Homework 9 Oracle

MATH 220 Spring 2021

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Section 7.5

Problem 9 (this is graded)

Let us solve the following system of ODEs

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \mathbf{x}$$

Solution Let $\mathbf{A}$ be the matrix above, so let us find the characteristic polynomial of our system

$$\det(\mathbf{A} - \lambda \mathbf{I}_3) = \det \begin{pmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{pmatrix} = -\lambda^3 + 2\lambda^2 + 5\lambda - 6 = 0$$

$$\implies \lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -2 \quad \text{and} \quad \lambda_3 = 3$$

Substitute the eigenvalues above to find the corresponding eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

$$\lambda = \lambda_1 \implies \begin{pmatrix} 1-\lambda_1 & -1 & 4 \\ 3 & 2-\lambda_1 & -1 \\ 2 & 1 & -1-\lambda_1 \end{pmatrix} \mathbf{v}_1 \implies \mathbf{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

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$$\lambda = \lambda_2 \implies \begin{pmatrix} 1-\lambda_2 & -1 & 4 \\ 3 & 2-\lambda_2 & -1 \\ 2 & 1 & -1-\lambda_2 \end{pmatrix} \mathbf{v_2} \implies \mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = \lambda_3 \implies \begin{pmatrix} 1-\lambda_3 & -1 & 4 \\ 3 & 2-\lambda_3 & -1 \\ 2 & 1 & -1-\lambda_3 \end{pmatrix} \mathbf{v_3} \implies \mathbf{v_3} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Therefore the final solution has the form

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{\lambda_1 t}\mathbf{v_1} + C_2 e^{\lambda_2 t}\mathbf{v_2} + C_3 e^{\lambda_3 t}\mathbf{v_3} \\ &= C_1 e^{t} \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{3t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \end{aligned}$$

where C_1 and C_2 and C_3 are arbitrary constants.

Problem 10

The solution strategy is the same as **Problem 9**. We find that eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$ with their corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, such that the solution is

$$\mathbf{x}(t) = C_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

By knowing the initial value $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, we solve the system

$$\begin{cases} C_1 + C_2 = 2 \\ C_1 + 3C_2 = -1 \end{cases} \implies \begin{cases} C_1 = \frac{7}{2} \\ C_2 = -\frac{3}{2} \end{cases}$$

Finally, the solution to the initial value problem is

$$\mathbf{x}(t) = \frac{7}{2} e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + -\frac{3}{2} e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Problem 15 (this is graded)

Recall the eigenvalues and their eigenvectors we found in **Problem 10**. Applying **Problem 13**, the solution is

$$\mathbf{x}(t) = C_1 t^4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 t^2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Problem 16

Similarly to all problems above, we find eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$, where their respective eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Applying **Problem 13**, we have the solution

$$\mathbf{x}(t) = C_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 t^{-2} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Section 7.6

Problem 9 and 10

Solutions are omitted for the sake of expediting the grading before Midterm II.