NUMBER GAME REVISITED-NUMGAME2

Problem Statement: Alice and Bob play the following game. They choose a number N to play with. The rules are as follows:
1. Bob plays first and the two players alternate.
2. In his/her turn, a player can subtract from N any prime number(including 1) . The number thus obtained is the new N.
3. The person who cannot make a move in his/her turn loses the game.
Assuming both play optimally, who wins the game ?

These type of problems frequently appear in contests and are usually considered to be easier and so we should never miss out on such problems and therefore the objective of this article is to analyse this game and tell who wins the game for a given N using minimal time.

Introduction: Without wasting much time on trivialities I will jump to the crux of this problem. But before that I would like to make a small observation.
If $N = 1$, it's obvious that Alice wins because Bob is helpless, he can't make a move as he can't subtract a positive number which is less than 1.
If $N = 2$, he subtracts 1 and gives Alice her turn and now Alice is helpless as we saw before.
If $N = 3$, he subtracts 2 and gives Alice her turn and now Alice is helpless as we saw before.
I hope few people got the idea of how to solve this problem. For those who din't get it yet here is a even larger list.

$$\begin{bmatrix} N & Winner \\ 1 & A \\ 2 & B \\ 3 & B \\ 4 & B \\ 5 & A \\ 6 & B \\ 7 & B \\ 8 & B \\ 9 & A \\ 10 & B \\ 11 & B \\ 12 & B \\ 13 & A \\ 14 & B \end{bmatrix}$$

Now I'm sure everyone is seeing the pattern. The $N's$ for which Alice wins are precisely of the form $4k + 1$. Infact this is the right answer. If you write a program which takes the input N, and if $N\%4$ is 1, prints $Alice$, otherwise Bob, will be accepted as the right answer. But Wait a second! How the hell do we know that this pattern won't break, say after one million terms. If the final answer is all

what you care about then do have a look at this problem
$http://www.codechef.com/COOK10/problems/BIGPIZA/$
Now for the above problem try to find out some pattern and I bet you'll get tired if you don't use proper arguments. So finally we are again at Box 1 as a problem solved without knowing the concept adds nothing to our experience.
Now, I will make a statement which is the gem of the day and this will infact help you in analysing such games and decide upon the outcome.
Imagine that you're playing this game and currently it's your turn. The state in which the game is *currently at* is decided by a parameter(or set of parameters), like N in this case. Let us call the set of all possible states to which you can move in one single step as X. In this problem, a winning position is one from where you can force a loss on the other player by playing optimally. And a losing position is a position from where you are bound to lose, i.e., loss is inevitable, if the other player plays optimally. Infact if the other player makes one wrong move the outcome may change, atleast in this example. That is why the extra condition "Assume that the players play optimally is given.
Now we have,
$X_N = \{N - p : p < N$ and p a prime$\}$
Crux: A position N is a winning position if there exists atleast one losing position in X_N. And a position is a losing position if all the positions in X_N are winning.
Now, let me explain what the above statement means. If currently the game is at state N and it is your turn to play. Assume for now that you've evaluated all the possibilities and come to a conclusion that you'll win the game no matter what he does, i.e., to say, you have a counter-move for every move made by your opponent. So it is quite clear that N is a winning position, so now if I'm asked what should you do now. My reply would be to subtract a prime p form N so that the final state you reach$(= N - p)$ is a losing position. Here I'm assuming that all possibilites for $N - p$ have been evaluated. Now it is your opponent's turn to play but he is helpless because he is at a losing position. No matter what he does you'll make a counter move and put him again at a losing position. To make it more clear I would like to put it this way. If in your X_N you have an element $N - p$ which is supposedly a losing position then jump to that position. Now as it is your opponents' turn he'll lose for sure as it is a losing position. I use the term helpless because his set X_{N-p} has only winning positions so no matter what he does he puts you in a winning position. And this is in fact the reason why the set X for a losing position doesn't have any losing positions, because if it had one then that player would choose to jump to it and put you into a losing position and thereby securing his win. A contradiction!
Let me illustrate this logic in the above problem. I'm just giving a table which has N, its corresponding set X_N and a remark on whether it is a winning or losing position.

$$\begin{bmatrix} N & X_N & Remark \\ 1 & \Phi & L \\ 2 & 1 & W \\ 3 & 1,2 & W \\ 4 & 1,2,3 & W \\ 5 & 2,3,4 & L \\ 6 & 1,3,4,5 & W \\ 7 & 2,4,5,6 & W \\ 8 & 1,3,5,6,7 & W \\ 9 & 2,4,6,7,8 & L \\ 10 & 3,5,7,8,9 & W \\ 11 & 4,6,8,9,10 & W \\ 12 & 1,5,7,9,10,11 & W \\ 13 & 2,6,8,10,11,12 & L \\ 14 & 1,3,7,9,11,12,13 & W \end{bmatrix}$$

Note that the claim made by me is true for this game and infact is true for games which belong to this class.

I would like to end this article by giving a proof by induction on why the losing positions are of the form $4k+1$.
It is easy to verify the claim for first few numbers, say upto 5. Now, assume that the claim is true for numbers upto $4m+1$. So we need to show that the claim is true for next four numbers i.e., $4m+2, 4m+3, 4m+4, 4m+5$ and then we're done. If $N \neq 4m+5$ then subtract $i-1$ from N where $i = N\%4$ and so you reach $4m+1$ which is a losing position and as it is your opponents' turn he'll be helpless as it is a losing position. And if instead you are at $4m+5$ and want to change your fate then you would desperately try to jump to some $4k+1$ so that you put your opponent in a losing position but the sad thing is, its a fact that $(4m+5)-(4k+1)$ is a multiple of 4 and hence not a prime so you cannot jump to any $4k+1$ and this leaves you helpless and so, no matter what you do you'll jump to a winning position and as it's his turn he'll jump to a losing position(by playing optimally) and again you're back on a square where your loss is inevitable.

I hope this article has helped atleast a few people in learning a new technique and I hope you'll surely use this technique when the situation is appropriate.

P.S.: It is because of this reason, i.e., winning and losing depends only on the starting positon, why you never find these games anywhere. And in fact this problem serves the motivation for a deep study of games so as to be sure that a particular game isn't dependent heavily on the starting position (Though in reality every game is biased). Anyone interested in this could watch the movie 21, based on the game *Blackjack* in which 21 is considered to be a special number. A group from MIT used a strategy called **card counting** to increase their odds of winning and eventually they made a lot of money by putting the casinos at loss. So it is very necessary for a game to not have any loopholes like this game does.

Have a nice day!