Homework 8 Oracle

MATH 220 Spring 2021

Sandy Urazayev*

107; 12021 H.E.

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Section 7.3

Problem 7

The three vectors are linearly dependent if there exists a nontrivial solution to

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

for c_1, c_2, and c_3. Rewrite this equation.

$$c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Calculate the determinant of the coefficient matrix.

$$\det\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} - 0\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + 0\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

Since it's zero, there are infinitely many solutions for c_1, c_2, and c_3.

*University of Kansas (ctu@ku.edu)

$$2c_1 - c_3 = 0$$
$$c_1 + c_2 + 2c_3 = 0$$

Solve this first equation for c_3

$$c_3 = 2c_1$$

and plug it into the second one.

$$c_1 + c_2 + 2(2c_1) = 0$$

Solve for c_2

$$c_2 = -5c_1$$

In terms of the free variable c_1, the solution to the system of equations is

$$\{c_1, -5c_1, 2c_1\}$$

For example, choose $c_1 = 1$. Then

$$\mathbf{x}^{(1)} - 5\mathbf{x}^{(2)} + 2\mathbf{x}^{(3)} = \mathbf{0}$$

Therefore, the three given vectors are linearly dependent.

Problem 16 [FOR GRADE]

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

The eigenvalues satisfy

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix} = 0$$
$$(-2-\lambda)(-2-\lambda)-1=0$$
$$\lambda^2+4\lambda+3=0$$
$$(\lambda+3)(\lambda+1)=0$$
$$\lambda=\{-3,-1\}$$

Therefore, the eigenvalues are

$$\lambda_1=-3 \text{ and } \quad \lambda_2=-1$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$

$$(\mathbf{A}-\lambda_1\mathbf{I})\,\mathbf{x}_1=\mathbf{0}$$
$$(\mathbf{A}-\lambda_2\mathbf{I})\,\mathbf{x}_2=\mathbf{0}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \end{pmatrix}$$
$$x_1+x_2=0$$
$$-x_1+x_2=0$$
$$x_1+x_2=0\}$$
$$x_1-x_2=0$$
$$x_2=-x_1$$
$$x_2=x_1$$
$$\mathbf{x}_1=\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\mathbf{x}_2=\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.

Problem 17

The aim is to solve the eigenvalue problem,

$$\mathbf{Ax}=\lambda\mathbf{x}$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

The eigenvalues satisfy

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

Evaluate the determinant and solve for λ.

$$\begin{gathered}
\det\begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix}=0 \\
(1-\lambda)(-1-\lambda)-3=0 \\
\lambda^2-4=0 \\
(\lambda+2)(\lambda-2)=0 \\
\lambda=\{-2,2\}
\end{gathered}$$

Therefore, the eigenvalues are

$$\lambda_1=-2 \text{ and } \quad \lambda_2=2$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\begin{gathered}
(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{x}_1=\mathbf{0} \\
\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \end{pmatrix} \\
\left.\begin{aligned} 3x_1+\sqrt{3}x_2&=0 \\ \sqrt{3}x_1+x_2&=0 \end{aligned}\right\} \\
x_2=-\sqrt{3}x_1 \\
\mathbf{x}_1=\begin{pmatrix} x_1 \\ -\sqrt{3}x_1 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}
\end{gathered}$$

and

$$\begin{gathered}
(\mathbf{A}-\lambda_2\mathbf{I})\mathbf{x}_2=\mathbf{0} \\
\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \end{pmatrix}
\end{gathered}$$

$$\left.\begin{aligned} -x_1+\sqrt{3}x_2 &= 0 \\ \sqrt{3}x_1-3x_2 &= 0 \end{aligned}\right\}$$

$$x_2 = \frac{1}{\sqrt{3}}x_1$$

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ \frac{1}{\sqrt{3}}x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.

Problem 18

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

The eigenvalues satisfy

$$\det(\mathbf{A}-\lambda\mathbf{I}) = 0$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)(1-\lambda)+4] = 0$$

$$1-\lambda = 0 \quad \text{or} \quad \lambda^2-2\lambda+5 = 0$$

$$5-7\lambda+3\lambda^2-\lambda^3 = 0$$

$$(1-\lambda)\left(\lambda^2-2\lambda+5\right) = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = \frac{2\pm\sqrt{4-20}}{2} = 1\pm 2i$$

Therefore, the eigenvalues are

$\lambda_1 = 1$	$\lambda_2 = 1 - 2i$	$\lambda_3 = 1 + 2i$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_1$.

$$(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{x}_1 = \mathbf{0}$$

$$\begin{pmatrix} 1-(1) & 0 & 0 \\ 2 & 1-(1) & -2 \\ 3 & 2 & 1-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left. \begin{aligned} 2x_1 - 2x_3 &= 0 \\ 3x_1 + 2x_2 &= 0 \end{aligned} \right\}$$

Solve for x_2 and x_3 in terms of the free variable x_1.

$x_3 = x_1$

$x_2 = -\frac{3}{2}x_1$

This means

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{3}{2}x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

Since x_1 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\mathbf{x}_1 = x_1' \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

Problem 20 [FOR GRADE]

The aim is to solve the eigenvalue problem,

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

where $\mathbf{A}$ is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

The eigenvalues satisfy

$$\det(\mathbf{A}-\lambda\mathbf{I})=0$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 11/9-\lambda & -2/9 & 8/9 \\ -2/9 & 2/9-\lambda & 10/9 \\ 8/9 & 10/9 & 5/9-\lambda \end{pmatrix}=0$$

$$(11/9-\lambda)[(2/9-\lambda)(5/9-\lambda)-100/81]+(2/9)[(-2/9)(5/9-\lambda)-80/81]$$

$$+(8/9)[-20/81-(8/9)(2/9-\lambda)]=0$$

$$(11/9-\lambda)\begin{vmatrix} 2/9-\lambda & 10/9 \\ 10/9 & 5/9-\lambda \end{vmatrix}-(-2/9)\begin{vmatrix} -2/9 & 10/9 \\ 8/9 & 5/9-\lambda \end{vmatrix}+(8/9)\begin{vmatrix} -2/9 & 2/9-\lambda \\ 8/9 & 10/9 \end{vmatrix}=0$$

$$-2+\lambda+2\lambda^2-\lambda^3=0$$

$$(\lambda+1)(\lambda-1)(2-\lambda)=0$$

Therefore, the eigenvalues are

$$\boxed{\lambda_1=1}\ \boxed{\lambda_2=2}\ \boxed{\lambda_3=-1}$$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_1$.

$$(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{x}_1=\mathbf{0}$$

$$\begin{pmatrix} 11/9-(1) & -2/9 & 8/9 \\ -2/9 & 2/9-(1) & 10/9 \\ 8/9 & 10/9 & 5/9-(1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2/9 & -2/9 & 8/9 \\ -2/9 & -7/9 & 10/9 \\ 8/9 & 10/9 & -4/9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} 2/9 & -2/9 & 8/9 & 0 \\ -2/9 & -7/9 & 10/9 & 0 \\ 8/9 & 10/9 & -4/9 & 0 \end{array}\right)$$

Multiply each row by 9

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ -2 & -7 & 10 & 0 \\ 8 & 10 & -4 & 0 \end{array}\right)$$

Multiply the first row by -4 and add it to the third row.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ -2 & -7 & 10 & 0 \\ 0 & 18 & -36 & 0 \end{array}\right)$$

Add the first row to the second row.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 18 & -36 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_2 in terms of the free variable x_3

$$\left.\begin{array}{r} 2x_1 - 2x_2 + 8x_3 = 0 \\ -9x_2 + 18x_3 = 0 \\ 18x_2 - 36x_3 = 0 \end{array}\right\} \quad \rightarrow \quad \begin{array}{l} x_1 = -2x_3 \\ x_2 = 2x_3 \end{array}$$

This means

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{pmatrix}$$

Therefore,

$$\mathbf{x}_1 = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

Substitute λ_2 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_2$.

$$(\mathbf{A}-\lambda_2\mathbf{I})\,\mathbf{x}_2=\mathbf{0}$$

$$\begin{pmatrix} 11/9-(2) & -2/9 & 8/9 \\ -2/9 & 2/9-(2) & 10/9 \\ 8/9 & 10/9 & 5/9-(2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -7/9 & -2/9 & 8/9 \\ -2/9 & -16/9 & 10/9 \\ 8/9 & 10/9 & -13/9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{rrr|r} -7/9 & -2/9 & 8/9 & 0 \\ -2/9 & -16/9 & 10/9 & 0 \\ 8/9 & 10/9 & -13/9 & 0 \end{array}\right)$$

Multiply each row by 9

$$\left(\begin{array}{rrr|r} -7 & -2 & 8 & 0 \\ -2 & -16 & 10 & 0 \\ 8 & 10 & -13 & 0 \end{array}\right)$$

Multiply the second row by 4 and add it to the third row.

$$\left(\begin{array}{rrr|r} -7 & -2 & 8 & 0 \\ -2 & -16 & 10 & 0 \\ 0 & -54 & 27 & 0 \end{array}\right)$$

Multiply the first row by -8 and add it to the second row.

$$\left(\begin{array}{rrr|r} -7 & -2 & 8 & 0 \\ 54 & 0 & -54 & 0 \\ 0 & -54 & 27 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_2 in terms of the free variable x_3

$$\left.\begin{array}{r} -7x_1-2x_2+8x_3=0 \\ 54x_1-54x_3=0 \\ -54x_2+27x_3=0 \end{array}\right\} \quad \rightarrow \quad \begin{array}{l} x_1=x_3 \\ x_2=\dfrac{1}{2}x_3 \end{array}$$

This means

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Since x_3 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$\mathbf{x}_2 = x_3' \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

Substitute λ_3 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_3$.

$$(\mathbf{A} - \lambda_3 \mathbf{I})\,\mathbf{x}_3 = \mathbf{0}$$

$$\begin{pmatrix} 11/9-(-1) & -2/9 & 8/9 \\ -2/9 & 2/9-(-1) & 10/9 \\ 8/9 & 10/9 & 5/9-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 20/9 & -2/9 & 8/9 \\ -2/9 & 11/9 & 10/9 \\ 8/9 & 10/9 & 14/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} 20/9 & -2/9 & 8/9 & 0 \\ -2/9 & 11/9 & 10/9 & 0 \\ 8/9 & 10/9 & 14/9 & 0 \end{array}\right)$$

Multiply each row by 9

$$\left(\begin{array}{ccc|c} 20 & -2 & 8 & 0 \\ -2 & 11 & 10 & 0 \\ 8 & 10 & 14 & 0 \end{array}\right)$$

Multiply the second row by 4 and add it to the third row.

$$\left(\begin{array}{ccc|c} 20 & -2 & 8 & 0 \\ -2 & 11 & 10 & 0 \\ 0 & 54 & 54 & 0 \end{array}\right)$$

Multiply the second row by 10 and add it to the first row.

$$\left(\begin{array}{ccc|c} 0 & 108 & 108 & 0 \\ -2 & 11 & 10 & 0 \\ 0 & 54 & 54 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_3 in terms of the free variable x_2

$$\left.\begin{array}{r} 108x_2 + 108x_3 = 0 \\ -2x_1 + 11x_2 + 10x_3 = 0 \\ 54x_2 + 54x_3 = 0 \end{array}\right\} \quad \rightarrow \quad \begin{array}{l} x_1 = \frac{1}{2}x_2 \\ x_3 = -x_2 \end{array}$$

This means

$$\mathbf{x}_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \\ -x_2 \end{pmatrix} = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \end{pmatrix}$$

Since x_2 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\mathbf{x}_3 = x_2' \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

Section 7.4

Problem 5(a)

$$t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} 2t - t \\ 3t - 2t \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix}$$

$$t\begin{pmatrix} -t^{-2} \\ -3t^{-2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} t^{-1} \\ 3t^{-1} \end{pmatrix} = \begin{pmatrix} 2t^{-1} - 3t^{-1} \\ 3t^{-1} - 6t^{-1} \end{pmatrix} = \begin{pmatrix} -t^{-1} \\ -3t^{-1} \end{pmatrix}$$

Problem 6(a) [FOR GRADE]

$$t\begin{pmatrix} -t^{-2} \\ -2t^{-2} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} t^{-1} \\ 2t^{-1} \end{pmatrix} = \begin{pmatrix} 3t^{-1} - 4t^{-1} \\ 2t^{-1} - 4t^{-1} \end{pmatrix} = \begin{pmatrix} -t^{-1} \\ -2t^{-1} \end{pmatrix}$$

$$t\begin{pmatrix} 4t \\ 2t \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2t^2 \\ t^2 \end{pmatrix} = \begin{pmatrix} 6t^2 - 2t^2 \\ 4t^2 - 2t^2 \end{pmatrix} = \begin{pmatrix} 4t^2 \\ 2t^2 \end{pmatrix}$$