

Approximation of the limit of Divergent Series

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<2018-05-15 Tue>

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"Divergent series are an invention of the devil and it is shameful to base on them any demonstration whatsoever." [1] – N.H. Abel. 1826

Abstract

Mathematics is the science of skilful operations, which are dealing with the logic of shape, quantity and rearrangement. Mathematics is using its core principles and rules to solve problems and get a reasonable answer. In my opinion, mathematics will soon run out of interesting theories if all new theories will formulate in terms of invented concept, which appear in axioms. This exploration is about finding finite values, limits for famous divergent series and later analytically extending Euler-Riemann's zeta function to find a limit for any divergent series that can be formed by Euler- Riemann's zeta function.

1 Introduction

I tend to think that in order to create innovative theories, mathematics as a whole should be pushed to its boundaries and limits, new axioms should be invented and unexplored topics should be studied. With this kind of approach, I got acquainted with the divergent series. It all has started in my former school when I was in 7 th grade and we have started the topic of geometric series. When my teacher asked a question – "What will be the infinite sum of one, a half, a quarter and so on?" Initially, I thought the answer was infinity, because at that time it made sense to me, an infinite amount of values equals to always growing sum until it reaches infinity. When the teacher told us the answer is two, I was quite shocked by the fact that it has an end and all these values are converging to one finite value. I remember how I was walking around and trying to understand, why this series does not go beyond that finite value? It was logical to me that if it keeps summing every element, every element should contribute to the goal of getting closer to that value and there should be one element, which after summing up will go beyond two. I started reading and researching additional material to understand the fundamental logic of convergent series. When I finally understood why it works, I found a new topic, that struck my mind – "Divergent series" and not the series in general, but how the most famous mathematicians: Leonhard Euler, Bernhard Riemann and Srinivasa Ramanujan proved that these series are in nature convergent too. Now it all made a perfect sense.

In this exploration I will explore the most famous and well-known divergent series, proving with multiple ways that they have a finite value that

they are converging too and at the end, I will show how it is possible to find limit almost for any divergent series.

2 Grandi's Series

This is probably one of the famous and trickiest-at-the-beginning divergent series. It looks like this:

$$1 - 1 + 1 - 1 + 1 - 1 + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \quad (1)$$

Strange, isn't? For the first time, it has been reported by mathematician Guido Grandi in 1703. He noticed that inserting parenthesis into the series will return a different result, but these parentheses are being inserted in a way, that it should not affect the series value. Let me show you:

Parenthesis before even elements: $1 - (1 - 1) - (1 - 1) - (1 - 1) \dots = 1$

Parenthesis before odd elements: $(1 - 1) + (1 - 1) + (1 - 1) + (1 \dots = 0$

Grandi explained this phenomenon by saying that - "Mettendo in modo diverso le parentesi nell'espressione $1 - 1 + 1 - 1 + \dots$ io posso, volendo, ottenere 0 o 1. Ma allora l'idea della creazione ex nihilo è perfettamente plausibile." [2] Which translates from Italian to English - "By putting parentheses into the expression $1 - 1 + 1 - 1 + \dots$ in different ways, I can, if I want, obtain 0 or 1. But then the idea of the creation ex nihilo is perfectly plausible." Where "ex nihilo" means "out of nothing". [3] As a true mathematician, Leonhard Euler did not think that the value is either 0 or 1, he thought that the true limit was 1/2. He proved it through the geometric reasoning of the "Witch of Agnesi". We will prove it arithmetically.

2.1 First Solution

This is pretty easy and convenient solution for rearranging and ordering. Let us denote Grandi's series by S , so $S = 1 - 1 + 1 - 1 + \dots$. Further, we will subtract S from 1, and then we will get:

$$\begin{aligned}
1 - S &= 1 - (1 + 1 - 1 + 1 - 1 + \dots) \\
1 - S &= 1 - 1 + 1 - 1 + 1 - \dots \\
1 - S &= S \\
2S &= 1 \\
S &= 1/2
\end{aligned}$$

2.2 Second Solution

We can formulate the second solution by using the infinite formula for converging geometric series - $\frac{a}{1-r}$, where a is the first element of the series and r is the common ratio. Generally, the formula is applicable only if absolute value of the common ratio is less than one, $|r| < 1$ but it is also applicable in our case, where $r = -1$. In this case, let us denote Grandi's series as S , and

$$S = \frac{1}{1 - (-1)} = \frac{1}{1 + 1} = \frac{1}{2}$$

This proves how the series converges to $1/2$. This is the solution that I thought about, when I imagined the Grandi's series as a geometric series.

2.3 Third Solution

Third solution can be achieved by using partial sums. So that the series's sum will be:

$$\begin{array}{cccccccc}
1 & -1 & +1 & -1 & +1 & -1 & +1 & \dots \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots
\end{array}$$

By averaging the sums, we can conclude that:

$$\begin{array}{cccccc}
1 & (1+0)/2 & (1+0+1)/3 & (1+0+1+0)/4 & (1+0+1+0+1)/5 \\
1 & 1/2 & 2/3 & 1/4 & 3/5
\end{array}$$

Therefore, we can evaluate that with each even indexes the series equal to $1/2$ and with odd indexes, the partial sums converge to $1/2 + 1/2n$, where n is the amount of elements summed up. If the amount of indexes tends to infinity then $1/2n$ will be infinitely small, resulting that the infinite partial sums will converge to $1/2$.

3 Wolff's series $(1 - 2 + 3 - 4 + 5 - 6 + \dots)$

This series is quite fascinating too. When Guido Grandi has published his work on his infinite series that we explored above, famous mathematician Christian Wolff doubted the logic of Grandi's series and because of that, he wrote a letter to Gottfield Wilhelm Leibniz. After receiving a reply [4] from Leibniz, Wolff was so pleased with the solution that he wanted to extend the series to $1 - 2 + 3 - 4 + 5 - 6 + \dots$

This series can be also represented like

$$\sum_{n=1}^{\infty} (-1)^{n+1} n$$

This series as goes, seems to tend to negativeinfinity, as the progression will look like this:

$$1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, \dots$$

By analytical continuation and mathematical rules, we can find the limit of this series. The simplest solution follows the rule of infinite geometric series and derivation.

If we have a series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots$$

let us denote it as s , where the absolute value of the common ration is less than 1: $|x| < 1$, then we can find the sum using the formula of infinite convergent geometric series - $\frac{1}{1-x}$. It can be derived by using fundamental mathematical operations:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \dots = 1 + x(1 + x + x^2 + x^3 + x^4 + x^5 + \dots)$$

$$s = 1 + xs$$

$$s - xs = 1$$

$$s(1 - x) = 1$$

$$s = \frac{1}{1 - x} \tag{2}$$

It was applicable to the Grandi's series but in order to make it suitable for Wolff's series, we will derivate the equation from both parts, so we will get:

$$\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \dots) = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

By applying basic differential methods on the left side and quotient rule on the right side of the equation, the equation will be:

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \dots = \frac{1}{(1-x)^2}$$

Now, if we will substitute x as -1 , so $x = -1$, on the left side we will get the Wolff's series and some finite value on the right side.

$$1 - 2 + 3 - 4 + 5 - 6 + \dots = \frac{1}{(1+1)^2} = \frac{1}{2^2} = \frac{1}{4} \quad (3)$$

Generally, the infinite sum formula for geometric series is valid only when the absolute value of the common ratio is less than one, but in our case, it is applicable too.

4 $1 + 2 + 3 + 4 + 5 + \dots$

It is safe to say that this is a fascinating series with a fascinating result. $1 + 2 + 3 + \dots$ is a particular case of Euler-Riemann zeta function, where the Euler-Riemann zeta function can be represented in the following way:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots \quad (4)$$

This function of a complex variable s that analytically continues from Dirichlet series, so our series from above can be written in terms of Euler-Riemann zeta function:

$$\sum_{n=1}^{\infty} n \text{ or } \zeta(-1) \quad (5)$$

I will talk about Euler-Riemann zeta function more in the next part. Back to our series. Interestingly enough, the finite value of this series is negative value of one over twelve or mathematically speaking, $-1/12$.

In this part, one solution with rearrangements and logic will be used, as the other one will be explored in the next chapter- "Euler-Riemann Zeta Function".

The first solution involves the series itself and the Wolff's series. So to start solving it, let us denote the series $1 + 2 + 3 + 4 + 5 + \dots$ as s_1 and the

series $1 - 2 + 3 - 4 + 5 - 6 + 7 - \dots$ as s_2 . So then we can write the following equation:

$$\begin{aligned} s_1 - s_2 &= 1 + 2 + 3 + 4 + 5 + 6 + \dots \\ &\quad - (1 - 2 + 3 - 4 + 5 - 6 + \dots) = \\ &= 0 + 4 + 0 + 8 + 0 + 16 + \dots = \\ &= 4(1 + 2 + 3 + 4 + 5 + \dots) = \end{aligned}$$

Now we can rewrite the equation by substituting $1 + 2 + 3 + 4 + 5 + \dots$ as s_1 and s_2 with the value from 3

$$\begin{aligned} s_1 - s_2 &= 4s_1 \\ -3s_1 &= s_2 \\ s_1 &= -\frac{1}{12} \end{aligned}$$

5 Euler-Riemann Zeta Function

I would like to talk more about Euler-Riemann zeta function. Euler-Riemann zeta function is a beautiful function in the world of complex numbers and later applied to real numbers. The notation of zeta Euler-Riemann function is $\zeta(s)$, where s is some complex number and is defined only when the real part of s is bigger than 1. From equation 4 we know the notation of Euler-Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$$

It is called Euler-Riemann because Euler was the first mathematician who made a notable contribution to the function, but Euler was making his works on the Euler-Riemann zeta function using real values as s , instead of complex. Almost after a decade famous mathematician Bernhard Riemann has published his works about Euler-Riemann zeta function by using complex number.

There are many solutions to this series that can be achieved by different approaches and formulas. I will explore some solutions to it and then I will try to find the functional equation of the Euler-Riemann zeta function and with that, we can prove that almost any divergent series formed by

Euler-Riemann zeta converges and has a finite value. One of the "Millenium Prize Problems" is called "Riemann Hypothesis", which says that the Euler-Riemann zeta function returns zeros only at the negative even integers and complex numbers with real part $1/2$.

This math exploration is called "Approximation of the limit of Divergent Series" and we are not going to explore Euler-Riemann zeta function, where s , Euler-Riemann zeta function's argument is a complex number. In this exploration I will research how to find finite values and trivial zeros for divergent series, which are particular cases of the Euler-Riemann zeta function, thus proving that almost all divergent series formed by Euler-Riemann zeta function are quite convergent.

In the last part of this exploration, I will show my way of presenting the Euler-Riemann zeta function in the form of a product, how to find convergent values for any zeta-function-formed divergent series and the connection of it with Bernoulli numbers.

5.1 Product notation

Firstly, we should know how to present the Euler-Riemann zeta function in the form of a product, so:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \quad (6)$$

When I introduced my exploration topic to my math teacher, he struck me with a fact that the sum can be presented in the form of a product. So I have been trying to experiment with the Euler-Riemann zeta function in different ways and saw one easy way of transforming it.

We can divide every element of the Euler-Riemann zeta function by 2^s , so:

$$\frac{\zeta(s)}{2^s} = \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots \right) \frac{1}{2^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \dots$$

Now, all denominators are multiples of two, so now we can subtract the series above from the Euler-Riemann zeta function in order to get new series, only with odd denominators, as subtraction excluded all denominators, which are multiples of 2.

$$\zeta(s) \left[1 - \frac{1}{2^s} \right] = \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots$$

Now we have only odd denominators and we can repeat the same process and exclude all elements, which denominators are multiples of 3.

$$\frac{\zeta(s)}{3^s} [1 - \frac{1}{2^s}] = (\frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots) \frac{1}{3^s} = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \dots$$

$$\zeta(s) [1 - \frac{1}{2^s}] [1 - \frac{1}{3^s}] = \frac{1}{1^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots$$

We can repeat this process for an infinite amount of prime numbers, thus excluding all denominators, which are multiples of kth prime. After repeating it an indefinite amount of times, on the right side, only 1 will be left, as 1 is not a multiple of any prime number.

$$\zeta(s) [1 - \frac{1}{2^s}] [1 - \frac{1}{3^s}] [1 - \frac{1}{5^s}] \dots [1 - \frac{1}{p_k^s}] \dots = 1$$

$$\zeta(s) \prod_{p \text{ prime}}^{\infty} (1 - \frac{1}{p^s}) = 1$$

$$\zeta(s) = \frac{1}{\prod_{p \text{ prime}}^{\infty} (1 - \frac{1}{p^s})} = \prod_{p \text{ prime}}^{\infty} \frac{1}{1 - \frac{1}{p^s}} = \prod_{p \text{ prime}}^{\infty} (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$$

After some rearranging, we can see how the Euler-Riemann zeta function can be presented as sum and product. With knowledge of that, we can go further and find a value for Euler-Riemann zeta function with every negative integer, thus finding a finite value for any divergent series. We will need this identity to find values for Euler-Riemann zeta function with negative odd integers.

From now on I will be using Wolfram Alpha and MATLAB to plot Euler-Riemann zeta function and find values for arguments.

5.2 Trivial Zeros

Firstly, we will find "trivial zeros" for Euler-Riemann zeta function. "Trivial zeros" are all zeros, which are a result of function $\zeta(-2n)$, where n is a non-negative integer. In order to find these zeros, we should know the Euler-Riemann zeta function's functional equation. Basically, the functional equation [5] is an equation of the form $f(x, y, \dots) = 0$, where f contains a

finite number of independent variables, known function, and unknown functions which are to be solved for. I found the Euler-Riemann zeta function's functional equation in a fantastic book "Divergent Series" by G.H.Hardy [6] In the function below, Γ is the gamma function, where $\Gamma(n) = (n-1)!$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (7)$$

Now, in order to prove that for any negative integer the Euler-Riemann zeta function will return zero, we should use an argument of $-2n$, where n is a non-negative integer.

$$\zeta(-2n) = 2^{-2n} \pi^{-2n-1} \sin(-\pi n) \Gamma(1+2n) \zeta(1+2n) \quad (8)$$

Subsequently, we can ignore all variables in the equation above, except $\sin(-\pi n)$. As sine function is an even function, so then:

$$\sin(-\pi n) = -\sin(\pi n) \quad (9)$$

Referring to the simplest trigonometric rules, we know that sine function with an angle, which is a multiple of π will always return a zero. Now proceeding next we can conclude that:

$$\zeta(-2n) = -2^{-2n} \pi^{-2n-1} \sin(\pi n) \Gamma(1+2n) \zeta(1+2n) = 0$$

Thereafter we can bring examples of Euler-Riemann zeta function, where the argument will be a negative even integer, thus finding a finite value for these divergent series:

$$\zeta(-2) = \sum_{n=1}^{\infty} n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \dots = 1 + 4 + 9 + 16 + 25 + \dots = 0$$

$$\zeta(-4) = \sum_{n=1}^{\infty} n^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4 + \dots = 1 + 16 + 81 + 256 + 625 + \dots = 0$$

$$\zeta(-2n) = \sum_{n=1}^{\infty} n^{2n} = 1^{2n} + 2^{2n} + 3^{2n} + 4^{2n} + 5^{2n} + \dots = 0 \quad (10)$$

In addition, we can plot the Euler-Riemann zeta function and prove the zero values for all even negative integers geometrically. I will use Wolfram Alpha Open Code to plot the Euler-Riemann zeta function and MATLAB to find the Euler-Riemann zeta function's zeros. Figure 1 shows this graph.

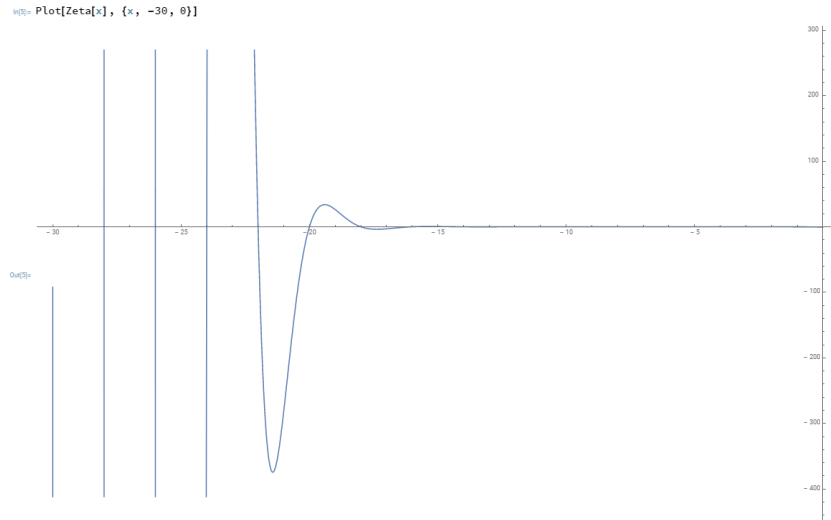


Figure 1: Graph of Euler-Riemann zeta function using Wolfram|Alpha Open Code and built-in math functions.

In the graph above we can see how the Euler-Riemann zeta function intersects x-axis on negative even integers. However, each time the absolute value of an argument increases, this feature of Euler-Riemann zeta function will be discussed next. Also, we can ensure the values of Euler- Riemann zeta function by using MATLAB as shown in Figure 2.

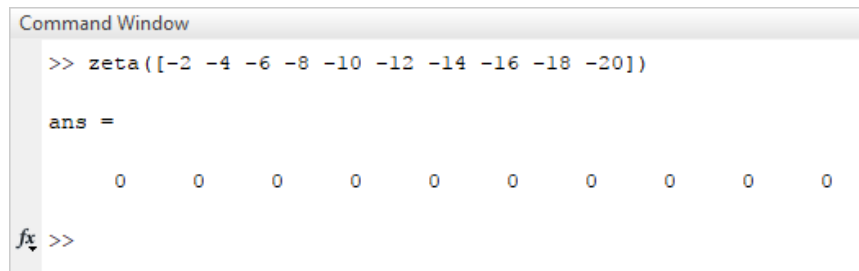


Figure 2: Euler-Riemann zeta function values for negative even integers, calculated in MATLAB.

5.3 Negative Odd Arguments

Secondly, as I discussed the “trivial zeros” for Euler-Riemann and as we have proved the product form for Euler-Riemann function, now we can find the values for negative odd integers. We know three ways of presenting Euler-Riemann zeta function:

$$\zeta(s) = \prod_{p \text{ prime}}^{\infty} (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad (11)$$

By using the notations below and Euler’s formula that we learned during our course of complex numbers, we can see how they lead [7] to the following formula for negative integer arguments:

$$\zeta(1-n) = -\frac{B_n}{n}, \quad n \in \mathbb{N} \quad (12)$$

B_n is the n^{th} Bernoulli number. Bernoulli numbers are a series of rational numbers, which are mainly used in number theory. The values for n^{th} value of Bernoulli numbers can be defined in the following equation [8], where for every odd n , $B_n = 0$:

$$B_n = \frac{\sum_{k=1}^n \sum_{j=1}^k \frac{(-1)^j (j^n \binom{1+n}{-j+k})}{\binom{n}{k}}}{1+n}$$

The table below shows the values for the first 10 even Bernoulli numbers’ arguments. These values have been calculated by using Wolfram|Alpha Open Code service. Table 1 will show first 10 non-zero elements’ values of Bernoulli numbers.

Table 1: First 10 Non-Zero Elements of Bernoulli Numbers.

n	0	1	2	4	6	8	10	12	14	16
B_N	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$

Subsequently, we can now find finite values for Euler-Riemann zeta function with negative odd integer arguments. Referring to the previous chapter with the series $1 + 2 + 3 + 4 + 5 + \dots$, now we can present the following series in the form of Euler-Riemann zeta function with Bernoulli numbers and find a convergent value for the series.

$$\sum_{n=1}^{\infty} n = 1+2+3+4+5+\dots = \zeta(-1) = \zeta(1-2) = -\frac{B_2}{2} = -\frac{\frac{1}{6}}{2} = -\frac{1}{12} \quad (13)$$

We can apply equation 12 to any natural even number and by that forming a divergent series, where it would be possible to find a value to which the series converges. As an example, I would like to test out powers 3, 5 and 7:

$$\begin{aligned} \sum_{n=1}^{\infty} n^3 &= 1^3 + 2^3 + 3^3 + 4^3 + \dots = \zeta(-3) = \zeta(1-4) = -\frac{B_4}{4} = -\frac{-\frac{1}{30}}{4} = \frac{1}{120} \\ \sum_{n=1}^{\infty} n^5 &= 1^5 + 2^5 + 3^5 + 4^5 + \dots = \zeta(-5) = \zeta(1-6) = -\frac{B_6}{6} = -\frac{\frac{1}{42}}{6} = -\frac{1}{252} \\ \sum_{n=1}^{\infty} n^7 &= 1^7 + 2^7 + 3^7 + 4^7 + \dots = \zeta(-7) = \zeta(1-8) = -\frac{B_8}{8} = -\frac{-\frac{1}{30}}{8} = \frac{1}{240} \end{aligned}$$

6 Conclusion

In the conclusion, throughout this exploration, we have explored the methods of finding the limit for divergent series. We should know that these limits are found using mathematical manipulations with infinity, meaning that we assume that we have an infinite amount of integers and elements, we should rather know that these convergent values have been found because we analytically extended Euler-Riemann zeta function. People tend to think of these results as "mathematical tricks" and theories to play around, however, these results are actively used in physics, as in physics we can not have an infinity or indefinite results, results should be finite, precise and exact. For example, the limit of $\zeta(-1)$ is used the "String Theory" by Joseph Polchinski [9] in order to compute energy levels of a single string. Also, the series $1 + 2 + 3 + 4 + \dots = -1/12$ has been mentioned in the media by Numberphile in video - "ASTOUNDING: $1+2+3+4+5+\dots = -1/12$ " [10]. Sure, at the end we came up with a rather extraordinary result, which makes Mathematics as a science beautiful and encouraging for further researches and explorations.

I would like to end this exploration with an idea that relatively to its full potential, Mathematics is quite young science, even if it may seem otherwise. From the very beginning of its existence, Mathematics helped us and guided us through the mysteries and secrets of the world around us.

People were working on Mathematics by using their imagination, reasoning and emotions. So that Mathematics became a result of humanity's thoughts and hope for the better future. We should never stop improving and working on Mathematics, no matter how difficult or unsolvable the future challenges may seem.

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