Dynamics

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Generalized coordinate: p_A^x, l_1, l_2, l_3

Terrain function: $h(x)$

$$p_B^x \text{ solved from: } (p_B^x - p_A^x)^2 + (h(p_B^x) - h(p_A^x))^2 = \frac{l_2^4}{l_2^2 + (l_3^2 - l_1^2)^2}$$

$$\frac{\partial p_B^x}{\partial p_A^x} = \frac{(p_B^x - p_A^x) \; + \; \left[h(p_B^x) - h(p_A^x)\right] h'(p_A^x)}{(p_B^x - p_A^x) \; + \; \left[h(p_B^x) - h(p_A^x)\right] h'(p_B^x)}$$

$$\frac{\partial p_B^x}{\partial l_1} = \frac{2\, l_1 \big(l_3^2 - l_1^2\big)\, l_2^4}{\left[l_2^2 + \big(l_3^2 - l_1^2\big)^2\right]^2 \big((p_B^x - p_A^x) + \left[\, h(p_B^x) - h(p_A^x)\right] h'(p_B^x)\big)},$$

$$\frac{\partial p_B^x}{\partial l_2} = \frac{2\, l_2^3 \left[l_2^2 + \big(l_3^2 - l_1^2\big)^2\right] - 1\, l_2^5}{\left[l_2^2 + \big(l_3^2 - l_1^2\big)^2\right]^2 \big((p_B^x - p_A^x) + \left[\, h(p_B^x) - h(p_A^x)\right] h'(p_B^x)\big)},$$

$$\frac{\partial p_B^x}{\partial l_3} = \frac{2\, l_3 \big(l_3^2 - l_1^2\big)\, l_2^4}{\left[l_2^2 + \big(l_3^2 - l_1^2\big)^2\right]^2 \big((p_B^x - p_A^x) + \left[\, h(p_B^x) - h(p_A^x)\right] h'(p_B^x)\big)}.$$

β solved from:

$$\omega = \beta = \arcsin\left(\frac{(p_B^x + \cos(\beta) l_3) - (p_A^x + \cos(\beta) l_1)}{l_2}\right)$$

$$\dot{\beta} = \frac{\dot{p}_B^x - \dot{p}_A^x + \cos\beta(\dot{l}_3 - \dot{l}_1) - \dot{l}_2 \sin\beta}{l_2 \cos\beta + \sin\beta(l_3 - l_1)}$$

$$\ddot{\omega} = \ddot{\beta} = \frac{\dot{N} D - N \dot{D}}{D^2}$$

$$N \;=\; \dot{p}_B^x - \dot{p}_A^x + \cos\beta\big(\dot{l}_3 - \dot{l}_1\big) - \dot{l}_2 \sin\beta, \tag{1}$$

$$D \;=\; l_2 \cos\beta + \sin\beta\big(l_3 - l_1\big), \tag{2}$$

$$\dot{N} \;=\; \ddot{p}_B^x - \ddot{p}_A^x - \sin\beta\, \dot{\beta}\big(\dot{l}_3 - \dot{l}_1\big) + \cos\beta\big(\ddot{l}_3 - \ddot{l}_1\big) - \ddot{l}_2 \sin\beta - \dot{l}_2 \cos\beta\, \dot{\beta}, \tag{3}$$

$$\dot{D} \;=\; \dot{l}_2 \cos\beta - l_2 \sin\beta\, \dot{\beta} + \cos\beta\, \dot{\beta}\big(l_3 - l_1\big) + \sin\beta\big(\dot{l}_3 - \dot{l}_1\big). \tag{4}$$

$$\mathbf{r}_{\text{com}} = \begin{bmatrix} p_A^x \\ h(p_A^x) \end{bmatrix} + l_1 \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + \frac{1}{2} l_2 \begin{bmatrix} \sin(\beta) \\ -\cos(\beta) \end{bmatrix}$$

$$\dot{\mathbf{r}}_{\text{com}} = \begin{bmatrix} \dot{p}_A^x \\ h'(p_A^x) \cdot \dot{p}_A^x \end{bmatrix} + \dot{l}_1 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + l_1 \dot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} + \frac{1}{2}\dot{l}_2 \begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix} + \frac{1}{2} l_2 \dot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$$

$$\begin{aligned}\ddot{\mathbf{r}}_{\text{com}} = & \begin{bmatrix} \ddot{p}_A^x \\ h''(p_A^x)(\dot{p}_A^x)^2 + h'(p_A^x) \cdot \ddot{p}_A^x \end{bmatrix} \\ & + \ddot{l}_1 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + 2\dot{l}_1 \dot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \\ & + l_1 \ddot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} + l_1 (\dot{\beta})^2 \begin{bmatrix} -\cos\beta \\ -\sin\beta \end{bmatrix} \\ & + \frac{1}{2}\ddot{l}_2 \begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix} + \dot{l}_2 \dot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \\ & + \frac{1}{2} l_2 \ddot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + \frac{1}{2} l_2 (\dot{\beta})^2 \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix}\end{aligned}$$

Projection Vector:

$$\frac{\partial \beta}{\partial p_A^x} = \frac{\frac{1}{l_2}\left(\frac{\partial p_B^x}{\partial p_A^x} - 1\right)}{\sqrt{1 - \left(\frac{(p_B^x + \cos(\beta) l_3) - (p_A^x + \cos(\beta) l_1)}{l_2}\right)^2} + \frac{\sin(\beta)(l_3 - l_1)}{l_2}}$$

$$\frac{\partial \beta}{\partial l_1} = \frac{\frac{1}{l_2}\left(\frac{\partial p_B^x}{\partial l_1} - \cos(\beta)\right)}{\sqrt{1 - \left(\frac{(p_B^x + \cos(\beta) l_3) - (p_A^x + \cos(\beta) l_1)}{l_2}\right)^2} + \frac{\sin(\beta)(l_3 - l_1)}{l_2}}$$

$$\frac{\partial \beta}{\partial l_2} = \frac{\frac{1}{l_2} \cdot \frac{\partial p_B^x}{\partial l_2} - \frac{1}{l_2^2}(p_B^x - p_A^x + \cos(\beta)(l_3 - l_1))}{\sqrt{1 - \left(\frac{(p_B^x + \cos(\beta) l_3) - (p_A^x + \cos(\beta) l_1)}{l_2}\right)^2} + \frac{\sin(\beta)(l_3 - l_1)}{l_2}}$$

$$\frac{\partial \beta}{\partial l_3} = \frac{\frac{1}{l_2}\left(\frac{\partial p_B^x}{\partial l_3} + \cos(\beta)\right)}{\sqrt{1 - \left(\frac{(p_B^x + \cos(\beta) l_3) - (p_A^x + \cos(\beta) l_1)}{l_2}\right)^2} + \frac{\sin(\beta)(l_3 - l_1)}{l_2}}$$

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial p_A^x} = \begin{bmatrix} 1 \\ h'(p_A^x) \end{bmatrix} + l_1 \frac{d\beta}{dp_A^x} \begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2} l_2 \frac{d\beta}{dp_A^x} \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial l_1} = \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + l_1 \frac{d\beta}{dl_1} \begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2} l_2 \frac{d\beta}{dl_1} \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial l_2} = \frac{1}{2}\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix} + \frac{1}{2}l_2\frac{d\beta}{dl_2}\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} + l_1\frac{d\beta}{dl_2}\begin{bmatrix}-\sin(\beta)\\ \cos(\beta)\end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial l_3} = l_1 \cdot \frac{\partial \beta}{\partial l_3}\begin{bmatrix}-\sin(\beta)\\ \cos(\beta)\end{bmatrix} + \frac{1}{2}l_2 \cdot \frac{\partial \beta}{\partial l_3}\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix}$$

d'lambere formula:

$$\sum_i (F_g - m\ddot{\mathbf{r}}_{\mathbf{com}})\delta_i + \sum_i (\tau_{fA} + \tau_{fB} + \tau_{A1} + \tau_{A2} + \tau_{B1} + \tau_{B2} - I\ddot{\omega})\delta_i = 0$$

for torque, we have:

$$\mathbf{r_A} = -l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}$$

$$\mathbf{r_B} = \begin{bmatrix}p_B^x\\ h(p_B^x)\end{bmatrix} - \begin{bmatrix}p_A^x\\ h(p_A^x)\end{bmatrix} - l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}$$

And:

$$F_g = \begin{bmatrix}0\\ mg\end{bmatrix}$$

$$\tau_{fA} = (-l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}) \times f_A\begin{bmatrix}1\\ h'(x_P^x)\end{bmatrix}$$

$$\tau_{A1} = (-l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}) \times \lambda_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} = -\frac{1}{2}\lambda_1 l_2$$

$$\tau_{A2} = (-l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}) \times \lambda_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix} = \lambda_2 l_1$$

$$\tau_{fB} = (\begin{bmatrix}p_B^x\\ h(p_B^x)\end{bmatrix} - \begin{bmatrix}p_A^x\\ h(p_A^x)\end{bmatrix} - l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}) \times f_A\begin{bmatrix}1\\ h'(x_P^x)\end{bmatrix}$$

$$\tau_{B1} = (\begin{bmatrix}p_B^x\\ h(p_B^x)\end{bmatrix} - \begin{bmatrix}p_A^x\\ h(p_A^x)\end{bmatrix} - l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}) \times \lambda_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} = \frac{1}{2}\lambda_1 l_2$$

$$\tau_{B2} = (\begin{bmatrix}p_B^x\\ h(p_B^x)\end{bmatrix} - \begin{bmatrix}p_A^x\\ h(p_A^x)\end{bmatrix} - l_1\begin{bmatrix}\cos(\beta)\\ \sin(\beta)\end{bmatrix} - \frac{1}{2}l_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix}) \times \lambda_2\begin{bmatrix}\sin(\beta)\\ -\cos(\beta)\end{bmatrix} = \lambda_2 l_3$$