Minimum Vertex Cover

Classical and Quantum Approaches

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Problem Statement

1. The problem involves finding a subset of nodes of a Graph, *G = (V,E)* such that each edge of the graph has at least one corner represented in this subset.

2. Multiple solutions, of different sizes, satisfy the above criterion and unless the graph has some very special properties, the solution is not unique. A trivial solution is the entire vertex set, ***V***.

3. Much more difficult problem to find the minimum vertex cover, which is the smallest set of nodes that satisfy the criterion above. One may naively attempt generating all possible subsets of a graph's nodes and testing each to see if it makes a valid cover, in the hopes of reaching a vertex cover of the smallest size, but total number of possible subsets of a list containing N elements is $O(2^N)$ ⇒ intractable.

4. In fact, this problem is *NP-Complete* and only approximate algorithms exist in classical computing.

Classical Approach

1. Inspired from CLRS, Section 35.1

```
# APPROX-VERTEX-COVER

Given : Dictionary{
                    Vertices : Corresponding Edges
                  }

Vertex-Cover = list of Booleans-Corresponding-to-Nodes, all initialized False

for node in Nodes:
    if node is not already in Vertex-Cover:
        for edge in Edges[node]:
            if other-node of the edge is not already in Vertex-Cover:
                Vertex-Cover[node] = True
                Vertex-Cover[other-node] = True
                break

```

2. Cannot return the minimum vertex cover in all cases but guaranteed to return a cover of size no worse than twice the minimum possible cover.

Quantum Approach

1. Map the graph vertices to Qubits. ***$O(2^N)$*** possibilities can be handled by ***$O(N)$*** qubits because of quantum superposition.

2. Mapping to an Adiabatic Quantum Computer (AQC), such as D-WAVE, requires reformulation of the problem to either a Binary Quadratic Model ***(BQM)*** or an Ising-type Hamiltonian.

3. ***Ising Model*** : Well studied example from many-body quantum mechanics. Hamiltonian of the form :

$$H(\sigma) = -\sum_{\langle i\ j\rangle} J_{ij}\sigma_i\sigma_j - \mu\sum_j h_j\sigma_j,$$

4. Here, $J_{i,j}$ is the interaction strength between qubits ***i*** and ***j***. h_j is the external magnetic field on qubit ***j***. This is emulated by the Quantum Processing Unit (QPU).

Quantum Approach (...contd)

1. Problem Hamiltonian has 2 parts : H_A and H_B. H_A is the part that refers to the number of vertices in our solution set and H_B is the part that enforces that a given edge has to be in the cover set. H_B is weighted using a penalty term (fixed at twice the coefficient of H_A) to ensure that, minimizing the set size does not violate the H_B term.

$$H_A = A \sum_{u,v \in E} (1 - x_v)(1 - x_u)$$

$$H_B = B \sum_{v \in V} x_v$$

$$H = B \sum_{v \in V} x_v + A \sum_{u,v \in E} (1 - x_v)(1 - x_u)$$

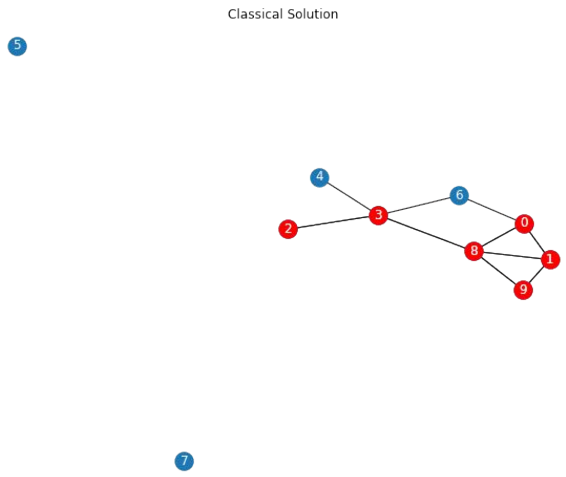
2. Set as an energy minimization problem or ground state estimation of the problem Hamiltonian.

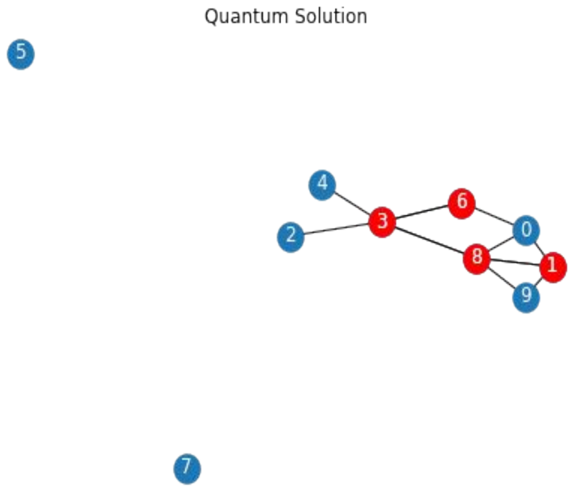
3. Coefficients : $J_{i,j}$ = *penalty* (set to 2) , $h_j = 1 - degree_j * penalty$

4. Graph sent as an '*Embedding*' to map logical qubits to physical qubits on the QPU and samples drawn after annealing terminates.

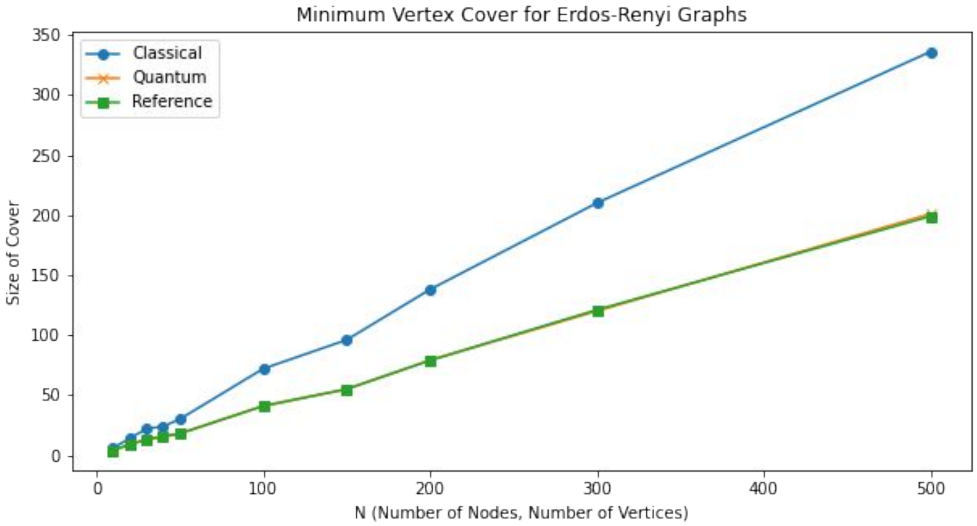
Results

1. Annealer finds the smallest cover all the time but the classical approach fails for most graph sizes. Both are verified against a reference solution from D-Wave.

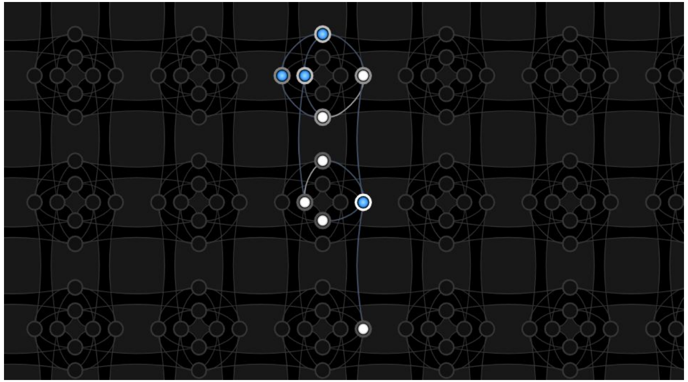




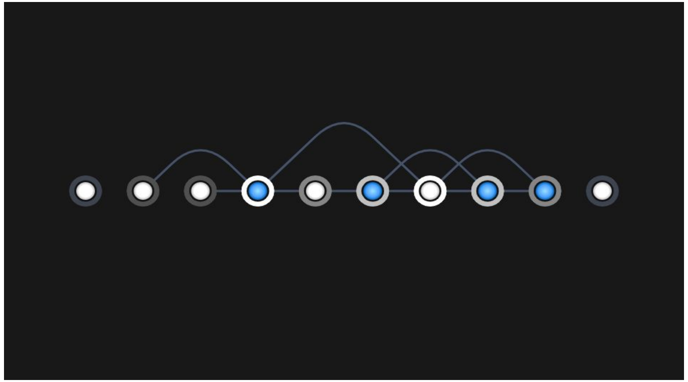
Results (...contd)



Results (...contd)



Mapping on the QPU (Chimera)



Ground State

Limitations

1. D-Wave systems have limited qubit availability (~4096 in the latest generation), so limited problem size for simulation.

2. Not free for QPU usage over cloud. Local simulation very slow as it does not utilize quantum mechanics.

3. QPU architecture has nuances such as ‘chain-breaks’, ‘chain-strength’ etc that can affect solution for large simulations.

4. From a logistical perspective : storage, maintenance and operating costs are significantly high as the qubits have to be maintained at fractions above 0K and very sophisticated measurement systems are needed for error-correction.

References

1. K. Corder, J. V. Monaco and M. M. Vindiola, "Solving Vertex Cover via Ising Model on a Neuromorphic Processor," 2018 IEEE International Symposium on Circuits and Systems (ISCAS), 2018, pp. 1-5, doi: 10.1109/ISCAS.2018.8351248.

2. A. Lucas, “Ising formulations of many NP problems”, https://doi.org/10.48550/arXiv.1302.5843

3. D-Wave Ocean Documentation : https://docs.ocean.dwavesys.com/en/stable/