Lagrange Method I

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Define Lagrange

$$T = 0.5 I_b \dot{\phi}^2 + \frac{0.5 I_w \dot{x}^2}{r^2} + 0.5 M_b \left(L_b^2 \dot{\phi}^2 + 2 L_b \dot{\phi} \dot{x} \cos(\phi) + \dot{x}^2 \right) + 0.5 M_w \dot{x}^2$$

$$T = L_b M_b g \cos(\phi)$$

$$L = 0.5 I_b \dot{\phi}^2 + \frac{0.5 I_w \dot{x}^2}{r^2} - L_b M_b g \cos(\phi) + 0.5 M_b \left(L_b^2 \dot{\phi}^2 + 2 L_b \dot{\phi} \dot{x} \cos(\phi) + \dot{x}^2 \right) + 0.5 M_w \dot{x}^2$$

X Coordinate

$$\frac{\partial L}{\partial \dot{x}} = \dot{\phi} \left(\frac{I_b}{r} + \frac{L_b^2 M_b}{r} + L_b M_b \cos(\phi) \right) + \dot{x} \left(\frac{I_w}{r^2} + \frac{L_b M_b \cos(\phi)}{r} + M_b + M_w \right)$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = & -L_b M_b \dot{\phi}^2 \sin(\phi) - \frac{L_b M_b \dot{\phi} \dot{x} \sin(\phi)}{r} \\ & + \ddot{\phi} \left(\frac{I_b}{r} + \frac{L_b^2 M_b}{r} + L_b M_b \cos(\phi) \right) + \ddot{x} \left(\frac{I_w}{r^2} + \frac{L_b M_b \cos(\phi)}{r} + M_b + M_w \right) \end{aligned}$$

$$\frac{\partial L}{\partial x} = -\frac{L_b M_b \dot{\phi} \dot{x} \sin(\phi)}{r} + \frac{L_b M_b g \sin(\phi)}{r}$$

$$- L_b M_b \dot{\phi}^2 \sin(\phi) - \frac{L_b M_b g \sin(\phi)}{r} + \ddot{\phi} \left(\frac{I_b}{r} + \frac{L_b^2 M_b}{r} + L_b M_b \cos(\phi) \right)$$

$$+ \ddot{x}\left(\frac{I_w}{r^2} + \frac{L_b M_b \cos(\phi)}{r} + M_b + M_w\right) = 0$$

θ Coordinate

$$\frac{\partial L}{\partial \dot{\theta}} = \dot{\phi}\left(I_b + L_b^2 M_b\right) + \dot{x} L_b M_b \cos(\phi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = -L_b M_b \dot{\phi}\dot{x}\sin(\phi) + \ddot{\phi}\left(I_b + L_b^2 M_b\right) + \ddot{x} L_b M_b \cos(\phi)$$

$$\frac{\partial L}{\partial \theta} = -L_b M_b \dot{\phi}\dot{x}\sin(\phi) + L_b M_b g \sin(\phi)$$

$$- L_b M_b g \sin(\phi) + \ddot{\phi}\left(I_b + L_b^2 M_b\right) + \ddot{x} L_b M_b \cos(\phi) = \tau$$

Solve The Equation(By Sympy)

$$\begin{aligned}\ddot{x} = -\frac{I_b r \tau}{D} - \frac{L_b^2 M_b^2 g r^2 \sin(\phi)\cos(\phi)}{D} - \frac{L_b^2 M_b r \tau}{D} \\ - \frac{L_b M_b r^2 \tau \cos(\phi)}{D} + \dot{\theta}^2\left(\frac{I_b L_b M_b r^2 \sin(\phi)}{D} + \frac{L_b^3 M_b^2 r^2 \sin(\phi)}{D}\right) \\ + \dot{x}^2\left(\frac{I_b L_b M_b \sin(\phi)}{D} + \frac{L_b^3 M_b^2 \sin(\phi)}{D}\right) \\ + \dot{x}\left(\frac{2 I_b L_b M_b \dot{\theta} r \sin(\phi)}{D} + \frac{2 L_b^3 M_b^2 \dot{\theta} r \sin(\phi)}{D}\right)\end{aligned}$$

$$D = I_b I_w + I_b M_b r^2 + I_b M_w r^2 + I_w L_b^2 M_b$$

$$\begin{aligned}\ddot{\theta} = \frac{I_b r \tau}{D_r} + \frac{I_w L_b M_b g r \sin(\phi)}{D_r} + \frac{I_w r \tau}{D_r} + \frac{L_b^2 M_b^2 g r^2 \sin(\phi)\cos(\phi)}{D_r} \\ + \frac{L_b^2 M_b r \tau}{D_r} + \frac{L_b M_b^2 g r^3 \sin(\phi)}{D_r} + \frac{L_b M_b M_w g r^3 \sin(\phi)}{D_r} + \frac{2 L_b M_b r^2 \tau \cos(\phi)}{D_r} \\ + \frac{M_b r^3 \tau}{D_r} + \frac{M_w r^3 \tau}{D_r} + \dot{\theta}^2\left(-\frac{I_b L_b M_b r^2 \sin(\phi)}{D_r} - \frac{L_b^3 M_b^2 r^2 \sin(\phi)}{D_r} - \frac{L_b^2 M_b^2 r^3 \sin(\phi)\cos(\phi)}{D_r}\right)\end{aligned}$$

$$+\dot{x}^2\left(-\frac{I_b L_b M_b \sin(\phi)}{D_r} - \frac{L_b^3 M_b^2 \sin(\phi)}{D_r} - \frac{L_b^2 M_b^2 r \sin(\phi)\cos(\phi)}{D_r}\right)$$

$$+\dot{x}\left(-\frac{2 I_b L_b M_b \dot{\theta} r \sin(\phi)}{D_r} - \frac{2 L_b^3 M_b^2 \dot{\theta} r \sin(\phi)}{D_r} - \frac{2 L_b^2 M_b^2 \dot{\theta} r^2 \sin(\phi)\cos(\phi)}{D_r}\right)$$

$$D_r = I_b I_w r + I_b M_b r^3 + I_b M_w r^3 + I_w L_b^2 M_b r - L_b^2 M_b^2 r^3 \cos^2(\phi) + L_b^2 M_b^2 r^3 + L_b^2 M_b M_w r^3$$

Linearized System

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & a_{43} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix}$$

$$\text{where} \quad D = I_b I_w + I_b M_b r^2 + I_b M_w r^2 + I_w L_b^2 M_b + L_b^2 M_b M_w r^2$$

$$D' = I_b I_w r + I_b M_b r^3 + I_b M_w r^3 + I_w L_b^2 M_b r + L_b^2 M_b M_w r^3$$

$$a_{21} = -\frac{L_b^2 M_b^2 g r}{D}, \quad a_{23} = -\frac{L_b^2 M_b^2 g r^2}{D}$$

$$a_{41} = \frac{I_w L_b M_b g + L_b^2 M_b^2 g r + L_b M_b^2 g r^2 + L_b M_b M_w g r^2}{D'}$$

$$a_{43} = \frac{I_w L_b M_b g r + L_b^2 M_b^2 g r^2 + L_b M_b^2 g r^3 + L_b M_b M_w g r^3}{D'}$$

$$b_2 = -\frac{I_b r + L_b^2 M_b r + L_b M_b r^2}{D}$$

$$b_4 = \frac{I_b r + I_w r + L_b^2 M_b r + 2 L_b M_b r^2 + M_b r^3 + M_w r^3}{D'}$$