Dynamics

Jerry Wu

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Generalized coordinate: p_A^x, l_1, l_2, l_3

Terrain function: $h(x)$

$$p_B^x \text{ solved from: } (p_B^x - p_A^x)^2 + (h(p_B^x) - h(p_A^x))^2 = \frac{l_2^4}{l_2^2 + (l_3^2 - l_1^2)^2}$$

β solved from:

$$\omega = \beta = \arcsin\left(\frac{(p_B^x + \cos(\beta)l_3) - (p_A^x + \cos(\beta)l_1)}{l_2}\right)$$

$$\dot{\omega} = \dot{\beta} = \frac{\begin{gathered} l_1 \cos\beta\, \dot{l}_2 - l_2 \cos\beta\, \dot{l}_1 + l_2 \cos\beta\, \dot{l}_3 \\ - l_2 \dot{p}_{A_x} + l_2 \dot{p}_{B_x} - l_3 \cos\beta\, \dot{l}_2 \\ + p_{A_x} \dot{l}_2 - p_{B_x} \dot{l}_2 \end{gathered}}{\left(\sqrt{\frac{-\left(l_1 \cos\beta - l_3 \cos\beta + p_{A_x} - p_{B_x}\right)^2 + l_2^2}{l_2^2}}\, l_2 - l_1 \sin\beta + l_3 \sin\beta\right) l_2}$$

$$\frac{\begin{gathered} -(l_1 \cos\beta\, \dot{l}_2 - l_2 \cos\beta\, \dot{l}_1 + l_2 \cos\beta\, \dot{l}_3 \\ -l_2 p\dot{A}_x + l_2 p\dot{B}_x - l_3 \cos\beta\, \dot{l}_2 + pA_x \dot{l}_2 - pB_x \dot{l}_2)\dot{l}_2 \end{gathered}}{\left(\sqrt{\frac{-(l_1 \cos\beta - l_3 \cos\beta + pA_x - pB_x)^2 + l_2^2}{l_2^2}} \cdot l_2 - l_1 \sin\beta + l_3 \sin\beta\right) l_2^2}$$

$$\begin{aligned} \ddot{\omega} = \ddot{\beta} = \quad & - l_1 \sin\beta\, \dot{\beta}\, \dot{l}_2 + l_1 \cos\beta\, \ddot{l}_2 + l_2 \sin\beta\, \dot{\beta}\, \dot{l}_1 \\ & - l_2 \sin\beta\, \dot{\beta}\, \dot{l}_3 - l_2 \cos\beta\, \ddot{l}_1 + l_2 \cos\beta\, \ddot{l}_3 \\ + & \frac{- l_2 p\ddot{A}_x + l_2 p\ddot{B}_x + l_3 \sin\beta\, \dot{\beta}\, \dot{l}_2 - l_3 \cos\beta\, \ddot{l}_2 + pA_x \ddot{l}_2 - pB_x \ddot{l}_2}{\left(\sqrt{\frac{-(l_1 \cos\beta - l_3 \cos\beta + pA_x - pB_x)^2 + l_2^2}{l_2^2}} \cdot l_2 - l_1 \sin\beta + l_3 \sin\beta\right) l_2} \\ + & \text{(third term omitted for brevity)} \end{aligned}$$

$$\mathbf{r}_{\text{com}} = \begin{bmatrix} p_A^x \\ h(p_A^x) \end{bmatrix} + l_1 \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + \frac{1}{2} l_2 \begin{bmatrix} \sin(\beta) \\ -\cos(\beta) \end{bmatrix}$$

$$\begin{aligned} \dot{\mathbf{r}}_{\text{com}} = & \begin{bmatrix} \dot{p}_A^x \\ h'(p_A^x) \cdot \dot{p}_A^x \end{bmatrix} + \dot{l}_1 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + l_1 \dot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \\ & + \frac{1}{2} \dot{l}_2 \begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix} + \frac{1}{2} l_2 \dot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\ddot{\mathbf{r}}_{\text{com}} = &\begin{bmatrix} \ddot{p}_A^x \\ h''(p_A^x)(\dot{p}_A^x)^2 + h'(p_A^x)\cdot \ddot{p}_A^x \end{bmatrix} \\
&+ \ddot{l}_1 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + 2\dot{l}_1\dot{\beta}\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \\
&+ l_1\ddot{\beta}\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} + l_1(\dot{\beta})^2\begin{bmatrix} -\cos\beta \\ -\sin\beta \end{bmatrix} \\
&+ \frac{1}{2}\ddot{l}_2\begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix} + \dot{l}_2\dot{\beta}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \\
&+ \frac{1}{2}l_2\ddot{\beta}\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + \frac{1}{2}l_2(\dot{\beta})^2\begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix}
\end{aligned}
$$

Projection Vector:

$$
\frac{\partial\beta}{\partial p_A^x} = \frac{1}{\sqrt{1-\left(\frac{(p_B^x+\cos(\beta)l_3)-(p_A^x+\cos(\beta)l_1)}{l_2}\right)^2}}\cdot\frac{1}{l_2}\cdot\left(\frac{\partial P_B^x}{\partial P_A^x} - 1 - \sin(\beta)(l_3-l_1)\cdot\frac{\partial\beta}{\partial P_A^x}\right)
$$

$$
\frac{\partial\beta}{\partial l_1} = \frac{1}{\sqrt{1-\left(\frac{(p_B^x+\cos(\beta)l_3)-(p_A^x+\cos(\beta)l_1)}{l_2}\right)^2}}\cdot\frac{1}{l_2}\cdot\left(\frac{\partial P_B^x}{\partial l_1} - \sin(\beta)(l_3-l_1)\cdot\frac{\partial\beta}{\partial l_1} - \cos(\beta)\right)
$$

$$
\frac{\partial\beta}{\partial l_2} = \frac{1}{\sqrt{1-\left(\frac{(p_B^x+\cos(\beta)l_3)-(p_A^x+\cos(\beta)l_1)}{l_2}\right)^2}}\cdot(-\frac{1}{l_2^2}\left(P_B^x - P_A^x + \cos(\beta)(l_3-l_1)\right) + \frac{1}{l_2}\left(\frac{\partial P_B^x}{\partial l_2} - \sin(\beta)(l_3-l_1)\cdot\frac{\partial\beta}{\partial l_2}\right))
$$

$$
\frac{\partial\beta}{\partial l_3} = \frac{1}{\sqrt{1-\left(\frac{(p_B^x+\cos(\beta)l_3)-(p_A^x+\cos(\beta)l_1)}{l_2}\right)^2}}\cdot\frac{1}{l_2}\cdot\left(\frac{\partial P_B^x}{\partial l_3} - \sin(\beta)(l_3-l_1)\cdot\frac{\partial\beta}{\partial l_3} + \cos(\beta)\right)
$$

$$
\frac{\partial\mathbf{r}_{\text{com}}}{\partial p_A^x} = \begin{bmatrix} 1 \\ h'(p_A^x) \end{bmatrix} + l_1\frac{d\beta}{dp_A^x}\begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2}l_2\frac{d\beta}{dp_A^x}\begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}
$$

$$
\frac{\partial\mathbf{r}_{\text{com}}}{\partial l_1} = \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + l_1\frac{d\beta}{dl_1}\begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2}l_2\frac{d\beta}{dl_1}\begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}
$$

$$
\frac{\partial\mathbf{r}_{\text{com}}}{\partial l_2} = \frac{1}{2}\begin{bmatrix} \sin(\beta) \\ -\cos(\beta) \end{bmatrix} + \frac{1}{2}l_2\frac{d\beta}{dl_2}\begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + l_1\frac{d\beta}{dl_2}\begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix}
$$

$$
\frac{\partial\mathbf{r}_{\text{com}}}{\partial l_3} = l_1\cdot\frac{\partial\beta}{\partial l_3}\begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2}l_2\cdot\frac{\partial\beta}{\partial l_3}\begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}
$$

d'lambere formula:

$$
\sum_i (F_g - m\ddot{\mathbf{r}}_{\mathbf{com}})\delta_i + \sum_i(\tau_{fA} + \tau_{fB} + \tau_{A1} + \tau_{A2} + \tau_{B1} + \tau_{B2} - I\ddot{\omega})\delta_i = 0
$$