Lagrange Method I

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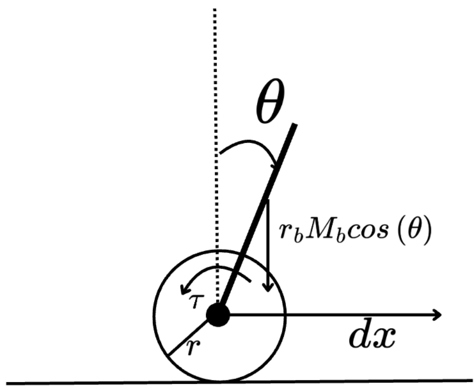

Figure 1: Lagrange Method 1 Diagram

Define Lagrange

$$T = 0.5 I_b \dot{\theta}^2 + \frac{0.5 I_w \dot{x}^2}{r^2} + 0.5 M_b \left(r_b^2 \dot{\theta}^2 + 2 r_b \dot{\theta} \dot{x} \cos\left(\theta\right) + \dot{x}^2 \right) + 0.5 M_w \dot{x}^2$$

$$T = r_b M_b g \cos\left(\theta\right)$$

$$L = 0.5 I_b \dot{\theta}^2 + \frac{0.5 I_w \dot{x}^2}{r^2} - r_b M_b g \cos\left(\theta\right) + 0.5 M_b \left(r_b^2 \dot{\theta}^2 + 2 r_b \dot{\theta} \dot{x} \cos\left(\theta\right) + \dot{x}^2 \right) + 0.5 M_w \dot{x}^2$$

X Coordinate

$$\frac{\partial L}{\partial \dot{x}} = \dot{\theta} \left(\frac{I_b}{r} + \frac{r_b^2 M_b}{r} + r_b M_b \cos\left(\theta\right) \right) + \dot{x} \left(\frac{I_w}{r^2} + \frac{r_b M_b \cos\left(\theta\right)}{r} + M_b + M_w \right)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = -r_b M_b \dot{\theta}^2 \sin(\theta) - \frac{r_b M_b \dot{\theta} \dot{x} \sin(\theta)}{r} + \ddot{\theta}\left(\frac{I_b}{r} + \frac{r_b^2 M_b}{r} + r_b M_b \cos(\theta)\right) + \ddot{x}\left(\frac{I_w}{r^2} + \frac{r_b M_b \cos(\theta)}{r} + M_b + M_w\right)$$

$$\frac{\partial L}{\partial x} = -\frac{r_b M_b \dot{\theta} \dot{x} \sin(\theta)}{r} + \frac{r_b M_b g \sin(\theta)}{r}$$

$$-r_b M_b \dot{\theta}^2 \sin(\theta) - \frac{r_b M_b g \sin(\theta)}{r} + \ddot{\theta}\left(\frac{I_b}{r} + \frac{r_b^2 M_b}{r} + r_b M_b \cos(\theta)\right) + \ddot{x}\left(\frac{I_w}{r^2} + \frac{r_b M_b \cos(\theta)}{r} + M_b + M_w\right) = -\frac{\tau}{r}$$

θ Coordinate

$$\frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}\left(I_b + r_b^2 M_b\right) + \dot{x} r_b M_b \cos(\theta)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = -r_b M_b \dot{\theta} \dot{x} \sin(\theta) + \ddot{\theta}\left(I_b + r_b^2 M_b\right) + \ddot{x} r_b M_b \cos(\theta)$$

$$\frac{\partial L}{\partial \theta} = -r_b M_b \dot{\theta} \dot{x} \sin(\theta) + r_b M_b g \sin(\theta)$$

$$- r_b M_b g \sin(\theta) + \ddot{\theta}\left(I_b + r_b^2 M_b\right) + \ddot{x} r_b M_b \cos(\theta) = \tau$$

Solve The Equation(By Sympy)

$$\ddot{x} = \frac{I_b M_b \dot{\theta}^2 r^2 r_b \sin(\theta)}{D} - \frac{I_b r \tau}{D} + \frac{M_b^2 \dot{\theta}^2 r^2 r_b^3 \sin(\theta)}{D} - \frac{M_b^2 g r^2 r_b^2 \sin(\theta)\cos(\theta)}{D} - \frac{M_b r^2 r_b \tau \cos(\theta)}{D} - \frac{M_b r r_b^2 \tau}{D}$$

$$\ddot{\theta} = \ddot{\theta} = \frac{I_w M_b g r_b \sin(\theta)}{D} + \frac{I_w \tau}{D} - \frac{M_b^2 \dot{\theta}^2 r^2 r_b^2 \sin(\theta)\cos(\theta)}{D} + \frac{M_b^2 g r^2 r_b \sin(\theta)}{D} + \frac{M_b M_w g r^2 r_b \sin(\theta)}{D} + \frac{M_b r^2 \tau}{D} + \frac{M_b r r_b \tau \cos(\theta)}{D} + \frac{M_w r^2 \tau}{D}$$

$$D = I_b I_w + I_b M_b r^2 + I_b M_w r^2 + I_w M_b r_b^2 - M_b^2 r^2 r_b^2 \cos^2(\theta) + M_b^2 r^2 r_b^2 + M_b M_w r^2 r_b^2$$

Linearized System

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{M_b^2 g r^2 r_b^2}{D} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{I_w M_b g r_b + M_b^2 g r^2 r_b + M_b M_w g r^2 r_b}{D} & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ -\frac{I_b r + M_b r^2 r_b + M_b r r_b^2}{D} \\ 0 \\ \frac{I_w + M_b r^2 + M_b r r_b + M_w r^2}{D} \end{bmatrix}$$

$$\text{where} \quad D = I_b I_w + I_b M_b r^2 + I_b M_w r^2 + I_w M_b r_b^2 + M_b M_w r^2 r_b^2$$