

Chapter 2.1: Quadratic functions - How do you sketch graphs and write equations of parabolas?

Notes Key

Polynomials and quadratic functions - review:

Polynomial Function:

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots$$

$\hookrightarrow n$: integer, ≥ 0 , not fraction, nonnegative

Degree:

the largest power of x

Quadratic function:

$$1) f(x) = ax^2 + bx + c \text{ (general form)}$$

$$2) f(x) = a(x-h)^2 + k \text{ (vertex form)}$$

$$3) f(x) = a(x-p)(x-q) \text{ (intercept form)}$$

$\hookrightarrow x\text{-intercepts: } (p, 0), (q, 0)$

Parabolas:

Standard (vertex) form, vertex, and a value:

$$\text{vertex form: } f(x) = a(x-h)^2 + k$$

vertex: (h, k)

a -value: gives direction

$\hookrightarrow a > 0$: opens up

$\hookrightarrow a < 0$: opens down

Axis of symmetry:

$$x=h$$

Solutions/ x -intercepts/roots/zeros:

$$f(x)=0$$

factor or use the quadratic equation

write as coordinates

Example: Write the function in vertex form: $f(x) = -3x^2 + 12x + 2$.

$$f(x) = -3(x^2 - 4x + 4) + 2 - 12 \quad (-\frac{4}{2})^2 = 2^2 = 4$$

$$f(x) = -3(x-2)^2 + 14$$

vertex: $(2, 14)$

Example: Write the function's equation.

vertex: $(-2, -1) = (h, k)$

$$f(x) = a(x-h)^2 + k$$

$$f(x) = a(x+2)^2 - 1$$

$$f(-1) = a(-1+2)^2 - 1 = 0$$

$$a=1$$

$$f(x) = (x+2)^2 - 1$$

OR

x -intercepts: $(-3, 0), (-1, 0)$

$$f(x) = a(x-p)(x-q)$$

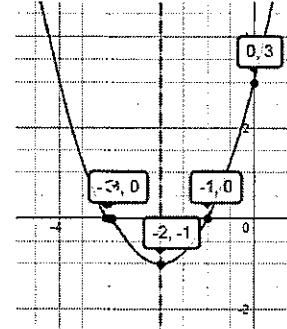
$$f(x) = a(x+3)(x+1)$$

$$f(0) = a(0+3)(0+1) = 3$$

$$3a = 3$$

$$a=1$$

$$f(x) = (x+3)(x+1)$$



Practice:

Write the function in vertex form: $f(x) = 2x^2 + 8x - 10$. Then graph it and identify the vertex, AOS, domain/range, and the x and y intercepts.

$$f(x) = 2(x^2 + 4x + 4) - 10 - 8$$

x -intercepts: $(-5, 0), (1, 0)$

$$\text{vertex form: } f(x) = 2(x+2)^2 - 18$$

$$f(x) = 0$$

$$2(x+2)^2 - 18 = 0$$

$$x+2 = \pm 3$$

$$x = -5, 1$$

vertex: $(-2, -18)$

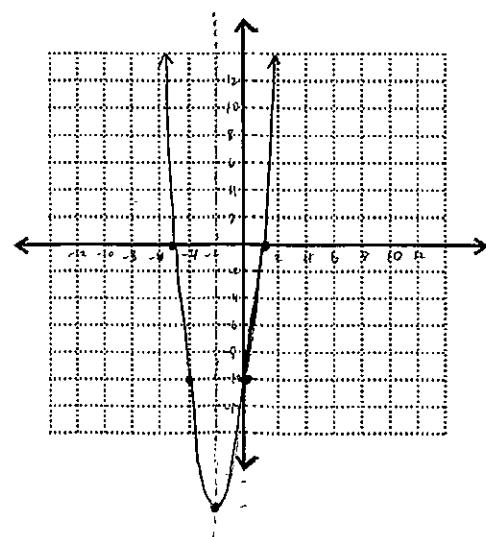
AOS: $x = -2$

domain: $(-\infty, \infty)$

range: $[-18, \infty)$

y -intercept: $(0, -10)$

$$f(0) = 2(0)^2 + 8(0) - 10 = -10$$



Writing equations, max/min values:

How many parabolas can go through the same vertex?
Ininitely many

What feature makes these parabolas different?

- vertical/horizontal stretch
- intercepts

What features will all these parabolas have in common?

- vertex
- AOs

Max and min values:

Fact: All parabolas have either an absolute max or an absolute min.

Question: When will a parabola have an absolute max? Min?

max: $a < 0$ (opens down) ↴

min: $a > 0$ (opens up) ↑

Example: A local newspaper has daily production costs of $C = 55,000 - 108x + 0.06x^2$ where c is the total cost in dollars and x is the # of newspapers printed. How many newspapers should be printed to yield a minimum cost? What is the minimum cost?

$$C = 0.06x^2 - 108x + 55000$$

$$x = \frac{-b}{2a} = \frac{108}{2(0.06)} = 900$$

$$C(900) = 0.06(900)^2 - 108(900) + 55000$$

$$C(900) = 6400$$

To yield a minimum cost, 900 newspapers should be printed. The minimum cost is \$6400.

Practice:

Find the standard form equation for the parabola described:

1. Vertex at (2,3), zero at (-1,0)

$$f(x) = a(x-h)^2 + k$$

$$f(x) = a(x-2)^2 + 3$$

$$f(-1) = a(-1-2)^2 + 3 = 0$$

$$-9a = -3$$

$$a = \frac{1}{3}$$

$$f(x) = \frac{1}{3}(x-2)^2 + 3$$

2. Vertex at (-2,0) and through the point (-4,6)

$$f(x) = a(x-h)^2 + k$$

$$f(x) = a(x+2)^2$$

$$f(-4) = a(-4+2)^2 = 6$$

$$4a = 6$$

$$a = \frac{3}{2}$$

$$f(x) = \frac{3}{2}(x+2)^2$$

3. Find any two quadratic functions, one that opens upwards and one that opens downward through the given zeroes (3,0) and (9,0)

$$\text{opens up: } f(x) = (x-3)(x-9)$$

$$\text{opens down: } f(x) = -(x-3)(x-9)$$

Chapter 2.2: Polynomial functions of higher degree - How do you sketch the graphs of polynomial functions?

Basics of the graphs of polynomials:

All polynomial graphs are *continuous* and contain only *smooth curves*.
End behavior is based on the leading coefficient and the degree.

Even degree: $f(x) = x^2$

$a > 0$

increase in both directions

$a < 0$

decrease in both directions

Odd degree: $f(x) = x^3$

$a > 0$

increase right
decrease left

$a < 0$

increase left
decrease right

Roots and factors:

If $x = a$ solves a polynomial function for $P(a) = 0$, then...

$x = a$ is a zero of the function

$x = a$ is a solution to $P(x) = 0$

$(x - a)$ must be a factor of $P(x)$

$(a, 0)$ is an x-intercept of the graph of $P(x)$.

Examples:

1. Determine the end behavior of the polynomial.

$$f(x) = -3x^5 + 2x^2$$

odd degree: $a < 0$

increase left, decrease right

2. Find the zeros of the polynomial $f(x) = x^4 - 4x^2$.

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x = 0, \pm 2 \Rightarrow x = \{-2, 0, 2\}$$

3. If 3, -2, and $\frac{1}{4}$ are zeroes of a polynomial, find a possible equation for the polynomial.

$$f(x) = (x-3)(x+2)(4x-1)$$

$$= (x^2 - x - 6)(4x - 1)$$

$$= 4x^3 - 4x^2 - 24x - x^2 + x + 6$$

$$f(x) = 4x^3 - 5x^2 - 23x + 6$$

*must have $f(x) = \dots$

Multiplicity and irrational roots:

Multiplicity - # of times a specific zero solves the polynomial

$$\text{ex: } f(x) = (x-2)^4 \text{ — multiplicity 4}$$

$$\text{Odd: } x = \{2 \text{ M4}\}$$

graph will cross x-axis at the zero

Even:

graph will not cross x-axis at the zero

*be able to differentiate between M1 and M>1 when graphing

Dealing with algebraic irrationals (roots)

irrational roots always come in pairs (conjugates)

ex: if $\sqrt{3}$ is a zero, $-\sqrt{3}$ is a zero

if $2+\sqrt{7}$ is a zero, $2-\sqrt{7}$ is a zero

*use conjugate

Practice:

4. Find the real zeros of $P(x) = x^5 + 2x^4 + x^3$.

$$x^5 + 2x^4 + x^3 = 0$$

$$x^3(x^2 + 2x + 1) = 0$$

$$x^3(x+1)^2 = 0$$

$$x = 0, -1$$

$$x = \{-1 \text{ M2}, 0 \text{ M3}\}$$

5. Write a polynomial function with zeros at -1 (m 2), $\sqrt{3}$, $-\sqrt{3}$.

$$f(x) = (x+1)^2(x-\sqrt{3})(x+\sqrt{3})$$

$$= (x^2 + 2x + 1)(x^2 - 3)$$

$$= x^4 + 2x^3 + x^2 - 3x^2 - 6x - 3$$

$$f(x) = x^4 + 2x^3 - 2x^2 - 6x - 3$$

6. Write a polynomial function with zeros at $2, 1 + \sqrt{2}$.

$$f(x) = (x-2)(x-(1+\sqrt{2}))(x-(1-\sqrt{2}))$$

$$= (x-2)(x-1-\sqrt{2})(x-1+\sqrt{2})$$

$$= (x-2)(x^2 - 2x + 1 - 2)$$

$$= (x-2)(x^2 - 2x - 1)$$

$$= x^3 + 2x^2 - x - 2x^2 + 4x + 2$$

$$f(x) = x^3 - 4x^2 + 3x + 2$$

Sketching graphs of polynomials:

What are some main features we can find?

- end behavior (leading coefficient & degree)
- zeroes
- multiplicity
- y-intercept

How can we figure out behavior *between* the zeroes?

- plug in values

Practice: Sketch a reasonable graph of the function.

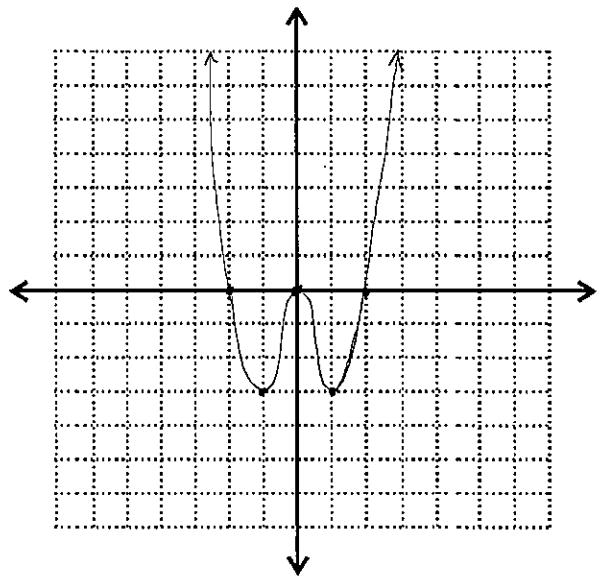
$$1. f(x) = x^4 - 4x^2 \quad \text{graph: } \cup$$

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x = 0, \pm 2$$

$$x = \{-2, 0, 2\}$$



$$2. f(x) = x^5 - 4x^3 + 8x^2 - 32$$

$$x^5 - 4x^3 + 8x^2 - 32 = 0$$

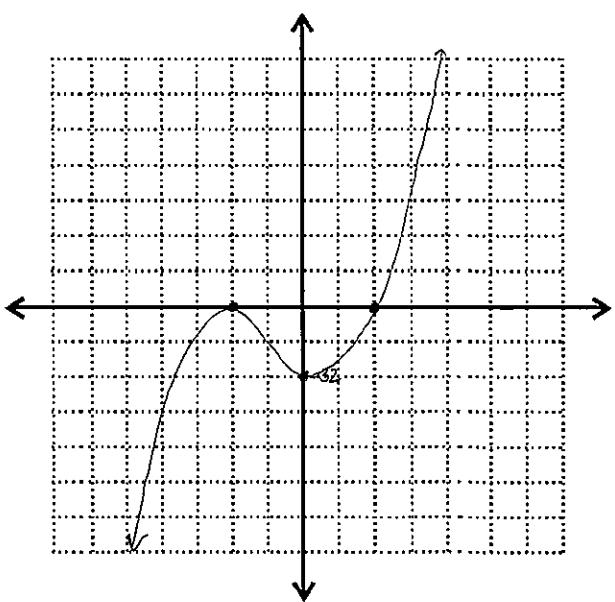
$$x^3(x^2 - 4) + 8(x^2 - 4) = 0$$

$$(x^3 + 8)(x + 2)(x - 2) = 0$$

$$(x + 2)^2(x - 2)(x^2 - 2x + 4) = 0$$

$$x = -2, 2$$

$$x = \{-2, 0, 2\}$$



Chapter 2.3 Using synthetic and polynomial division – How do you divide a polynomial by another polynomial and use polynomial division to find the rational and real zeros of polynomials?

Synthetic division:

When the divisor in a polynomial division problem is *linear* ($x - c$), there is a very nice algorithm we can use. It is called *Synthetic division*.

Synthetic division:

Example -

The set up: The *coefficients* of the dividend are written out on the top row.

Put the constant term of the divisor, c , in the box. Put a place holder 0 for any missing exponent.

$$\begin{array}{r} x^6 - 3x^5 + 14x + 4 \\ \hline x - 2 \end{array}$$

$$x - c \Rightarrow c = 2$$

zero

2	1	-3	0	0	0	14	9
↓							
2(1)							

							2 -2 -4 -8 -16 4
							1 -1 -2 -4 -8 -2 10

** remainder
↳ factor polynomial*

$x^5 - x^4 - 2x^3 - 4x^2 - 8x - 2$

The utility of synthetic division:

Synthetic division is an *algorithm* that works under certain constraints. It is not the only way to divide, but it's quick.

Factor theorem:

If $(x - k)$ is a factor of a polynomial, $P(x)$, then the quotient $\frac{P(x)}{(x-k)} = Q(x)$

$$P(x) = Q(x) \cdot (x - k)$$

If we can divide evenly by a linear factor, what will happen to the degree of the quotient compared to the degree of the dividend?

exponent decreases by 1

Example:

Factor $f(x) = 2x^3 + 11x^2 + 18x + 9$ completely if $(x + 3)$ is a factor of $f(x)$.

$$\begin{array}{r} -3 | 2 & 11 & 18 & 9 \\ \downarrow & -6 & -15 & -9 \\ 2 & 5 & 3 & 0 \checkmark \end{array}$$

$$2x^2 + 5x + 3 = (2x+3)(x+1)$$

$$f(x) = (x+3)(2x+3)(x+1)$$

Practice: Factor the polynomial equation completely using the given factors.

1. $f(x) = 3x^3 + 2x^2 - 19x + 6$; $(x + 3)$

$$\begin{array}{r} -3 | 3 & 2 & -19 & 6 \\ \downarrow & -9 & 21 & -6 \\ 3 & -7 & 2 & 0 \checkmark \end{array}$$

$$3x^2 - 7x + 2 = (3x-1)(x-2)$$

$$f(x) = (x+3)(3x-1)(x-2)$$

2. $f(x) = x^4 + 4x^3 - x^2 - 16x - 12$; $(x + 3)$, $(x + 1)$

$$\begin{array}{r} -3 | 1 & 4 & -1 & -16 & -12 \\ \downarrow & -3 & -3 & 12 & 12 \\ -1 | 1 & 1 & -4 & -4 & 0 \checkmark \\ \downarrow & -1 & 0 & 4 & \\ 1 & 0 & -4 & 0 \checkmark \end{array}$$

$$x^2 - 4 = (x+2)(x-2)$$

$$f(x) = (x+3)(x+1)(x+2)(x-2)$$

Polynomial long division: When we want to divide by a non-linear factor, we have to use long division.

$$Ex: \frac{x^4+5x^3+6x^2-x-2}{x+2} = x^3+3x^2-1$$

$$\begin{array}{r} x^3+3x^2-1 \\ x+2 \overline{) x^4+5x^3+6x^2-x-2} \\ -(x^4+2x^3) \downarrow \\ 3x^3+6x^2 \downarrow \\ -(3x^3+6x^2) \downarrow \\ -x-2 \\ -(-x-2) \\ \hline 0 \end{array}$$

$$Ex: \frac{x^4+5x^3+12x^2+17x+5}{x^2+3x+1} = x^2+2x+5$$

$$\begin{array}{r} x^2+2x+5 \\ x^2+3x+1 \overline{) x^4+5x^3+12x^2+17x+5} \\ -(x^4+3x^3+x^2) \downarrow \\ 2x^3+11x^2+17x \downarrow \\ -(2x^3+6x^2+2x) \downarrow \\ 5x^2+15x+5 \\ -(5x^2+15x+5) \\ \hline 0 \end{array}$$

Rational zero test:

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with integer coefficients and $\frac{p}{q}$ is a rational root of $f(x) \dots$

\uparrow
q (leading coefficient)

\uparrow
p (constant)

$\frac{p}{q} = \frac{\text{all factors of } p}{\text{all factors of } q}$

Practice: Determine the possible rational zeros of the polynomial.

1. $f(x) = 2x^5 + 4x^2 + 3$

$$\frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \boxed{\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}}$$

2. $f(x) = 4x^4 + 3x^2 + x + 6$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}}{\pm 1, \pm 2, \pm 4} = \boxed{\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}}$$

Use the rational zero test, and synthetic division, to factor the polynomial completely.

3. $f(x) = 2x^3 - 3x^2 - 3x + 2$

possible zeros: $\frac{\pm 1, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm \frac{1}{2}$

$$\begin{array}{r} 1 \ 2 \ -3 \ -3 \ 2 \\ \downarrow \ 2 \ \ 2 \ -1 \ -4 \\ 2 \ -1 \ -4 \ \boxed{-2} \times \end{array}$$

$$\begin{array}{r} -1 \ 2 \ -3 \ -3 \ 2 \\ \downarrow \ -2 \ \ 5 \ -2 \\ 2 \ -5 \ 2 \ \boxed{0} \checkmark \end{array}$$

$$2x^2 - 5x + 2 = (2x-1)(x-2)$$

$$\boxed{f(x) = (x+1)(2x-1)(x-2)}$$

4. $f(x) = 3x^4 + 4x^3 - 11x^2 - 16x - 4$

possible zeros: $\frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

$$\begin{array}{r} 1 \ 3 \ 4 \ -11 \ -16 \ -4 \\ \downarrow \ 3 \ 7 \ -4 \ -20 \ \boxed{-24} \times \\ 3 \ 7 \ -4 \ -20 \ \boxed{-24} \end{array}$$

$$\begin{array}{r} -1 \ 3 \ 4 \ -11 \ -16 \ -4 \\ \downarrow \ -3 \ 1 \ -12 \ -4 \ \boxed{0} \checkmark \\ 3 \ 1 \ -12 \ -4 \ \boxed{0} \checkmark \end{array}$$

$$\begin{array}{r} 1 \ 3 \ 1 \ -12 \ -4 \\ \downarrow \ 1 \ 4 \ 4 \\ 3 \ 7 \ 2 \ \boxed{0} \checkmark \end{array}$$

$$3x^2 + 7x + 2 = (3x+1)(x+2)$$

$$\boxed{f(x) = (x+1)(x-2)(3x+1)(x+2)}$$

Chapter 2.4: The complex number system – How do you perform operations with complex numbers?

Imaginary and complex numbers:

$$\text{ex: } \sqrt{-5} = \sqrt{-1} \cdot \sqrt{5} = i\sqrt{5}$$

The imaginary Number: $i = \sqrt{-1}$, we will frequently use a property of radicals, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

When a real number is multiplied by i , it is a number in the *complex number system*.

A *complex number* has two parts. Since real numbers and imaginary numbers are not like terms, then for **any** real numbers a and b , a complex number in standard form is defined as $a + bi$ * cannot do:

$$a = \text{real}, b = \text{imaginary}$$

When doing operations with complex numbers, they must be treated as...

Examples:

$$1. (3 + 6i) - (4 + 2i)$$

$$= 3 + 6i - 4 - 2i$$

$$= \boxed{-1 + 4i}$$

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$\text{ex: } i^{80}$$

$$= (i^4)^{20}$$

$$= \boxed{1}$$

$$3i + 2 \Rightarrow 2 + 3i$$

$$\frac{1+2i}{3} \Rightarrow \frac{1}{3} + \frac{2}{3}i$$

$$2. (3 + 6i)(4 + 2i)$$

$$= 12 + 6i + 24i + 12i^2$$

$$= 12 + 30i - 12$$

$$= \boxed{30i}$$

$$\text{ex: } i^{123}$$

$$= (i^4)^{30}i^3$$

$$= \boxed{-i}$$

Practice:

$$1. (9 + 2i) + (1 - 7i)$$

$$= \boxed{10 - 5i}$$

$$2. (6 - 11i)^2$$

$$= 36 - 132i + 121i^2$$

$$= \boxed{-85 - 132i}$$

$$3. (3 + 7i)(2 - 4i)$$

$$= 6 - 12i + 14i - 28i^2$$

$$= \boxed{34 + 2i}$$

$$4. \sqrt{-4} + (-4 - \sqrt{-4})$$

$$= 2i - 4 - 2i$$

$$= \boxed{-4}$$

$$5. \sqrt{-4} \cdot \sqrt{-16}$$

$$= 2i(4i)$$

$$= 8i^2$$

$$= \boxed{-8}$$

The complex conjugate, rationalization and graphing:

If $a + bi$ is a complex number, what is its conjugate?

$$a - bi$$

Stylistically, we are not allowed to have a complex number in a denominator of a fraction. So, you must *rationalize* any denominators that have an i in them.

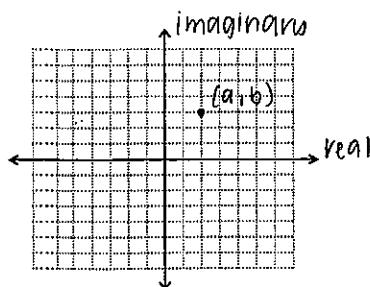
Example: rationalize the following:

$$\frac{6+i}{2-i} \cdot \frac{2+i}{2+i}$$

$$= \frac{12 + 8i - 1}{4 + 1}$$

$$= \frac{11 + 8i}{5} = \boxed{\frac{11}{5} + \frac{8}{5}i}$$

Graphing a complex number is easy, instead of using a Cartesian plane, you use a similar set up with axes labeled (*real, imaginary*), so to plot $a + bi$, you plot the ordered pair...



$$-2 + 3i \rightarrow (-2, 3)$$

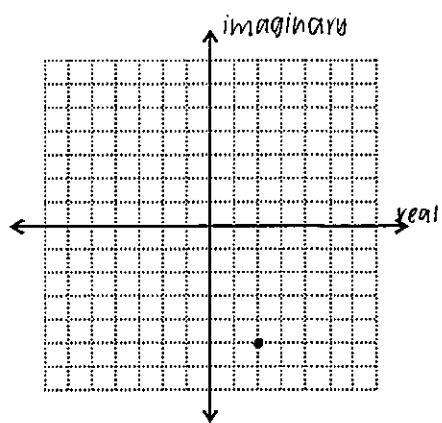
$$a + bi \Rightarrow (a, b)$$

Practice:

$$\begin{aligned}1. \quad & \frac{4+i}{3-2i} \cdot \frac{3+2i}{3+2i} \\&= \frac{(2+i)(i-2)}{9+4} \\&= \frac{16+11i}{13} \\&= \boxed{\frac{10}{13} + \frac{11}{13}i}\end{aligned}$$

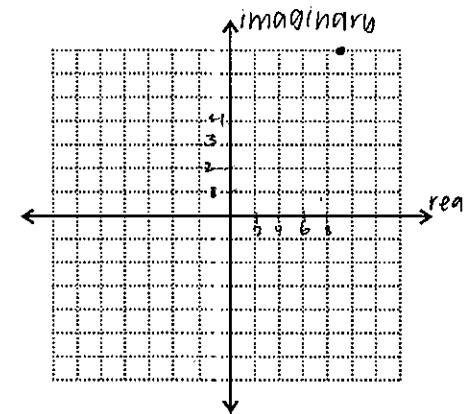
$$\begin{aligned}2. \quad & \text{Find the reciprocal of } 1+3i \\& \frac{1}{1+3i} \cdot \frac{1-3i}{1-3i} \\&= \frac{1-3i}{1+9} \\&= \boxed{\frac{1}{10} - \frac{3}{10}i}\end{aligned}$$

3. Graph $2-5i$



4. Graph $(3-i)(2+3i)$

$$\begin{aligned}& (3-i)(2+3i) \\&= 6+7i+3 \\&= \boxed{9+7i}\end{aligned}$$



Chapter 2.5: The fundamental theorem of algebra – How do you find all the zeros of a polynomial function?

The fundamental theorem of algebra and linear factorization theorem:

The fundamental theorem of algebra states...

every n^{th} degree polynomial has exactly n complex zeros

Linear factorization theorem states...

every n^{th} degree polynomial with n complex zeros has n linear factors

$$\hookrightarrow f(x) = a(x - c_1)(x - c_2)(x - c_n)$$

If a complex number is a zero of a polynomial, then...

the conjugate is also a zero

* all irrational and imaginary zeros come in pairs

Practice: Find the zeros of the function and write as a product of linear factors.

1. $f(x) = x^2 + 10x + 23$

$$x^2 + 10x + 23 = 0$$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(23)}}{2}$$

$$= \frac{-10 \pm \sqrt{8}}{2}$$

$$x = \{-5 \pm \sqrt{2}\}$$

$$f(x) = (x + 5 - \sqrt{2})(x + 5 + \sqrt{2})$$

2. $f(x) = x^4 - 81$

$$x^4 - 81 = 0$$

$$(x^2 - 9)(x^2 + 9) = 0$$

$$(x+3)(x-3)(x+3i)(x-3i) = 0$$

$$x = \{\pm 3, \pm 3i\}$$

$$f(x) = (x+3i)(x-3i)(x+3)(x-3)$$

3. $f(x) = x^4 - 5x^2 - 66$

$$x^4 - 5x^2 - 66 = 0$$

$$(x^2 - 11)(x^2 + 5) = 0$$

$$x = \{\pm\sqrt{11}, \pm i\sqrt{5}\}$$

$$f(x) = (x + \sqrt{11})(x - \sqrt{11})(x + i\sqrt{5})(x - i\sqrt{5})$$

Use the given zeroes to write a polynomial function with real coefficients

5. $3, 4i, -4i$

$$f(x) = (x - 3)(x + 4i)(x - 4i)$$

$$= (x - 3)(x^2 + 16)$$

$$f(x) = x^2 - 3x^2 + 16x - 48$$

6. $-1, -1, 2 + 5i, 2 - 5i$

$$f(x) = (x+1)^2(x-2-5i)(x-2+5i)$$

$$= (x^2 + 2x + 1)(x^2 - 4x + 4 + 25)$$

$$= (x^2 + 2x + 1)(x^2 - 4x + 29)$$

$$= x^4 + 4x^3 + 29x^2 - 2x^3 - 8x^2 + 58x + x^2 - 4x + 29$$

$$f(x) = x^4 + 2x^3 + 21x^2 + 54x + 29$$

Factoring and specific polynomials: $f(x) = x^4 - 5x^2 - 66$

Completely factoring over rational numbers:

* no irrational / imaginary

$$f(x) = (x^2 - 11)(x^2 + 6)$$

Completely factoring over real numbers:

* no imaginary

$$f(x) = (x + \sqrt{11})(x - \sqrt{11})(x^2 + 6)$$

Completely factoring over complex numbers:

* fully factored

$$f(x) = (x + \sqrt{11})(x - \sqrt{11})(x + i\sqrt{6})(x - i\sqrt{6})$$

Given zeros and a point on the function:

find a :

$$f(x) = a(x - c_1)(x + c_2)$$

→ use given point to find a

Practice: Given the zeros, multiplicities, and a point on the function, determine the equation of the polynomial.

1. Zeros at $3 - i, -2, 1$, degree 4, $f(2) = 4$

$$3+i$$

$$f(x) = a(x+2)(x-1)(x-i)(x-3-i)$$

$$f(x) = a(x^2 + x - 2)(x^2 - 6x + 10)$$

$$f(2) = a(2^2 + 2 - 2)(2^2 - 6(2) + 10) = 4$$

$$2a = 1$$

$$a = \frac{1}{2}$$

$$f(x) = \frac{1}{2}(x^4 - 6x^3 + 10x^2 + x^3 - 6x^2 + 10x - 2x^2 + 12x - 20)$$

$$\boxed{f(x) = \frac{1}{2}(x^4 - 6x^3 + 2x^2 + 22x - 20)}$$

2. Zeros at $\sqrt{3}, i, 2$, degree 5, $f(-1) = 4$

$$-\sqrt{3}, -i$$

$$f(x) = a(x + \sqrt{3})(x - \sqrt{3})(x + i)(x - i)(x - 2)$$

$$f(x) = a(x^2 - 3)(x^2 + 1)(x - 2)$$

$$f(-1) = a((-1)^2 - 3)((-1)^2 + 1)(-1 - 2) = 4$$

$$a(-2)(2)(-3) = 4$$

$$a = \frac{1}{3}$$

$$f(x) = \frac{1}{3}(x^4 - 2x^2 - 3)(x - 2)$$

$$f(x) = \frac{1}{3}(x^5 - 2x^3 - 3x - 2x^4 + 4x^2 + b)$$

$$\boxed{f(x) = \frac{1}{3}(x^5 - 2x^4 - 2x^3 + 4x^2 - 3x + b)}$$

Factor each polynomial a) Over the rational numbers b) over the real numbers c) over the complex numbers.

3. $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$, $1 + 3i$ is a root

$$1 - 3i$$

$$(x - 1 - 3i)(x - 1 + 3i)$$

$$= x^2 - 2x + 1 + 9$$

$$= x^2 - 2x + 10$$

$$\begin{array}{r} x^2 - x - 6 \\ \hline x^2 - 2x + 10 \quad | \quad x^4 - 3x^3 + 6x^2 + 2x - 60 \\ \quad - (x^4 - 2x^3 + 10x^2) \downarrow \\ \quad -x^3 - 4x^2 + 2x \downarrow \\ \quad - (-x^3 + 2x^2 - 10x) \downarrow \\ \quad -6x^2 + 12x - 60 \\ \quad - (-6x^2 + 12x - 60) \downarrow \\ \quad 0 \end{array}$$

$$a) f(x) = (x^2 - 2x + 10)(x - 3)(x + 2)$$

$$b) f(x) = (x^2 - 2x + 10)(x - 3)(x + 2)$$

$$c) f(x) = (x - 1 - 3i)(x - 1 + 3i)(x - 3)(x + 2)$$

4. $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$, $-2i$ and 1 are roots

$$2i$$

$$\begin{array}{r} 1 \ 0 \ 1 \ 2 \ -12 \ 8 \\ \downarrow \ 1 \ 1 \ 2 \ 4 \ -8 \\ 1 \ 1 \ 2 \ 4 \ -8 \ \boxed{0} \end{array}$$

$$x^4 + x^3 + 2x^2 + 4x - 8$$

possible zeros: $\pm 1, \pm 2, \pm 4, \pm 8$

$$\begin{array}{r} 1 \ 1 \ 2 \ 4 \ -8 \\ \downarrow \ 1 \ 2 \ 4 \ 8 \\ 1 \ 2 \ 4 \ 8 \ \boxed{0} \end{array}$$

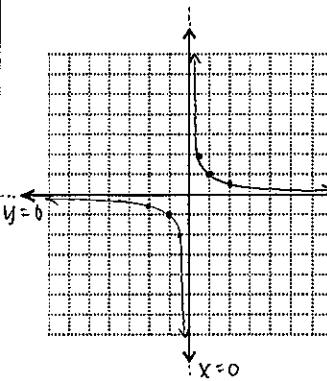
$$x^3 + 2x^2 + 4x + 8 = x^2(x + 2) + 4(x + 2) = (x^2 + 4)(x + 2)$$

$$a) f(x) = (x - 1)^2(x + 2)(x^2 + 4)$$

$$b) f(x) = (x - 1)^2(x + 2)(x^2 + 4)$$

$$c) f(x) = (x - 1)^2(x + 2)(x - 2i)(x + 2i)$$

Chapter 2.6: Rational functions and asymptotes – How do you find the domain and asymptotes of a rational function?

<p>Rational function: $f(x) = \frac{N(x)}{D(x)}$</p> <p>Parent function: $f(x) = \frac{1}{x}$</p>  <table border="1" data-bbox="489 337 652 718"> <thead> <tr> <th>x</th> <th>f(x)</th> </tr> </thead> <tbody> <tr><td>-2</td><td>-\$\frac{1}{2}\$</td></tr> <tr><td>-1</td><td>-1</td></tr> <tr><td>-\$\frac{1}{2}\$</td><td>-2</td></tr> <tr><td>\$\frac{1}{2}\$</td><td>2</td></tr> <tr><td>1</td><td>1</td></tr> <tr><td>2</td><td>\$\frac{1}{2}\$</td></tr> </tbody> </table>	x	f(x)	-2	-\$\frac{1}{2}\$	-1	-1	-\$\frac{1}{2}\$	-2	\$\frac{1}{2}\$	2	1	1	2	\$\frac{1}{2}\$	<p>Domain: $(-\infty, 0) \cup (0, \infty)$</p> <p>Range: $(-\infty, 0) \cup (0, \infty)$</p> <p>Vertical asymptote: $x = 0$</p> <p>Horizontal asymptote: $y = 0$</p> <p>Discontinuous: vertical asymptote</p>
x	f(x)														
-2	-\$\frac{1}{2}\$														
-1	-1														
-\$\frac{1}{2}\$	-2														
\$\frac{1}{2}\$	2														
1	1														
2	\$\frac{1}{2}\$														
<p>Asymptotes: ALWAYS an equation of a line ($x = \dots$ or $y = \dots$)</p> <p>Vertical asymptotes:</p> <ul style="list-style-type: none"> - create discontinuity in graph (break) - You can never cross a vertical asymptote - to find: denominator = 0 	<p>Horizontal asymptotes:</p> $y = \frac{a \cdot x^m}{b \cdot x^n}$ <ul style="list-style-type: none"> 1) $m > n$: none 2) $m = n$: $y = \frac{a}{b}$ 3) $m < n$: $y = 0$ <p>Holes: always coordinates</p> <ul style="list-style-type: none"> - removable discontinuity - occur when you reduce - plug in x-value into <u>remaining function</u> 														

Examples: Find any asymptotes, zeros, holes and use the behavior at critical points and extremes to graph the function.

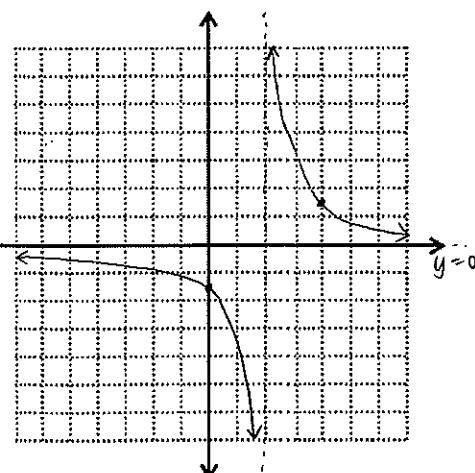
1. $f(x) = \frac{3}{x-2}$

vertical asymptote: $x = 2$

horizontal asymptote: $y = 0$

holes: none

x	f(x)
0	-\$\frac{3}{2}\$
4	\$\frac{3}{2}\$



STEPS:

- 1) factor numerator and denominator
↳ cross out same factors and write remaining function
- 2) find all asymptotes

2. $f(x) = \frac{x^2+2x+1}{2x^2-x-3} = \frac{(x+1)^2}{(2x-3)(x+1)}$

$$f(x) = \frac{x+1}{2x-3}, x \neq -1$$

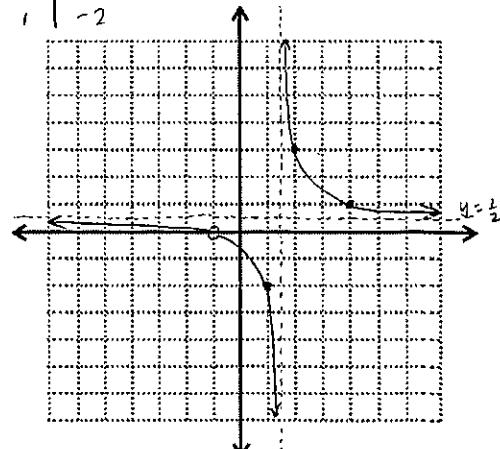
vertical asymptote: $x = \frac{3}{2}$

horizontal asymptote: $y = \frac{1}{2}$

hole: $(-1, 0)$

$$\frac{-1+1}{2(-1)+3} = 0$$

x	f(x)
4	1
1	-2



Practice: Find any asymptotes, zeroes, holes and use the behavior at critical points and extremes to graph the function.

$$1. f(x) = \frac{2x+3}{x-1}$$

VA: $x=1$

HA: $y=2$

Holes: none

x	$f(x)$
0	-3
2	7

$$2. f(x) = \frac{x^2+x-2}{x^2-x-6}$$

$$= \frac{(x+2)(x-1)}{(x-3)(x+2)}$$

$$f(x) = \frac{x-1}{x-3}, x \neq -2$$

VA: $x=3$

HA: $y=1$

Holes: $(-2, \frac{3}{5})$

$$\frac{-2-1}{-2-3} = \frac{3}{5}$$

x	$f(x)$
2	-1
5	2

$$3. f(x) = \frac{x-1}{x^2-9}$$

VA: $x=3, x=-3$

HA: $y=0$

Holes: none

x-intercept: $(1, 0)$

$$x-1=0$$

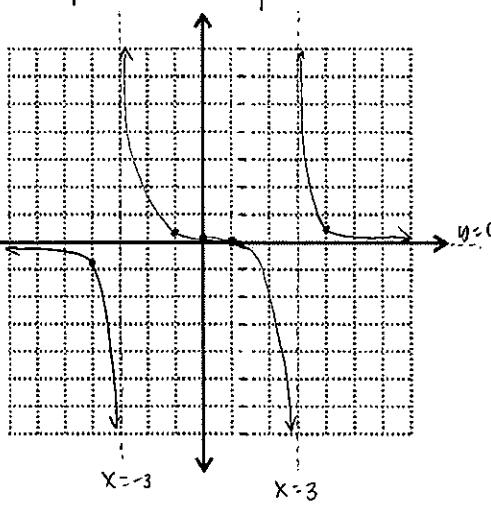
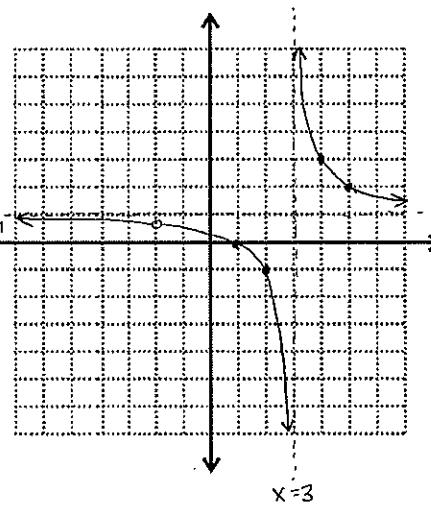
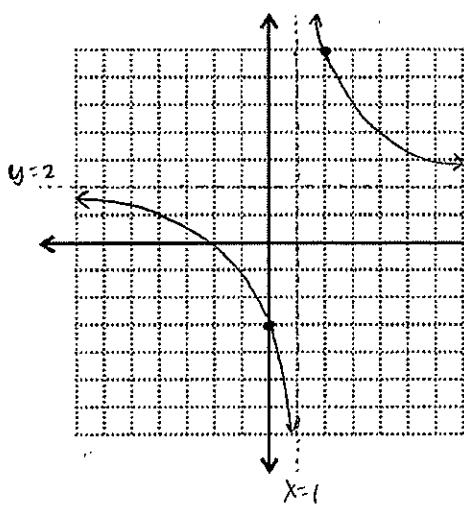
$$x=1$$

y-intercept: $(0, \frac{1}{9})$

$$f(0) = \frac{0-1}{0^2-9} = \frac{1}{9}$$

x	$f(x)$
-4	-5/7
2	-1/5

x	$f(x)$
-1	1/4
4	3/7



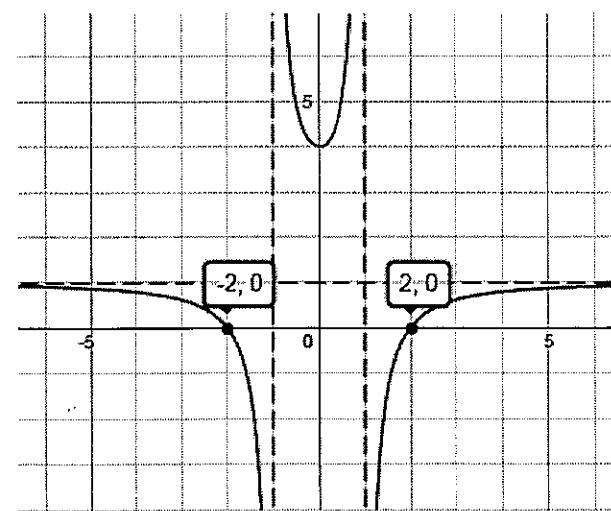
3. What could be the equation of the function?

VA: $x=1, x=-1$

HA: $y=0$

x-int.: $(\pm 2, 0)$

$$f(x) = \frac{(x+2)(x-2)}{(x+1)(x-1)}$$



4. Write the equation of any function that fits the following criteria:

a) Horizontal Asymptote: $y = 0$

b) Vertical Asymptote: $x = -1$

c) Zero: $x = 1$

$$f(x) = \frac{x-1}{(x+1)^2}$$

The slant asymptote:

Third case: The degree of the numerator is *exactly one more* than the degree of the denominator. Use long division!

Describe the slant asymptotes:

$$1. f(x) = \frac{2x}{1}$$

$\boxed{y = 2x}$

$$2. f(x) = \frac{2x^2+1}{x}$$

$$\times \begin{array}{r} 2x \\ \hline 2x^2 + 0x + 1 \\ -2x^2 + 0x \\ \hline 0 \quad | \end{array}$$

$$3. f(x) = \frac{2x^3}{2x^2 + 1}$$

$$\begin{array}{r} x \\ 2x^2 + 0x + 1 \quad | \quad 2x^3 + 0x^2 + 0x + 0 \\ \underline{- (2x^2 + 0x^2 + x)} \quad \quad \quad -x + 0 \end{array}$$

Graphing complex rational functions:

Things to note:

All horizontal and slant asymptotes can be found by finding the quotient through division and disregarding the remainder.

Ex:

$$1. f(x) = \frac{4x+5}{2x-1}$$

$$y = \frac{4}{2}$$

$y = 2$

$$2. f(x) = \frac{x^2+2x+1}{x-3}$$

$$\begin{array}{r} \boxed{3} \\ \downarrow \\ \begin{array}{r} 1 & 2 & 1 \\ - & 3 & 15 \\ \hline 1 & 5 & \boxed{16} \end{array} \end{array}$$

$y = x + 5$

1. We need to check behavior in between all critical x – values which include zeros and vertical asymptotes.

2. There will always be either a horizontal or a slant asymptote, not both.

3. Graphs of rational functions *can* cross a horizontal or slant asymptote when near the center of the graph, asymptotes only describe what happens at the extremes.

4. Each non removable domain restriction will give you a unique vertical asymptote.

5. Always find the y-value of a hole by plugging the removable domain restriction's x-value into the simplified function.

Practice: Sketch a graph of the rational function by finding the key features.

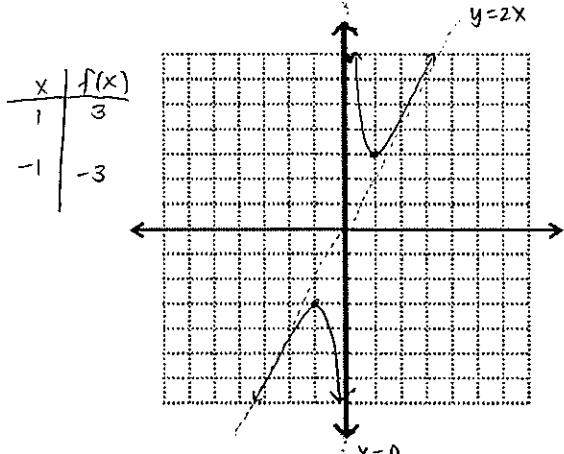
1. $f(x) = \frac{2x^2+1}{x}$

vertical asymptote : $x=0$

horizontal asymptote : none

slant asymptote : $y=2x$

$\underline{\underline{y}}$	2	0	1
\downarrow	0	0	0
2	0	0	0



$$2. f(x) = \frac{x^3}{2x^2 - 8}$$

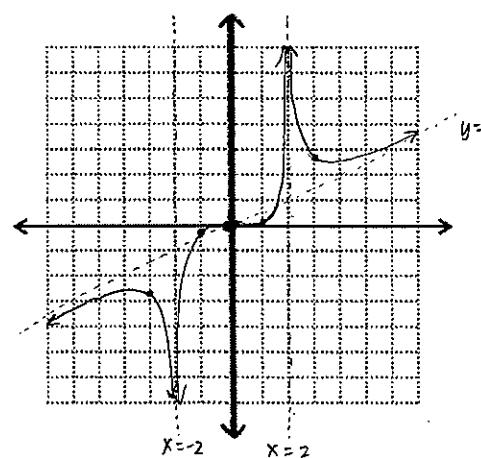
Vertical asymptote: $x = 2, x = -2$

$2x^2 - 8 = 0$
 $x^2 = 4$
 $x = \pm 2$

Horizontal asymptote: none

Slant asymptote: $y = \frac{1}{2}x$

$$\begin{array}{r} 2x^3 + 0x^2 - 8 \\ \underline{(x^3 + 0x^2 - 4x)} \\ \hline 4x \end{array}$$

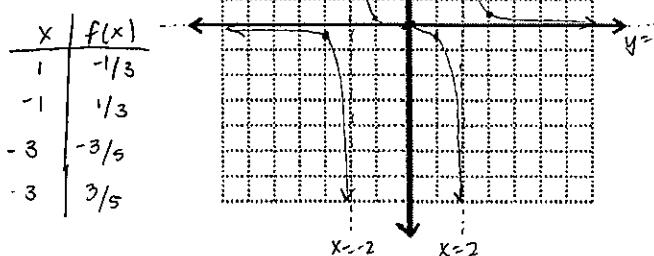


3. $f(x) = \frac{x}{x^2 - 4}$

Vertical asymptote: $x = 2, x = -2$ $x^2 - 4 = 0$
 $x^2 = 4$
 $x = \pm 2$

horizontal asymptote: $y = 0$

slant asymptote: none



Practice: Graph each rational function.

$$1. f(x) = \frac{x^2}{x^3 - x} = \frac{x^2}{x(x^2 - 1)} = \frac{x}{x^2 - 1}, x \neq 0$$

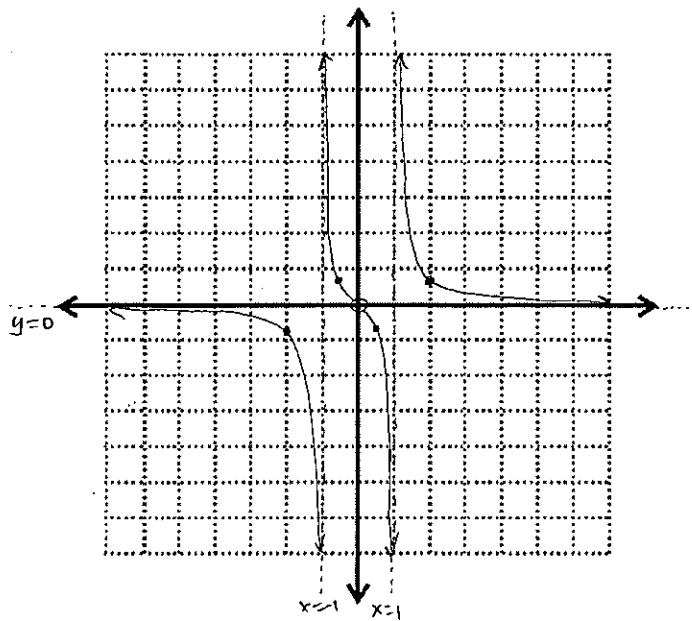
VA: $x = 1, -1$

HA: $y = 0$

SA: none

Hole: $(0, 0)$

x	$f(x)$
-2	-2/3
-1	0
0	0
1	0
2	2/3



$$3. f(x) = \frac{x^2 - 1}{2x - 3}$$

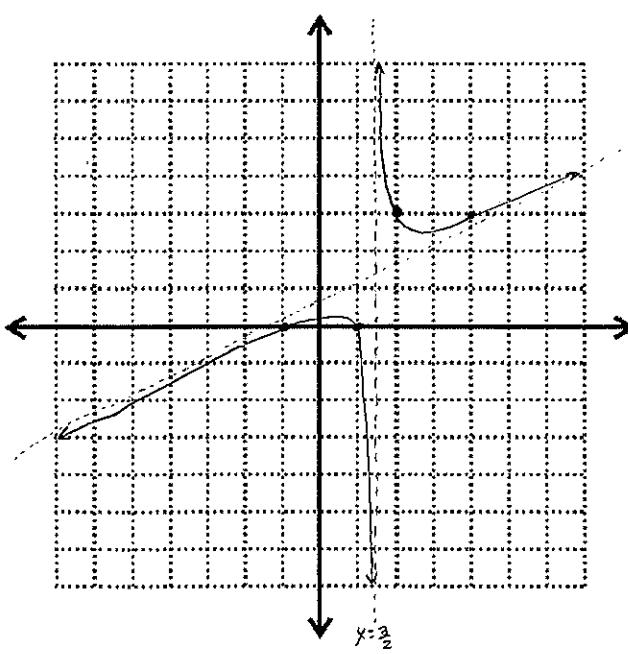
VA: $x = \frac{3}{2}$

HA: none

SA: $y = \frac{1}{2}x + \frac{3}{4}$

$$\frac{3}{2} \boxed{1} \quad D \quad \frac{3}{2} \quad -1$$

x	$f(x)$
1	0
-1	0
2	3
4	3



$$2. f(x) = \frac{x^2 - 4x + 4}{2x - 1} = \frac{(x-2)^2}{2x-1}$$

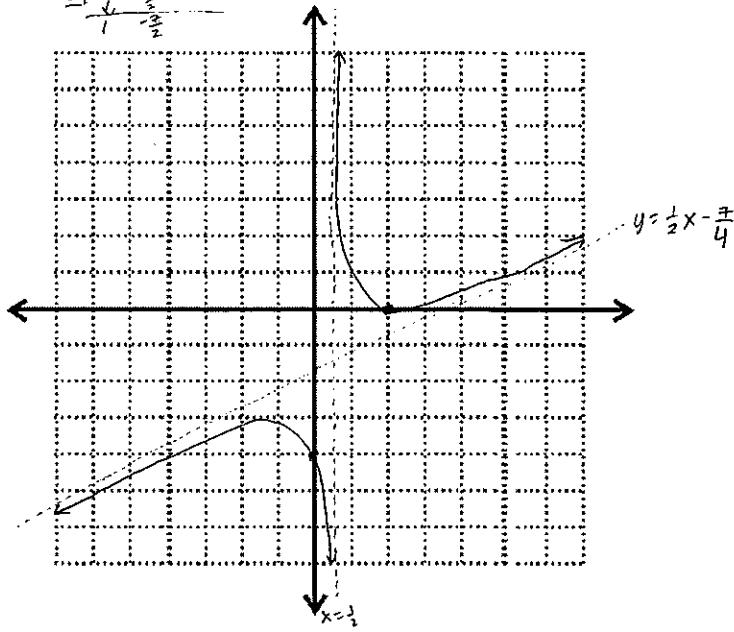
VA: $x = \frac{1}{2}$

HA: none

SA: $y = \frac{1}{2}x - \frac{7}{4}$

$$\frac{1}{2} \boxed{1} \quad -4 \quad \frac{7}{4}$$

x	$f(x)$
2	0
0	-4



$$4. f(x) = \frac{x-2}{x^2}$$

VA: $x = 0$

HA: $y = 0$

x	$f(x)$
2	0
-1	-3
1	-1
-2	-1

