

Polynomials and quadratic functions – review:

Polynomial Function:

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots$$

↳ n : integer, ≥ 0 , not fraction, nonnegative

Degree:

the largest power of x

Quadratic function:

1) $f(x) = ax^2 + bx + c$ (general form)

2) $f(x) = a(x-h)^2 + k$ (vertex form)

3) $f(x) = a(x-p)(x-q)$ (intercept form)

↳ x -intercepts: $(p, 0)$, $(q, 0)$

Parabolas:

Standard (vertex) form, vertex, and a value:

vertex form: $f(x) = a(x-h)^2 + k$

vertex: (h, k)

a -value: gives direction

↳ $a > 0$: opens up

↳ $a < 0$: opens down

Axis of symmetry:

$x = h$

Solutions/ x – intercepts/roots/zeros:

$f(x) = 0$

factor or use the quadratic equation

write as coordinates

Example: Write the function in vertex form: $f(x) = -3x^2 + 12x + 2$.

$$f(x) = -3(x^2 - 4x + \frac{4}{2}) + 2 - \frac{12}{2} \quad \left(\frac{-4}{2}\right)^2 = 2^2 = 4$$

$$f(x) = -3(x-2)^2 + 14$$

vertex: $(2, 14)$

Example: Write the function's equation.

vertex: $(-2, -1) = (h, k)$

$f(x) = a(x-h)^2 + k$

$f(x) = a(x+2)^2 - 1$

$f(-1) = a(-1+2)^2 - 1 = 0$

$a = 1$

$f(x) = (x+2)^2 - 1$

OR

x -intercepts: $(-3, 0)$, $(-1, 0)$

$f(x) = a(x-p)(x-q)$

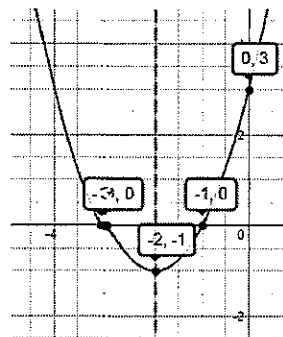
$f(x) = a(x+3)(x+1)$

$f(0) = a(0+3)(0+1) = 3$

$3a = 3$

$a = 1$

$f(x) = (x+3)(x+1)$



Practice:

Write the function in vertex form: $f(x) = 2x^2 + 8x - 10$. Then graph it and identify the vertex, AOS, domain/range, and the x and y intercepts.

$$f(x) = 2(x^2 + 4x + \frac{4}{2}) - 10 - \frac{8}{2}$$

vertex form: $f(x) = 2(x+2)^2 - 18$

vertex: $(-2, -18)$

AOS: $x = -2$

domain: $(-\infty, \infty)$

range: $[-18, \infty)$

x -intercepts: $(-5, 0)$, $(1, 0)$

$f(x) = 0$

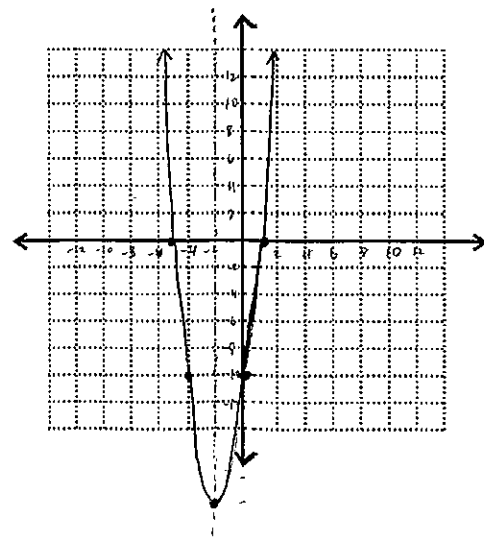
$2(x+2)^2 - 18 = 0$

$x+2 = \pm 3$

$x = -5, 1$

y -intercept: $(0, -10)$

$f(0) = 2(0)^2 + 8(0) - 10 = -10$



Writing equations, max/min values:

How many parabolas can go through the same vertex?

infinitely many

What feature makes these parabolas different?

- vertical / horizontal stretch
- intercepts

What features will all these parabolas have in common?

- vertex
- AOS

Example: find the equation for the quadratic function with a vertex at $(1, -3)$ that goes through the point $(2, 5)$.

$$f(x) = a(x-h)^2 + k$$

$$f(x) = a(x-1)^2 - 3$$

$$f(2) = a(2-1)^2 - 3 = 5$$


$$a = 8$$


$$f(x) = 8(x-1)^2 - 3$$

Max and min values:

Fact: All parabolas have either an absolute max or an absolute min.

Question: When will a parabola have an absolute max? Min?

max: $a < 0$ (opens down) 

min: $a > 0$ (opens up) 

Example: A local newspaper has daily production costs of $C = 55,000 - 108x + 0.06x^2$ where c is the total cost in dollars and x is the # of newspapers printed. How many newspapers should be printed to yield a minimum cost? What is the minimum cost?

$$C = 0.06x^2 - 108x + 55000$$

$$x = \frac{-b}{2a} = \frac{108}{2(0.06)} = 900$$

To yield a minimum cost, 900 newspapers should be printed. The minimum cost is \$6400.

$$C(900) = 0.06(900)^2 - 108(900) + 55000$$

$$C(900) = 6400$$

Practice:

Find the standard form equation for the parabola described:

1. Vertex at $(2, 3)$, zero at $(-1, 0)$

$$f(x) = a(x-h)^2 + k$$

$$f(x) = a(x-2)^2 + 3$$

$$f(-1) = a(-1-2)^2 + 3 = 0$$

$$-9a = -3$$

$$a = \frac{1}{3}$$

$$f(x) = \frac{1}{3}(x-2)^2 + 3$$

2. Vertex at $(-2, 0)$ and through the point $(-4, 6)$

$$f(x) = a(x-h)^2 + k$$

$$f(x) = a(x+2)^2$$

$$f(-4) = a(-4+2)^2 = 6$$

$$4a = 6$$

$$a = \frac{3}{2}$$

$$f(x) = \frac{3}{2}(x+2)^2$$

3. Find any two quadratic functions, one that opens upwards and one that opens downward through the given zeroes $(3, 0)$ and $(9, 0)$

opens up: $f(x) = (x-3)(x-9)$

opens down: $f(x) = -(x-3)(x-9)$

Chapter 2.2: Polynomial functions of higher degree - How do you sketch the graphs of polynomial functions?

Basics of the graphs of polynomials:

All polynomial graphs are *continuous* and contain only *smooth curves*.
End behavior is based on the leading coefficient and the degree.

Even degree: $f(x) = x^2$

$a > 0$ increase in both directions

$a < 0$ decrease in both directions

Odd degree: $f(x) = x^3$

$a > 0$ increase right decrease left

$a < 0$ increase left decrease right

Roots and factors:

If $x = a$ solves a polynomial function for $P(a) = 0$, then...

$x = a$ is a zero of the function

$x = a$ is a solution to $P(x) = 0$

$(x - a)$ must be a factor of $P(x)$

$(a, 0)$ is an x-intercept of the graph of $P(x)$.

Examples:

1. Determine the end behavior of the polynomial.

$$f(x) = -3x^5 + 2x^2$$

odd degree: $a < 0$

increase left, decrease right

2. Find the zeros of the polynomial $f(x) = x^4 - 4x^2$.

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x = 0, \pm 2 \Rightarrow x = \{-2, 0, 2\}$$

3. If 3, -2, and $\frac{1}{4}$ are zeroes of a polynomial, find a possible equation for the polynomial.

$$\begin{aligned} f(x) &= (x-3)(x+2)(4x-1) \\ &= (x^2-x-6)(4x-1) \\ &= 4x^3-4x^2-24x-x^2+x+6 \\ &= 4x^3-5x^2-23x+6 \end{aligned}$$

$$f(x) = 4x^3 - 5x^2 - 23x + 6$$

*must have $f(x) = \dots$

Multiplicity and irrational roots:

Multiplicity - # of times a specific zero solves the polynomial

ex: $f(x) = (x-2)^4$ — multiplicity

$$x = \{2 \text{ M4}\}$$

Odd:
graph will cross x-axis at the zero

Even:
graph will not cross x-axis at the zero

*be able to differentiate between M1 and M>1 when graphing

Dealing with algebraic irrationals (roots)

irrational roots always come in pairs (conjugates)

ex: if $\sqrt{3}$ is a zero, $-\sqrt{3}$ is a zero

if $2+\sqrt{7}$ is a zero, $2-\sqrt{7}$ is a zero

*use conjugate

Practice:

4. Find the real zeros of $P(x) = x^5 + 2x^4 + x^3$.

$$x^5 + 2x^4 + x^3 = 0$$

$$x^3(x^2 + 2x + 1) = 0$$

$$x^3(x+1)^2 = 0$$

$$x = 0, -1$$

$$x = \{-1 \text{ M2}, 0 \text{ M3}\}$$

5. Write a polynomial function with zeros at -1 (m 2), $\sqrt{3}$, $-\sqrt{3}$.

$$f(x) = (x+1)^2(x-\sqrt{3})(x+\sqrt{3})$$

$$= (x^2 + 2x + 1)(x^2 - 3)$$

$$= x^4 + 2x^3 + x^2 - 3x^2 - 6x - 3$$

$$f(x) = x^4 + 2x^3 - 2x^2 - 6x - 3$$

6. Write a polynomial function with zeros at 2, $1 + \sqrt{2}$.

$$f(x) = (x-2)(x-(1+\sqrt{2}))(x-(1-\sqrt{2}))$$

$$= (x-2)(x-1-\sqrt{2})(x-1+\sqrt{2})$$

$$= (x-2)(x^2 - 2x + 1 - 2)$$

$$= (x-2)(x^2 - 2x - 1)$$

$$= x^3 - 2x^2 - x - 2x^2 + 4x + 2$$

$$f(x) = x^3 - 4x^2 + 3x + 2$$

Sketching graphs of polynomials:

What are some main features we can find?

- end behavior (leading coefficient & degree)
- zeroes
- multiplicity
- y-intercept

How can we figure out behavior *between* the zeroes?

- plug in values

Practice: Sketch a reasonable graph of the function.

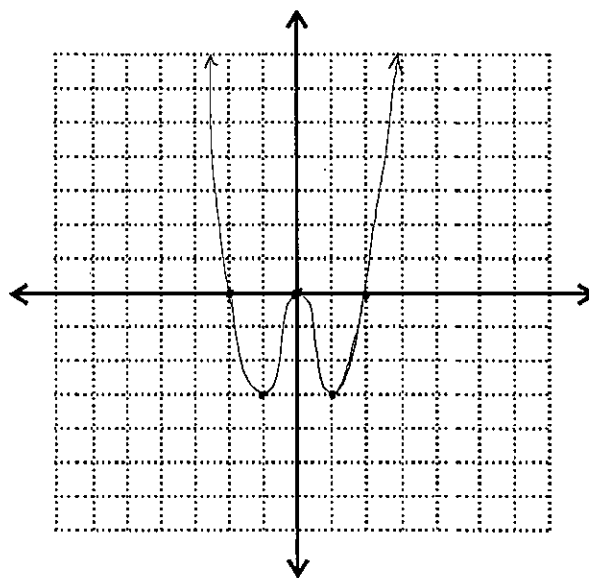
1. $f(x) = x^4 - 4x^2$ ↗ ↘

$$x^4 - 4x^2 = 0$$

$$x^2(x^2 - 4) = 0$$

$$x = 0, \pm 2$$

$$x = \{-2, 0 \text{ M } 2, 2\}$$



2. $f(x) = x^5 - 4x^3 + 8x^2 - 32$

$$x^5 - 4x^3 + 8x^2 - 32 = 0$$

$$x^3(x^2 - 4) + 8(x^2 - 4) = 0$$

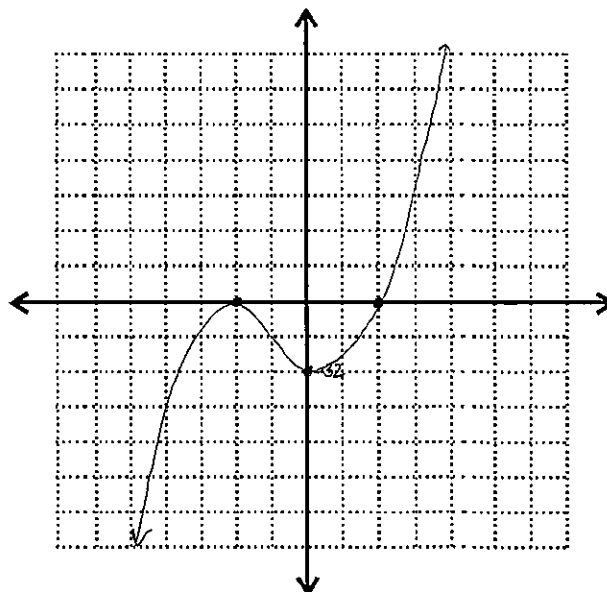
$$(x^3 + 8)(x^2 - 4) = 0$$

$$(x+2)^2(x-2)(x^2 - 2x + 4) = 0$$

$$x = -2, 2$$

$$x = \{-2 \text{ M } 2, 2\}$$

y-int: (0, -32)



Chapter 2.3 Using synthetic and polynomial division – How do you divide a polynomial by another polynomial and use polynomial division to find the rational and real zeros of polynomials?

Synthetic division:

When the *divisor* in a polynomial division problem is *linear* ($x - c$), there is a very nice algorithm we can use. It is called *Synthetic division*.

Synthetic division:

Example -

The set up: The *coefficients* of the dividend are written out on the top row.

Put the constant term of the divisor, c in the box. Put a place holder 0 for any missing exponent.

$$x^6 - 3x^5 + 14x + 4$$

$$x - 2$$

$$x - c \Rightarrow c = 2$$

zero \downarrow

2	1	-3	0	0	0	14	4	
	↓							
		2(1)						
		2	-2	-4	-8	-16	4	
		1	-1	-2	-4	-8	-2	0

$x^5 - x^4 - 2x^3 - 4x^2 - 8x - 2$

* remainder
→ factor
polynomial

The utility of synthetic division:

Synthetic division is an *algorithm* that works under certain constraints. It is not the only way to divide, but it's quick.

Factor theorem:

If $(x - k)$ is a factor of a polynomial, $P(x)$, then the quotient $\frac{P(x)}{(x-k)} = Q(x)$

$$P(x) = Q(x) \cdot (x - k)$$

If we can divide evenly by a linear factor, what will happen to the degree of the quotient compared to the degree of the dividend?

exponent decreases by 1

Example:

Factor $f(x) = 2x^3 + 11x^2 + 18x + 9$ completely if $(x + 3)$ is a factor of $f(x)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & \downarrow & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array} \checkmark$$

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

$$f(x) = (x + 3)(2x + 3)(x + 1)$$

Practice: Factor the polynomial equation completely using the given factors.

1. $f(x) = 3x^3 + 2x^2 - 19x + 6; (x + 3)$

$$\begin{array}{r|rrrr} -3 & 3 & 2 & -19 & 6 \\ & \downarrow & -9 & 21 & -6 \\ \hline & 3 & -7 & 2 & 0 \end{array} \checkmark$$

$$3x^2 - 7x + 2 = (3x - 1)(x - 2)$$

$$f(x) = (x + 3)(3x - 1)(x - 2)$$

2. $f(x) = x^4 + 4x^3 - x^2 - 16x - 12; (x + 3), (x + 1)$

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & -1 & -16 & -12 \\ & \downarrow & -3 & -3 & 12 & 12 \\ \hline -1 & 1 & 1 & -4 & -4 & 0 \end{array} \checkmark$$

$$\begin{array}{r|rrrr} & 1 & 0 & -4 & 4 \\ & \downarrow & -1 & 0 & 4 \\ \hline & 1 & -1 & -4 & 0 \end{array} \checkmark$$

$$x^2 - 4 = (x + 2)(x - 2)$$

$$f(x) = (x + 3)(x + 1)(x + 2)(x - 2)$$

Polynomial long division: When we want to divide by a non-linear factor, we have to use long division.

Ex: $\frac{x^4+5x^3+6x^2-x-2}{x+2} = x^3+3x^2-1$

$$\begin{array}{r} x^3+3x^2-1 \\ x+2 \overline{) x^4+5x^3+6x^2-x-2} \\ \underline{-(x^4+2x^3)} \\ 3x^3+6x^2 \\ \underline{-(3x^3+6x^2)} \\ -x-2 \\ \underline{-(-x-2)} \\ 0 \end{array}$$

Ex: $\frac{x^4+5x^3+12x^2+17x+5}{x^2+3x+1} = x^2+2x+5$

$$\begin{array}{r} x^2+2x+5 \\ x^2+3x+1 \overline{) x^4+5x^3+12x^2+17x+5} \\ \underline{-(x^4+3x^3+x^2)} \\ 2x^3+11x^2+17x \\ \underline{-(2x^3+6x^2+2x)} \\ 5x^2+15x+5 \\ \underline{-(5x^2+15x+5)} \\ 0 \end{array}$$

Rational zero test:

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with integer coefficients and $\frac{p}{q}$ is a rational root of $f(x)$...

\uparrow
q (leading coefficient)

\uparrow
p (constant)

$\frac{p}{q}$ = $\frac{\text{all factors of } p}{\text{all factors of } q}$

Practice: Determine the possible rational zeros of the polynomial.

1. $f(x) = 2x^5 + 4x^2 + 3$

$$\frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

2. $f(x) = 4x^4 + 3x^2 + x + 6$

$$\frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}$$

Use the rational zero test, and synthetic division, to factor the polynomial completely.

3. $f(x) = 2x^3 - 3x^2 - 3x + 2$

possible zeros: $\frac{\pm 1, \pm 2}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm \frac{1}{2}$

$$\begin{array}{r} 1 \mid 2 \quad -3 \quad -3 \quad 2 \\ \downarrow 2 \quad -1 \quad -4 \quad 2 \\ 2 \quad -1 \quad -4 \quad 2 \end{array}$$

$$\begin{array}{r} -1 \mid 2 \quad -3 \quad -3 \quad 2 \\ \downarrow 2 \quad -2 \quad 5 \quad -2 \\ 2 \quad -5 \quad 2 \quad 0 \end{array}$$

$$2x^2 - 5x + 2 = (2x-1)(x-2)$$

$$f(x) = (x+1)(2x-1)(x-2)$$

4. $f(x) = 3x^4 + 4x^3 - 11x^2 - 16x - 4$

possible zeros: $\frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm 3} = \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

$$\begin{array}{r} 1 \mid 3 \quad 4 \quad -11 \quad -16 \quad -4 \\ \downarrow 3 \quad 7 \quad -4 \quad -20 \quad -24 \\ 3 \quad 7 \quad -4 \quad -20 \quad -24 \end{array}$$

$$\begin{array}{r} -1 \mid 3 \quad 4 \quad -11 \quad -16 \quad -4 \\ \downarrow 3 \quad -3 \quad -1 \quad 12 \quad 4 \\ 3 \quad 1 \quad -12 \quad -4 \quad 0 \end{array}$$

$$\begin{array}{r} 2 \mid 3 \quad 1 \quad -12 \quad -4 \\ \downarrow 3 \quad 7 \quad 2 \quad 10 \\ 3 \quad 7 \quad 2 \quad 10 \end{array}$$

$$3x^2 + 7x + 2 = (3x+1)(x+2)$$

$$f(x) = (x+1)(x-2)(3x+1)(x+2)$$

Chapter 2.4: The complex number system – How do you perform operations with complex numbers?

Imaginary and complex numbers:

The imaginary Number: $i = \sqrt{-1}$, we will frequently use a property of radicals, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

$$\text{ex: } \sqrt{-5} = \sqrt{-1} \cdot \sqrt{5} = i\sqrt{5}$$

When a real number is multiplied by i , it is a number in the *complex number system*.

A *complex number* has two parts. Since real numbers and imaginary numbers are not like terms, then for **any** real numbers a and b , a complex number in standard form is defined as $a + bi$ * (often do:

$a = \text{real}$, $b = \text{imaginary}$

When doing operations with complex numbers, they must be treated as...

Examples:

$$1. (3 + 6i) - (4 + 2i)$$

$$= 3 + 6i - 4 - 2i$$

$$= \boxed{-1 + 4i}$$

$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$\text{ex) } i^{80}$$

$$= (i^4)^{20}$$

$$= \boxed{1}$$

$$3i + 2 \Rightarrow 2 + 3i$$

$$\frac{1+2i}{3} \Rightarrow \frac{1}{3} + \frac{2}{3}i$$

$$2. (3 + 6i)(4 + 2i)$$

$$= 12 + 6i + 24i + 12i^2$$

$$= 12 + 30i - 12$$

$$= \boxed{30i}$$

$$\text{ex) } i^{123}$$

$$= (i^4)^{30} i^3$$

$$= \boxed{-i}$$

Practice:

$$1. (9 + 2i) + (1 - 7i)$$

$$= \boxed{10 - 5i}$$

$$2. (6 - 11i)^2$$

$$= 36 - 132i + 121i^2$$

$$= \boxed{-85 - 132i}$$

$$3. (3 + 7i)(2 - 4i)$$

$$= 6 - 12i + 14i - 28i^2$$

$$= \boxed{34 + 2i}$$

$$4. \sqrt{-4} + (-4 - \sqrt{-4})$$

$$= 2i - 4 - 2i$$

$$= \boxed{-4}$$

$$5. \sqrt{-4} \cdot \sqrt{-16}$$

$$= 2i(4i)$$

$$= 8i^2$$

$$= \boxed{-8}$$

The complex conjugate, rationalization and graphing:

If $a + bi$ is a complex number, what is its conjugate?

$$a - bi$$

Stylistically, we are not allowed to have a complex number in a denominator of a fraction. So, you must *rationalize* any denominators that have an i in them.

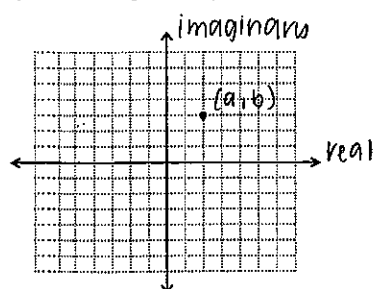
Example: rationalize the following:

$$\frac{6+i}{2-i} \cdot \frac{2+i}{2+i}$$

$$= \frac{12 + 8i - 1}{4 + 1}$$

$$= \frac{11 + 8i}{5} = \boxed{\frac{11}{5} + \frac{8}{5}i}$$

Graphing a complex number is easy, instead of using a Cartesian plane, you use a similar set up with axes labeled (real, imaginary), so to plot $a + bi$, you plot the ordered pair...



$$-2 + 3i \Rightarrow (-2, 3)$$

$$a + bi \Rightarrow (a, b)$$

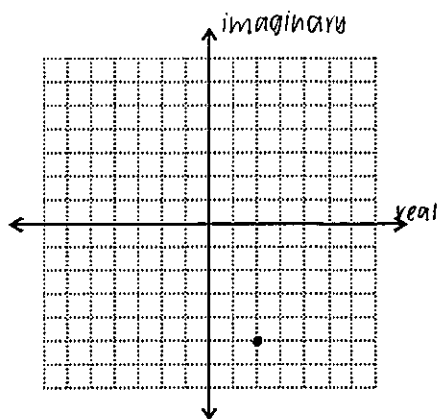
Practice:

$$\begin{aligned}
 1. \quad & \frac{4+i}{3-2i} \cdot \frac{3+2i}{3+2i} \\
 & = \frac{12+11i-2}{9+4} \\
 & = \frac{10+11i}{13} \\
 & = \boxed{\frac{10}{13} + \frac{11}{13}i}
 \end{aligned}$$

2. Find the reciprocal of $1 + 3i$

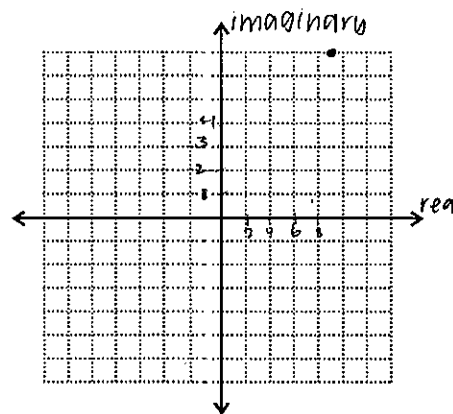
$$\begin{aligned}
 & \frac{1}{1+3i} \cdot \frac{1-3i}{1-3i} \\
 & = \frac{1-3i}{1+9} \\
 & = \boxed{\frac{1}{10} - \frac{3}{10}i}
 \end{aligned}$$

3. Graph $2 - 5i$



4. Graph $(3 - i)(2 + 3i)$

$$\begin{aligned}
 & (3-i)(2+3i) \\
 & = 6 + 7i + 3 \\
 & = \boxed{9 + 7i}
 \end{aligned}$$



Chapter 2.5: The fundamental theorem of algebra – How do you find all the zeros of a polynomial function?

The fundamental theorem of algebra and linear factorization theorem:

The fundamental theorem of algebra states...

every n^{th} degree polynomial has exactly n complex zeros

Linear factorization theorem states...

every n^{th} degree polynomial with n complex zeros has n linear factors

$$\hookrightarrow f(x) = a(x-c_1)(x-c_2)(x-c_3)$$

If a complex number is a zero of a polynomial, then...

the conjugate is also a zero

* all irrational and imaginary zeros come in pairs

Practice: Find the zeros of the function and write as a product of linear factors.

1. $f(x) = x^2 + 10x + 23$

$$x^2 + 10x + 23 = 0$$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(1)(23)}}{2}$$

$$= \frac{-10 \pm \sqrt{8}}{2}$$

$$x = \{-5 \pm \sqrt{2}\}$$

$$f(x) = (x + 5 - \sqrt{2})(x + 5 + \sqrt{2})$$

2. $f(x) = x^4 - 81$

$$x^4 - 81 = 0$$

$$(x^2 - 9)(x^2 + 9) = 0$$

$$(x+3)(x-3)(x+3i)(x-3i) = 0$$

$$x = \{\pm 3, \pm 3i\}$$

$$f(x) = (x+3i)(x-3i)(x+3)(x-3)$$

3. $f(x) = x^4 - 5x^2 - 66$

$$x^4 - 5x^2 - 66 = 0$$

$$(x^2 - 11)(x^2 + 6) = 0$$

$$x = \{\pm \sqrt{11}, \pm i\sqrt{6}\}$$

$$f(x) = (x + \sqrt{11})(x - \sqrt{11})(x + i\sqrt{6})(x - i\sqrt{6})$$

Use the given zeroes to write a polynomial function with real coefficients

5. $3, 4i, -4i$

$$f(x) = (x-3)(x+4i)(x-4i)$$

$$= (x-3)(x^2+16)$$

$$f(x) = x^3 - 3x^2 + 16x - 48$$

6. $-1, -1, 2+5i, 2-5i$

$$f(x) = (x+1)^2(x-2-5i)(x-2+5i)$$

$$= (x^2+2x+1)(x^2-4x+4+25)$$

$$= (x^2+2x+1)(x^2-4x+29)$$

$$= x^4 + 4x^3 + 29x^2 - 2x^3 + 8x^2 + 58x + x^2 - 4x + 29$$

$$f(x) = x^4 - 2x^3 + 21x^2 + 54x + 29$$

Factoring and specific polynomials: $f(x) = x^4 - 5x^2 - 66$

Completely factoring over rational numbers:

*no irrational / imaginary

$$f(x) = (x^2 - 11)(x^2 + 6)$$

Completely factoring over real numbers:

*no imaginary

$$f(x) = (x + \sqrt{11})(x - \sqrt{11})(x^2 + 6)$$

Completely factoring over complex numbers:

*fully factored

$$f(x) = (x + \sqrt{11})(x - \sqrt{11})(x + i\sqrt{6})(x - i\sqrt{6})$$

Given zeros and a point on the function:

find a:

$$f(x) = a(x - c_1)(x + c_2)$$

→ use given point to find a

Practice: Given the zeros, multiplicities, and a point on the function, determine the equation of the polynomial.

1. Zeros at $3 - i, -2, 1$, degree 4, $f(2) = 4$

$3 + i$

$$f(x) = a(x + 2)(x - 1)(x - 3 + i)(x - 3 - i)$$

$$f(x) = a(x^2 + x - 2)(x^2 - 6x + 10)$$

$$f(2) = a(2^2 + 2 - 2)(2^2 - 6(2) + 10) = 4$$

$$2a = 1$$

$$a = \frac{1}{2}$$

$$f(x) = \frac{1}{2}(x^4 - 6x^3 + 10x^2 - 2x - 20)$$

$$f(x) = \frac{1}{2}(x^4 - 6x^3 + 10x^2 - 2x - 20)$$

2. Zeros at $\sqrt{3}, i, 2$, degree 5, $f(-1) = 4$

$-\sqrt{3}, -i$

$$f(x) = a(x + \sqrt{3})(x - \sqrt{3})(x + i)(x - i)(x - 2)$$

$$f(x) = a(x^2 - 3)(x^2 + 1)(x - 2)$$

$$f(-1) = a((-1)^2 - 3)((-1)^2 + 1)(-1 - 2) = 4$$

$$a(-2)(2)(-3) = 4$$

$$a = \frac{1}{3}$$

$$f(x) = \frac{1}{3}(x^4 - 2x^2 - 3)(x - 2)$$

$$f(x) = \frac{1}{3}(x^5 - 2x^3 - 3x - 2x^4 + 4x^2 + 6)$$

$$f(x) = \frac{1}{3}(x^5 - 2x^4 - 2x^3 + 4x^2 - 3x + 6)$$

Factor each polynomial a) Over the rational numbers b) over the real numbers c) over the complex numbers.

3. $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$, $1 + 3i$ is a root

$1 - 3i$

$$(x - 1 - 3i)(x - 1 + 3i)$$

$$= x^2 - 2x + 10$$

$$= x^2 - 2x + 10$$

$$\begin{array}{r} x^2 - 2x + 10 \overline{) x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{-(x^4 - 2x^3 + 10x^2)} \\ -x^3 - 4x^2 + 2x \\ \underline{-(-x^3 + 2x^2 - 10x)} \\ -6x^2 + 12x - 60 \\ \underline{-(-6x^2 + 12x - 60)} \\ 0 \end{array}$$

$$a) f(x) = (x^2 - 2x + 10)(x - 3)(x + 2)$$

$$b) f(x) = (x^2 - 2x + 10)(x - 3)(x + 2)$$

$$c) f(x) = (x - 1 - 3i)(x - 1 + 3i)(x - 3)(x + 2)$$

4. $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$, $-2i$ and 1 are roots

$2i$

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & \downarrow & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array}$$

$$x^4 + x^3 + 2x^2 + 4x - 8$$

possible zeros: $\pm 1, \pm 2, \pm 4, \pm 8$

$$\begin{array}{r|rrrrrr} 1 & 1 & 1 & 2 & 4 & -8 \\ & \downarrow & 1 & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 8 & 0 \end{array}$$

$$x^3 + 2x^2 + 4x + 8 = x^2(x + 2) + 4(x + 2) = (x^2 + 4)(x + 2)$$

$$a) f(x) = (x - 1)^2(x + 2)(x^2 + 4)$$

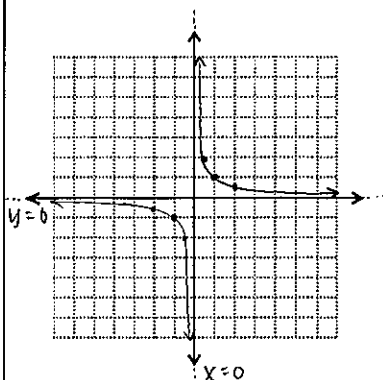
$$b) f(x) = (x - 1)^2(x + 2)(x^2 + 4)$$

$$c) f(x) = (x - 1)^2(x + 2)(x - 2i)(x + 2i)$$

Chapter 2.6: Rational functions and asymptotes – How do you find the domain and asymptotes of a rational function?

Rational function: $f(x) = \frac{N(x)}{D(x)}$

Parent function: $f(x) = \frac{1}{x}$



x	f(x)
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Discontinuous: vertical asymptote

Asymptotes: ALWAYS an equation of a line ($x = \dots$ or $y = \dots$)

Vertical asymptotes:

- create discontinuity in graph (break)
- you can never cross a vertical asymptote
- to find: denominator = 0

Horizontal asymptotes:

$$y = \frac{a \cdot x^m}{b \cdot x^n}$$

1) $m > n$: none

2) $m = n$: $y = \frac{a}{b}$

3) $m < n$: $y = 0$

Holes: always coordinates

- removable discontinuity
- occur when you reduce
- plug in x-value into remaining function

Examples: Find any asymptotes, zeros, holes and use the behavior at critical points and extremes to graph the function.

1. $f(x) = \frac{3}{x-2}$

vertical asymptote: $x = 2$

horizontal asymptote: $y = 0$

holes: none

x	f(x)
0	$-\frac{3}{2}$
4	$\frac{3}{2}$

STEPS:

- 1) factor numerator and denominator
→ cross out same factors and write remaining function
→ remaining: graph
- 2) find all asymptotes

2. $f(x) = \frac{x^2+2x+1}{2x^2-x-3} = \frac{(x+1)^2}{(2x-3)(x+1)}$

$f(x) = \frac{x+1}{2x-3}, x \neq -1$

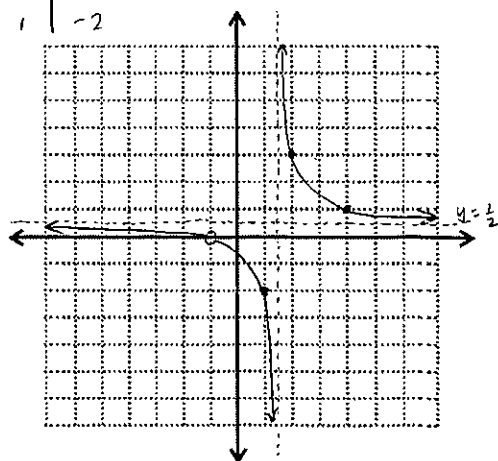
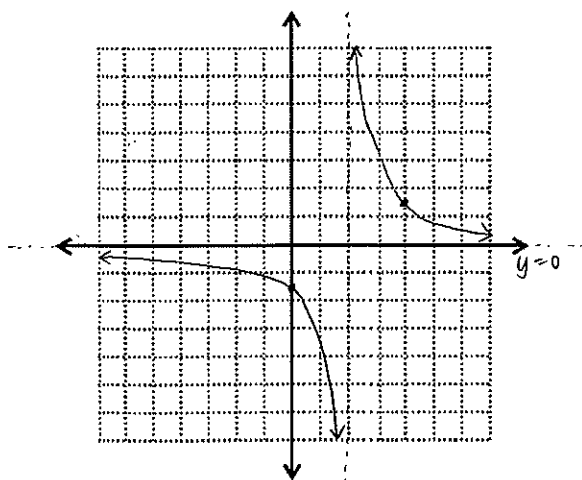
vertical asymptote: $x = \frac{3}{2}$

horizontal asymptote: $y = \frac{1}{2}$

hole: $(-1, 0)$

$\frac{-1+1}{2(-1)+3} = 0$

x	f(x)
4	1
1	-2



Practice: Find any asymptotes, zeroes, holes and use the behavior at critical points and extremes to graph the function.

1. $f(x) = \frac{2x+3}{x-1}$

VA: $x=1$

HA: $y=2$

holes: none

x	f(x)
0	-3
2	7

2. $f(x) = \frac{x^2+x-2}{x^2-x-6}$

$= \frac{(x+2)(x-1)}{(x-3)(x+2)}$

$f(x) = \frac{x-1}{x-3}, x \neq -2$

VA: $x=3$

HA: $y=1$

holes: $(-2, \frac{3}{5})$

$\frac{-2-1}{-2-3} = \frac{3}{5}$

x	f(x)
2	-1
5	2

3. $f(x) = \frac{x-1}{x^2-9}$

VA: $x=3, x=-3$

HA: $y=0$

holes: none

x-intercept: $(1, 0)$

$x-1=0$

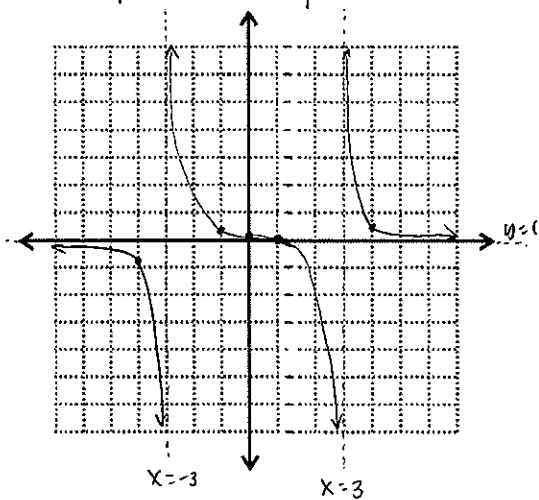
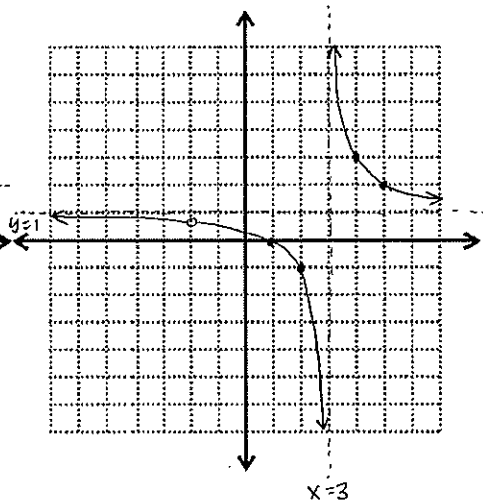
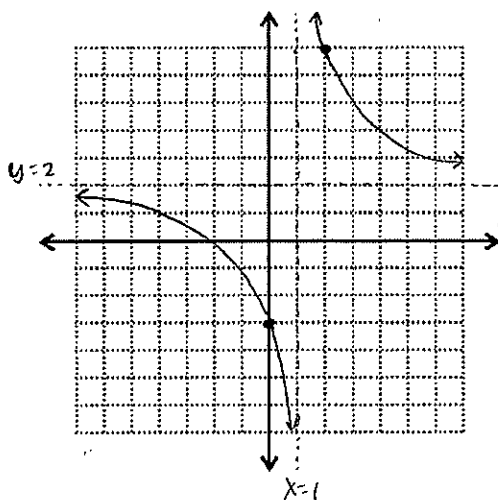
$x=1$

y-intercept: $(0, \frac{1}{9})$

$f(0) = \frac{0-1}{0^2-9} = \frac{1}{9}$

x	f(x)
-4	-5/7
-1	1/4

x	f(x)
2	-1/5
4	3/7



3. What could be the equation of the function?

VA: $x=1, x=-1$

HA: $y=1$

x-int: $(\pm 2, 0)$

$f(x) = \frac{(x+2)(x-2)}{(x+1)(x-1)}$

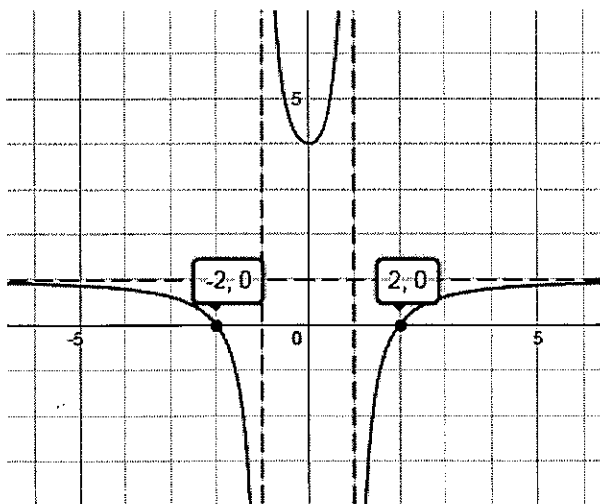
4. Write the equation of any function that fits the following criteria:

a) Horizontal Asymptote: $y=0$

b) Vertical Asymptote: $x=-1$

c) Zero: $x=1$

$f(x) = \frac{x-1}{(x+1)^2}$



The slant asymptote:

Third case: The degree of the numerator is *exactly one more* than the degree of the denominator. Use long division!

Describe the slant asymptotes:

1. $f(x) = \frac{2x}{1}$

$$\begin{array}{r} 2x \\ 1 \overline{) 2x} \\ \underline{2x} \\ 0 \end{array}$$

$y = 2x$

2. $f(x) = \frac{2x^2+1}{x}$

$$\begin{array}{r} 2x \\ x \overline{) 2x^2 + 0x + 1} \\ \underline{2x^2 + 0x} \\ 0 + 1 \\ 1 \end{array}$$

$y = 2x$

3. $f(x) = \frac{2x^3}{2x^2+1}$

$$\begin{array}{r} x \\ 2x^2+0x+1 \overline{) 2x^3+0x^2+0x+0} \\ \underline{2x^3+0x^2+x} \\ 0 + 0 + 0 \\ -x+0 \end{array}$$

$y = x$

Graphing complex rational functions:

Things to note:

All horizontal and slant asymptotes can be found by finding the quotient through division and disregarding the remainder.

Ex:

1. $f(x) = \frac{4x+5}{2x-1}$

$$\begin{array}{r} 2 \\ 2 \overline{) 4} \\ \underline{4} \\ 0 \end{array}$$

$y = 2$

2. $f(x) = \frac{x^2+2x+1}{x-3}$

$$\begin{array}{r} x+5 \\ x \overline{) x^2+2x+1} \\ \underline{x^2-3x-15} \\ 5x+16 \end{array}$$

$y = x+5$

1. We need to check behavior in between all critical x - values which include zeros and vertical asymptotes.

2. There will always be either a horizontal or a slant asymptote, not both.

3. Graphs of rational functions *can* cross a horizontal or slant asymptote when near the center of the graph, asymptotes only describe what happens at the extremes.

4. Each non removable domain restriction will give you a unique vertical asymptote.

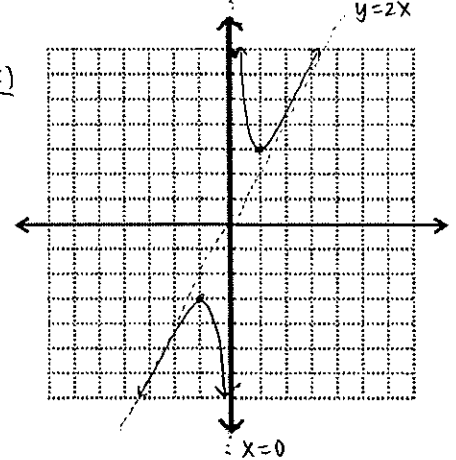
5. Always find the y -value of a hole by plugging the removable domain restriction's x -value into the simplified function.

Practice: Sketch a graph of the rational function by finding the key features.

1. $f(x) = \frac{2x^2+1}{x}$

vertical asymptote: $x=0$
horizontal asymptote: none
slant asymptote: $y=2x$

x	$f(x)$
1	3
-1	-3



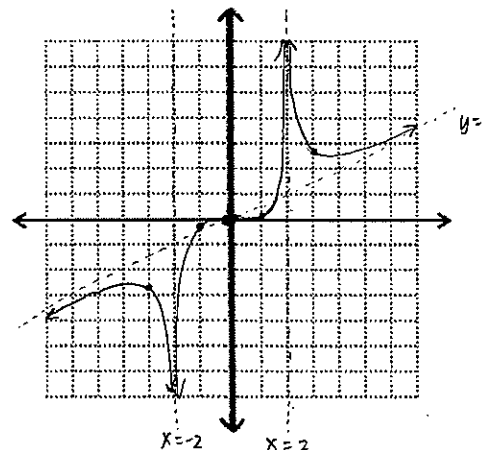
2. $f(x) = \frac{x^3}{2x^2-8}$

vertical asymptote: $x=2, x=-2$
horizontal asymptote: none
slant asymptote: $y=\frac{1}{2}x$

$$\begin{aligned} 2x^2-8 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

$$\begin{array}{r} \frac{1}{2}x \\ 2x^2+0x-8 \overline{) x^3+0x^2+0x+0} \\ \underline{x^3+0x^2+x} \\ 0 + 0 + 0 \\ -x-8 \end{array}$$

x	$f(x)$	x	$f(x)$
1	1/6	3	2.7
-1	-1/6	-3	-2.7

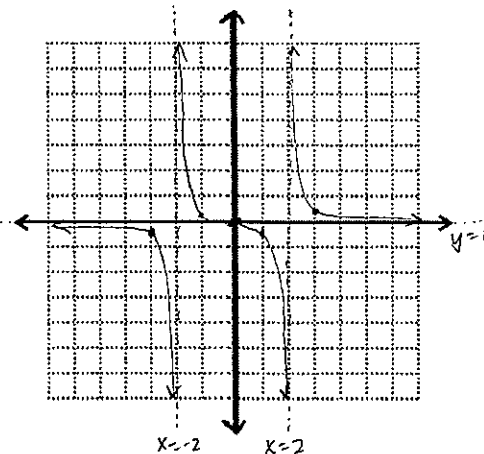


3. $f(x) = \frac{x}{x^2-4}$

vertical asymptote: $x=2, x=-2$
horizontal asymptote: $y=0$
slant asymptote: none

$$\begin{aligned} x^2-4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

x	$f(x)$
1	-1/3
-1	1/3
-3	-3/5
3	3/5



Practice: Graph each rational function.

$$1. f(x) = \frac{x^2}{x^3 - x} = \frac{x^2}{x(x^2 - 1)} = \frac{x}{x^2 - 1}, x \neq 0$$

VA: $x = 1, -1$

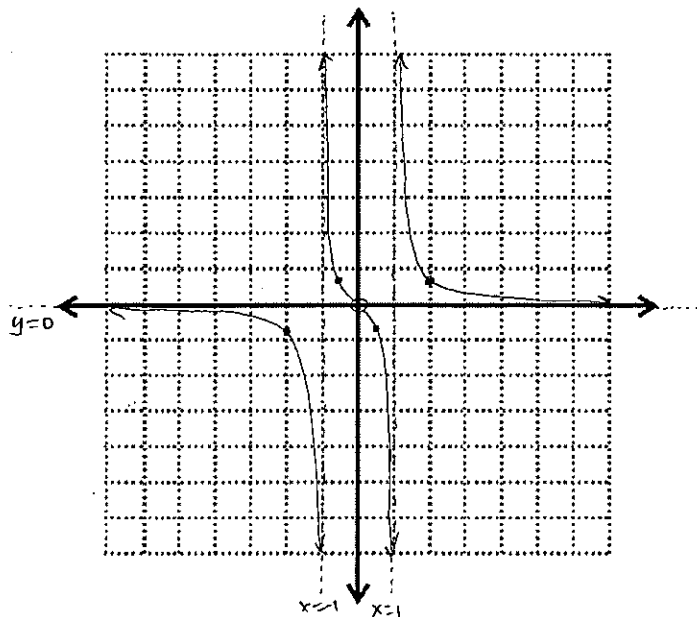
HA: $y = 0$

SA: none

Hole: $(0, 0)$

x	f(x)
-2	-2/3
-1/2	2/3

x	f(x)
1/2	-2/3
2	2/3



$$2. f(x) = \frac{x^2 - 4x + 4}{2x - 1} = \frac{(x-2)^2}{2x-1}$$

VA: $x = \frac{1}{2}$

HA: none

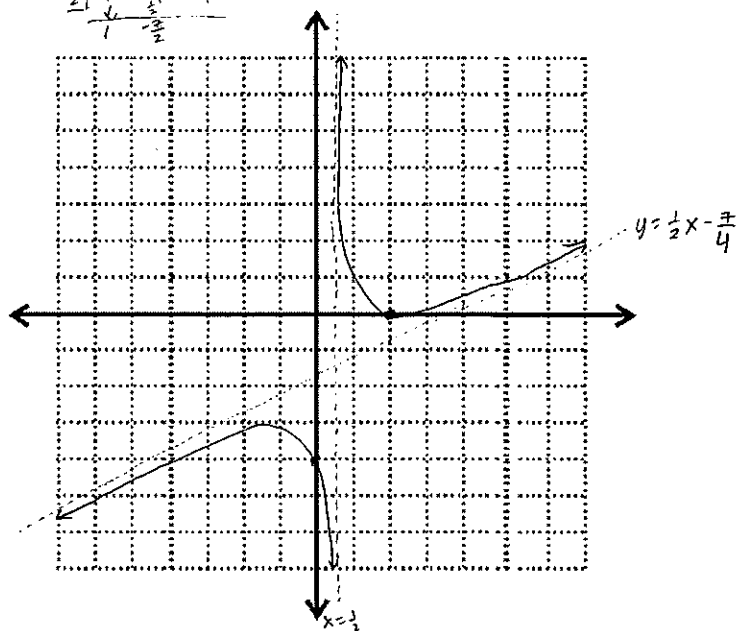
SA: $y = \frac{1}{2}x - \frac{7}{4}$

$$\frac{1}{2} \overline{) 1 \quad -4 \quad 4}$$

$$\underline{1 \quad -2 \quad 2}$$

$$0 \quad 0 \quad 0$$

x	f(x)
2	0
0	-4



$$3. f(x) = \frac{x^2 - 1}{2x - 3}$$

VA: $x = \frac{3}{2}$

HA: none

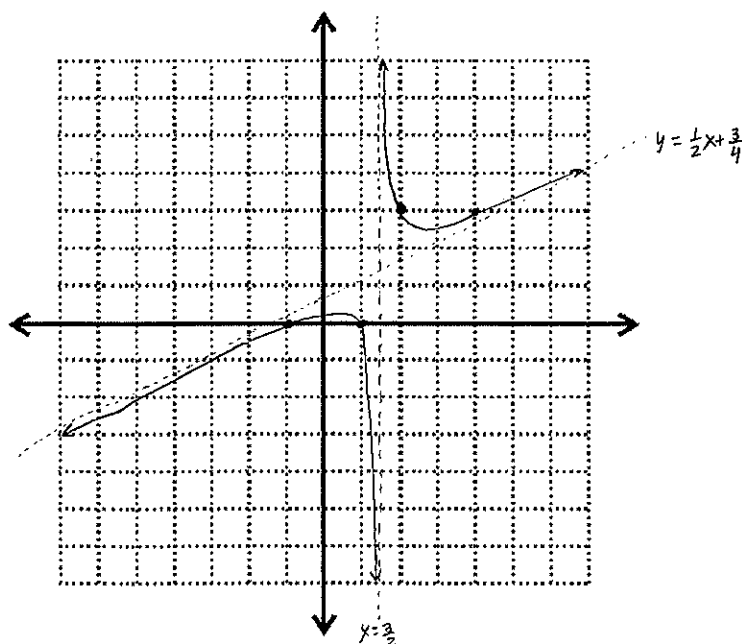
SA: $y = \frac{1}{2}x + \frac{3}{4}$

$$\frac{1}{2} \overline{) 1 \quad 0 \quad -1}$$

$$\underline{1 \quad 3/2}$$

$$0 \quad -3/2$$

x	f(x)
1	0
-1	0
2	3
4	3



$$4. f(x) = \frac{x-2}{x^2}$$

VA: $x = 0$

HA: $y = 0$

x	f(x)
2	0
-1	-3
1	-1
-2	-1

