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A credibilistic decision support system for portfolio optimization

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ABSTRACT

In this paper, a Decision Support System (DSS) for generating a suitable portfolio for an investor in an uncertain multi-criteria framework is proposed. We model uncertain parameters like return and illiquidity of various assets using L–R fuzzy numbers belonging to a power reference function family. Such usage of L–R fuzzy numbers is more generic as compared to the conventional triangular or trapezoidal fuzzy numbers and is a closer representation of uncertain behavior of the asset parameters. The credibility measure which has an advantage of being self-dual as compared to usual possibility measure marks the uncertain context of the entire setup and adds a new dimension to existing studies. We use an "Entropy-Cross Entropy (ECE) Algorithm", for finding the solution of the optimization problem meant for finding the best fit L–R fuzzy number corresponding to uncertain asset parameters. This automates the entire subjective exercise where, otherwise, a human intervention is required for feeding the parameters required for fitting L–R fuzzy number. Once the L–R fuzzy number are created around various assets available for the portfolio formation, the portfolio optimization problem is solved using Hybrid Intelligent Algorithm (HIA). HIA is designed by embedding fuzzy simulation within the "MIBEX-SM" genetic algorithm. To demonstrate the entire solution approach, four portfolio optimization models are solved using historical data from the National Stock Exchange (NSE) of India. The performance of the models is compared using a modified Sharpe ratio in the fuzzy context, namely the "Credibilistic Sharpe Ratio (CrSR)". To carry out the empirical study, firstly we use the data from the period of 2008–2013 for training the optimization models. Thereafter, to test the performance of the models, the data set from the two year period 2013–2015 is used. The process of validation of the results is then carried out using the data set of the one year period from 2015 to 2016.

1. Introduction

Investment decisions belong to a complex multi-criteria decision making area that has benefited a great deal from soft computing techniques prevalent in literature [1–10]. Fuzzy set theory [11] has contributed a lot in order to handle vagueness and ambiguity such as "high risk", "low returns" and "low liquidity", embedded in financial investment markets. Specifically, while modeling fuzzy portfolio selection problems an extensive usage of possibility measure has been seen in literature [12–15]. However, the possibility measure has a limitation that it is not self-dual, a property that is quite desirable. Liu and Liu [16] proposed a self-dual credibility measure that has recently become popular for modeling the uncertain parameters like return and illiquidity (antagonistic to liquidity) of assets involved in forming suitable portfolios [17–20]. The usual convention is to treat membership functions of uncertain asset parameters as linear, trapezoidal, triangular and S-shaped [21–26]. However, these shapes may not be the best available fit to the historical data related to uncertain asset parameters. Membership functions corresponding to more generic L–R fuzzy numbers proves to be a better choice for such modeling processes [20,26–29]. Fitting of L–R fuzzy number can be visualized as a bi-objective optimization problem in which one simultaneously seeks to maximize the entropy that measures the ambiguity within the information and minimize the cross-entropy that quantify the deviation of the fuzzy membership function from the historical data [30–33].

The review of literature so far marks some selected studies [20,26,28] that discuss L–R fuzzy fitting of uncertain parameters under study. These are accompanied by a limitation where parameters required for L–R fuzzy fitting are feeded manually. This involves domain expert's knowledge and his/her specific inputs. It is not possible that each time the experience articulated by the domain experts in the form of these parameter values turns out to be pre-

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cise and authentic. Thus, arises a need for an intelligent system that can automatically fit the most appropriate (best fit) L–R fuzzy number around the uncertain parameters by extracting the information inbuilt within the data set. Moreover, there are very few studies [20,29] that use credibility measure for fitting L–R fuzzy number. This study attempts to create a decision support system where we firstly automate the fitting process of L–R fuzzy numbers around uncertain asset parameters in a credibilistic framework. Thereafter, these fuzzy boundaries are used to simulate uncertain values of asset return and illiquidity. Finally, portfolio optimization models are solved to obtain the most relevant portfolio of assets. This is done through a soft computing approach namely HIA, wherein fuzzy simulation is embedded within the "MIBEX-SM" genetic algorithm.

The key differences between closely related studies [20,29] of L–R fuzzy fitting in a credibilistic setup and this study is that in [20] the parameters for L–R fuzzy fitting need human intervention thus involving subjectivity, whereas in this study we automate the process by solving a bi-objective optimization problem consisting of entropy and cross-entropy functions. The entropy and cross-entropy functions to fit an L–R fuzzy number without subjective human intervention have been used earlier by Jalota et al. [29]. However, distinct differences between [29] and this work are as follows:

i) In both the approaches ECE methodology is used to create an L–R fuzzy number around uncertain parameters in the context of portfolio formation. Note that ECE methodology in itself relies on solving a bi-objective optimization problem. [29]: The objectives and unknowns of ECE method form a part of the portfolio selection problem being solved. Thus, as one solves the portfolio selection model for the resultant portfolio, simultaneously he/she is able to create L–R fuzzy numbers corresponding to uncertain portfolio return and illiquidity. Note that, in [29] the individual asset parameters are not fuzzy. Proposed study: There are two stages in the proposed methodology. We are dealing with a context where parameters of individual assets are represented as L–R fuzzy numbers. We run ECE methodology individually on each asset and obtain L–R fuzzy numbers for uncertain asset parameters like asset return and illiquidity. Later we use these fuzzy bounds on uncertain parameters computed for each asset, to solve the portfolio selection problem. The main advantage of using L–R fuzzy numbers for individual assets is that better information regarding the behavior of each asset and hence, its contribution in forming the portfolio gets captured in this approach. Thus, the methodology used in the present work is more suitable to capture the uncertain behavior of financial markets.

ii) [29]: Crisp equivalents of various functions of return, risk and illiquidity have been used for computing their functional values in various portfolio selection models. This makes the approach in [29] as less time (computational time) consuming. Proposed study: Once L–R fuzzy numbers have been generated around the individual asset parameters, one needs information on the return and illiquidity of the portfolio arising from combination of these assets. It may not be possible to assess the nature of the fuzzy number arising as the sum of different L–R fuzzy numbers. Thus, we resort to the use of fuzzy simulation for simulating the return, risk and illiquidity between the bounds specified by the L–R fuzzy numbers for the parameters of the individual assets. These simulated values are then used to compute the functional values of the portfolio selection model. However, the use of fuzzy simulation increases the computational time required to run this approach. To summarize this point we may say that approach in [29] is faster but the one proposed in this work carries more information on assets and hence results in better portfolios.

iii) [29]: ECE is run within the MIBEX-SM GA. Proposed study: ECE is used independently outside the MIBEX-SM GA. In fact the algorithm that is used in the proposed work is a Hybrid Intelligent Algorithm, in which we run the fuzzy simulation within the MIBEX-SM GA. Even though the MIBEX-SM GA used in this work uses the same set of genetic operators as in [29], but the one used in this work distinguishes itself from [29] on the basis of fitness function evaluation. The functional evaluation in this work is not a crisp equivalent as in [29] but is handled differently through the process of fuzzy simulation.

iv) When compared to [29] another difference is that once the portfolio optimization models have been solved and compared, the results are validated by taking a future data set. Such a process of validation was missing in [29]. A modified version of the Sharpe ratio, namely the Credibilistic Sharpe Ratio (CrSR) is used for comparing the performance of various portfolio optimization models.

The organization of this paper is as follows: In Section 2, the basics of L–R fuzzy numbers and credibility theory is presented. The Entropy and Cross-Entropy (ECE) methodology to automatically fit L–R fuzzy numbers is also discussed in this section. A mathematical formulation of the four portfolio selection models handled in this study is given in Section 3. The details of HIA are provided in Section 4. Section 5 contains the experimental set-up followed by Section 6 with discussion of results in detail. We use a data set from the NSE in India, from the period 2008 to 2013 for training the models and 2013 to 2015 for testing the performance of the models. Moreover, a comparison of ECE methodology with the existing methodology is also made in Section 5 and 6. A novel approach of validating the results using a data set of a one year period from 2015 – 2016, to select the most robust portfolio selection model, is also presented in Section 6. Finally, conclusions from the current study are drawn in Section 7.

2. Preliminaries

In this section we present some preliminary discussion on the L–R power fuzzy number, possibility, necessity and credibility of a fuzzy event, as well as the ECE methodology.

2.1. L–R power fuzzy number

The reference functions $\Phi_p(.), \Phi_q(.) : [0, 1] \rightarrow [0, 1]$ of an L–R fuzzy number are strictly non-increasing and upper semi-continuous functions such that [34]:

$$\Phi_p(a), \Phi_q(a) = \begin{cases} 0 & \text{if } a = 1 \\ 1 & \text{if } a = 0 \end{cases}$$

The expression of membership function μ, of an L–R-type fuzzy number, i.e., $\tilde{A} = (\zeta_a, \zeta_b, \zeta_c, \zeta_d)_{pq}$ with $\zeta_a < \zeta_b < \zeta_c < \zeta_d$, is as follows:

$$\mu_{\tilde{A}}(\omega) = \begin{cases} \Phi_p\left(\dfrac{\zeta_b - \omega}{\zeta_b - \zeta_a}\right), & \text{if } \zeta_a \le \omega < \zeta_b \\ 1, & \text{if } \zeta_b \le \omega \le \zeta_c \\ \Phi_q\left(\dfrac{\omega - \zeta_c}{\zeta_d - \zeta_c}\right), & \text{if } \zeta_c < \omega \le \zeta_d \end{cases}$$

where $(\zeta_b - \zeta_a)$ and $(\zeta_d - \zeta_c)$ are the left and right spreads respectively and $p, q > 0$ are the shape parameters. The left and right

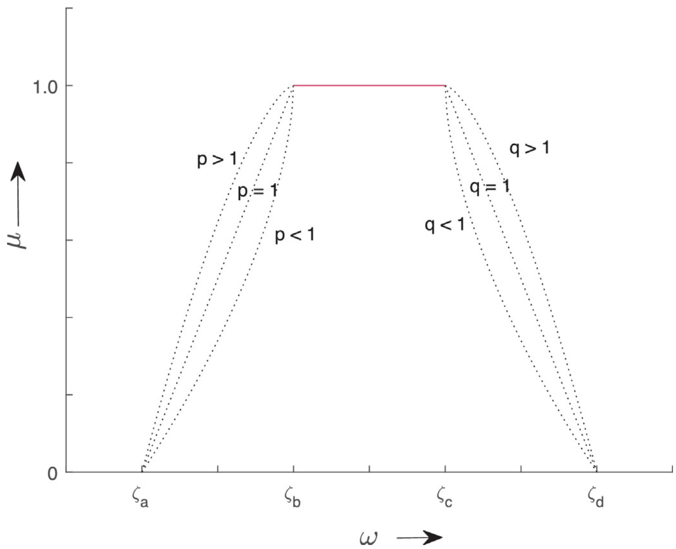


Fig. 1. L–R fuzzy number.

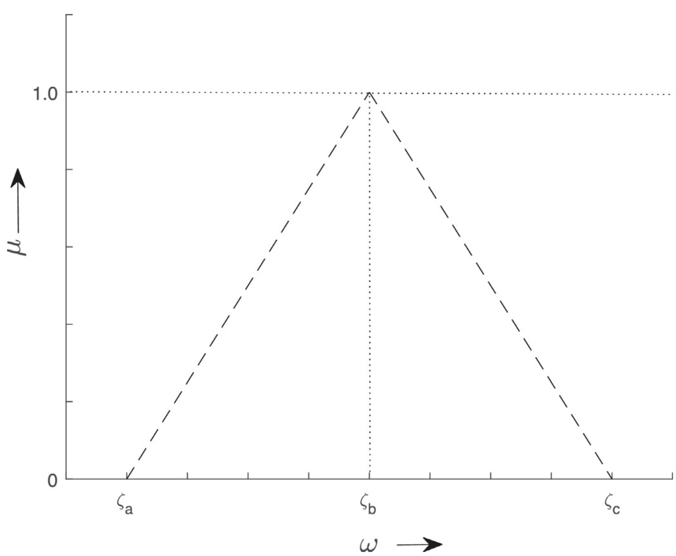


Fig. 2. Triangular fuzzy number.

shapes of an L–R fuzzy number are given by $\Phi_p(\omega) = 1 - \omega^p$, and $\Phi_q(\omega) = 1 - \omega^q$, respectively. Fig. 1 represents the shape of L–R fuzzy number for various values that p and q may take. As a special case when $p = 1$ and $q = 1$, the L–R fuzzy number becomes a trapezoidal fuzzy number. In case $p = 1$ and $q = 1$ and $\xi_b = \xi_c$, it turns out to be a triangular fuzzy number.

2.2. Possibility, necessity and credibility of a fuzzy event

The possibility measure [35] of a fuzzy event ξ for any set A of real numbers, with membership function as μ, is defined as

$$Pos\left\{\xi \in A\right\} = \sup_{\omega \in A} \mu(\omega). \tag{1}$$

Though possibility is a well established measure for a fuzzy event, it lacks self-dual property and is at times difficult and chaotic to understand. Consider a triangular fuzzy variable ξ, represented by the triplet (ξ_a, ξ_b, ξ_c) with $\xi_a < \xi_b < \xi_c$, whose membership function (see Fig. 2) is given as:

$$\mu(\omega) = \begin{cases} \dfrac{\omega - \xi_a}{\xi_b - \xi_a}, & \text{if}\, \xi_a \le \omega \le \xi_b \\[2ex] \dfrac{\omega - \xi_c}{\xi_b - \xi_c}, & \text{if}\, \xi_b \le \omega \le \xi_c \\[2ex] 0, & \text{otherwise.} \end{cases}$$

Now, consider a scenario where fuzzy return of a portfolio is best described by a triangular fuzzy variable $\xi = (-0.7, 0.3, 1.3)$. Note that, the chance of occurring of the fuzzy event in which the investor attains obtains a return of atleast 0.5, is given by $Pos\{\xi \ge 0.5\} = 0.8$. Whereas, the chance of occurring of the fuzzy event where the investor attains a return less than 0.5, is given by $Pos\{\xi < 0.5\} = 1$. In this case

$$Pos\left\{\xi \in A\right\} + Pos\left\{\xi \notin A\right\} \neq 1. \tag{2}$$

In literature, the necessity measure has been defined as the dual of possibility measure [34] as follows:

$$Nec\left\{\xi \in A\right\} = 1 - \sup_{\omega \notin A} \mu(\omega). \tag{3}$$

However, the necessity measure is also not self-dual. To overcome this difficulty, Liu and Liu [16] proposed the concept of credibility to measure fuzzy events. The credibility measure of a fuzzy event is defined as:

$$Cr\left\{\xi \in A\right\} = \frac{1}{2}\left(\sup_{\omega \in A} \mu(\omega) + 1 - \sup_{\omega \notin A} \mu(\omega)\right). \tag{4}$$

In fact, the credibility measure is the arithmetic mean of possibility measure and necessity measure and can also be expressed as:

$$Cr\left\{\xi \in A\right\} = \frac{1}{2}\left(Pos\left\{\xi \in A\right\} + Nec\left\{\xi \in A\right\}\right). \tag{5}$$

Thus, for a triangular fuzzy variable ξ, represented by the triplet (ξ_a, ξ_b, ξ_c) with $\xi_a < \xi_b < \xi_c$, the credibility of $\xi \ge \omega$ is given as follows:

$$Cr\{\xi \ge \omega\} = \begin{cases} 1, & \text{if}\, \omega \le \xi_a \\[2ex] \dfrac{\xi_a - 2\xi_b + \omega}{2(\xi_a - \xi_b)}, & \text{if}\, \xi_a \le \omega \le \xi_b \\[2ex] \dfrac{\omega - \xi_c}{2(\xi_b - \xi_c)}, & \text{if}\, \xi_b \le \omega \le \xi_c \\[2ex] 0, & \omega \ge \xi_c. \end{cases}$$

The credibility of $\xi \le \omega$ is given as follows:

$$Cr\{\xi \le \omega\} = \begin{cases} 0, & \text{if}\, \omega \le \xi_a \\[2ex] \dfrac{\omega - \xi_a}{2(\xi_b - \xi_a)}, & \text{if}\, \xi_a \le \omega \le \xi_b \\[2ex] \dfrac{\xi_c - 2\xi_b + \omega}{2(\xi_c - \xi_b)}, & \text{if}\, \xi_b \le \omega \le \xi_c \\[2ex] 1, & \omega \ge \xi_c. \end{cases}$$

In the previous example, when the fuzzy return of a portfolio is expressed as a triangular fuzzy variable $\xi = (-0.7, 0.3, 1.3)$, then using the credibility measure we can represent the chance of occurring of the fuzzy event in which the investor attains obtains a return of atleast 0.5, by $Cr\{\xi \geq 0.5\} = 0.4$ and the chance of occurring of the fuzzy event that the investor attains a return value less than 0.5, as $Cr\{\xi < 0.5\} = 0.6$. Note that, as credibility is a self-dual measure, the following result holds true:

$$Cr\left\{\xi \in A\right\} + Cr\left\{\xi \notin A\right\} = 1. \tag{6}$$

Definition 1. Let ξ be a normalized fuzzy variable, then the expected value $E[\xi]$ of ξ is defined as [16]:

$$E[\xi] = \int_0^{\infty} Cr\left\{\xi \geq x\right\} dx - \int_{-\infty}^{0} Cr\left\{\xi \leq x\right\} dx \tag{7}$$

provided that at least one integral is finite.

Note that, a fuzzy variable is said to be normal if there exits a real number r such that, $\mu(r) = 1$ [36].

Definition 2. Let ξ be a normalized fuzzy variable with finite expected value e, then the variance $V[\xi]$ is defined as [36]:

$$\begin{aligned} V[\xi] &= E\left[\left(\xi - e\right)^2\right], \\ &= \int_0^{\infty} Cr\left\{\left(\xi - e\right)^2 \geq x\right\} dx. \end{aligned} \tag{8}$$

Definition 3. Let ξ be a normalized fuzzy variable with finite expected value e, then the semi-variance $SV[\xi]$ is defined as [36]

$$SV[\xi] = \int_0^{\infty} Cr\left\{\left(S_v\left[\xi - e\right]\right)^2 \geq x\right\} dx, \tag{9}$$

where

$$S_v\left[\xi - e\right] = \begin{cases} \xi - e, & \text{if } \xi \leq e, \\ 0, & \text{otherwise.} \end{cases}$$

2.3. Entropy-Cross Entropy Algorithm

In this section we show how the subjectivity involved in fitting an L–R fuzzy number can be done away with, by automating the entire process. We attempt to find the best fit L–R fuzzy number belonging to power reference family corresponding to the uncertain asset parameters like return and illiquidity. Let the return and the illiquidity of an asset assumed to be a L–R fuzzy number having a structure $(\zeta_a, \zeta_b, \zeta_c, \zeta_d)_{pq}$ where, $\zeta_a < \zeta_b < \zeta_c < \zeta_d$. An algorithm involving entropy and cross-entropy measures, called as ECE algorithm [29], is used to find this most appropriate L–R fuzzy number.

The fitting of an L–R fuzzy number using the ECE algorithm to the historical data set H, reflecting the return of the assets at various point in time, is as follows:

Step-1: The historical data set $H = [H_{\min}, H_{\max}]$ is partitioned into n equidistant intervals $H_1, H_2, \ldots, H_n$.

Step-2: Compute frequency (number of data items) $\mathfrak{F}_i, i \in \{1, 2, \ldots, n\}$ in each interval.

Step-3: Let $\mathfrak{F}_{\max} = \max_{1 \leq i \leq n}\{\mathfrak{F}_i\}$

Step-4: Assign $\mathfrak{F}'_i = \frac{\mathfrak{F}_i}{\mathfrak{F}_{\max}}, i \in \{1, 2, \ldots, n\}$ to each interval. $\mathfrak{F}'_i$ accounts for the membership value for any randomly generated $s \in [H_{min}, H_{max}]$, once s is located in relevant interval i. Thus, $\rho(s) = \mathfrak{F}'_i$, where ρ is the membership value corresponding to randomly generated data point $s \in H$.

Step-5: Now, using Eqs. (10) and (11), the support of the L–R fuzzy number $(\zeta_a, \zeta_b, \zeta_c, \zeta_d)_{pq}$ from historical data H can be computed as follows:

$$\zeta_a = H_{\min} + x_1\left(H_{avg} - H_{\min}\right), \tag{10}$$

$$\zeta_d = H_{\max} - x_4\left(H_{\max} - H_{avg}\right). \tag{11}$$

Here x_1 and x_4 are the unknowns and H_{avg} is the average value of set H.

Step-6: Eqs. (12) and (13) are used to evaluate the core of the L–R fuzzy number:

$$\zeta_b = H_{avg} - x_2\left(H_{avg} - H_{\min}\right), \tag{12}$$

$$\zeta_c = H_{avg} + x_3\left(H_{\max} - H_{avg}\right). \tag{13}$$

Here x_2 and x_3 are the unknowns.

Step-7: The reverse rating procedure [37] is applied to find the shape parameters p and q of $(\zeta_a, \zeta_b, \zeta_c, \zeta_d)_{pq}$.

$$p = \frac{\ln(1 - x_5)}{\ln\left(\zeta_b - m_p/\zeta_b - \zeta_a\right)}, \zeta_a < m_p < \zeta_b, \tag{14}$$

$$q = \frac{\ln(1 - x_6)}{\ln\left(m_q - \zeta_c/\zeta_d - \zeta_c\right)}, \zeta_c < m_q < \zeta_d. \tag{15}$$

For $m_p = \zeta_b - x_5(\zeta_b - \zeta_a)$ and $m_q = \zeta_c + x_6(\zeta_d - \zeta_c)$ equations (14) and (15) becomes

$$p = \frac{\ln(1 - x_5)}{\ln(x_5)}, \tag{16}$$

$$q = \frac{\ln(1 - x_6)}{\ln(x_6)}. \tag{17}$$

Here $x_5, x_6 \in (0, 1)$ form another pair of unknowns and act as the estimation of the fuzzy coefficients that specify the degree of the fuzzy number.

For ensuring $\zeta_a < \zeta_b < \zeta_c < \zeta_d$, the following must hold:

$$x_1 + x_2 < 1, \tag{18}$$

$$x_3 + x_4 < 1. \tag{19}$$

Moreover,

$$0 < x_5 < 1, \tag{20}$$

$$0 < x_6 < 1. \tag{21}$$

The above procedure defines the computation of the L–R fuzzy return of an asset.

Out of several L–R fuzzy numbers that would fit around uncertain return of an asset, we are interested in the one that gives us a relatively good fit as compared to others. In this process, we need to look into two aspects: (i) the measure of the vagueness within the information and (ii) the deviation of the L–R fuzzy number being generated, from historical data set H. We adapt concepts of entropy and cross-entropy from information theory for doing so. Entropy is a popular measure that measures the information contained within a fuzzy event [30,31,33]. Higher value of entropy is associated with the higher chances of fuzzy event being informative. The Shannon entropy [33] is used in this study to compute the entropy of an L–R fuzzy number. For a randomly generated t_j, within the support (see Step-4) of an L–R fuzzy number and the corresponding membership value as $\mu(t_j)$ (see Section 2.1), the entropy can be computed as:

$$\Gamma\left(t_j\right) = -\left(\mu\left(t_j\right)\ln\left(\mu\left(t_j\right)\right) + \left(1 - \mu\left(t_j\right)\right)\ln\left(1.0 - \mu\left(t_j\right)\right)\right). \tag{22}$$

The cross-entropy [38] is applied to measure the deviation of an L–R fuzzy number from the historical data set H. In order to minimize the deviation of the L–R fuzzy number from the historical data H, we need to minimize the cross-entropy term. The cross-entropy can be computed as:

$$\Pi\left(t_j, h_j\right) = \mu\left(t_j\right) \ln\left(\frac{\mu\left(t_j\right)}{0.5\left(\mu\left(t_j\right)+\rho\left(h_j\right)\right)}\right) + \left(1-\mu\left(t_j\right)\right) \ln\left(\frac{1-\mu\left(t_j\right)}{1-0.5\left(\mu\left(t_j\right)+\rho\left(h_j\right)\right)}\right). \tag{23}$$

Here h_j is randomly generated within the range of the historical data set H, s.t. $h_j \in [H_{min}, H_{max}]$, and $\rho(h_j)$ is its corresponding membership value (see Step-4).

We now summarize the steps (Step 8–Step 12) required to evaluate the net entropy and the net cross-entropy of the L–R fuzzy number.

Step-8: Randomly generate $t_j \in [\zeta_a, \zeta_d]$ and compute the corresponding membership $\mu(t_j)$ (Section 2.1).

Step-9: Use Eq. (22) to compute the entropy $\Gamma\left(t_j\right)$.

Step-10: Randomly generate $h_j \in [H_{min}, H_{max}]$, and the corresponding membership $\rho(h_j)$ (Step 4).

Step-11: Use Eq. (23) to compute the cross-entropy $\Pi\left(t_j, h_j\right)$.

Step-12: Repeat Step 8–Step 11 for a sufficiently large number F_N.

In the above steps we simulate several pairs (F_N in count) of randomly generated points, each within the corresponding L–R fuzzy number and historical data set H. The entropy and cross-entropy is computed for each such simulated pair. Then, we average out these entropy values and the cross-entropy values to compute the net entropy and net cross-entropy of the L–R fuzzy number using all such simulated pairs as follows:

Step-13: Net entropy:

$$\Gamma(x) = \frac{1}{F_N}\sum_{j=1}^{F_N}\Gamma\left(t_j\right).$$

Step-14: Net cross-entropy:

$$\Pi(x, h) = \frac{1}{F_N}\sum_{j=1}^{F_N}\Pi\left(t_j, h_j\right).$$

Recall from our previous discussion that we want an L–R fuzzy number that can fetch maximum information and at same time will have minimum deviation from historical data set H. Thus, the fitting of an L–R fuzzy number turns out to be an optimization problem where the two objectives are maximizing the entropy and minimizing the cross-entropy. The constraints of the problem are the unknowns required to find the most appropriate L–R fuzzy number. Note that the objectives of maximizing entropy and minimizing cross-entropy are equally important in order to find the L–R fuzzy number that best fits the uncertain asset parameter. Thus, we use weighted sum approach, with equal weights, to convert the bi-objective problem to a single objective problem as follows:

$$\begin{aligned}
&\text{(P1)Maximize} \quad F(x) = \Gamma(x) - \Pi(x, h),\\
&\text{subjectto}:\\
&\qquad x_1 + x_2 \le 1,\\
&\qquad x_3 + x_4 \le 1,\\
&\qquad x_i \ge 0, \qquad i = 1, 2, \ldots, 4,\\
&\qquad 0 < x_i < 1, \qquad i = 5, 6.
\end{aligned} \tag{24}$$

This ECE algorithm can likewise be applied to the data sets of other available assets to create the L–R fuzzy boundaries around them. Thus, for each asset we will be able to create an L–R fuzzy return and an L–R fuzzy illiquidity number. As a special case, the ECE methodology can also generate the trapezoidal and the triangular fuzzy numbers.

3. Mathematical formulation of portfolio selection models

In this section, four mathematical models of a credibilistic multi-objective portfolio selection problem are discussed. Let there be n assets available for the purpose of forming a portfolio. The fuzzy return and the fuzzy illiquidity of these assets is marked by L–R fuzzy numbers as computed in the previous section, using the ECE algorithm. Models formulated in this work have two or more objectives. These objectives quantify the return, illiquidity and risk associated with the portfolio. To compute these attributes, two widely used operators namely, expectation and chance constraint programming are used. Expectation is generally applied to infer about the aggregate information about these attributes while chance constraints are used to infer about the chances of having the value of these attributes to be greater or less than any aspiration value set by the investor. Note that, we use a self-dual credibility measure for such computations.

3.1. Objectives

- **Portfolio Return**
  - When the return of n assets is represented by the L–R fuzzy numbers η_i, $i = 1, 2, \ldots, n$, the maximization type objective of expected return of the portfolio is expressed as

$$\text{Maximize}\, E\left[\sum_{i=1}^{n} w_i\eta_i\right], \tag{25}$$

where w_i = proportion of money invested in the ith asset.

  - In case the investor desires to attain a portfolio with a high credibility for the return of portfolio to be greater than or equal to the target return r_1, the resultant expression becomes

$$\text{Maximize}\, Cr\left(\sum_{i=1}^{n} w_i\eta_i \ge r_1\right). \tag{26}$$

- **Illiquidity** Apart from the return another important factor that needs consideration is the liquidity of the assets being selected for the portfolio [6]. Liquidity relates to the ease of trading an asset. Liquidity has been measured by various methods in the literature [39,40,2]. Among these, illiquidity [41] as a measure of liquidity is used in this study. Illiquidity acts as a dual of liquidity. Thus, a lesser illiquidity value indicates higher liquidity and vice versa [5].

  - When the illiquidity of n assets is represented by the L–R fuzzy numbers λ_i, $i = 1, 2, \ldots, n$, the minimization type objective of expected illiquidity of the portfolio is expressed as

$$\text{Minimize}\, E\left[\sum_{i=1}^{n} w_i\lambda_i\right]. \tag{27}$$

  - In the cases where an investor wishes to maximize the credibility that the illiquidity of the portfolio is not more than an upper threshold value of r_2, the resultant expression becomes

$$\text{Maximize}\, Cr\left(\sum_{i=1}^{n} w_i\lambda_i \le r_2\right). \tag{28}$$

- **Risk** Semi-variance has been applied in this study to model the portfolio risk in two of the proposed models. Semi-variance is one of the popular down-side risk measure and unlike variance

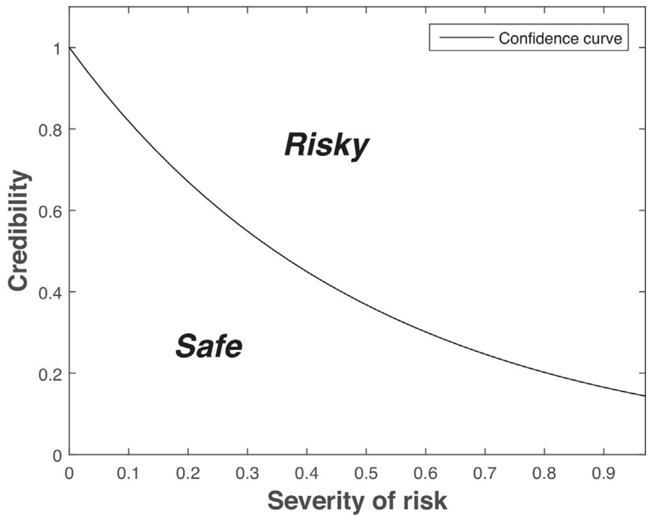


Fig. 3. Risk curve.

it takes into account the downside moment only [42]. Thus, in cases where the returns are asymmetric in nature [43,44,18,45] i.e., when the membership functions of fuzzy returns are not symmetric and have different shapes on both sides, semi-variance is preferred as a risk measure over variance. The L–R fuzzy number for the fuzzy return of an asset, being generated by ECE methodology has two shape parameters p and q for the left and the right side respectively. With the shape parameters taking any value other than 1, there are high chances of the L–R fuzzy number being generated for fuzzy return of an asset, to be asymmetric (see Fig. 1). Thus, we consider semi-variance as a risk measure in this work. The minimization type objective involving the semi variance of the portfolio is expressed as

$$\text{Minimize}\, SV\left[\sum_{i=1}^{n} w_i\eta_i\right] \tag{29}$$

where

$$SV\left[\sum_{i=1}^{n} w_i\eta_i\right] = \begin{cases} E\left(\sum_{i=1}^{n} w_i\eta_i - E\left[\sum_{i=1}^{n} w_i\eta_i\right]\right)^2, & \text{if} \sum_{i=1}^{n} w_i\eta_i \le E\left[\sum_{i=1}^{n} w_i\eta_i\right], \\ 0, & \text{otherwise.} \end{cases}$$

3.2. Constraints

- **Risk curve** In two of the four models presented in this study, a risk curve [46] is used as the downside risk of the portfolio, which is in the form of credibility based chance constraints, i.e.,

$$Cr\left(T_R - \sum_{i=1}^{n} w_i\eta_i \ge r\right) \le \varphi(r), \forall r \ge 0. \tag{30}$$

Here r is the severity indicator of the risk and T_R represents the return that an investor seeks from the portfolio. For a safe portfolio, the left side of Eq. (30) representing the portfolio's risk curve should be below an investor specified confidence curve ($\varphi(r), \forall r > 0$) (Fig. 3).

- **Cardinality constraint** If the investor wishes, he/she may restrict the number of assets selected for investment.

$$\sum_{i=1}^{n} y_i = k, \tag{31}$$

where $y_i \in \{0, 1\}$ and represents the inclusion or exclusion of corresponding i-th asset in the portfolio. k is the number of assets desired in a portfolio.

- **Capital budget constraint** A constraint corresponding to the available budget is represented as

$$\sum_{i=1}^{n} w_i = 1. \tag{32}$$

- **Minimal and maximal proportion of the budget to be invested in a single asset** The following equations define the lower and upper bounds of investment in individual assets:

$$w_i \ge l_i y_i, \tag{33}$$

$$w_i \le u_i y_i. \tag{34}$$

Here l_i is the lower bound and u_i is the upper bound corresponding to the proportion of budget allocated to the ith asset.

- **No short selling** An equation representing no short selling of assets is expressed as

$$w_i \ge 0. \tag{35}$$

3.3. Portfolio selection models

The four portfolio selection models formulated using the objectives and constraints discussed in the previous section are as follows:

3.3.1. Model-1

In this model, portfolio return and illiquidity are quantified using expectation based on credibility measure. Return and illiquidity of available assets is an L–R fuzzy number as computed using the ECE algorithm. According to the risk taking ability of the investor, a risk curve is used to obtain a safe portfolio. The corresponding portfolio selection problem is modeled as follows:

$$\text{Maximize} \quad E\left[\sum_{i=1}^{n} w_i\eta_i\right],$$

$$\text{Minimize} \quad E\left[\sum_{i=1}^{n} w_i\lambda_i\right],$$

subject to:

$$Cr\left(T_R - \sum_{i=1}^{n} w_i\eta_i \ge r\right) \le \varphi(r), \forall r \ge 0, \tag{36}$$

$$\sum_{i=1}^{n} w_i = 1, \tag{37}$$

$$\sum_{i=1}^{n} y_i = k, \tag{38}$$

$$l_i y_i \le w_i \le u_i y_i,\ i = 1, 2, \ldots, n, \tag{39}$$

$$w_i \ge 0,\ i = 1, 2, \ldots, n, \tag{40}$$

$$y_i \in \{0, 1\},\ i = 1, 2, \ldots, n. \tag{41}$$

3.3.2. Model-2

The two objectives of this model are where credibility of obtaining the portfolio return and illiquidity are constrained by the investor specified threshold levels. Risk curve is used to contain the risk of the portfolio. Thus, the portfolio selection problem can be stated as follows:

$$\text{Maximize} \quad Cr\left(\sum_{i=1}^{n} w_i \eta_i \ge r_1\right),$$

$$\text{Maximize} \quad Cr\left(\sum_{i=1}^{n} w_i \lambda_i \le r_2\right),$$

subject to:

$$Cr\left(T_R - \sum_{i=1}^{n} w_i \eta_i \ge r\right) \le \varphi(r), \forall r \ge 0, \tag{42}$$

$$\sum_{i=1}^{n} w_i = 1, \tag{43}$$

$$\sum_{i=1}^{n} y_i = k, \tag{44}$$

$$l_i y_i \le w_i \le u_i y_i,\ i = 1, 2, \ldots, n, \tag{45}$$

$$w_i \ge 0,\ i = 1, 2, \ldots, n, \tag{46}$$

$$y_i \in \{0, 1\},\ i = 1, 2, \ldots, n. \tag{47}$$

3.3.3. Model-3

In this model, one of the key objectives is the risk of the portfolio quantified as semi-variance based on a credibility measure. Other objectives are related to optimizing the credibility of the return and the illiquidity of the portfolio constrained by the pre-specified thresholds. The resultant portfolio selection problem is given as:

$$\text{Maximize} \quad Cr\left(\sum_{i=1}^{n} w_i \eta_i \ge r_1\right),$$

$$\text{Maximize} \quad Cr\left(\sum_{i=1}^{n} w_i \lambda_i \le r_2\right),$$

$$\text{Minimize} \quad SV\left[\sum_{i=1}^{n} w_i \eta_i\right],$$

subject to:

$$\sum_{i=1}^{n} w_i = 1, \tag{48}$$

$$\sum_{i=1}^{n} y_i = k, \tag{49}$$

$$l_i y_i \le w_i \le u_i y_i,\ i = 1, 2, \ldots, n, \tag{50}$$

$$w_i \ge 0,\ i = 1, 2, \ldots, n, \tag{51}$$

$$y_i \in \{0, 1\},\ i = 1, 2, \ldots, n. \tag{52}$$

3.3.4. Model-4

In this model the expected return and the illiquidity of the portfolio along with the risk quantified as a semi-variance measure, forms the three objectives of the portfolio selection problem. The resultant model is formulated as:

$$\text{Maximize} \quad E\left[\sum_{i=1}^{n} w_i \eta_i\right],$$

$$\text{Minimize} \quad E\left[\sum_{i=1}^{n} w_i \lambda_i\right],$$

$$\text{Minimize} \quad SV\left[\sum_{i=1}^{n} w_i \eta_i\right],$$

subject to:

$$\sum_{i=1}^{n} w_i = 1, \tag{53}$$

$$\sum_{i=1}^{n} y_i = k, \tag{54}$$

$$l_i y_i \le w_i \le u_i y_i,\ i = 1, 2, \ldots, n, \tag{55}$$

$$w_i \ge 0,\ i = 1, 2, \ldots, n, \tag{56}$$

$$y_i \in \{0, 1\},\ i = 1, 2, \ldots, n. \tag{57}$$

4. Hybrid Intelligent Algorithm to solve portfolio optimization models

The credibility based models of portfolio optimization, as formulated in the previous section, are multi-objective, NP-hard, optimization problems containing the fuzzy parameters approximated by the L–R fuzzy numbers (using the ECE algorithm). It will not be possible to handle the models discussed in this work using the traditional algorithms meant for handling multi-objective optimization problems. Use of Genetic Algorithm (GA) for handling such unconventional problems has seen a great advancement in literature [18,23,29,32]. The GA is a popular random search method based on the idea of natural selection and evolution. Instead of depending on the gradient information, it searches the optimal solution by following the process of natural evolution. The GA has been established as a suitable method for solving large-scale optimization problems which are nonlinear, non-convex, NP-hard and discrete.

In the present study, as a solution approach, we embed the process of fuzzy simulation within a larger framework of the MIBEX-SM genetic algorithm. This hybrid approach involving the fuzzy simulation and the GA is known as Hybrid Intelligent Algorithm (HIA). In the proposed approach, first we use the fuzzy simulation process to compute the expected values and the credibility based chance constraints, and then the MIBEX-SM GA is used to find solutions to the four portfolio selection problems.

4.1. MIBEX-SM genetic algorithm

The MIBEX-SM GA is applied to find solution of the four combinatorial portfolio selection problems formulated in this study. Main operators that form a part of this GA are BEX crossover, Swap mutation (SM) and truncation methodology meant for handling Mixed Integer (MI) problems. The algorithm is advantageous as all solutions produced are always within the bounds defined for the corresponding variable. This reduces the computational complexity arising everytime a new solution violates the bounds.

- **BEX crossover** BEX crossover [47] uses two parents π_1 and π_2 to produce a pair of offsprings θ_1 and θ_2. It has the advantage of producing the offspring solutions near parents while preserving the restrictions over the bounds of the decision variables. There is only one parameter α which controls the exploration capability of this operator and gives rise to both parent and mean centric characteristics in a non symmetric manner around the parents. BEX operator is defined as:

$$\theta_1 = \pi_1 + \alpha_1 \left|\pi_2 - \pi_1\right|, \tag{58}$$

$$\theta_2 = \pi_2 + \alpha_2 \left|\pi_2 - \pi_1\right|. \tag{59}$$

Here,

$$\alpha_i = \begin{cases} \alpha \ \ln\left\{ e^{\left(\dfrac{m-\pi_k}{\alpha(\pi_2-\pi_1)}\right)} + \tau\left(1 - e^{\left(\dfrac{m-\pi_k}{\alpha(\pi_2-\pi_1)}\right)}\right)\right\}, & \text{if } \upsilon \le 0.5, \\ -\alpha \ \ln\left\{1 - \tau\left(1 - e^{\left(\dfrac{M-\pi_k}{\alpha(\pi_2-\pi_1)}\right)}\right)\right\}, & \text{if } \upsilon > 0.5. \end{cases}$$

$\tau, \upsilon \in U[0, 1]$, $k \in \{1, 2\}$, $\alpha > 0$ is the scale parameter and $m, M \in \mathbb{R}^n$ are lower and upper bounds of the decision variables respectively.

- **Swap mutation** Swap mutation is applied to switch the allocation of asset proportions among the active and inactive assets by shuffling the asset weights. It allows the inactive assets to be a part of the portfolio construction by replacing some of the active assets by inactive assets. This process results in creating new portfolio solutions and does not change the number of active and inactive assets in the portfolio as it only shuffles the positions.
- **Truncation technique** All the decision variables undergo the truncation procedure suggested by Deep et al. [48]. In this process, variables y_i, $i = 1, 2, \ldots, n$ are allowed to have values 0 or 1 with equal probabilities. The aim of applying the truncation process is to have sufficient diversification in the population during evolution over their generations.

As all the models in the present work involves multiple objectives, we use the classical weighted-sum approach of combining these multiple objective functions into a scalar fitness solution. The main motivation in selecting a weighted sum approach comes from the fact that a single objective is required while evaluating fitness within a GA. Clubbing all the objectives using weighted sum can be achieved with minimum modification. In addition, this approach is computationally efficient [49]. Note that if we consider the constant weights in the fitness function, the search direction in the GA is also constant. To explore various search directions in order to find good solutions, we adopt the random weight approach proposed by Murata and Ishibuchi [50]. In this approach, the weights are randomly generated for the fitness evaluation of each chromosome. This approach is efficient in terms of exploring the entire solution space and avoids local optima. Thus, giving a uniform chance to all, while searching for possible pareto solutions along the efficient (pareto) frontier. The approached used here is simple to implement and for more details about other mathematically elegant approaches readers may refer [51].

In the present work, we assign random weights $\delta_i \ge 0$ to obtain the single objective model having objective value F_Σ as follows:

$$F_\Sigma = \sum_{i=1}^{N_f} \delta_i \rho_i f_i, \tag{60}$$

where, N_f is the number of functions. $\delta_i \in U[0, 1]$, $i = 1, 2, \ldots, N_f$ are the weights that are assigned randomly to each of the assets such that $\sum_i^{N_f} \delta_i = 1$ and $\rho_i \in \{-1, 1\}$ indicates whether function f_i is to be minimized ($\rho_i = -1$) or maximized ($\rho_i = 1$), respectively.

Deb's [52] parameter less approach is used to deal with constraints. The fitness value $F(\varsigma)$ of a solution ς is given by the following expression

$$F(\varsigma) = \begin{cases} F_\Sigma(\varsigma), & \text{if } \varsigma \text{ is feasible}, \\ F_\Sigma^{worst} + \displaystyle\sum_{l=1}^{p} < g_l(\varsigma) > + \sum_{s=p+1}^{p+q} |h_e(\varsigma)|, & \text{otherwise}, \end{cases}$$

where F_Σ^{worst} is the objective value of the worst feasible solution, $g_l(\varsigma) \ge 0$ $(l = 1, 2, \ldots, p)$ are inequality constraints and $h_s(\varsigma) = 0$ $(s = p+1, p+2, \ldots, p+q)$ are equality constraints. The bracket function $<g(\varsigma)>$ is assigned the value equal $g(\varsigma)$ if ς is infeasible, otherwise it is assigned a zero value.

Deb [52] suggested that the proposed constraint handling mechanism can be easily embedded with the selection operator without having any explicit implementation for calculating the fitness function with the help of following rules

- Select feasible solution over an infeasible solution.
- Select feasible solution with better objective value among all the feasible solutions.
- Select infeasible solution having minimum constraint violation among all the infeasible solutions.

4.2. Fuzzy simulation for computing credibility based portfolio parameters

We now present various fuzzy simulation techniques in a credibility based framework, applied in this study.

4.2.1. Fuzzy simulation for computing credibility (L)

In order to compute credibility based chance constraints we employ the following algorithm:

1. Set $j = 1$.
2. Generate random number g_i within the support for each asset such that $\mu(g_i) > 0$. Calculate $f_j(w, g)$ as $f_j(w, g) = \sum_{i=1}^{n} w_i g_i$, $i = 1, 2, \ldots, n$, where n is the number of assets.
3. Find $U_j = \min_{1 \le i \le n} \mu(g_i)$.
4. Increase $j \longleftarrow j + 1$. Repeat step 1 to step 4 till $j = F_N$, where F_N is a very large number.
5. Find $Pos(f(w, g \ge r) = \left(\max_{1 \le j \le F_N} U_j | f_j(w, g) \ge r\right)$.
6. Find $Nec(f(w, g) \ge r) = 1 - \left(\max_{1 \le j \le F_N} U_j | f_j(w, g) \le r\right)$.
7. Find $L = Cr(f(w, g) \ge r) = \frac{1}{2}\left[Pos(f(w, g) \ge r) + Nec(f(w, g) \ge r)\right]$.

4.2.2. Fuzzy simulation for calculating expectation (E)

In order to compute credibility based expected values of the return and illiquidity, we employ the following algorithm:

1. Set $j = 1$.
2. Generate random number g_i within the support for each asset such that $\mu(g_i) > 0$. Calculate $f_j(w, g)$ as $f_j(w, g) = \sum_{i=1}^{n} w_i g_i$, $i = 1, 2, 3, \ldots, n$ where, n is the number of assets.

3. Increase $j \longleftarrow j+1$. Repeat step 2 and step 3 till $j = F_N$, where F_N is a very large number.
4. Set $E_{\min} = Min\left\{f_j(w,g)|j \in \left\{1,2,\ldots,F_N\right\}\right\}$ and $E_{\max} = Max\left\{f_j(w,g)|j \in \left\{1,2,\ldots,F_N\right\}\right\}$.
5. Set $j=1$ and $E=0$.
6. Generate random number $e_j \in (E_{\min}, E_{\max})$.
7. If $e_j \geq 0$ then $E = E + Cr(f(w,g) \geq e_j)$ otherwise, $E = E - Cr(f(w,g) \leq e_j)$.
8. Increase $j \longleftarrow j+1$. Repeat step 6 to step 8 till $j = F_N$, where F_N is a very large number.
9. Find $E = E\frac{(E_{\max}-E_{\min})}{F_N} + Max(E_{\min}, 0) + Min(0, E_{\max})$.

4.2.3. Fuzzy simulation for calculating semi-variance (SV)

In order to compute credibility based semi-variance measure, we employ the following algorithm:

1. Set $j=1$.
2. Generate random number g_i within the support for each asset such that $\mu(g_i) > 0$. Calculate $f_j(w,g)$ as $f_j(w,g) = \sum_{i=1}^{n} w_i g_i$, $i = 1, 2, \ldots, n$ where, n is the number of assets.
3. If $f_j(w,g) \geq E$ then $f_j(w,g) = 0$ otherwise, $f_j(w,g) = \left(f_j(w,g) - E\right)^2$.
4. Increase $j \longleftarrow j+1$. Repeat step 2 to step 4 till $j = F_N$, where F_N is a very large number.
5. Set $SV_{\min} = Min\left\{f_j(w,g)|j \in \left\{1,2,\ldots,F_N\right\}\right\}$ and $SV_{\max} = Max\left\{f_j(w,g)|j \in \left\{1,2,\ldots,F_N\right\}\right\}$.
6. Set $j=1$ and $SV=0$.
7. Generate random number $s_j \in (SV_{\min}, SV_{\max})$.
8. If $s_j \geq 0$ then $SV = SV + Cr(f(w,g) \geq e_i)$ otherwise, $SV = SV - Cr(f(w,g) \leq e_i)$.
9. Increase $j \longleftarrow j+1$. Repeat step 7 to step 9 till $j = F_N$.
10. Find $SV = SV\frac{(SV_{\max}-SV_{\min})}{F_N} + Max(SV_{\min}, 0) + Min(0, SV_{\max})$.

4.2.4. Fuzzy simulation for calculating variance (V)

In order to compute credibility based measure of variance, we employ the following algorithm:

1. Set $j=1$.
2. Generate random number g_i within the support for each asset such that $\mu(g_i) > 0$. Calculate $f_j(w,g)$ as $f_j(w,g) = \sum_{i=1}^{n} w_i g_i$, $i = 1, 2, \ldots, n$. Here, n is the number of assets.
3. Compute $f_j(w,g) = \left(f_j(w,g) - E\right)^2$.
4. Increase $j \longleftarrow j+1$. Repeat step 2 to step 4 till $j = F_N$.
5. Set $V_{\min} = Min\left\{f_j(w,g)|j \in \left\{1,2,\ldots,F_N\right\}\right\}$ and $V_{\max} = Max\left\{f_j(w,g)|j \in \left\{1,2,\ldots,F_N\right\}\right\}$.
6. Set $j=1$ and $V=0$.
7. Generate random number $s_j \in (V_{\min}, V_{\max})$.
8. If $s_j \geq 0$ then $V = V + Cr(f(w,g) \geq e_i)$ otherwise, $V = V - Cr(f(w,g) \leq e_i)$.
9. Increase $j \longleftarrow j+1$. Repeat step 7 to step 9 till $j = F_N$.
10. Find $V = V\frac{(V_{\max}-V_{\min})}{F_N} + Max(V_{\min}, 0) + Min(0, V_{\max})$.

5. The experimental set up

In order to demonstrate the applicability and efficiency of the proposed DSS, monthly adjusted historical data of fifteen stocks (stated in Table 1) from NSE of India for the financial block of 2008–2016 is considered. The entire data is divided into three parts. First five years data for 2008–2013, is considered as the training data (DS1) and is used to generate the results (portfolio weights) for all the models. Data for two years from 2013–2015, is considered as testing data (DS2) and is used for testing and comparing the performance of all the models. The last one year data from 2015–2016 (DS3) is used to validate the performance of the model. The testing and validation is done on the basis of modified Sharpe ratio in the credibilistic context. Such a modified version of Sharpe ratio is called "Credibilistic Sharpe Ratio" [29] and can be computed as

Table 1
Name of the stocks.

Stock	Name	Code
1	Axis Bank Ltd.	S_1
2	Bank of Baroda	S_2
3	Bharat Heavy Electricals Ltd.	S_3
4	HCL Technologies Ltd.	S_4
5	Hindalco Industries Ltd.	S_5
6	IDFC Ltd.	S_6
7	Jindal Steel & Power Ltd.	S_7
8	JSW Steel Ltd.	S_8
9	Reliance Communications Ltd.	S_9
10	State Bank of India	S_{10}
11	Sesa Sterlite (Vedanta) Ltd.	S_{11}
12	Tata Motors Ltd.	S_{12}
13	Tata Steel Ltd.	S_{13}
14	Tech Mahindra Ltd.	S_{14}
15	Yes Bank Ltd.	S_{15}

$$\text{Credibilistic Sharpe Ratio} = CrSR = \frac{E\left[\sum_{i=1}^{n} w_i \eta_i\right]}{\sqrt{V\left[\sum_{i=1}^{n} w_i \eta_i\right]}}, \tag{61}$$

where $E\left[\sum_{i=1}^{n} w_i \eta_i\right]$ is the expected fuzzy return of the portfolio and $\sqrt{V\left[\sum_{i=1}^{n} w_i \eta_i\right]}$ is the fuzzy standard deviation of the portfolio. Fuzzy simulation process, as discussed in Section 4 has been used to compute the expected return and variance of the obtained portfolio.

To begin with, the training data set (DS1) is divided into seven parts (extremely low, very low, low, medium, high, very high, extremely high) [53]. There after, the ECE method discussed in Section 2 is used and the resulting optimization problem (P1) is solved to generate the L–R fuzzy numbers corresponding to the return and illiquidity of the assets as listed in Table 1.

5.1. Comparison with the existing approach

For the purpose of making comparisons, we also generate the L–R fuzzy numbers for returns and illiquidity, for the same set of assets (Table 1) using another popular approach proposed by Vercher and Bermudez [20] (named as VB methodology/approach). As discussed earlier, the ECE approach is an attempt to automate the otherwise subjective VB approach.

In VB approach [20], percentile of data is used to determine the parameters of an L–R fuzzy number $(\zeta_a, \zeta_b, \zeta_c, \zeta_d)_{pq}$ where, $\zeta_a < \zeta_b < \zeta_c < \zeta_d$. Parameters of the L–R fuzzy number are then fixed as: ζ_a = minimumreturn, $\zeta_b = p_{45}$, $\zeta_c = p_{55}$, and $\zeta_d = p_{97}$. The shape parameters are computed using Eqs. (62) and (63)

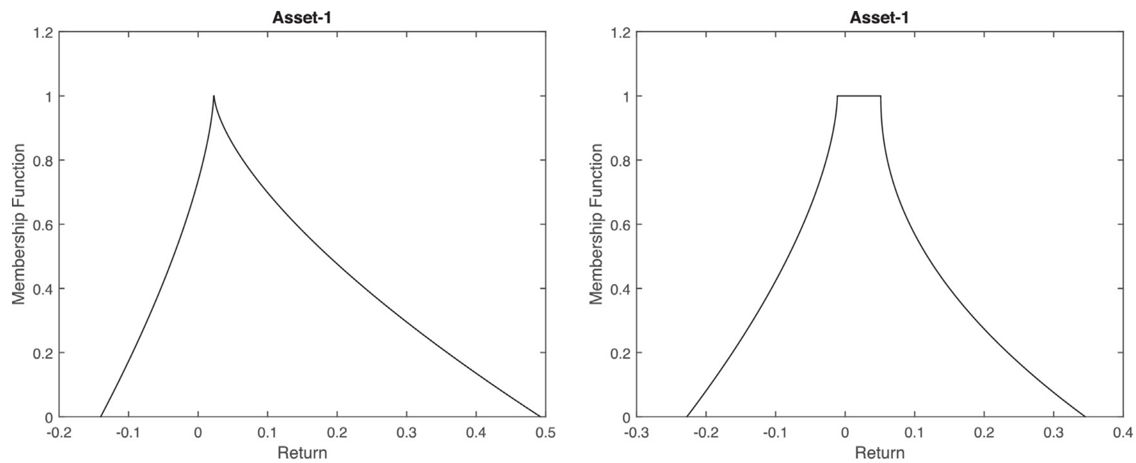
$$p = \frac{\ln(0.5)}{\ln\left(\zeta_b - p_{25}/\zeta_b - \zeta_a\right)}, \tag{62}$$

$$q = \frac{\ln(0.5)}{\ln\left(p_{75} - \zeta_c/\zeta_d - \zeta_c\right)}, \tag{63}$$

where p_j, $j = 1, 2, \ldots, 100$ is jth percentile.

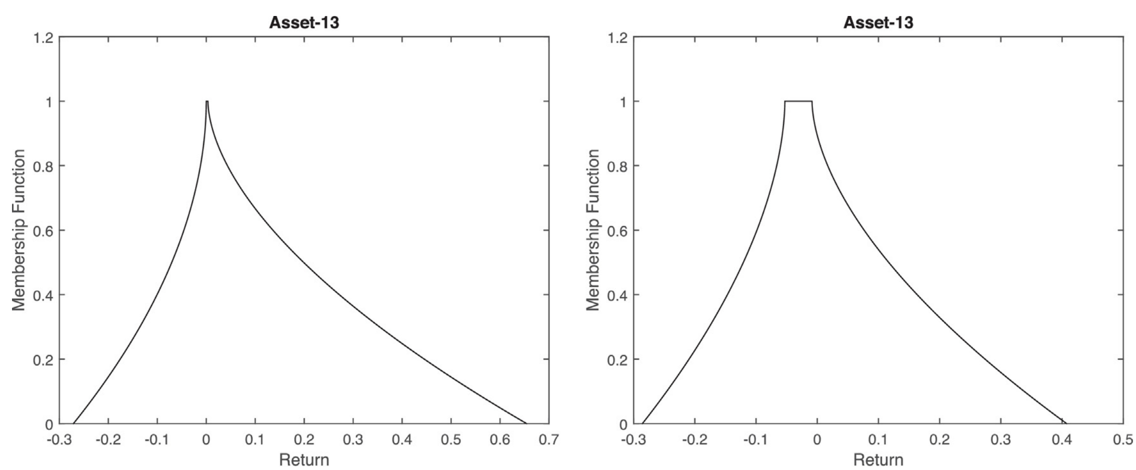
Tables 2 and 3 present the L-R fuzzy returns for 15 assets generated by ECE and VB approach respectively. Similarly, Tables 4 and 5 present the L-R fuzzy illiquidity for 15 assets generated by ECE and VB approach respectively.

The L–R fuzzy numbers generated by the ECE and the VB approaches (Tables 2–5) are used as bound within which we simulate the return and the illiquidity of the individual asset, while



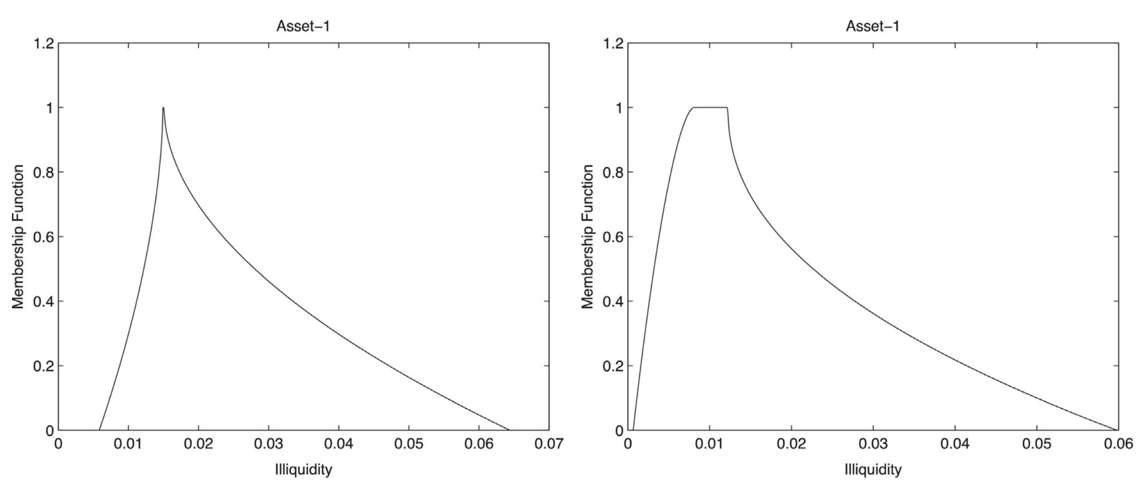
(a) ECE Method (b) VB Method

Fig. 4. L–R fuzzy return of asset S_1.



(a) ECE Method (b) VB Method

Fig. 5. L–R fuzzy return of asset S_{13}.

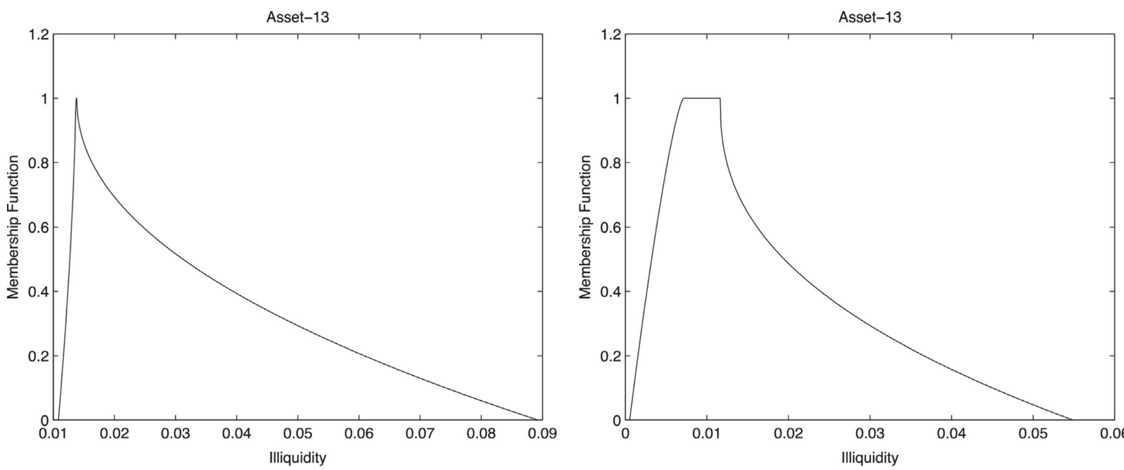


(a) ECE Method (b) VB Method

Fig. 6. L–R fuzzy illiquidity of asset S_1.

solving the portfolio selection model discussed in Section 3. For the purpose of demonstration, the membership function generated by using the ECE and the VB approaches for the return and illiquidity of assets S_1 and S_{13} are shown in Figs. 4–7 .

Next, we solve the four portfolio selection problems as formulated in Section 3 using MIBEX-SM GA [29]. The parameter setting of MIBEX-SM GA and various other parameters of the portfolio selection problem are presented in Table 6. Note that cardinality k of the



(a) ECE Method (b) VB Method

Fig. 7. L–R fuzzy illiquidity of asset S_{13}.

Table 2
L-R fuzzy number for return of each asset using ECE method.

Asset	ζ_a	ζ_b	ζ_c	ζ_d	p	q
S_1	−0.14013	0.02234	0.02303	0.49244	0.67131	0.66294
S_2	−0.2063	0.02163	0.02254	0.34932	0.63726	0.6009
S_3	−0.11145	−0.00224	−0.0016	0.30454	0.69177	0.58945
S_4	−0.06132	0.02365	0.02394	0.18515	0.78658	0.80623
S_5	−0.22167	0.00141	0.00202	0.26476	0.51723	0.63914
S_6	−0.17177	0.00866	0.00955	0.3589	0.56483	0.58352
S_7	−0.34603	0.00439	0.00678	0.34395	0.68911	0.65178
S_8	−0.11619	0.01297	0.01373	0.41199	0.7006	0.63632
S_9	−0.17408	−0.00989	−0.00687	0.63744	0.58874	0.67893
S_{10}	−0.09381	0.02461	0.02513	0.35595	0.62739	0.60401
S_{11}	−0.16709	0.01001	0.01261	0.43205	0.59738	0.59293
S_{12}	−0.31862	0.03484	0.03648	0.24574	0.64215	0.59754
S_{13}	−0.27127	0.00001	0.00337	0.65425	0.51424	0.57615
S_{14}	−0.20905	0.01244	0.01395	0.52571	0.54617	0.58598
S_{15}	−0.29956	0.03173	0.03222	0.31057	0.54508	0.52954

Table 3
L-R fuzzy number for return of each asset using VB method.

Asset	ζ_a	ζ_b	ζ_c	ζ_d	p	q
S_1	−0.22762	−0.01123	0.05126	0.34536	0.61795	0.46850
S_2	−0.20083	−0.00141	0.05174	0.35116	0.60941	0.35208
S_3	−0.16555	−0.02468	0.01328	0.22432	0.97462	0.43428
S_4	−0.20248	−0.01087	0.07043	0.28431	0.60891	0.37478
S_5	−0.24279	−0.03855	0.02115	0.33248	0.61938	0.56276
S_6	−0.21182	−0.02347	0.00467	0.44188	0.71323	0.38755
S_7	−0.30337	−0.04764	0.03664	0.27295	0.45020	0.62845
S_8	−0.36623	−0.03604	0.04095	0.44785	0.47501	0.48350
S_9	−0.32456	−0.09387	−0.00710	0.41994	0.52655	0.58350
S_{10}	−0.21119	−0.01138	0.03248	0.27811	0.51765	0.57568
S_{11}	−0.23452	−0.02860	0.01749	0.33547	0.74312	0.58270
S_{12}	−0.25130	0.00264	0.06547	0.36667	0.89729	0.56580
S_{13}	−0.28569	−0.05293	−0.00853	0.40746	0.56217	0.57731
S_{14}	−0.29188	−0.01841	0.04795	0.38919	0.48408	0.38679
S_{15}	−0.24190	−0.00935	0.05953	0.50236	0.71814	0.36314

portfolio is considered to be 5 which is not greater than one third of the number of total assets as suggested by Chang et al. [54].

We solve the portfolio optimization models for two different cases pertaining to the risk taking/aversion tendency of the investor. The parameter settings for these two cases are given in Table 7. Note that case-1 relates to a more risk-averse investor as compared to case-2. Since Model-4 does not have any term related to investor preferences, hence there will be no impact of the change in these parameters on the performance of Model-4.

Table 4
L-R fuzzy number for illiquidity of each asset using ECE method.

Asset	ζ_a	ζ_b	ζ_c	ζ_d	p	q
S_1	0.00583	0.01492	0.01509	0.06442	0.57674	0.51734
S_2	0.00745	0.01572	0.0159	0.13468	0.33587	0.46865
S_3	0.01015	0.0119	0.01202	0.04679	0.58818	0.56375
S_4	0.01525	0.01614	0.01631	0.08663	0.4074	0.52991
S_5	0.00374	0.01497	0.01509	0.0801	0.53655	0.4521
S_6	0.01194	0.01441	0.01447	0.05813	0.47473	0.53184
S_7	0.00875	0.01381	0.01383	0.04565	0.47072	0.56541
S_8	0.01029	0.01473	0.01486	0.13938	0.49513	0.45987
S_9	0.01215	0.0151	0.01523	0.04229	0.5871	0.63227
S_{10}	0.01237	0.01504	0.01526	0.13856	0.61234	0.46321
S_{11}	0.0051	0.01342	0.01349	0.0584	0.5984	0.5852
S_{12}	0.00405	0.01606	0.01624	0.1841	0.38895	0.44476
S_{13}	0.01085	0.01367	0.01386	0.08931	0.64091	0.47165
S_{14}	0.00888	0.01632	0.01649	0.17782	0.57019	0.45856
S_{15}	0.00214	0.01596	0.01644	0.13524	0.43424	0.45601

Table 5
L-R fuzzy number for illiquidity of each asset using VB method.

Asset	ζ_a	ζ_b	ζ_c	ζ_d	p	q
S_1	0.00064	0.00807	0.01223	0.05979	1.64909	0.45633
S_2	0.00046	0.00741	0.01088	0.09573	0.94150	0.31064
S_3	0.00024	0.00771	0.01237	0.03859	1.40139	0.52393
S_4	0.00047	0.00599	0.01144	0.06342	0.91294	0.44260
S_5	0.00039	0.00583	0.00954	0.09138	1.11451	0.25030
S_6	0.00049	0.00665	0.01070	0.05808	2.09740	0.70009
S_7	0.00070	0.00785	0.01260	0.05292	1.31250	0.46763
S_8	0.00032	0.00416	0.00770	0.08014	0.95923	0.34696
S_9	0.00055	0.01068	0.01478	0.04028	0.78522	0.71297
S_{10}	0.00024	0.00528	0.00929	0.08182	1.14505	0.33540
S_{11}	0.00034	0.00593	0.01215	0.06260	1.42841	0.38971
S_{12}	0.00079	0.00410	0.00645	0.08038	1.69592	0.41590
S_{13}	0.00052	0.00710	0.01162	0.05490	1.32118	0.40608
S_{14}	0.00013	0.00404	0.00903	0.15623	1.17673	0.23525
S_{15}	0.00045	0.00361	0.00625	0.14124	1.04852	0.25860

Table 6
Parameter setting for portfolio selection problems.

Parameters	Value
Number of pairs $(r, \varphi(r))$	35
k	5
l_i	0.08
u_i	0.3
Population size	30
Generation size	2000
F_N	3000
Tournament size	2
Elitism size	1

Table 7
Parameter setting for two different cases.

	Case-1			Case-2		
Parameters	Model-1	Model-2	Model-3	Model-1	Model-2	Model-3
$\varphi(r)$	$\frac{1}{\exp(3r)}$	$\frac{1}{\exp(3r)}$	–	$\frac{1}{\exp(2r)}$	$\frac{1}{\exp(2r)}$	–
r_1	–	0.10	0.10	–	0.11	0.11
r_2	–	0.025	0.025	–	0.03	0.03
T_R	0.20	0.20	–	0.25	0.25	–

Table 8
Objective function values for Model-1 (case-1).

Method	cp	mp	$E\left[\sum_{i=1}^{n} w_i\eta_i\right]$	$E\left[\sum_{i=1}^{n} w_i\lambda_i\right]$
ECE	0.6	0.4	0.117923	0.033911
VB	0.6	0.1	0.051829	0.024296

Table 9
Objective function values for Model-1 (case-2).

Method	cp	mp	$E\left[\sum_{i=1}^{n} w_i\eta_i\right]$	$E\left[\sum_{i=1}^{n} w_i\lambda_i\right]$
ECE	0.9	0.2	0.1349	0.0381
VB	0.7	0.1	0.0669	0.0262

Table 10
Objective function values for Model-2 (case-1).

Method	cp	mp	$Cr\left(\sum_{i=1}^{n} w_i\eta_i \geq r_1\right)$	$Cr\left(\sum_{i=1}^{n} w_i\lambda_i \leq r_2\right)$
ECE	0.7	0.3	0.5793	0.7702
VB	0.6	0.1	0.5709	0.7856

Table 11
Objective function values for Model-2 (case-2).

Method	cp	mp	$Cr\left(\sum_{i=1}^{n} w_i\eta_i \geq r_1\right)$	$Cr\left(\sum_{i=1}^{n} w_i\lambda_i \leq r_2\right)$
ECE	0.9	0.1	0.5197	0.8094
VB	0.7	0.2	0.5139	0.8163

6. Results: analysis and discussion

This section is devoted to the analysis and discussion of the results obtained, when the four portfolio optimization models are solved for the two different cases. We use both ECE and VB approaches for this purpose and draw a comparison between the two.

In Section 6.1 we analyze the performance of portfolios obtained using ECE and VB methods, for each portfolio selection model. It would be inappropriate to compare the performance of both approaches viz. ECE and VB on the basis of obtained return, risk or fitness value of each model since the fuzzy return and illiquidity obtained by both the methods are different. Thus, in Section 6.2 we use modified Sharpe ratio, namely Credibilistic Sharpe Ratio (CrSR), for comparing the performance of ECE and VB approaches for each model. In Section 6.3 we discuss a novel testing procedure for comparing the four portfolio selection models. This testing procedure is based on validation of results on futuristic data set. This guides the investor in picking up the most robust and reliable model out of the four portfolio selection models discussed in this study.

6.1. Risk-Return Analysis with respect to two different cases

6.1.1. Analyzing Model-1

A risk curve bounded above by investor's confidence curve defines the risk measure in Model-1. For the obtained portfolio to be safe, the corresponding risk curve should fall below the investor defined confidence curve. Figs. 8 and 9 shows that the attained portfolio in each case, for both the approaches (ECE and VB), is a safe portfolio.

Note that in case-2, the investor has higher risk taking tendency as compared to the investor in case-1. One can easily verify this by comparing the parameter settings of the $\varphi(r)$ in Table 7. This is also evident from Figs. 8 and 9 when we compare the area under the confidence curve for two cases. Thus, it is expected that case-1 should result in a less risky portfolio as compared to the portfolio generated in case-2. Higher returns are usually associated with the selection of assets that are more risky, whereas a portfolio that is less risky is usually associated with the selection of assets that will generate lesser returns. Thus, an investor in case-2 may set an expectation of higher return (T_R) from the portfolio. Illiquidity is another criteria that gets impacted as assets that are associated with good returns usually involve a longer investment period. Thus, a high yield portfolio is usually accompanied by a low liquidity value, i.e., higher illiquidity.

We solve Model-1, with respect to both ECE and VB methods for the two cases (case-1 and case-2). Tables 8 and 9 present the objective function values corresponding to best found solution in case-1 and case-2, respectively. When we compare the objective function values as obtained for case-2 with those obtained for case-1, we can draw parallels from our previous discussion. Thus, we can see a rise in the expected return and expected illiquidity of the portfolio when the risk bearing ability of the investor increases (case-2). This is true for both ECE and VB approaches.

6.1.2. Analyzing Model-2

Model-2 is solved with respect to both ECE and VB methods for the two cases (case-1 and case-2). Tables 10 and 11 present the objective function values corresponding to best found solution in case-1 and case-2, respectively.

Note that for both the ECE and VB approaches, as we move from case-1 to case-2 raising the threshold value r_1 of anticipated portfolio return, the credibility of the portfolio return being more than this threshold value r_1 goes down. Similarly, as we move from case-1 to case-2 increasing the upper threshold value r_2 (higher return r_1 relates to a longer investment period), the credibility of the portfolio illiquidity being less than this threshold value r_2 goes up or in turn we may say the credibility of portfolio being more liquid goes down.

For each case we are able to generate a safe portfolio as the risk curve is contained below the confidence curve (see Figs. 10 and 11).

6.1.3. Analyzing Model-3

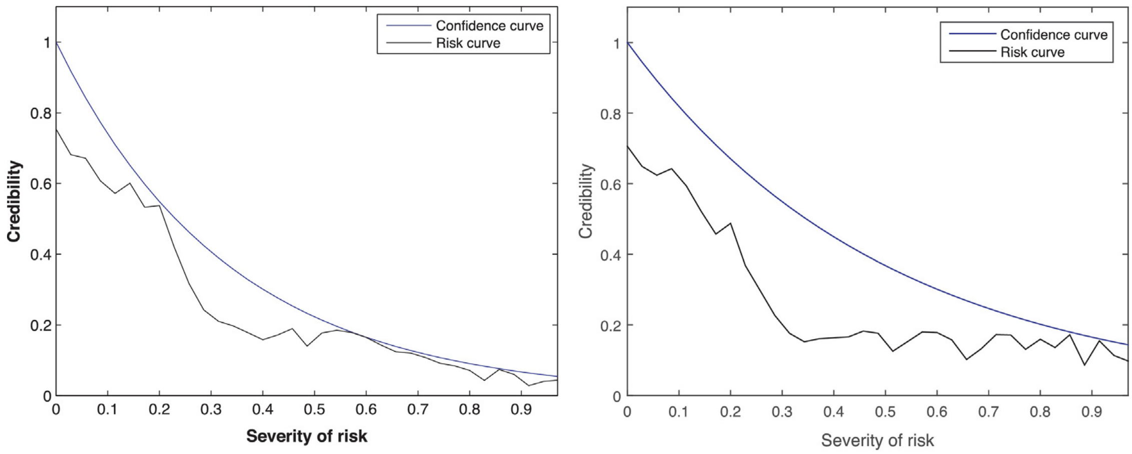
Tables 12 and 13 present the objective function values for both the ECE and the VB approach, in case of portfolio selection problem given by Model-3.

Note that as we move from case-1 to case-2, raising the lower threshold value r_1 with respect to the portfolio return, the credibility of the portfolio return being more than r_1 goes down. Also note that, this raise in r_1 is associated with an increased risk (i.e. semi-variance) in case-2, as can be verified from Table 13. Similarly, as we move from case-1 to case-2 increasing the upper threshold value r_2 for illiquidity of the portfolio, the credibility of this goes up in case-2.

6.1.4. Analyzing Model-4

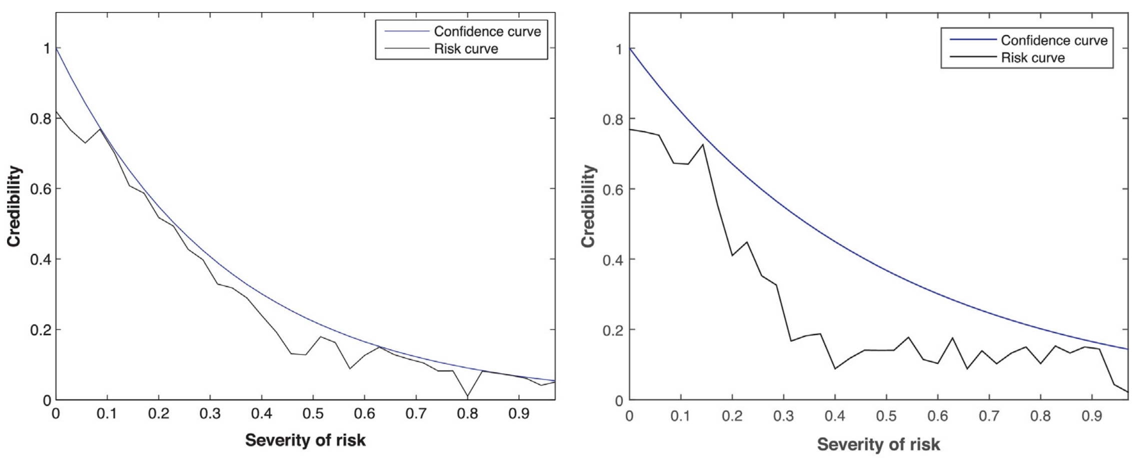
Table 14 reflects objective function values of Model-4 for both ECE and VB approach. As this model is independent of parameters that are investor driven, the objective function attainments will not change for the two cases.

The optimal allocation among assets corresponding to all the models for both the cases for the ECE and the VB approaches are shown in Tables 15–18.



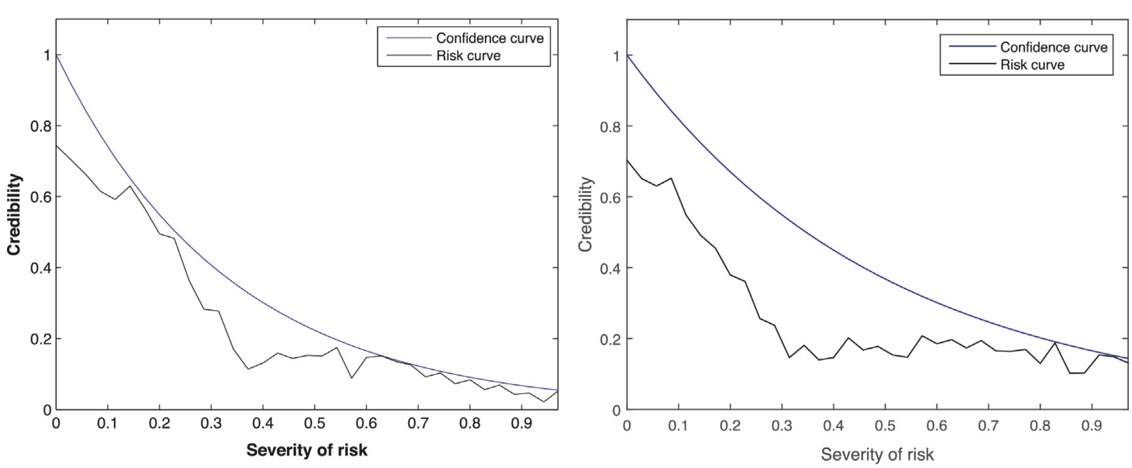
(a) Case-1 (b) Case-2

Fig. 8. Risk curves of Model-1 using ECE method.



(a) Case-1 (b) Case-2

Fig. 9. Risk curves of Model-1 using VB method.

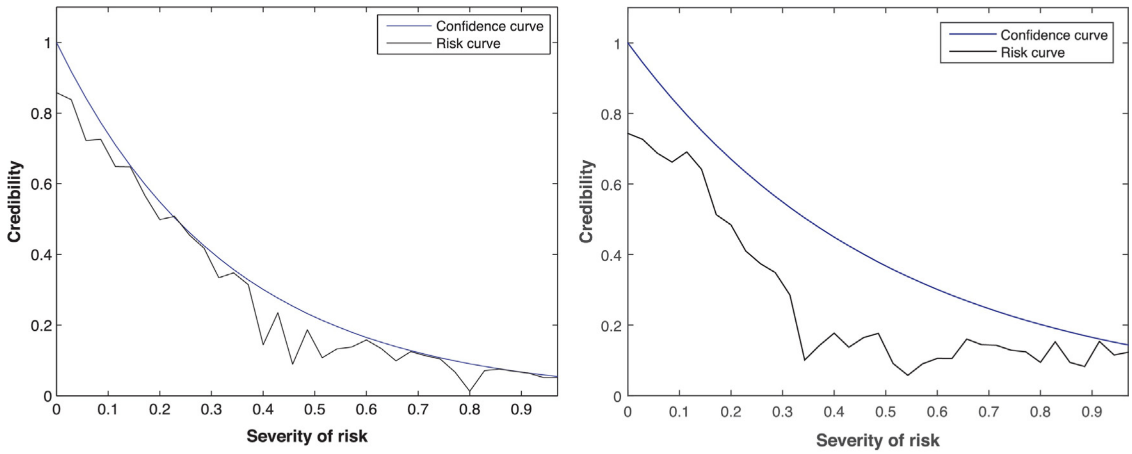


(a) Case-1 (b) Case-2

Fig. 10. Risk curves of Model-2 using ECE method.

Table 12
Objective function values for Model-3 (case-1).

Method	cp	mp	$Cr\left(\sum_{i=1}^{n} w_i\eta_i \geq r_1\right)$	$Cr\left(\sum_{i=1}^{n} w_i\lambda_i \leq r_2\right)$	$SV\left[\sum_{i=1}^{n} w_i\eta_i\right]$
ECE	0.9	0.1	0.5689	0.7238	0.0106
VB	0.5	0.2	0.5678	0.7594	0.0147



(a) Case-1 (b) Case-2

Fig. 11. Risk curves of Model-2 using VB method.

Table 13
Objective function values for Model-3 (case-2).

Method	cp	mp	$Cr\left(\sum_{i=1}^{n} w_i\eta_i \geq r_1\right)$	$Cr\left(\sum_{i=1}^{n} w_i\lambda_i \leq r_2\right)$	$SV\left[\sum_{i=1}^{n} w_i\eta_i\right]$
ECE	0.7	0.3	0.5185	0.7642	0.0139
VB	0.6	0.3	0.5108	0.7961	0.0207

Table 14
Objective function values for Model-4.

Method	cp	mp	$E\left[\sum_{i=1}^{n} w_i\eta_i\right]$	$E\left[\sum_{i=1}^{n} w_i\lambda_i\right]$	$SV\left[\sum_{i=1}^{n} w_i\eta_i\right]$
ECE	0.6	0.4	0.1287	0.0358	0.0174
VB	0.6	0.4	0.1226	0.0394	0.0181

Table 15
Asset allocation for case-1 using ECE method.

Assets	Model-1	Model-2	Model-3	Model-4
S_1	0.18757	0.189779	0.148699	0.165227
S_2	0.00000	0.22280	0.00000	0.00000
S_3	0.00000	0.18978	0.00000	0.00000
S_4	0.00000	0.00000	0.27007	0.28597
S_5	0.00000	0.00000	0.00000	0.00000
S_6	0.21908	0.00000	0.00000	0.00000
S_7	0.00000	0.22280	0.00000	0.00000
S_8	0.00000	0.00000	0.00000	0.23047
S_9	0.18131	0.17484	0.25980	0.20341
S_{10}	0.19295	0.00000	0.00000	0.00000
S_{11}	0.21908	0.00000	0.13536	0.00000
S_{12}	0.00000	0.00000	0.00000	0.00000
S_{13}	0.00000	0.00000	0.00000	0.11492
S_{14}	0.00000	0.00000	0.18609	0.00000
S_{15}	0.00000	0.00000	0.00000	0.00000

6.2. *Comparison of results obtained for ECE and VB approaches*

Sharpe [55] developed Sharpe Ratio (SR) to measure the performance of portfolios using average and standard deviation of historical returns. Since then, it has become a standard criteria in industry and academic literature for comparing the performance of different portfolios in probabilistic as well as in fuzzy environment [56–58]. The portfolio having higher SR is preferred as compared to the portfolio with lower SR.

We use Credibilistic Sharpe Ratio (CrSR) [29] to compare the results obtained using the ECE and the VB approaches for the two cases, corresponding to different portfolio selection models discussed in this work. Once the expected return and the variance of the obtained portfolio have been computed using L–R fuzzy num-

Table 16
Asset allocation for case-2 using ECE method.

Assets	Model-1	Model-2	Model-3	Model-4
S_1	0.17921	0.22439	0.00000	0.165227
S_2	0.00000	0.17597	0.00000	0.00000
S_3	0.00000	0.00000	0.18521	0.00000
S_4	0.00000	0.20386	0.00000	0.28597
S_5	0.00000	0.00000	0.00000	0.00000
S_6	0.00000	0.00000	0.00000	0.00000
S_7	0.00000	0.18099	0.10043	0.00000
S_8	0.24447	0.00000	0.00000	0.23047
S_9	0.20542	0.21479	0.24099	0.20341
S_{10}	0.18510	0.00000	0.00000	0.00000
S_{11}	0.18581	0.00000	0.25724	0.00000
S_{12}	0.00000	0.00000	0.00000	0.00000
S_{13}	0.00000	0.00000	0.00000	0.11492
S_{14}	0.00000	0.00000	0.21614	0.00000
S_{15}	0.00000	0.00000	0.00000	0.00000

Table 17
Asset allocation for case-1 using VB method.

Assets	Model-1	Model-2	Model-3	Model-4
S_1	0.18851	0.20734	0.00000	0.15042
S_2	0.00000	0.00000	0.00000	0.00000
S_3	0.00000	0.18630	0.00000	0.00000
S_4	0.00000	0.00000	0.00000	0.29286
S_5	0.19864	0.00000	0.00000	0.00000
S_6	0.00000	0.00000	0.00000	0.00000
S_7	0.00000	0.23057	0.00000	0.00000
S_8	0.21324	0.00000	0.00000	0.00000
S_9	0.19643	0.19519	0.16451	0.21235
S_{10}	0.00000	0.00000	0.20541	0.00000
S_{11}	0.20318	0.00000	0.24995	0.00000
S_{12}	0.00000	0.00000	0.00000	0.00000
S_{13}	0.00000	0.18061	0.23343	0.09441
S_{14}	0.00000	0.00000	0.00000	0.25006
S_{15}	0.00000	0.00000	0.14662	0.00000

Table 18
Asset allocation for case-2 using VB method.

Assets	Model-1	Model-2	Model-3	Model-4
S_1	0.00000	0.00000	0.25176	0.15042
S_2	0.23739	0.00000	0.00000	0.00000
S_3	0.00000	0.00000	0.00000	0.00000
S_4	0.00000	0.17151	0.00000	0.29286
S_5	0.00000	0.00000	0.00000	0.00000
S_6	0.00000	0.00000	0.00000	0.00000
S_7	0.00000	0.00000	0.00000	0.00000
S_8	0.00000	0.18474	0.00000	0.00000
S_9	0.00000	0.00000	0.24839	0.21235
S_{10}	0.00000	0.00000	0.00000	0.00000
S_{11}	0.17795	0.00000	0.25339	0.00000
S_{12}	0.22270	0.24706	0.00000	0.00000
S_{13}	0.17547	0.19592	0.24647	0.09441
S_{14}	0.00000	0.00000	0.00000	0.25006
S_{15}	0.18649	0.20077	0.00000	0.00000

bers generated around the assets, Eq. (61) is used to compute the CrSR.

Note that the models are first evaluated using training data set (DS1). The weights of portfolio so obtained are used with testing data set (DS2) to compute the return and risk of the portfolio for the time horizon meant for testing the models. CrSR is computed for portfolios running on DS1 and DS2 for both the approaches and for both the cases. It is evident from Tables 19 and 20 that CrSR for ECE methodology is higher (values in bold) than VB methodology for both the cases of each model for training period (using training data set (DS1)) and testing period (using testing data set (DS2)). Hence, it can be concluded that ECE is showing better performance than VB for the considered set of data for each model.

6.3. Comparing the performance of models based on ECE approach

Usually on lines of Sharpe ratio, one will like to pick a model with highest CrSR. However, for an investor it is important to select a model which is robust and reliable with respect to the changes in the financial markets. In order to choose most reliable model out of four models proposed in this study, the ratio $RT_j = CrSR_j(DS2)/CrSR_j(DS1)$ for each jth model is considered. The idea behind considering this ratio is that after using the training data set (DS1), once the weights of assets in the obtained portfolio are fixed, then the model which has similar performance on testing data (DS2) should be preferred. Thus, a model corresponding to a ratio closer to value of 1 should be preferred for investment decision making. The model selected using this simple heuristic may be validated using the validation data (DS3) by finding $RT' = CrSR(DS3)/CrSR(DS2)$.

Table 21
CrSR and RT for testing stability for case-1.

Model	CrSR		RT
	DS1	DS2	
1	1.0205	1.4422	1.4133
2	1.0373	1.5076	1.4534
3	1.4525	1.6151	**1.1120**
4	1.3794	1.7469	1.2664

Table 22
CrSR and RT for testing stability for case-2.

Model	CrSR		RT
	DS1	DS2	
1	1.0525	1.3308	1.2643
2	1.0373	1.4946	1.4409
3	1.3178	1.5086	**1.1447**
4	1.3794	1.7469	1.2664

Table 23
Validation of Model-3.

Case	CrSR			RT	RT'	Relative change (%)
	DS1	DS2	DS3			
1	1.4525	1.6151	1.7145	1.1120	1.0615	4.54%
2	1.3178	1.5086	1.8698	1.1447	1.2395	8.28%

Tables 21 and 22 demonstrate the value of RT corresponding to each model for each case. Note that, Model-3 has a RT value closer to one (value in bold) as compared to other models. This is true for both the cases. Hence, Model-3 is considered to be a better model (in terms of reliability of performance in future) among other models considered in this study.

To validate the performance of selected model, i.e. Model-3, data set for year 2015–2016 is taken and CrSR is computed for case-1 and case-2. We next compute $RT' = CrSR(DS3)/CrSR(DS2)$, to check for the stability in performance of Model-3. As shown in Table 23, the relative change between RT and RT' is 4.54% for case-1 and 8.28% for case-2 which are not very high and quite acceptable values. From these results, it can be concluded that the solution obtained for Model-3 for both cases is stable and the model shows a promising performance for the considered set of assets.

Fig. 12 illustrates the entire framework of the proposed DSS.

7. Concluding remarks

In this study a DSS which guides an investor to pick a robust portfolio of assets is developed. An automation process, namely the ECE approach, forms an integral part of this DSS. This approach

Table 19
CrSR for training data set (DS1) using VB and ECE methods.

Methods	Case-1				Case-2			
	Model-1	Model-2	Model-3	Model-4	Model-1	Model-2	Model-3	Model-4
ECE	**1.0205**	**1.0373**	**1.4525**	**1.3794**	**1.0525**	**1.0373**	**1.3178**	**1.3794**
VB	0.3641	0.2967	0.4290	0.3749	0.4197	0.2681	0.4497	0.3749

Table 20
CrSR for testing data set (DS2) using VB and ECE methods.

Methods	Case-1				Case-2			
	Model-1	Model-2	Model-3	Model-4	Model-1	Model-2	Model-3	Model-4
ECE	**1.4422**	**1.5076**	**1.6151**	**1.7469**	**1.3308**	**1.4946**	**1.5086**	**1.7469**
VB	0.9843	1.0624	1.0884	0.9588	1.0010	1.0786	1.1685	0.9588

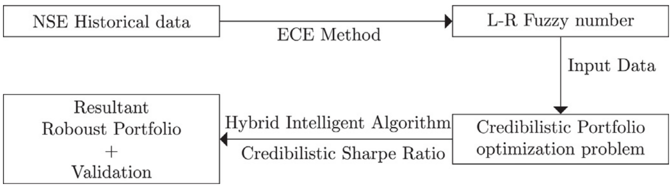


Fig. 12. Framework of proposed DSS.

has an ability of deriving the information regarding uncertain asset parameters that is embedded within the historical data sets and helps fitting L–R fuzzy numbers around them. Owing to its self-dual property, credibility measure is preferred over possibility measure while formulating four portfolio selection models in multi-criteria framework for this study. A Hybrid Intelligent Algorithm is then designed by running fuzzy simulation within the MIBEX-SM GA, so as to solve each of these formulated models. The models are initially trained using historical data of 2008–2013 for 15 assets from the NSE of India. All the models are then subjected to a test data of 2013–2015 for evaluating their performance. Model-3 is selected as the best performing model based on the stability analysis done using the modified Sharpe ratio, namely the Credibilistic Sharpe Ratio. Model-3 is further validated using data set of 2015–2016 and found to be robust and stable in future performance for two different cases pertaining to two different parameter settings of the problem. The models formulated in this study can be extended further by adding more realistic criteria of investment market like lot-size and transaction costs. The ECE approach can also be used along with portfolio selection problems belonging to fuzzy random or random fuzzy environments. Such an approach of automation of fitting a fuzzy number around uncertain parameters can also be utilized in areas of study other than portfolio selection problems for an example pattern recognition, predictive soil mapping etc.

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