Dynamics

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Generalized coordinate: p_A^x, l_1, l_2, l_3

Terrain function: $h(x)$

$$p_B^x \text{ solved from: } (p_B^x - p_A^x)^2 + (h(p_B^x) - h(p_A^x))^2 = \frac{l_2^4}{l_2^2 + (l_3^2 - l_1^2)^2}$$

β solved from:

$$\beta = \arcsin\left(\frac{(p_B^x + \cos(\beta)l_3) - (p_A^x + \cos(\beta)l_1)}{l_2}\right)$$

$$\mathbf{r}_{\text{com}} = \begin{bmatrix} p_A^x \\ h(p_A^x) \end{bmatrix} + l_1 \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + \frac{1}{2} l_2 \begin{bmatrix} \sin(\beta) \\ -\cos(\beta) \end{bmatrix}$$

$$\begin{aligned} \dot{\mathbf{r}}_{\text{com}} = & \begin{bmatrix} \dot{p}_A^x \\ h'(p_A^x) \cdot \dot{p}_A^x \end{bmatrix} + \dot{l}_1 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + l_1 \dot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \\ & + \frac{1}{2} \dot{l}_2 \begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix} + \frac{1}{2} l_2 \dot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \ddot{\mathbf{r}}_{\text{com}} = & \begin{bmatrix} \ddot{p}_A^x \\ h''(p_A^x)(\dot{p}_A^x)^2 + h'(p_A^x) \cdot \ddot{p}_A^x \end{bmatrix} \\ & + \ddot{l}_1 \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + 2\dot{l}_1 \dot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \\ & + l_1 \ddot{\beta} \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} + l_1 (\dot{\beta})^2 \begin{bmatrix} -\cos\beta \\ -\sin\beta \end{bmatrix} \\ & + \frac{1}{2} \ddot{l}_2 \begin{bmatrix} \sin\beta \\ -\cos\beta \end{bmatrix} + \dot{l}_2 \dot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \\ & + \frac{1}{2} l_2 \ddot{\beta} \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} + \frac{1}{2} l_2 (\dot{\beta})^2 \begin{bmatrix} -\sin\beta \\ \cos\beta \end{bmatrix} \end{aligned}$$

Projection Vector:

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial p_A^x} = \begin{bmatrix} 1 \\ h'(p_A^x) \end{bmatrix} + l_1 \frac{d\beta}{dp_A^x} \begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2} l_2 \frac{d\beta}{dp_A^x} \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial l_1} = \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + l_1 \frac{d\beta}{dl_1} \begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2} l_2 \frac{d\beta}{dl_1} \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial l_2} = \frac{1}{2} \begin{bmatrix} \sin(\beta) \\ -\cos(\beta) \end{bmatrix} + \frac{1}{2} l_2 \frac{d\beta}{dl_2} \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix} + l_1 \frac{d\beta}{dl_2} \begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix}$$

Notation:
l_1 refers to left leg, l_2 refers to spine, l_3 refers to right leg

$\dot{\beta}$ and $\ddot{\beta}$ can also be derived with the help from computer

$$\frac{\partial \mathbf{r}_{\text{com}}}{\partial l_3} = l_1 \cdot \frac{\partial \beta}{\partial l_3} \begin{bmatrix} -\sin(\beta) \\ \cos(\beta) \end{bmatrix} + \frac{1}{2} l_2 \cdot \frac{\partial \beta}{\partial l_3} \begin{bmatrix} \cos(\beta) \\ \sin(\beta) \end{bmatrix}$$

Define the intermediate variable:

$$u := \frac{p_B^x - p_A^x + \cos\beta(l_3 - l_1)}{l_2}$$

The partial derivative of F with respect to β is:

$$\frac{\partial F}{\partial \beta} = 1 + \frac{\sin\beta(l_3 - l_1)}{l_2\sqrt{1 - u^2}}$$

The partial derivatives of β with respect to each parameter are:

$$\frac{\partial \beta}{\partial p_A^x} = -\frac{1}{l_2\sqrt{1 - u^2} \cdot \left(\frac{\partial F}{\partial \beta}\right)}$$

$$\frac{\partial \beta}{\partial l_1} = \frac{\cos\beta}{l_2\sqrt{1 - u^2} \cdot \left(\frac{\partial F}{\partial \beta}\right)}$$

$$\frac{\partial \beta}{\partial l_3} = -\frac{\cos\beta}{l_2\sqrt{1 - u^2} \cdot \left(\frac{\partial F}{\partial \beta}\right)}$$

$$\frac{\partial \beta}{\partial l_2} = \frac{u}{l_2\sqrt{1 - u^2} \cdot \left(\frac{\partial F}{\partial \beta}\right)}$$