Homework 8 Oracle

MATH 220 Spring 2021

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Section 7.3

Problem 7

The three vectors are linearly dependent if there exists a nontrivial solution to

$$c_1 \mathrm{x}^{(1)} + c_2 \mathrm{x}^{(2)} + c_3 \mathrm{x}^{(3)} = 0$$

for c_1, c_2, and c_3. Rewrite this equation.

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Calculate the determinant of the coefficient matrix.

$$\det \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = 0 \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} - 0 \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} + 0 \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 0$$

Since it's zero, there are infinitely many solutions for c_1, c_2, and c_3.

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$$2c_1 - c_3 = 0$$

$$c_1 + c_2 + 2c_3 = 0$$

Solve this first equation for c_3

$$c_3 = 2c_1$$

and plug it into the second one.

$$c_1 + c_2 + 2(2c_1) = 0$$

Solve for c_2

$$c_2 = -5c_1$$

In terms of the free variable c_1, the solution to the system of equations is

$$\{c_1, -5c_1, 2c_1\}$$

For example, choose $c_1 = 1$. Then

$$x^{(1)} - 5x^{(2)} + 2x^{(3)} = 0$$

Therefore, the three given vectors are linearly dependent.

Problem 16 [FOR GRADE]

The aim is to solve the eigenvalue problem,

$$Ax = \lambda x$$

where A is the given matrix. Bring λx to the left side and combine the terms.

$$(A - \lambda I)x = 0$$

The eigenvalues satisfy

$$\det(A - \lambda I) = 0$$

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Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{pmatrix}=0$$
$$(-2-\lambda)(-2-\lambda)-1=0$$
$$\lambda^2+4\lambda+3=0$$
$$(\lambda+3)(\lambda+1)=0$$
$$\lambda=\{-3,-1\}$$

Therefore, the eigenvalues are

$$\lambda_1=-3 \text{ and } \quad \lambda_2=-1$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, x_1 and x_2

$$(A-\lambda_1 I)x_1=0$$
$$(A-\lambda_2 I)x_2=0$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \end{pmatrix}$$
$$x_1+x_2=0$$
$$-x_1+x_2=0$$
$$x_1+x_2=0\}$$
$$x_1-x_2=0$$
$$x_2=-x_1$$
$$x_2=x_1$$
$$x_1=\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$x_2=\begin{pmatrix} x_1 \\ x_1 \end{pmatrix}=x_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.

Problem 17

The aim is to solve the eigenvalue problem,

$$Ax = \lambda x$$

where A is the given matrix. Bring λx to the left side and combine the terms.

$$(A - \lambda I)x = 0$$

The eigenvalues satisfy

$$\det(A - \lambda I) = 0$$

Evaluate the determinant and solve for λ.

$$\det \begin{pmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{pmatrix} = 0$$
$$(1-\lambda)(-1-\lambda) - 3 = 0$$
$$\lambda^2 - 4 = 0$$
$$(\lambda + 2)(\lambda - 2) = 0$$
$$\lambda = \{-2, 2\}$$

Therefore, the eigenvalues are

$$\lambda_1 = -2 \text{ and } \lambda_2 = 2$$

Substitute λ_1 and λ_2 back into equation (1) to determine the corresponding eigenvectors, x_1 and x_2

$$(A - \lambda_1 I)x_1 = 0$$
$$\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$\left. \begin{matrix} 3x_1 + \sqrt{3}x_2 = 0 \\ \sqrt{3}x_1 + x_2 = 0 \end{matrix} \right\}$$

$$x_2 = -\sqrt{3}x_1$$
$$x_1 = \begin{pmatrix} x_1 \\ -\sqrt{3}x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

and

$$(A - \lambda_2 I)x_2 = 0$$

$$\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

$$\left. \begin{array}{c} -x_1 + \sqrt{3}x_2 = 0 \\ \sqrt{3}x_1 - 3x_2 = 0 \end{array} \right\}$$

$$x_2 = \frac{1}{\sqrt{3}} x_1$$

$$x_2 = \begin{pmatrix} x_1 \\ \frac{1}{\sqrt{3}} x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Note that x_1 is a free variable, or arbitrary constant.

Problem 18

The aim is to solve the eigenvalue problem,

$$Ax = \lambda x$$

where A is the given matrix. Bring λx to the left side and combine the terms.

$$(A - \lambda I)x = 0$$

The eigenvalues satisfy

$$\det(A - \lambda I) = 0$$

Evaluate the determinant and solve for λ.

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)(1-\lambda)+4] = 0$$

$$1-\lambda = 0 \quad \text{or} \quad \lambda^2 - 2\lambda + 5 = 0$$

$$5 - 7\lambda + 3\lambda^2 - \lambda^3 = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$\lambda = 1 \quad \text{or} \quad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Therefore, the eigenvalues are

$\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 1 - 2i}$ and $\boxed{\lambda_3 = 1 + 2i}$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector x_1.

$$(\mathrm{A} - \lambda_1 \mathrm{I})\mathrm{x}_1 = 0$$

$$\begin{pmatrix} 1-(1) & 0 & 0 \\ 2 & 1-(1) & -2 \\ 3 & 2 & 1-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Write the implied system of equations.

$$\left. \begin{array}{r} 2x_1 - 2x_3 = 0 \\ 3x_1 + 2x_2 = 0 \end{array} \right\}$$

Solve for x_2 and x_3 in terms of the free variable x_1.

$$x_3 = x_1$$

$$x_2 = -\frac{3}{2} x_1$$

This means

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -\frac{3}{2}x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{pmatrix}$$

Since x_1 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\mathbf{x}_1 = x_1' \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

Problem 20 [FOR GRADE]

The aim is to solve the eigenvalue problem,

$$\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$$

where A is the given matrix. Bring $\lambda\mathbf{x}$ to the left side and combine the terms.

$$(\mathrm{A} - \lambda\mathrm{I})\mathbf{x} = 0$$

The eigenvalues satisfy

$$\det(\mathrm{A} - \lambda\mathrm{I}) = 0$$

Evaluate the determinant and solve for λ.

$$\det\begin{pmatrix} 11/9 - \lambda & -2/9 & 8/9 \\ -2/9 & 2/9 - \lambda & 10/9 \\ 8/9 & 10/9 & 5/9 - \lambda \end{pmatrix} = 0$$

$$(11/9 - \lambda)[(2/9 - \lambda)(5/9 - \lambda) - 100/81] + (2/9)[(-2/9)(5/9 - \lambda) - 80/81]$$
$$+ (8/9)[-20/81 - (8/9)(2/9 - \lambda)] = 0$$

$$(11/9 - \lambda)\begin{vmatrix} 2/9 - \lambda & 10/9 \\ 10/9 & 5/9 - \lambda \end{vmatrix} - (-2/9)\begin{vmatrix} -2/9 & 10/9 \\ 8/9 & 5/9 - \lambda \end{vmatrix} + (8/9)\begin{vmatrix} -2/9 & 2/9 - \lambda \\ 8/9 & 10/9 \end{vmatrix} = 0$$

$$-2 + \lambda + 2\lambda^2 - \lambda^3 = 0$$

$$(\lambda + 1)(\lambda - 1)(2 - \lambda) = 0$$

Therefore, the eigenvalues are

$$\begin{array}{|c|c|c|} \hline \lambda_1 = 1 & \text{and} & \lambda_2 = 2 \\ \hline \text{and} & \lambda_3 = -1 \\ \hline \end{array}$$

Substitute λ_1 back into equation (1) to determine the corresponding eigenvector x_1.

$$(A - \lambda_1 I)x_1 = 0$$

$$\begin{pmatrix} 11/9-(1) & -2/9 & 8/9 \\ -2/9 & 2/9-(1) & 10/9 \\ 8/9 & 10/9 & 5/9-(1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2/9 & -2/9 & 8/9 \\ -2/9 & -7/9 & 10/9 \\ 8/9 & 10/9 & -4/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} 2/9 & -2/9 & 8/9 & 0 \\ -2/9 & -7/9 & 10/9 & 0 \\ 8/9 & 10/9 & -4/9 & 0 \end{array} \right)$$

Multiply each row by 9

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ -2 & -7 & 10 & 0 \\ 8 & 10 & -4 & 0 \end{array} \right)$$

Multiply the first row by -4 and add it to the third row.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ -2 & -7 & 10 & 0 \\ 0 & 18 & -36 & 0 \end{array} \right)$$

Add the first row to the second row.

$$\left(\begin{array}{ccc|c} 2 & -2 & 8 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 18 & -36 & 0 \end{array} \right)$$

Write the implied system of equations and solve for x_1 and x_2 in terms of the free variable

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x_3

$$\left.\begin{array}{r} 2x_1 - 2x_2 + 8x_3 = 0 \\ -9x_2 + 18x_3 = 0 \\ 18x_2 - 36x_3 = 0 \end{array}\right\} \quad \rightarrow \quad \begin{array}{l} x_1 = -2x_3 \\ x_2 = 2x_3 \end{array}$$

This means

$$x_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{pmatrix}$$

Therefore,

$$x_1 = x_3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

Substitute λ_2 back into equation (1) to determine the corresponding eigenvector x_2.

$$(A - \lambda_2 I)x_2 = 0$$
$$\begin{pmatrix} 11/9 - (2) & -2/9 & 8/9 \\ -2/9 & 2/9 - (2) & 10/9 \\ 8/9 & 10/9 & 5/9 - (2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -7/9 & -2/9 & 8/9 \\ -2/9 & -16/9 & 10/9 \\ 8/9 & 10/9 & -13/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write the augmented matrix.

$$\left(\begin{array}{ccc|c} -7/9 & -2/9 & 8/9 & 0 \\ -2/9 & -16/9 & 10/9 & 0 \\ 8/9 & 10/9 & -13/9 & 0 \end{array} \right)$$

Multiply each row by 9

$$\left(\begin{array}{ccc|c} -7 & -2 & 8 & 0 \\ -2 & -16 & 10 & 0 \\ 8 & 10 & -13 & 0 \end{array} \right)$$

Multiply the second row by 4 and add it to the third row.

$$\left(\begin{array}{ccc|c} -7 & -2 & 8 & 0 \\ -2 & -16 & 10 & 0 \\ 0 & -54 & 27 & 0 \end{array}\right)$$

Multiply the first row by -8 and add it to the second row.

$$\left(\begin{array}{ccc|c} -7 & -2 & 8 & 0 \\ 54 & 0 & -54 & 0 \\ 0 & -54 & 27 & 0 \end{array}\right)$$

Write the implied system of equations and solve for x_1 and x_2 in terms of the free variable x_3

$$\left.\begin{array}{r} -7x_1-2x_2+8x_3=0 \\ 54x_1-54x_3=0 \\ -54x_2+27x_3=0 \end{array}\right\} \quad \rightarrow \quad \begin{array}{l} x_1=x_3 \\ x_2=\dfrac{1}{2}x_3 \end{array}$$

This means

$$\mathbf{x}_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

Since x_3 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$\mathbf{x}_2 = x_3' \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Substitute λ_3 back into equation (1) to determine the corresponding eigenvector $\mathbf{x}_3$.

$$(A-\lambda_3 I)\mathbf{x}_3 = 0$$

$$\begin{pmatrix} 11/9-(-1) & -2/9 & 8/9 \\ -2/9 & 2/9-(-1) & 10/9 \\ 8/9 & 10/9 & 5/9-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 20/9 & -2/9 & 8/9 \\ -2/9 & 11/9 & 10/9 \\ 8/9 & 10/9 & 14/9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Write the augmented matrix.

$$
\left(
\begin{array}{ccc|c}
20/9 & -2/9 & 8/9 & 0 \\
-2/9 & 11/9 & 10/9 & 0 \\
8/9 & 10/9 & 14/9 & 0
\end{array}
\right)
$$

Multiply each row by 9

$$
\left(
\begin{array}{ccc|c}
20 & -2 & 8 & 0 \\
-2 & 11 & 10 & 0 \\
8 & 10 & 14 & 0
\end{array}
\right)
$$

Multiply the second row by 4 and add it to the third row.

$$
\left(
\begin{array}{ccc|c}
20 & -2 & 8 & 0 \\
-2 & 11 & 10 & 0 \\
0 & 54 & 54 & 0
\end{array}
\right)
$$

Multiply the second row by 10 and add it to the first row.

$$
\left(
\begin{array}{ccc|c}
0 & 108 & 108 & 0 \\
-2 & 11 & 10 & 0 \\
0 & 54 & 54 & 0
\end{array}
\right)
$$

Write the implied system of equations and solve for x_1 and x_3 in terms of the free variable x_2

$$
\left.
\begin{array}{r}
108x_2 + 108x_3 = 0 \\
-2x_1 + 11x_2 + 10x_3 = 0 \\
54x_2 + 54x_3 = 0
\end{array}
\right\}
\quad \rightarrow \quad
\begin{array}{l}
x_1 = \frac{1}{2}x_2 \\
x_3 = -x_2
\end{array}
$$

This means

$$
\mathrm{x}_3 = \left(
\begin{array}{c}
x_1 \\
x_2 \\
x_3
\end{array}
\right)
=
\left(
\begin{array}{c}
\frac{1}{2}x_2 \\
x_2 \\
-x_2
\end{array}
\right)
= x_2
\left(
\begin{array}{c}
\frac{1}{2} \\
1 \\
-1
\end{array}
\right)
$$

Since x_2 is arbitrary, the eigenvector can be multiplied by 2 to get rid of the fraction.

$$
\mathrm{x}_3 = x_2'
\left(
\begin{array}{c}
1 \\
2 \\
-2
\end{array}
\right)
$$

Section 7.4

Problem 5(a)

$$t\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} 2t - t \\ 3t - 2t \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix}$$

$$t\begin{pmatrix} -t^{-2} \\ -3t^{-2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} t^{-1} \\ 3t^{-1} \end{pmatrix} = \begin{pmatrix} 2t^{-1} - 3t^{-1} \\ 3t^{-1} - 6t^{-1} \end{pmatrix} = \begin{pmatrix} -t^{-1} \\ -3t^{-1} \end{pmatrix}$$

Problem 6(a) [FOR GRADE]

$$t\begin{pmatrix} -t^{-2} \\ -2t^{-2} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} t^{-1} \\ 2t^{-1} \end{pmatrix} = \begin{pmatrix} 3t^{-1} - 4t^{-1} \\ 2t^{-1} - 4t^{-1} \end{pmatrix} = \begin{pmatrix} -t^{-1} \\ -2t^{-1} \end{pmatrix}$$

$$t\begin{pmatrix} 4t \\ 2t \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2t^2 \\ t^2 \end{pmatrix} = \begin{pmatrix} 6t^2 - 2t^2 \\ 4t^2 - 2t^2 \end{pmatrix} = \begin{pmatrix} 4t^2 \\ 2t^2 \end{pmatrix}$$