}}_{ab} = S_{ab} - \frac{1}{3}\gamma_{ab}\gamma^{cd}S_{cd} = t_{ab} - \frac{1}{3}\gamma_{ab}\gamma^{cd}t_{cd} $$

of equation (4.111), for which we have applied relation (4.97). So, we have to now use the world stress-energy tensor (4.98) again. Using the 1+3-split (2.116), we obtain

$$S^{\mathrm{TF}}_{ab} = \nabla_a A_b + \nabla_b A_a - 2A_a A_b - \frac{2}{3}\gamma_{ab}\gamma^{cd}\left(\nabla_c A_d - A_c A_d\right)$$

Here, interestingly the matter field λ does not play a role. To continue rewriting the above expression, we use equation (4.108) such that

$$\begin{aligned} S^{\mathrm{TF}}_{ab} &= \partial_a A_b + \partial_b A_a - 2e^{-6B}K_{ab}\phi - 2\left[\bar{\Gamma}^c_{ab} + 2\left(\delta^c_a\partial_b + \delta^c_b\partial_a - \bar{\gamma}^{cd}\bar{\gamma}_{ab}\partial_d\right)B\right]A_c - 2A_aA_b \\ &\quad -\frac{2}{3}\bar{\gamma}_{ab}\left[\bar{\gamma}^{cd}\left(\partial_c - A_c\right)A_d - e^{-2B}K\phi - \left(\bar{\Gamma}^c - 2\bar{\gamma}^{cd}\partial_d B\right)A_c\right] \end{aligned} \tag{4.119}$$

where we have applied definition (4.82) together with relations (4.42) and (4.48).

We have now the means to write down the final form of equation (4.111). To this end, we adopt our three results (4.112), (4.113) and (4.119) together with relation (4.42) and the split (4.77), which leads to the $\bar{\boldsymbol{A}}_{\boldsymbol{ab}}$**-evolution equation**

$$\boxed{\begin{aligned} \partial_t\bar{A}_{ab} =& e^{-4B}\Big[{}^3R_{ab} - \partial_{ab} + 2\left(\partial_a B\partial_b + \partial_b B\partial_a\right) + \left(\bar{\Gamma}^c_{ab} - \frac{1}{3}\bar{\gamma}_{ab}\bar{\Gamma}^c\right)\partial_c \\ &\qquad + \frac{1}{3}\bar{\gamma}_{ab}\bar{\gamma}^{cd}\left(\partial_{cd} - {}^3R_{cd} - 4\partial_c B\partial_d\right)\Big]\alpha \\ &+ \alpha\left(K\bar{A}_{ab} - 2\bar{A}_{ac}\bar{A}^c_b\right) + \beta^c\partial_c\bar{A}_{ab} + \bar{A}_{ac}\partial_b\beta^c + \bar{A}_{bc}\partial_a\beta^c - \frac{2}{3}\bar{A}_{ab}\partial_c\beta^c \\ &+ 2\alpha e^{-4B}\Big\{A_aA_b - \frac{1}{2}\left(\partial_a A_b + \partial_b A_a\right) + e^{-2B}\bar{A}_{ab}\phi + \bar{\Gamma}^c_{ab}A_c \\ &\qquad + 2\left(A_a\partial_b + A_b\partial_a\right)B + \frac{1}{3}\bar{\gamma}_{ab}\left[\bar{\gamma}^{cd}\left(\partial_c - A_c - 4\partial_c B\right) - \bar{\Gamma}^d\right]A_d\Big\} \end{aligned}} \tag{4.120}$$

4.6.4 $\bar{\Gamma}^a$-evolution

We come now to the evolution equation of the field $\bar{\Gamma}^a$, for which

$$\begin{aligned} \partial_t\bar{\Gamma}^a &= -2\bar{A}^{ab}\partial_b\alpha + 2\alpha\left(\bar{\Gamma}^a_{bc}\bar{A}^{bc} - \frac{2}{3}\bar{\gamma}^{ab}\partial_b K - \bar{\gamma}^{ab}S_b + 6\bar{A}^{ab}\partial_b B\right) \\ &\quad + \beta^b\partial_b\bar{\Gamma}^a - \bar{\Gamma}^b\partial_b\beta^a + \frac{2}{3}\bar{\Gamma}^a\partial_b\beta^b + \frac{1}{3}\bar{\gamma}^{ab}\partial_{bc}\beta^c + \bar{\gamma}^{bc}\partial_{bc}\beta^a \end{aligned} \tag{4.121}$$

(see equation (11.55) of Baumgarte & Shapiro 2010). The only quantity above to be evaluated here is

$$S_a = -n_\beta t^\beta_a = \alpha t^t_a \tag{4.122}$$

(see equations (2.138) of Baumgarte & Shapiro 2010), where we have used definition (2.117). So, using the world stress-energy tensor (4.98), we have to look at the components

$$t^t_a = \nabla_a A^t + \nabla^t A_a - 2A_a A^t \tag{4.123}$$

in which by the way the matter field λ disappears again. The covariant derivatives appearing above can be rewritten to

$$\nabla_a A^t + \nabla^t A_a = 2g^{t\beta}\nabla_{(a}A_{\beta)} = \frac{2}{\alpha^2}\left(\beta^b\nabla_{(a}A_{b)} - \nabla_{(a}A_{t)}\right) \tag{4.124}$$

where we have applied the 1+3-split (2.119), which is also used in the following without mentioning it explicitly. For the second last covariant derivative above, we use relation (4.107) such that

$$\nabla_{(a}A_{b)} = \partial_{(a}A_{b)} - e^{-6B}K_{ab}\phi - {}^3\Gamma^c_{ab}A_c \tag{4.125}$$

In addition to that, we have

$$\nabla_{(a}A_{t)} = \partial_{(a}A_{t)} - \Gamma^\beta_{at}A_\beta \tag{4.126}$$

for which equations (4.54) and (4.59) give

$$\partial_{(a}A_{t)} = -\alpha\left\{\frac{1}{2}e^{-2B}\overline{\gamma}_{ab}E^b + e^{-6B}\left[\partial_a\phi + \phi\left(\frac{1}{\alpha}\partial_a\alpha - 6\partial_a B\right)\right]\right\} + \beta^b\partial_{(a}A_{b)} + A_b\partial_a\beta^b \tag{4.127}$$

Moreover, using the components (4.71) and (4.104), we find

$$\Gamma^\beta_{at}A_\beta = \Gamma_{tat}A^t + \Gamma_{bat}A^b = \frac{1}{\alpha}e^{-6B}\left(\Gamma_{tat} - \Gamma_{bat}\beta^b\right)\phi + \Gamma_{bat}\gamma^{bc}A_c \tag{4.128}$$

The first Christoffel symbols appearing at the end are

$$\Gamma_{tat} = \frac{1}{2}\partial_a g_{tt} = \frac{1}{2}\partial_a\left(\gamma^{bc}\beta_b\beta_c - \alpha^2\right) = -\frac{1}{2}\beta^b\beta^c\partial_a\gamma_{bc} + \beta^b\partial_a\beta_b - \alpha\partial_a\alpha$$

where we have used the analog

$$\partial_a\gamma^{bc} = -\gamma^{bd}\gamma^{ce}\partial_a\gamma_{de}$$

of relation (4.55) for the 3-metric γ_{ab}. And, the other Christoffel symbols are

$$\Gamma_{bat} = \frac{1}{2}\left(\partial_a g_{bt} + \partial_t g_{ab} - \partial_b g_{at}\right) = \partial_a\beta_b - \Gamma_{cab}\beta^c - \alpha K_{ab} \tag{4.129}$$

based on equation (4.106). Applying

$$\Gamma_{cab}\beta^c\beta^b = \frac{1}{2}\beta^c\beta^b\partial_a g_{cb} = \frac{1}{2}\beta^b\beta^c\partial_a\gamma_{bc}$$

we then obtain

$$\Gamma_{tat} - \Gamma_{bat}\beta^b = \alpha\left(K_{ab}\beta^b - \partial_a\alpha\right) \tag{4.130}$$

We can also use

$$\partial_a\beta_b = \partial_a\left(\gamma_{bc}\beta^c\right) = \beta^c\partial_a\gamma_{bc} + \gamma_{bc}\partial_a\beta^c$$

and thus write the Christoffel symbols (4.129) as

$$\Gamma_{bat} = \gamma_{bc}\partial_a\beta^c + \frac{1}{2}\left(\partial_a\gamma_{cb} - \partial_b\gamma_{ac} + \partial_c\gamma_{ab}\right)\beta^c - \alpha K_{ab} \tag{4.131}$$

The two results (4.130) and (4.131) then change expression (4.128) to

$$\Gamma^\beta_{at}A_\beta = e^{-6B}\left(K_{ab}\beta^b - \partial_a\alpha\right)\phi + \left(\partial_a\beta^c + {}^3\Gamma^c_{ad}\beta^d - \alpha\gamma^{bc}K_{ab}\right)A_c$$

Now, we can go back to equation (4.126), for which our above outcome and relation (4.127) show

$$\begin{aligned}\nabla_{(a}A_{t)} \;&=\; -\alpha\left[\frac{1}{2}e^{-2B}\overline{\gamma}_{ab}E^b + e^{-6B}\left(\partial_a - 6\partial_a B\right)\phi - \gamma^{bc}K_{ab}A_c\right] \\ &\quad +\beta^b\left(\partial_{(a}A_{b)} - e^{-6B}K_{ab}\phi - {}^3\Gamma^c_{ab}A_c\right)\end{aligned}$$

Then, relation (4.125) allows us to write equation (4.124) as

$$\nabla_a A^t + \nabla^t A_a = \frac{2}{\alpha}\left[\frac{1}{2}e^{-2B}\overline{\gamma}_{ab}E^b + e^{-6B}\left(\partial_a - 6\partial_a B\right)\phi - \gamma^{bc}K_{ab}A_c\right]$$

We use this together with the component (4.104) in the world stress-energy tensor (4.123) such that the quantity (4.122) becomes

$$S_a = 2\left[\frac{1}{2}e^{-2B}\overline{\gamma}_{ab}E^b + e^{-6B}\left(\partial_a - 6\partial_a B - A_a\right)\phi - \left(\bar{A}_a^b + \frac{1}{3}\delta_a^b K\right)A_b\right] \tag{4.132}$$

where we have also used relations (4.42) and (4.48) together with the split (4.77). Finally, we go back to equation (4.121) and get the $\overline{\boldsymbol{\Gamma}}^{\boldsymbol{a}}$**-evolution equation**

$$\boxed{\begin{aligned}\partial_t\overline{\Gamma}^a =& -2\bar{A}^{ab}\partial_b\alpha + 2\alpha\left(\overline{\Gamma}^a_{bc}\bar{A}^{bc} - \frac{2}{3}\overline{\gamma}^{ab}\partial_b K + 6\bar{A}^{ab}\partial_b B\right)\\ &+ \beta^b\partial_b\overline{\Gamma}^a - \overline{\Gamma}^b\partial_b\beta^a + \frac{2}{3}\overline{\Gamma}^a\partial_b\beta^b + \frac{1}{3}\overline{\gamma}^{ab}\partial_{bc}\beta^c + \overline{\gamma}^{bc}\partial_{bc}\beta^a\\ &+ 4\alpha\left[e^{-6B}\overline{\gamma}^{ab}\left(A_b + 6\partial_b B - \partial_b\right)\phi + \left(\bar{A}^{ab} + \frac{1}{3}\overline{\gamma}^{ab}K\right)A_b - \frac{1}{2}e^{-2B}E^a\right]\end{aligned}} \tag{4.133}$$

4.6.5 Hamiltonian constraint

We have now found all evolution equations of the BSSN-variables (4.83) up to the gauge variables α and β^a. Before addressing how these gauge variables are fixed by slicing conditions, we will now look at the gravitational constraints. These constraints are the Hamiltonian and momentum constraint, and we begin with the former in this section. The Hamiltonian constraint is

$$\overline{\gamma}^{ab}\overline{\nabla}_a\overline{\nabla}_b e^B - \frac{e^B}{8}\overline{R} + \frac{e^{5B}}{4}\left(\frac{1}{2}\bar{A}_{ab}\bar{A}^{ab} - \frac{1}{3}K^2 + \rho\right) = 0 \tag{4.134}$$

(see equation (11.48) of Baumgarte & Shapiro 2010). In the following, we will go through its terms.

Recalling that the BSSN-derivative $\overline{\nabla}_a$ treats the quantity B and thus also e^B as a scalar, we find

$$\overline{\nabla}_a\overline{\nabla}_b e^B = \overline{\nabla}_a\partial_b e^B = \partial_{ab}e^B - \overline{\Gamma}^c_{ab}\partial_c e^B$$

Using

$$\partial_a e^B = e^B\partial_a B$$

and

$$\partial_{ab}e^B = e^B\left(\partial_{ab}B + \partial_a B\partial_b B\right)$$

we then obtain

$$\overline{\gamma}^{ab}\overline{\nabla}_a\overline{\nabla}_b e^B = e^B\left[\overline{\gamma}^{ab}\left(\partial_{ab}B + \partial_a B\partial_b B\right) - \overline{\Gamma}^a\partial_a B\right] \tag{4.135}$$

where we have used relation (4.82). This is the first term in the constraint (4.134).

The second term contains the scalar

$$\overline{R} = \overline{\gamma}^{ab}\overline{R}_{ab} \tag{4.136}$$

Adopting definition (4.116), we can write it as

$$\boxed{\overline{R} = \overline{\gamma}^{ab}\left\{\overline{\gamma}^{cd}\left[-\frac{1}{2}\partial_{cd}\overline{\gamma}_{ab} + \overline{\Gamma}^e_{ac}\left(2\overline{\Gamma}_{bde} + \overline{\Gamma}_{ebd}\right)\right] + \overline{\Gamma}^c\overline{\Gamma}_{abc}\right\} + \partial_a\overline{\Gamma}^a} \tag{4.137}$$

Note that similar to expression (4.118), we do not write the scalar $\overline{R}$ out in the final form of the Hamiltonian constraint obtained at the end of this section.

In the constraint (4.134), only the quantity ρ is now left and must be written out. So, recalling definition (4.96), we see that we have to evaluate the world stress-energy tensor component t^{tt}, but in contrast to Section 4.6.2 not together with the quantity S. Using definition (4.98), we then have

$$\rho = \alpha^2 \left[2\nabla^t A^t - 2\left(A^t\right)^2 - g^{tt}\left(2\nabla_\alpha A^\alpha + A^2 + \lambda\right)\right]$$

For the terms quadratic in the electromagnetic vector potential, we apply relations (4.54), (4.100) and (4.104) together with the 1+3-split (2.119) such that

$$\begin{aligned} -2\left(A^t\right)^2 - g^{tt} A^2 \quad = \quad & \frac{1}{\alpha^2}\left[-\frac{1}{\alpha^2}\left(-\alpha e^{-6B}\phi + A_a\beta^a\right)^2 + 2\frac{\beta^a}{\alpha^2} A_a \left(-\alpha e^{-6B}\phi + A_b \beta^b\right) \right. \\ & \left. + \left(\gamma^{ab} - \frac{\beta^a \beta^\beta}{\alpha^2}\right) A_a A_b \right] - \frac{2}{\alpha^2} e^{-12B}\phi^2 \end{aligned}$$

and thus

$$\rho = 2\alpha^2 \nabla^t A^t + \gamma^{ab} A_a A_b - 3e^{-12B}\phi^2 + \lambda + 2\nabla_\alpha A^\alpha$$

We have already evaluated the first term on the right hand side above in relation (4.109), which allows us to continue with

$$\rho = e^{-4B}\left[\overline{\gamma}^{ab}\left(A_b + 2\partial_b + 4\partial_b B\right) - 2\overline{\Gamma}^a\right] A_a - \phi\left(3e^{-12B}\phi + 2e^{-6B}K\right) + \lambda \tag{4.138}$$

using relation (4.48). Inserting the two results (4.135) and (4.138) in equation (4.134), we obtain the **Hamiltonian constraint**

$$\boxed{\begin{aligned} \lambda = & e^{-4B}\left\{\frac{\overline{R}}{2} + 2\overline{\Gamma}^a\left(A_a + 2\partial_a B\right) - \overline{\gamma}^{ab}\left[2\left(\partial_a A_b + 2\partial_{ab} B\right) + 4\left(A_a + \partial_a B\right)\partial_b B + A_a A_b\right]\right\} \\ & + \frac{1}{3}K^2 - \frac{1}{2}\bar{A}_{ab}\bar{A}^{ab} + \phi\left(3e^{-12B}\phi + 2e^{-6B}K\right) \end{aligned}} \tag{4.139}$$

4.6.6 Momentum constraint

We proceed with the momentum constraint

$$\overline{\nabla}_b\left(e^{6B}\bar{A}^{ba}\right) - \frac{2}{3}e^{6B}\overline{\nabla}^a K - e^{10B}S^a = 0$$

(see equation (11.49) of Baumgarte & Shapiro 2010, where we have corrected the typo $e^{6\phi}S^a$ in the last term there to $e^{10\phi}S^a$). Using

$$\overline{\nabla}_b\left(e^{6B}\bar{A}^{ba}\right) = e^{6B}\left(6\bar{A}^{ba}\partial_b B + \overline{\nabla}_b \bar{A}^{ba}\right)$$

we can also write it as

$$6\bar{A}^{ab}\partial_b B + \overline{\nabla}_b \bar{A}^{ba} - \frac{2}{3}\overline{\nabla}^a K - e^{4B}S^a = 0$$

For the second term, we find

$$\overline{\nabla}_b \bar{A}^{ba} = \overline{\gamma}^{ac}\overline{\gamma}^{bd}\overline{\nabla}_b \bar{A}_{cd} = \overline{\gamma}^{ac}\overline{\gamma}^{bd}\left(\partial_b \bar{A}_{cd} - \overline{\Gamma}^e_{bc}\bar{A}_{ed} - \overline{\Gamma}^e_{bd}\bar{A}_{ce}\right)$$

and for the third one

$$\overline{\nabla}^a K = \overline{\gamma}^{ab}\partial_b K$$

In addition to that, relations (4.48) and (4.132) give

$$e^{4B}S^a = e^{4B}\gamma^{ab}S_b = e^{-2B}E^a + 2\left[e^{-6B}\overline{\gamma}^{ab}\left(\partial_b - 6\partial_b B - A_b\right)\phi - \left(\bar{A}^{ab} + \frac{1}{3}\overline{\gamma}^{ab}K\right)A_b\right]$$

That way, we arrive at the **momentum constraint**

$$\boxed{\begin{aligned} E^a =&2\left[e^{-4B}\overline{\gamma}^{ab}\left(A_b + 6\partial_b B - \partial_b\right)\phi + e^{2B}\left(\bar{A}^{ab} + \frac{1}{3}\overline{\gamma}^{ab}K\right)A_b\right] \\ &+ e^{2B}\left[\overline{\gamma}^{ac}\overline{\gamma}^{bd}\left(\partial_b \bar{A}_{cd} - \overline{\Gamma}_{ebc}\bar{A}^e_d - \overline{\Gamma}_{ebd}\bar{A}^e_c\right) - \frac{2}{3}\overline{\gamma}^{ab}\partial_b K + 6\bar{A}^{ab}\partial_b B\right]\end{aligned}} \tag{4.140}$$

4.6.7 BSSN-constraints

As we have chosen the BSSN-formalism for the gravitational field equation (3.2), there are three additional constraints. We have already encountered them in equations (4.43), (4.78) and (4.82), but we want to summarize them here. The **BSSN-constraints** are

$$\boxed{\begin{aligned} \overline{\gamma} =&1 \\ \bar{A} =&0 \\ \overline{\Gamma}^a =&\overline{\gamma}^{bc}\overline{\Gamma}^a_{bc}\end{aligned}} \tag{4.141}$$

These constraints were introduced in the BSSN-formalism to stabilize the numeric evolution. Choosing a different formalism, like the ADM-formalism, the above constraints can be removed. So, these constraints are artificial constraints that are not counted as actual constraints of the gravitational field equation. The Hamiltonian and momentum constraint in contrast are always present, independent of how we write the evolution equations. These two constraints can therefore not be removed and are thus true constraints.

4.6.8 Gravitational constraint cure

Looking at the Hamiltonian constraint (4.139) and the momentum constraint (4.140), we see that we have the same situation as for the electromagnetic constraint (4.62). We can use the gravitational constraints to directly compute two evolution variables. For the Hamiltonian constraint (4.139), this is the matter field λ, and the momentum constraint (4.140) gives the electric field E^a.

So, we actually do not need the evolution equations (4.46) and (4.68). Similar to Section 4.5.4, it is by the way clear that these two evolution equations are consistent with the gravitational constraints (4.139) and (4.140). The evolution equation (4.46) was anyway not derived from a field equation. It was an immediate consequence of the definition of the conjugate momentum μ. In contrast, the evolution equation (4.68) was derived from the spatial contravariant components of the electromagnetic field equation (3.8). However, due to Section 3.2.2, we know that this field equation results from the gravitational field equation (3.2). So, consistency is guaranteed.

It is also important that there is no electric field E^a in the Hamiltonian constraint (4.139) and no matter field λ in the momentum constraint (4.140). We can therefore really compute the matter field λ and the electric field E^a from the remaining evolution variables in an analytic manner. That way, we arrive at the following result:

Gravitational constraint cure:

> Electromatter cures the Hamiltonian and momentum constraints present in vacuum general relativity.

The above constraint cure together with the electromagnetic one (4.69) have by the way an important side effect. Let us assume that all three constraints (4.62), (4.139) and (4.140) hold initially in a simulation. This can, for instance, be achieved by choosing the initial values of the fields μ, λ and E^a according to the constraints. As constraint (4.62) depends on the fields λ and E^a, the conjugate momentum μ has to, of course, be computed last. Then, the constraints will also hold when evolved in time with the evolution equations. In general relativity, such a statement is nontrivial and has to be validated explicitly. For that purpose, one has to evaluate the time-derivative of the constraints by using the evolution equations. The constraints are then still obeyed if they were so initially. However, for world theory, such a proof is far simpler. We just have to use the constraints to not only set the values of the fields μ, λ and E^a initially, but also at all remaining points in time.

Note that the gravitational constraint cure in fact needs both the matter field λ and the electromagnetic vector potential A_α. Leaving electromagnetism away, we would have only the matter field λ, a scalar which could never cure the vectorial momentum constraint. If instead there were no matter field λ, then the situation would be a little bit more tricky. The Hamiltonian constraint (4.139) could then, for instance, be used as an equation to compute the electric potential ϕ. Unfortunately, the constraint (4.139) contains that potential both linearly and quadratically such that solving for the potential ϕ will give complex values in some cases. However, the electric potential is real by definition. It does also not help to instead look at the spatial components A_a of the electromagnetic vector potential, as they appear linearly and quadratically, too.

The gravitational constraint cure distinguishes electrogravitation from Einstein-Maxwell theory in a crucial point. As we have just seen, the Hamiltonian constraint (4.139) is not cured in electrogravitation. However, there is at least a cure for the momentum constraint (4.140), because that constraint allows us to directly evaluate the electric field E^a. The difference to Einstein-Maxwell theory is now that the momentum constraint has the form

$$F_{ab}E^b = \cdots$$

there, where the right hand side does not contain electromagnetic evolution variables. We do not derive the above constraint. It is sufficient to see that the momentum constraint of Einstein-Maxwell theory does not allow us to directly compute the electric field E^a, because of the factor F_{ab}. This situation is similar to the factor $F_\alpha{}^\beta$ present in the anomaly (2.6). So, we see that Einstein-Maxwell theory contains an uncured momentum constraint, in contrast to electrogravitation.

4.6.9 Constraint freedom

It is obvious that the matter field λ is required to cure the electromagnetic constraint, and the electromagnetic vector potential A_α is unavoidable for a cure of the gravitational constraints. So, we can get rid of all constraints in world theory, where artificial constraints like the BSSN-constraints mentioned in Section 4.6.7 do not count here:

Constraint freedom:

World theory does not contain constraints. (4.142)

This result is unexpected. There does not seem to be any way to directly deduce the constraint freedom of world theory from initial principles, like demanding that all known classical fields should be geometric properties of spacetime. Instead, the constraint freedom is a feature of world theory that is rather hidden, but elegant and important. It demonstrates a kind of completeness of world theory (see also Section 4 of Gorbatenko 2005 for conformal geometrodynamics).

At this point, it is worth to recall our general approach to world theory. We have explained that approach in the third paragraph of Section 1.3. There, we have realized that anomalies may be caused by a hidden lack of simplicity. The anomaly cured by world theory is of course the general relativistic anomaly 2.6. However, now we also see that there is a lack of simplicity in general relativity which is cured by world theory. It is the presence of constraints.

Similar to the drawing (3.25), we can visualize the development of physics to world theory in the following. Why is it important to become aware of this development? We do not know whether there is a world formula (see Section 3.3.5). Yet, if there is one, this leads to two questions: How does the world formula look like and why does it look like that? In principle, it could be that the world formula is an arbitrary equation. As there are no limits on how to construct an equation, this would lead to infinitely many possibilities. Some of these possibilities are so complicated that it is impossible for us humans to handle them. However, what if the world formula is not arbitrary? What if the world formula is a very special equation within the set of all possible equations? The question would then be, what distinguishes that equation. And, here we come to the following drawing:

$$\begin{array}{rc} & \text{Ignorance} \\ \text{Observation} \longrightarrow & \downarrow \\ & \text{Newtonian gravitation} \\ \text{Coordinate cure} \longrightarrow & \downarrow \\ & \text{General relativity} \\ \text{Constraint cure} \longrightarrow & \downarrow \\ & \text{World theory} \end{array}$$

We do not know whether world theory contains the world formula. Yet, world theory may be a further step in that direction. If this is the case, the world formula would not only be based on the coordinate cure introduced by general relativity but also on the constraint cure of world theory.

Both cures are very general principles. Although not in the early days of physics, it is nowadays pretty clear that coordinate independence is unavoidable. Nature does not know coordinates. Coordinates are artificial entities introduced by thinking beings to describe nature. For the constraint cure, we can argue in a similar manner. As long as we have constraints, our physical equations split into evolution and constraint equations. Actually, the evolution equations are evolutions in time, whereas the constraint equations can be seen as evolutions in space, though typically complicated and hidden ones. So, the constraint cure tells us that nature does not know evolutions in space.

Based on the above considerations, we generally want to postulate that constraints are a sign of incompleteness. If world theory were in fact realized in nature, this statement would at least hold in classical physics. In quantum physics, two additional interactions come into play, the weak and the strong force. Both interactions are governed by constraints (see, e.g., equation (15.51) of Peskin & Schroeder 1997). From the theoretical perspective, it is therefore sound to demand a cure for them. This leads to physics beyond the standard model of particle physics.[12] Unfortunately, there is currently no experimental evidence of such a modification.

4.6.10 Slicing conditions

We now eventually come to the gauge variables, the lapse α and the shift β^a. These variables have to be fixed by slicing conditions. We choose the **1+log-slicing condition**

$$\boxed{\partial_t \alpha = \beta^a \partial_a \alpha - 2\alpha K} \tag{4.144}$$

for the lapse and the **hyperbolic Gamma-driver slicing condition**

$$\boxed{\begin{aligned} \partial_t \beta^a &= b^a \\ \partial_t b^a &= \frac{3}{4} \partial_t \overline{\Gamma}^a - \eta b^a \end{aligned}} \tag{4.145}$$

for the shift (see equations (4.87) and (4.89) of Baumgarte & Shapiro 2010). Both slicing conditions are evolution equations, and the hyperbolic Gamma-driver even introduces the

[12] The constraint of the weak force can be cured in the following way. Let B^A_α be the weak vector potential, with the gauge indices $A, B, \ldots \in \{1, 2, 3\}$. The weak field strength is then $B^A_{\alpha\beta} = \partial_\alpha B^A_\beta - \partial_\beta B^A_\alpha + \epsilon^{ABC} B^B_\alpha B^C_\beta$, where the units of the weak vector potential are chosen such that the weak coupling constant is unity: $g = 1$. The cure of the weak constraint is now achieved by generalizing the field equation $\partial^\beta B^A_{\beta\alpha} + \epsilon^{ABC} B^{B\beta} B^C_{\beta\alpha} = 0$ of the vacuum weak force to the field equation

$$\partial^\beta B^A_{\beta\alpha} + \epsilon^{ABC} B^{B\beta} B^C_{\beta\alpha} = \partial_\alpha \lambda^A + \epsilon^{ABC} B^B_\alpha \lambda^C \tag{4.143}$$

The so-called weak matter field $\lambda^A \in \mathbb{R}$ is a newly introduced field which is not known from the standard model of particle physics. Note that the derivation of the above field equation is unambiguous and the details are straightforward (for the strong force, the corresponding gauge indices and structure constants have to be used instead, which yields a strong matter field).

At first glance, one may easily mistake the weak matter field λ^A for the Faddeev-Popov ghost $c^A \in \mathbb{C}$ (see equation (16.32) of Peskin & Schroeder 1997). However, the ghost field is complex and not real as the new quantity λ^A. Readers wondering whether the field λ^A has anything to do with the Higgs field should take into account the following. Equation (4.143) can be written for, e.g., $A = 1$ as $\Box B^1_\alpha + \ldots = g\left(B^2_\alpha \lambda^3 - B^3_\alpha \lambda^2\right)$. This differs from $\Box B^1_\alpha + \ldots = m^2 B^1_\alpha$ according to the Higgs mechanism, where m is the mass of the vector boson described by the potential B^1_α (see equation (20.62) of Peskin & Schroeder 1997). So, the field λ^A does not make the vector potential B^1_α massive. *(This comment is separated from the remaining text in a footnote and without intermediate mathematical steps, because it concerns elementary particle physics)* new evolution variable b^a. This variable is controlled by the constant η, which is of the order $1/(2M)$. Here, the quantity M is the total mass of spacetime. So, if we have, for instance, just a black hole in an astrophysical context, then the quantity M is simply the mass of that black hole.

It is known from general relativity that numeric stability depends significantly on the chosen slicing conditions. The choice made in this section is typical for astrophysical applications of vacuum gravitation. We therefore chose these slicing conditions as the starting point for world theory. However, I emphasize that our whole approach to numeric world theory is merely a demonstration that numerics works at all for that theory. I in fact expect that some modifications of our evolution equations are required for applications beyond vacuum gravitation.

4.6.11 Physical metric

A consequence of conformal gauge invariance is that the metric $g_{\alpha\beta}$ can be determined only up to a transformation $g_{\alpha\beta} \overset{\text{cg}}{\to} e^{-2\chi} g_{\alpha\beta}$, see box (2.70). However, the metric perceived in nature appears to be fixed, except, of course, the unavoidable coordinate freedom. Let us call this metric, which is in fact the one used in general relativity, the **physical metric** $\tilde{g}_{\alpha\beta}$. So, what is the relationship between the metric $g_{\alpha\beta}$ and the physical metric $\tilde{g}_{\alpha\beta}$? As conformal gauge invariance implies $\lambda \overset{\text{cg}}{\to} e^{2\chi}\lambda$ due to box (3.3), one of the easiest possibilities is to assume

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} \frac{\lambda}{\Lambda} \tag{4.146}$$

as then

$$\tilde{g}_{\alpha\beta} \overset{\text{cg}}{\to} \tilde{g}_{\alpha\beta}$$

The reason for the usage of the cosmological constant $\Lambda > 0$ in the denominator is explained further below. Unfortunately, there is no real justification of the relationship (4.146). There are infinitely many ways to construct a conformally gauge invariant, physical metric. For example, due to the results (2.97) and (2.71), $\frac{1}{\lambda^2} F_{\alpha\beta} F^{\alpha\beta} \overset{\text{cg}}{\to} \frac{1}{\lambda^2} F_{\alpha\beta} F^{\alpha\beta}$. Hence, multiplying the right hand side of equation (4.146) by an arbitrary function of the expression $\frac{1}{\lambda^2} F_{\alpha\beta} F^{\alpha\beta}$ yields a valid alternative physical metric. Therefore, the choice (4.146) should be taken with a pinch of salt. However, it is a very easy choice and therefore worth to be studied further.

Let us now investigate the implications of replacing the metric $g_{\alpha\beta}$ with the physical metric $\tilde{g}_{\alpha\beta}$. Such a replacement can be achieved by applying a conformal gauge transformation. To this end, we divide spacetime into three, not necessarily connected regions, where $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. For the region $\lambda > 0$, the conformal gauge transformation $\chi = \frac{1}{2} \ln \frac{\Lambda}{\lambda}$ causes

$$g_{\alpha\beta} \overset{\text{cg}}{\to} g^{\bullet}_{\alpha\beta} = e^{-2\chi} g_{\alpha\beta} = g_{\alpha\beta} \frac{\lambda}{\Lambda} \tag{4.147}$$

Hence, the metric $g^{\bullet}_{\alpha\beta}$ resulting from the conformal gauge transformation is the physical metric according to definition (4.146). The electromagnetic vector potential becomes

$$A_\alpha \overset{\text{cg}}{\to} A^{\bullet}_\alpha = A_\alpha + \partial_\alpha \chi = A_\alpha - \frac{\partial_\alpha \lambda}{2\lambda} \tag{4.148}$$

due to box (2.70). The expression $A_\alpha - \partial_\alpha \lambda/(2\lambda)$ is conformally gauge invariant. Therefore, the result (4.148) tells us that the electromagnetic vector potential $A^{\bullet}_\alpha$ is a special vector. Let us call it **physical electromagnetic vector potential**. Eventually, the matter field transforms according to

$$\lambda \overset{\mathrm{cg}}{\to} \lambda^{\bullet} = e^{2\chi}\lambda = \Lambda \tag{4.149}$$

So, we get a constant value here. And, it is obviously most natural to demand the cosmological constant Λ, which requires the usage of it in the denominator of the relationship (4.146).

What about the region where $\lambda = 0$. In this region, the idea (4.146) is not applicable. The physical metric would be zero, which is not a reasonable metric. So, if the relationship (4.146) is truly the right one, then there is no physical metric in the region $\lambda = 0$. The region $\lambda < 0$ also causes trouble. Here, we proceed as follows. Due to equations (3.1), (1.17), (1.16), (1.14), (1.13) and (1.4), the world equation and thus world theory is invariant under the transformation $g_{\alpha\beta} \to -g_{\alpha\beta}$ and $\lambda \to -\lambda$. We perform first this transformation and then the conformal gauge transformation $\chi = \frac{1}{2}\ln\frac{\Lambda}{\lambda}$ in the region $\lambda < 0$. The problem is then that the metric $g^{\bullet}_{\alpha\beta}$ in line (4.147) does not have the signature $(-,+,+,...)$ demanded in Section 1.6.1. Instead, we have now the signature $(+,-,-,...)$. However, we have to just look at the relationship (4.146) to see that both sides cannot have the same signature for $\lambda < 0$. So, either we accept that also the region $\lambda < 0$ fails. Or, we are less restrictive about the signature and allow regions where it is $(+,-,-,...)$. This has to be done for at least either the metric $g_{\alpha\beta}$ or the physical metric $\tilde{g}_{\alpha\beta}$.

Anyway, the key step to get to the physical metric is the result (4.149). However, now we can proceed in a different manner. Instead of starting with an arbitrary solution $g_{\alpha\beta}, A_\alpha, \lambda$ of the world equation and then going to the transformed fields $g^{\bullet}_{\alpha\beta}, A^{\bullet}_\alpha, \lambda^{\bullet}$, we can also simply demand $\lambda = \Lambda$. This is a gauge condition which automatically implies the metric $g_{\alpha\beta}$ to be the physical one, up to the issues of the last paragraph. In so doing, equation (3.2) becomes

$$G_{\alpha\beta} = \nabla_\alpha A_\beta + \nabla_\beta A_\alpha - 2A_\alpha A_\beta - g_{\alpha\beta}\left(2\nabla_\gamma A^\gamma + A^2 + \Lambda\right) \tag{4.150}$$

For equation (3.8), we find

$$\nabla^\beta F_{\beta\alpha} = -2A_\alpha\Lambda \tag{4.151}$$

So, the electromagnetic vector potential has itself as the source, up to the constant -2Λ. Finally, equation (3.11) becomes

$$\nabla^\alpha A_\alpha = 0 \tag{4.152}$$

which is just the Lorentz gauge (see equations (60) of Gorbatenko 2009, up to the limitation to flat spacetime there). Fortunately, this fits to our earlier choice (4.70).

A side note: In the three equations (4.150), (4.151) and (4.152), not only the metric $g_{\alpha\beta}$ is the physical version but also the electromagnetic vector potential A_α. It is therefore even possible to write world theory in a gauge invariant way, such that we get box (2.65). We have to just replace the physical electromagnetic vector potential A_α with the expression $A_\alpha + \partial_\alpha l$. And, then we obtain gauge invariance for the electromagnetic vector potential A_α, i.e. $A_\alpha \overset{\mathrm{g}}{\to} A_\alpha + \partial_\alpha\chi$, by demanding that the newly introduced scalar l transforms according to $l \overset{\mathrm{g}}{\to} l - \chi$. The field equation of the scalar l is then by the way given via equation (4.152), because it becomes $\nabla^\alpha\left(A_\alpha + \nabla_\alpha l\right) = 0$. However, as we have to anyway choose a gauge condition for the numeric evolution, we stick to the form (4.150), (4.151) and (4.152). The introduction of the scalar l is just necessary in case one wants the physical metric but not the Lorentz gauge (4.152).

All in all, we can thus now say the following. The only thing we have to do in the numeric formalism presented in this paper to get to the simple physical metric according to relationship (4.146) is to apply the additional constraint

$$\boxed{\lambda = \Lambda} \tag{4.153}$$

The metric $g_{\alpha\beta}$ is then the physical metric. Note that condition (4.153) is not obligatory. It is like a special choice of coordinates. One may just as well ignore the constraint (4.153). Then, the metric $g_{\alpha\beta}$ is not the one measured in nature with rods and clocks, i.e. it is a nonphysical metric. However, it is a valid alternative way to represent space and time.

Unfortunately, although appearing very easy at first glance, the constraint (4.153) is far more involved than the previously encountered ones. It is not possible to get it obeyed at the beginning of a simulation like, for instance, the Hamiltonian constraint. For the latter, we have to use just relation (4.139) to directly evaluate the matter field λ. However, we can now not simply use equation (4.153) to set the matter field. The problem is that then the Hamiltonian constraint (4.139) may be violated. And, inserting $\lambda = \Lambda$ into the Hamiltonian constraint (4.139) yields a complicated equation, which cannot be solved so readily. Additionally, the conjugate momentum μ would have to be studied in this context. To avoid all these issues, it is best to ignore the constraint (4.153) when setting up and running a simulation. And then, if desired, one retrieves the physical metric and the associated fields simply via the conformal gauge transformation $\chi = \frac{1}{2}\ln\frac{\Lambda}{\lambda}$ mentioned earlier. Due to Sections 2.6.6 and 2.6.7 we by the way remember that the electric field E^a and the magnetic one B^a are unaffected by a conformal gauge transformation. Hence, these two fields are already given from the perspective of the physical metric.

4.6.12 Summary

Let us summarize our approach to numeric world theory in this section. We use the following **13 evolution variables in our formalism**

λ	μ	A_a	E^a	$[\phi]$	B	$\overline{\gamma}_{ab}$	K	$\bar{A}_{ab}$	$\overline{\Gamma}^a$	$[\alpha]$	$[\beta^a]$	$[b^a]$
4.46	4.56	4.59	4.68	4.72	4.84	4.85	4.110	4.120	4.133	4.144	4.145	4.145
4.139	4.62		4.140									

(4.154)

The squared brackets have the same meaning as in Section 4.5.6, i.e. they denote gauge variables. The numbers are the references to the equations required to compute the respective evolution variables (summarized in the Numeric formulary). The first row of references are evolution equations and the second one direct computation equations. So, for the three fields λ, μ and E^a, we can respectively choose either an evolution equation or a direct computation. The choice does, in principle, not matter, except that numeric stability may depend on it. Note that the conjugate momentum μ appears on the right hand side of only equations (4.46) and (4.56). Therefore, if we decide for the direct computation (4.139) of the matter field λ, then we can leave the conjugate momentum μ away as an evolution variable.

The above 13 evolution variables are also governed by constraints. We always have the three BSSN-constraints (4.141). Depending on the choice made in box (4.154), we may additionally have one or more of the constraints given in the second row of numbers of that box. These are the Hamiltonian, electromagnetic and momentum constraints

(4.139), (4.62) and (4.140). So, if we, for instance, decide for the evolution equation (4.46) of the matter field λ, then we also have to care for the Hamiltonian constraint (4.139). However, if we instead compute the matter field λ directly with that constraint such that the evolution equation (4.46) becomes superfluous, the meaning of the Hamiltonian constraint changes. It is no longer considered as a constraint. Instead, it is a direct computation equation for the matter field λ. This variable is then also no longer an evolution variable but an ancillary field.

Whatever constraints we have, they have to be obeyed at the beginning of a numeric evolution. In addition to that, we have to supervise whether and how strongly they are violated when performing a simulation. In general relativity, the initial constraints typically require some extra amount of work. Looking at the term $\overline{\gamma}^{ab}\partial_{ab}B$ in the Hamiltonian constraint (4.139), we, for example, see that this constraint is an elliptic differential equation. So, we have to solve such a differential equation here in general relativity. Fortunately, there are no true constraints in world theory due to box (4.142). There are merely formalism-dependent constraints, which can be addressed much easier. Let us have a closer look at all this. If we, for instance, decide for the evolution equation (4.46), then we have to obey the Hamiltonian constraint (4.139) initially. However, it is a trivial task to get that constraint obeyed at the beginning of a simulation. We solely have to set the matter field λ according to the direct computation equation (4.139). The same method holds for the electromagnetic and momentum constraint. What about the BSSN-constraints (4.141)? For the third constraint there, we merely have to set the initial value of the evolution variable $\overline{\Gamma}^a$ by just using that constraint as a direct computation of the variable $\overline{\Gamma}^a$. That way, only the other two constraints of box (4.141) remain. These constraints are far simpler than something like an elliptic differential equation. All in all, setting up initial models for world theory is therefore a straightforward task, in contrast to general relativity.

4.6.13 Degrees of freedom

Let us finally count the minimal number of evolution variables in world theory. First, we choose the Hamiltonian constraint (4.139) in box (4.154) to directly compute the matter field λ. As already mentioned in the last section, that way the conjugate momentum μ becomes superfluous. The matter field λ itself is then an ancillary variable. Inserting its analytic expression in all other involved equations, we can also remove the matter field λ. We proceed in the same way for the electric field E^a. So, we take the momentum constraint (4.140), and insert this expression of the electric field E^a in the other equations. That way, the electric field E^a does no longer play a role. We can also remove the quantity $\overline{\Gamma}^a$. This evolution variable was introduced in the BSSN-formalism to stabilize the numeric evolution. Ignoring stability issues, we can write the variable $\overline{\Gamma}^a$ in terms of the BSSN-metric $\overline{\gamma}_{ab}$ according to equation (4.82). For a more compact description, we return to equations (4.42) and (4.77). So, we can unify the fields B and $\overline{\gamma}_{ab}$ again to the 3-metric γ_{ab}. In addition to that, the fields K and $\bar{A}_{ab}$ can be merged to the exterior curvature K_{ab}. We by the way have the evolution variables of the ADM-formalism now (see equations (2.134) and (2.135) of Baumgarte & Shapiro 2010). The final reduction concerns the gauge variable b^a. This field is merely required for the hyperbolic Gamma-driver slicing condition (4.145). We could also choose a different slicing condition for which the shift β^a is sufficient. In total, we then arrive at the following **minimal number of**

evolution variables

$$\boxed{A_a, \gamma_{ab}, K_{ab}, [\phi], [\alpha], [\beta^a]}$$

of world theory.

Note that the gauge fields in the squared brackets above could also be set without evolution equations. Anyway, the three fields A_a, γ_{ab} and K_{ab} have to be evolution variables and contain $3+6+6=15$ independent components, because the quantities γ_{ab} and K_{ab} are symmetric. So, world theory has 15 degrees of freedom (without constraints) and is governed by the same number of evolution equations. In addition to that, there are the gauge variables ϕ, α and β^a with $1+1+3=5$ independent components. Due to world invariance, solutions differing merely by a combination of a conformal gauge transformation and a coordinate transformation, a so-called **world transformation**, are physically equivalent. The number of true dynamic degrees of freedom of world theory is therefore $15-5=10$, just like the number of components of the symmetric world tensor $\Omega_{\alpha\beta}$. Thinking not in terms of equations that are first order in time but second order, the number of degrees of freedom reduces to $10/2=5$.

The number 5 splits into degrees of freedom for gravitation, electromagnetism and matter. For vacuum gravitation, we have the evolution variables γ_{ab}, K_{ab}, α and β^a. Not taking the gauge variables α and β^a into account, this gives $6+6=12$ components. That number is then reduced by the Hamiltonian and momentum constraint present in vacuum general relativity, which are described by $1+3=4$ components. Moreover, solutions differing by a coordinate transformation are physically the same. Hence, vacuum gravitation consists of $12-4-4=4$ degrees of freedom in first order or $4/2=2$ in second order in time. For vacuum electromagnetism in flat spacetime, we proceed in the same way. There, we have the evolution variables A_a, E^a and ϕ. The electric potential ϕ is a gauge variable, and the other two fields have $3+3=6$ components. The electromagnetic constraint (4.63) then causes a reduction by 1 and gauge invariance by another 1 such that we get $6-1-1=4$. This gives $4/2=2$ degrees of freedom in second order in time. Eventually, we look at matter without gravitation and electromagnetism. In that case, the evolution variables are λ and μ. However, there are no constraints nor invariances under any transformations, like gauge invariance or covariance. Hence, we have $1+1=2$ components, which leads to $2/2=1$ degree of freedom in second order in time. All in all, we therefore see that the 5 degrees of freedom of world theory split into 2 gravitational, 2 electromagnetic and 1 material degree of freedom.

4.7 Alpha code

4.7.1 Description

In the last sections, we have derived a formalism to perform numeric simulations of world theory. This formalism is based on the variables and equations summarized in box (4.154). It is first and foremost important to realize that our formalism is well-defined, i.e. we have not encountered any problems above. This is not automatically guaranteed. Writing down an arbitrary field equation could always lead to problems like strange constraints or missing evolution equations for fields that should not be gauge variables. However, despite the lack of any problems, it is still reasonable to go one step further and actually perform simulations.

For that purpose, I have created a new program from scratch, the "Alpha-code". Note that the name of that code is not a sign of incompleteness. The code is not an alpha version. Instead, the first letter in the Greek alphabet “Alpha” is used in the metaphorical sense of “beginning”. So, it stands for the first attempt to simulate world theory (as far as I know, conformal geometrodynamics has not yet been simulated). Simultaneously, it emphasizes that the code deals with fundamental physics, where all the phenomena of nature have their beginning.

The Alpha-code is written in C++ under Linux, and it is heavily based on using the GPU (graphics processing unit). The GPU is used for two purposes. The first one is computation. The equations summarized in box (4.154) are directly evaluated on the GPU via OpenCL. These computations are based on the following settings. We use 3-dimensional Cartesian grids with periodic boundary conditions and finite differences. The time-integrations required for that approach can be done with either forward Euler or fourth order Runge-Kutta integration. For the space-integrations, second or fourth order may be chosen. The numeric precision can be single or double.

The second purpose of the GPU is visualization. The Alpha-code contains an OpenGL interface based on OpenCL interoperability. This approach allows the user to directly supervise the evolution of the fields (4.154) in realtime. So, one merely has to start the simulation by pressing a key and can then observe the evolution of the fields until that key is pressed again. It is possible to observe the visualized fields during the evolution in various ways. The fields can be rotated, zoomed in and out, and so on. The constraints belonging to the fields (4.154) are also visualized. All in all, the advantage and reason for the OpenGL visualization is that it allows a rapid investigation of the simulated models. One can, for instance, immediately see where and how instabilities occur.

The number of frames visualized per second depends mostly on the chosen resolution. The Alpha-code is currently simulated on an NVidia GTX 970 GPU with 4 GB graphics memory. For resolutions around 100^3 or below, the simulations run without lags. The maximal resolution is set by the GPU memory and limited to about 270^3 for the mentioned type of GPU (for forward Euler time-integration, second order space-integration and single-precision). As the power of GPUs will massively increase in the future (and has already gone beyond my GTX 970), these limitations are not a real problem. We are anyway not interested in performing simulations with large resolutions. The primary goal is to simulate world theory at all for the first time. After that, we are interested in studying the numeric stability of our formalism (4.154).

The Alpha-code is able to simulate the individual cases shown in box (3.81). In addition to that, I have also implemented the field equations (1.15) and (1.20) of Einstein-Maxwell theory. The evolution equations belonging to that theory are not written down in this paper. Deriving them is straightforward and does not contain new knowledge. The implementation of Einstein-Maxwell theory allows us to compare that theory with electrogravitation in the following section. That way, we can even numerically verify that both theories are in fact different.

4.7.2 Einstein-Maxwell theory versus electrogravitation

For the comparison of Einstein-Maxwell theory and electrogravitation, we consider the following model. We take a uniform Cartesian grid with the resolution 150^3 and the edge length 1000. Note that we do not express model properties like this length in terms of familiar units, for instance meters. The reason is that the models presented in this paper show only qualitative considerations and not quantitative ones, which could then be actually observed in nature. For a quantitative comparison, it would be unavoidable to find out the value of the electromagnetic constant c_{EM} in standard units, like SI-units.

Anyway, the chosen resolution and edge length lead to a grid cell size of $\Delta x \approx 6.7$ $(= 1000/150)$. The corresponding timestep size is $\Delta t \approx 0.2$ $(= 0.05 \cdot 6.7/\sqrt{3})$, and it is determined by the Courant number 0.05 (the Courant number is defined as $\sqrt{D-1}\Delta t/\Delta x$ here, see also equation (19.3.11) of Press *et al.* 2002). The maximally allowed Courant number which yields a stable simulation (crashes due to other reasons are not excluded here, see further below) is determined for simplicity by trial and error. The value 0.05 is below that maximum. Moreover, we use forward Euler time-integration, second order space-integration and single precision. These settings are sufficient for the comparison shown in this section. Concerning the options available in box (4.154), we can either choose equation (4.68) such that the electric field E^a in electrogravitation is evaluated with an evolution equation. The alternative is the direct computation equation (4.140). Note that the choice can have an influence on the numeric stability, as already mentioned in Section 4.6.12. However, we are not interested in a quantitative study of this influence here, for simplicity. Instead, we just take into account the following. In Einstein-Maxwell theory, the electric field E^a can be computed only with an evolution equation and not a direct computation. To make the comparison between Einstein-Maxwell theory and electrogravitation as meaningful as possible, we have to decide for the same procedure in both theories. We therefore evaluate the electric field E^a in electrogravitation with the evolution equation (4.68).

The initial configuration chosen for the comparison of Einstein-Maxwell theory and electrogravitation is the following one. We take a flat spacetime, which leads to the initial values

$$\begin{aligned} B, K, \bar{A}_{ab}, \overline{\Gamma}^a, \beta^a, b^a &\overset{\text{i}}{=} 0 \\ \overline{\gamma}_{ab} &\overset{\text{i}}{=} \operatorname{diag}(1,1,1) \\ \alpha &\overset{\text{i}}{=} 1 \end{aligned} \tag{4.155}$$

of the gravitational evolution variables shown in box (4.154). Note that the letter "i" emphasizes initial values. For the electromagnetic evolution variables, we use

$$\begin{aligned} \phi, E^a, A_x, A_y &\overset{\text{i}}{=} 0 \\ A_z &\overset{\text{i}}{=} 5 \cdot 10^{-2} e^{-\left(\frac{r}{100}\right)^2} \end{aligned} \tag{4.156}$$

where the radius r is the distance from the grid center. So, the non-vanishing value of the component A_z prevents our model from just being an undeformed spacetime. Also, note that we do not consider the evolution variables λ and μ in this section, because they are not part of Einstein-Maxwell theory. For electrogravitation, we therefore use $\lambda = \mu = 0$ at all points in time.

What about the initial constraints of our model? Without matter, the electromagnetic constraint (4.62) of electrogravitation reduces to $\partial_a E^a \overset{\text{i}}{=} 0$. So, electrogravitation contains true initial constraints, in contrast to world theory. The mentioned constraint has by the way the same form as the electromagnetic constraint of Einstein-Maxwell theory. The constraint is therefore obeyed for both theories due to our choice $E^a \overset{\text{i}}{=} 0$. It is also evident that the momentum constraint (4.140) of electrogravitation is fulfilled by our initial values. For Einstein-Maxwell theory, the momentum constraint has a form that differs from equation (4.140). We do not show that form in this paper, but the momentum constraint is obeyed even for Einstein-Maxwell theory. That way, merely the Hamiltonian constraint remains, because the BSSN-constraints (4.141) are anyway valid.

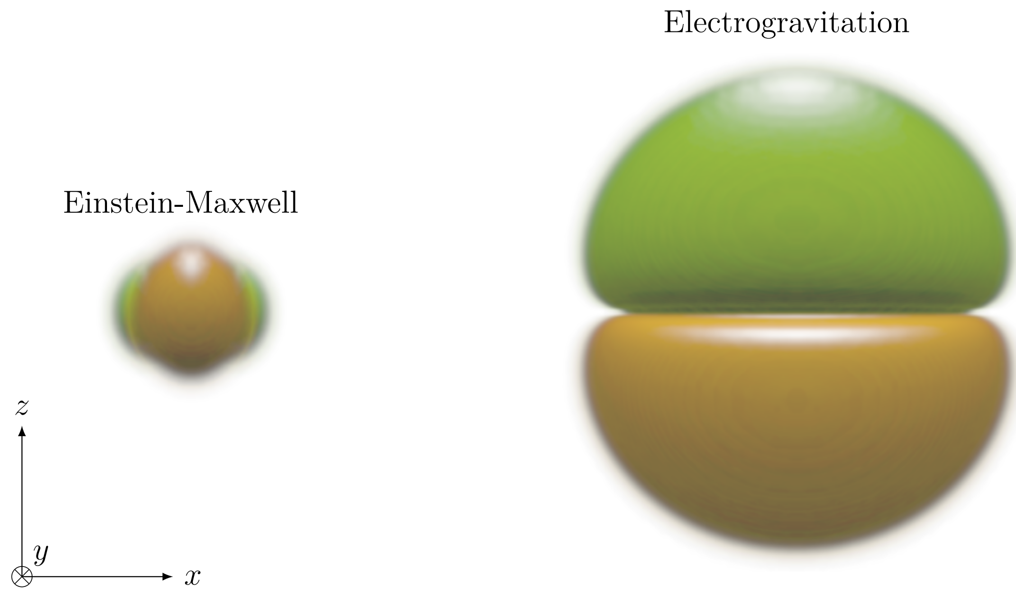

Figure 2: **Comparison of Einstein-Maxwell theory and electrogravitation**: The two plots show the BSSN-metric component $\overline{\gamma}_{xx}$ of Einstein-Maxwell theory (left) and electrogravitation (right) for the model described in the text of Section 4.7.2 at the timestep 100. The visualization works in the following manner. We consider the expression $5 \cdot 10^4\,(\overline{\gamma}_{xx} - 1)$, and show negative values in brown and positive ones in green color. The 3-dimensional objects seen above are the less transparent, the larger the absolute value of the mentioned mathematical expression. For a better volumetric visualization, the objects are also lit by light coming from above. This explains the white shimmer and the darkness. Note that both plots are shown with exactly the same visualization settings, i.e. the zoom factors and so on do not differ. The y-axis of the perspective projection by the way points into the figure plane (visualized above by the circle with a cross inside).

Looking at equation (4.139), we realize that the Hamiltonian constraint is violated for electrogravitation. This constraint is also violated for Einstein-Maxwell theory, where, similar to the momentum constraint, it has a form that differs from equation (4.139). However, these two constraint violations have to be accepted, because Einstein-Maxwell theory and electrogravitation are governed by differing initial constraints. So, it is surely possible to find initial values that fit to the one theory, but there is no guarantee that they also work for the other one. There might be some special cases where nontrivial initial values fit to both theories, but finding such cases does not seem to be straightforward. For our simple comparison of Einstein-Maxwell theory and electrogravitation, it is therefore sufficient to accept the violation of the Hamiltonian constraint.

Having set up the initial configuration, we come now to the actual simulation. We perform 100 timesteps on the one hand for Einstein-Maxwell theory and on the other one for electrogravitation. It is not possible to perform a much larger number of timesteps for the comparison, because our model begins to crash at the timestep 115 for electrogravitation. This crash can neither be cured by choosing a smaller Courant number (the crash occurs then of course at a later timestep, because the timestep size Δt is smaller, but the simulation time of the crash is roughly the same) nor going to fourth order Runge-Kutta time-integration, fourth order space-integration or double precision. Even using all these four options at once does not help (the resolution must be lowered to 140^3 in this case, because now the memory required per grid point is higher than before and for 150^3 it would exceed the GPU memory). Choosing the direct computation (4.140) instead of the evolution equation (4.68), the beginning of the crash can be delayed to the timestep 135. Einstein-Maxwell theory in contrast has no problem with much larger numbers of timesteps. The outcome of the simulations at the timestep 100 is shown in Figure 2. The plots presented there are just two screenshots of the realtime OpenGL interface of the Alpha-code. The right plot does by the way not change significantly when switching between the direct computation (4.140) and the evolution equation (4.68). Also, note that we merely compare the BSSN-metric component $\overline{\gamma}_{xx}$ in Figure 2 for both theories, because it is already evident in this manner that for electrogravitation the electromagnetic vector potential A_α has a different influence on the metric $g_{\alpha\beta}$ than for Einstein-Maxwell theory. The influence is in fact significantly stronger in case of electrogravitation. The metric in contrast influences the electromagnetic vector potential in exactly the same way for the two compared theories, because both are governed by the same electromagnetic field equation (1.15). For at least the model considered in this section and ignoring the mentioned problem with the electromagnetic constant c_{EM}, we can therefore say that **electromagnetism couples stronger to gravitation if we switch from Einstein-Maxwell theory to electrogravitation**.

The reason is evident. Einstein-Maxwell theory is based on Einstein's equation (1.20), where the stress-energy tensor contains only terms quadratic in the electromagnetic vector potential A_α. For electrogravitation in contrast, the stress-energy tensor is also composed of terms linear in that potential, which can be seen in equation (2.21). Electromagnetic vector potential values far below unity, like our initial values (4.156), are therefore reduced less for electrogravitation when coupling to gravitation.

4.7.3 Simulation of world theory

Let us now include the matter field λ and that way in fact simulate world theory for the first time. The advantage of world theory is that we do not have to care for true initial constraints. There are merely the BSSN-constraints (4.141). We fulfill them by taking the flat spacetime (4.155) of the last section. That way, just the initial values of the electromaterial evolution variables λ, μ, A_a, E^a and ϕ remain in box (4.154). By choosing the direct computation equations (4.139), (4.62) and (4.140) in that box, the initial values of the variables λ, μ and E^a are set automatically. Hence, only the two evolution variables A_a and ϕ remain.

Our goal is now to construct a very simple model. Note that we are not interested here in the question what kind of physical phenomenon is described by that model. So, we do not ask whether our model represents, for instance, a star or an elementary particle. The problem is simply that we neither know the distribution of the matter field λ inside of astrophysical or other objects nor do we know how far world theory is valid when approaching the microcosm.

The easiest way to construct such a starting model is to set

$$A_a \overset{\text{i}}{=} 0$$

That way, the model is completely defined by the electric potential ϕ. Looking at the constraints (4.139), (4.140) and (4.62), we by the way see that the matter field, the

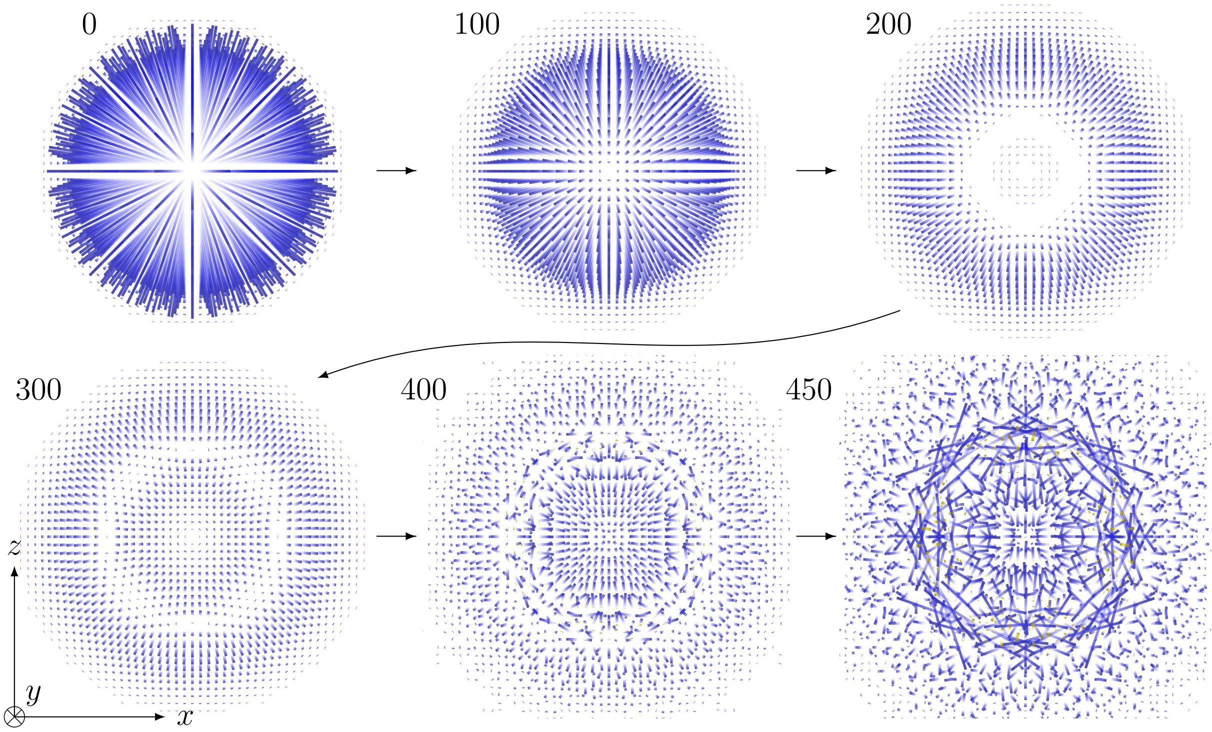

Figure 3: **First simulation of world theory**: The plots show the electric (blue) and magnetic field (yellow) for the model described in Section 4.7.3 at the timesteps specified by the numbers above. The magnetic field is nearly not visible, only in the last plot it can be spotted somewhat. The visualization works in the following way. We limit ourselves to the (x, z)-plane through the center of the Cartesian grid. In that plane, we visualize vectors on every second grid point. The heads of the vectors are by definition the darker sides of the lines shown above. Due to the symmetry of the considered model, the electric field vectors lie in the (x, z)-plane, and the magnetic ones are perpendicular to it. The magnetic field is still visible at the timestep 450, because we do not use an orthogonal but a perspective projection.

conjugate momentum and the electric field are then initially

$$
\begin{aligned}
\lambda &\overset{i}{=} 3\phi^2 \\
\mu &\overset{i}{=} -2\left(\Delta + 3\phi^2\right)\phi \\
E^a &\overset{i}{=} -2\partial_a\phi
\end{aligned}
\tag{4.157}
$$

So, choosing, for instance, the distribution $\phi = q/(2r)$, we obtain the electric field E^a of a charged point particle at the beginning of our simulation. Unfortunately, singular distributions like $1/r$ can cause numeric problems such that we instead decide for the smoother version

$$\phi \overset{i}{=} 5 \cdot 10^{-2} e^{-\left(\frac{r}{100}\right)^2}$$

All other parameters required to define the above model, like the grid resolution (up to a marginal change to 151^3 for a better visualization) and so on, are equal to those of the last section. Figure 3 then presents the temporal evolution of our model. The plots in that figure are again just screenshots of the Alpha-code OpenGL interface. The top, left plot shows our initial configuration (4.157) of the electric field. During the first half of the simulation, that field reduces in strength, especially in the grid center. After that, it appears as if a spherical wave begins to develop. Unfortunately, the electric field starts to grow dramatically then, and the simulation crashes somewhat beyond the timestep 450. The last two plots in the second row of Figure 3 show some nontrivial patterns. They appear at about the timestep 400, which is much larger than our grid resolution 150 along one dimensions. That way, waves can propagate multiply through the grid until the mentioned patterns begin to develop. It is therefore possible that these patterns are the result of the anisotropy caused by the Cartesian grid with periodic boundary conditions. However, they may also just be numeric problems. Switching to fourth order Runge-Kutta time-integration, fourth order space-integration or double precision (each of the three options is activated separately) does not prevent the crash of the simulation. Also, a smaller Courant number does not lead to a stabilization. However, choosing the two evolution equations (4.46) and (4.68) in box (4.154) instead of the direct computations (4.139) and (4.140) (the conjugate momentum μ is still evaluated by the direction computation (4.62)) increases the timestep where the crash occurs by about a factor two. So, evolution equations appear to be more stable here, in contrast to the model of the last section where it was just vice versa.

Anyway, I currently do not know what is actually going on just before the crash of the simulation occurs. The reason may be numeric, but it could also be a problem of our formalism and especially the slicing conditions. Another possibility is that the electric field has to really grow for our model and that some kind of collapse occurs. This would not be unexpected for a theory that contains vacuum general relativity. So, how does the metric look like? Is the model itself reasonable at all? What does it mean physically? We begin to realize that much more work is required here to really obtain a physically interpretable and then maybe observable simulation result. However, that way we would leave the realm of the uttermost fundamental physics and the scope of this paper. The reader has at least gained a first glance at numeric world theory here, and it is also clear now where future research in that theory is necessary.

5 Conclusion

5.1 World theory

5.1.1 Fields and field equation

We have developed world theory in two successive steps in this paper. The first one was the geometric unification of electromagnetism and gravitation to electrogravitation. The second one was the geometric unification of electrogravitation and a scalar form of matter. Electrogravitation is a theory on its own. It may be realized in nature even if the second step plays no role there. In the following, we will summarize world theory in a manner as compact as our introduction to electromagnetism and gravitation at the beginning of this paper (Sections 1.5 and 1.6.1). For simplicity, we limit ourselves to four dimensions throughout this chapter.

World theory is based on the three fundamental fields

$$g_{\alpha\beta}, A_\alpha, \lambda$$

which are the metric, the electromagnetic vector potential and the matter field. These fields describe the geometry of spacetime, which is the actual unified field. To get to the unified field equation governing that geometry, we have to introduce two ancillary quantities. The unification of gravitation and electromagnetism to electrogravitation is done in the electrogravitational connection

$$I^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma} \right) - \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma}$$

(equation (3.32)). For the unification of electrogravitation and matter to world theory, we use the world tensor

$$\Omega_{\alpha\beta} = \partial_\gamma I^\gamma_{\alpha\beta} - \partial_\beta I^\gamma_{\alpha\gamma} + I^\gamma_{\delta\gamma} I^\delta_{\alpha\beta} - I^\gamma_{\delta\beta} I^\delta_{\alpha\gamma} - g_{\alpha\beta}\lambda$$

(equation (3.33)). The world equation is then

$$\Omega_{\alpha\beta} = 0$$

5.1.2 Geometric interpretation

World theory is a purely geometric theory based on three distinct geometric phenomena. These phenomena express different ways how spacetime can be manipulated. We call manipulations of such a kind **deformations**. One of these deformations is already known from general relativity, namely gravitation. Gravitation is the interior curvature of spacetime. It is based on the metric $g_{\alpha\beta}$ and characterized by the curvature tensor

$$R^\alpha{}_{\beta\gamma\delta} = 2 \left(\partial_{[\gamma} \Gamma^\alpha_{\beta\delta]} + \Gamma^\alpha_{\epsilon[\gamma} \Gamma^\epsilon_{\beta\delta]} \right)$$

where the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ are just the electrogravitational connection $I^\alpha_{\beta\gamma}$ for the limit $A_\alpha = 0$. The second way to manipulate spacetime is electromagnetism, founded on the electromagnetic vector potential A_α. In contrast to gravitation, electromagnetism is a mixture of two deformations. It consists of the interior torsion

$$T^\alpha_{\beta\gamma} = 4\delta^\alpha_{[\gamma} A_{\beta]}$$

and interior nonmetricity

$$\Delta_\gamma g_{\alpha\beta} = 2A_\gamma g_{\alpha\beta}$$

of spacetime (box (2.48)). These two deformations can also be described in a unified manner by using the electrogravitational deviation

$$L^\alpha_{\beta\gamma} = -\delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma}$$

(equation (2.40)), i.e. the difference between the electrogravitational connection $I^\alpha_{\beta\gamma}$ and the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$. The split of the electrogravitational deviation into torsion and nonmetricity is actually not directly motivated by world theory. We have performed that split merely due to the traditional picture of geometry. Finally, there is the scalar form of matter used in world theory. This kind of matter is based on the matter field λ and again a single deformation, like gravitation. Matter is the interior inflation

$$I^\alpha{}_{\beta\gamma\delta} = -\frac{2}{3}\delta^\alpha_{[\gamma} g_{\beta\delta]}\lambda$$

of spacetime (equations (3.22) and (3.23)).

We can then summarize the geometric interpretation of world theory as

$$\begin{aligned} \text{gravitation} &= \text{interior curvature} \\ \text{electromagnetism} &= \text{interior torsion and nonmetricity} \\ \text{matter} &= \text{interior inflation} \end{aligned}$$

To get a more natural, but unconventional viewpoint, we consider the curvature tensor encoded in the metric. The three deformations of world theory are then the metric $g_{\alpha\beta}$, the electrogravitational deviation $L^\alpha_{\beta\gamma}$ and the interior inflation $I^\alpha{}_{\beta\gamma\delta}$. Each deformation operates at a different derivative level and therefore requires a different number of indices. As all three phenomena are interior manipulations of spacetime, we also call them **interior deformations**. It is of course also possible to construct spacetimes where **exterior deformations** are present. However, it is not clear how they could be dynamically evolved in time with a field equation. This is the reason why world theory uses only interior deformations.

5.2 World theory versus general relativity

World theory contains vacuum general relativity as a limit. This is the case where only pure gravitation is present. Yet, world theory contradicts full general relativity. So, one of these two theories cannot be realized in nature. In this paper, it is not possible to answer which theory has to be abandoned. We can, however, at least perform a theoretical comparison of the two theories.

Let us begin with the advantages of world theory compared to full general relativity. World theory is philosophically more satisfying, because it contains a geometric interpretation of electromagnetism, in contrast to general relativity (box (2.50)). Our theory also obeys the world principle, which means that there are no non-geometric constituents in it (box (2.59)). An astonishing side effect of world theory is that it contains a fundamental criterion which tells us why we live in just four dimensions (box (2.127)). Pure electric fields are impossible otherwise. Moreover, the matter field λ allows a geometric interpretation of dark energy (box (3.74)). That way, world theory does not contain the cosmological constant as an artificial parameter in the field equation like general relativity. Instead, the cosmological constant is set by the solutions of the field equation, namely by the cosmological value of the matter field λ. So, world theory is free of parameters. The world equation may also be an at least partial explanation of dark matter (box (3.75)), but this point is merely a proposal and not yet worked out sufficiently to count it in favor of world theory. We were also able to derive a classical Klein-Gordon-like equation from the world equation. This has allowed us to give a geometric interpretation of the spin 0 wave function (box (3.133)). Unfortunately, the statistical nature of that wave function remains obscure. Finally, there are no constraints in world theory (box (4.142)). From the viewpoint of world theory, the Hamiltonian and momentum constraints of general relativity even get a totally new interpretation. These constraints are a sign that there is something wrong in full general relativity. All in all, we thus have

- Geometric interpretation of electromagnetism
- Geometric interpretation of dark energy
- Geometric interpretation of simplest, classical wave function
- Purely geometric theory
- Free of parameters
- No constraints
- Fundamental cause of four dimensions

where these points are ordered in descending importance.

Unfortunately, there are currently also open problems in world theory. The central problem is that it is not clear what matter really is. The easiest way would just be if the matter field λ were a complete description of matter. However, we do not know whether this is true. If it were, then there would still be the problem to create a bridge between the matter field λ and our current understanding of matter, based on elementary particle physics. We refer to this issue as the **matter bridge**. If instead the matter field λ were insufficient, then it would anyway be clear that we have to proceed beyond world theory. The second problem is that we do not know how world theory behaves in the microcosm. This is also the reason why the full name of our theory is classical world theory. So, the quantization of world theory and the microscopic forces, i.e. the weak and strong force, are an open issue. Finally, there is the problem that we have not managed to perform a nontrivial, stable, long-term evolution of world theory. We have also found that the variation principle fails for world theory (box (2.155). However, we do not count this lack as a point for general relativity, because there is no obligatory philosophical reason why the variation principle should hold everywhere in physics. We can then summarize the open issues as

- Matter bridge
- Microcosm
- Numeric stability

again in descending order of relevance.

The issue about the microcosm is also present for general relativity. Quantum gravitation is still under development by the physics community. However, we have anyway limited this paper to classical physics right at the beginning. Numeric stability has to be investigated in the future. The experience with numeric relativity will surely be very helpful for that purpose. Yet, the stability problems encountered in this paper were only the result of not having invested enough time in numeric world theory, because this is mainly a theoretical work. So, numeric stability is probably not a real problem of world theory. That way, solely the bridge between the matter field λ and the standard description of matter remains. We cannot rule out that world theory has to be extended here. However, if such a procedure were necessary, the resulting new theory would probably be just an extension of world theory and contain it as a limit. Realizing all these considerations, I want to propose world theory as a replacement of full general relativity. Or in other words, world theory is suggested as an extension of vacuum general relativity and thus the geometrically interpreted part of physics.

5.3 Outlook

With the fundamentals of the new theory done, the next chapter begins: the falsification of either world theory or full general relativity. We need a solid criterion that rules one of the two theories out. This is a quest for you, the readers of this paper.

At several locations, I have developed world theory to a point with an open end. This was done on purpose. Readers interested in starting their own research in world theory can use these open ends as starting points. Let us now go through them, in ascending order of relevance. For theorists, it may be tempting to study whether the variation principle can somehow be modified in an unknown manner to work for electrogravitation (box (2.155)). Mathematicians could try to get a deeper understanding of the geometric phenomenon inflation, especially its meaning from the viewpoint of an embedding space (equation (3.23)). This viewpoint may also give rise to new insight into why we have to discern interior and exterior torsion, nonmetricity and inflation. Researchers with experience in particle physics may want to investigate the relationship between the geodesic equation of electrogravitation and particles (equation (2.172)). There is also the question which spin gauge weights are realized in nature (Section 2.9.4). And, it would be very helpful to know whether there are stationary, axisymmetric solutions of world theory which are not already part of vacuum general relativity (box (4.31)). To enable a quantitative comparison with general relativity, the electromagnetic constant c_{EM} is an open issue (equation (2.20) and Section 4.7.2). For astrophysicists, a central question will be the further clarification of dark matter (box (3.75)). With the discovery of a Higgs particle at the LHC in 2012, I guess that there might be people who want to study the impact of the constraint cure on microphysics (as a start, see footnote 12). Eventually, we come to a topic expected to be relevant for a larger audience: numeric world theory. To make the start as simple as possible here, I have included a detailed description in this paper how world theory can be simulated (Section 4.6.12 and Numeric formulary). Readers doing studies in numeric relativity can therefore quickly change their codes to test numeric world theory. It may be exciting to see nontrivial examples of how world theory evolves spacetime.

That way, only the second part of physics remains, the quantum world. Yet, this realm lies far beyond the scope of this paper. Our expedition therefore ends, but the voyage of the theory has just begun.

Acknowledgment

This paper is the result of a short-term, self-financed, independent research position. An approach like that is unorthodox. It is a personal experiment, not in science itself, but in the manner how science is conducted. The subject of the experiment is the maximization of scientific freedom. This was made possible by private savings earned from my former positions at the Max Planck Institute for Astrophysics.

Since my official time at the institute, I was by the way a frequent visitor there to remain in contact with the physics community. I am grateful to the institute for this hospitality. My gratitude also goes to Bernhard Müller for critical remarks before the publication of the paper.

References

BAUMGARTE, T. W. & SHAPIRO, S. L., **2003**, *General relativistic magnetohydrodynamics for the numerical construction of dynamical spacetimes* (Astrophysical Journal, 585, 921)

BAUMGARTE, T. W. & SHAPIRO, S. L., **2010**, *Numerical Relativity: Solving Einstein's Equations on the Computer* (Cambridge: Cambridge University Press)

BERTONE, G., HOOPER, D., & SILK, J., **2005**, *Particle dark matter: evidence, candidates and constraints* (Physics Reports, 405, 279)

BUCHERT T. ET AL., **2015**, *Is there proof that backreaction of inhomogeneities is irrelevant in cosmology?* (arXiv:1505.07800v1 [gr-qc])

CALDWELL, R. R., DAVE, R., & STEINHARDT, P. J., **1998**, *Cosmological Imprint of an Energy Component with General Equation of State* (Physical Review Letters, 80, 8, 1582)

CARMELI, M., **2001**, *Classical Fields: General Relativity and Gauge Theory* (New Jersey: World Scientific)

EINSTEIN, A., **1916**, *Die Grundlage der allgemeinen Relativitätstheorie* (Annalen der Physik, 4, 49, 769)

GOENNER, H. F. M., **2004**, *On the History of Unified Field Theories* (Living Reviews in Relativity, 7, 2. URL, cited on 4.8.2015)

GOENNER, H. F. M., **2014**, *On the History of Unified Field Theories. Part II. (ca. 1930 - ca. 1965)* (Living Reviews in Relativity, 17, 5. URL, cited on 4.8.2015)

GORBATENKO, M. V., **2002**, *Fermions in conformally invariant geometrodynamics* (arXiv:gr-qc/0206031v1)

GORBATENKO, M. V., **2003**, *Discontinuity surface instead of singularity* (arXiv:gr-qc/0306117v1)

GORBATENKO, M. V., **2005**, *Some consequences of the conformally invariant generalization of Einstein's equations* (General Relativity and Gravitation, 37, 1, 81)

GORBATENKO, M. V., **2007**, *Conformal Geometrodynamics: Exact Nonstationary Spherically Symmetric Solutions* (arXiv:0711.1584v1 [gr-qc])

GORBATENKO, M. V., **2009**, *Relationship between Conformal Geometrodynamics and Dirac Equations* (arXiv:0907.4558v1 [gr-qc])

GORBATENKO, M. V., **2010**, *On physical meaning of Weyl vector* (arXiv:1002.2748v1 [gr-qc])

GORBATENKO, M. V. & KOCHEMASOV, G. G., **2007**, *A Conformally Invariant Approach to Estimation of Relations Between Physical Quantities* (arXiv:0711.2013v1 [gr-qc])

GORBATENKO, M. V. & PUSHKIN, A. V., **2002**, *Conformally Invariant Generalization of Einstein Equations and the Causality Principle* (General Relativity and Gravitation, 34, 2, 175)

Gorbatenko, M. V., Pushkin, A. V., & Schmidt, H. J., **2002**, *On a Relation between the Bach Equation and the Equation of Geometrodynamics* (General Relativity and Gravitation, 34, 1, 9)

Greiner, W., **2000**, *Relativistic Quantum Mechanics: Wave Equations* (Berlin: Springer)

Guth, A. H., **1981**, *Inflationary universe: A possible solution to the horizon and flatness problems* (Physical Review D, 23, 2, 347)

Jackson, J. D., **1999**, *Classical Electrodynamics* (New York: John Wiley & Sons)

Kan, N., Maki, T., & Shiraishi, K., **2010**, *Weyl invariant Dirac-Born-Infeld-Einstein theory* (arXiv:1012.5375v1 [gr-qc])

Landau, L. D. & Lifshitz, E. M., **2003**, *The Classical Theory of Fields* (Oxford: Butterworth-Heinemann)

Maki, T., Kan, N., & Shiraishi, K., **2012**, *Dirac-Born-Infeld-Einstein Theory with Weyl Invariance* (Journal of Modern Physics, 3, 1081)

Misner, C. W., Thorne, K. S., & Wheeler, J. A., **2002**, *Gravitation* (New York: W.H. Freeman and Company)

Perlmutter S. et al., **1999**, *Measurements of Ω and Λ from 42 High-Redshift Supernovae* (Astrophysical Journal, 517, 565)

Peskin, M. E. & Schroeder, D. V., **1997**, *An Introduction to Quantum Field Theory* (Massachusetts: Addison-Wesley)

Planck Collaboration, **2014**, *Planck 2013 results. XVI. Cosmological parameters* (Astronomy & Astrophysics, 571, A16)

Press W. H. et al., **2002**, *Numerical Recipes in C: The Art of Scientific Computing* (Cambridge: Cambridge University Press)

Pushkin, A. V., **2000**, *On Properties of the Vacuum Polarization Tensor Leading to Conformally Invariant Equations of the General Relativity Theory* (The Ninth Marcel Grossmann Meeting, 2047)

Riess A. G. et al., **1998**, *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant* (Astronomical Journal, 116, 1009)

Stephani H. et al., **2006**, *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

Straumann, N., **2004**, *General Relativity: With Applications to Astrophysics* (Berlin: Springer)

Tangherlini, F. R., **1963**, *Schwarzschild Field in n Dimensions and the Dimensionality of Space Problem* (Il Nuovo Cimento, 27, 3, 636)

Wald, R. M., **1984**, *General Relativity* (Chicago: The University of Chicago Press)

Weyl, H., **1918**, *Gravitation und Elektrizität* (Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, 465)

Symbols

Abbreviations used below
EM=electromagnetic
GR=general relativistic
EG=electrogravitation(al)

Multiple usage below
G, h

Constants

c	$=1$	Speed of light, 13
G	$=1$	Gravitational constant, 13
c_{EM}	$=1$	EM constant, (2.20)
c_{M}	$=1$	Material constant, 91
Λ		Cosmological constant, (3.64)
H		Hubble constant, (3.67)

General

D	Number of dimensions, 10
W	Tensorial weight, (1.7)
w	Gauge weight, (2.68)
L	Lagrangian, (2.130)
$\mathcal{L}$	Lagrangian density, (2.131)
A	Action, (2.132)
$R(t)$	Scale factor, (3.56)
$\mathcal{A}_{\alpha\beta\ldots}$	Antisymmetrization, 114
$[\alpha\beta\ldots]$	Completely antisym. symbol, 16

General tensors and densities

$\eta_{\alpha\beta}$	Minkowksi metric, 28
δ^{α}_{β}	Kronecker tensor, 11
$\delta^{\gamma\delta\ldots}_{\alpha\beta\ldots}$	Generalized Kronecker te., (2.8)
$\epsilon_{\alpha\beta\ldots}$	Levi-Civita tensor, 16
$\varepsilon_{\alpha\beta\ldots}$	Levi-Civita tensor density, 16
X_{α}	Arbitrary vector, (1.11)
$Y_{\alpha\beta}$	Arbitrary tensor, (2.111)

Fields

λ	Matter field, 70
A_{α}	EM vector potential, 10
$g_{\alpha\beta}$	Metric (signature -,+,+,...), 11
g	Determinant of $g_{\alpha\beta}$, 16
h	Hubble field, (3.22)
$h_{\alpha\beta}$	Metric perturbation, (3.82)
h	Trace of $h_{\alpha\beta}$, 99
$\bar{h}_{\alpha\beta}$	Dual metric perturb., (3.86)
$\bar{h}$	Trace of $\bar{h}_{\alpha\beta}$, (3.87)
Φ	Scalar grav. density, (3.96)
$\varphi_{\alpha\beta}$	Tensorial grav. density, (3.94)

Connections and deviations

$\Gamma^{\alpha}_{\beta\gamma}$	Christoffel symbols, (1.4)
$L^{\alpha}_{\beta\gamma}$	EG deviation, (2.45)
$I^{\alpha}_{\beta\gamma}$	EG connection, (2.24)
$l^{\alpha}_{\beta\gamma}$	World deviation, (2.78)
$i^{\alpha}_{\beta\gamma}$	World connection, (2.77)
${}^{\mathrm{W}}L^{\alpha}_{\beta\gamma}$	Weyl deviation, (2.61)
${}^{\mathrm{W}}I^{\alpha}_{\beta\gamma}$	Weyl connection, (2.73)
${}^{\varphi}\Gamma^{\alpha}_{\beta\gamma}$	Connection of $\varphi_{\alpha\beta}$, (3.100)
${}^{3}\Gamma^{a}_{bc}$	3-Christoffel symbols, (4.88)
$\overline{\Gamma}^{a}_{bc}$	BSSN-connection, (4.81)
$\overline{\Gamma}^{a}$	Contraction of $\overline{\Gamma}^{a}_{bc}$, (4.82)

Derivatives

∂_{α}	Partial derivative, 10
D_{α}	Gauge derivative, (2.68)
∇_{α}	Covariant derivative, (1.7)
Δ_{α}	EG derivative, (2.26)
δ_{α}	World derivative, (2.80)
${}^{\mathrm{W}}\Delta_{\alpha}$	Weyl derivative, (2.79)
${}^{\varphi}\nabla_{\alpha}$	Covariant derivative of $\varphi_{\alpha\beta}$, 104
${}^{3}\nabla_{a}$	3-covariant derivative, 141
$\overline{\nabla}_{a}$	BSSN-derivative, 145
$\delta/\delta\ldots$	Variation derivative, 53
$\partial/\delta\ldots$	Euler-Lagrange derivative, (2.136)
$\Box$	d'Alembertian, 73

Curvatures

$R^{\alpha}{}_{\beta\gamma\delta}$	Riemann tensor, (1.13)
$Z^{\alpha}{}_{\beta\gamma\delta}$	EG curvature, (2.33)
$\Omega^{\alpha}{}_{\beta\gamma\delta}$	World curvature, (3.13)

Contractions and Companions

$F_{\alpha\beta}$	EM field strength, (1.1)

$R_{\alpha\beta}$	Ricci tensor, (1.14)
R	Ricci scalar, (1.16)
$G_{\alpha\beta}$	Einstein tensor, (1.17)
G	Contraction of $G_{\alpha\beta}$, 18
$Z_{\alpha\beta}$	EG tensor, (2.34)
Z	EG scalar, (3.111)
$\overline{Z}_{\alpha\beta}$	Dual EG tensor, (3.158)
$\Omega_{\alpha\beta}$	World tensor, (3.18)
Ω	World scalar, (3.27)
$\overline{\Omega}_{\alpha\beta}$	Dual world tensor, (3.28)
$I^{\alpha}_{\ \beta\gamma\delta}$	Inflation, (3.23)
${}^{\varphi}R_{\alpha\beta}$	Ricci tensor of $\varphi_{\alpha\beta}$, (3.103)
${}^{\varphi}R$	Ricci scalar of $\varphi_{\alpha\beta}$, (3.107)
${}^{3}R_{ab}$	3-Ricci tensor, (4.114)
$\overline{R}_{ab}$	BSSN-Ricci tensor, (4.116)
$\overline{R}$	BSSN-Ricci scalar, (4.136)
R^{B}_{ab}	Ricci tensor difference, (4.117)

Conserved quantities

$T^{\mathrm{EM}}_{\alpha\beta}$	GR stress-energy tensor, (1.19)
$t^{\mathrm{EM}}_{\alpha\beta}$	EG stress-energy tensor, (3.45)
$t_{\alpha\beta}$	World stress-energy tensor, (3.46)
J_{α}	General relativistic current, (3.51)
j^{EM}_{α}	EG current, (3.54)
j_{α}	World current, (3.53)
$t^{\alpha\beta}_{\mathrm{G}}$	Landau-Lif. pseudotensor, (3.50)

New Geometry

$T^{\alpha}_{\beta\gamma}$	Torsion, (2.28)
$\Delta_{\gamma}g_{\alpha\beta}$	Nonmetricity, (2.28)
$t^{\alpha}_{\beta\gamma}$	World torsion, (2.84)
$\delta_{\gamma}g_{\alpha\beta}$	World nonmetricity, (2.84)
${}^{\mathrm{W}}T^{\alpha}_{\beta\gamma}$	Weyl torsion, (2.60)
${}^{\mathrm{W}}\Delta_{\gamma}g_{\alpha\beta}$	Weyl nonmetricity, (2.60)

Foliation

α	Lapse, (2.116)
β^{a}	Shift, (2.116)
b^{a}	Time derivative of β^{a}, (4.145)
γ_{ab}	(D-1)-metric, (2.116)
γ	Determinant of γ_{ab}, (4.41)
n^{α}	Unit normal vector, (2.117)
${}^{\mathrm{GR}}E^{\alpha}$	GR electric field, (2.118)
${}^{\mathrm{GR}}B^{a}$	GR magnetic field, (2.123)
E^{α}	EG electric field, (2.121)
B^{a}	EG magnetic field, (2.124)

Transport

x^{α}	Coordinates, 61
u^{α}	Velocity D-vector, (2.165)
a^{α}	Acceleration (no tensor), (2.157)
λ	Curve parameter, 61
v^{α}	Velocity (no tensor), (2.168)
S^{α}	Spin D-vector, 65
s^{α}	Spin direction (no tensor), (2.183)
S^{2}	Quadratic spin length, 66

Numerics

μ	Conjugate momentum, (4.39)
B	Metric exponent, (4.42)
$\overline{\gamma}_{ab}$	BSSN-metric, (4.42)
$\overline{\gamma}=1$	Determinant of $\overline{\gamma}_{ab}$, (4.43)
ϕ	Electric potential, (4.52)
K_{ab}	Exterior curvature, (4.75)
K	Contraction of K_{ab}, (4.79)
$\bar{A}_{ab}$	Exterior BSSN-curvature, (4.80)
$\bar{A}=0$	Contraction of $\bar{A}_{ab}$, (4.78)
ρ	n^{α}-projection of $t_{\alpha\beta}$, (4.93)
S_{ab}	γ^{β}_{a}-projection of $t_{\alpha\beta}$, (4.95)
S	Contraction of S_{ab}, (4.94)
S_{a}	n^{α}-projection of $t_{\alpha\beta}$, (4.122)

Analytic formulary

Colors used below

Electrogravitational view

World view

$$h = \tfrac{\lambda}{3} \qquad (3.22)$$

$$D_\alpha = \partial_\alpha + wA_\alpha \qquad (2.68)$$

$$\begin{aligned} L^\alpha_{\beta\gamma} &= -\left(\tfrac{D}{2}-1\right)\delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma} && (2.45)\\ l^\alpha_{\beta\gamma} &= \qquad\qquad \delta^\alpha_\beta A_\gamma + \delta^\alpha_\gamma A_\beta - A^\alpha g_{\beta\gamma} && (2.78)\end{aligned}$$

$$\begin{aligned} L^\beta_{\beta\alpha} &= -\left(\tfrac{D}{2}-1\right) DA_\alpha && (2.82)\\ l^\beta_{\beta\alpha} &= \qquad\qquad DA_\alpha && (2.81)\end{aligned}$$

$$\begin{aligned} I^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\beta\gamma} + L^\alpha_{\beta\gamma} && (2.24)\\ i^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\beta\gamma} + l^\alpha_{\beta\gamma} && (2.77)\\ &= \tfrac{1}{2} g^{\alpha\delta}\left(D_\beta g_{\delta\gamma} + D_\gamma g_{\beta\delta} - D_\delta g_{\beta\gamma}\right) && (2.75)\end{aligned}$$

$$\begin{aligned} \Delta_\gamma T^{\beta\ldots}_{\alpha\ldots} &= \partial_\gamma T^{\beta\ldots}_{\alpha\ldots} - I^\delta_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + I^\beta_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots - W\, I^\delta_{\delta\gamma} T^{\beta\ldots}_{\alpha\ldots} && (2.26)\\ &= \nabla_\gamma T^{\beta\ldots}_{\alpha\ldots} - L^\delta_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + L^\beta_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots - W L^\delta_{\delta\gamma} T^{\beta\ldots}_{\alpha\ldots} && (2.27)\\ \delta_\gamma T^{\beta\ldots}_{\alpha\ldots} &= D_\gamma T^{\beta\ldots}_{\alpha\ldots} - i^\delta_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + i^\beta_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots - W\, i^\delta_{\delta\gamma} T^{\beta\ldots}_{\alpha\ldots} && (2.80)\\ &= \nabla_\gamma T^{\beta\ldots}_{\alpha\ldots} - l^\delta_{\alpha\gamma} T^{\beta\ldots}_{\delta\ldots} - \cdots + l^\beta_{\delta\gamma} T^{\delta\ldots}_{\alpha\ldots} + \cdots + (w - WD)\, A_\gamma T^{\beta\ldots}_{\alpha\ldots} && (2.83)\end{aligned}$$

$$\begin{aligned} T^\alpha_{\beta\gamma} &= 2I^\alpha_{[\beta\gamma]} && (2.28)\\ &= 2L^\alpha_{[\beta\gamma]} && (2.29)\\ &= D\delta^\alpha_{[\gamma} A_{\beta]} && (2.48)\\ \Delta_\gamma g_{\alpha\beta} &= \partial_\gamma g_{\alpha\beta} - 2\, I_{(\alpha\beta)\gamma} && (2.28)\\ &= \qquad -2L_{(\alpha\beta)\gamma} && (2.29)\\ &= (D-2)\, A_\gamma g_{\alpha\beta} && (2.48)\end{aligned}$$

$$\begin{aligned} t^\alpha_{\beta\gamma} &= 2i^\alpha_{[\beta\gamma]} &&= 0\\ \delta_\gamma g_{\alpha\beta} &= D_\gamma g_{\alpha\beta} - 2i_{(\alpha\beta)\gamma} &&= 0 \end{aligned} \qquad (2.84),\ (2.85)$$

$$\begin{aligned} \Delta_\gamma \delta^\beta_\alpha &= 0 && (3.153)\\ \Delta_\gamma g^{\alpha\beta} &= -(D-2)\, A_\gamma g^{\alpha\beta} && (3.155)\\ \delta_\gamma \delta^\beta_\alpha &= \delta_\gamma g^{\alpha\beta} = 0 && (3.156),\ (3.157)\end{aligned}$$

$$[D_\alpha, D_\beta] = wF_{\alpha\beta} \tag{2.87}$$

$$\begin{aligned}
Z^\alpha{}_{\beta\gamma\delta} X_\alpha &= \left(T^\alpha_{\gamma\delta}\Delta_\alpha - [\Delta_\gamma, \Delta_\delta]\right) X_\beta && (2.32)\\
&= -[\delta_\gamma, \delta_\delta] X_\beta && (2.91)\\
-Z^\alpha{}_{\beta\delta\gamma} \equiv Z^\alpha{}_{\beta\gamma\delta} &= 2\left(\partial_{[\gamma} I^\alpha_{\beta\delta]} + I^\alpha_{\epsilon[\gamma} I^\epsilon_{\beta\delta]}\right) && (3.14),\ (2.66)\\
&= R^\alpha{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma} L^\alpha_{\beta\delta]} + L^\alpha_{\epsilon[\gamma} L^\epsilon_{\beta\delta]}\right) && (2.33)\\
&= 2\left(\partial_{[\gamma} i^\alpha_{\beta\delta]} + i^\alpha_{\epsilon[\gamma} i^\epsilon_{\beta\delta]}\right) - \tfrac{D}{2}\delta^\alpha_\beta F_{\gamma\delta} && (2.90)\\
&= R^\alpha{}_{\beta\gamma\delta} + 2\left(\nabla_{[\gamma} l^\alpha_{\beta\delta]} + l^\alpha_{\epsilon[\gamma} l^\epsilon_{\beta\delta]}\right) - \tfrac{D}{2}\delta^\alpha_\beta F_{\gamma\delta} && (2.92)
\end{aligned}$$

$$\begin{aligned}
-\Omega^\alpha{}_{\beta\delta\gamma} \equiv \Omega^\alpha{}_{\beta\gamma\delta} = Z^\alpha{}_{\beta\gamma\delta} - 2\delta^\alpha_{[\gamma} g_{\beta\delta]} h && (3.15),\ (3.16)\\
I^\alpha{}_{\beta\gamma\delta} = \Omega^\alpha{}_{\beta\gamma\delta} - Z^\alpha{}_{\beta\gamma\delta} && (3.23)
\end{aligned}$$

$$\begin{aligned}
Z_{\alpha\beta} &= Z^\gamma{}_{\alpha\gamma\beta} && (2.34)\\
&= R_{\alpha\beta} + 2\left(\nabla_{[\gamma} L^\gamma_{\alpha\beta]} + L^\gamma_{\delta[\gamma} L^\delta_{\alpha\beta]}\right) && (2.35)\\
\Omega_{\beta\alpha} \equiv \Omega_{\alpha\beta} &= \Omega^\gamma{}_{\alpha\gamma\beta} && (3.147),\ (3.18)\\
&= Z_{\alpha\beta} - (D-1)\, g_{\alpha\beta} h && (3.19)
\end{aligned}$$

$$\begin{aligned}
Z = g^{\alpha\beta} Z_{\alpha\beta} && (3.111)\\
\Omega = g^{\alpha\beta}\Omega_{\alpha\beta} && (3.27)
\end{aligned}$$

$$\begin{aligned}
\overline{Z}_{\alpha\beta} = Z_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} Z && (3.158)\\
\overline{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}\Omega && (3.28)
\end{aligned}$$

$$\begin{aligned}
\left([\Delta_\gamma, \Delta_\delta] - T^\epsilon_{\gamma\delta}\Delta_\epsilon\right) Y_{\alpha\beta} = -Z^\epsilon{}_{\alpha\gamma\delta} Y_{\epsilon\beta} - Z^\epsilon{}_{\beta\gamma\delta} Y_{\alpha\epsilon} && (3.138)\\
[[\delta_\gamma, \delta_\delta] + (D - w_Y)\, F_{\gamma\delta}]\, Y_{\alpha\beta} = -Z^\epsilon{}_{\alpha\gamma\delta} Y_{\epsilon\beta} - Z^\epsilon{}_{\beta\gamma\delta} Y_{\alpha\epsilon} && (3.139)\\
\left[\delta^\alpha, \delta^\beta\right] Y_{\alpha\beta} \equiv (D - 4 + w_Y)\, F^{\alpha\beta} Y_{\alpha\beta} && (3.164)
\end{aligned}$$

$$\begin{aligned}
E^\alpha = \sqrt{-g}^{\frac{4}{D}} F^{t\alpha} && (2.121)\\
B^a \overset{D=4}{\equiv} (\partial_2 A_3 - \partial_3 A_2, \partial_3 A_1 - \partial_1 A_3, \partial_1 A_2 - \partial_2 A_1) && (2.124)
\end{aligned}$$

$$\begin{aligned}
A_\alpha &\overset{\text{cg}}{\to} A_\alpha + \partial_\alpha \chi && (2.70)\\
g_{\alpha\beta} &\overset{\text{cg}}{\to} e^{-2\chi} g_{\alpha\beta} && (2.70)\\
\left(g^{\alpha\beta}, \lambda\right) &\overset{\text{cg}}{\to} e^{2\chi}\left(g^{\alpha\beta}, \lambda\right) && (2.71),\ (3.3)\\
\left(i^\alpha_{\beta\gamma}, F_{\alpha\beta}, Z^\alpha{}_{\beta\gamma\delta}, \Omega^\alpha{}_{\beta\gamma\delta}, Z_{\alpha\beta}\right) &\overset{\text{cg}}{\to} \left(i^\alpha_{\beta\gamma}, F_{\alpha\beta}, Z^\alpha{}_{\beta\gamma\delta}, \Omega^\alpha{}_{\beta\gamma\delta}, Z_{\alpha\beta}\right) && (2.76),\ (2.97),\ (3.17)
\end{aligned}$$

Numeric formulary

Colors used below

Constraints and symmetries

Gauge and slicing conditions

Formulas shown below (usage is explained in Section 4.6.12)

All blue boxes of Chapter 4 (terms shown grayed out vanish for gauge used below)

Equations (1.1), (4.81), (4.118), (4.137) and (4.141)

Undeformed spacetime (= trivial initial values, see list (4.1))

$(\lambda, \mu, A_a, E^a, \phi, B, \overline{\gamma}_{ab}, K, \bar{A}_{ab}, \overline{\Gamma}^a, \alpha, \beta^a, b^a) \overset{\mathrm{i}}{=} (0, 0, 0, 0, 0, 0, \eta_{ab}, 0, 0, 0, 1, 0, 0)$

Matter

$$\begin{aligned}
\partial_t \lambda &= \alpha e^{-6B} \mu + \beta^a \partial_a \lambda \\
&\text{or} \\
\lambda &= e^{-4B} \Big\{ \overline{R}/2 + 2\overline{\Gamma}^a (A_a + 2\partial_a B) \\
&\qquad - \overline{\gamma}^{ab} [2 (\partial_a A_b + 2\partial_{ab} B) + 4 (A_a + \partial_a B) \partial_b B + A_a A_b] \Big\} \\
&\quad + K^2/3 - \bar{A}_{ab} \bar{A}^{ab}/2 + \phi \left(3e^{-12B} \phi + 2e^{-6B} K \right)
\end{aligned}$$

$$\begin{aligned}
\partial_t \mu &= \mu \partial_a \beta^a + \beta^a \partial_a \mu - 2\alpha e^{-6B} \phi \mu - 2\lambda \sqrt{-g} \nabla_\alpha A^\alpha \\
&\quad + \alpha e^{2B} \overline{\gamma}^{ab} \left[\partial_{ab} \lambda + \partial_a \lambda \left(\frac{1}{\alpha} \partial_b \alpha - \overline{\gamma}^{cd} \partial_c \overline{\gamma}_{bd} + 2\partial_b B - 2A_b \right) \right] \\
&\text{or} \\
\mu &= \partial_a E^a - 2\phi \lambda
\end{aligned}$$

Electromagnetism

$$\partial_t A_a = -\alpha \left\{ e^{-2B} \overline{\gamma}_{ab} E^b + e^{-6B} \left[\partial_a \phi + \phi \left(\frac{1}{\alpha} \partial_a \alpha - 6\partial_a B \right) \right] \right\} + \beta^b \partial_b A_a + A_b \partial_a \beta^b$$

$$\begin{aligned}
\partial_t E^a &= \partial_b \left(E^a \beta^b - E^b \beta^a + \alpha e^{-2B} \overline{\gamma}^{ac} \overline{\gamma}^{bd} F_{cd} \right) + \beta^a \partial_b E^b + \alpha e^{2B} \overline{\gamma}^{ab} (\partial_b - 2A_b) \lambda \\
&\text{or} \\
E^a &= 2 \left[e^{-4B} \overline{\gamma}^{ab} (A_b + 6\partial_b B - \partial_b) \phi + e^{2B} \left(\bar{A}^{ab} + \overline{\gamma}^{ab} K/3 \right) A_b \right] \\
&\quad + e^{2B} \left[\overline{\gamma}^{ac} \overline{\gamma}^{bd} \left(\partial_b \bar{A}_{cd} - \overline{\Gamma}_{ebc} \bar{A}^e_d - \overline{\Gamma}_{ebd} \bar{A}^e_c \right) - 2\overline{\gamma}^{ab} \partial_b K/3 + 6\bar{A}^{ab} \partial_b B \right]
\end{aligned}$$

$$\partial_t \phi = \beta^a \partial_a \phi + \phi \partial_a \beta^a + \alpha e^{2B} \overline{\gamma}^{ab} \left[A_a \left(\overline{\gamma}^{cd} \partial_c \overline{\gamma}_{bd} - 2\partial_b B - \frac{1}{\alpha} \partial_b \alpha \right) - \partial_a A_b \right]$$

Gravitation

$$\partial_t B = (\partial_a \beta^a - \alpha K)/6 + \beta^a \partial_a B$$

$$
\begin{aligned}
\partial_t \overline{\gamma}_{ab} &= -2\alpha \bar{A}_{ab} + \beta^c \partial_c \overline{\gamma}_{ab} + \overline{\gamma}_{ac} \partial_b \beta^c + \overline{\gamma}_{bc} \partial_a \beta^c - 2\overline{\gamma}_{ab} \partial_c \beta^c /3 \qquad \overline{\gamma}_{ab} = \overline{\gamma}_{ba} \\
\partial_t K &= \alpha \left(\bar{A}_{ab} \bar{A}^{ab} + K^2/3 \right) + \beta^a \partial_a K - e^{-4B} \left[\overline{\gamma}^{ab} \left(\partial_{ab} + 2\partial_a B \partial_b \right) - \overline{\Gamma}^a \partial_a \right] \alpha \\
&\quad + 2\alpha \left\{ e^{-4B} \left[\overline{\gamma}^{ab} \left(\partial_b + 2\partial_b B - A_b \right) - \overline{\Gamma}^a \right] A_a - e^{-6B} K \phi - \lambda/2 \right\} - 3\alpha \nabla_\alpha A^\alpha \\
\partial_t \bar{A}_{ab} &= e^{-4B} \Big[{}^3R_{ab} - \partial_{ab} + 2 \left(\partial_a B \partial_b + \partial_b B \partial_a \right) \qquad \bar{A}_{ab} = \bar{A}_{ba} \\
&\qquad + \left(\overline{\Gamma}^c_{ab} - \overline{\gamma}_{ab} \overline{\Gamma}^c /3 \right) \partial_c + \frac{1}{3} \overline{\gamma}_{ab} \overline{\gamma}^{cd} \left(\partial_{cd} - {}^3R_{cd} - 4\partial_c B \partial_d \right) \Big] \alpha \\
&\quad + \alpha \left(K \bar{A}_{ab} - 2\bar{A}_{ac} \bar{A}^c_b \right) + \beta^c \partial_c \bar{A}_{ab} + \bar{A}_{ac} \partial_b \beta^c + \bar{A}_{bc} \partial_a \beta^c - 2\bar{A}_{ab} \partial_c \beta^c /3 \\
&\quad + 2\alpha e^{-4B} \Big\{ A_a A_b - \left(\partial_a A_b + \partial_b A_a \right)/2 + e^{-2B} \bar{A}_{ab} \phi + \overline{\Gamma}^c_{ab} A_c \\
&\qquad + 2 \left(A_a \partial_b + A_b \partial_a \right) B + \overline{\gamma}_{ab} \left[\overline{\gamma}^{cd} \left(\partial_c - A_c - 4\partial_c B \right) - \overline{\Gamma}^d \right] A_d /3 \Big\} \\
\partial_t \overline{\Gamma}^a &= -2\bar{A}^{ab} \partial_b \alpha + 2\alpha \left(\overline{\Gamma}^a_{bc} \bar{A}^{bc} - 2\overline{\gamma}^{ab} \partial_b K/3 + 6\bar{A}^{ab} \partial_b B \right) \\
&\quad + \beta^b \partial_b \overline{\Gamma}^a - \overline{\Gamma}^b \partial_b \beta^a + \left(2\overline{\Gamma}^a \partial_b \beta^b + \overline{\gamma}^{ab} \partial_{bc} \beta^c \right)/3 + \overline{\gamma}^{bc} \partial_{bc} \beta^a \\
&\quad + 4\alpha \left[e^{-6B} \overline{\gamma}^{ab} \left(A_b + 6\partial_b B - \partial_b \right) \phi + \left(\bar{A}^{ab} + \overline{\gamma}^{ab} K/3 \right) A_b - e^{-2B} E^a /2 \right]
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \alpha &= \beta^a \partial_a \alpha - 2\alpha K \\
\partial_t \beta^a &= b^a \\
\partial_t b^a &= \frac{3}{4} \partial_t \overline{\Gamma}^a - \eta b^a \qquad (\eta \text{ is a constant, see Section 4.6.10})
\end{aligned}
$$

$$
\begin{aligned}
1 &= \det \overline{\gamma}_{ab} \qquad (\text{for Cartesian coordinates}) \\
0 &= \overline{\gamma}^{ab} \bar{A}_{ab} \\
\overline{\Gamma}^a &= \overline{\gamma}^{bc} \overline{\Gamma}^a_{bc} \qquad (\text{do not use this to replace } \overline{\Gamma}^a, \text{ see below equation (4.82)})
\end{aligned}
$$

Auxiliary

$$
\begin{aligned}
F_{ab} &= \partial_a A_b - \partial_b A_a \\
\overline{\gamma}^{ab} &= \text{inverse of } \overline{\gamma}_{ab} \\
\overline{\Gamma}_{abc} &= \left(\partial_b \overline{\gamma}_{ac} + \partial_c \overline{\gamma}_{ba} - \partial_a \overline{\gamma}_{bc} \right)/2 \\
\overline{\Gamma}^a_{bc} &= \overline{\gamma}^{ad} \overline{\Gamma}_{dbc} \\
\bar{A}^a_b &= \overline{\gamma}^{ac} \bar{A}_{cb} \\
\bar{A}^{ab} &= \overline{\gamma}^{bc} \bar{A}^a_c \\
{}^3R_{ab} &= -\overline{\gamma}^{cd} \partial_{cd} \overline{\gamma}_{ab}/2 + \overline{\gamma}_{c(a} \partial_{b)} \overline{\Gamma}^c + \overline{\Gamma}^c \overline{\Gamma}_{(ab)c} + \overline{\gamma}^{cd} \left(2\overline{\Gamma}^e_{c(a} \overline{\Gamma}_{b)de} + \overline{\Gamma}^e_{ac} \overline{\Gamma}_{ebd} \right) \\
&\quad - 2 \left[\partial_{ab} B - \overline{\Gamma}^c_{ab} \partial_c B + \overline{\gamma}_{ab} \left(\overline{\gamma}^{cd} \partial_{cd} B - \overline{\Gamma}^c \partial_c B \right) \right] + 4 \left(\partial_a B \partial_b B - \overline{\gamma}_{ab} \overline{\gamma}^{cd} \partial_c B \partial_d B \right) \\
\overline{R} &= \overline{\gamma}^{ab} \left\{ \overline{\gamma}^{cd} \left[-\partial_{cd} \overline{\gamma}_{ab}/2 + \overline{\Gamma}^e_{ac} \left(2\overline{\Gamma}_{bde} + \overline{\Gamma}_{ebd} \right) \right] + \overline{\Gamma}^c \overline{\Gamma}_{abc} \right\} + \partial_a \overline{\Gamma}^a
\end{aligned}
$$

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